PURE DIMENSIONALITY PROBLEM IN THE PRYM SETTING

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Abstract. We study metric graphs, Jacobians, and in particular recap the pure dimensionality of the image of the Abel-Jacobi map. We investigate the behaviour in the double cover (Prym) case and investigate analogous objects that may yield us a solution to the pure dimensionality of the image of the Abel-Prym map.

1. Metric Graphs, Divisors, Abel-Jacobi

1.1. **Metric Graphs.** A metric graph is a generalization of the usual combinatorial graphs to include edge lengths. However, it differs from a weighted graph in a key way: In a weighted graph we have a fixed vertex set and a weighted edge connects two vertices. However, in a metric graph we do not distinguish vertices from edges: instead we think of a metric graph as a collection of intervals glued at some special points that may have degree higher than 2.

Definition 1.1. More formally, a *metric graph* Γ is the compact metric space obtained from a finite graph G, when we provide the edge set $E(G)$ with a length function $l : E(G) \to \mathbb{R}^+$, and identify each edge $e \in E(G)$ with a closed interval of length $l(e)$. We then endow the graph with the shortest path metric and obtain the metric graph. The pair (G, l) is also called a *model* of the metric graph. The points of degree greater than 2 are also known as branch points. At points of Γ with degree greater than 2, we glue the edge segments. A *canonical model* (G, l) of a metric graph Γ is one where the vertex set of G consists precisely of the set of all branch points of Γ.

1.2. **Divisors.**

Definition 1.2. A Divisor on a metric graph is a placement of integers on the graph. Formally a *divisor* on a metric graph Γ is a finite linear combination of the form

$$D = a_1p_1 + \cdots + a_np_n,$$

where $a_i \in \mathbb{Z}$ and $p_i \in \Gamma$. A *degree* of a divisor is the total sum of these numbers, $\deg D = a_1 + \cdots + a_n$. An *effective* divisor is one where all $a_i \geq 0$. Note that we have a formal linear combination, and therefore this forms a group. We denote $\mathrm{Div}(\Gamma)$ as the group of all divisors, and $\mathrm{Div}^k(\Gamma)$ as the set (not necessarily a group!) of all divisors of degree k. Note that $\mathrm{Div}^0(\Gamma)$ is a subgroup of $\mathrm{Div}(\Gamma)$.

Classically, for Riemann surfaces (1-dimension complex manifolds), a divisor is a co-dimension 1 (0 dimensional) object. One of the basic example of a divisor is one associated to a meromorphic function f via the following: A divisor records the order of vanishing of f at each point on the Riemann surface. Divisors of this form are denoted (f) (order of vanishing of a meromorphic function) and are called principal divisors. Observe that $(fg) = (f) + (g)$, so principal divisors form a subgroup inside the free abelian group of all divisors.

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Definition 1.3. In the same spirit we consider the space of *rational functions* $R(\Gamma)$ of continuous, piecewise linear functions on Γ, with integer slopes. (It turns out these are the correct tropical analogues of meromorphic functions). Then for such a function $\phi \in R(\Gamma)$, we define the corresponding *principal divisor* of ϕ via

$$(\phi) = \operatorname{div}(\phi) = \sum_{p\in\Gamma} \sigma_p(\phi)p$$

which is the sum of incoming slopes at each point $p \in \Gamma$. The collection of all principal divisors is denoted by $\operatorname{Prin}(\Gamma)$. Two divisors D_1 and D_2 are said to be *equivalent* denoted $D_1 \sim D_2$ when $D_1 - D_2 = \operatorname{div}(\phi)$ for some piecewise linear function $\phi \in R(\Gamma)$.

As mentioned above, we denote divisors of degree 0 by $\operatorname{Div}^0(\Gamma)$, observe that principal divisors are always of degree 0, however, they are not all, we define the *Picard group* as the quotient $\operatorname{Pic}^0(\Gamma) = \operatorname{Div}^0(\Gamma)/\operatorname{Prin}(\Gamma)$. In general we use $\operatorname{Pic}^d(\Gamma)$ to mean degree d divisors upto principal divisors, i.e., $\operatorname{Pic}^d(\Gamma) = \operatorname{Div}^d(\Gamma)/\operatorname{Prin}(\Gamma)$.

Before we introduce the Abel-Jacobi map, we will need to define the Jacobian. To introduce the Jacobian we will need the notion of *tropical abelian varieties* (see [FRSS18] and [LU21] for more). On a first reading one may read Section 1.3 after Section 1.4.

1.3. **Tropical Abelian Varieties.**

1.3.1. *Real Tori.*

Definition 1.4. Let Λ, Λ' be finitely generated free abelian groups of the same rank (for example $\Lambda, \Lambda' \simeq \mathbb{Z}^k$), and suppose we have a pairing (a bilinear form) $[\cdot,\cdot] : \Lambda' \times \Lambda \to \mathbb{R}$. The triple $(\Lambda, \Lambda', [\cdot,\cdot])$ defines a *real torus with integral structure* $\Sigma = \operatorname{Hom}(\Lambda, \mathbb{R})/\Lambda'$, where by *integral structure* we mean the lattice $\operatorname{Hom}(\Lambda, \mathbb{Z}) \subset \operatorname{Hom}(\Lambda, \mathbb{R})$.

