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END SEM EXAM

MATHS - 4

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Q1 Find diagonal and modal matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} = A$

ANS let λ be Eigen value and x be corresponding Eigen vector of matrix A

The characteristic equation is $\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$

$\therefore \lambda^3 - (-9+3+7)\lambda^2 + (-11-1+5)\lambda - 3 = 0$

$\therefore \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

$\therefore (1+\lambda)(1+\lambda)(3-\lambda) = 0$

$\therefore \lambda = -1, -1, 3$

(i) For $\lambda = -1$, $[A - \lambda_1 I]\, X = 0$ gives

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$, $R_3 - 2R_1$ and $-(1/4)R_1$: $\begin{bmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore 2x_1 - x_2 - x_3 = 0$

The rank of coefficient matrix is 1. The number of unknowns is 3. Hence there are 2 linearly independent solutions putting $x_2 = 2t$ and $x_3 = 2s$

we get $2x_1 = x_2 + x_3$

$\therefore 2x_1 = 2t + 2s$

$\therefore x_1 = t + s$

$$\therefore X_1 = \begin{bmatrix} s+t \\ 0+2t \\ 2s+0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore$ Corresponding to the eigenvalue -1 we get the following two linearly dependent eigenvectors

$$X_1 = [1, 0, 2]' \text{ and } X_2 = [1, 2, 0]'$$

For $\lambda = -1$, the algebraic multiplicity is 2 and the geometric multiplicity is 2.

(ii) For $\lambda = 3$, $[A - \lambda_2 I]X = 0$ gives

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$, $R_3 - R_1$ $\begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ -4 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_3 + R_2$ $\begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

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By $(-1/4)R_1$, $(1/4)R_2$
$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore 3x_1 - x_2 - x_3 = 0$

$x_1 - x_2 = 0$

$\therefore x_1 = x_2$

Putting $x_2 = t$, we get $x_1 = t$ and $x_3 = 3x_1 - x_2$

$= 3t - t$

$= 2t$

$$\therefore X_3 = \begin{bmatrix} t \\ t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ corresponding to eigenvalue 3, we get the eigenvector $X_3 = [1, 1, 2]'$

For $\lambda = 3$, the algebraic multiplicity is 1 and the geometric multiplicity is 1.

Although eigenvalues of A are not distinct the geometric multiplicity of each eigenvalues is equal to the algebraic multiplicity.

$\therefore$ A is diagonalizable.

Now, $M = [X_1, X_2, X_3]$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

Since $M^{-1} A M = D$, the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

will be diagonalized to the diagonal matrix

$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the modal matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$

$\therefore$ original matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

Diagonal matrix $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Modal matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$

Q5 a] Six coins are tossed 6400 times. Determine the approximate probability of getting six heads x times.

ANS considering single coin,
we know probability of getting one head with one coin $= 1/2$

$n(E) = 1$, $n(S) = 2$, $P(E) = 1/2$

Considering all coins,
$\therefore$ Probability of getting six heads with six coins,

$$P(E) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

If six coins are tossed 6400 times, then $n\,P(E) = 6400 \times \frac{1}{64}$

let $m = n\,P(E) = 100$

$\therefore$ According to poisson's distribution

$$P(n = x) = \frac{e^{-m}\, m^x}{x!}$$

Here probability of getting six heads in x times,
so $x = 2$, $m = 100$

Hence $P(n=2) = \dfrac{e^{-100}\, 100^2}{2!} = e^{-100} \times 5000$.

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Q5 b] A sample of hundred dry battery cells tested to find the length of life produced the following results: $\bar{x} = 12$ hours, $\sigma = 3$ hours

what % of battery cells are expected to have life

(i) more than 15 hours

(ii) less than 6 hours

(iii) Between 10 and 14 hours

ANS Here x denotes the length of life of dry battery cell.

