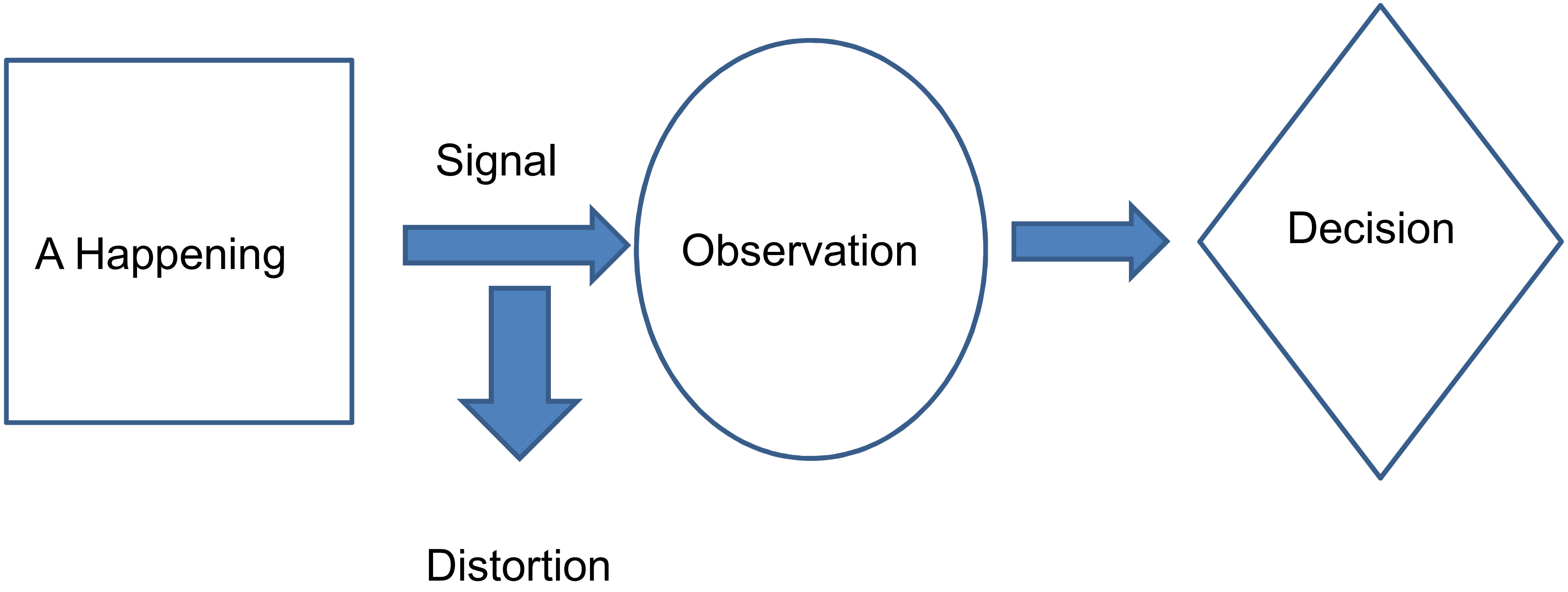
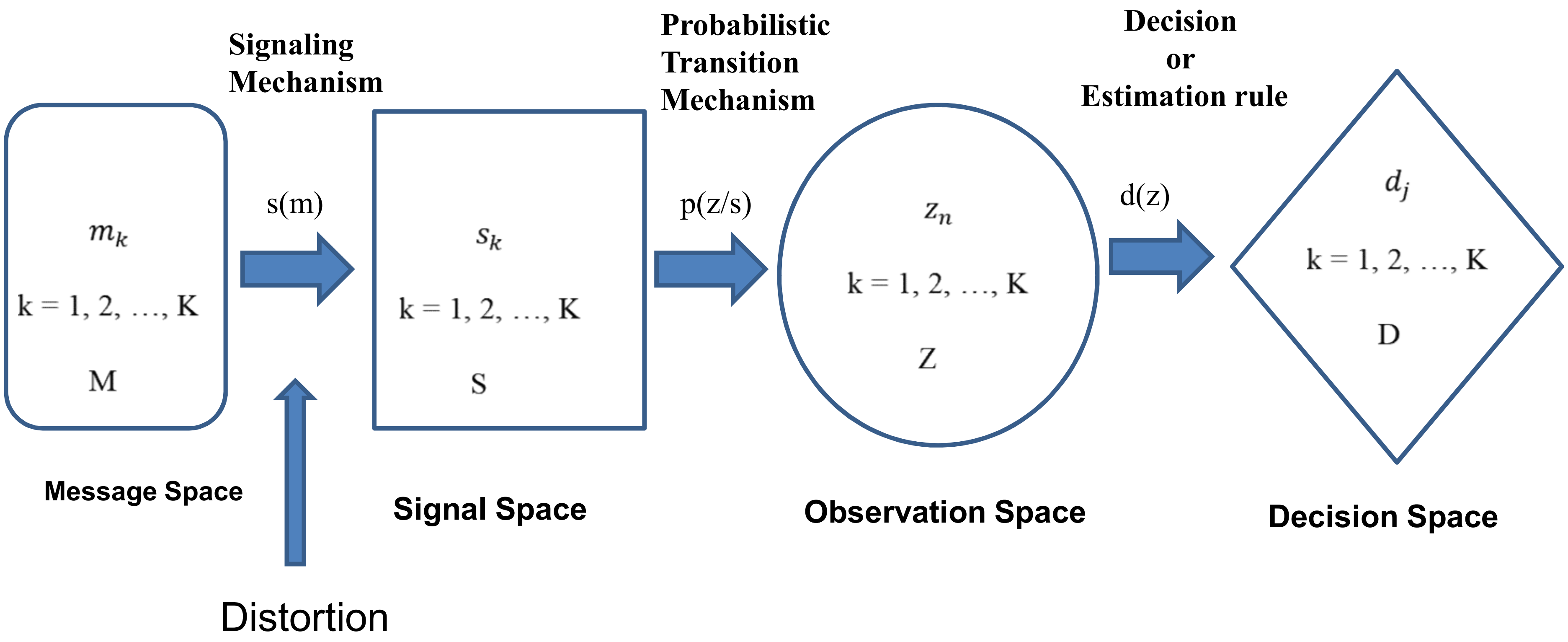
STATISTICAL INFERENCE
PART VI