The quotient $\operatorname{Hom}(\Lambda, \mathbb{R})/\Lambda'$ is well defined because we think of Λ' embedded inside $\operatorname{Hom}(\Lambda, \mathbb{R})$ via $\lambda' \mapsto [\lambda', \cdot]$.

1.3.2. *Homomorphisms of Real Tori.* Suppose we have two tori $\Sigma_1 = (\Lambda_1, \Lambda_1', [\cdot,\cdot]_1)$ and $\Sigma_2 = (\Lambda_2, \Lambda_2', [\cdot,\cdot]_2)$ with integral structure. A *homomorphism* $f : \Sigma_1 \to \Sigma_2$ is given by two maps $f = (f_*, f^*)$ such that $f_* : \Lambda_1' \to \Lambda_2'$ and $f^* : \Lambda_2 \to \Lambda_1$ such that

$$[\lambda_1', f^*(\lambda_2)]_1 = [f_*(\lambda_1'), \lambda_2]_2,$$

for all $\lambda_1' \in \Lambda_1'$ and $\lambda_2 \in \Lambda_2$.

- Given a homomorphism $f = (f_*, f^*) : \Sigma_1 \to \Sigma_2$ of integral tori, the connected component of identity (0) of the kernel of f, denoted $(\operatorname{Ker} f)_0$, is obtained via: let $K = (\operatorname{Coker}(f^*))^{\mathrm{tf}}$ be the torsion free part of $\operatorname{Coker} f^*$, and $K' = \operatorname{Ker} f_*$, then $(\operatorname{Ker} f)_0 = (K, K', [\cdot,\cdot]_K)$, where $[\cdot,\cdot] : K' \times K \to \mathbb{R}$ is the pairing induced from $[\cdot,\cdot]_1$. This is well-defined because, given $\lambda_1' \in K'$ and $\lambda_2 \in \Lambda_2$,
$$[\lambda_1', f^*(\lambda_2)]_1 = [f_*(\lambda_1'), \lambda_2]_2 = [0, \lambda_2]_2 = 0.$$
- Similarly, we define the $\operatorname{Coker} f = (C, C', [\cdot,\cdot]_C)$, where $C = \operatorname{Ker} f^*$, $C' = (\operatorname{Coker} f_*)^{\mathrm{tf}}$.

1.3.3. *Polarization of Tori.* Given a real torus Σ with integral structure, a polarization (a way of measuring angles!) on $\Sigma = (\Lambda, \Lambda', [\cdot,\cdot])$ is an injective map $\xi : \Lambda' \to \Lambda$ such that the induced bilinear form

$$(\cdot,\cdot) : \Lambda' \times \Lambda' \to \mathbb{R} : \qquad (\lambda', \mu') = [\lambda', \xi(\mu')] \qquad \text{is}$$

- *symmetric*
$$(\lambda', \mu') = [\lambda', \xi(\mu')] = [\mu', \xi(\lambda')] = (\mu', \lambda')$$

- *positive definite*

$$(\lambda', \lambda') = [\lambda', \chi(\lambda')] > 0 \qquad \text{for all } \lambda' \neq 0.$$

Definition 1.5. If $\xi : \Lambda' \to \Lambda$ is an isomorphism, then the pair (Σ, ξ) is called a *principally polarized tropical abelian variety.* If ξ fails to be an isomorphism then we call the pair (Σ, ξ) a polarized tropical abelian variety.

Definition 1.6. Let $f : \Sigma_1 \to \Sigma_2$ be a homomorphism of real tori with integral structures given via $f^* : \Lambda_2 \to \Lambda_1$ and $f_* : \Lambda_1' \to \Lambda_2'$ and assume f_* is injective. Furthermore, given a polarization $\xi_2 : \Lambda_2' \to \Lambda_2'$ with an associated bilinear form $(\cdot, \cdot)_2$, we define the *induced polarization* $\xi_1 : \Lambda_1' \to \Lambda_1$ by $\xi_1 = f^* \circ \xi_2 \circ f_*$. This works because f_* is injective and

$$(\lambda_1', \mu_1')_1 = [\lambda_1', \xi_1(\mu_1')]_1 = [\lambda_1', f^*(\xi_2(f_*(\mu_1')))]_2 = [f_*(\lambda_1'), \xi_2(f_*(\mu_1'))]_2 = (f_*(\lambda_1'), f_*(\mu_1'))_2.$$

