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MATHS

TUTORIAL-1

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Q1 Use dual simplex method to solve the following LPP :

Maximize $Z = -4x_1 - 3x_2$

$x_1 + x_3 \le 2$

$x_2 \geq 1$

$-x_1 + 2x_2 \le 1$

$x_1, x_2 \geq 0$

ANS MAXIMIZE $Z = -4x_1 - 3x_2$

$x_1 + x_2 \le 2$

$-x_2 \le -1$

$-x_1 + 2x_2 \le 1$

$x_1, x_2 \geq 0$

Standard form :

Maximize $Z = -4x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3$

$-x_1 + x_2 + s_1 = 2$

$-x_2 + s_2 = -1$

$-x_1 + 2x_2 + s_3 = 1$

$x_1, x_2 \geq 0$ (Basis variable) $s_1, s_2, s_3 \geq 0$ (slack variable)

ITERATION 1

	c_j		-4	-3	0	0	0	
C_B	x_B	Solution	x_1	x_2	S_1	S_2	S_3	
0	S_1	2	1	1	1	0	0	
0	S_2	-1	0	-1	0	1	0	→
0	S_3	1	-1	2	0	0	1	
$Z_j - C_j$			4	3	0	0	0	
Max ratio				-3	—	—	—	
				↑				

$\because \forall\ z_j - c_j \geq 0$ but $\forall\ x_{Bi} \ngtr 0$

$\therefore$ For next iteration

s_2 is departing variable

x_2 is entering variable

-1 is key element

ITERATION 2:

C_B	x_B	Solution	c_j: -4	-3	0	0	0	
0	s_1	1	1	0	1	1	0	
-3	x_2	1	0	1	0	-1	0	
0	s_3	-1	-1	0	0	2	1	$\rightarrow$
$z_j - c_j$			4	0	0	3	0	
Max ratio			-4	–	–	–	–	
			$\uparrow$					

$\because \forall\ z_j - c_j \geq 0$ but $\forall\ x_{Bi} \ngtr 0$

$\therefore$ For next iteration

s_3 is departing variable

x_1 is entering variable

-1 is key element

ITERATION 3:

	c_j		-4	-3	0	0	0
C_B	x_B	solution	x_1	x_2	s_1	s_2	s_3
0	s_1	0	0	0	1	3	1
-3	x_2	1	0	1	0	-1	0
-4	x_1	1	1	0	0	-2	-1
$z_j - c_j$			0	0	0	11	4
Max ratio							

$\because \forall\ Z_j - C_j \geq 0$ and $\forall\ x_{Bi} \geq 0$

$\therefore$ Current solution is optimal solution

$\therefore\ x_1 = 1$

$x_2 = 1$

$\therefore\ Z_{max} = -4(1) - 3(1)$

$= -7$

$\therefore\ \boxed{Z_{max} = -7}$

Q2 Use dual simplex method to solve the following LPP.

Maximize $Z = -4x_1 - 3x_2$

$x_1 + x_2 \leq 1$

$x_2 \geq 1$

$-x_1 + 2x_2 \leq 1$

$x_1, x_2 \geq 0$

ANS Maximize $Z = -4x_1 - 3x_2$

$x_1 + x_2 \leq 1$

$-x_2 \leq -1$

$-x_1 + 2x_2 \leq 1$

$x_1, x_2 \geq 0$

Standard form:

Maximize $Z = -4x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3$

$x_1 + x_2 + s_1 = 01$

$-x_2 + s_2 = -1$

$-x_1 + 2x_2 + s_3 = 1$

$x_1, x_2 \geq 0$ [Basis variables]

$s_1, s_2, s_3 \geq 0$ [Slack variables]

ITERATION 1:

	C_j		-4	-3	0	0	0	
C_B	x_B	Solution	x_1	x_2	S_1	S_2	S_3	
0	S_1	1	1	1	1	0	0	
0	S_2	-1	0	-1	0	1	0	$\rightarrow$
0	S_3	1	-1	2	0	0	1	
$Z_j - C_j$			4	3	0	0	0	
Max ratio			–	-3	–	–	–	
				$\uparrow$				