Also $z = \dfrac{x - \bar{x}}{\sigma} = \dfrac{x - 12}{3}$

(i) when $x = 15$, $z = 1$

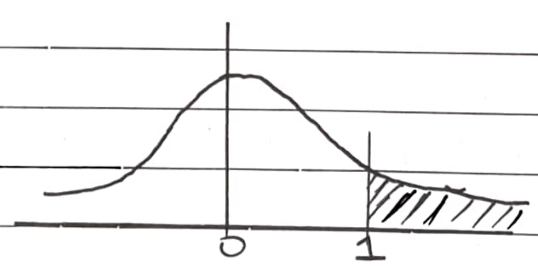
$\therefore P(x > 15) = P(z > 1)$

$= P(0 < z < \infty) - P(0 < z < 1)$

$= 0.5 - 0.3413$

$= 0.1587$

$= 15.27\ \%$



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(ii) when $x = 6$, $z = -2$

$\therefore P(x < 6) = P(z < -2)$

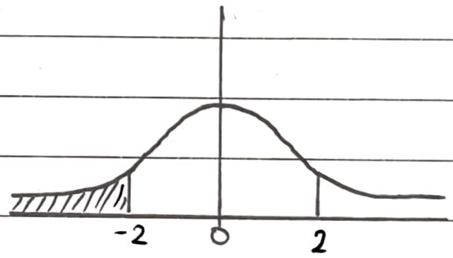
$= P(z > 2)$

$= P(0 < z < \infty) - P(0 < z < 2)$

$= 0.5 - 0.4772$

$= 0.0228$

$= 2.28\%$



(iii) when $x = 10$, $z = \frac{-2}{3} = -0.67$

when $x = 14$, $z = \frac{2}{3} = 0.67$

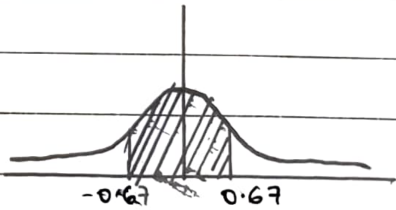
$\therefore P(10 < x < 14) = P(-0.67 < z < 0.67)$

$= 2P(0 < z < 0.67)$

$= 2 \times 0.2487$

$= 0.4974$

$= 49.74\%$



ANS) (i) 15.87 %

(ii) 2.28 %

(iii) 49.74 %

Q4 using Dual - Simplex to solve the following LPP

Maximize $Z = -2x_1 - 2x_2 - 3x_3$

Subject to

$-3x_1 + x_2 - 2x_3 - x_4 = 1$

$x_1 - 2x_2 + x_3 - x_5 = 2$

$x_i \geq 0 \ \forall i$

ANS standard Form :

Maximize $Z = -2x_1 - x_2 - 3x_3 + 0x_4 + 0x_5 - Ma_1 - Ma_2$

$\therefore$ $-3x_1 + x_2 - 2x_3 - x_4 + a_1 = 1$

$x_1 - 2x_2 + x_3 - x_5 + a_2 = 2$

and $x_i \geq 0 \ \forall i$ and $a_1, a_2 \geq 0$

ITERATION 1

	c_j		-2	-1	-3	0	0	-m	-m	
C_B	x_B	sol^n	x_1	x_2	x_3	x_4	x_5	a_1	a_2	Min Ratio
-m	a_1	1	-3	1	-2	-1	0	1	0	
-m	a_2	2	1	-2	1	0	-1	0	1	
	$Z_j - C_j$		2m+2	m+1	m+3	m	m	0	0	

Since all $Z_j - C_j \geq 0$ and all $x_{Bi} > 0$ thus the current solution is the optimal solution.

Hence optimal solution is: $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$

Max $Z = 0$

But this solution is not feasable as it violates the 1st constraint $-3x_1 + x_2 - 2x_3 - x_4 = 1$ and the artificial variable a_1 appears in the basis with positive value 1.

Q6 The following table gives the number of good and bad parts produced by each of the three shifts in a factory.

	GOOD PARTS	BAD PARTS	TOTAL
Day shift	960	40	1000
Evening shift	940	50	990
Night shift	950	45	995
Total	2850	135	2985

Test whether or not the production of bad parts is independent of the shift on which they were produced. use chi square test

ANS Null hypothesis H_0 : The production of bad parts is independent of the shift on which they were produced.

under the null hypothesis expected frequencies can be calculated using $E_{ij} = \frac{R_i \times C_j}{N}$ $(i = 1,2,3, j = 1,2)$

Expected Frequencies are:

	GOOD PARTS	BAD PARTS	TOTAL
DAY SHIFT	$\frac{1000 \times 2850}{2985} = 954.774$	$\frac{1000 \times 135}{2985} = 45.226$	1000
EVENING SHIFT	$\frac{990 \times 2850}{2985} = 945.226$	$\frac{990 \times 135}{2985} = 44.774$	990
NIGHT SHIFT	$\frac{995 \times 2850}{2985} = 950$	$\frac{995 \times 135}{2985} = 45.$	995
TOTAL	2850	135	2985

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$$\chi^2 = \sum_{i=1}^{3} \sum_{j=1}^{2} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

$$= \frac{(960 - 954.774)^2}{954.774} + \frac{(40 - 45.226)^2}{45.226} + \frac{(940 - 945.226)^2}{945.226}$$

$$+ \frac{(50 - 44.774)^2}{44.774} + \frac{(950 - 950)^2}{950} + \frac{(45 - 45)^2}{45}$$

$$= 0.0286 + 0.6039 + 0.0289 + 0.6099 + 0 + 0$$

$$= 1.2713$$

Tabulated value of χ^2 for $2\,[(3-1)(2-1) = 2]$ degrees of freedom at 5% level of significance is 5.991

Since calculated value of χ^2 is less than the tabulated value of χ^2, so H_0 is accepted i.e. the production of bad parts is independent of the shift on which they were produced.

