-:Linear Algebra:-

Exercise 1.4

(1) $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$

Matrix Ax is not defined because the number of columns (2) in 3×2 matric does ~~do~~ not match the number of entries (3) in the matrix x.

(3) $\begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$$Ax = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$= (-2) \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 18 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 8 \\ 16 \end{bmatrix}$$

(5) $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

L.H.S

$$= 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ -3 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 2 - 3 - 1 \\ -4 + 3 + 1 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

L.H.S = R.H.S

Hence, proved.

(7) $x_1\begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2\begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$

$4x_1 - 5x_2 + 7x_3 = 6$

$-x_1 + 3x_2 - 8x_3 = -8$

$7x_1 - 5x_2 + 0 = 0$

$-4x_1 + x_2 + 2x_3 = -7$

$$\begin{bmatrix} 4x_1 \\ -x_1 \\ 7x_1 \\ -4x_1 \end{bmatrix} + \begin{bmatrix} -5x_2 \\ 3x_2 \\ -5x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} 7x_3 \\ -8x_3 \\ 0x_3 \\ 2x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

(9) $5x_1 + x_2 - 3x_3 = 8$

$2x_2 + 4x_3 = 0$

First of all we write in vector equation

$$x_1 \begin{bmatrix} 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

Now, write in matrix equation

$$\begin{bmatrix} 5 & 1 & -3 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

(11)

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & 6 \end{bmatrix}, b = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 1 & 5 & 2 & 4 \\ -3 & -7 & 6 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 2 & 6 & 6 \\ -3 & -7 & 6 & 12 \end{bmatrix} \quad R_2 - R_1$$

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 2 & 6 & 6 \\ 0 & 2 & -6 & 6 \end{bmatrix} \quad R_3 + 3R_1$$

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 2 & 6 & 6 \\ 0 & 0 & -12 & 0 \end{bmatrix} \quad R_3 - R_2$$

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -12 & 0 \end{bmatrix} \quad R_2 \text{ divide with } 2$$

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_3 \text{ divide with } -12$$

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_2 - 3R_3$$

$$\begin{bmatrix} 1 & 0 & -4 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_1 - 3R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_1 + 4R_3$$

$x_1 = -11$

$x_2 = 3$

$x_3 = 0$

In a vector solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 0 \end{bmatrix}$

(13) Let $u = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix} \quad R_1 + R_2$$

$$\begin{bmatrix} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{Exchange } R_1 \,\&\, R_3$$

$$\begin{bmatrix} 1 & 1 & 4 \\ -1 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix} \quad R_2 \text{ divide with } 2$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & 6 \\ 1 & 1 & 0 \end{bmatrix} \quad R_2 + R_1$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & -4 \end{bmatrix} \quad R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \text{ divide with } -4.$$

The equation $Ax = 4$ having no solution. So u is not spanned by column by A.

(14) Let $u = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 5 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 4 \end{bmatrix} \quad 2R_3 - R_1$$

$$\begin{bmatrix} 2 & 5 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad R_3 + R_2$$

$Ax = u$ has no solution because u is not spanned by columns of A.

(15) Let $A = \begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\begin{bmatrix} 3 & -1 & b_1 \\ -9 & 3 & b_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & b_1 \\ -3 & 1 & \frac{b_2}{3} \end{bmatrix} \quad R_2 \text{ divide with } 3$$

$$\begin{bmatrix} 3 & -1 & b_1 \\ 0 & 0 & b_1 + \frac{b_2}{3} \end{bmatrix} \quad R_2 + R_1$$

$$b_1 + \frac{b_2}{3} = 0$$

$$\frac{3b_1 + b_2}{3} = 0$$

$$3b_1 + b_2 = 0$$

This equation shows that it is inconsistent and does not having any solution in it.

(17) $\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ $R_2 + R_1$

$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix}$ $R_4 - 2R_1$

$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ $R_4 + 3R_2$

$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -2 & 1 & -4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ R_3 divide with 2

$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & -2 & 1 & -4 \end{bmatrix}$ Exchange R_3 with R_4

$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_4 + R_2$

Every row is not on pivot position. Row 4 is free. So $Ax=b$ does not have any solution. It is inconsistent.

(18) $\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 0 & 1 & 3 & -11 \end{bmatrix}$ $R_4 - 2R_1$

$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 15 \\ 0 & 1 & 3 & -11 \end{bmatrix}$ $R_3 - 2R_2$

$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & -7 \end{bmatrix}$ $R_4 - R_2$

No, Every vector in R^4 does not written as a linear combination of the column of B matrix. Because each row does not contain a pivot position.
No, the column of B does not span R^3.

(22) $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad R_3 + R_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad R_4 + R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad R_1 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad R_2 \text{ divide with} (-1)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_4 + R_3$$

(22) $u_1 \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}, u_3 = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

$$[u_1, v_2, u_3] = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & 6 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 9 & 6 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix} \text{ Interseet } R_3 \,\&\, R_1$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix} \quad R_3 \div (-3)$$

R_3 spane the matrix.

(25) $\begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 10 \end{bmatrix}$

$Ax = b$

So,

$$-3 \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} - 1 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 10 \end{bmatrix}$$

$c_1 = -3, \; c_2 = -1, \; c_3 = 2$

$$= \begin{bmatrix} -12 \\ -15 \\ 18 \end{bmatrix} - \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 10 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} -12+3+2 \\ -15+2+10 \\ 18-2-6 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$$

So, L.H.S = R.H.S

(26) $u = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}, \; v = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \; w = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$

$$2u - 3v - w = 0 \longrightarrow (1)$$

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

from equ (1)

$$2 \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

So, $x_1 = 2$, $x_2 = 3$

Proof:-

$$= x_1 \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} + (-3) \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -9 \\ 4 & -3 \\ 10 & -9 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

So,

$L \cdot H \cdot S = R \cdot H \cdot S$

(27) $$\begin{bmatrix} -3 & 5 & -4 & 7 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \end{bmatrix}$$

$$C_1 V_1 + C_2 V_2 + \dots + C_n V_n = b$$

So,

$$3 \begin{bmatrix} -3 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 5 \\ 8 \end{bmatrix} + 2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 9 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \end{bmatrix}$$

$C_1 = -3$, $V_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

$C_2 = 1$, $V_2 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$C_3 = 2$, $V_3 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$C_4 = -1$, $V_4 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$

$C_5 = 2$, $V_5 = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$

$V_6 = \begin{bmatrix} 11 \\ 11 \end{bmatrix}$

So,

$$C_1 V_1 + C_2 V_2 + C_3 V_3 + C_4 V_4 + C_5 V_5 = V_6$$