$\because \forall\ Z_j - C_j \geq 0$ but $\forall\ x_{Bi} \ngeq 0$

$\therefore$ For next iteration,

S_2 is departing variable

x_2 is entering variable

-1 is key element

ITERATION 2:

	C_j		-4	-3	0	0	0	
C_B	x_B	Solution	x_1	x_2	S_1	S_2	S_3	
0	S_1	0	1	0	1	1	0	
-3	S_2	1	0	1	0	-1	0	
0	S_3	-1	-1	0	0	2	1	$\rightarrow$
$Z_j - C_j$			4	0	0	3	0	
Max ratio			-4	–	–	–	–	
			$\uparrow$					

$\therefore \forall\ Z_j - C_j > 0$ but $\forall\ x_{Bi} \ngeq 0$

$\therefore$ For next iteration,

S_3 is departing variable

x_1 is entering variable

-1 is key element

ITERATION 3 :

	C_j		-4	-3	0	0	0
C_B	x_B	solution	x_1	x_2	S_1	S_2	S_3
0	S_1	-1	0	0	1	3	1
-3	x_2	1	0	1	0	-1	0
4	x_1	1	1	0	0	-2	-1
	$Z_j - C_j$		0	0	0	11	4
Max Ratio			–	–	–	–	–

$\because \forall\ z_j - c_j \geq 0$ but $\forall\ x_{Bi} \ngtr 0$

and Max ratio cannot be found.

$\because$ Key row has all +ve elements

$\therefore$ Given LPP is unbounded

i.e. No feasable solution.

Q3 Use duality to solve the LPP

Minimize $Z = x_1 - x_2$

$x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1, x_2 \geq 0$

ANS Maximize $T = 2y_1 + y_2$

s/t $y_1 - y_2 \leq 1$ [Due to x_1]

$y_1 - y_2 \leq -1$ [Due to x_2]

Standard Form :

Maximize $T = 2y_1 + y_2$

s/t $y_1 - y_2 \leq 1$

$-y_1 + y_2 \geq 1$

$y_1, y_2 \geq 0$

$$\text{Max } T = 2y_1 + y_2 + 0x_1 + 0x_2 - Ma_1$$

S/t
$$y_1 - y_2 + x_1 = 1$$
$$-y_1 + y_2 - x_2 + a_1 = 1$$

$y_1, y_2 \geq 0$ [Basic variable]
$x_1 \geq 0$ [slack variable]
$x_2 \geq 0$ [slack variable]
$a_1 \geq 0$ [Artificial variable]

	C_j		2	1	0	0	$-M$	
C_B	x_B	Sol^n	y_1	y_2	x_1	x_2	a_1	Min Ratio
0	x_1	1	1	-1	1	0	0	—
-1	a_1	1	-1	1	0	-1	1	1 →
	$Z_j - C_j$		M-2	M-1	0	M	0	
				↑				

	C_j		2	1	0	0	
C_B	x_B	Sol^n	y_1	y_2	x_1	x_2	Min ratio
0	x_1	2	0	0	1	-1	—
1	y_2	1	-1	1	0	-1	—
	$Z_j - C_j$		-3	0	0	-1	
			↑				

$\therefore \forall Z_j - C_j \neq 0$ & Minimum ratio can't be found as element of key column are not positive

$\therefore$ Given LPP has unbounded collection

i.e. NO feasable solution.

Q6 let $A_1 = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$; $A_2 = \begin{bmatrix} 11 & 0 \\ 7 & 11 \end{bmatrix}$; $A_3 = \begin{bmatrix} 11 & -7 \\ 14 & -10 \end{bmatrix}$

a) Check which of matrices A_1, A_2, A_3 are diagonalizable.

b) For each diagonalizable matrix A_i, find a matrix $C_i \in M(2\times 2; R)$ such that $C_i^{-1} A_i C_i = \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix}$ where $a_i \geq b_i$

