PROBABILITY OF ERROR CRITERION

- $$\begin{aligned} P_e &= P(make\ an\ \in correct\ decision) \\ &= P(decide\, d_2\, when\, m_1\, is\, true\ \text{or}\ decide\, d_1\, when\, m_2\, is\, true) \\ &= P(d_2\, \text{and}\, m_1)\ \text{or}\ (d_1\, \text{and}\, m_2) \\ &= P(d_2, m_1) + P(d_1, m_2) \\ &= P(d_2|m_1)P(m_1) + P(d_1|m_2)P(m_2) \end{aligned}$$

- Here $P(m_1)$ and $P(m_2)$ are referred to as the a priori probabilities;

- $P(m_1) + P(m_2) = 1.0$
- The a priori probabilities are dependent on the source coding procedure used.
- Probability-of-error criterion => the ideal observer criterion

- $$\begin{aligned} P(d_1|m_2) &= \int_{Z_1} p(z|m_2)dz \\ &= \int_{Z_1} p(z|m_2)dz + \int_{Z_2} p(z|m_2)dz - \int_{Z_2} p(z|m_2)dz \\ &= 1 - \int_{Z_2} p(z|m_2)dz \\ &= 1 - P(d_2|m_2) \end{aligned}$$
- We have been able to write $P(d_1|m_2)$ as a function of Z_2 .
- To write $P(d_1|m_2)$ as a function of Z_2, we use
$$P(d_2|m_1) = \int_{Z_2} p(z|m_1)dz$$

- $$\begin{aligned} P(d_2|m_1) &= P\{z \in Z_2 | m_1\} = \int_{Z_2} p(z|m_1)dz \\ P(d_1|m_2) &= P\{z \in Z_1 | m_2\} = \int_{Z_1} p(z|m_2)dz \\ P(d_1|m_1) &= P\{z \in Z_1 | m_1\} = \int_{Z_1} p(z|m_1)dz \\ P(d_2|m_2) &= P\{z \in Z_2 | m_2\} = \int_{Z_2} p(z|m_2)dz \end{aligned}$$

- $$\begin{aligned} P(d_1|m_1) + P(d_2|m_1) &= 1.0 \\ P(d_1|m_2) + P(d_2|m_2) &= 1.0 \end{aligned}$$

$$
\begin{aligned}
P_e = &\ P(d_2|m_1)P(m_1) + P(d_1|m_2)P(m_2) \\
= &\ P(m_1)\int_{Z_2} p(z|m_1)dz + P(m_2)\int_{Z_1} p(z|m_2)dz \\
= &\ P(m_1)\int_{Z_2} p(z|m_1)dz + P(m_2)(1 - \int_{Z_2} p(z|m_2)dz \\
= &\ P(m_1) + \int_{Z_2} \left[P(m_1)p(z|m_1)dz - P(m_2)p(z|m_2) \right] dz.
\end{aligned}
$$

To minimize P_e by the selection of Z_2, we should put into the decision region Z_2 the value of z for which the integrand in the above equation is negative!

Therefore Z_2 is given by

$$Z_2 = \{ z : [P(m_1)p(z|m_1) - P(m_2)p(z|m_2)] < 0 \}$$

and Z_1 is the set z's not in Z_2, or

$$Z_1 = \{ z : [P(m_1)p(z|m_1) - P(m_2)p(z|m_2)] > 0 \}$$

In terms of the likelihood ratio

$$\Lambda(z) = \frac{p(z|m_1)}{p(z|m_2)}$$

the decision regions can be written as

$$Z_2 = \left\{ z : \Lambda(z) > \frac{P(m_1)}{P(m_2)} \right\}$$

$$Z_1 = \left\{ z : \Lambda(z) < \frac{P(m_1)}{P(m_2)} \right\}$$

and the decision rule is

$$\Lambda(z) \underset{d_1}{\overset{d_2}{\gtrless}} \frac{P(m_1)}{P(m_2)}$$

The decision rule consists of comparing the likelihood ratio to the threshold.
The threshold is determined by the ratio of a priori probabilities.

