Example 2

Given the conditional probability

$$p(z|m_1)=\begin{cases}\frac{3}{2}(1-z)^2 & 0\leq z\leq 2\\ 0 & otherwise\end{cases},\quad p(z|m_2)=\begin{cases}\frac{3}{4}z(2-z) & 0\leq z\leq 2\\ 0 & otherwise\end{cases}$$

The likelihood ratio

$$\Lambda(z)=\frac{\frac{3}{3}z(2-z)}{\frac{3}{2}(1-z)^2}=\frac{z(2-z)}{2(1-z)^2}=\frac{1}{2}\left[\frac{1}{(1-z)^2}-1\right]\quad where\, z(2-z)=1-(1-z)^2$$

If ζ is defined by $\Lambda(\zeta)=\lambda$, then the decision regions are given by

$$\Lambda(z)=\frac{1}{2}\left[\frac{1}{(1-z)^2}-1\right]\underset{d_1}{\overset{d_2}{\gtrless}}\lambda$$

$$Z_1=\{z:\Lambda(z)<\lambda\},$$
$$Z_2=\{z:\Lambda(z)>\lambda\}$$

$$Z_1=\{z:0\leq z\leq\zeta\ \text{or}\ 12-\zeta\leq z\leq 2\},$$
$$Z_2=\{z:\zeta\leq z\leq 2-\zeta\}$$

Now we must select λ and hence ζ such that $P(d_2|m_1)=\alpha_0$ in order to be able to calculate a specific value of λ.

The error probability

$$P(d_2|m_1)=\int_{\zeta}^{2-\zeta}\frac{3}{2}(1-z)^2dz=(1-\zeta)^3=0.2 \Rightarrow \zeta=0.415\quad 0.2^{1/3}=0.5848$$

주의>> 0.2^(1/3)

ans =   0.58480

$$Z_1=\{z:0\leq z\leq 0.5848\ \text{or}\ 2-0.5848\leq z\leq 2\}$$
$$Z_2=\{z:0.5848\leq z\leq 2-0.5848\}$$

$$Z_1=\{z:|z-1|>0.5848\}$$
$$Z_2=\{z:|z-1|<0.5848\}$$

It is not necessary to calculate λ in order to determine the decision rule.

One would like to simultaneously maximize $P(d_2|m_2)$ and minimize $P(d_2|m_1)$.
However it is not possible!
The Neyman-Pearson criterion was offered as an approach to the design of decision rules.
By the use of the Neyman-Pearson criterion, the decision rule is completely specified by the selection of α_0; however, no logical procedure has been provided for selecting α_0.

The use of a receiver operating characteristic (ROC) can provide some guidance in the selection of α_0.

The receiver operating characteristic is a plot of the probability of detection (power) $P(d_2|m_2)$ for the Neyman-Pearson decision rule versus the probability of false alarm (level of significance) $P(d_2|m_1)$ as a function of one or more parameters.

The ROC may be obtained by determining $P(d_2|m_2)$ as a function $\alpha_0 = P(d_2|m_1)$ and varying α_0 from zero to one.

Alternatively, one may determine both $P(d_2|m_2)$ and $P(d_2|m_1)$ as functions of the threshold λ and then vary λ from zero to infinity.

Example 3

In order to illustrate this procedure, let us determine the ROC for the problem of Example 1. Here, however, we will assume that the expected value of z when m_2 is true is $\mu > 0$ rather than unity. Hence the conditional densities of z are given by

$$p(z|m_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},$$

$$p(z|m_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2}}$$

The Neymann-Pearson rule

$$z \underset{d_2}{\overset{d_1}{\gtrless}} \frac{\ln\lambda}{\mu} + \frac{\mu}{2}$$

The false-alarm probability

$$P(d_2|m_1) = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln\lambda}{\mu}+\frac{\mu}{2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = Q(\frac{\ln\lambda}{\mu} + \frac{\mu}{2})$$

The probability of detection

$$P(d_2|m_2) = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln\lambda}{\mu}+\frac{\mu}{2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2}} dz$$

Let $v = z - \mu$

$$P(d_2|m_2) = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln\lambda}{\mu}-\frac{\mu}{2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv = Q(\frac{\ln\lambda}{\mu} - \frac{\mu}{2}).$$

For a given value μ, we may vary λ from zero to infinity and obtain $P(d_2|m_1)$ and $P(d_2|m_2)$ can be obtained for $\mu = 0, 1, 2$.