ANS $A_1 = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

Characteristic equation is given by $|A - \lambda I| = 0$

$\therefore \lambda^2 - 0\lambda + 1 = 0$

$\therefore \lambda^2 + 1 = 0$

$\therefore \lambda = \pm i$

$\therefore$ Eigen values are $\lambda = i, -i$

For $\lambda_1 = i$

$(A - iI)X_1 = 0$

$\begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} X_1 = 0$

$R_2 \to R_2 + \left(\frac{1+i}{2}\right) R_1$

$\therefore \begin{bmatrix} 1-i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(1-i)x_1 + 2x_2 = 0$

$x_2 = -\frac{(1-i)x_1}{2}$

$\therefore X_1 = \begin{bmatrix} x_1 \\ -\left(\frac{1-i}{2}\right)x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -\frac{(1-i)}{2} \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -(1-i) \end{bmatrix}$

For $\lambda_2 = -i$

$(A + iI)X_2 = 0$

$\begin{bmatrix} 1+i & 2 \\ -1 & -1+i \end{bmatrix} X_2 = 0$

$R_2 \to R_2 + \left(\frac{1-i}{2}\right) R_2$

$\therefore \begin{bmatrix} 1+i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(1+i)x_1 + 2x_2 = 0$

$x_2 = -\frac{(1+i)x_1}{2}$

$\therefore X_2 = \begin{bmatrix} x_1 \\ -\frac{(1+i)x_1}{2} \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -\frac{(1+i)}{2} \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -(1+i) \end{bmatrix}$

$\therefore$ Eigenvectors are $\begin{bmatrix} 2 \\ -(1-i) \end{bmatrix}$, $\begin{bmatrix} 2 \\ -(1+i) \end{bmatrix}$ for $i, -i$ respectively.

$\because$ For each eigenvalue, the algebraic multiplicity is equal to geometric multiplicity [i.e. AM = GM = 1]

$\therefore$ Given matrix is diagonalizable only over ($\mathbb{Z}/\mathbb{C}$) i.e complex

But given matrix is not diagonalizable over $\mathbb{R}$ i.e. Real Numbers.

Consider $C_1 = [X_1 \; X_2] = \begin{bmatrix} 2 & 2 \\ -1+i & -1-i \end{bmatrix} \notin \mathbb{R}$

$\therefore$ Given matrix A_1 is not diagonalizable over $\mathbb{R}$.

But it is diagonalizable over $\mathbb{C}/\mathbb{Z}$.

b $A_2 = \begin{bmatrix} 11 & 0 \\ 7 & 11 \end{bmatrix}$

Characteristic equation $|A - \lambda I| = 0$

$\therefore \lambda^2 - 22\lambda + 121 = 0$

$\therefore \lambda = 11, 11$

For $\lambda_1 = 11$

$(A - \lambda_1 I) X_1 = 0$

$(A - 11 I) X_1 = 0$

$\begin{bmatrix} 0 & 0 \\ 7 & 0 \end{bmatrix} X_1 = 0$

$7x_1 = 0$

$\boxed{x_1 = 0}$

$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$

$x_2 \in \mathbb{R}/\{0\}$

The algebraic multiplicity of $\lambda = 11$ is 2 & Geometric multiplicity for $\lambda = 11$ is 1 $\therefore$ AM $\neq$ GM

Given matrix is not diagonalizable.

c $A_3 = \begin{bmatrix} 11 & -7 \\ 14 & -10 \end{bmatrix}$

Characteristic equation $|A - \lambda I| = 0$

$\lambda^2 - \lambda - 12 = 0$

$\lambda^2 = 4, -3$;

The given matrix is diagonalizable since eigenvalue are distinct.

For $\lambda_1 = 4$

$(A - \lambda_1 I) X_1 = 0$

$(A - 4I) X_1 = 0$

$\begin{bmatrix} 7 & -7 \\ 14 & -14 \end{bmatrix} X_1 = 0$

$R_2 \to R_2 - 2R_1$ $\begin{bmatrix} 7 & -7 \\ 0 & 0 \end{bmatrix} X_1$

$\boxed{x_1 = x_2}$

$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ $x_2 \in R / \{0\}$

For $\lambda_2 = -3$

$(A - \lambda_2 I) X_2 = 0$

$(A + 3I) X_2 = 0$

$\begin{bmatrix} 14 & -7 \\ 14 & -7 \end{bmatrix} X_2 = 0$

$R_2 = R_2 - R_1$

$\begin{bmatrix} 14 & -7 \\ 0 & 0 \end{bmatrix} X_2 = 0$

$14 x_1 = 7 x_2$

$x_2 = 2x_1$

$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{\therefore X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

