Bayes Risk Criterion

Three as hoc methods: ML, Neyman-Pearson, Probability of Error, MAP
Bayes decision criterion:
1. Assigning a cost to correct and incorrect decision
2. Minimizing the total average cost

C_{11} = Cost of deciding d_1 when m_1 is true

C_{12} = Cost of deciding d_1 when m_2 is true

C_{21} = Cost of deciding d_2 when m_1 is true

C_{22} = Cost of deciding d_2 when m_2 is true

The number of messages
The number of decisions

The expected and average cost

$$\begin{aligned} B = E(C_{jk}) &= C_{11}P(d_1,m_1) + C_{21}P(d_2,m_1) + C_{12}P(d_1,m_2) + C_{22}P(d_2,m_2) \\ &= [C_{11}P(d_1|m_1) + C_{21}P(d_2|m_1)]P(m_1) + [C_{12}P(d_1|m_2) + C_{22}P(d_2|m_2)]P(m_2) \\ &= B_1P(m_1) + B_2P(m_2) \end{aligned}$$

$B_{1,}B_2$: conditional costs

$$B_1 = C_{11}P(d_1|m_1) + C_{21}P(d_2|m_1)$$

$$B_2 = C_{12}P(d_1|m_2) + C_{22}P(d_2|m_2)$$

$$P(d_1|m_1) = 1 - P(d_2|m_1)$$

$$P(d_1|m_2) = 1 - P(d_2|m_2)$$

$$\begin{aligned} B_1 &= C_{11}(1 - P(d_2|m_1)) + C_{21}P(d_2|m_1) \\ &= C_{11} - C_{11}P(d_2|m_1) + C_{21}P(d_2|m_1) \\ &= C_{11} + [-C_{11} + C_{21}]P(d_2|m_1) \\ &= C_{11} + [C_{21} - C_{11}]P(d_2|m_1) \end{aligned}$$

$$B_2 = C_{12} - (C_{12} - C_{22})P(d_2|m_2)$$

The average cost

$$
\begin{aligned}
B &= B_1 P(m_1) + B_2 P(m_2) \\
&= \{C_{11} + [C_{21} - C_{11}] P(d_2|m_1)\} P(m_1) + \{C_{12} + [C_{12} - C_{22}] P(d_2|m_2)\} P(m_2) \\
&= C_{11} P(m_1) + [C_{21} - C_{11}] P(d_2|m_1) P(m_1) + C_{12} P(m_2) - [C_{12} - C_{22}] P(d_2|m_2) P(m_2) \\
&= C_{11} P(m_1) + [C_{21} - C_{11}] P(m_1) P(d_2|m_1) + C_{12} P(m_2) - [C_{12} - C_{22}] P(m_2) P(d_2|m_2)
\end{aligned}
$$

$$
\begin{aligned}
B &= C_{11} P(m_1) + [C_{21} - C_{11}] P(m_1) P(d_2|m_1) + C_{12} P(m_2) - [C_{12} - C_{22}] P(m_2) P(d_2|m_2) \\
&= C_{11} P(m_1) + C_{12} P(m_2) + [C_{21} - C_{11}] P(m_1) \int_{Z_2} p(z|m_1) dz - [C_{12} - C_{22}] P(m_2) \int_{Z_2} p(z|m_2) dz \\
&= C_{11} P(m_1) + C_{12} P(m_2) + \int_{Z_2} \{[C_{21} - C_{11}] P(m_1) p(z|m_1) dz - [C_{12} - C_{22}] P(m_2) p(z|m_2)\} dz
\end{aligned}
$$

Bayes decision criterion

Select the decision region Z_2 in order to minimize the average cost B.

The first two terms are not a function of Z_2 and hence are not involved in minimization.

We can minimize the integral if we assign to Z_2 all the values of z for which

$(C_{21} - C_{11}) P(m_1) p(z|m_1) dz - (C_{12} - C_{22}) P(m_2) p(z|m_2) < 0$

Therefore the decision rule can be written as

$$
(C_{21} - C_{11}) P(m_1) p(z|m_1) dz \ \underset{d_2}{\overset{d_1}{\gtrless}} \ (C_{12} - C_{22}) P(m_2) p(z|m_2)
$$

Now if we assume that $(C_{12} - C_{22}) > 0$, and then if we divide both sides of the Eq. by $(C_{12} - C_{22}) P(m_2)$, the Bayes decision rule takes the familiar form of likelihood-ratio test:

$$
\frac{(C_{21} - C_{11}) P(m_1)}{(C_{12} - C_{22}) P(m_2)} \ \underset{d_2}{\overset{d_1}{\gtrless}} \ \frac{p(z|m_2)}{p(z|m_1)} = \Lambda(z)
$$

Usually for the most problem both $(C_{21} - C_{11}) > 0$, $(C_{12} - C_{22}) > 0$ since the cost of an incorrect decision is generally greater than the cost of a correct decision.

