CHAPTER 1

# Set Theory

## 1. Introduction

You have already studied set theory to some extent. Here, we shall take a brief review of what you have learnt.

A set is a collection having the property that given 'anything' we can say whether 'that thing' is in the set or not. Although a **set is an undefined term**, given an element we should be able to say whether the element is in the set or not. The editice of modern mathematics rests on the concept of 'Sets'. The pioneering work on set theory was done by German Mathematician George Cantor (1845-1918).

**Georg Cantor (1845 - 1918)**

The eminent German mathematician was born in St. Petersburg. His father a successful merchant and broker wanted his son to be an engineer. But Georg Cantor pursued his studies in Mathematics and got his Ph.D. from Berlin University at the age of 22 in number theory. In 1869 he started his career as an unsalaried lecturer at University of Halle and published his revolutionary work on set theory five years later. Deeply religious, Cantor took deep interest in art, music and philosophy.

Bertrand Russel, another well known mathematician and philosopher, described him as "one of the greatest intellect of the nineteenth century". Although his theory is now used in many theoretical and practical fields, in his own time it was not accepted by contemporary mathematicians. This intensified his manic depression and he died in a mental hospital in Halle in 1918.

## 2. Notation

Consider a set $A$ whose members are 2, 3, 6, 8. The fact that 2 is in the set is stated as **2 belongs to $A$** and is written symbolically as $2 \in A$. The fact that 5 is not in the set is stated as **5 does not belong to $A$** and is written as $5 \notin A$.

We usually denote elements of sets by small letters $a, b, c, \ldots, x, y, z$ and sets by capital letters $A, B, C, \ldots, X, Y, Z$. If $x$ is an element and $A$ is a set, the statement that $x$ is an element of $A$ (or $x$ belongs to $A$) is denoted by $x \in A$. The statement that $x$ is not an element of $A$ (or $x$ does not belong to $A$) is denoted by $x \notin A$.

**Designation of a set :** There are two standard notations for designating a particular set (i) List form and (ii) Property form.

### (a) List Form

Whenever possible we specify the set by listing the elements of the set in braces. Thus, a set whose elements are $a, b, c, d$ is denoted by $\{a, b, c, d\}$. The set of week days can be denoted as $D$ = {SUNDAY, MONDAY, TUESDAY, WEDNESDAY, THURSDAY, FRIDAY, SATURDAY}.

It may be noted that the order in which elements of the set are written is not important. With this convention. The sets $\{a, b, c, d\}$, $\{b, c, d, a\}$, $\{a, c, b, d\}$ are the same. Further if an element occurs more than once in a set, it is treated once only. The set $\{x, x\}$ does not have two elements, the set $\{a, a, a\}$ does not have three elements. Both the sets have one element each.

### (b) Property Form

The method of listing all the elements of a set is obviously inconvenient if there are large number of elements. In many circumstances it is even unworkable. If there are infinite elements in the set obviously we cannot list all of them. Under these circumstances we denote the set by stating a property which is possessed by all the elements of the set. We usually use the letter $x$ to represent an arbitrary element. If the property is $P(x)$ then the set of elements $x$ which possess the property $P(x)$ is denoted by $\{x \mid P(x)\}$. It is read as the set of $x$ such that $x$ satisfies the property $P(x)$. Thus, the set of all irrational numbers can be denoted as $B = \{x \mid x$ is real and irrational$\}$. We read this as the set of all $x$ such that $x$ is real and irrational.

A set containing only one element is called a **singleton** set or a **unit set.** Thus, $A = \{a\}$ or $B = \{x \mid x$ is the positive root of $x^2 - x - 6 = 0\}$ or $C$ = {The present principal of a college} are singleton sets.

The following are the common notations for denoting the various sets of real numbers :



$Z$ = Set of all integers
$= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$Z^+ = N$ = Set of all positive integers
$= \{1, 2, 3, 4, \ldots\}$

$Z^-$ = Set of negative integers
$= \{\ldots -4, -3, -2, -1\}$

$W$ = Set of whole numbers
$= \{0, 1, 2, 3, 4, \ldots\}$

$Q$ = Set of rational numbers
$= \left\{\frac{p}{q} \middle| p, q \in Z, q \neq 0\right\}$

$R$ = Set of real numbers

$R^+$ = Set of positive real numbers
$= \{x \mid x \in R, x > 0\}$

$R^-$ = Set of negative real numbers
$= \{x \mid x \in R, x < 0\}$

$C$ = Set of complex numbers

Finite Intervals are denoted as sets as follows :

Closed interval $[a, b] = \{x \mid x \in R, a \leq x \leq b\}$

Closed-open interval $[a, b) = \{x \mid x \in R, a \leq x < b\}$

Open-closed interval $(a, b] = \{x \mid x \in R, a < x \leq b\}$

Open interval $(a, b) = \{x \mid x \in R, a < x < b\}$

Infinite intervals are denoted as follows :-

$[a, \infty) = \{x \mid x \in R, x \geq a\}$

$(-\infty, a] = \{x \mid x \in R, x \leq a\}$

## 3. Finite And Infinite Set

A set which has finite elements is called a **finite set**, *e.g.* the set $\{a_1, a_2, a_3, \ldots, a_n\}$ is a finite set. A set which is not finite is called **infinite** set *e.g.* the sets described above are **infinite sets.**

## 4. Null Set

It is convenient to introduce the set which contains **no elements**. Such a set is empty or void and hence it is also called the **null** set, the **empty** set or the **void** set. The null set (derived from the Latin word nullus = not any) is denoted by the symbol $\Phi$ (pronounced as phi) or { }. It should be noted that $\Phi$ is not nothing; an empty box is very much different from no box at all.

**Definition** : The set having no elements is called the **null set**. It should be noted that $\Phi$ is not identical to { $\Phi$ }, because $\Phi$ is a set with no elements while { $\Phi$ } is a set with $\Phi$ as its element. Thus, $\Phi \neq \{ \Phi \}$. Also for the same reason $\Phi \neq \{0\}$.

Examples of the null set can be given by stating any condition such that no object satisfies the condition.

(i) The set of women presidents of India till 2000 is the null set,

(ii) $\{x \mid x$ is an odd number and divisible by $2\}$ is the null set,

(iii) $\{x \mid x \neq x\}$ is the null set.

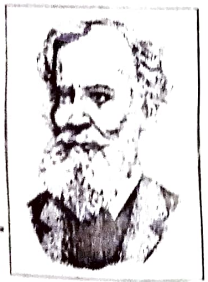
## 5. Universal Set

While talking about any set we usually need a 'bigger' set or a 'reference' set. **Such a set is called a universal set**. The universal set is denoted by $U$ or S. It serves the purpose of providing a frame of reference or the universe for our discussion. The universal set naturally depends upon what ideas we are considering. If we are considering sets of students of a college, then the set of all the students of the college is sufficient as a universal set. If we are studying sets of real numbers, then the universal set $U$ will be the set of all real numbers.

