$$L^{-1}\left\{\frac{}{s^2+49}\right\} = \left(\frac{}{s^2+(7)^2}\right)$$

$$= \frac{5}{7} L^{-1}\left\{\frac{7}{s^2+7^2}\right\}$$

$$= \frac{5}{7}\sin 7t$$

9. $L^{-1}\left\{\dfrac{4s}{4s^2+1}\right\}$

$$L^{-1}\left\{\frac{4s}{4s^2+1}\right\} = 4L^{-1}\left\{\frac{s}{4s^2+1}\right\}$$

$$= 4L^{-1}\left\{\frac{s}{4(s^2+1/4)}\right\}$$

$$= L^{-1}\left\{\frac{s}{s^2+(1/2)^2}\right\}$$

$$= \cos\frac{1}{2}t$$

10. $L^{-1}\left\{\dfrac{2s-6}{s^2+9}\right\}$

$$L^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = 2L^{-1}\left\{\frac{s}{s^2+9}\right\} - 6L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= 2\cos 3t - \frac{6}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= 2\cos 3t - 2\sin 3t.$$

Partial Fractions:

Evaluate $L^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$

Solution:

Let

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+A)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4} \quad \text{—(1)}$$

Multiply (1) with $(s-1)(s-2)(s+4)$

$$\Rightarrow s^2+6s+9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2) \rightarrow (*)$$

Put $s=2$ in $(*)$

$$\Rightarrow 4+12+9 = B(1)(6) \Rightarrow 6B = 25$$

$$\Rightarrow \boxed{B = \frac{25}{6}}$$

(a) Put $s=1$ in (*)

$1+6+9 = A(-1)(5) \Rightarrow -5A = 16 \Rightarrow \boxed{A = -\frac{16}{5}}$

Put $s=-4$ in (*)

$16-24+9 = C(-5)(-6) \Rightarrow +30C = 1 \Rightarrow \boxed{C = \frac{1}{30}}$

So ① becomes

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{-16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}$$

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16}{5}\cdot\frac{1}{s-1} + \frac{25}{6}\cdot\frac{1}{s-2} + \frac{1}{30}\cdot\frac{1}{s+4}$$

Taking inverse Laplace transform

$$L^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}L^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

(b) $L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\}$

Solution:

Let

$$L\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2} \quad — ①$$

$$\Rightarrow \quad 1 = (As+B)(s^2+b^2) + (Cs+D)(s^2+a^2) \quad — ②$$

$$1 = As^3 + Asb^2 + Bs^2 + Bb^2 + Cs^3 + Csa^2 + Ds^2 + Da^2 \quad —(*)$$

Comparing coefficient of s^3

$\Rightarrow \quad A + C = 0 \Rightarrow \boxed{A = -C}$

Comparing coefficient of s^2

$B + D = 0 \Rightarrow \boxed{B = -D}$

Comparing coefficient of s

$Ab^2 + Ca^2 = 0 \Rightarrow Ab^2 - Aa^2 = 0$

$\Rightarrow A(b^2-a^2) = 0$

$\Rightarrow \boxed{A=0}$ and $\boxed{C=0}$

Comparing coefficient of constant

$1 = Bb^2 + Da^2 \Rightarrow Bb^2 - Ba^2 = 1$

$B(b^2-a^2) = 1 \Rightarrow \boxed{B = \frac{1}{b^2-a^2}}$

As $B = -D \Rightarrow \boxed{D = \frac{-1}{b^2-a^2}}$

So eq. (1) becomes

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1/b^2-a^2}{s^2+a^2} + \frac{\left[-1/b^2-a^2\right]}{s^2+b^2}$$

$$= \frac{1}{b^2-a^2}\left(\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right)$$

Taking inverse Laplace transform

$$L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{b^2-a^2}\left\{L^{-1}\left[\frac{1}{s^2+a^2}\right] - L^{-1}\left[\frac{1}{s^2+b^2}\right]\right\}$$

$$= \frac{1}{b^2-a^2}\left\{\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right] - \frac{1}{b}L^{-1}\left[\frac{b}{s^2+b^2}\right]\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{b^2-a^2}\left\{\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right\}$$

3. $L^{-1}\left\{\frac{2s^2+5s+7}{(s-2)(s^2+4s+13)}\right\}$

Solution.

