DATA STRUCTURES AND ALGORITHMS

Lecture 10: Trees

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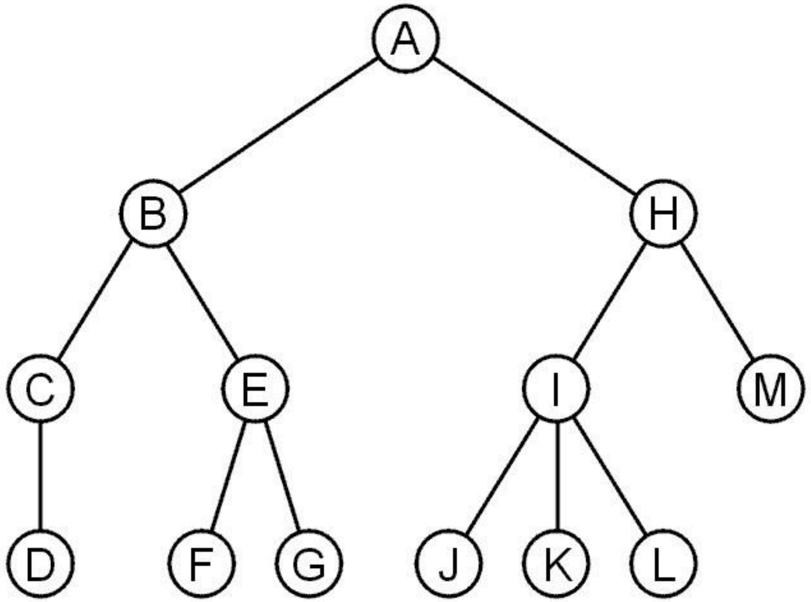
TREES

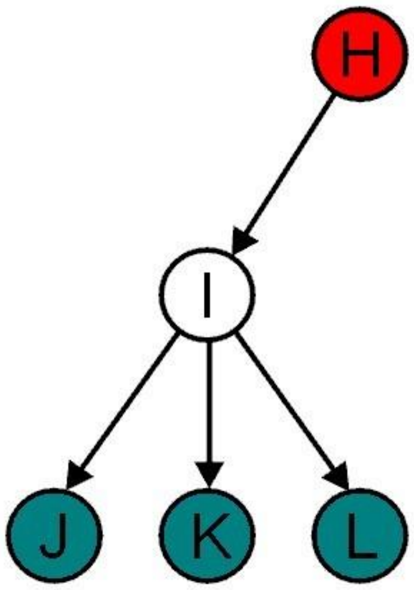
- A rooted tree data structure stores information in nodes
- Similar to linked lists:
  - There is a first node, or root
  - Each node has variable number of references to successors
  - Each node, other than the root, has exactly one node pointing to it



TERMINOLOGY: PARENT CHILD RELATIONS

- All nodes have zero or more child nodes or children
  - I has three children: J, K and L

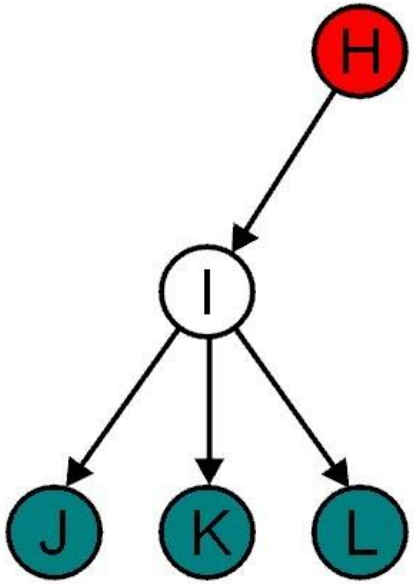
- For all nodes other than the root node, there is one parent node
  - H is the parent I



TERMINOLOGY: DEGREE

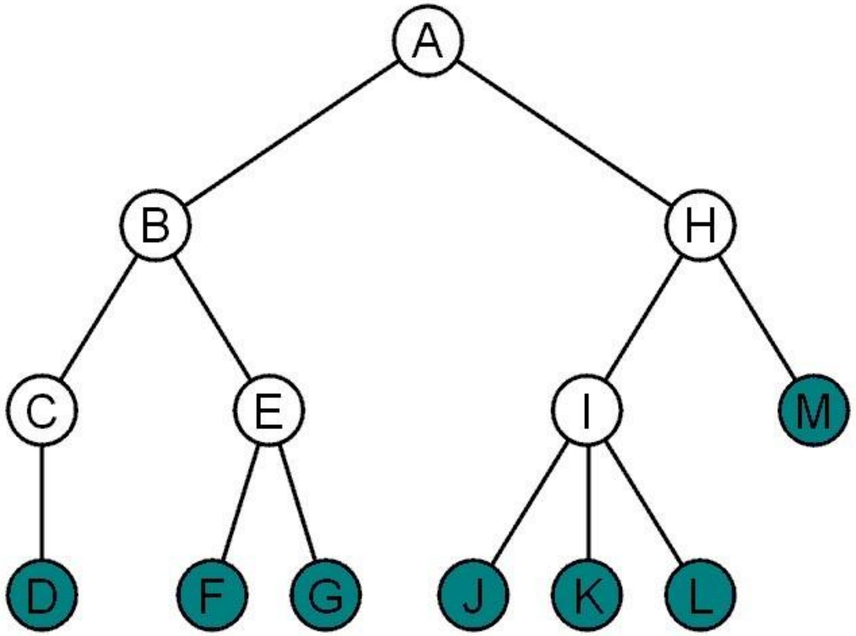
- The degree of a node is defined as the number of its children
  - `deg(I) = 3`

- Nodes with the same parent are siblings
  - J, K, and L are siblings



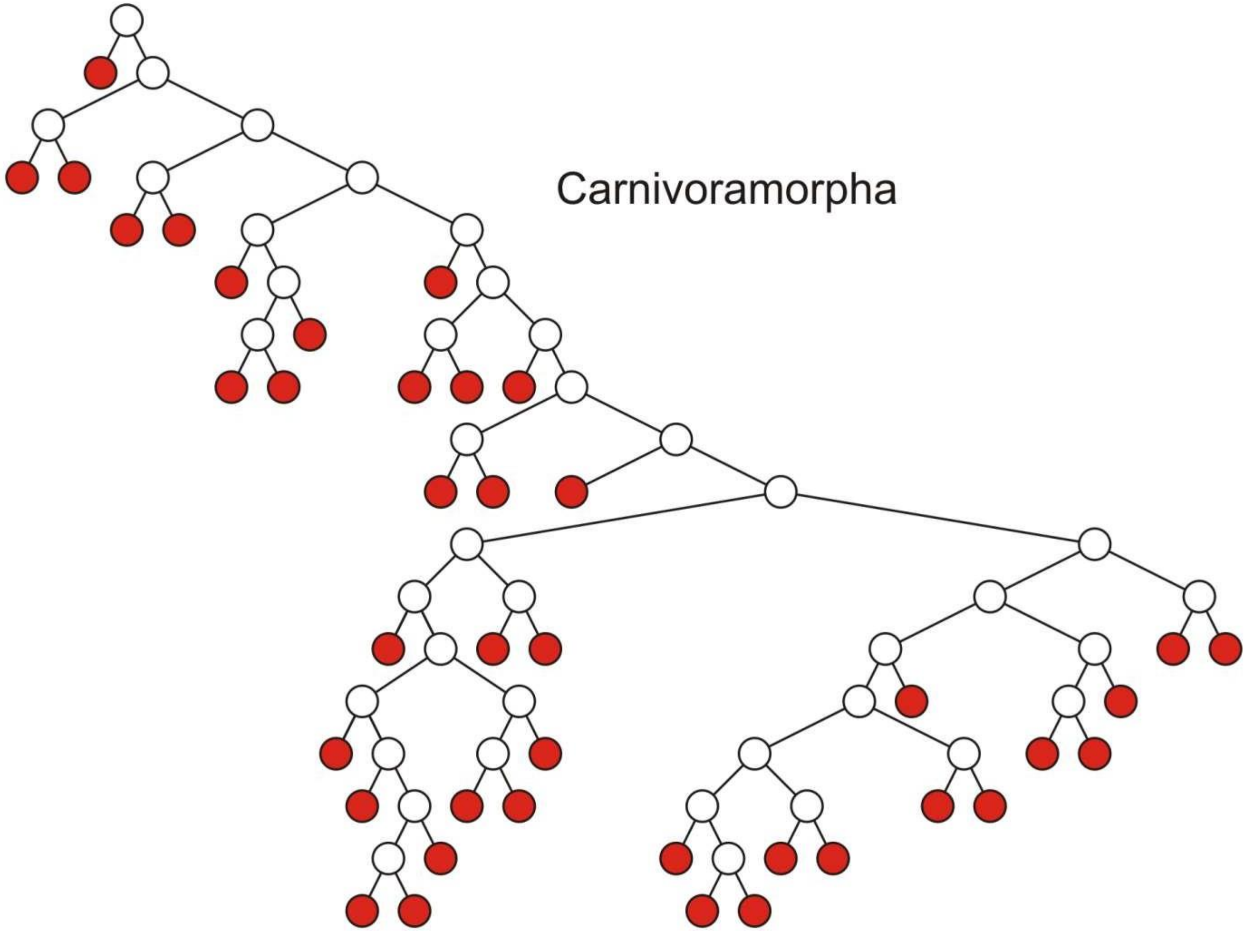
TERMINOLOGY: LEAF AND INTERNAL NODES

- Nodes with degree zero are also called leaf nodes
- All other nodes are said to be internal nodes, that is, they are internal to the tree