1.4. **Jacobian.**

1.4.1. *Idea.* Given a metric graph, we can think of a harmonic 1-form on the graph, by assigning a real-valued number (slope) to each edge, such that the sum of the incoming numbers is 0 at each vertex. We denote it by $\Omega(\Gamma)$. In fact, there is a convenient way to think of these 1-forms as 1-cycles i.e, as elements in $H_1(\Gamma, \mathbb{R})$ (this works, because we have assumed the sum of outgoing slopes to add to 0). The dual space of harmonic 1-forms $\Omega^*(\Gamma)$ are those objects that integrate these 1-forms to give a real number. We observe that the homology group $H^1(\Gamma, \mathbb{Z})$ embeds as a lattice in $\Omega^*(\Gamma)$, by integration of the 1-forms along the 1-cycles. Therefore we can define the Jacobian as a quotient lattice:

$$\mathrm{Jac}(\Gamma) = \Omega^*(\Gamma)/H_1(\Gamma, \mathbb{Z}).$$

The beauty of such a quotient is that it inherits the metric structure from $\Omega^*(\Gamma)$.

1.4.2. *Construction of the Jacobian.* We work in a model (G, l) of a metric graph Γ. Furthermore, we fix an orientation on G by defining source and target maps : s and t respectively from $E(G) \to V(G)$ which mark the source and target of each oriented edge. Let $C_0(G, \mathbb{R}) = \mathbb{R}^{V(G)}$ and $C_1(G, \mathbb{R}) = \mathbb{R}^{E(G)}$ denotes simplicial 0-chain and simplicial 1-chain groups with values in $\mathbb{R}$ respectively.

- We have the *boundary* maps

$$d_{\mathbb{R}} = C_1(G, \mathbb{R}) \to C_0(G, \mathbb{R}), \qquad \sum_{e \in E(G)} a_e e \mapsto \sum_{e \in E(G)} a_e[t(e) - s(e)],$$

  then the first *simplicial homology group* with coefficient in $\mathbb{R}$ is given by $H_1(G, \mathbb{R}) := \operatorname{Ker} d_{\mathbb{R}}$.
- We can also define the group of $\mathbb{R}$ valued harmonic 1-forms $\Omega(G, \mathbb{R})$ as

$$\Omega(G, \mathbb{R}) = \left\{ \sum_{e \in E(G)} \omega_e de \quad \middle| \quad \sum_{e : t(e) = v} \omega_e = \sum_{e : s(e) = v} \omega_e \quad \text{for all } v \in V(G) \right\}$$

- Observe that $\Omega(G, \mathbb{R}) \simeq H_1(G, \mathbb{R})$.
- The *integration pairing* for the Jacobian is given by $[\cdot, \cdot] : C_1(G, \mathbb{R}) \times \Omega(G, \mathbb{R}) \to \mathbb{R}$

$$[\gamma, \omega] = \int_\gamma \omega = \sum_{e \in E(G)} \gamma_e \omega_e l(e),$$

  where $\gamma = \sum_{e \in E(G)} \gamma_e e$ is a 1-chain and $\omega = \sum_{e \in E(G)} \omega_e de$ is a 1-form.

- Because of the isomorphism $H_1(G,\mathbb{R}) \simeq \Omega(G,\mathbb{R})$, the above pairing restricts to a *perfect pairing* on $H_1(G,\mathbb{R}) \times \Omega(G,\mathbb{R})$.
- Refinement: If G' is a model obtained from G by subdividing an edge e into two edges e_1 and e_2, because $l(e) = l(e_1) + l(e_2)$, the embedding $C_1(G,\mathbb{R}) \mapsto C_1(G',\mathbb{R})$ that sends $e \mapsto e_1 + e_2$ gives isomorphisms $H_1(G,\mathbb{R}) \mapsto H_1(G',\mathbb{R})$ and $\Omega(G,\mathbb{R}) \mapsto \Omega(G',\mathbb{R})$. Therefore it makes sense to define $H_1(\Gamma,\mathbb{R})$ and $\Omega(\Gamma,\mathbb{R})$ and the above pairing for the metric graph instead of the model.
- All the steps above can be repeated for $\mathbb{Z}$ instead of $\mathbb{R}$.

Definition 1.7. Let $\Lambda = \Omega(\Gamma,\mathbb{Z})$ and $\Lambda' = H_1(\Gamma,\mathbb{Z})$, and $[\cdot,\cdot] : \Lambda' \times \Lambda$ be the integration pairing defined above, and $\xi : H_1(\Gamma,\mathbb{Z}) \to \Omega(\Gamma,\mathbb{Z})$ be the isomorphism sending a 1-cycle to a 1-form. We denote $\Omega^*(\Gamma) = \mathrm{Hom}(\Lambda,\mathbb{R}) = \mathrm{Hom}(\Omega(\Gamma,\mathbb{Z}),\mathbb{R})$. The *Jacobian* is a principally polarized tropical abelian variety with the integral structure:

$$\mathrm{Jac}(\Gamma) = \Omega^*(\Gamma)/H_1(\Gamma,\mathbb{Z})$$

Note that the $\mathrm{rank}(\Lambda) = \mathrm{rank}(\Lambda') = \mathrm{rank}(H_1(\Gamma,\mathbb{Z})) =$ genus of Γ, and the polarization on the Jacobian is given by $(\cdot,\cdot) = [\cdot,\xi(\cdot)] : H_1(\Gamma,\mathbb{Z}) \times H_1(\Gamma,\mathbb{Z}) \to \mathbb{R}$

$$\left(\sum_{e\in E(G)} \gamma_e e, \sum_{e\in E(G)} \delta_e e \right) = \sum_{e\in E(G)} \gamma_e \delta_e l(e).$$