Now let M be a non-singular matrix

$$M = [X_1 \quad X_2] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$M^{-1} = 1 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M^{-1}AM = 1 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 11 & -7 \\ 14 & -10 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} = D$$

Matrix is Diagonalizable

Q7 If $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$, Find (i) e^A
(ii) 4^A

ANS The characteristic equation of A is $\begin{bmatrix} 3/2-\lambda & 1/2 \\ 1/2 & 3/2-\lambda \end{bmatrix} = 0$

$\therefore \left(\frac{3}{2} - \lambda\right)^2 - \frac{1}{4} = 0$

$\therefore \frac{9}{4} - 3\lambda + \lambda^2 - \frac{1}{4} = 0$

$\therefore \lambda^2 - 3\lambda + 2 = 0$

$\therefore (\lambda - 1)(\lambda - 2) = 0$

$\therefore \lambda = 1, 2.$

(i) For $\lambda = 1$, $[A - \lambda I]x = 0$ gives

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} 2R_1 \\ 2R_2 \end{matrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 - R_1$ $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore x_1 + x_2 = 0$

Putting $x_2 = -t$, we get $x_1 = t$

$\therefore X_1 = \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence, the eigen vector is $[1, -1]$.

(ii) For $\lambda = 2$, $(A - \lambda I)X = 0$ gives

$$\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} 2R_1 \\ 2R_2 \end{matrix}$ $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 + R_1$ $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore -x_1 + x_2 = 0$

$\therefore x_1 = x_2$

$\therefore X_2 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence, the eigen vector is $[1, 1]$.

$\therefore M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\therefore |M| = 2$

$\therefore M^{-1} = \dfrac{Adj(M)}{|M|} = \dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Now $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

If $f(A) = e^A$, $f(D) = e^D = \begin{bmatrix} e^t & 0 \\ 0 & e^2 \end{bmatrix}$

If $f(A) = 4^A$, $f(D) = 4^D = \begin{bmatrix} 4^1 & 0 \\ 0 & 4^2 \end{bmatrix}$

$\therefore e^A = M f(D) M^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}\dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$= \dfrac{1}{2}\begin{bmatrix} e & e^2 \\ -e & e^2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$e^A = \frac{1}{2}\begin{bmatrix} e + e^2 & -e + e^2 \\ -e + e^2 & e + e^2 \end{bmatrix}$$

Similarly, replacing e by 4 we get,

$$4^A = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix}$$

$$4^A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} //$$

Q8 Is the following matrix diagonalizable ? If so, find the diagonal and modal matrix.

ANS
$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

The characteristic equation of matrix A :

$\lambda^3 - (\text{trace of } A)\lambda^2 + (\text{sum of minors of diagonal elements})\lambda - \det(A) = 0$

$\therefore \lambda^3 - 7\lambda + (-21 + 20 + 21 - 15 + -9 + 20)\lambda - (3(-1) - 10(-2) + 5(-1)) = 0$

$\therefore \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

$\lambda = 2, 2, 3$

The eigenvalues of matrix A are 2, 2, 3

For $\lambda_1 = 2$

$(A - \lambda_1 I)X_1 = 0$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} X_1 = 0$$

$R_2 + 2R_1$, $R_3 - 3R_1$ $\Rightarrow$
$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & -10 \end{bmatrix} X_1 = 0$$

$R_2 \div 3$, $R_3 \div 5$, $R_2 + R_3$:
$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + 10x_2 + 5x_3 = 0$

$\therefore 5x_2 + 2x_3 = 0$

$\therefore x_2 = -2/5\, x_3$

$x_1 = -x_3$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -2/5\, x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2/5 \\ 1 \end{bmatrix}$$

For $\lambda = 2$

AM $= 2$ [Since eigen value 2 is obtained twice]

GM = linearly independent eigenvector corresponding eigenvalue $\lambda = 1$ since AM $\neq$ GM for $\lambda = 2$.

