MIN-MAX CRITERION

When the a priori probabilities of the messages m_1 and m_2 may not known, there are many practical problems.

In this case, the min-max criterion may often be successfully employed.

Basically the min-max criterion says that one should use the Bayses decision rule which corresponds to the least favorable $P(m_1)$.

The average risk $B(P(m_1), Z_2)$:

$$\begin{aligned} B = E(C_{jk}) &= C_{11}P(d_1,m_1) + C_{21}P(d_2,m_1) + C_{12}P(d_1,m_2) + C_{22}P(d_2,m_2) \\ &= [C_{11}P(d_1|m_1) + C_{21}P(d_2|m_1)]P(m_1) + [C_{12}P(d_1|m_2) + C_{22}P(d_2|m_2)]P(m_2) \\ &= B_1P(m_1) + B_2P(m_2) \end{aligned}$$

$$B(P(m_1), Z_2) = B_1(Z_2)P(m_1) + B_2(Z_2)(1 - P(m_1))$$

$$\begin{aligned} B_1 &= C_{11}(1 - P(d_2|m_1)) + C_{21}P(d_2|m_1) \\ &= C_{11} - C_{11}P(d_2|m_1) + C_{21}P(d_2|m_1) \\ &= C_{11} + [-C_{11} + C_{21}]P(d_2|m_1) \\ &= C_{11} + [C_{21} - C_{11}]P(d_2|m_1) \end{aligned}$$

$$B_2 = C_{12} - (C_{12} - C_{22})P(d_2|m_2)$$

$$\begin{aligned} B &= C_{11}P(m_1) + [C_{21} - C_{11}]P(m_1)P(d_2|m_1) + C_{12}P(m_2) - [C_{12} - C_{22}]P(m_2)P(d_2|m_2) \\ &= C_{11}P(m_1) + C_{12}P(m_2) + [C_{21} - C_{11}]P(m_1)\int_{Z_2} p(z|m_1)dz - [C_{12} - C_{22}]P(m_2)\int_{Z_2} p(z|m_2)dz \\ &= C_{11}P(m_1) + C_{12}P(m_2) + \int_{Z_2} \{[C_{21} - C_{11}]P(m_1)p(z|m_1)dz - [C_{12} - C_{22}]P(m_2)p(z|m_2)\}dz \\ &= \left(C_{11} + (C_{21} - C_{11})\int_{Z_2} p(z|m_1)dz\right)P(m_1) + \left(C_{12} + (C_{12} - C_{22})\int_{Z_2} p(z|m_2)dz\right)P(m_2) \end{aligned}$$

$$B_1(z_2) = \left(C_{11} + (C_{21} - C_{11})\int_{Z_2} p(z|m_1)dz\right)$$

$$B_2(z_2) = \left(C_{12} + (C_{12} - C_{22})\int_{Z_2} p(z|m_2)dz\right)$$

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Min-max decision criterion

The min-max decision region Z_2^* is defined by

$$\max_{P(m_1)} B\left(P(m_1), Z_2^*\right) = \min_{Z_2} \max_{P(m_1)} B\left(P(m_1), Z_2\right)$$
$$< \max_{P(m_1)} B\left(P(m_1), Z_2\right)$$

for all $Z_2 \neq Z_2^*$.

In other words, Z_2^* is the decision region which yields the minimum cost for the least favorable $P(m_1)$.

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Under the mild restrictions, it is possible to show that the minimization and maximization operations are interchangeable so that

$$\min_{Z_2} \max_{P(m_1)} B\left(P(m_1), Z_2\right) = \max_{P(m_1)} \min_{Z_2} B\left(P(m_1), Z_2\right)$$

The minimization of $B(P(m_1), Z_2)$ with respect to Z_2 is simply the Bayes criterion, so that

$$\min_{Z_2} B\left(P(m_1),\ Z_2\right) = B^0(P(m_1))$$

where $B^0(P(m_1))$ is the minimum Bayes cost associated with the a priori probability $P(m_1)$.

We may find the min-max decision rule by finding the Bayes decision rule for the least favorable $P(m_1)$, that is, the $P(m_1)$ which maximizes $B^0(P(m_1))$.