## 6. Venn-Euler Diagram

To have a geometric picture, we represent the universal set $U$ by a rectangular area in a plane and the members of $U$ by the points of the rectangular area. Sets then can be pictured by areas within this rectangle. The areas used to denote the sets may be enclosed by any simple curve (*i.e.*, not intersecting itself) although circles are preferred by many. Such a diagram is known as Venn-Euler diagram or **Venn-diagram**, named after the English logician John Venn (1834-1923) who first used them.

**John Venn (1834 - 1923)**



John Venn was born in a charitable family in England. He got his degree in mathematics in 1856 from Cambridge. In 1883 he got his D.Sc. from Cambridge and was elected a fellow of Royal Society of London. He was greatly influenced by Boole's work in Symbolic Logic. He wrote a book "Symbolic Logic" in which he clarified inconsistencies and ambiguities in Boole's ideas and



notations. He used geometric diagrams to represent logical arguments. The technique was first suggested by Leibnitz and was developed by Euler. His two more famous books are "The Logic Of Chance" and "The Principles Of Empirical Logic".

**Leonhard Euler (1707 - 1783)**

One of the great mathematicians of Switzerland. His father wanted him to become a pastor (a priest). But Bernoulli persuaded his father to allow his son to pursue mathematics. He studied under his fellow countryman, mathematician Bernoulli and had published his first paper when he was 18. The word function first suggested by Leibnitz was generalised further by Bernoulli and Euler. Euler is supposed to be the most prolific mathematical writer in history. He has written a number of text books which are known for clarity, detail and completeness. Although he had lost his eye-sight for the last 17 years of his life, he did not allow his work to be hampered because all the formulae from trigonometry and analysis (and many poems including the entire Latin epic-Aeneid) were on the tip of his tongue.

## 7. Subsets

Let $A$ be the set of the students of F.E. class of your college and $B$ be the set of all students of your college. We see that all the elements of $A$ are in $B$. We describe such situations as $A$ is a subset of $B$.

**Definition :** Given two sets $A$ and $B$ if every element of $A$ is an element of $B$, then **$A$ is called a subset of $B$**. We denote this as $A \subseteq B$. It is read as $A$ is a subset of $B$ or $A$ is included in $B$ or $A$ is contained in $B$. If $A$ is not a subset of $B$, it is denoted as $A \not\subseteq B$. $A \subseteq B$ means if $x \in A$ then $x \in B$.

We may also say that $B$ is a superset of $A$ or $B$ includes $A$ or $B$ contains $A$. This we denote as $B \supseteq A$. Thus, the notations $A \subseteq B$ and $B \supseteq A$ mean one and the same thing. By the by, can we say, $A \subseteq A$ *i.e.*, every set is a subset of itself ? [See Fig. 1.1 (Yes)]

**Example 1 :** If $A$ is the set of black balls in a box and $B$ is the set of all balls in the box then $A \subseteq B$.

**Example 2 :** If $A = \{x \mid x$ is a positive integer divisible by $6\}$ and $B = \{x \mid x$ is a positive integer divisible by $3\}$, then since

$A = \{6, 12, 18, 24, .....\}$

and $B = \{3, 6, 9, 12, 15, 18, 21, 24, 27, ....\}$, $A \subseteq B$.

**Proper subset :** If $A$ is a subset of $B$ and $A$ is not equal to $B$, we say that **$A$ is a proper subset of $B$.** This relation is denoted by $A \subset B$.

If the sets $N, Z, Q, R, C$ are as defined on page 1-3, then it is clear that $N \subset Z \subset Q \subset R \subset C$.

## 8. Equality

By equality of sets we mean that the sets have the same or identical elements.

**Definition :** If every element of $A$ is an element of $B$ and if every element of $B$ is an element of $A$, then the set $A$ is **equal** to the set $B$. This is denoted by $A = B$.

In other words using the notation of set inclusion,

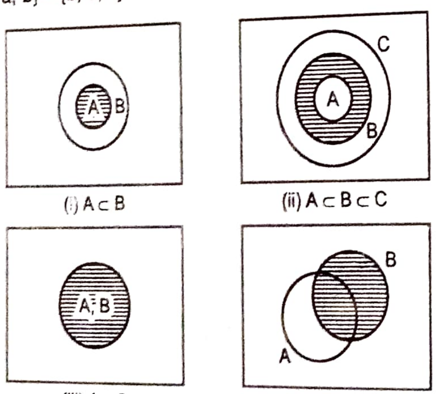
$$\boxed{\textbf{if } A \subseteq B \textbf{ and if } B \subseteq A\textbf{, then } A = B \textbf{ and conversely}}$$

**Example 1 :** If $A = \{3, 4\}$ and $B = \{x \mid x^2 - 7x + 12 = 0\}$, then $A = B$. It is clear that the roots of the equation $x^2 - 7x + 12 = 0$ are 3 and 4.

$\therefore$ $B = \{3, 4\}$ if written in the list form. Since each element 3, 4 of $A$ is an element of $B$ and each element 3, 4 of $B$ is in $A$, $A = B$.

**Example 2 :** If $A = \{x \mid x$ is a letter in 'spot'$\}$, $B = \{x \mid x$ is a letter in 'tops'$\}$, then $A = B$.

The elements of the set $A$ are $s, p, o, t$ and those of $B$ are $t, o, p, s$. Since, every element of $A$ is in $B$ and every element of $B$ is in $A$. $\therefore$ $A = B$. Note that because of this reason **order of elements in a set is immaterial.** Thus, $\{a, b, c\} = \{c, a, b\} = \{b, a, c\}$.




(i) $A \subset B$ (ii) $A \subset B \subset C$ (iii) $A = B$ (iv) $A \not\subseteq B$

**Fig. 1.1**



Using Venn diagrams set inclusion and equality can be represented as shown above. If $A$, $B$ and $C$ are sets, then Fig. (i) represents the situation that $A$ is subset of $B$, and Fig. (ii) represents the situation that $A$ is a subset of $B$ and $B$ is a subset of $C$, Fig. (iii) represents the situation $A = B$, Fig. (iv) represents the situation that $A \not\subseteq B$.

## 9. Operations On Sets

Just as we have operations of addition, subtraction, multiplication and division on real numbers, we can operate on given sets and produce new sets. The operations are (i) Complementation, (ii) Union, (iii) Intersection.

## 10. Complement of A Set

Consider the set $U$ of all students of your college and the set $A$ of all students of F.E. class, and the set $B$ of all students of all other classes. We see that the elements of $B$ are in $U$ but not in $A$. Such a set is called the **complement** of the given set.

**Definition :** Let $A$ be a subset of the universal set $U$. The set of all elements of $U$ which do not belong to $A$ is called the **complement of $A$.** It is denoted by $A'$ or $\bar{A}$ or $A^c$. In symbols,

$$\boxed{\bar{A} = \{x \mid x \in U \text{ and } x \notin A\}}$$

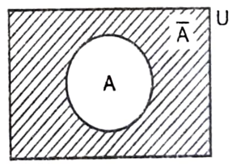



**Fig. 1.2 (a)**

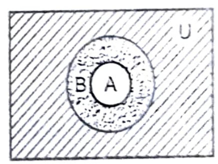



**Fig. 1.2 (b)**

In the Fig. 1.2 (a), the shaded area represents the complement of $A$.