예제3.3-1

조건 확률밀도함수가 다음과 같이 주어졌을 때

$$p(z|m_1) = \frac{1}{\sqrt{2\pi}} exp(-\frac{z^2}{2})$$

$$p(z|m_2) = \frac{1}{\sqrt{2\pi}} exp(-\frac{(z-1)^2}{2}), \qquad P(m_1) = 0.25,\ P(m_2) = 0.75$$

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$$\Lambda(z) = \frac{p(z|m_2)}{p(z|m_1)} \qquad (\underset{d_1}{\overset{d_2}{\gtrless}} \frac{P(m_1)}{P(m_2)} = \frac{0.25}{0.75} = \frac{1}{3})$$

$$= \frac{\frac{1}{\sqrt{2\pi}} exp(-\frac{(z-1)^2}{2})}{\frac{1}{\sqrt{2\pi}} exp(-\frac{z^2}{2})}$$

$$= \exp(-\frac{z^2-2z+1}{2} + \frac{z^2}{2})$$

$$= \exp(z - \frac{1}{2})$$

$$= \exp(\frac{2z-1}{2})$$

$$\ln\Lambda(z) = \frac{2z-1}{2} \underset{d_1}{\overset{d_2}{\gtrless}} \ln(\frac{1}{3}) = -1.0986 = -1.1$$

$$z = (-2.2+1)/2 = -0.6$$

The decision region Z_1 is $Z_1 = \{z : z > -0.6\}$.

The decision region Z_2 is $Z_1 = \{z : z < -0.6\}$.

$$P(d_1|m_2) = \int_{Z_1} p(z|m_1)dz = \int_{-\infty}^{-0.6} \frac{1}{\sqrt{2\pi}} exp(-\frac{(z-1)^2}{2})dz = 0.0048$$

$$P(d_2|m_1) = \int_{Z_2} p(z|m_1)dz = \int_{-0.6}^{\infty} \frac{1}{\sqrt{2\pi}} exp(-\frac{z^2}{2})dz = 0.7258$$

$$\begin{aligned} P_e &= P(d_2|m_1)P(m_1) + P(d_1|m_2)P(m_2) \\ &= 0.25(0.7258) + 0.75(0.0048) \\ &= 0.1851 \end{aligned}$$

The maximum Likelihood

$$P(d_1|m_2) = \int_{Z_1} p(z|m_1)dz = \int_{-\infty}^{0.5} \frac{1}{\sqrt{2\pi}} exp(-\frac{(z-1)^2}{2})dz = 0.3085$$

$$P(d_2|m_1) = \int_{Z_2} p(z|m_1)dz = \int_{0.5}^{\infty} \frac{1}{\sqrt{2\pi}} exp(-\frac{z^2}{2})dz = 0.3085$$

$P(m_1) = 0.25,\ P(m_2) = 0.75$

$$\begin{aligned} P_e = &\ P(d_2|m_1)P(m_1) + P(d_1|m_2)P(m_2) \\ = &\ 0.25(0.3085) + 0.75(0.3085) \\ = &\ 0.3085 \end{aligned}$$

The Neyman-Pearson Decision

The error probability $P(d_2|m_1)$ is constrained to be 0.25 and

$$P(d_1|m_2) = \int_{Z_1} p(z|m_1)dz = \int_{-\infty}^{-0.674} \frac{1}{\sqrt{2\pi}} exp(-\frac{(z-1)^2}{2})dz = 0.3722$$

$$P_e = \ P(d_2|m_1)P(m_1) + P(d_1|m_2)P(m_2)$$

$$= 0.25(0.25) + 0.75(0.3722)$$

$$= 0.3417$$

- Both the maximum-likelihood and Neyman-Pearson decision rules lead to a higher value for P_e.
- Neither of these decision rules was based on minimizing P_e.
- If, however, we had selected a Neyman-Pearson decision rule with $\alpha_0 = 0.7258$, then the Neyman-Pearson and the probability-of-error decision rules would be equivalent.
- In general, we do not know the value of α_0 which will establish this equivalence until the probability-of-error decision rule is determined.
- Such a value always exists.