1.4.3. *Abel-Jacobi Map.* Moreover, if we fix a base point $q \in \Gamma$, then we can define the *Abel-Jacobi* map $\Phi_q : \Gamma \to \mathrm{Jac}(\Gamma)$ as follows:

Definition 1.8. Fixing a base point $q \in \Gamma$, a path γ_p from q to p, gives us a map known as the *Abel-Jacobi* map $\Phi_q : \Gamma \to \mathrm{Jac}(\Gamma)$ that takes a 1-form ω and integrates it along the 1-chain corresponding to the path γ_p from q to p.

$$p \mapsto \left(\omega \mapsto \int_{\gamma_p} \omega \right)$$

The Abel-Jacobi map [MZ08] is very nice, in particular

- Φ_q is continuous, and moreover, piecewise linear.
- Φ_q extends by linearity to $\mathrm{Div}(\Gamma)$ and its restriction to $\mathrm{Div}^0(\Gamma)$ does not depend on the base-point q chosen.
- Φ_q collapses any segment that is not part of a cycle, and it maps interior segments of cycles to straight segments.
- Furthermore, this means that we can use addition in $\mathrm{Jac}(\Gamma)$ to define a d-fold (for d a positive integer), Abel-Jacobi map:

  $$\Phi_q^{(d)} : \mathrm{Sym}^d \to \mathrm{Jac}(\Gamma),$$

  where $\mathrm{Sym}^d = \Gamma^d/S_d$ is the d-fold product of Γ upto permutations.
- Furthermore [MZ08] showed that the map $\Phi_q^{(d)} : \mathrm{Div}^d \to \mathrm{Jac}(\Gamma)$ factors through $\mathrm{Pic}^d(\Gamma)$, where $i : \mathrm{Div}^d(\Gamma) \to \mathrm{Pic}^d(\Gamma)$ is the quotient map (modulo linear equivalence of divisors)

  $$\begin{tikzcd} \mathrm{Div}^d(\Gamma) \arrow[r, "i"] \arrow[dr, "\Phi_q^d"] & \mathrm{Pic}^d(\Gamma) \arrow[d, "\rho"] \\ & \mathrm{Jac}(\Gamma) \end{tikzcd}$$

  and in fact, $\rho : \mathrm{Pic}^d(\Gamma) \to \mathrm{Jac}(\Gamma)$ is a bijection.

- This means that we can endow Pic^d with the topology inherited from this bijection. Then the image of *effective divisors* in $\mathrm{Pic}^d(\Gamma)$ corresponds to those divisor classes W_d that contain at least one effective divisor representative, i.e., those divisor classes $[D] \in \mathrm{Pic}^d(\Gamma)$ such that $D \sim E$ for some effective E.

2. Polyhedral structures, Semi-break divisors, Pure dimensionality

Given a model (G, l) of a metric graph Γ, the space $\mathrm{Sym}^d(\Gamma)$ has a natural cellular decomposition with top dimensional cells $C(F)$ indexed by d-tuples $F \subset E(G)$. If the d-tuple $F = \{e_1, \ldots, e_d\}$ contains neither loops, nor repeated edges, then $C(F)$ is a prallelotope obtained by taking the Cartesian product of e_i inside $\mathbb{R}^d$. For a more complete description of polyhedral structure on $\mathrm{Sym}^d(\Gamma)$ one can take a look at [BU22].

To study the effective divisor classes W_d we need the notion of semi-break divisor. To introduce this lets introduce the notion of a break divisor first:

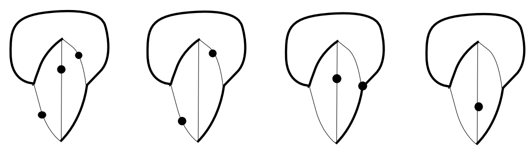

Figure 1. Break (a) and Semi-Break (b), (c), (d) divisors

(1) Suppose we have a graph Γ of genus g, and we pick g disjoint open edge segments, such that if we remove them from Γ we get a tree (genus 0). A break-divisor on Γ is a divisor which corresponds to putting an integer 1 on the closure of each of the g open edge segments.