TERMINOLOGY: LEAF NODES EXAMPLES

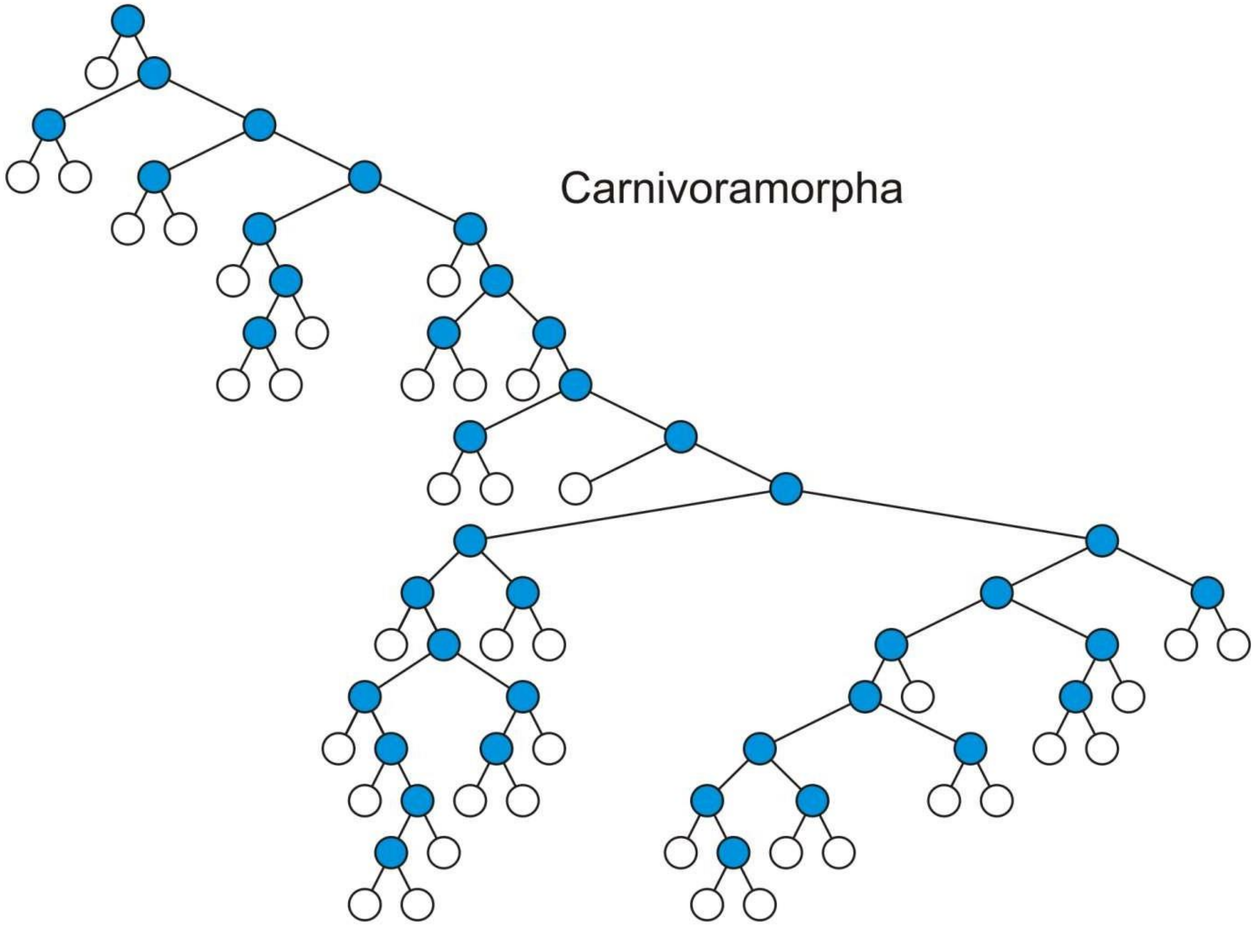
- Leaf nodes



Wesley-Hunt, G. D.; Flynn, J. J. "Phylogeny of the Carnivora: basal relationships among the Carnivoramorphans, and assessment of the position of 'Miacoidea'

TERMINOLOGY: INTERNAL NODES EXAMPLE

- Internal nodes

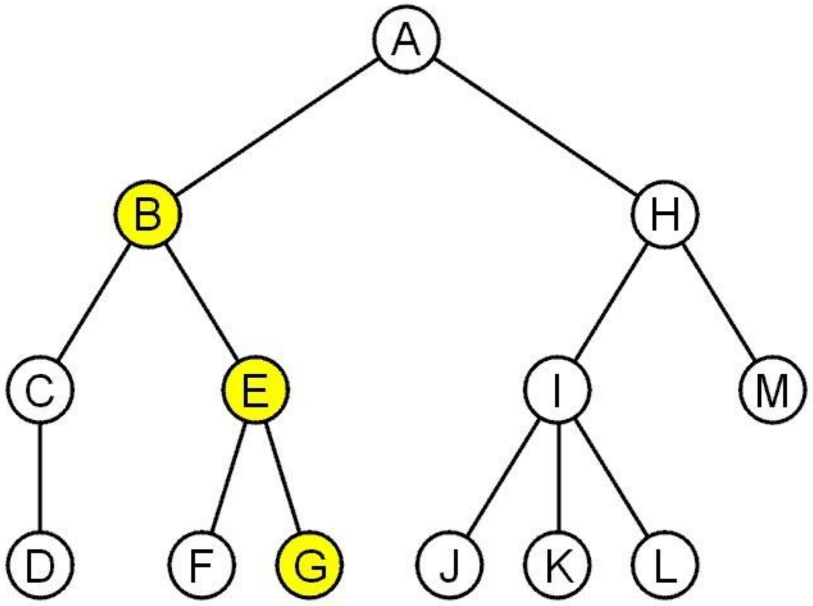


Wesley-Hunt, G. D.; Flynn, J. J. “Phylogeny of the Carnivora: basal relationships among the Carnivoramorphans, and assessment of the position of ‘Miacoidea’

TERMINOLOGY: PATH

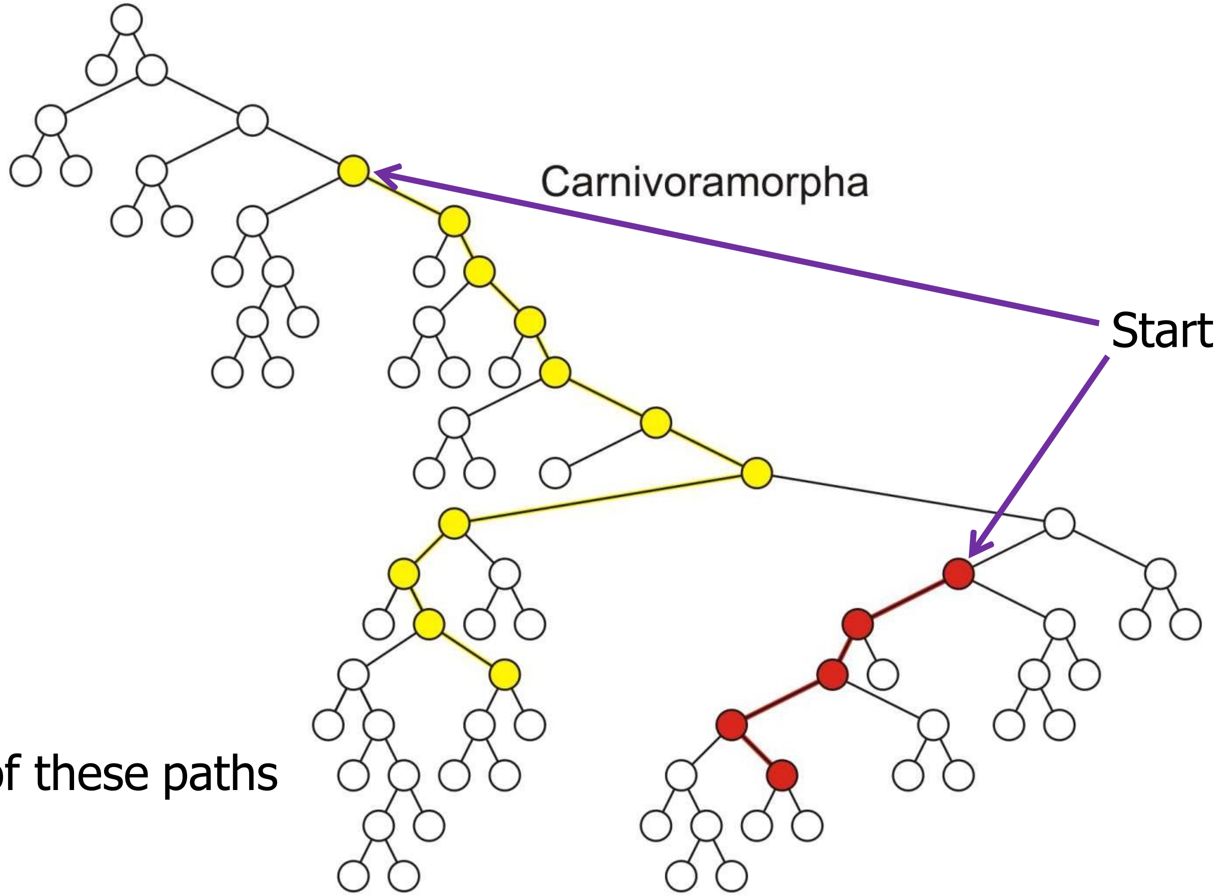
- A path is a sequence of nodes $(a_0, a_1, ..., a_n)$
  - Where $a_k + 1$ is a child of a_k is

- The length of this path is: $n = |\text{nodes in the path}| - 1$
  - For example, the path (B, E, G) has length 2



TERMINOLOGY: PATH EXAMPLE

- Paths of length 10 (11 nodes) and 4 (5 nodes)



Start of these paths

End of these paths

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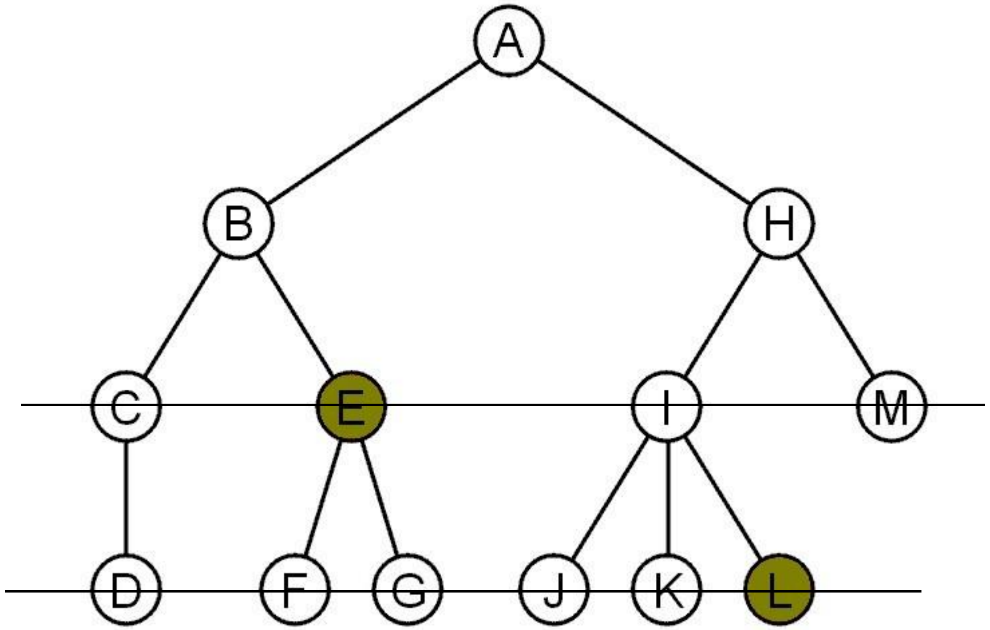
TERMINOLOGY: DEPTH (OR LEVEL)