The procedure is therefore to find $B^0(P(m_1))$ either numerically or analytically and then to determine the $P(m_1)$ for which $B^0(P(m_1))$ is maximum.

The Bayes decision rule for $P(m_1)$ which maximize $B^0(P(m_1))$ is the min-max decision rule.

$$T = 1 => \frac{p(z|m_2)}{p(z|m_1)} \overset{d_2}{\underset{d_1}{\gtrless}} 1.0 \quad \Leftarrow \quad p(z|m_2) \overset{d_2}{\underset{d_1}{\gtrless}} p(z|m_1)$$ 참조

Example 3.5-1

Build the min-max design procedure for a decision problem with the following conditional probabilities and the following costs:

$p(z|m_1) = e^{-z}$ for $z > 0$ for $z > 0$ 영역 z는 0보다 큰 것이 조건이다!

$p(z|m_2) = 2e^{-2z}$ for $z > 0$ for $z > 0$

$C_{11} = C_{22} = 0,\ C_{12} = 2,\ C_{21} = 1.$

In terms of $P(m_1)$, the Bayes decision rule takes the following form:

$$\Lambda(z) = \frac{2e^{-2z}}{e^{-z}} = 2e^{-z} \gtrless \frac{P(m_1)}{1-P(m_1)}$$

$$\Lambda(z) = \frac{2e^{-2z}}{e^{-z}} = 2e^{-z} \underset{d_1}{\overset{d_2}{\gtrless}} \left(\frac{1}{2}\right)\frac{P(m_1)}{1-P(m_1)}$$

$$e^{-z} \underset{d_1}{\overset{d_2}{\gtrless}} \left(\frac{1}{4}\right)\frac{P(m_1)}{1-P(m_1)} \Rightarrow \quad -z \underset{d_1}{\overset{d_2}{\gtrless}} \ln\left(\left(\frac{1}{4}\right)\frac{P(m_1)}{1-P(m_1)}\right)$$

$$z \underset{d_1}{\overset{d_2}{\lessgtr}} -\ln\left(\left(\frac{1}{4}\right)\frac{P(m_1)}{1-P(m_1)}\right) = \ln\left(\frac{4(1-P(m_1))}{P(m_1)}\right) = T$$

$$z \underset{d_2}{\overset{d_1}{\gtrless}} \ln\left(\frac{4(1-P(m_1))}{P(m_1)}\right) \Rightarrow 0 < \left(\frac{4(1-P(m_1))}{P(m_1)}\right) < 1.0 \qquad \text{for } z < 0\ ,$$

$$\underline{1.0 < \left(\frac{4(1-P(m_1))}{P(m_1)}\right) \qquad \text{for } z > 0}$$

$$\frac{4[1-P(m_1)]}{P(m_1)} > 1.0 \Rightarrow 4 - 4P(m_1) > P(m_1) \Rightarrow 4 > 5P(m_1) \Rightarrow P(m_1) < 0.8$$

This decision rule is valid if $P(m_1) \leq 0.8$.

For $P(m_1) > 0.8$, T becomes negative and $Z_1 = Z$: that is, we always, decide d_1.

$\ln x => x > 1.0$ 되어야 $\ln x$가 양의 수가 된다.
$\ln x => 0 < x < 1.0$ 되면 $\ln x$는 음의 수가 된다.
$e^x = y \quad \Rightarrow \quad \ln e^x = \ln y \;\; y \geqq 0, \quad x = \ln y => y = \ln x \;\; => \;\; x \geqq 0$

Now for $P(m_1) \leq 0.8$, we may express the minimum Bayes cost as a function of $P(m_1)$ as

$$
\begin{aligned}
B^0(P(m_1)) &= P(m_1)\int_0^T e^{-z}dz + 2(1-P(m_1))\int_T^\infty 2e^{-2z}dz \\
&= P(m_1)(1-e^{-T}) + 2(1-P(m_1))e^{-2T} \\
&= P(m_1)(1-e^{-\ln\left(\frac{4(1-P(m_1))}{P(m_1)}\right)}) + 2(1-P(m_1))e^{-2\ln\left(\frac{4(1-P(m_1))}{P(m_1)}\right)},
\end{aligned}
$$