(2) A semi-break divisor on Γ of degree $d \leq g$ is one which corresponds to putting an integer 1 on the closure of d (which may be less than or equal to g) of the open-edge segments, above.

   Equivalently, a semi-break divisor of degree d can be obtained by taking a break-divisor and replacing $g - d$ of the 1s by 0s.

(3) Now, for $0 \leq d \leq g$, let $\mathbb{SB}_d \subseteq \mathrm{Div}_+^d(\Gamma)$ denote the set of all semi-break divisors of degree d, under the map described above, its pre-image in Γ^d is the union of all sets of the form $\overline{e_1} \times \cdots \times \overline{e_d}$, where $\{e_1, \ldots, e_d\} \subseteq \{e_1, \ldots, e_g\}$, the g many edges that when removed from Γ gives us a tree.

(4) If Σ_d is the interior of $(\mathbb{SB}_d)$, then the pre-image of Σ_d in Γ^d is the union of sets of the form $\{e_1 \times \cdots \times e_d\}$, in particular Σ_d is *open* in $\mathbb{SB}_d$. (This follows from the closed map lemma, because Γ^d is compact, and $\mathrm{Jac}(\Gamma)$ is Hausdorff.)

The following was proven in Theorem 8.3 (b) in [GST22], which roughly says that the map $\Phi_q^{(d)}$ is of degree 1, over $\{e_1, \ldots, e_d\}$.

Lemma 2.1. *For any $D \in \Sigma_d$, we have $|D| = \{D\}$. Here by $|D|$ we mean the set of all effective divisors equivalent to D. The statement then reads that each semi-break divisor is the unique effective divisor in its divisor equivalence class.*

After, a little bit of thought, the above lemma yields the following impressive theorem:

Theorem 2.2. *$W_d \subseteq \mathrm{Pic}^d(\Gamma)$ is of pure-dimension d.*

3. Double Covers and Prym Varieties

In this section, and the rest of the paper, we will discuss double covers of graphs (analogs of classical étale double covers), and pursue efforts to generalize and prove similar statements as in the previous section.

To keep the exposition simple, we will work with free double covers (however, some ideas can be generalized to the more general *dilated* double covers).

Definition 3.1. A local isometry $\pi : \widetilde{\Gamma} \to \Gamma$ is called a *free cover* of degree d if the preimage of every $x \in \Gamma$ consists of d distinct points of $\widetilde{\Gamma}$. A free cover of degree 2 is known as a *free double cover.*

Equivalently, this means that we have a non-trivial ι involution that swaps the fibres $\pi(x) = \iota(\pi(x))$.

Since we want both Γ and in particular its free double cover $\widetilde{\Gamma}$ to be connected, via a computation of Euler characteristic we can check that the genus of $\widetilde{\Gamma}$ is $2g-1$, where g is the genus of Γ.

3.1. Prym Varieties.

For a free double cover $\pi : \widetilde{\Gamma} \to \Gamma$, we have an induced map on divisors, which sends a divisor $\widetilde{D} = \sum_i a_i \cdot y_i$ on $\widetilde{\Gamma}$ to the corresponding divisor $D = \sum_i a_i \cdot \pi(y_i)$ on Γ. Furthermore, this map respects the equivalence on divisors, and therefore we obtain

$$\pi_* : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Jac}(\Gamma)$$

a surjective map between the Jacobian of the free double cover and the graph. The map π_* is known as the *norm* map.

Definition 3.2. For a free double cover $\pi : \widetilde{\Gamma} \to \Gamma$, the kernel of the norm map has two connected components:

- The component that contains 0 is a subgroup of the Jacobian, and called the *even Prym variety* denoted $\mathrm{Prym}_0(\widetilde{\Gamma}/\Gamma)$, while
- The other component is called the *odd Prym variety* and is denoted $\mathrm{Prym}_1(\widetilde{\Gamma}/\Gamma)$.

3.1.1. *Prym Variety as a Tropical Abelian Variety.*

Let $\widetilde{\Lambda} = \Omega(\widetilde{\Gamma}, \mathbb{Z})$, $\widetilde{\Lambda}' = H_1(\widetilde{\Gamma}, \mathbb{Z})$ and $\Lambda = \Omega(\Gamma, \mathbb{Z})$, $\Lambda' = H_1(\Gamma, \mathbb{Z})$, we pick on oriented model $p : \widetilde{G} \to G$ for π, and consider the pushforward $\pi_* : H_1(\widetilde{\Gamma}, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z})$ and pullback $\pi^* : \Omega(\Gamma, \mathbb{Z}) \to \Omega(\widetilde{\Gamma}, \mathbb{Z})$ defined as

$$\pi_* \left(\sum_{\widetilde{e} \in E(\widetilde{G})} a_{\widetilde{e}} \widetilde{e} = \sum_{\widetilde{e} \in E(\widetilde{G})} a_{\widetilde{e}} \pi(\widetilde{e}) \right), \qquad \pi^* \left(\sum_{e \in E(G)} a_e de \right) = \sum_{e \in E(G)} a_e (d\widetilde{e}^+ + d\widetilde{e}^-).$$