If $P(m_1) = 0.5$, $P(m_2) = 0.5$, then the probability-of-error decision rule is equivalent to the maximum-likelihood decision rule.

The maximum-likelihood decision rule will yield a minimum probability-of-error decision only if the two messages are equally likely.

$$\Lambda(z) \underset{d_1}{\overset{d_2}{\gtrless}} \frac{P(m_1)}{P(m_2)} \Rightarrow \Lambda(z) \underset{d_1}{\overset{d_2}{\gtrless}} 1.0$$

$$\Lambda(z) = \frac{p(z|m_2)}{p(z|m_1)} \underset{d_1}{\overset{d_2}{\gtrless}} \frac{P(m_1)}{P(m_2)}$$

$$\frac{p(z|m_2)}{p(z|m_1)} \frac{P(m_2)}{P(m_1)} \underset{d_1}{\overset{d_2}{\gtrless}} \frac{P(m_1)}{P(m_2)} \frac{P(m_2)}{P(m_1)} = 1.0$$

$$\frac{p(z)}{p(z)} \frac{P(m_2|z)}{P(m_1|z)} = \frac{P(m_2|z)}{P(m_1|z)} \underset{d_1}{\overset{d_2}{\gtrless}} 1.0 \Rightarrow \begin{cases} d_2 : P(m_2|z) > P(m_1|z) \\ d_1 : P(m_2|z) < P(m_1|z) \end{cases}$$

Choose the message with the larger a posteriori probability:
the probability of m_k given z

The probability-of-error criterion is identical to the maximum a posteriori (MAP) decision criterion.

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$$d(z) = \begin{cases} d_1 & p(m_1|z) > p(m_2|z) \\ d_2 & p(m_1|z) < p(m_2|z) \end{cases}$$

Select the message with the larger posteriori probability

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MAP decision criterion

Given an observation z, select d_1 if m_1 is more likely than m_2

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Maximum-likelihood decision criterion

Given an observation $z \in Z$, let $d(z) = d_1$ if it is more likely that m_1 generated z than that m_2 generated z.

$$d(z) = \begin{cases} d_1 & p(z|m_1) > p(z|m_2) \\ d_2 & p(z|m_1) < p(z|m_2) \end{cases}$$

$p(z|m_1)$: Probability of receiving an observation in the range $(z, z+dz)$ when the message is m_1

$p(z|m_2)$: Probability of receiving an observation in the range $(z, z+dz)$ when the message is m_2

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$$p(z|m_1) = \frac{1}{\sqrt{2\pi}} exp(-\frac{(z-\mu_1)^2}{2\sigma^2})$$

$$p(z|m_2) = \frac{1}{\sqrt{2\pi}} exp(-\frac{(z-\mu_2)^2}{2\sigma^2})$$

$(\mu_2 - \mu_1) > 0$

$$\Lambda(z) = \frac{\exp[-(z-\mu_2)^2/2\sigma^2]}{\exp[-(z-\mu_1)^2/2\sigma^2}$$
$$= \exp[\frac{2z(\mu_2-\mu_1)-(\mu_2^2-\mu_1^2)}{2\sigma^2}]$$

$$\exp[\frac{2z(\mu_2-\mu_1)-(\mu_2^2-\mu_1^2)}{2\sigma^2}] \begin{array}{c} d_2 \\ < \\ > \\ d_1 \end{array} \frac{P(m_1)}{P(m_2)}$$

$$\Lambda(z) = \frac{P(m_1)}{P(m_2)}$$

$$\frac{2z(\mu_2-\mu_1)-(\mu_2^2-\mu_1^2)}{2\sigma^2} \begin{array}{c} d_2 \\ < \\ > \\ d_1 \end{array} \ln\Lambda(z)$$