예제:

Determine the Bayes decision rule associated with the following conditional probabilities:

$$p(z|m_1) = \frac{1}{2} exp(-|z|), \quad p(z|m_2) = \exp(-2|z|)$$

The costs are given by
$C_{11} = C_{22} = 0,\ C_{12} = 1, C_{21} = 2,$ and $P(m_2) = 0.75.$

The likelihood ratio is

$$\frac{(C_{21}-C_{11})P(m_1)}{(C_{12}-C_{22})P(m_2)} \underset{d_2}{\overset{d_1}{\gtrless}} \frac{p(z|m_2)}{p(z|m_1)} = \Lambda(z) = 2\exp(-|z|) = \frac{(C_{21}-C_{11})P(m_1)}{(C_{12}-C_{22})P(m_2)} =$$

$$= \frac{(2-0)(1-0.75)}{(1-0)0.75} = (2)\frac{0.25}{0.75} = \frac{2}{3}$$

$$\Lambda(z) = 2e^{-|z|} \underset{d_1}{\overset{d_2}{\gtrless}} \frac{2}{3} \quad => \quad e^{-|z|} \underset{d_1}{\overset{d_2}{\gtrless}} \frac{1}{3} \quad => \quad \ln e^{-|z|} \underset{d_1}{\overset{d_2}{\gtrless}} \ln\frac{1}{3} \quad =>$$

$$-|z| \underset{d_1}{\overset{d_2}{\gtrless}} \ln\frac{1}{3} \quad => \quad |z| \underset{d_1}{\overset{d_2}{\lessgtr}} -\ln\frac{1}{3} = 1.0986 \Rightarrow |z| \underset{d_1}{\overset{d_2}{\lessgtr}} 1.0986$$

In order to determine the Bayes risk, we need to find the false-alarm and detection probabilities.

The false-alarm probability is given by

$$P(d_2|m_1) = \int_{-1.0986}^{1.0986} \frac{1}{2} e^{-|z|} dz = 2\int_{0}^{1.0986} \frac{1}{2} e^{-z} dz$$

$$= \frac{1}{-1} e^{-z}\Big|_0^{1.0986} = -(e^{-1.0986} - 1) = -(0.3334 - 1) = 0.6666 = 0.67$$

$$P(d_2|m_2) = \int_{-1.0986}^{1.0986} e^{-2|z|} dz = 2\int_{0}^{1.0986} e^{-2z} dz$$

$$= \frac{2}{-2} e^{-2z}\Big|_0^{1.0986} = -(e^{-2(1.0986)} - 1) = -(0.1111 - 1) = 0.88888 = 0.89$$

$$B_1 = C_{11}P(d_1|m_1) + C_{21}P(d_2|m_1)$$
$$B_2 = C_{12}P(d_1|m_2) + C_{22}P(d_2|m_2)$$

$C_{11} = C_{22} = 0$, $C_{12} = 1$, $C_{21} = 2$, and $P(m_2) = 0.75$, $P(m_1) = 0.25$.

$$P(d_2|m_1) = 0.67,\ P(d_2|m_2) = 0.89$$
$$P(d_1|m_1) = 1 - 0.67 = 0.33,\ P(d_1|m_2) = 1 - 0.89 = 0.11$$

$B_1 = 0(0.33) + 2(0.67) = 1.34$0.33)+ 2(0.67)=1.34

$$B_2 = 1(0.11) + 0(0.89) = 0.11$$

$$B = B_1 P(m_1) + B_2 P(m_2)$$
$$B = 1.34(0.25) + 0.11(0.75) = 0.42$$

The Bayes cost formulation can be viewed as a generalization of the probability-of-error method.

If $C_{11} = C_{22} = 0$, $C_{12} = 1$, $C_{21} = 1$, then B is identical to P_e.