One can check that the maps π_* and π^* give a homomorphism $\mathrm{Nm} : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Jac}(\Gamma)$ of tori with integral structure. Therefore the Prym variety $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ is a real torus with integral structure $(\mathrm{Ker}\,\mathrm{Nm})_0 = (K, K', [\cdot,\cdot]_K)$, where $K = (\mathrm{Coker}\,\pi^*)^{\mathrm{tf}}$ and $K' = \mathrm{Ker}\,\pi_*$, and $[\cdot,\cdot]_K$ is the induced pairing from the integration pairing on $\widetilde{\Gamma}$.

Remark. One can use the polarization $\widetilde{\xi} : H_1(\widetilde{\Gamma}, \mathbb{Z}) \to \Omega(\widetilde{\Gamma}, \mathbb{Z})$ on $\mathrm{Jac}(\widetilde{\Gamma})$ to induce a polarization $i^*\widetilde{\xi} : K' \to K$ on $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$. However, (from Theorem 2.2.7 [LU21] one observes that this induced polarization is not principal. The correct principal polarization turns out to be half the induced polarization i.e., $\psi : K' \to K$ given by $\psi = \frac{1}{2}i^*\widetilde{\xi}$. Therefore we obtain an inner product $(\cdot,\cdot)_P)$ on $\operatorname{Ker}\pi_*$, namely for $\gamma,\delta \in \operatorname{Ker}\pi_*$ and $\gamma = \sum_{\widetilde{e}\in E(\widetilde{\Gamma})}\gamma_{\widetilde{e}}\widetilde{e}, \quad \delta = \sum_{\widetilde{e}\in E(\widetilde{\Gamma})}\delta_{\widetilde{e}}\widetilde{e}$

$$(\cdot,\cdot)_P = [\gamma,\psi(\delta)] = \frac{1}{2}[\gamma,\widetilde{\xi}(\delta)] = \frac{1}{2}\sum_{\widetilde{e}\in E(\widetilde{\Gamma})}\gamma_{\widetilde{e}}\delta_{\widetilde{e}}l(\widetilde{e}).$$

3.2. **Abel-Prym Map and Pure Dimensionality problem.** Do we have nice representatives for a divisor $\widetilde{D} \in \operatorname{Prym}(\widetilde{\Gamma}/\Gamma)$? The answer is yes. It is clear that for a an effective divisor $E \in \operatorname{Div}(\widetilde{\Gamma})$, the divisor $E - \iota E \in \operatorname{Prym}(\widetilde{\Gamma}/\Gamma)$. Moreover, we also have the following nice fact [LZ22]:

Lemma 3.1. *If $\widetilde{D} \in \operatorname{Prym}(\widetilde{\Gamma}/\Gamma)$, then $\widetilde{D} \sim E - \iota E$, for some effective divisor $E \in \operatorname{Div}(\widetilde{\Gamma})$.*

This allows us to define the Abel-Prym map

Definition 3.3. Let $\pi : \widetilde{\Gamma} \to \Gamma$ be a free double cover of metric graphs. The *Abel-Prym* map of degree d is the map

$$\Psi^{(d)} : \operatorname{Sym}^d(\widetilde{\Gamma}) \to \operatorname{Prym}_\epsilon(\widetilde{\Gamma}/\Gamma)$$

where $\epsilon \equiv d \mod 2$ given by $\Psi^{(d)}(p_1 + \cdots + p_d) = [p_1 + \cdots + p_d - \iota p_1 - \cdots - \iota p_d]$.

Equivalently, if we fix a point q, then the Abel-Prym map is the map

$$\begin{aligned}\Psi_q : &\widetilde{\Gamma} \to \operatorname{Prym}(\widetilde{\Gamma}/\Gamma)\\ &p \mapsto \left(\omega \mapsto \int_{\gamma_p}\omega - \int_{\iota_*(\gamma_p)}\omega\right),\end{aligned}$$

where γ_p is a path in $\widetilde{\Gamma}$ from q to p. The former definition above is the d-fold version of the latter definition.

Conjecture 3.2 (Conjecture 4.28 in [LZ22])**.** *For $1 \le d \le g-1$, let Y_d denote the image of the d-fold Abel-Prym map $\Psi_q^{(d)} : \operatorname{Sym}^d(\widetilde{\Gamma}) \to \operatorname{Prym}(\widetilde{\Gamma}/\Gamma)$, then Y_d is pure-dimensional.*

In the rest of the paper, we focus our attention to this conjecture, and discuss potential tools that can help resolve the conjecture. Observe that this is remarkably similar to the pure dimensionality of the Abel-Jacobi map in Theorem 2.2.

In fact, the above conjecture would allow one to prove a tropical version of the Poincaré-Prym formula.