Let

$$L^{-1}\left\{\frac{2s^2+5s+7}{(s-2)(s^2+4s+13)}\right\} = \frac{A}{s-2} + \frac{Bs+C}{s^2+4s+13} \quad \text{— (1)}$$

$$\Rightarrow 2s^2+5s+7 = A(s^2+4s+13) + (Bs+C)(s-2) \quad \text{— (2)}$$

$$2s^2+5s+7 = As^2+4As+13A+Bs^2+Cs-2Bs-2C \quad \text{— (*)}$$

Put $s=2$ in (*)

$$8+10+7 = A(4+8+13) \Rightarrow 25A = 25$$

$$\Rightarrow \boxed{A=1}$$

Comparing coefficient of s^2 in (*)

$$2 = A+B \Rightarrow \boxed{A = B-2} \quad \because A = 1 \Rightarrow \boxed{B=1}$$

Comparing coefficient of s in (*)

$$5 = 4A+C-2B \Rightarrow$$

$$5 = 4+C-2$$

$$\Rightarrow \boxed{C=3}$$

(1.

So ① becomes

$$\frac{2s^2}{(s-2)(s^2+4s+13)} = \frac{1}{s-2} + \frac{s+3}{s^2+4s+13}$$

$$= \frac{1}{s-2} + \frac{s+3}{s^2+4s+4+13-4}$$

$$= \frac{1}{s-2} + \frac{s+3}{(s+2)^2+9}$$

Taking inverse Laplace transform

$$L^{-1}\left\{\frac{2s^2}{(s-2)(s^2+4s+13)}\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} + L^{-1}\left\{\frac{s+3}{(s^2+2)^2+9}\right\}$$

$$= e^{2t} + L^{-1}\left\{\frac{s+2+1}{(s^2+2)^2+3^2}\right\}$$

$$= e^{2t} + L^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\} + L^{-1}\left\{\frac{1}{(s+2)^2+3^2}\right\}$$

$$= e^{2t} + e^{-2t}\cos 3t + \frac{1}{3} L^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{2s^2}{(s-2)(s^2+4s+13)}\right\} = e^{2t} + e^{-2t}\cos 3t + \frac{1}{3} e^{-2t}\sin 3t.$$

4. $L^{-1}\left\{\frac{s+7}{s^2+2s+5}\right\}$

Solution:

$$\frac{s+7}{s^2+2s+5} = \frac{s+7}{s^2+2s+1+4}$$

$$= \frac{s+7}{(s+1)^2+2^2}$$

$$= \frac{s+1+6}{(s+1)^2+2^2} = \frac{s+1}{(s+1)^2+2^2} + \frac{6}{(s+1)^2+2^2}$$

Taking inverse Laplace transform

$$L^{-1}\left\{\frac{s+7}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + 3\,L^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$= e^{-t}\cos 2t + 3e^{-t}\sin 2t.$$

Transforms of Derivatives:

We will use Laplace transform to solve differential equations. For this we need to evaluate quantities such as $L\left\{\frac{dy}{dt}\right\}$ and $L\left\{\frac{d^2y}{dt^2}\right\}$. (11)

Thus,

$$L\{f'(t)\} = sF(s) - f(0)$$

Similarly

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Also

$$L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

In general

$$L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0),$$

where $F(s) = L\{f(t)\}$.