**Example 1 :** If $U$ is the set of all students who appeared for F.E. examination and $A$ is the set of students who passed, then $\bar{A}$ is the set of all students who failed at the F.E. examination.

**Example 2 :** If $U = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{1, 3, 5\}$, then

$$\bar{A} = \{2, 4, 6, 7\}.$$

The following properties of the operation of forming complements are obvious :

| | |
|---|---|
| **(i)** $\bar{\Phi} = U$ | **(ii)** $\bar{U} = \Phi$ |
| **(iii)** $(\bar{\bar{A}}) = A$ | **(iv) If** $A \subseteq B$ **then** $\bar{B} \subseteq \bar{A}$ |

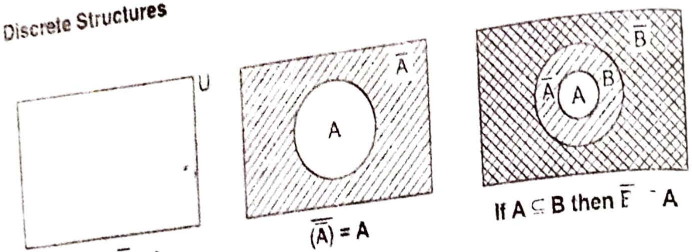



Fig. 1.3

## 11. Union of Sets

Let $U$ be the set of all persons in Maharashtra, $A$ be the set of persons who speak Marathi and $B$ be the set of persons who speak English. Then the set of persons who speak either Marathi or English is clearly the set of all persons in the two sets taken together. Some persons may speak both Marathi and English but we know that they are to be taken into account only once. This set is called the **union**.

**Definition :** The **union** of two sets $A$ and $B$ is defined as the set of all elements which are either in $A$ or in $B$ (including those which are in both). It is denoted by $A \cup B$ and read as $A$ union $B$ or $A$ cup $B$. In symbols,

$$\boxed{A \cup B = \{ x \mid x \in A \text{ or } x \in B \text{ or both} \}}$$

It is clear that $A \cup B$ is formed by lumping together the elements of $A$ and of $B$ and treating them as forming a single set.

**Example 1 :** If $A$ is the set of persons who drink tee and $B$ is the set of those persons who drink coffee then $A \cup B$ is the set of persons who drink either tea or coffee including those who drink both.

**Example 2 :** If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then

$$A \cup B = \{1, 2, 3, 4, 5, 6\}.$$

**Example 3 :** If $A = \{a, b, d, f\}$, $B = \{b, c, d, e\}$, then

$$A \cup B = \{a, b, c, d, e, f\}$$

The following properties of the operation of forming union are obvious

| | |
|---|---|
| **(i)** $A \cup B = B \cup A$ | **(ii)** $A \cup (B \cup C) = (A \cup B) \cup C$ |
| **(iii)** $A \cup A = A$ | **(iv)** $A \cup \Phi = A$ |
| **(v)** $A \cup U = U$ | **(vi)** If $A \subseteq B$, then $A \cup B = B$ |
| **(vii)** $A \cup \overline{A} = U$ | **(viii)** $A \subseteq (A \cup B)$ and $B \subseteq (A \cup B)$ |



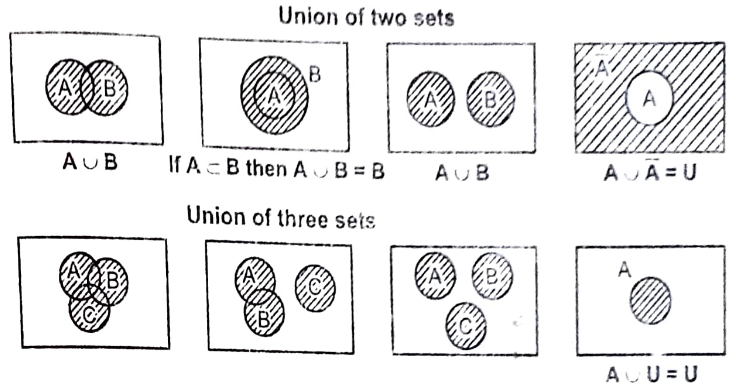



Fig. 1.4

**Note ....**

Note that the usual law of 'cancellation' does not hold for union of two sets. For example, if $A = \{a, b\}$, $B = \{a, c\}$ and $C = \{b, c\}$, then $A \cup B = A \cup C = \{a, b, c\}$ and still $B \neq C$.

## 12. Intersection of Sets

The next operation on sets is that of forming intersection. Consider the example of the previous article. $A$ is the set of persons who speak Marathi and $B$ is the set of persons who speak English. Then the set of persons who speak both Marathi and English is the set of persons common to both the sets $A$ and $B$. This set is called the **intersection** of $A$ and $B$.

**Definition :** The **intersection** of two sets $A$ and $B$ is defined as the set of elements which are both in $A$ and in $B$. It is denoted by $A \cap B$ and read as $A$ intersection $B$ or $A$ cap $B$. In symbols,

$$\boxed{A \cap B = \{ x \mid x \in A \text{ and } x \in B \}}$$

In the Figs. 1.6 given below, $A \cap B$ is denoted by shaded area. It is clear that $A \cap B$ is obtained by taking the common part *i.e.*, the common elements of $A$ and $B$.

**Example 1 :** If $A = \{1, 2, 5, 7\}$, $B = \{1, 3, 4, 5, 6, 8\}$, then $A \cap B = \{1, 5\}$.

**Example 2 :** If $A = \{a, b, c, d\}$ and $B = \{b, d, g, h\}$, then $A \cap B = \{b, d\}$.

**Notes ....**

Note that the usual law of 'cancellation' does not hold for the intersection of sets. For example, if $A = \{a, b\}$, $B = \{a, c\}$ and $C = \{a, d\}$ then $A \cap B = A \cap C = \{a\}$ and still $B \neq C$.

## 13. Disjoint Sets (M.U. 2010)

**Definition :** If $A$ and $B$ are two sets such that $A \cap B = \Phi$ *i.e.* their intersection is empty then $A$ and $B$ are called **disjoint sets.**

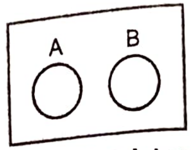


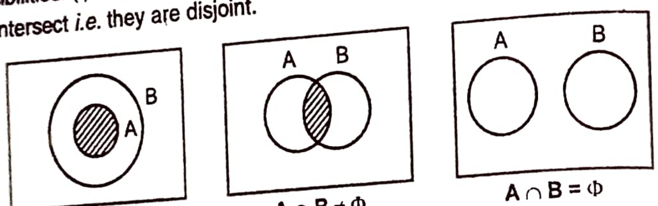

**Fig. 1.5 : Disjoint sets.**

**Example 1 :** The sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ are disjoint sets.

**Example 2 :** The sets $A = \{a, b, c\}$, $B = \{d, e, f\}$ are disjoint sets.

**Example 3 :** The set $A = \{1, 3, 5, 7, .....\}$, $B = \{2, 4, 6, 8, .....\}$ are disjoint.

If $A$ and $B$ are any two sets with universal set $U$ then there are three possibilities. (i) one set is a subset of the other, (ii) they intersect, (iii) they do not intersect *i.e.* they are disjoint.