HYPOTHESIS TESTING

Structure of Decision and Estimation Problem



Decision and Estimation Problem



TESTS OF HYPOTHESIS

- A hypothesis is a statement about a population parameter.

- The goal of a hypothesis test is to decide which of two complementary hypothesis is true, based on a sample from a population.

TESTS OF HYPOTHESIS

- **STATISTICAL TEST:** The statistical procedure to draw an appropriate conclusion from sample data about a population parameter.
- **HYPOTHESIS:** Any statement concerning an unknown population parameter.
- **Aim of a statistical test:** test a hypothesis concerning the values of one or more population parameters.

Binary Decision: Single Observation

Maximum-Likelihood Decision Criterion

$$d(z) = \begin{cases} d_1 & if \;\; p(z/m_1) \;\; > \;\; p(z/m_1) \\ d_2 & if \;\; p(z/m_2) \;\; > \;\; p(z/m_2) \end{cases}$$

$$Z_1 \;=\; \{z\colon \; p(z/m_1) \;\; > \;\; p(z/m_1)\,\}$$
$$Z_2 \;=\; \{z\colon \; p(z/m_2) \;\; > \;\; p(z/m_2)\,\}$$

$$\Lambda(z) = \frac{p(z/m_2)}{p(z/m_1)}$$

$$Z_1 \;=\; \{z\colon \; \Lambda(z) < 1\}$$
$$Z_2 \;=\; \{z\colon \; \Lambda(z) > 1\}$$

$$\Lambda(z) \underset{d_1}{\overset{d_2}{\gtrless}} 1$$

Binary Decision: Single Observation

Neyman-Pearson Criterion

$P\{d_2/m_1\} = Probability\ of\ making\ decision\ d_2\ \ when\ m_1\ is\ true$
$P\{d_1/m_2\} = Probability\ of\ making\ decision\ d_1\ \ when\ m_2\ is\ true$

$$P\{d_2/m_1\} = P\{z\ \in Z_2\,|\,m_1\} = \int_{Z_2} p(z|\,m_1)dz$$

$$P\{d_1/m_2\} = P\{z\ \in Z_1\,|\,m_2\} = \int_{Z_1} p(z|\,m_2)dz$$

$P\{d_1/m_1\} = Probability\ of\ making\ decision\ d_1\ \ when\ m_1\ is\ true$
$P\{d_2/m_2\} = Probability\ of\ making\ decision\ d_2\ \ when\ m_2\ is\ true$

$$P\{d_1/m_1\} = P\{z\ \in Z_1\,|\,m_1\} = \int_{Z_1} p(z|\,m_1)dz$$

$$P\{d_2/m_2\} = P\{z\ \in Z_2\,|\,m_2\} = \int_{Z_2} p(z|\,m_2)dz$$

Binary Decision: Single Observation

Neyman-Pearson Criterion

$$P\{d_1|m_1\} + P\{d_2|m_1\} = 1.0$$
$$P\{d_1|m_2\} + P\{d_2|m_2\} = 1.0$$

$$P\{d_1 \mid m_1\} + P\{d_2 \mid m_1\}$$
$$= \int_{Z_1} p(z|\, m_1)dz + \int_{Z_2} p(z|\, m_1)dz$$
$$= \int_{Z} p(z|\, m_1)dz$$
$$=1.0$$

$$P\{d_1 \mid m_2\} + P\{d_2 \mid m_2\}$$
$$= \int_{Z_1} p(z|\, m_2)dz + \int_{Z_2} p(z|\, m_2)dz$$
$$= \int_{Z} p(z|\, m_2)dz$$
$$=1.0$$