1. Use the Laplace transform to solve the IVP

$$\frac{dy}{dt} + 3y = 13\sin 2t, \quad y(0) = 6$$

Solution:

Given that

$$\frac{dy}{dt} + 3y = 13\sin 2t \quad -(1) \qquad y(0) = 6$$

Taking Laplace transform

$$L\left\{\frac{dy}{dt}\right\} + 3L\{y\} = 13L\{\sin 2t\}$$

$$sF(s) - f(0) + 3F(s) = 13 \cdot \frac{2}{s^2+2^2}$$

$$sF(s) - 6 + 3F(s) = \frac{26}{s^2+4}$$

$$(s+3)F(s) = 6 + \frac{26}{s^2+4}$$

$$\Rightarrow \quad F(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)}$$

Taking inverse Laplace transform

$$L^{-1}\{F(s)\} = 6L^{-1}\left\{\frac{1}{s+3}\right\} + 26L^{-1}\left\{\frac{1}{(s+3)(s^2+4)}\right\}$$

(12)

$$y(t) = 6e^{-3t} + 26\,L^{-1}\left\{\frac{1}{(s+3)(s^2+4)}\right\} \quad — (*)$$

Let

$$\frac{1}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4} \rightarrow (1)$$

$$\Rightarrow \quad 1 = A(s^2+4) + (Bs+C)(s+3) \quad — (2)$$

$$1 = As^2 + 4A + Bs^2 + 3Bs + Cs + 3C \quad — (3)$$

Put $s = -3$ in (2)

$$1 = 13A + 0 \Rightarrow \boxed{A = \frac{1}{13}}$$

By comparing coefficient of s^2 in (3)

$$A + B = 0$$

$$\frac{1}{13} + B = 0 \Rightarrow \boxed{B = \frac{-1}{13}}$$

By comparing coefficient of s

$$3B + C = 0$$

$$\boxed{C = +\frac{3}{13}}$$

So (1) becomes

$$\frac{1}{(s+3)(s^2+4)} = \frac{1/13}{s+3} + \frac{s(-1/13) + 3/13}{\cancel{(s+3)}(s^2+4)}$$

$$= \frac{1}{13(s+3)} + \frac{3-s}{13(s^2+4)}$$

So (*) becomes

$$y(t) = 6e^{-3t} + 26\,L^{-1}\left\{\frac{1}{13(s+3)} + \frac{3-s}{13(s^2+4)}\right\}$$

$$= 6e^{-3t} + 2L^{-1}\left\{\frac{1}{s+3}\right\} + L^{-1}\left\{\frac{6}{s^2+4}\right\} - 2L^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$= 6e^{-3t} + 2e^{-3t} + \frac{6}{2}L^{-1}\left\{\frac{2}{s^2+4}\right\} - 2\cos 2t$$

$$\Rightarrow \boxed{y(t) = 8e^{-3t} + 3\sin 2t - 2\cos 2t.}$$

Solve $y'' - 3y' + 2y = e^{-4t}$, $y(0)=1$, $y'(0)=5$.

Solution:

Given that

$$y'' - 3y' + 2y = e^{-4t} \quad - (1) \qquad y(0)=1, \quad y'(0)=5$$

Taking Laplace transform

$$L\{y''\} - 3\{y'\} + 2L\{y\} = L\{e^{-4t}\}$$

$$s^2F(s) - sf(0) - f'(0) - 3\{sF(s) - f(0)\} + 2F(s) = \frac{1}{s+4}$$

$$s^2F(s) - s(1) - 5 - 3sF(s) + 3(1) + 2F(s) = \frac{1}{s+4}$$

$$s^2F(s) + 2F(s) - 3sF(s) - s - 2 = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)F(s) = s + 2 + \frac{1}{s+4}$$

$$\Rightarrow \quad F(s) = \frac{s+2}{s^2-3s+2} + \frac{1}{(s^2-3s+2)(s+4)}$$

$$= \frac{s+2}{(s-1)(s-2)} + \frac{1}{(s-1)(s-2)(s+4)}$$

$$= \frac{(s+2)(s+4)+1}{(s-1)(s-2)(s+4)}$$

$$F(s) = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$$

Taking inverse Laplace transform

$$\Rightarrow \quad y(t) = L^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

By Partial fraction it can be written as

$$y(t) = L^{-1}\left\{\frac{16}{5(s-1)}\right\} + L^{-1}\left\{\frac{25}{6(s-2)}\right\} + L^{-1}\left\{\frac{1}{30(s+4)}\right\}$$

$$= \frac{16}{5}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}L^{-1}\left\{\frac{1}{s+4}\right\}$$

$$\Rightarrow \quad \boxed{y(t) = \frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}}$$