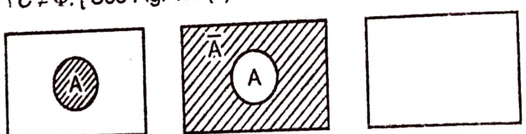



If $A \subset B$ then $A \cap B = A$ $A \cap B \neq \Phi$ $A \cap B = \Phi$

**Fig. 1.6**

If $A, B, C$ are three sets then there are various possibilities. We consider some of them.

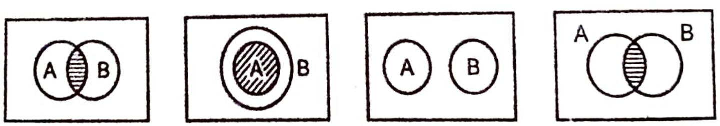
(i) $A$ and $B$ intersect but $A, C$ and $B, C$ are disjoint, (ii) $A, B$ intersect; $B, C$ intersect but $A, C$ are disjoint, (iii) $A, B, C$ are all disjoint, (iv) $A, B, C$ all intersect *i.e.* $A \cap B \cap C \neq \Phi$. [ See Fig. 1.7 (c) on the next page. ]




$A \cap A = A$ $A \cap \overline{A} = \Phi$ $A \cap \Phi = \Phi$

$A \cap U = A$

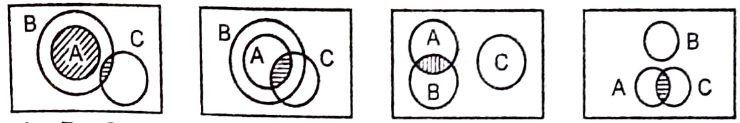
**Fig. 1.7 (a) : Intersertion of one set.**



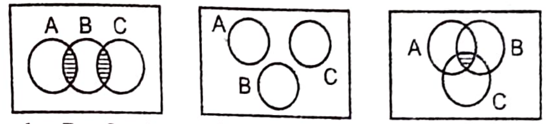

$A \cap B = B \cap A$ If $A \subset B$ then $A \cap B = A$ $A \cap B = \Phi$ $(A \cap B) \subset A$ $(A \cap B) \subset B$

**Fig. 1.7 (b) : Intersertion of two sets.**






$A \cap B \cap C = \Phi$ $A \cap B \cap C \neq \Phi$ $A \cap B \cap C = \Phi$ $A \cap B \cap C = \Phi$




$A \cap B \cap C = \Phi$ $A \cap B \cap C = \Phi$ $A \cap B \cap C \neq \Phi$

**Fig. 1.7 (c) : Intersertion of three sets.**

Observe that in the last case $A, B, C$ all intersect and they create $2^3 = 8$ regions. These regions can be described as follows. If $x$ is an element of the universal set $U$ then (i) $x \in A$ only, (ii) $x \in B$ only, (iii) $x \in C$ only, (iv) $x \in A$ and $x \in B$ but $x \notin C$, (v) $x \in B$, $x \in C$ but $x \notin A$, (vi) $x \in C$, $x \in A$ but $x \notin B$, (vii) $x \in A$, $x \in B$ and also $x \in C$, (viii) $x \notin A$, $x \notin B$, $x \notin C$.

Since for an element of $U$ there are two possibilities, either $x$ belongs to the set $A$ or $B$ or $C$ or $x$ does not belong to these set $A$ or $B$ or $C$. Since there are three sets there are $2^3$ possibilities.

**Example 3 :** If $A = \{ x \mid x$ is real and $x^2 - 5x + 6 = 0\}$ and $B = \{ x \mid x$ is real and $x^2 - 7x + 10 = 0\}$ *i.e.*, $A = \{2, 3\}$ and $B = \{2, 5\}$ then $A \cap B = \{ 2 \}$.

The following properties of the operation of forming intersection are obvious :

| | |
|---|---|
| **(i)** $A \cap B = B \cap A$ | **(ii)** $A \cap (B \cap C) = (A \cap B) \cap C$ |
| **(iii)** $A \cap A = A$ | **(iv)** $A \cap \Phi = \Phi$ |
| **(v)** $A \cap U = A$ | **(vi) if** $A \subseteq B$ **then** $A \cap B = A$ |
| **(vii)** $A \cap \overline{A} = \Phi$ | **(viii)** $(A \cap B) \subseteq A$ **and** $(A \cap B) \subset B$ |

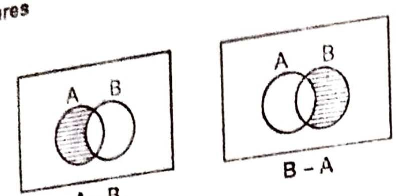
## 14. Difference of Two Sets

If $A$ and $B$ are two sets then the set of elements in $A$ which are not in $B$ is called the **difference** and is denoted by $A - B$. Some authors denote the difference by $A \setminus B$ or $A - B$.

$\therefore \quad A - B = \{ x \mid x \in A$ and $x \notin B \}$

Similarly, $\quad B - A = \{ x \mid x \in B$ and $x \notin A \}$

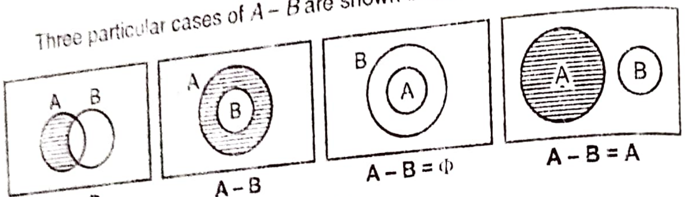
The Venn-diagrams of $A - B$ and $B - A$ are shown below.



A − B  B − A

Fig. 1.8

Three particular cases of $A - B$ are shown below.



A − B  A − B  A − B = Φ  A − B = A

Fig. 1.9

The difference of two sets is also sometimes called the relative complement.

It is easy to see that .......... (I)

$$A - B = A \cap \overline{B} \quad \text{and} \quad B - A = B \cap \overline{A}$$

A − B  $A \cap \overline{B}$

Fig. 1.10

**Example 1 :** If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 6, 7\}$, $B = \{2, 3, 4, 8\}$, then $A - B = \{1, 6, 7\}$ and $B - A = \{4, 8\}$.

**Example 2 :** If $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, b, c, e, f\}$, $B = \{b, c, d, g, h\}$, then $A - B = \{a, e, f\}$ and $B - A = \{d, g, h\}$.