Binary Decision: Single Observation

Neyman-Pearson Criterion

Fix $P\{d_2 \mid m_1\}$ *at a preselected value* α_0, *then maximize* $P\{d_2|m_2\}$

$$P\{d_1 \mid m_1\} + P\{d_2 \mid m_1\}$$
$$= \int_{Z_1} p(z|m_1)dz + \int_{Z_2} p(z|m_1)dz$$
$$= \int_{Z} p(z|m_1)dz$$
$$=1.0$$

$$P\{d_1 \mid m_2\} + P\{d_2 \mid m_2\}$$
$$= \int_{Z_1} p(z|m_2)dz + \int_{Z_2} p(z|m_2)dz$$
$$= \int_{Z} p(z|m_2)dz$$
$$=1.0$$

Level of significance: $P\{d_2|m_1\}$
Power of the test: $P\{d_2|m_2\}$

m_1 -> No object signal : False alarm
m_2 -> Object signal : Missing Probability

H_0: Null Hypothesis: *No change occured* : m_1
H_1: Alternative Hypothesis: *Change occured* : m_2

THE RADAR PROBLEM:

Binary Decision: Single Observation

Neyman-Pearson Criterion

Fix $P\{d_2 \mid m_1\}$ *at a preselected value* α_0, *then maximize* $P\{d_2|m_2\}$

- We should maximize the power($P\{d_2|m_2\}$) for a given level of significance ($P\{d_2 \mid m_1\} = \alpha_0$).

- Out of all the decision regions z_2 for which $P\{d_2 \mid m_1\} = \alpha_0$,
- we are to select the one for which $P\{d_2|m_2\}$ is maximum.

This is a classical problem in optimization theory: maximizing a function subject to a constraint, which can be solved by the use of Lagrange multipliers.

$$\Gamma = P\{d_2|m_2\} - \lambda[P\{d_2|m_1\} - \alpha_0]$$

Therefore we now wish to select the decision Z_2 in order to maximize Γ.

Binary Decision: Single Observation

Neyman-Pearson Criterion

Fix $P\{d_2 \mid m_1\}$ $at\ a\ preselected\ value\ \alpha_0, then\ maximize$ $P\{d_2|m_2\}$

To use this approach we append the constraint $P\{d_2 \mid m_1\} = \alpha_0$ to $P\{d_2|m_2\}$ by the use of an undetermined Lagrange multiplier λ.

Therefore we now wish to select the decision region Z_2 in order to maximize:

$$\Gamma = P\{d_2|m_2\} - \lambda[P\{d_2|m_1\} - \alpha_0]$$

The problem is now treated as unconstrained but with λ as a parameter so that the remaining Z_2 will be a function of λ.

The Lagrange multiplier is then chosen to satisfy the constraint.

Therefore we now wish to select the decision Z_2 in order to maximize Γ.

Binary Decision: Single Observation

Neyman-Pearson Criterion

Therefore we now wish to select the decision Z_2 in order to maximize Γ.

$$\boxed{\text{Fix } P\{d_2 \mid m_1\} \textit{ at a preselected value } \alpha_0, \textit{ then maximize } P\{d_2|m_2\}}$$

$$\Gamma = P\{d_2|m_2\} - \lambda[P\{d_2|m_1\} - \alpha_0]$$

The problem is now treated as unconstrained but with λ as a parameter so that the remaining Z_2 will be a function of λ.

The Lagrange multiplier is then chosen to satisfy the constraint.

Once the constraint is satisfied, the second term of Eq. is zero and we have truly maximized $P\{d_2|m_2\}$ subject to the constraint.

The minus sign has been chosen in front of λ in Eq. in oder to simplify the final result.

Binary Decision: Single Observation

Neyman-Pearson Criterion

Therefore we now wish to select the decision Z_2 in order to maximize Γ.

Fix $P\{d_2 \mid m_1\}$ *at a preselected value* α_0, *then maximize* $P\{d_2|m_2\}$

$$\Gamma = P\{d_2|m_2\} - \lambda[P\{d_2|m_1\} - \alpha_0]$$

$$\Gamma = \int_{Z_2} P\{z|m_2\}\, dz - \lambda[\int_{Z_2} P\{z|m_1\} dz - \alpha_0]$$

$$\Gamma = \int_{Z_2} [\ P\{z|m_2\} - \lambda P\{z|m_1\}]\, dz + \lambda\alpha_0]$$

To maximize Γ selection of Z_2 the values of z for which the integrand is positive.
Therefore
Z_2 is given by $Z_2 = \{\text{z}: [\text{p}(\text{z}|m_2) - \lambda\, p(z|m_1)] > 0\}$
Z_1 is given by $Z_1 = \{\text{z}: [\text{p}(\text{z}|m_2) - \lambda\, p(z|m_1)] < 0\}$

Binary Decision: Single Observation

Neyman-Pearson Criterion

Therefore we now wish to select the decision Z_2 in order to maximize Γ.