Conjecture 3.3 (Conjecture 4.26 in [LZ22])**.** *If $\widetilde{Y}_d$ is the image of the d-fold Abel-Prym map $\Psi^d : \widetilde{\Gamma}^d \to \operatorname{Prym}(\widetilde{\Gamma}/\Gamma)$ then*

$$[\widetilde{Y}_d] = \frac{2^d}{(g_0-d)!}[\Xi]^{g_0-d} \quad \in \quad H_{d,d}(\operatorname{Prym}(\widetilde{\Gamma}/\Gamma)),$$

where $[\Xi]$ is the class of the principal polarization of $\operatorname{Prym}(\widetilde{\Gamma}/\Gamma)$.

The following is a nice theorem proven Theroem 4.1 in [LZ22].

Theorem 3.4. *Let $\widetilde{F} = \{\widetilde{f}_1,\ldots,\widetilde{f}_d \subset E(\widetilde{G})\}$ be a multiset (repetitions allowed) of edges of $\widetilde{G}$, let $C(\widetilde{F}) \subset \operatorname{Sym}^d(\widetilde{\Gamma})$ be the corresponding top dimensional cell*

- *If the edges in $F = \{f_1, \ldots, f_d\}$ where $f_i = p(\widetilde{f}_i)$ are not distinct, then the Abel-Prym map Ψ^d contracts the cell $C(\widetilde{F})$.*
- *If the edges in F are distinct, then Ψ^d does not contract $C(\widetilde{F})$ if and only if the preimage under p of each connected component of $E(G) \setminus F$ is connected.*

3.2.1. *Abel-Prym map on highest degree.* As a special case if we focus on $d = g - 1$, and we denoted the Abel-Prym map Ψ^{g-1} by

$$\Psi : \mathrm{Sym}^{g-1}(\widetilde{\Gamma}) \to \mathrm{Prym}^{[g-1]}(\widetilde{\Gamma}/\Gamma) \text{ given by } \Psi(\widetilde{D}) = \widetilde{D} - \iota(\widetilde{D}).$$

Before we write a corollary of the previous theorem, we make a definition:

Definition 3.4. Given a connected free double cover $p : \widetilde{G} \to G$ of a graph G of genus g, a subset $F \subset E(G)$ of $g - 1$ edges is known as *relative spanning tree* or *odd genus one decomposition* (abbreviated OGOD) of rank r if $E(G) \setminus F$ consists of r connected components of genus one, each having a connected preimage in $\widetilde{G}$.

Corollary 3.4.1 ([LZ22] Corollary 4.2)**.** *The degree of the Abel-Prym map over a top-dimensional cell $\widetilde{F}$ corresponding to a multiset $\widetilde{F} = \{\widetilde{f}_1, \ldots, \widetilde{f_{g-1}} \subset \mathrm{Sym}^{g-1}(\widetilde{\Gamma})\}$ is equal to*

$$\deg_\Psi(\widetilde{F}) = \begin{cases} 2^{r-1} & \text{when edges of } F = \{f_1, \ldots, f_{g-1}\} \text{ are distinct and form an OGOD of rank } r, \\ 0 & \text{otherwise.} \end{cases}$$

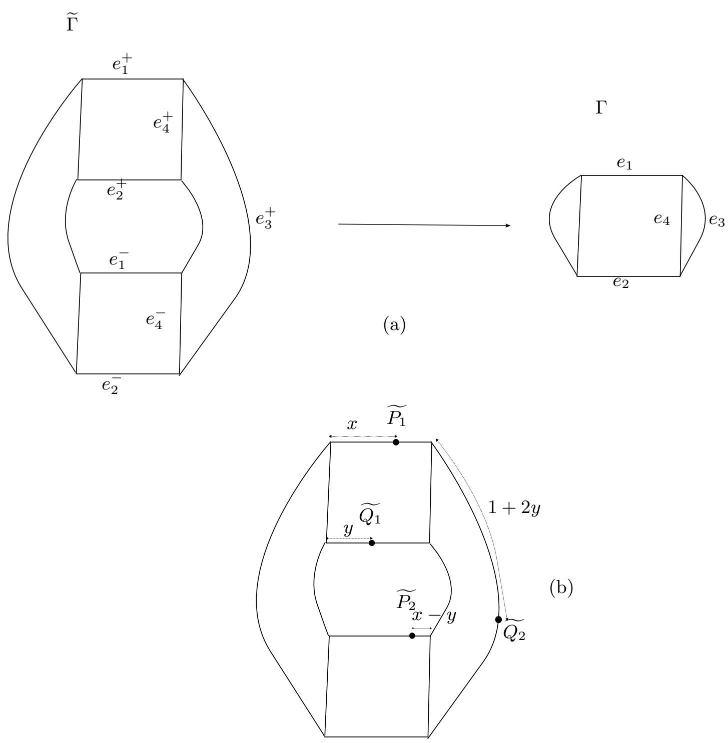

FIGURE 2. (a) An example of a double cover genus 5 of a graph of genus 3 and (b) divisors on the double cover.