$$
\ln\left(\frac{4(1-P(m_1))}{P(m_1)}\right) = T
$$

$$
\begin{aligned}
B^0(P(m_1)) &= P(m_1)(1-\frac{P(m_1)}{4[1-P(m_1)]}) + 2[1-P(m_1)]\frac{P(m_1)^2}{4^2[1-P(m_1)]^2} \\
&= 2P(m_1)(\frac{4-5P(m_1)}{(2)4[1-P(m_1)]}) + \frac{P^2(m_1)}{8[1-P(m_1)]} \\
&= \frac{8P(m_1) - 10P^2(m_1) + P^2(m_1)}{8[1-P(m_1)]} \\
&= \frac{-9P^2(m_1) + 8P(m_1)}{8[1-P(m_1)]} \\
&= \frac{9P^2(m_1) - 8P(m_1)}{8[P(m_1)-1]}
\end{aligned}
$$

for $P(m_1) < 0.8$.

For $P(m_1) > 0.8$, $B^0(P(m_1))$ is given by $B^0(P(m_1)) = 2[1-P(m_1)]$.
$C_{11} = C_{22} = 0, \; C_{12} = 2, C_{21} = 1$: 참조

The value of $P(m_1) < 0.8$ maximizes $B^0(P(m_1))$, which can be obtained by $\dfrac{dB^0(P(m_1))}{d(P(m_1))} = 0$ and solving for $P(m_1)$.

The resulting value of $P(m_1)$ is $\dfrac{2}{3}$.

Now we substitute this value into the expression for T to obtain the following min-max decision rule:

$$z \underset{d_1}{\overset{d_2}{\lessgtr}} -\ln\left((\frac{1}{4})\frac{P(m_{1)}}{1-P(m_1)}\right)=\ln\left(\frac{4(1-P(m_{1)})}{P(m_1)}\right)=\ln\left(\frac{4(1-2/3}{2/3}\right)=\ln(\frac{4/3}{2/3})=\ln 2=0.69315=T$$

The following theorem offers an alternative procedure for finding the min-max decision rule.

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Theorem 3.5-1

If there exists a decision region Z_2^* such that the conditional risks $B_1(Z_2^*)$ and $B_2(Z_2^*)$ are equal and Z_2^* is a Bayes decision region for some $P(m_1)$, the Z_2^* is a min-max decision region.

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Assume that there exists a Z_2^* satisfying the conditions of the theorem which is not a min-max decision region.

Then there exists a $Z_2^{'} \neq Z_2^*$ such that

The min-max decision region Z_2^* is defined by

$$\max_{P(m_1)} B(P(m_1), Z_2^{'}) < \max_{P(m_1)} B(P(m_1), Z_2^*)$$

by definition of a min-max decision region.

Now using

$$B(P(m_1),\, Z_2) = B_1(Z_2)P(m_1) + B_2(Z_2)(1 - P(m_1),$$

we can write $B(P(m_1),\, Z_2^*)$ as

$$B(P(m_1),\, Z_2^*) = B_1(Z_2^*)P(m_1) + B_2(Z_2^*)(1 - P(m_1)$$

But $B_1(Z_2^*) = B_2(Z_2^*)$, so that (주어진 조건)

$$B(P(m_1),\, Z_2^*) = B_1(Z_2^*)P(m_1) + B_2(Z_2^*)(1 - P(m_1) = B_1(Z_2^*) = B_2(Z_2^*)$$

which is independent of $P(m_1)$.

Therefore

$$\max_{P(m_1)} B(P(m_1), Z_2^*) = B_1(Z_2^*) = B_2(Z_2^*)$$

so that

$$\max_{P(m_1)} B(P(m_1), Z_2^{'}) < \max_{P(m_1)} B(P(m_1), Z_2^*)$$

implies that

$$\max_{P(m_1)} B(P(m_1), Z_2^{'}) < B_1(Z_2^*)$$

$$\max_{P(m_1)} B(P(m_1), Z_2^{'}) < B_2(Z_2^*)$$

Therefore we can conclude that
$B\left(P(m_1), Z_2'\right) < B\left(P(m_1), Z_2^*\right)$ for all $P(m_1)$.