Therefore, the given matrix is non - diagonalizable.

Q9 Find eigen values and eigen vectors of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

ANS $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The characteristic equation of matrix A is

$\lambda^3 - (\text{trace of } A)\lambda^2 + (\text{sum of minors of diagonal elements})\lambda - \det(A) = 0$

$\lambda^3 - 0\lambda^2 + (-1 + -1 + -1)\lambda - [0 - 1(-1) + 1(1-0)] = 0$

$\therefore \lambda^3 - 3\lambda - 2 = 0$

$\therefore \lambda = -1, -1, 2$

$\therefore$ Eigen values of matrix A are $-1, -1, 2$

Eigenvector for $\lambda_1 = 2$

$(A - 2I) X_1 = 0$

$$\therefore \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} X_1 = 0$$

$R_3 \to R_1 + R_2 + R_3$

$R_2 \to R_2 + R_1/2$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3/2 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -x_2 + x_3 = 0 \Rightarrow x_2 = x_3$

$\therefore -2x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = x_3$

$$\therefore X_1 = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eigenvector for $\lambda = -1$

$(A + I) X = 0$

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} X = 0$$

$R_3 - R_2$

$R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$

$$\therefore X = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Eigenvectors are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ for eigenvalues $2, -1, -1$ respectively.

Q10 $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$, Prove that $3 \tan A = A \tan 3$

ANS $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$\therefore \tan A = A + \frac{A^3}{3} + \frac{2A^5}{15} + \cdots$$

If A is diagonalizable $\exists$ some non-regular matrix M such that $M^{-1} A M = D$ i.e. $A = MDM^{-1}$, $A^2 = MD^2M^{-1}$

$$\therefore A^n = MD^nM^{-1}$$

$$\therefore \tan A = MDM^{-1} + \frac{MD^3M^{-1}}{3} + \frac{2MD^5M^{-1}}{15} + \cdots$$

$$= M\left[D + \frac{D^3}{3} + \frac{2D^5}{15} + \cdots\right]M^{-1}$$

$$= M \tan D \, M^{-1}$$

$$\therefore f(A) = \tan A = M f(D) M^{-1} = M \tan D \, M^{-1}$$

$A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$

The characteristic equation of matrix A is

$$\lambda^2 - (\text{trace of } A)\lambda + \det(A) = 0$$

$$\therefore \lambda^2 - (-1+1)\lambda + (-1-8) = 0$$

$$\lambda^2 - 9 = 0$$

$$\therefore \lambda = 3, -3$$

For $\lambda_1 = 3$

$(A - \lambda_1 I)X_1 = 0$

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_1 + 2R_2$

For $\lambda_2 = -3$

$(A - \lambda_2 I)X_2 = 0$

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$

$$\begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 - 2x_2 = 0$

$\therefore x_1 = x_2$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 + 4x_2 = 0$

$x_1 = -2x_2$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Since eigenvalues are distinct, matrix A is diagonalizable

Let $M\,[X_1 \;\; X_2] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

$$M^{-1} = \frac{-1}{3} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$D = M^{-1} A M = \frac{-1}{3} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\tan A = M \tan D\, M^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{-1}{3} \begin{bmatrix} \tan 3 & -2\tan 3 \\ \tan 3 & -\tan 3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} \tan 3 & -4\tan 3 \\ -2\tan 3 & -\tan 3 \end{bmatrix}$$

$$\therefore \text{LHS} = 3\tan A = 3 \begin{bmatrix} \tan 3 & -4\tan 3 \\ -2\tan 3 & -\tan 3 \end{bmatrix} \times \frac{-1}{3} = \begin{bmatrix} -\tan 3 & 4\tan 3 \\ 2\tan 3 & \tan 3 \end{bmatrix}$$

$$= \tan 3 \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$$

$$= \tan 3 \cdot A$$

$$= A \tan 3$$

$$= \text{RHS}.$$

$\because$ LHS = RHS $\Rightarrow$ $3\tan A = A \tan 3$ for given matrix A.