The Laplace Transform

In linear mathematical models for a physical system such as spring/mass system or a series electrical circuit, the right-hand member, or driving function, of the differential equations $mx'' + \beta x' + kx = f(x)$ or $Lq'' + Rq' + \frac{q}{C} = E(t)$ represents either an external force $f(t)$ or an impressed voltage $E(t)$. We have already solved problems in which the functions f and E were continuous were continuous. However, in practice discontinuous driving functions are not uncommon. The Laplace transform is an especially valuable tool that simplifies the solution of such equations.

Introduction: In elementary calculus we have learned that differentiation and integration are transforms; this means, roughly speaking, that these operations transform a function into another function. For example, the function $f(x) = x^2$ is transformed, in turn, into a linear function and a family of cubic polynomial functions by the operation of differentiation and integration:

$$\frac{d}{dx} x^2 = 2x, \quad \text{and} \quad \int x^2 dx = \frac{x^3}{3} + C.$$

In this section, we will study a special type of integral transform called the Laplace transform.

Definition: Let f be a function defined for $t \geq 0$. Then the integral

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$$

is said to be the Laplace transform of f provided that the integral converges. When the defining integral (1) converges, the result is a function of s.

We use a lowercase letter to denote the function being transformed and the corresponding capital letter to denote its Laplace transform. For example:

$$L\{f(t)\} = F(s), \quad L\{g(t)\} = G(s), \quad L\{y(t)\} = Y(s).$$

① Examples:

1. Evaluate $L\{1\}$.

Solution:

$$L\{1\} = \int_0^\infty e^{-st}(1)\,dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty} \frac{e^{-st}}{-s}\Big|_0^b$$

$$= -\frac{1}{s} \lim_{b\to\infty} e^{-st}\Big|_0^b$$

$$= -\frac{1}{s} \left\{ \lim_{b\to\infty} e^{-sb} - 1 \right\}$$

$$= -\frac{1}{s} \left\{ \lim_{b\to\infty} \frac{1}{e^{sb}} - 1 \right\}$$

$$= -\frac{1}{s} \left\{ 0 - 1 \right\}$$

$$\Rightarrow L\{1\} = \frac{1}{s} \quad \text{for } s > 0$$

The integral diverges for $s < 0$.

2. Evaluate $L\{t\}$

Solution:

$$L\{t\} = \int_0^\infty e^{-st} . t\,dt = \lim_{b\to\infty} \int_0^b e^{-st} t\,dt$$

$$= t\frac{e^{-st}}{-s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}.(1)\,dt$$

$$= -\frac{1}{s} . \lim_{b\to\infty} \left\{ te^{-st}\Big|_0^b \right\} - \lim_{b\to\infty} \frac{e^{-st}}{s^2}\Big|_0^b$$

$$= -\frac{1}{s} \lim_{b\to\infty} \left\{ \frac{t}{e^{st}}\Big|_0^b \right\} - \frac{1}{s^2} \lim_{b\to\infty} \left\{ \frac{1}{e^{st}}\Big|_0^b \right\}$$

$$= -\frac{1}{s} \lim_{b\to\infty} \left\{ \frac{1}{se^{st}} - 0 \right\} - \frac{1}{s^2} \left\{ \frac{1}{\infty} - 1 \right\}$$

$$= 0 + \frac{1}{s^2}$$

$$\Rightarrow L\{t\} = \frac{1}{s^2}$$

$$L\{t^2\} = \frac{2}{s^3}, \quad L\{t^3\} = \frac{6}{s^4} = \frac{3!}{s^4} \quad \ldots, \quad L\{t^n\} = \frac{n!}{s^{n+1}}$$

3. Evaluate $L\{e^{at}\}$

Solution:

$$L\{e^{at}\} = \int_0^\infty e^{-st}\cdot e^{at}\,dt = \lim_{b\to\infty}\int_0^b e^{-(s-a)t}\,dt$$

$$= \lim_{b\to\infty}\left\{ \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^b \right\}$$

$$= -\frac{1}{(s-a)} \lim_{b\to\infty}\left\{ e^{-(s-a)b} - e^0 \right\}$$

$$= -\frac{1}{s-a} \lim_{b\to\infty}\left\{ \frac{1}{e^{(s-a)b}} - 1 \right\}$$

$$= -\frac{1}{s-a}\{0-1\}$$

$$\Rightarrow \quad L\{e^{at}\} = \frac{1}{s-a}$$

$$L\{e^{-at}\} = \frac{1}{s+a}.$$

$$L\{e^{-3t}\} = \frac{1}{s+3}, \quad L\{e^{5t}\} = \frac{1}{s-5}.$$

Laplace Transforms of Some Basic Functions:

1. $L\{1\} = \frac{1}{s}$
2. $L\{t^n\} = \frac{n!}{s^{n+1}}$
3. $L\{e^{at}\} = \frac{1}{s-a}$
4. $L\{\sin kt\} = \frac{k}{s^2+k^2}$
5. $L\{\cos kt\} = \frac{s^2}{s^2+k^2}$
6. $L\{\sinh kt\} = \frac{k}{s^2-k^2}$
7. $L\{\cosh kt\} = \frac{s}{s^2-k^2}$
8. $L\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}$
9. $L\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}$
10. $L\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}$

Example 2:

Evaluate the following:

1. $L\{1+5t\}$

Solution:

$$L\{1+5t\} = L\{1\} + L\{5t\}$$

$$= \frac{1}{s} + 5L\{t\}$$

$$= \frac{1}{s} + \frac{5}{s^2}$$

2. $L\{4e^{5t} - 10\sin 2t\}$

Solution:

$$L\{4e^{5t} - 10\sin 2t\} = L\{4e^{5t}\} - L\{10\sin 2t\}$$

$$= 4L\{e^{5t}\} - 10L\{\sin 2t\}$$

$$= \frac{4}{s-5} - \frac{20}{s^2+4}$$

3. Evaluate $L\{f(t)\}$ where

$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3. \end{cases}$$

Solution:

$$L\{f(t)\} = \int_0^3 e^{-st}(0)\,dt + \int_3^\infty e^{-st}\cdot 2\,dt$$

$$= 0 + 2\lim_{b\to\infty} \int_3^b e^{-st}\,dt$$

$$= 2\lim_{b\to\infty} \left. \frac{e^{-st}}{-s} \right|_3^b$$

$$= -\frac{2}{3} \lim_{b\to\infty} \left\{ \frac{1}{e^{bs}} - e^{-3s} \right\}$$

$$= -\frac{2}{3} \left\{ 0 - e^{-3s} \right\}$$

$$\Rightarrow \quad L\{f(t)\} = \frac{2e^{-3s}}{3}$$

Inverse Laplace Transform:

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $L\{f(t)\} = F(s)$, we say that $f(t)$ is the inverse Laplace transform of $F(s)$ and write $f(t) = L^{-1}\{F(s)\}$.

Example:

Transform	Inverse Transform
$L\{1\} = \frac{1}{s}$	$1 = L^{-1}\left\{\frac{1}{s}\right\}$
$L\{t\} = \frac{1}{s^2}$	$t = L^{-1}\left\{\frac{1}{s^2}\right\}$
$L\{e^{-3t}\} = \frac{1}{s+3}$	$e^{-3t} = L^{-1}\left\{\frac{1}{s+3}\right\}$

Theorem: Some Inverse Transforms:

(1) $1 = L^{-1}\left\{\frac{1}{s}\right\}$ (2) $t^n = L^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$, $n = 1, 2, 3, \dots$ (3) $e^{at} = L^{-1}\left\{\frac{1}{s-a}\right\}$

(4) $\sin kt = L^{-1}\left\{\frac{k}{s^2+k^2}\right\}$ (5) $\cos kt = L^{-1}\left\{\frac{s}{s^2+k^2}\right\}$ (6) $\sinh kt = L^{-1}\left\{\frac{k}{s^2-k^2}\right\}$

(7) $\cosh kt = L^{-1}\left\{\frac{s}{s^2-a^2}\right\}$

In evaluating the inverse transforms, it often happens that the function of s under consideration does not match exactly the form of a Laplace transform F(s) given in a table. It may be necessary to "fix up" the function of s by multiplying and dividing by appropriate constant.