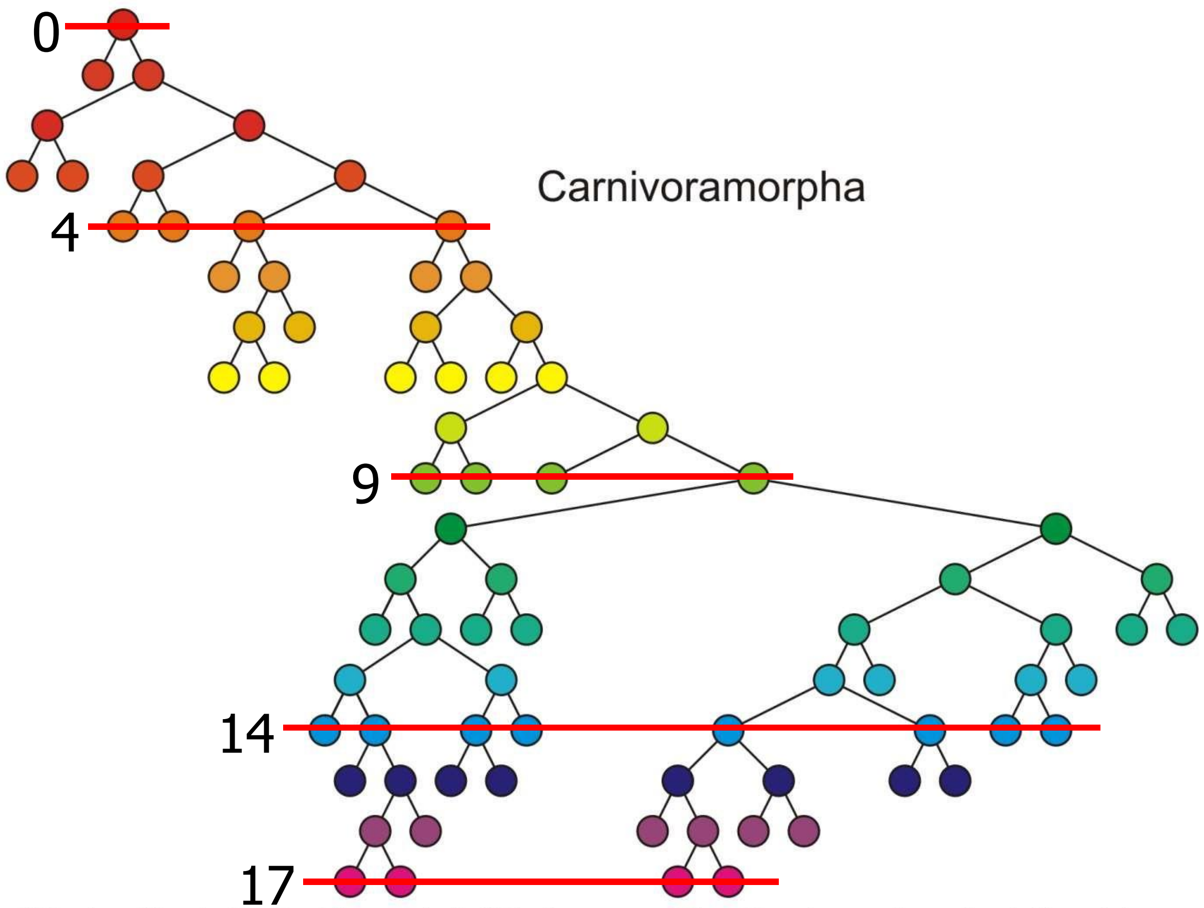
- For each node in a tree, there exists a unique path from the root node to that node

- The length of this path is the depth of the node, e.g.,
  - E has depth 2
  - L has depth 3



TERMINOLOGY: DEPTH EXAMPLE

- Nodes of depth up to 17



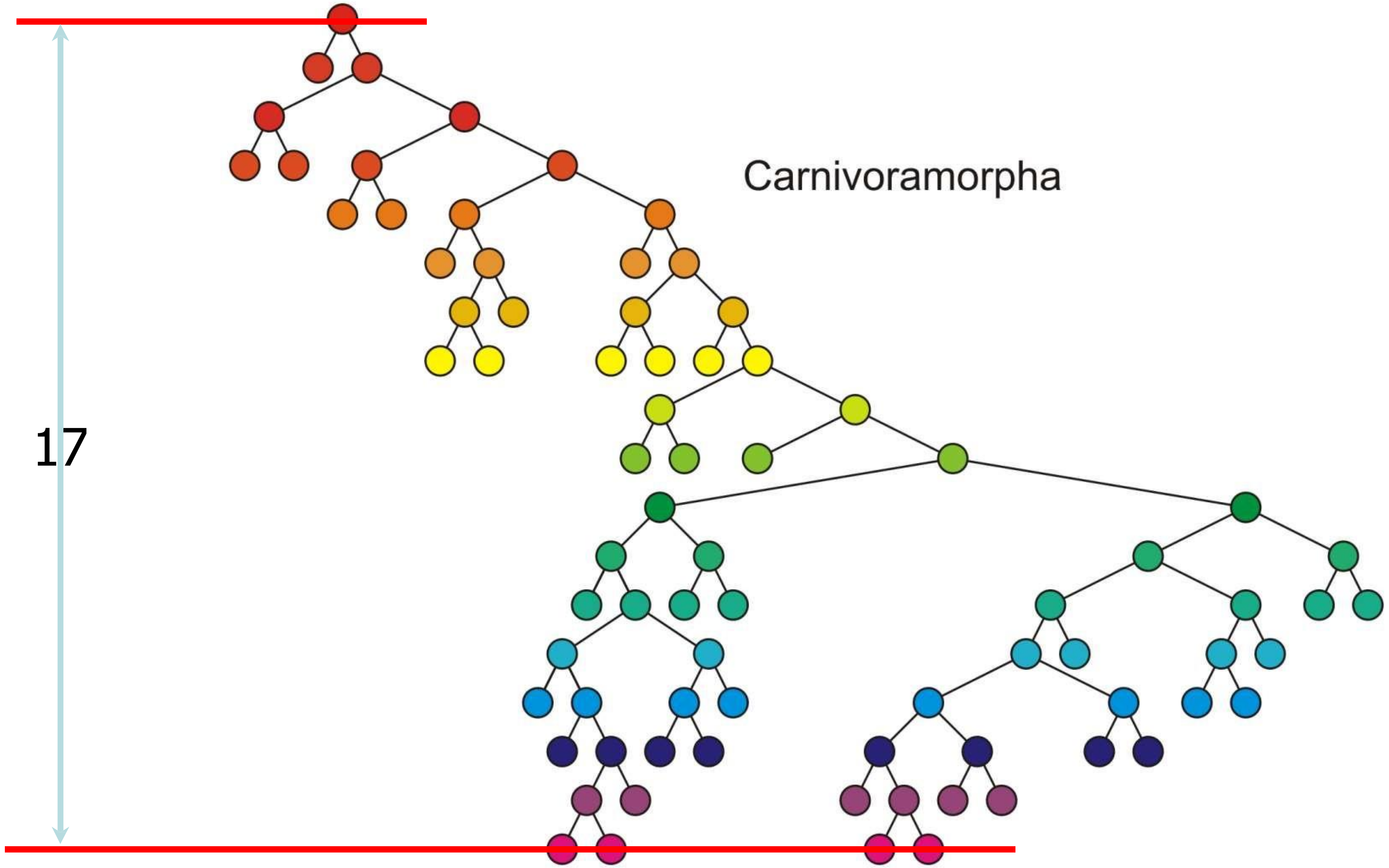
Wesley-Hunt, G. D.; Flynn, J. J. "Phylogeny of the Carnivora: basal relationships among the Carnivoramorphans, and assessment of the position of 'Miacoidea'

TERMINOLOGY: HEIGHT

- The height of a tree is defined as the maximum depth of any node within the tree
- The height of a tree with one node is 0
  - Just the root node
- For convenience, we define the height of the empty tree to be -1

TERMINOLOGY: HEIGHT EXAMPLE

- Height of this tree is 17



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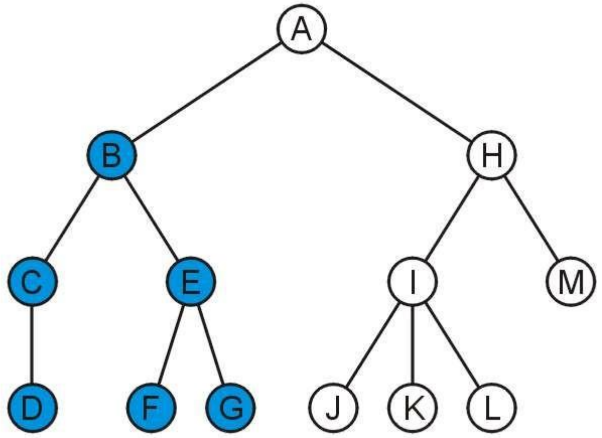
TERMINOLOGY: ANCESTORS AND DESCENDANTS

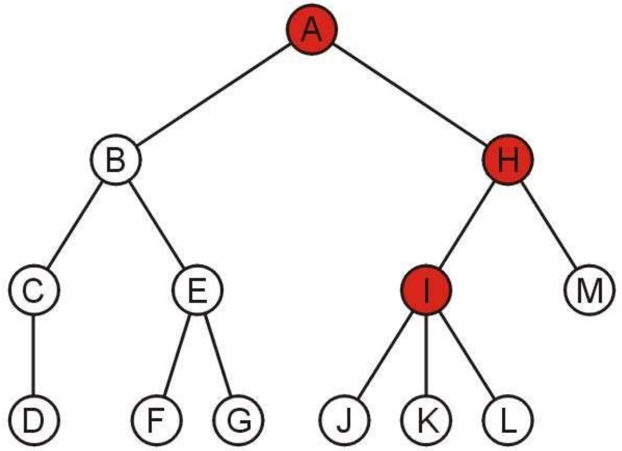
- If a path exists from node a to node b
  - a is an ancestor of b
  - b is a descendent of a
- Thus, a node is both an ancestor and a descendant of itself
  - We can add the adjective strict to exclude equality
  - a is a strict descendent of b if a is a descendant of b but a $\neq$ b
- The root node is an ancestor of all nodes