$$z \begin{array}{c} d_2 \\ < \\ > \\ d_1 \end{array} [2\sigma^2 \ln\Lambda(z)+(\mu_2^2-\mu_1^2)]/[2(\mu_2-\mu_1)]$$

$$z \begin{array}{c} d_2 \\ < \\ > \\ d_1 \end{array} \sigma[\frac{\sigma\ln\Lambda(z)}{\mu_2-\mu_1}+\frac{(\mu_2+\mu_1)}{\sigma}]=\eta$$

$$P(d_2|m_1) = \int_\eta^\infty \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(z-\mu_1)^2}{2\sigma^2}\right)dz$$

$$\frac{z-\mu_1}{\sigma}=\xi,\ z=\xi\sigma+\mu_1=\eta,\ \xi=\frac{\eta-\mu_1}{\sigma},\ dz=\sigma d\xi$$
$$P(d_2|m_1) = \int_{\frac{\eta-\mu_1}{\sigma}}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} exp(\xi)\sigma d\xi = Q(\frac{\sigma}{\mu_2-\mu_1} ln\Lambda_0 + \frac{\mu_2-\mu_1}{\sigma})$$

$$\delta = \frac{\mu_2 - \mu_1}{\sigma}$$

$$\begin{aligned} P(d_2|m_1) &= \int_{\frac{\eta-\mu_1}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} exp(\xi)\sigma d\xi \\ &= Q(\frac{\sigma}{\mu_2-\mu_1} ln\Lambda_0 + \frac{\mu_2-\mu_1}{\sigma}) \\ &= Q(\frac{\delta}{2} + \frac{\ln\Lambda_0}{\delta}) \end{aligned}$$

$$P(d_1|m_2) = Q(\frac{\delta}{2} - \frac{\ln\Lambda_0}{\delta})$$

$$\begin{aligned} P_e &= P(d_2|m_1)P(m_1) + P(d_1|m_2)P(m_2) \\ &= P(m_1)\,P(d_2|m_1) + P(m_2)P(d_1|m_2) \\ &= P(m_1)\,Q(\frac{\delta}{2} + \frac{\ln\Lambda_0}{\delta}) + P(m_2)Q(\frac{\delta}{2} - \frac{\ln\Lambda_0}{\delta}) \end{aligned}$$

$$\Lambda_0 = \frac{P(m_1)}{P(m_2)}, \quad P(m_1) + P(m_2) = 1.0$$

$$\Lambda_0 = \frac{P(m_1)}{1-P(m_1)},\ \Lambda_0 - \Lambda_0 P(m_1) = P(m_1), \quad \Lambda_0 = P(m_1)(1+\Lambda_0)$$

$$P(m_1) = \frac{\Lambda_0}{1+\Lambda_0},$$

$$P(m_2) = 1.0 - P(m_1) = 1.0 - \frac{\Lambda(z)}{1+\Lambda(z)} = \frac{1}{1+\Lambda(z)}$$

$$
\begin{aligned}
P_e =&\ P(d_2|m_1)P(m_1)+P(d_1|m_2)P(m_2) \\
=&\ P(m_1)\,P(d_2|m_1)+P(m_2)P(d_1|m_2) \\
=&\ \frac{\Lambda_0}{1+\Lambda_0}\,Q(\frac{\delta}{2}+\frac{\ln\Lambda_0}{\delta})+\frac{1}{1+\Lambda_0}\,Q(\frac{\delta}{2}-\frac{\ln\Lambda_0}{\delta}) \\
=&\ \frac{1}{1+\Lambda_0}\left[\Lambda_0\,Q(\frac{\delta}{2}+\frac{\ln\Lambda_0}{\delta})+Q(\frac{\delta}{2}-\frac{\ln\Lambda_0}{\delta})\right]
\end{aligned}
$$

The minimum probability of error depends only on Λ_0 and δ.

SNR(signal-to-noise-ratio) : $\delta=\frac{\mu_2-\mu_1}{\sigma}$.

For any given Λ_0, P_e decrease as δ increase.

SNR이 증가할수록, 오류확률은 줄어든다.