Example 4

Determine the ROC for the decision problem with the conditional probability densities

$$p(z|m_1) = \begin{cases} e^{-z} & z \geqq 0 \\ 0 & otherwise \end{cases}$$

$$p(z|m_2) = \begin{cases} \tau e^{-\tau z} & z \geqq 0 \\ 0 & otherwise \end{cases}$$

where $\tau > 1$.

Sol.

The likelihood ratio for the problem is given by

$$\Lambda(z) = \frac{p(z|m_2)}{p(z|m_1)} = \tau e^{-(\tau-1)z}$$

and the Neyman-Pearson decision rule is

$$\tau e^{-(\tau-1)z} \underset{d_1}{\overset{d_2}{\gtrless}} \lambda \Rightarrow z \underset{d_1}{\overset{d_2}{\gtrless}} \frac{1}{1-\tau} ln(\frac{\lambda}{\tau}) = \tau'$$

Hence the false-alarm probability is given by

$$P(d_2|m_1) = \int_0^{\lambda'} e^{-z} dz = 1 - e^{-\lambda'}$$

while the probability of detection is

$$P(d_2|m_2) = \int_0^{\lambda'} \tau e^{-\tau z} dz = 1 - e^{-\tau\lambda'}$$

The slope of the ROC is

$$\frac{dP(d_2|m_2)}{dP(d_2|m_1)} = \frac{dP(d_2|m_2)/d\lambda'}{dP(d_2|m_1)/d\lambda'} = \frac{\tau e^{\tau\lambda'}}{e^{-\lambda'}} = \tau e^{-(\tau-1)\lambda'}$$

From the definition of λ' we have

$$\lambda = \tau e^{-(\tau-1)\lambda'}$$

and so the slope is exactly λ.

$$\frac{P(d_2|m_2)}{P(d_2|m_1)} = \frac{1-e^{-\tau\lambda'}}{1-e^{-\lambda'}}, \quad \frac{1}{1-\tau} ln(\frac{\lambda}{\tau}) = \tau'$$

$\lambda = \tau e^{-(\tau-1)\lambda'}$ with $\frac{1}{1-\tau} ln(\frac{\lambda}{\tau}) = \tau'$

$$
\begin{aligned}
&\tau e^{-(\tau-1)(\frac{1}{1-\tau})\ln\frac{\lambda}{\tau}} \\
&= \tau e^{\ln(\frac{\lambda}{\tau})} \\
&= \tau(\frac{\lambda}{\tau}) \\
&= \lambda
\end{aligned}
$$

The use of ROC for determining the α_0 parameter of the Neyman-Pearson decision rule.

- By examining the ROC associated with a given decision problem, we can obtain some feeling of the trade-off of decreasing $P(d_2|m_1)$ versus increasing $P(d_2|m_2)$.

- In general, we would like to select a value for $P(d_2|m_2)$ which generates a test at the "knee" of the ROC.

- If $P(d_2|m_1)$ is increased above *that value*, this will result in a smaller increase in $P(d_2|m_2)$ and hence a poor trade-off.

- Correspondingly, if $P(d_2|m_1)$ is decreased, $P(d_2|m_2)$ will decrease even more.

- For example, consider the ROC above. If $\tau = 16$, then a reasonable value for $P(d_2|m_1)$ would be 0.1 to 0.2.

- Making $P(d_2|m_1)$ <u>larger than 0.2</u> will cause only <u>a small increase</u> in $P(d_2|m_2)$, while $P(d_2|m_1)$ <u>smaller than 0.1</u> will cause <u>a large decrease</u> in $P(d_2|m_2)$.

- There are problems in which it may be necessary to make $P(d_2|m_1)$ smaller than the value at the knee.

- In this case, one must simply pay the penalty in a reduced value of $P(d_2|m_2)$ or try to modify the problem in some way, for example, increasing τ in the previous example.

- Note that the ROC also gives us some information concerning the effect of changing various parameters.

- In example, increasing τ above 16 would be desirable only if a very small value of $P(d_2|m_1)$ is required.