TERMINOLOGY: ANCESTORS AND DESCENDANTS EXAMPLE

- The descendants of node B are C, D, E, F, and G

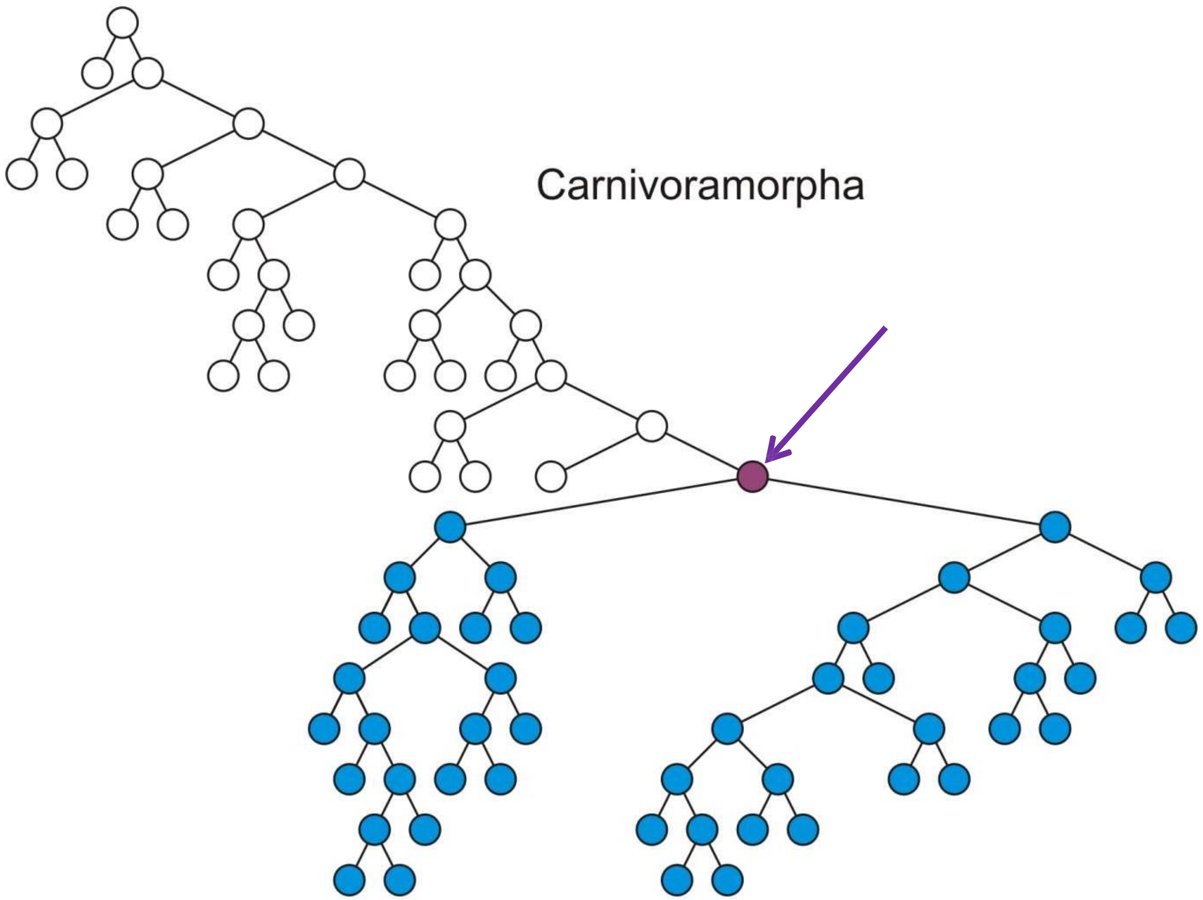


- The ancestors of node I are H and A



TERMINOLOGY: DESCENDANTS EXAMPLE

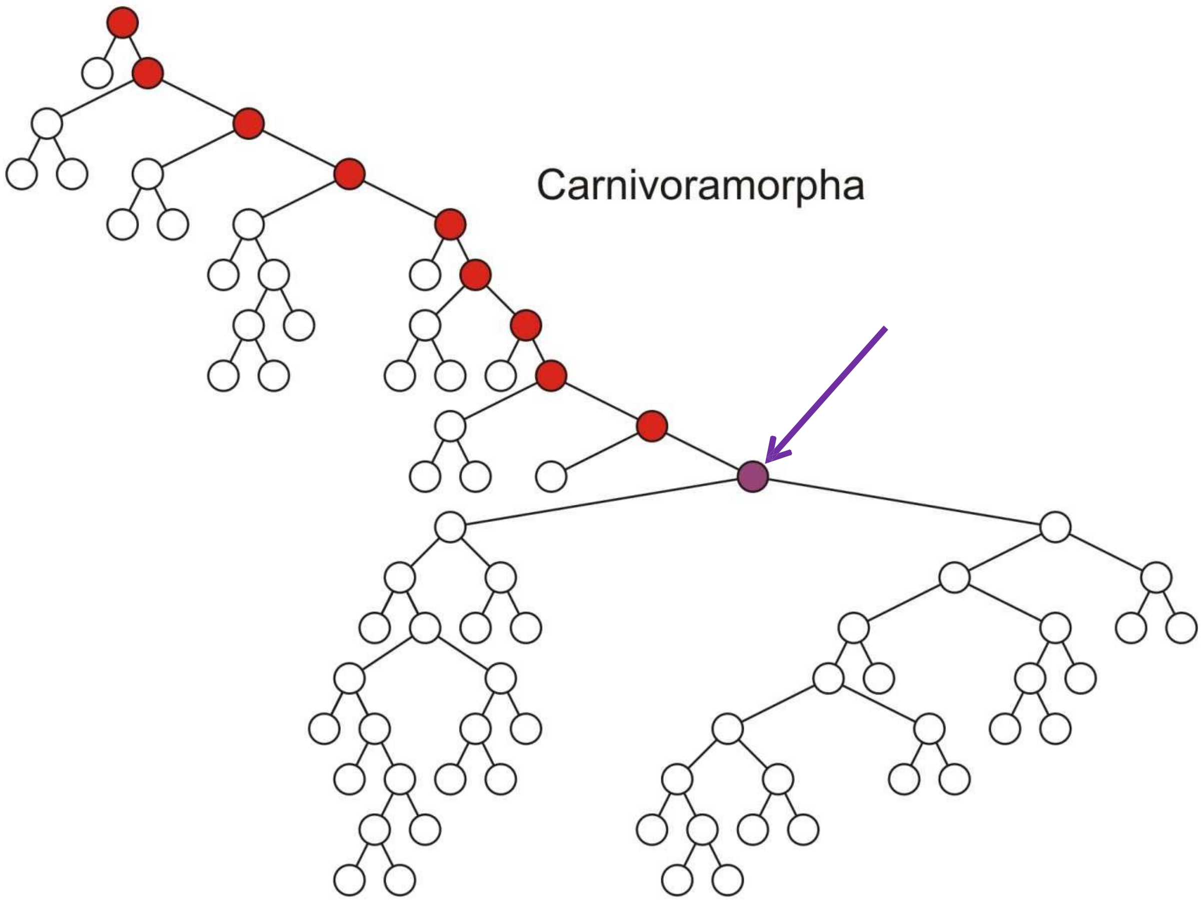
- All descendants (including itself) of the indicated node



Wesley-Hunt, G. D.; Flynn, J. J. “Phylogeny of the Carnivora: basal relationships among the Carnivoramorphans, and assessment of the position of ‘Miacoidea’

TERMINOLOGY: ANCESTORS EXAMPLE

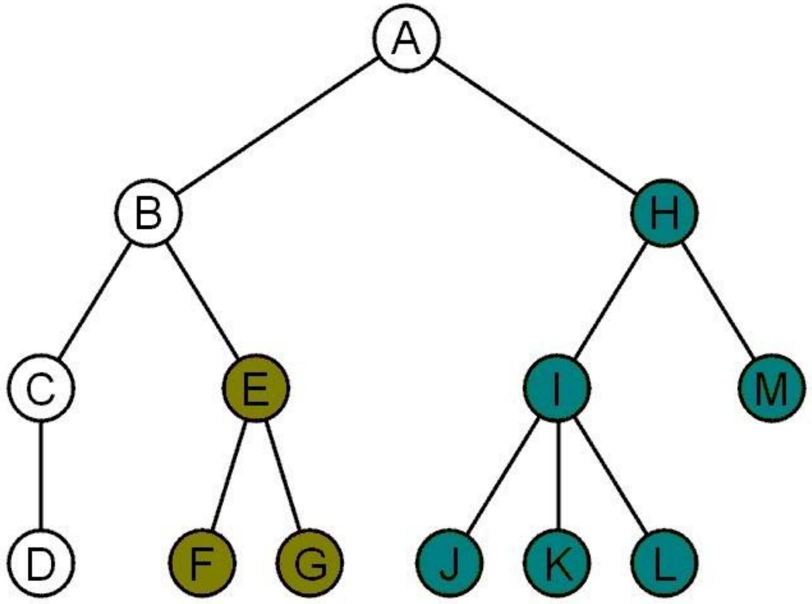
- All ancestors (including itself) of the indicated node



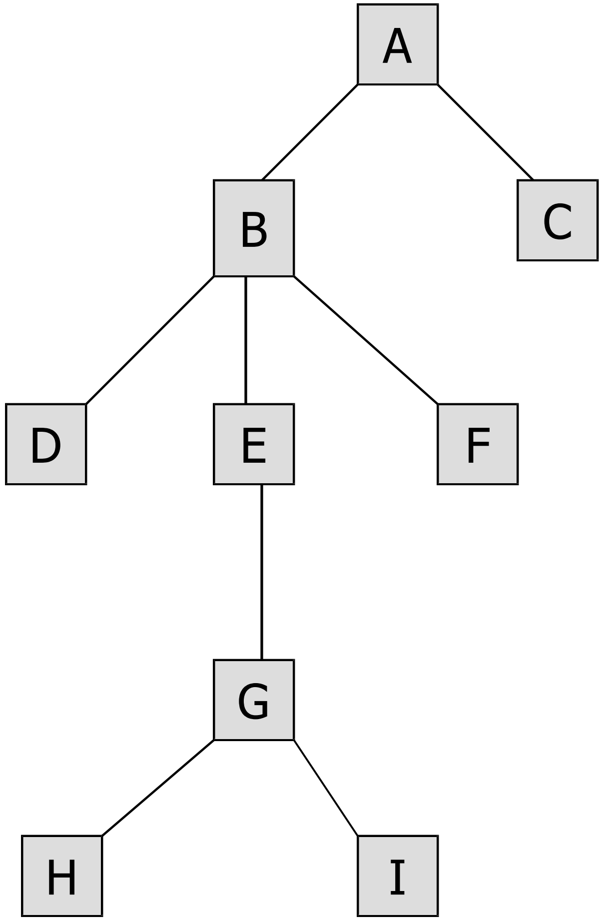
Wesley-Hunt, G. D.; Flynn, J. J. "Phylogeny of the Carnivora: basal relationships among the Carnivoramorphans, and assessment of the position of 'Miacoidea'