But Z_2^* is a Bayes decision region for some $P(m_1)$, say, α^*.
Therefore
$B\left(\alpha^*, Z_2^*\right) < B\left(\alpha^*, Z_2'\right)$

This yields a contradiction, which means Z_2^* must be a min-max decision region.

Example 3.5-2

Example 3.5-1

Build the min-max design procedure for a decision problem with the following conditional probabilities and the following costs:

$p(z|m_1) = e^{-z}$ for $z > 0$ $\quad C_{11} = C_{22} = 0,\ C_{12} = 2,\ C_{21} = 1.$

$p(z|m_2) = 2e^{-2z}$ for $z > 0$

영역 z는 0보다 큰 것이 조건이다!

In terms of $P(m_1)$, the Bayes decision rule takes the following form:

$$\Lambda(z) = \frac{2e^{-2z}}{e^{-z}} = 2e^{-z} \underset{d_1}{\overset{d_2}{\gtrless}} T$$

where $T > 0$.

We can simplify the decision rule to the form

$$z \underset{d_1}{\overset{d_2}{\lessgtr}} \ln\frac{T}{2} = T'$$

Therefore $Z_2 = (0, T')$, and the conditional costs are given by

$$B_1(Z_2) = \int_0^{T'} e^{-z} dz = 1 - e^{-T'}$$

and

$$B_2(Z_2) = 2\int_{T'}^{\infty} 2e^{-2z} dz = 2e^{-2T'}$$

Equating the conditional costs yields the following expression for T':

$$1 - e^{-T'} = 2e^{-2T'}$$

By letting $T' = \ln\frac{1}{x}$, we obtain a simple quadratic equation for x:

$1 - e^{-\ln\frac{1}{x}} = 2e^{-2\ln\frac{1}{x}}$ => $1 - x = 2x^2$, such that $x = 1/2$ and -1.

The -1 value is meaningless since it would lead to an undefined value for T'.

Using $x = \frac{1}{2}$, we find that $T' = \ln 2$ as obtained previously.

The likelihood-ratio threshold $T = 2e^{-T'} = 1.0$, and we have a valid Bayes decision rule.

This method is easier method for determining the nin-max decision rule than the basic definition.

드물지만 이 방법이 적용될 수 없는 경우도 있다.
이러한 경우는 equal condition risk를 갖춘 Bayes decision rule이 없기 때문인 것이다.
이것은 $P(m_1)$에 대하여 $B^0(P(m_1))$의 변화율, $\frac{dB^0(P(m_1))}{d(P(m_1)}$, 이 0이 아니거나 정의되지 않는 경우로서 $B^0(P(m_1))$이 maximum 값을 가질 경우이다.

$P(m_1)$이 0이거나 1인 경우에 최대치가 발생하는 경우로서 미분이 불연속인 경우의 내부 지점에서 발생한다.

If such a region exists, it is tangent to $B^0(P(m_1))$ for $P(m_1) = P^*$ where P^* is defined by

$$\left.\frac{dB^0(P(m_1))}{d(P(m_1)}\right|_{p(m_1) = P^*} = 0$$

In other words, minimizing the maximum cost(i.e., finding the horizontal line) is identical to maximizing the minimum cost(i.e., finding the peak of the $B^0(P(m_1)))$ curve.

Clearly this method will not work if there is no $P*$ which satisfies the above equation.

Summary

5가지 서로 다른 방법

Threshold를 선택하는 방법에 따라 다르다.

이 선택의 방법은 정보의 양과 문제의 스펙에 사용 된다.

$$\frac{p(z|m_2)}{p(z|m_1)} = \Lambda(z) \underset{d_1}{\overset{d_2}{\gtrless}} T$$

ML: 관측된 조건 확률만을 비교

NP: 관측된 조건 확률 + false-alarm probability

POE: 관측되 조건 확률 + a priori probabilities 사용

Bayes risk CR: 관측된 조건 확률 + specification of costs for all decisions

Min-Max: Bayes Costs with assumption that the priori probabilities are unknown.

모르는 것으로 해 주세요~