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Q2 Find spearman's coefficient of rank correlation, will the result change if the marks of all the students are increased by 5 and 10 respectively? will the result change if marks in the two subjects of all the students are ~~halfed~~. halved.

ANS

STUDENT NO	ACCOUNTANCY (x)	STATISTICS (y)	R_x	R_y	d_i	d_i^2
1	45	35	8	10	-2	4
2	70	90	3	12	1	1
3	65	70	4	5	-1	1
4	30	40	10	8.5	1.5	2.25
5	90	95	1	1	0	0
6	40	40	9	8.5	0.5	0.25
7	50	60	7	6	1	1
8	57	80	6	3.5	2.5	6.25
9	85	80	2	3.5	-1.5	2.25
10	60	50	5	7	-2	4

$\Sigma d_i^2 = 22$

$n = 10$

40 repeated twice, $m_1 = 2$

80 repeated twice, $m_2 = 2$

$\therefore$ Spearman's rank correlation $(r) = 1 - \dfrac{6\left[\Sigma d_i^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2)\right]}{n^3 - n}$

$$r = 1 - \frac{6\left[22 + \frac{1}{12}(2^3-2) + \frac{1}{12}(2^3-2)\right]}{10^3 - 10}$$

$= 1 - 138/990$

$\boxed{\therefore\ r = 0.86}$

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If the marks in both subjects are changed by 5 and 10 respectively, it would not affect the result as the result is calculated based on ranks and since each student's score is changed on equal scale, there is no change in rank

Same goes for when we half the marks of each student in each subject, the ranks remain unchanged and hence the result wont change

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Q3 Using principle of duality to solve the following LPP

Minimize : $z = 3x_1 + 4x_2 - x_3$

subject to : $2x_1 + 3x_2 + 5x_3 \geq 10$

$3x_1 + 10x_3 \leq 14$

$x_1 - x_2 \leq 0$

$x_1 \geq 0, \; x_2 \leq 0, \; x_3 \geq 0$

ANS The Dual problem :

Maximize $T = 10y_1 + 14y_2$

subject to : $2y_1 + 3y_2 + y_3 \geq 3$

$3y_1 - y_3 \leq 4$

$5y_1 + 10y_2 \geq -1$

$y_1 \leq 0, \; y_2 \geq 0, \; y_3 \geq 0$

Standard form, put $y_0 = -y_1$

Maximize $T = -10y_0 + 14y_2$

subject to $-2y_0 + 3y_2 + y_3 \geq 3$

$-3y_0 - y_3 \leq 4$

$5y_0 - 10y_2 \leq 1$

$y_0 \geq 0, \; y_2 \geq 0, \; y_3 \geq 0$

Now adding variables

Maximize $T = -10y_0 + 14y_2 + 0S_1 + 0S_2 + 0S_3 - MQ_1$

subject to $-2y_0 + 3y_2 + y_3 - S_1 + a_1 = 3$

$-3y_0 - y_3 + S_2 = 4$

$5y_0 - 10y_2 + S_3 = 1$

$y_0, y_2, y_3 \geq 0$

$S_1 \geq 0$ (surplus valiable)

$S_2, S_3 \geq 0$ (slack valiable)

a_1 (artificial variable)

ITERATION - 1

	C_j		-10	14	0	-m	0	0	0	
C_B	x_B	sol^n	y_0	y_2	y_3	a_1	S_1	S_2	S_3	Min Ratio
-m	a_1	3	-2	3	1	1	-1	0	0	1 →
0	S_2	4	-3	0	-1	0	0	1	0	—
0	S_3	1	5	-10	0	0	0	0	1	—
$z_j - c_j$			2m+10	-3m-14	-m	0	m	0	0	

↑

y_2 : entering variable ; a_1 : departing variable
key value = 3

ITERATION - 2

	C_j		-10	14	0	0	0	0	
C_B	x_B	sol^n	y_0	y_2	y_3	S_1	S_2	S_3	min ratio
14	y_2	1	-2/3	1	1/3	-1/3	0	0	
0	S_2	4	-3	0	-1	0	1	0	
0	S_3	-9	-5/3	0	10/3	-10/3	0	1	
$z_j - c_j$			2/3	0	14/3	-14/3	0	0	

Since $z_j - c_j$ of all ij variables ≥ 0, it is an optimal solution

$\therefore$ $y_0 = 0$, $y_2 = 1$, $y_3 = 0$

Max $T = -10y_0 + 14y_2$

$= -10(0) + 14(1)$

$= 14$

Hence using principle of duality

$\boxed{\text{Min } Z = 14}$