Constraint Optimization Problem

$f(x,y) = x^2y$

$g(x,y) = x^2 + y^2 = 1.0$

$$\begin{bmatrix} \dfrac{\partial f(x,y)}{\partial x} \\ \dfrac{\partial f(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x^2 y}{\partial x} \\ \dfrac{\partial x^2 y}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial g(x,y)}{\partial x} \\ \dfrac{\partial g(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x^2 + y^2}{\partial x} \\ \dfrac{\partial x^2 + y^2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$$\begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$2xy = \lambda 2x$, $x^2 = \lambda 2y$, Constraint: $x^2 + y^2 = 1.0$

$y = \lambda$, $x^2 = \lambda 2\lambda = 2\lambda^2$, such that $2\lambda^2 + \lambda^2 = 1.0$. $\lambda^2 = \dfrac{1}{3,}$ $\lambda = \pm\sqrt{\dfrac{1}{3}}$

$x^2 = 2(\dfrac{1}{3}) = \dfrac{2}{3}$, $x = \pm\sqrt{\dfrac{2}{3}}$, $y = \pm\sqrt{\dfrac{1}{3}}$

$p_1 = (\sqrt{\dfrac{2}{3}}, \sqrt{\dfrac{1}{3}})$, $p_2 = (-\sqrt{\dfrac{2}{3}}, \sqrt{\dfrac{1}{3}})$, $p_3 = (\sqrt{\dfrac{2}{3}}, -\sqrt{\dfrac{1}{3}})$, $p_4 = (-\sqrt{\dfrac{2}{3}}, -\sqrt{\dfrac{1}{3}})$

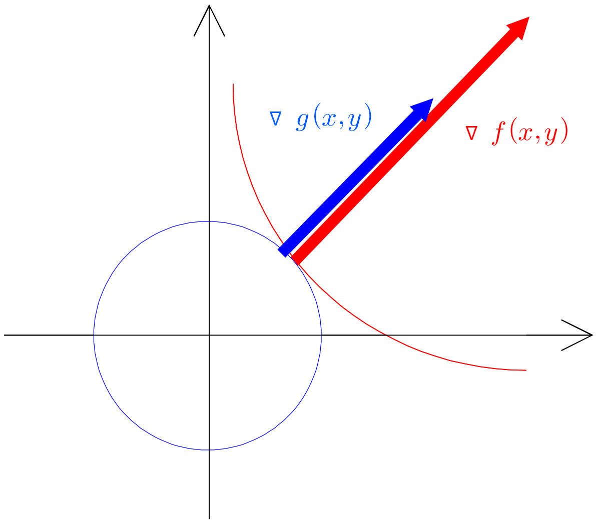
$Max\{f(p_1), f(p_2), f(p_3), f(p_4)\}$, $f(x,y) = x^2y$

$Max\{ \dfrac{2}{3\sqrt{3}}, \dfrac{2}{3\sqrt{3}}, -\dfrac{2}{3\sqrt{3}}, -\dfrac{2}{3\sqrt{3}})\} = \dfrac{2}{3\sqrt{3}}$

$f(x,y) = x^2 e^y y = C$

$g(x,y) = x^2 + y^2 = b$

$\nabla f(x,y) = \lambda \nabla g(x,y)$



$L(x,y,\lambda) = f(x,y) - \lambda[g(x,y) - b]$

$$\nabla L(x,y,\lambda) = \begin{bmatrix} \frac{\partial L(x,y,\lambda)}{\partial x} \\ \frac{\partial L(x,y,\lambda)}{\partial y} \\ \frac{\partial L(x,y,\lambda)}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} - \lambda \frac{\partial g(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} - \lambda \frac{\partial g(x,y)}{\partial y} \\ -g(x,y) + b \end{bmatrix}$$

$\nabla L(x,y,\lambda) = \begin{bmatrix} 0 \\ 0 \\ 0 RIGHT \end{bmatrix}$, $\nabla L(x,y,\lambda) = 0$, We get $\nabla L(x^*, y^*, \lambda^*) = 0$.

Then $Max^* = f(x^*, y^*)$. $Max^*(b) = f(x^*(b), y^*(b))$

$\lambda^* = \frac{\partial Max^*(b)}{\partial b}$ => 이것이 주어진 제약조건에 따라 최적치의 변화를 알 수 있다.

제약조건의 변경은 최적치의 변화를 가져오는데, 그 변화률을 말하고 있다.