L^{-1} is a linear transform: The inverse Laplace transform is a linear transform, for constants α and β

$$L^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha L^{-1}\{F(s)\} + \beta L^{-1}\{G(s)\}$$

Example:

Evaluate the following

1. $L^{-1}\left\{\frac{1}{s^5}\right\}$

Solution:

$$L^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!} L^{-1}\left\{\frac{4!}{s^5}\right\}$$

$$= \frac{1}{4!} \cdot t^4$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{24}$$

2. $L^{-1}\left\{\frac{1}{s^2+7}\right\}$

Solution: Here $k^2 = 7 \Rightarrow k = \sqrt{7}$

So

$$L^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}} L^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}} \sin\sqrt{7}t.$$

$L^{-1}\left\{\frac{1}{s^3}\right\}$

$$L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2} L^{-1}\left\{\frac{2}{s^3}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2} t^2.$$

4. $L^{-1}\left\{ \frac{1}{s^2} - \frac{48}{s^5} \right\}$

$$L^{-1}\left\{ \frac{1}{s^2} - \frac{48}{s^5} \right\} = L^{-1}\left\{ \frac{1}{s^2} \right\} - 48\, L^{-1}\left\{ \frac{1}{s^5} \right\}$$

$$= t - \frac{48}{4!} L^{-1}\left\{ \frac{4!}{s^5} \right\}$$

$$= t - \frac{48}{24} t^4$$

$$= t - 2t^4$$

5. $L^{-1}\left\{ \frac{(s+1)^3}{s^4} \right\}$

$$L^{-1}\left\{ \frac{(s+1)^3}{s^4} \right\} = L^{-1}\left\{ \frac{s^3 + 1 + 3s(s+1)}{s^4} \right\}$$

$$= L^{-1}\left\{ \frac{s^3 + 1 + 3s^2 + 3s}{s^4} \right\}$$

$$= L^{-1}\left\{ \frac{1}{s} + \frac{1}{s^4} + \frac{3}{s^2} + \frac{3}{s^3} \right\}$$

$$= L^{-1}\left\{ \frac{1}{s} \right\} + L^{-1}\left\{ \frac{1}{s^4} \right\} + 3L^{-1}\left\{ \frac{1}{s^2} \right\} + 3L^{-1}\left\{ \frac{1}{s^3} \right\}$$

$$= 1 + \frac{1}{3!} L^{-1}\left\{ \frac{3!}{s^4} \right\} + 3t + \frac{3}{2!} L^{-1}\left\{ \frac{2!}{s^3} \right\}$$

$$= 1 + \frac{1}{6} t^3 + 3t + \frac{3}{2} t^2$$

6. $L^{-1}\left\{ \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-2} \right\}$

$$L^{-1}\left\{ \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-2} \right\} = L^{-1}\left\{ \frac{1}{s^2} \right\} - L^{-1}\left\{ \frac{1}{s} \right\} + L^{-1}\left\{ \frac{1}{s-2} \right\}$$

$$= t + 1 + e^{2t}$$

7. $L^{-1}\left\{ \frac{1}{4s+1} \right\}$

$$L^{-1}\left\{ \frac{1}{4s+1} \right\} = L^{-1}\left\{ \frac{1}{4(s+1/4)} \right\}$$

$$= \frac{1}{4} L^{-1}\left\{ \frac{1}{s+1/4} \right\}$$

$$= \frac{1}{4} e^{-t/4}.$$

8. $L^{-1}\left\{\frac{5}{s^2+49}\right\}$

$L^{-1}\left\{\frac{5}{s^2+49}\right\} = 5L^{-1}\left\{\frac{1}{s^2+(7)^2}\right\}$

$= \frac{5}{7}L^{-1}\left\{\frac{7}{s^2+7^2}\right\}$

$= \frac{5}{7}\sin 7t$

9. $L^{-1}\left\{\frac{4s}{4s^2+1}\right\}$

$L^{-1}\left\{\frac{4s}{4s^2+1}\right\} = 4L^{-1}\left\{\frac{s}{4s^2+1}\right\}$

$= 4L^{-1}\left\{\frac{s}{4(s^2+1/4)}\right\}$

$= L^{-1}\left\{\frac{s}{s^2+(1/2)^2}\right\}$

$= \cos\frac{1}{2}t$

10. $L^{-1}\left\{\frac{2s-6}{s^2+9}\right\}$

$L^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = 2L^{-1}\left\{\frac{s}{s^2+9}\right\} - 6L^{-1}\left\{\frac{1}{s^2+9}\right\}$

$= 2\cos 3t - \frac{6}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\}$

$= 2\cos 3t - 2\sin 3t.$