## 15. Symmetric Difference (M.U. 2010)

**Definition :** If $A$ and $B$ are two sets then the **symmetric difference** of $A$ and $B$ is defined as the set of elements which belong to either $A$ or to $B$ **but not** to both. It is denoted by $A \Delta B$ or $A \oplus B$ and is also called the Boolean sum of $A$ and $B$. Thus,

$$A \Delta B = \{x \mid x \in A \text{ or } x \in B\} \quad \text{.......... (1)}$$

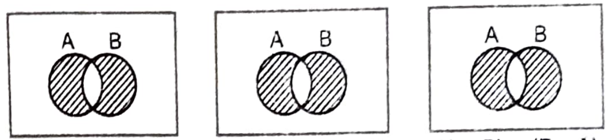
**Example 1 :** If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 4, 6\}$, then $A \Delta B = \{1, 3, 5, 6\}$.



**Example 2 :** If $U = \{a, b, c, d, e, f, g, h\}$ and $A = \{a, c, e, f\}$, $B = \{c, e, g, h\}$, then $A \Delta B = \{a, f, g, h\}$.

It is easy to see that

$$A \Delta B = (A \cup B) - (A \cap B) \quad \text{.......... (2)}$$



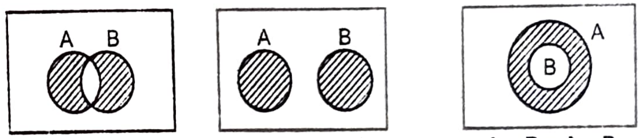
$A \Delta B$  $(A \cup B) - (A \cap B)$  $(A - B) \cup (B - A)$

Fig. 1.11

Also we can show that

$$A \Delta B = (A - B) \cup (B - A) \quad \text{.......... (3)}$$

Three particular cases of symmetric difference are shown below.



$A \Delta B = (A \cup B) - (A \cap B)$  $A \Delta B = A \cup B$ $(\because A \cap B = \Phi)$  $A \Delta B = A - B$ $(\because A \cup B = A, A \cap B = B)$

Fig. 1.12

**Example 3 :** Prove that $\overline{A} \Delta \overline{B} = A \Delta B$.

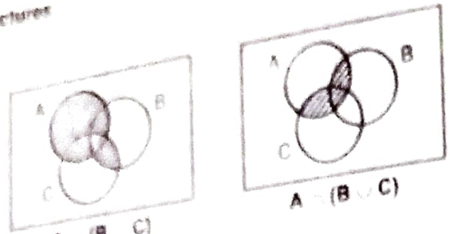
**Sol. :**

$$\begin{aligned}
\text{l.h.s.} &= (\overline{A} - \overline{B}) \cup (\overline{B} - \overline{A}) && [\text{ By (3) }] \\
&= (\overline{A} \cap \overline{\overline{B}}) \cup (\overline{B} \cap \overline{\overline{A}}) && [\text{ By (I), page 1-12 }] \\
&= (\overline{A} \cap B) \cup (\overline{B} \cap A) && [\because (\overline{\overline{A}}) = A, (\overline{\overline{B}}) = B] \\
&= (B - A) \cup (A - B) && [\text{ By (I) }] \\
&= (A - B) \cup (B - A) \\
&= A \Delta B. && [\text{ By (3) }]
\end{aligned}$$

## 16. Distribution

We have defined above the two fundamental operations on sets *viz.* union and intersection. We shall now see how they are related to each other. They are known as **distributive laws**. These laws are :

$$\boxed{\begin{aligned} &A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ \text{and} \quad &A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{aligned}}$$



$A \cup (B \cap C)$ $A \cap (B \cup C)$

Fig. 1.13

It may be seen that the above laws are analogous to the law of **multiplication over addition** viz. $a \times (b + c) = a \times b + a \times c$.

**Example** : If $A = \{1, 3, 4\}$, $B = \{2, 3, 5\}$, $C = \{1, 5, 6, 7\}$

verify $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Sol. $A \cup B = \{1, 2, 3, 4, 5\}$, $A \cup C = \{1, 3, 4, 5, 6, 7\}$, $B \cap C = \{5\}$

$A \cup (B \cap C) = \{1, 3, 4, 5\} = (A \cup B) \cap (A \cup C)$

Now, $B \cup C = \{1, 2, 3, 5, 6, 7\}$, $A \cap B = \{3\}$, $A \cap C = \{1\}$

$A \cap (B \cup C) = \{1, 3\} = (A \cap B) \cup (A \cap C)$.

## EXERCISE - I

1. If $U = \{a, b, c, d, e, f, g, h, i\}$ and $A = \{b, e, f, i\}$,

   $B = \{a, b, c, e, f, h, i\}$ then verify that

   $\overline{(\bar{A})} = A$, $A \subseteq B$ and $\bar{B} \subseteq \bar{A}$.

2. In the above example, verify that

   $A \cup B = B$, $A \subseteq A \cup B$,

   $B \subseteq A \cup B$, $A \cap B = A$, $A \cap B \subseteq A$, $A \cap B \subseteq B$.

3. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 4, 6\}$,

   $B = \{2, 3, 4, 5, 7\}$, $C = \{2, 4, 5, 9, 10\}$, verify that

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. If $U = \{x \mid x(x^2 - 1)(x^2 - 4)(x^2 - 4)(x^2 - 9) = 0\}$,

   $A = \{x \mid x(x^2 - 1)(x - 2) = 0\}$, $B = \{x \mid x(x - 1)(x^2 - 4) = 0\}$,

   $C = \{x \mid x(x^2 - 4)(x^2 - 9) = 0\}$ verify that

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,

   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.



## 17. De Morgan's Laws

Lastly we state how the operation of forming complements is related to the formation of unions and intersection. These laws are known as **De Morgan's laws**.

If $A$ and $B$ are any two sets, then

$$\boxed{\overline{(A \cup B)} = \bar{A} \cap \bar{B} \quad \text{and} \quad \overline{(A \cap B)} = \bar{A} \cup \bar{B}}$$

The first equation states that the **complement of the union is equal to the intersection of the complements** and the second equation states that **the complement of the intersection is equal to the union of the complements**.

### Augustus De Morgan (1806 - 1871)

Augustus De Morgan was born in Madurai, Tamil Nadu. His father was a colonel in the Indian army. His family returned to England when he was 7 months old. When in schools he mastered Latin, Greek and Hebrew and developed strong interest in mathematics.

He was a fellow of the Astronomical Society and a founder of London Mathematical Society. De Morgan greatly influenced the development of mathematics in the 19th century. He was a prolific writer and wrote over 1000 articles in more than 15 journals, in addition to a number of books known for clarity, logical presentation and minute details. He made original contributions to analysis and logic. He coined the term mathematical induction and gave first precise definition of limit in his book "The Differential And Integral Calculus".

**Example :** If $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, b, c, d\}$ and $B = \{c, d, e, f\}$, verify the De Morgan's Laws.

**Sol. :** We have $A \cup B = \{a, b, c, d, e, f\}$

$\therefore \quad \overline{(A \cup B)} = \{g, h\}$

But $\quad \bar{A} = \{e, f, g, h\}$ and $\bar{B} = \{a, b, g, h\}$

$\therefore \quad \bar{A} \cap \bar{B} = \{g, h\} \quad \therefore \quad \overline{(A \cup B)} = \bar{A} \cap \bar{B} = \{g, h\}$

Further, $A \cap B = \{c, d\}$

$\therefore \overline{(A \cap B)} = \{a, b, e, f, g, h\}$

But $\quad \bar{A} \cup \bar{B} = \{a, b, e, f, g, h\} \quad \therefore \quad \overline{A \cap B} = \bar{A} \cup \bar{B}$.