TERMINOLOGY: SUBTREE

- Another approach to a tree is to define the tree recursively
  - A degree-0 node is a tree
- A node with degree n is a tree if it has n children
  - All of its children are disjoint trees (i.e., with no intersecting nodes)
- Given any node a within a tree with root r, the collection of a and all of its descendants is said to be a subtree of the tree with root a



TREE PROPERTIES



Property	Value
Number of nodes	
Height	
Root Node	
Leaves	
Ancestors of H	
Descendants of B	
Siblings of E	
Left subtree	

EXAMPLE: HTML

- HTML document has a tree structure

```
<html>
    <head>
        <title>Hello World!</title>
    </head>
    <body>
        <h1>This is a <u>Heading</u></h1>

        <p>This is a paragraph with some
        <u>underlined</u> text.</p>
    </body>
</html>
```

EXAMPLE: HTML

- HTML document has a tree structure

title

heading

body of page

```
<html>
    <head>
        <title>Hello World!</title>
    </head>
    <body>
        <h1>This is a <u>Heading</u></h1>

        <p>This is a paragraph with some
        <u>underlined</u> text.</p>
    </body>
</html>
```

underlining

paragraph

EXAMPLE: HTML

- The nested tags define a tree rooted at the HTML tag

```
<html>
    <head>
        <title>Hello World!</title>
    </head>
    <body>
        <h1>This is a <u>Heading</u></h1>

        <p>This is a paragraph with some
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    </body>
</html>
```
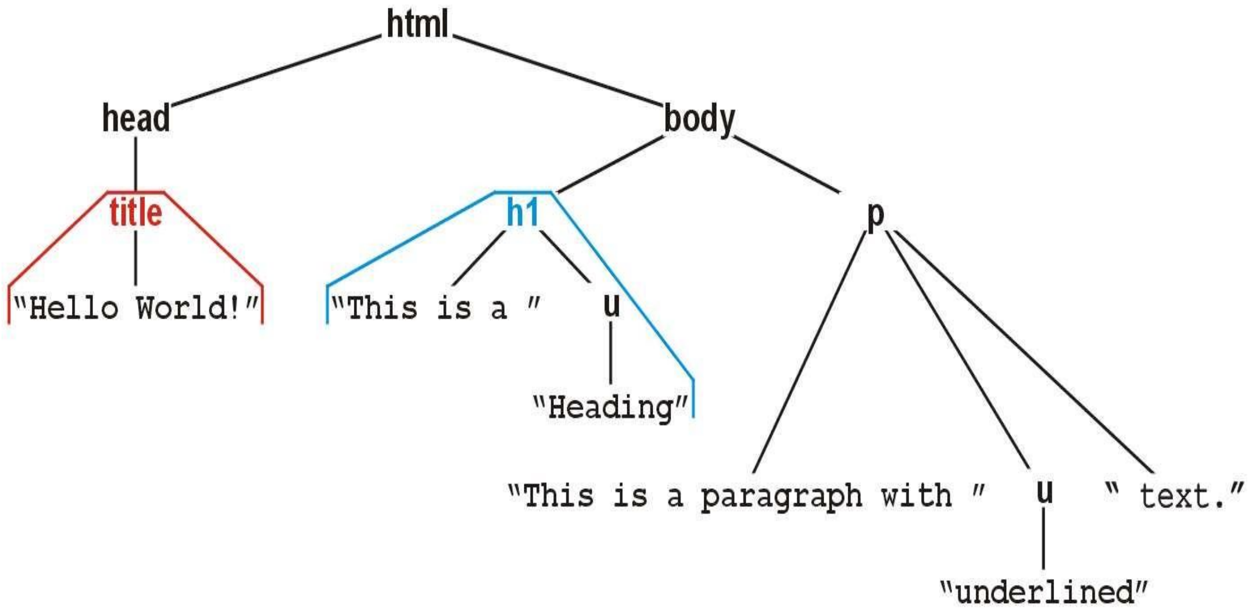
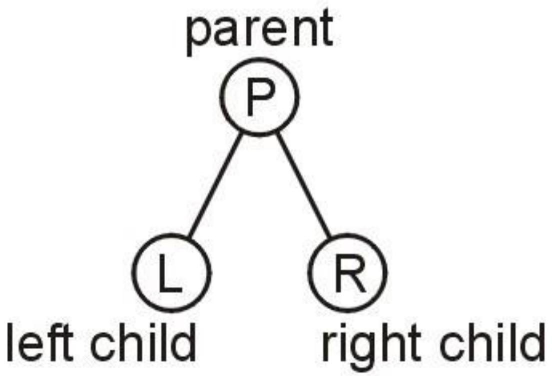



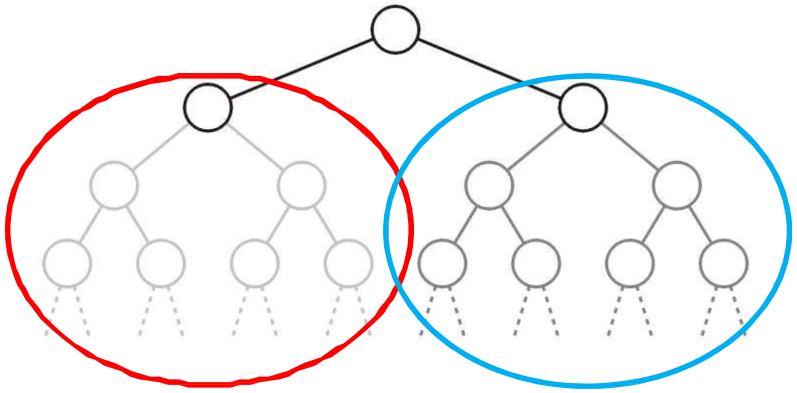
BINARY TREE

BINARY TREE

- In a binary tree each node has at most two children
  - Allows to label the children as left and right

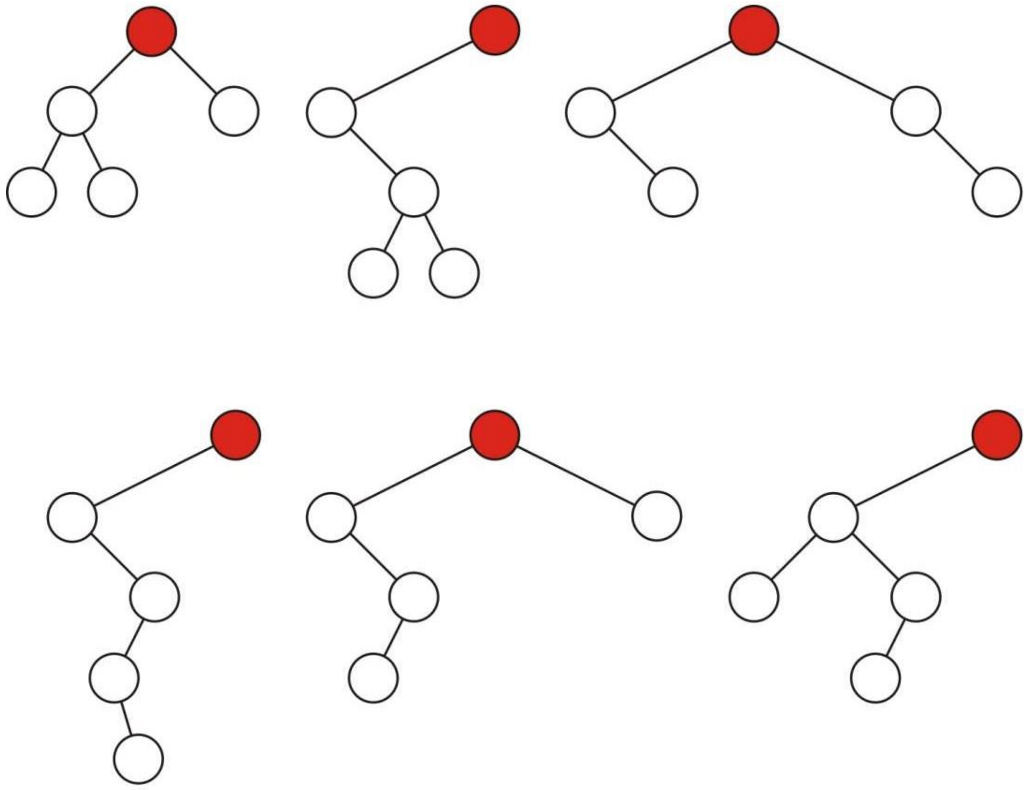


- Likewise, the two sub-trees are referred as
  - Left-hand subtree
  - Right-hand subtree