4. Examples and Observations

(1) Consider the double cover in Figure 2. In Γ all edges are length 1, except edge e_3 which we take to be of length 3. Each edge has two copies in the double cover, which we label by $+$ and $-$ respectively.

Let $\widetilde{P_1}, \widetilde{Q_1}, \widetilde{P_2}, \widetilde{Q_2}$ be points on $\widetilde{\Gamma}$ as shown in (b). If $\widetilde{D_1} = \widetilde{P_1} + \widetilde{Q_1}$ and $\widetilde{D_2} = \widetilde{P_2} + \widetilde{Q_2}$, are two divisors, then the divisors

$$\widetilde{D_1} - \iota\widetilde{D_1} = \widetilde{D_2} - \iota\widetilde{D_2}$$

in the $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$, because $\left(\widetilde{D_1} - \iota\widetilde{D_1}\right) - \left(\widetilde{D_2} - \iota\widetilde{D_2}\right)$ is a Principal divisor in $\widetilde{\Gamma}$.

(2) In this case $g = 3$, so an *odd genus one decomposition* will consist of $g - 1 = 2$ edges from Γ. We write two examples below:
- Let us inspect Figure 3. $F = \{e_1, e_2\}$ (edges colored brown). Then the complement $E(G) \setminus F$ consists of the edges colored blue. In particular it contains two connected components of genus 1, each of whose pre-image (again colored blue) is connected.

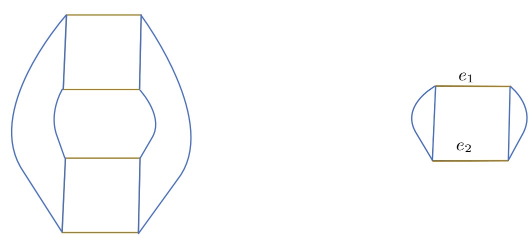

Figure 3. $\{e_1, e_2\}$ is an odd genus one decomposition

- Let us now inspect Figure 4. $F = \{e_1, e_3\}$ (edges colored green). The complement $E(G) \setminus F$ consists of a 1 connected componenent of genus 1, whose pre-image (again colored blue) is connected.

Figure 4. $\{e_1, e_3\}$ is an odd genus one decomposition

If we vary the values of x, y such that $0 < y < x < 1$, we obtain two polyhedral cells in $\mathrm{Sym}^2(\widetilde{\Gamma})$ having the same image in $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$:
- $C_1 = \{(x, y) \quad | \quad 0 < y < x < 1\}$ of $e_1^+ \times e_2^+$. The volume of C_1 is equal to $\frac{1}{2}$, but its rank is 2; and
- $C_2 = \{(x - y, 1 + 2y) \quad | \quad 0 < y < x < 1\}$ of $e_1^- \times e_3^+$. The volume of C_2 is equal to 1 but its rank is 1.

The volume times the rank is the same which is because both the cells correspond to the same cell in $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$.

4.0.1. *Ongoing Work.*

- Instead of the double cover, one can equivalently work with *signed graphs* as observed in [Zas82]. In fact, one can go a step further, and introduce *signed metric graphs*. The odd genus one decomposition, then becomes an analog for the circuit (smallest dependent set) of these signed graphs.
- The scenario between break divisors, and divisors supported on odd genus one decompositions is remarkably similar. The difficulty in working with odd genus one decomposition is that different cells collapse to the same cell in the Prym variety, and they collapse with different ranks. One can think about the lower degree analogs of the odd genus one decompositions, by taking a subset of the $g-1$ edges that build the odd genus one decomposition.
- Even in the subset of the odd genus one decomposition, because the rank is not 1, the cells collapse, and its hard to show pure dimensionality. One needs to construct an object that lies in the middle between the cells supported by the odd genus one decomposition and the Prym which would remove the collapsing. This can be done if one finds a unique representative in each odd genus one decomposition class.
- Perhaps we can take some inspiration from what we know classically, in Corollary A.15 of [LZ22] they show that in the classical case for $d < \frac{g}{2}$, the degree d Abel-Prym map is indeed of rank 1. Therefore one can inspect the lower degree odd genus one decompositions for $d < \frac{g}{2}$ and hope for uniqueness.
- If one is able to show Pure Dimensionality of the Abel-Prym map, then via Conjecture 4.26 of [RZ22] one has the tropical Poincaré-Prym formula:

  If $\widetilde{Y}_d$ is the image of the d-fold Abel-Prym map $\Psi^d : \widetilde{\Gamma}^d \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ then

  $$[\widetilde{Y}_d] = \frac{2^d}{(g_0 - d)!}[\Xi]^{g_0 - d} \quad \in \quad H_{d,d}(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)),$$

  where $[\Xi]$ is the class of the principal polarization of $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$.

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