## EXERCISE - II

1. If $A = \{1, 3, 5, 7, 9, \ldots\}$, $B = \{2, 4, 6, 8, 10, \ldots\}$
   $U = \{1, 2, 3, 4, 5, \ldots\}$.
   What is $A \cap B$, $A \cup B$, $A - B$, $B - A$? How are $A$ and $B$?
   **[ Ans. :** $\Phi$, $U$, $A$, $B$, Disjoint ]
2. If $A = \{a, b, c, d, e, f, k\}$, $B = \{b, c, d, g, h\}$, $C = \{a,$ [illegible]$, f, g, i, j\}$
   verify that $A \cap (B - C) = (A \cap B) - (A \cap C)$.
3. In the example 3 if $U$ is the set of letters from $a$ to $m$, verify that
   $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$, $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$.
4. In the example 1, verify that
   $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$, $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$.
5. In Ex. 2, verify that $A - B = A - (A \cap B)$
   $= A \cap \bar{B}$

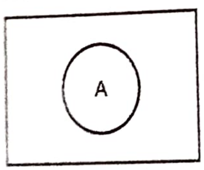
## 18. Algebra of Sets

We have seen above how new sets can be produced from given sets by doing certain operations on them. These laws of operations on sets are known as algebra of sets. They are listed below along with their diagrams.

### Laws of Sets

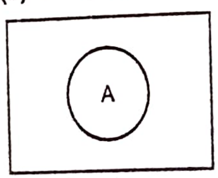
1. **Indempotent Laws**

(a) $A \cup A = A$




**A ∪ A = A**

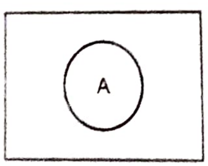
(b) $A \cap A = A$




**A ∩ A = A**

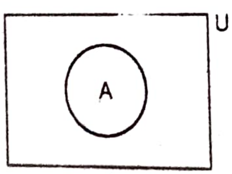
2. **Identity Laws**

(a) $A \cup \Phi = A$




**A ∪ Φ = A**

(b) $A \cap U = A$

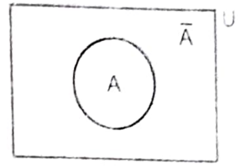



**A ∩ U = A**



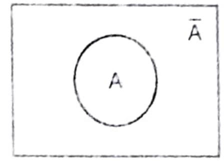
3. **Inverse Laws**

(a) $A \cup \bar{A} = U$



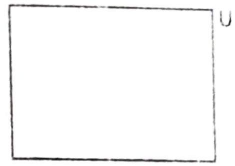

**A ∪ Ā = U**

(b) $A \cap \bar{A} = \Phi$




**A ∩ Ā = Φ**

(c) $\bar{U} = \Phi$



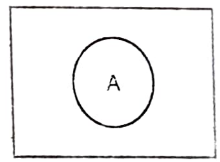

**Ū = Φ**

(d) $\bar{\Phi} = U$


**$\bar{\Phi}$ = U**

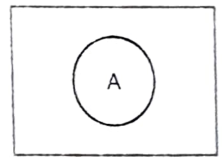
4. **Domination Laws**

(a) $A \cup U = U$




**A ∪ U = U**

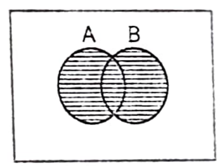
(b) $A \cap \Phi = \Phi$




**A ∩ Φ = Φ**

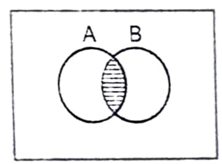
5. **Commutative Laws**

(a) $A \cup B = B \cup A$




**A ∪ B = B ∪ A**

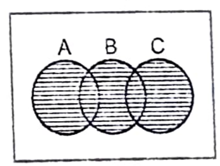
(b) $A \cap B = B \cap A$




**A ∩ B = B ∩ A**

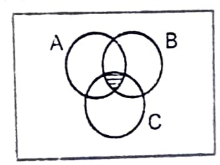
6. **Associative Laws**

(a) $A \cup (B \cup C) = (A \cup B) \cup C$




**A ∪ (B ∪ C) = (A ∪ B) ∪ C**

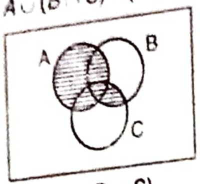
(b) $A \cap (B \cap C) = (A \cap B) \cap C$



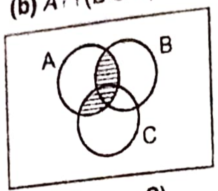

**A ∩ (B ∩ C) = (A ∩ B) ∩ C**

7. **Distributive Laws**

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



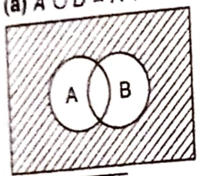

$A \cup (B \cap C)$



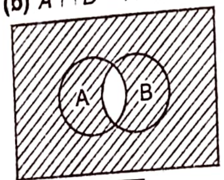

$A \cap (B \cup C)$

8. **De-Morgan's Laws**

(a) $\overline{A \cup B} = \bar{A} \cap \bar{B}$ (b) $\overline{A \cap B} = \bar{A} \cup \bar{B}$



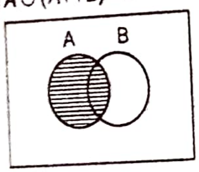

$\overline{A \cup B}$



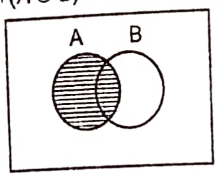

$\overline{A \cap B}$

9. **Absorption Laws**

(a) $A \cup (A \cap B) = A$ (b) $A \cap (A \cup B) = A$

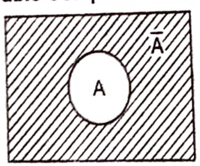



$A \cup (A \cap B) = A$




$A \cap (A \cup B) = A$

10. **Double Complementation (or Involution)**

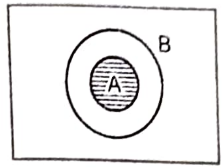



$\overline{\overline{(A)}} = A$

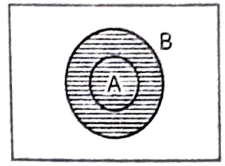
11. **The following are some more laws.**

1. If $A \subseteq B$ then $A \cap B = A$ [ Fig. (a) below ]
2. If $A \subseteq B$ then $A \cup B = B$ [ Fig. (b) below ]
3. If $A \subseteq B$ then $\bar{B} \subseteq \bar{A}$ [ Fig. (c) below ]
4. $A - B = A \cap \bar{B}$ [ Fig. (d) below ]
5. $A \Delta B = (A \cup B) - (A \cap B)$ [ Fig. (e) below ]

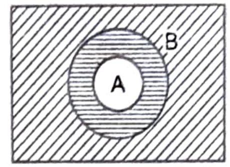




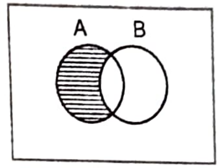

(a) $A \cap B = A$



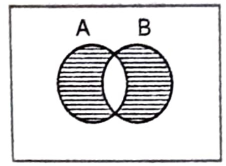

(b) $A \cup B = B$




(c) $\bar{B} \subset \bar{A}$




(d)




(e)

## 19. Duality

In the above laws (1), (2), (3), (4), (5), (6), (7), (8) and (9) you might have observed certain pattern in the laws stated in (a) and (b). This pattern is known as duality. If in (a) (or in b) if you change $\cup$ by $\cap$, $\cap$ by $\cup$, $U$ by $\Phi$ and $\Phi$ by $U$ you get b (or a).