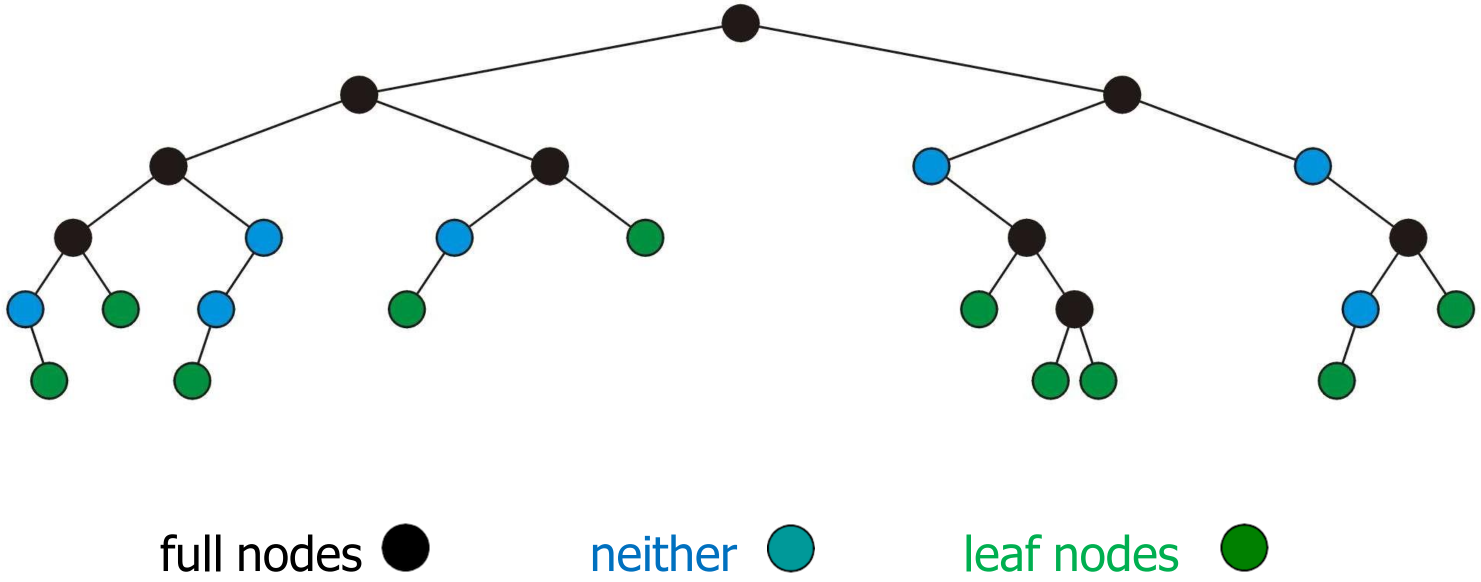
BINARY TREE: EXAMPLE

- Some variations on binary trees with five nodes



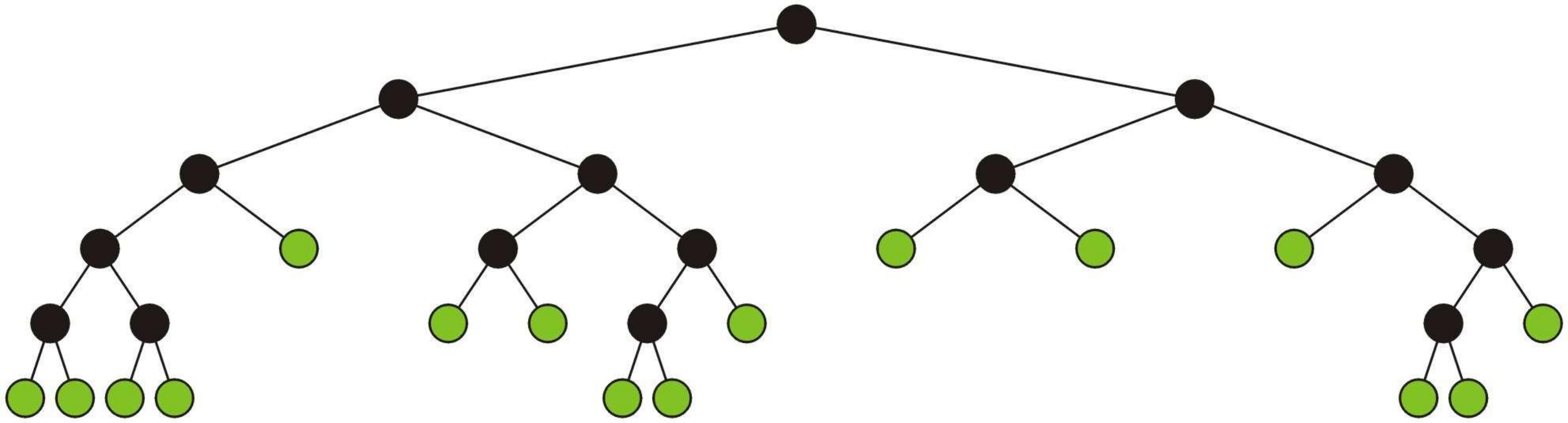
BINARY TREE: FULL NODE

- A full node is a node where both the left and right sub-trees are non-empty trees



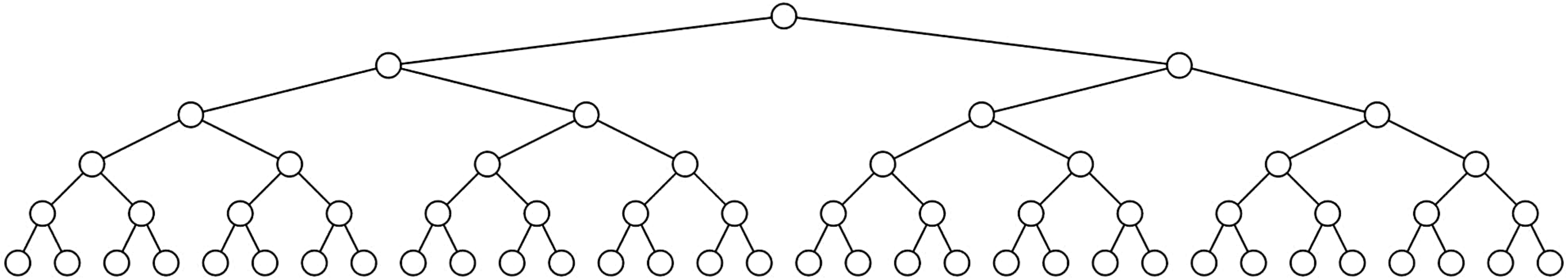
FULL BINARY TREE

- A full binary tree is where each node is:
  - A full node, or
  - A leaf node
- Full binary tree is also called proper binary tree, strictly binary tree or 2-tree



COMPLETE (OR PERFECT) BINARY TREE

- A complete binary tree of height h is a binary tree where
  - All leaf nodes have the same depth h
  - All other nodes are full

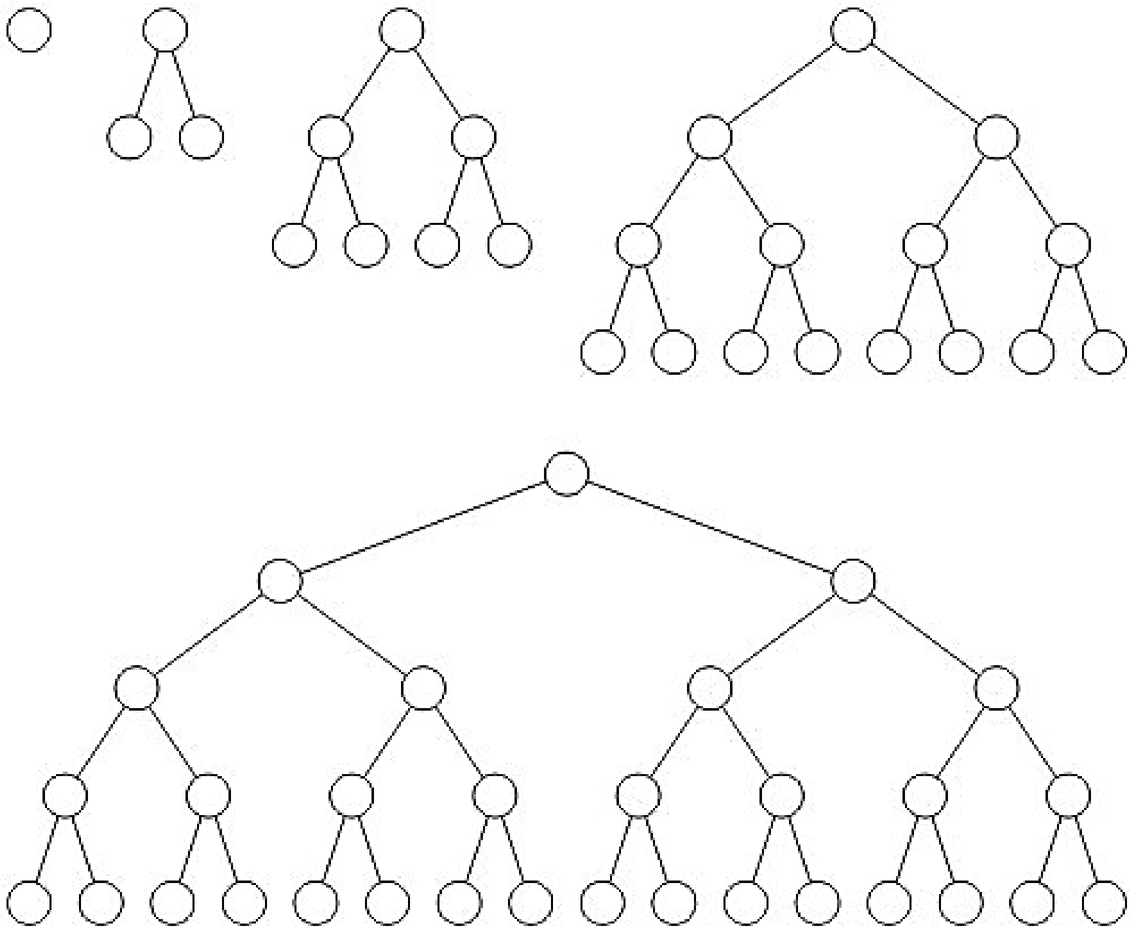


COMPLETE BINARY TREE: RECURSIVE DEFINITION

- A binary tree of height $h = 0$ is perfect
- A binary tree with height $h > 0$ is perfect
  - If both sub-trees are prefect binary trees of height $h - 1$

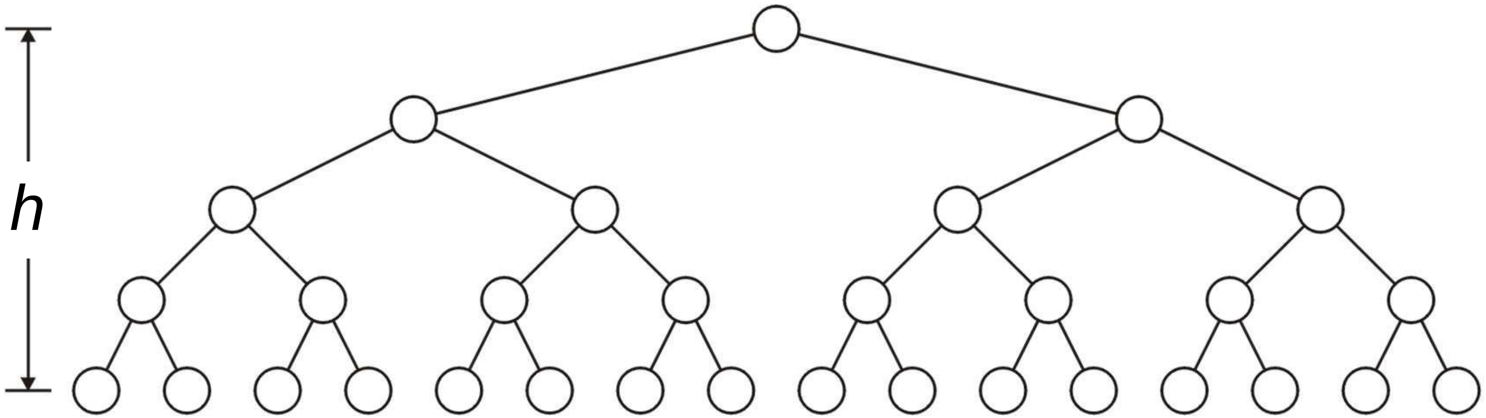
COMPLETE BINARY TREE: EXAMPLE

- Complete binary trees of height h = 0, 1, 2, 3 and 4



BINARY TREE: PROPERTIES

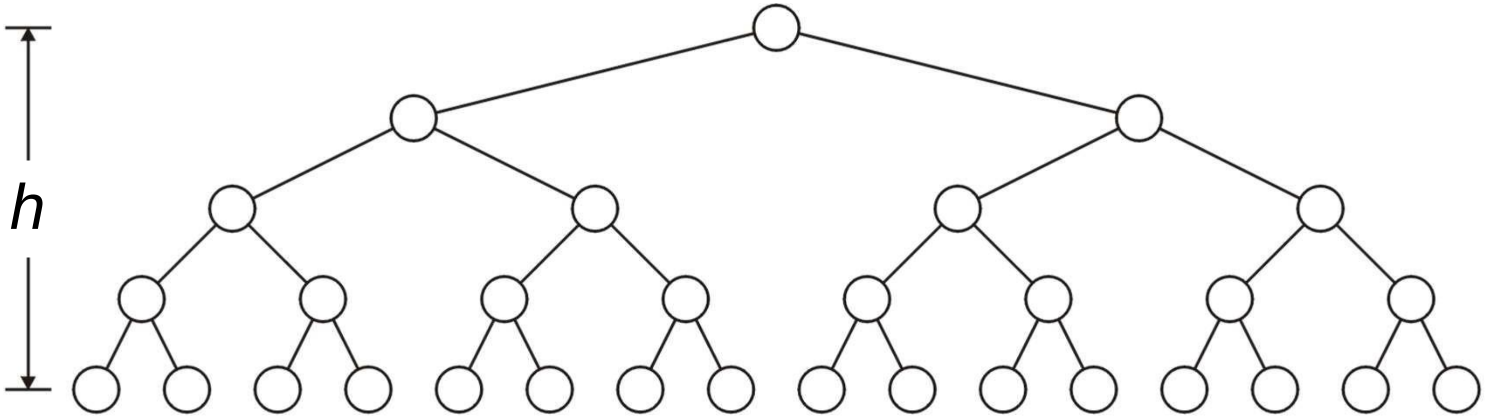
- A complete binary tree with height h has 2^h leaf nodes