Fix $P\{d_2 \mid m_1\}$ *at a preselected value* α_0, *then maximize* $P\{d_2|m_2\}$

$$\Gamma = P\{d_2|m_2\} - \lambda[P\{d_2|m_1\} - \alpha_0]$$

To maximize Γ selection of Z_2 the values of z for which the integrand is positive. Therefore

Z_2 is given by $Z_2 = \{\mathrm{z}: [\mathrm{p}(\mathrm{z}|m_2) - \lambda\, p(z|m_1)] > 0\}$

Z_1 is given by $Z_1 = \{\mathrm{z}: [\mathrm{p}(\mathrm{z}|m_2) - \lambda\, p(z|m_1)] < 0\}$

$$\int_{Z_2} p(z|m_1)dz$$

We must select λ such that the constraint

$$P\{d_2|m_1\} = \alpha_0 = \int_{Z_2} p(z|m_1)dz.$$

The basic approach is to compute $P(d_2|m_1) = \alpha_0 = \int_{Z_2} p(z|m_1)dz$ as a function of λ and then find the value of λ which makes $P(d_2|m_1)$ equal to α_0.

Binary Decision: Single Observation

Neyman-Pearson Criterion

Therefore we now wish to select the decision Z_2 in order to maximize Γ.

Fix $P\{d_2 \mid m_1\}$ $at\ a\ preselected\ value\ \alpha_0, then\ maximize$ $P\{d_2|m_2\}$

$$\Gamma = P\{d_2|m_2\} - \lambda[P\{d_2|m_1\} - \alpha_0]$$

A test of likelihood ratio against a threshold.

If we make use of the likelihood ratio

$$\Lambda(z) = \frac{p(z|m_2)}{p(z|m_1)}$$

In the **maximum-likelihood** decision rule, the threshold is always **unity**, while in the **Neyman-Pearson** decision rule we must select the threshold $\boldsymbol{\lambda}$.

We can write Z_1 and Z_2 as

$$Z_1 = \{z: \Lambda(z) < \lambda\}$$
$$Z_2 = \{z: \Lambda(z) > \lambda\}$$

The decision rule can be represented as $\Lambda(z) \underset{d_1}{\overset{d_2}{\gtrless}} \lambda$

Binary Decision: Single Observation

Neyman-Pearson Criterion

$$p(z|m_1) = \frac{1}{\sqrt{2\pi}} exp\frac{-z^2}{2}$$

$$p(z|m_2) = \frac{1}{\sqrt{2\pi}} exp\frac{-(z-1)^2}{2}$$

$$z \underset{d_1}{\overset{d_2}{\gtrless}} ln\,\lambda + \frac{1}{2}$$

$P\{d_2 \mid m_1\} = 0.25$

If we make use of the likelihood ratio

$$\Lambda(z) = \frac{p(z|m_2)}{p(z|m_1)} \qquad \Lambda(z) = \exp\frac{(2z-1)}{2}$$ the likelihood ratio

$$\Lambda(z) \underset{d_1}{\overset{d_2}{\gtrless}} \lambda \qquad \Lambda(z) = \exp\frac{(2z-1)}{2} \underset{d_1}{\overset{d_2}{\gtrless}} \lambda \qquad \frac{(2z-1)}{2} \underset{d_1}{\overset{d_2}{\gtrless}} ln\,\lambda$$

Binary Decision: Single Observation

Neyman-Pearson Criterion

$$z \underset{d_1}{\overset{d_2}{\gtrless}} \ln \lambda + \frac{1}{2}$$

Now, we need to select such that $P\{d_2 \mid m_1\} = 0.25$.

$$P\{d_2 \mid m_1\} = \int_{\ln\lambda \; +1/2}^{\infty} \frac{1}{\sqrt{2\pi}} exp \frac{-z^2}{2} dz\} = 0.25$$

$$Q(0.674) = \int_{0.674}^{\infty} \frac{1}{\sqrt{2\pi}} exp \frac{-z^2}{2} dz\} = 0.25$$

$$z \underset{d_1}{\overset{d_2}{\gtrless}} 0.674$$

$$ln\lambda + 0.5 = 0.674$$

$$\lambda = \exp(0.174) = 1.19$$

Decision and Estimation Problem