**Definition :** If E is an equation in set operations then equation obtained by replacing $\cup$ by $\cap$, $\cap$ by $\cup$, $U$ by $\Phi$ and $\Phi$ by $U$ is called the dual of E and is denoted by E*. Further if E is an identity then E* is also an identity. This is known as the **principal of duality**.

For example, the dual of $A = (A \cap B) \cup (A \cap \bar{B})$ is $A = (A \cup B) \cap (A \cup \bar{B})$ and dual of $A = (A \cup B) \cap (A \cup \Phi)$ is $A = (A \cap B) \cup (A \cap U)$.

Observe that in the list of laws of sets given in (1) to (9) in each pair one is a dual of the other.

## 20. Proof of Laws

There are three methods of proving the above laws given on page 1-17, 1-18. The first method is based on definitions and the second is based on Venn-diagram and the third uses laws.

**Method 1 :** To prove a law in set operations you take an element in the set on the l.h.s. Now, prove that $x$ belongs to the set on the r.h.s. Similarly, take an element $x$ in the set on the r.h.s and prove that $x$ belongs to the set on the l.h.s. This means if $x \in$ {l.h.s.} then $x \in$ {r.h.s.} and if $x \in$ {r.h.s.} then $x \in$ {l.h.s.} *i.e.* the two sides are equal.

**Method 2 :** This method is based on the concept of Venn-diagram. By shading the sets as given in l.h.s., you obtain the set on the l.h.s In the same manner by shading given sets as given on r.h.s. you obtain the set on the r.h.s.

The two shaded areas are then found to be the same. ... proves the equality of the two sides.

**Method 3 :** In this method we use the basic definitions of the terms and the laws of sets such as distributivity, De'Morgan's laws, etc. This is illustrated below in Ex. 4 and 5.

**Example 1 :** Prove that if $A \subseteq B$ then $A \cap B = A$.

**Proof : Method 1 :** Let $x \in A \cap B$

Hence, $x \in A$ and $x \in B$ $\therefore$ $x \in A$.

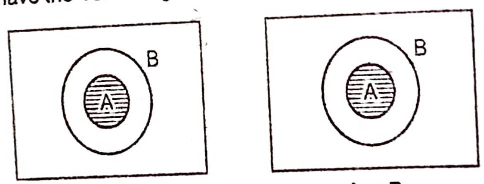
Now, let $x \in A$. Since $A \subseteq B$, $x \in B$

$\therefore$ $x \in A$ and $x \in B$ $\therefore$ $x \in A \cap B$

Thus every element of $A$ is an element of $A \cap B$ and every element of $A \cap B$ is an element of $A$.

$\therefore$ $A \cap B = A$.

**Method 2 :** We have on the l.h.s. Venn-diagram of $A$ (when $A \subseteq B$). On the r.h.s. we have the Venn-diagram of $A \cap B$ (when $A \subseteq B$).




A $\qquad$ A ∩ B

**Fig. 1.15**

The two shaded areas are the same.

**Remark ....**

Since the above equality is not an identity bu[t] conditional statement, it has no dual.

**Example 2 :** Prove the De Morgan's Laws $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

**Proof : Method 1 :**

(i) Let $x \in \overline{A \cup B}$ $\therefore$ $x \notin A \cup B$

$\therefore$ $x \notin A$ or $x \notin B$

$\therefore$ $x \in \overline{A}$ and $x \in \overline{B}$ $\therefore$ $x \in \overline{A} \cap \overline{B}$.

(ii) Let $x \in \overline{A} \cap \overline{B}$ $\therefore$ $x \in \overline{A}$ and $x \in \overline{B}$.

$\therefore$ $x \notin A$ or $x \notin B$

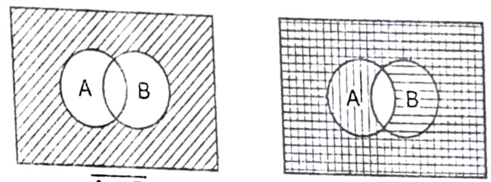
$\therefore$ $x \notin A \cup B$ $\therefore$ $x \in \overline{A \cup B}$



Thus, we have proved that if $x \in \overline{A \cup B}$ then $x \in \overline{A} \cap \overline{B}$ and if $x \in \overline{A} \cap \overline{B}$ then $x \in \overline{A \cup B}$. This, means $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

$\therefore$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

**Method 2 :** We first draw the Venn-diagram of $\overline{A \cup B}$. The shaded area represents $\overline{A \cup B}$.




$\overline{A \cup B}$ $\qquad$ $\overline{A} \cap \overline{B}$

**Fig. 1.16**

Now, we show $\overline{A}$ by **horizontal lines** and $\overline{B}$ by **vertical lines**.

Clearly the doubly shaded area is the intersection $\overline{A}$ and $\overline{B}$ *i.e.* $\overline{A} \cap \overline{B}$.

Singly shaded area in the first figure is the same as the doubly shaded area in the second figure.

Hence, the result.

**Example 3 :** Prove that if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

**(M.U. 2007, 08)**

**Sol. : (i) Method 1**

Let $x \in A$ or $x \in B$

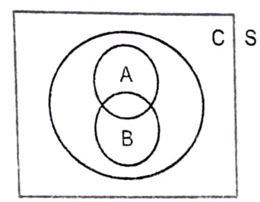
$\therefore$ $x \in A \cup B$.

Since, $A \subseteq C$, $x \in C$

and since $B \subseteq C$, $x \in C$

$\therefore$ $A \cup B \subseteq C$

(ii) Method 2 : We show in the Fig. 1.17 by Venn-diagram that $A \cup B \subseteq C$.