BINARY TREE: PROPERTIES

- A complete binary tree with height h has 2^h leaf nodes
- A complete binary tree of height h has $2^{h+1} - 1$ nodes

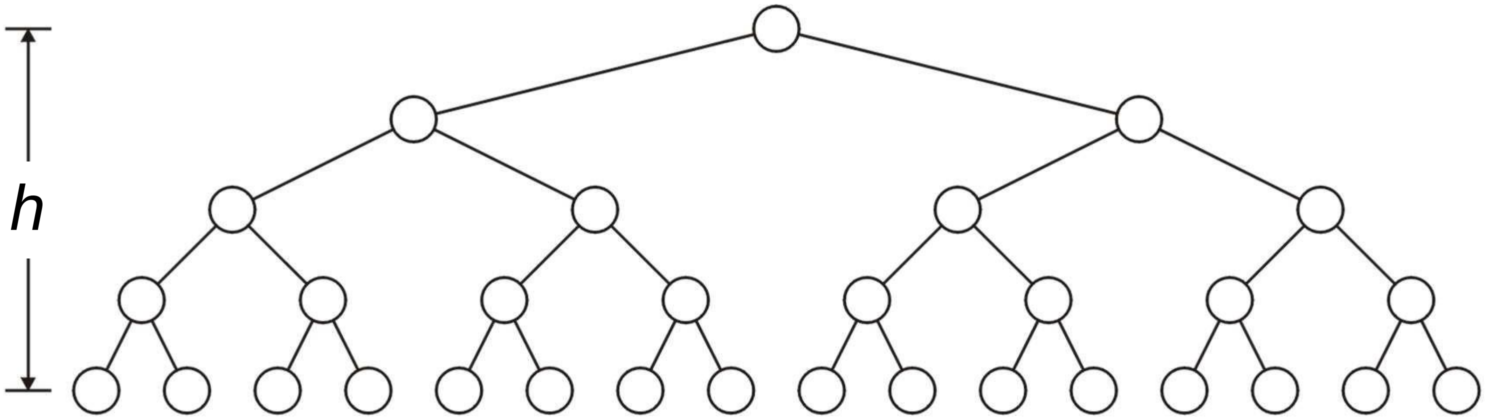
$$n = 2^0 + 2^1 + 2^2 + . . . + 2^h = \sum_{j=0}^{h} 2^j = 2^{h+1} - 1$$



BINARY TREE: PROPERTIES

- A complete binary tree with height h has 2^h leaf nodes

- A complete binary tree of height h has $2^{h+1} - 1$ nodes
  - Number of leaf nodes: $L = 2^h$
  - Number of internal nodes: $2^h - 1$
  - Total number of nodes: $2L-1 = 2^{h+1} - 1$



BINARY TREE: PROPERTIES

- A complete binary tree with height h has 2^h leaf nodes

- A complete binary tree of height h has $2^{h+1} - 1$ nodes
  - Number of leaf nodes: $L = 2^h$
  - Number of internal nodes: $2^h - 1$
  - Total number of nodes: $2L-1 = 2^{h+1} - 1$

- A complete binary tree with n nodes has height $\log_2(n+1) - 1$

$$n = 2^{h+1} - 1$$

$$2^{h+1} = n + 1$$

$$h + 1 = \log_2(n+1)$$

$$\Rightarrow h = \log_2(n+1) - 1$$

BINARY TREE: PROPERTIES

- A complete binary tree with height h has 2^h leaf nodes

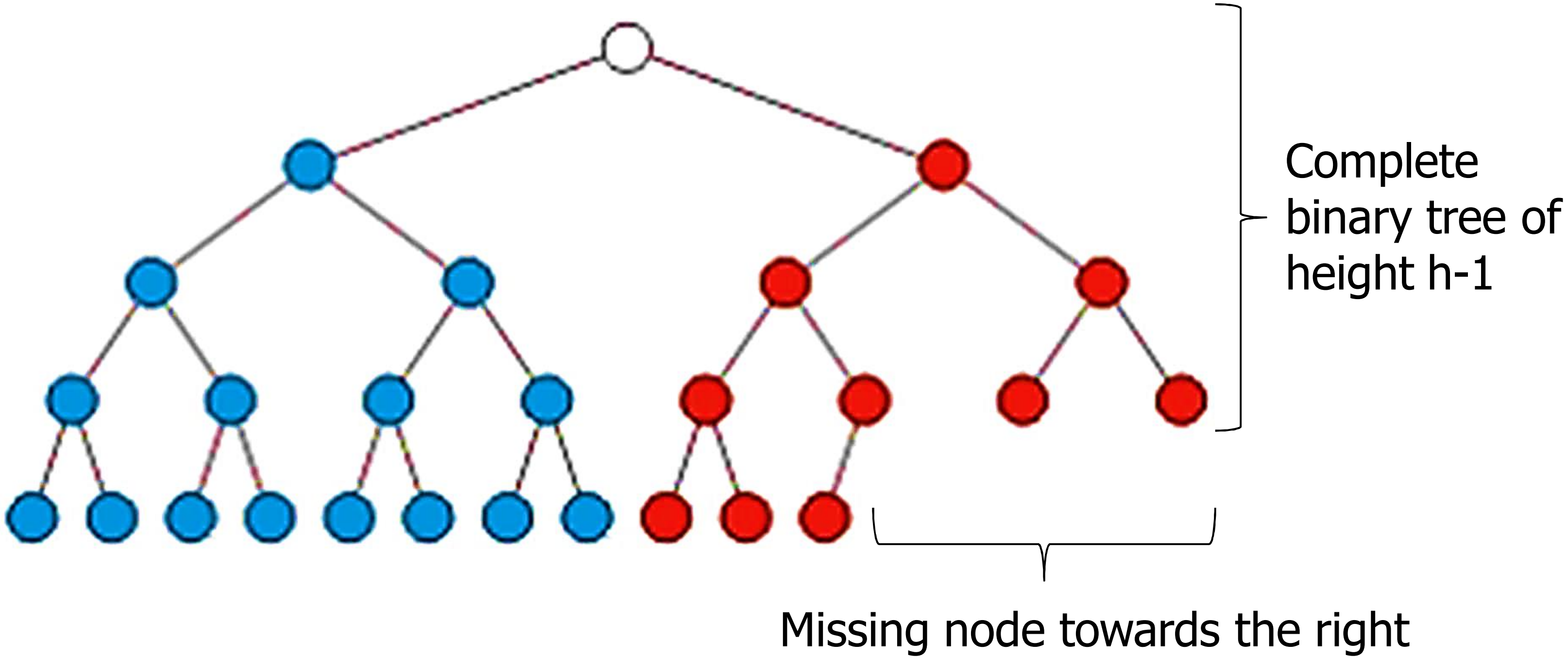
- A complete binary tree of height h has $2^{h+1} - 1$ nodes
  - Number of leaf nodes: $L = 2^h$
  - Number of internal nodes: $2^h - 1$
  - Total number of nodes: $2L-1 = 2^{h+1} - 1$

- A complete binary tree with n nodes has height $\log_2(n + 1) - 1$

- Number n of nodes in a binary tree of height h is at least $h+1$ and at most $2^{h+1} - 1$

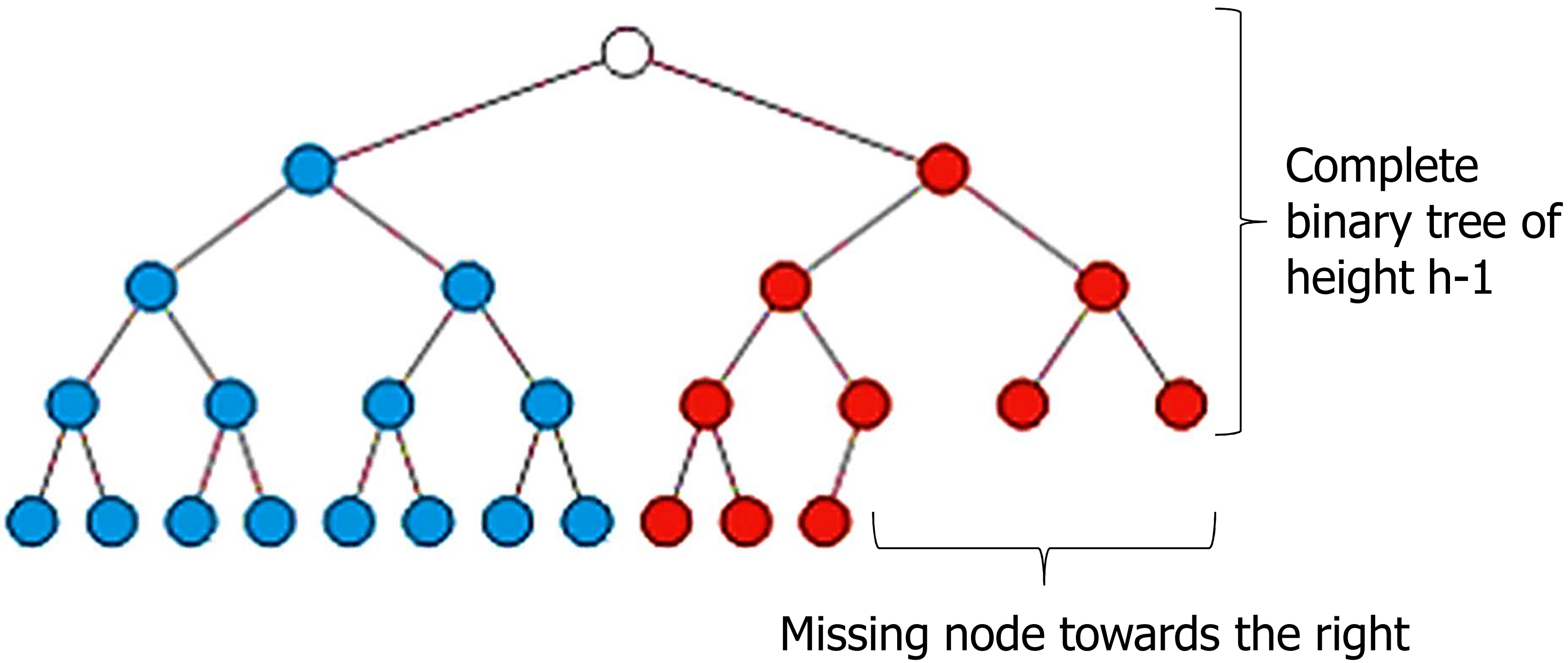
ALMOST (OR NEARLY) COMPLETE BINARY TREE

- Almost complete binary tree of height h is a binary tree in which
  1. There are 2^d nodes at depth d for $d = 1, 2, \dots, h-1$
     - Each leaf in the tree is either at level h or at level $h-1$
  2. The nodes at depth h are as far left as possible