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$$\nabla L(x,y,\lambda) = \begin{bmatrix} \frac{\partial L(x,y,\lambda)}{\partial x} \\ \frac{\partial L(x,y,\lambda)}{\partial y} \\ \frac{\partial L(x,y,\lambda)}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} = \lambda \frac{\partial g(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} = \lambda \frac{\partial g(x,y)}{\partial y} \\ g(x,y) = b \end{bmatrix}$$

$g(x,y) = b$ is the constraint given!

이 수학적 표현은 위의 함수로 얻어지는 관계식을 하나의 식으로 표현한 것이라 볼 수 있다.

$L(x*,y*,\lambda*) = f(x*,y*) - \lambda*[g(x*,y*) - b]$
$[g(x*,y*) - b] = 0$ by constraint

$L(x*,y*,\lambda*) = f(x*,y*)$
여기서 b는 상수이므로, 다르게 표현해야 한다.
$L(x*(b),\ y*(b),\ \lambda*(b),\ b) = f(x*(b),\ y*(b))$
이것은 b에 대한 함수로 표시한 것이다.

이것이 암시하는 것은 b가 정해지면, 모든 것이 바뀌게 된다는 것이다.

예제: Widgets Problem

Labor $\$20/h$
Steel $\$2{,}000/ton$

$$R(h,s) = 100h^{2/3}s^{1/3}$$

Budget=\$20,000

$$g(h,s) = 20h + 2{,}000s = 20{,}000$$

$\nabla\ R(h,s) = \lambda\nabla\ g(h,s)$ $\lambda : Lagrange\ Mltiplier$

$$\nabla\ R(h,s) = \begin{bmatrix} \dfrac{\partial R(h,s)}{\partial h} \\ \dfrac{\partial R(h,s)}{\partial s} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial 100(\frac{2}{3})h^{\frac{2}{3}}s^{\frac{1}{3}}}{\partial h} \\ \dfrac{\partial 100\frac{2}{3}h^{\frac{2}{3}}s^{\frac{1}{3}}}{\partial s} \end{bmatrix} = \begin{bmatrix} 100(\frac{2}{3})h^{-\frac{1}{3}}s^{\frac{1}{3}} \\ 100(\frac{2}{3})h^{\frac{2}{3}}s^{-\frac{2}{3}} \end{bmatrix}$$

$$\nabla\ g(h,s) = \begin{bmatrix} \dfrac{\partial g(h,s)}{\partial h} \\ \dfrac{\partial g(h,s)}{\partial s} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial(20h + 2.000s)}{\partial h} \\ \dfrac{\partial(20h + 2{,}000s)}{\partial s} \end{bmatrix} = \begin{bmatrix} 20 \\ 2{,}000 \end{bmatrix}$$

$$\nabla\ R(h,s) = \lambda\nabla\ g(h,s)$$

$$\begin{bmatrix} 100(\frac{2}{3})h^{-\frac{1}{3}}s^{\frac{1}{3}} \\ 100(\frac{2}{3})h^{\frac{2}{3}}s^{-\frac{2}{3}} \end{bmatrix} = \lambda\begin{bmatrix} 20 \\ 2{,}000 \end{bmatrix},\quad \begin{bmatrix} (\frac{200}{3})\frac{s^{1/3}}{h^{1/3}} \\ (\frac{200}{3})\frac{h^{2/3}}{s^{2/3}} \end{bmatrix} = \begin{bmatrix} 20\lambda \\ 2{,}000\lambda \end{bmatrix}$$

$u = s/h,\ \ \dfrac{200}{3}u^{1/3} = 20\lambda,\ \ \dfrac{100}{3}u^{-2/3} = 2000\lambda$

$u^{1/3} = 3/10\lambda,\ \ u^{-2/3} = 60\lambda$

$u = 3/10\lambda u^{2/3},\ \ 1 = 60\lambda u^{2/3}$

$200u = 60\lambda u^{2/3},\ \ 1 = 60\lambda u^{2/3}$

$200u = 1,\ \ \ 200(s/h) = 1,\ \ h = 200s$

$20h + 2{,}000s = 20{,}000$
$20(200s) + 2{,}000s = 20{,}000$
$4000s + 2{,}000s = 20{,}000$
$6{,}000s = 20{,}000$
$s = \dfrac{10}{3},\ \ h = 200s = \dfrac{2000}{3}$