ALMOST (OR NEARLY) COMPLETE BINARY TREE

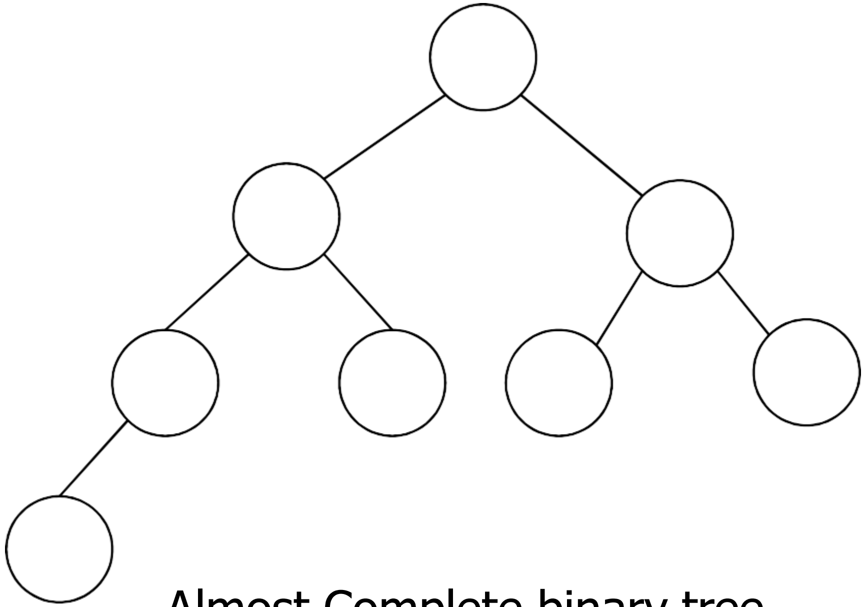
- Almost complete binary tree of height h is a binary tree in which
  1. There are 2^d nodes at depth d for $d = 1, 2, \ldots, h-1$
     - Each leaf in the tree is either at level h or at level h– 1
  2. **The nodes at depth h are as far left as possible (Formal ?)**



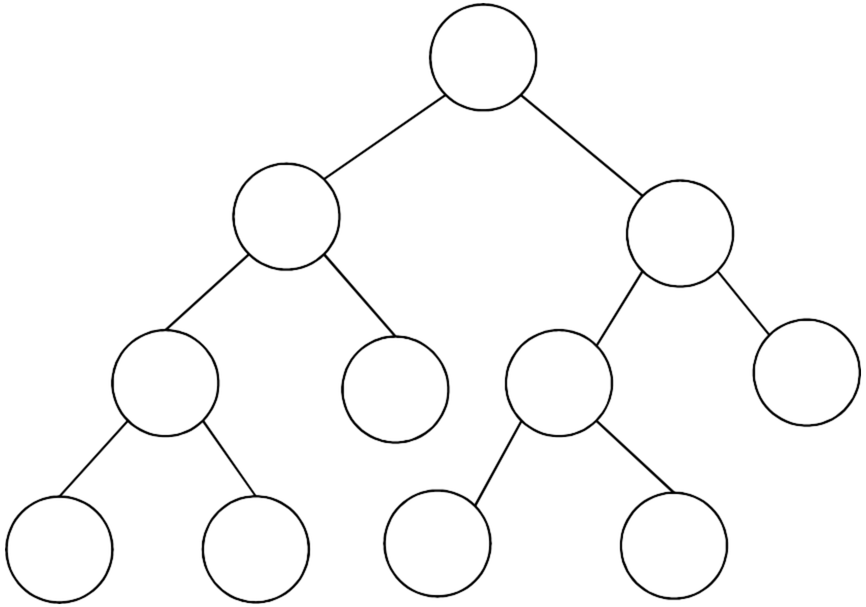
ALMOST (OR NEARLY) COMPLETE BINARY TREE

Condition 2: The nodes at depth h are as far left as possible

- If a node p at depth h−1 has a left child
  - Every node at depth h−1 to the left of p has 2 children
- If a node at depth h−1 has a right child
  - It also has a left child



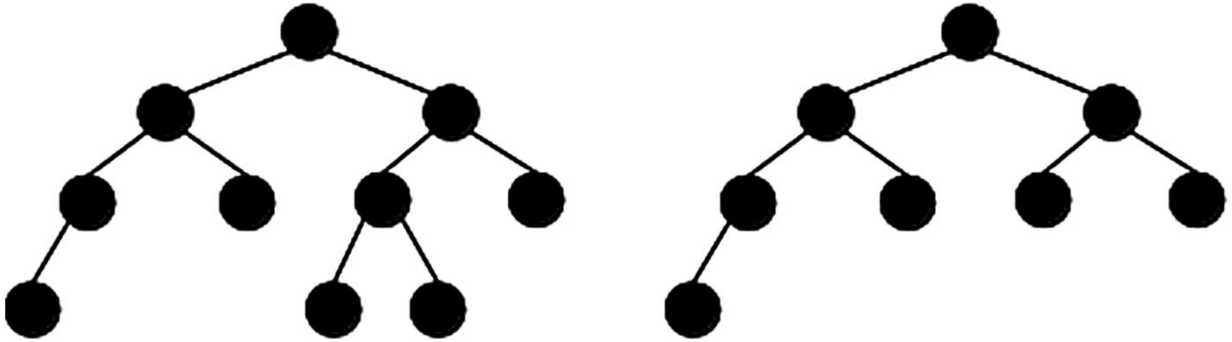
Almost Complete binary tree



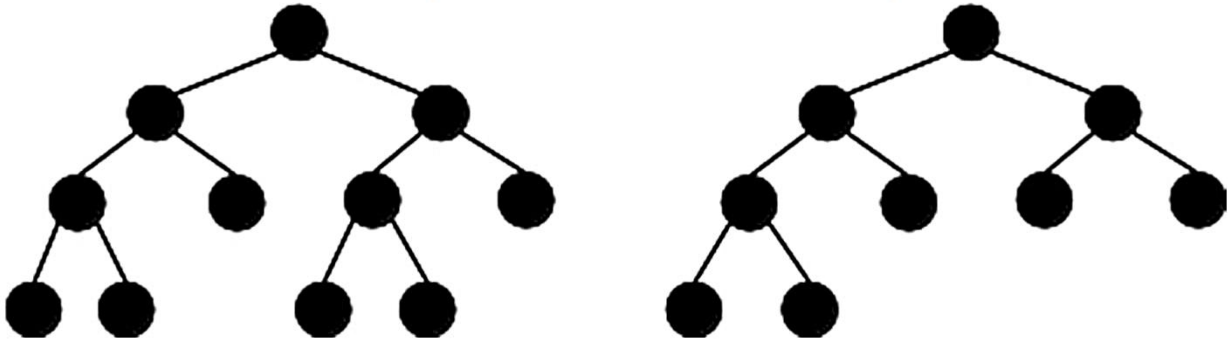
Not Almost Complete binary tree (condition 2 violated)

FULL VS. ALMOST COMPLETE BINARY TREE

Almost Complete



Full



ALMOST COMPLETE BINARY TREE: PROPERTIES

- Total number of nodes n are between
  - Complete binary tree of height h-1, i.e., 2^h nodes
  - Complete binary tree of height h, i.e., $2^{h+1} - 1$ nodes

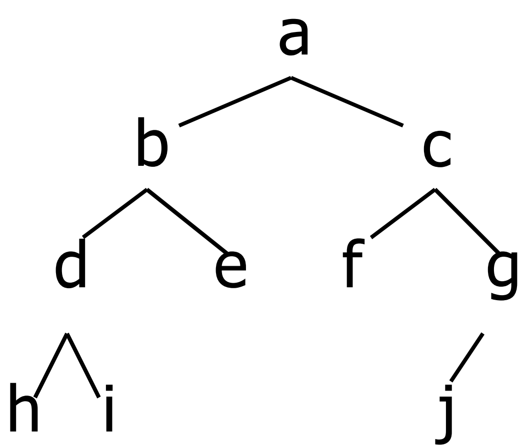
- Height h is the largest integer less than or equal to $\log_2(n)$

(COMPLETELY) BALANCED BINARY TREE

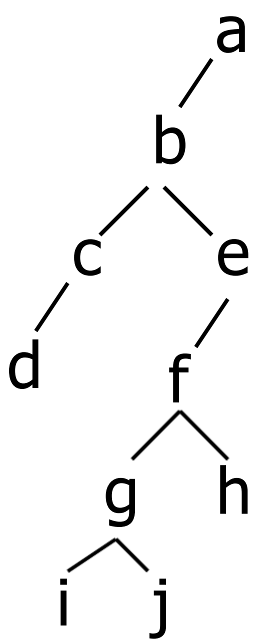
- Balanced binary tree
  - For each node, the difference in height of the right and left sub-trees is no more than one

- Completely balance binary tree
  - Left and right sub-trees of every node have the same height

BALANCED BINARY TREE: EXAMPLE



A balanced binary tree



An unbalanced binary tree

CONCLUSION

- In this lecture we have studied:
  - Tree Data Structure
  - Terminologies of Tree
  - Binary Tree
  - Full, Complete and Almost Complete Binary Tree
  - Balanced and Unbalanced Tree

Question?