A ⊆ C, B ⊆ C, then A ∪ B ⊆ C

**Fig. 1.17**

**Example 4 :** Prove that $\overline{A} \Delta \overline{B} = A \Delta B$.

**Sol. :**

| | |
|---|---|
| l.h.s. $= (\overline{A} - \overline{B}) \cup (\overline{B} - \overline{A})$ | [ By (3), page 1-13 ] |
| $= (\overline{A} \cap \overline{\overline{B}}) \cup (\overline{B} \cap \overline{\overline{A}})$ | [ By (1), page 1-12 ] |
| $= (\overline{A} \cap B) \cup (\overline{B} \cap A)$ | [ $\because$ $\overline{\overline{A}} = A$ and $\overline{\overline{B}} = B$ ] |
| $= (B - A) \cup (A - B)$ | [ By (1), page 1-12 ] |
| $= (A - B) \cup (B - A)$ | [ Commutativity ] |
| $= A \Delta B$ | [ By (3), page 1-13 ] |

**Example 5** : Prove that $A \cap \overline{B} = A \cup \overline{B}$

**Proof** : The result follows from the principal of duality (or you can prove.)

**Example 6** : Using set laws prove that $(A - B) - C = A - (B - C)$. **(M.U. 1994)**

**Proof** : We have

$$
\begin{aligned}
(A - B) - C &= (A - B) \cap \overline{C} && [\because A - B = A \cap \overline{B}] \\
&= (A \cap \overline{B}) \cap \overline{C} && [\because A - B = A \cap \overline{B}] \\
&= A \cap (\overline{B} \cap \overline{C}) && [\text{Associativity}] \\
&= A \cap (\overline{B \cup C}) && [\text{De Morgan}] \\
&= A - (B \cup C) && [\because A - B = A \cap \overline{B}]
\end{aligned}
$$

Diagrammatically, we have

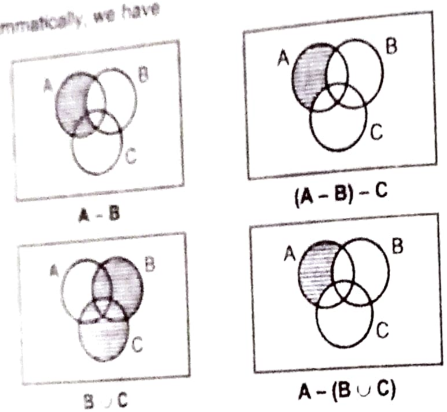



Fig. 1.18

**Example 7** : Using set laws, prove that

$$(A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) = \overline{A} \cup \overline{B}$$

**Proof** : We have

$$
\begin{aligned}
&(A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) \\
&= (A \cap \overline{B}) \cup [(\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})] && [\text{Associativity}] \\
&= (A \cap \overline{B}) \cup [\overline{A} \cap (B \cup \overline{B})] && [\text{Distributivity}] \\
&= (A \cap \overline{B}) \cup (\overline{A} \cap U) && [\because B \cup \overline{B} = U] \\
&= (A \cap \overline{B}) \cup \overline{A} && [\because A \cap U = A] \\
&= \overline{A} \cup (A \cap \overline{B}) && [\text{Commutativity}] \\
&= (\overline{A} \cup A) \cap (\overline{A} \cup \overline{B}) && [\text{Distributivity}] \\
&= U \cap (\overline{A} \cup \overline{B}) && [\overline{A} \cup A = U] \\
&= \overline{A} \cup \overline{B} && [\because A \cap U = A]
\end{aligned}
$$





Diagrammatically, we have

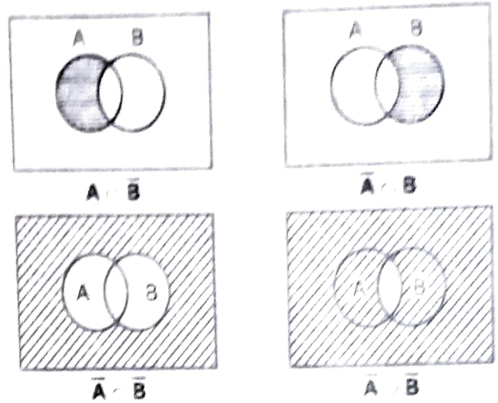



Fig. 1.19

**Example 8 : Prove the following**

**(i)** $\overline{A} - \overline{B} = \overline{A} \cap B$ **(ii)** $A \cup (A \cap B) = A$ **(iii)** $(A - B) \cup (A \cap B) = A$

**Sol. : (i)** $\overline{A} - \overline{B} = \overline{A} \cap B$

$$
\begin{aligned}
\text{l.h.s.} &= \overline{A} \cap \overline{\overline{B}} && [\text{Definition (1) of } A - B, \text{ page 1-12}] \\
&= \overline{A} \cap B = \text{r.h.s.} && [\because \overline{\overline{B}} = B]
\end{aligned}
$$

**(ii)** $A \cup (A \cap B) = A$

$$
\begin{aligned}
\text{l.h.s.} &= (A \cup A) \cap (A \cup B) && [\text{Distributivity}] \\
&= A \cap U && [\because A \cup A = A \text{ and } A \cup B = U] \\
&= A && [\because A \cup A = A]
\end{aligned}
$$

**(iii)** $(A - B) \cup (A \cap B) = A$

$$
\begin{aligned}
\text{l.h.s} &= (A \cap \overline{B}) \cup (A \cap B) && [\text{Definition (1) of } A - B] \\
&= (A \cap \overline{B} \cup A) \cap (A \cap \overline{B} \cup B) && [\text{Distributivity}] \\
&= [(A - B) \cup A] \cap [A \cap U] && [\because B \cup \overline{B} = U] \\
&= A \cap A && [\because (A - B) \cup A = A \text{ and } A \cap U = A] \\
&= A && [\text{Idempotent}]
\end{aligned}
$$

## EXERCISE - III

Prove or show by Venn diagrams the following relations

**(i)** $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$ **(M.U. 2005)**

**(ii)** $A \cap (B - C) = (A \cap B) - (A \cap C)$

**(iii)** $\overline{A} - \overline{B} = B - A$

(iv) $(A-B)-C=(A-C)-(B-C)$

(v) If $A \subseteq B$ then $A \cup (B-A)=B$

(vi) If $A \subseteq B$, then $(B \cup A) \cup (B-A)=B$

(vii) $A-(B \cup C)=(A-B) \cap (A-C)$

(viii) $A-(A-B) \subseteq B$

(ix) $A \cap (B-C)=A \cap B-A \cap C$

(x) $A \cup (\overline{B}-C)=(A-B) \cap (A-C)$

(xi) $(A \cup B)-(A \cap C)=B-(A \cup C)$

(xii) $A \cap (B \Delta C)=(A \cap B) \Delta (A \cap C)$

## 21. Fundamental Products

**Definition :** If $A_1, A_2, \ldots, A_n$ are given sets then their product of the form $A_1^* \cap A_2^* \cap A_3^* \cap \ldots \cap A_n^*$ where $A^*$ denotes either **the set $A$ or its** complement $\overline{A}$ is called a **fundamental product** of the sets $A_1, A_2, \ldots, A_n$.

**Example 1 :** If $A$ and $B$ are two sets, write down all the fundamental products.

**Sol. :** If $A$ and $B$ are given sets then we have $2^2 = 4$ fundamental products. They are $A \cap B$, $A \cap \overline{B}$, $\overline{A} \cap B$, $\overline{A} \cap \overline{B}$.

We show below the Venn-diagram of these fundamental products

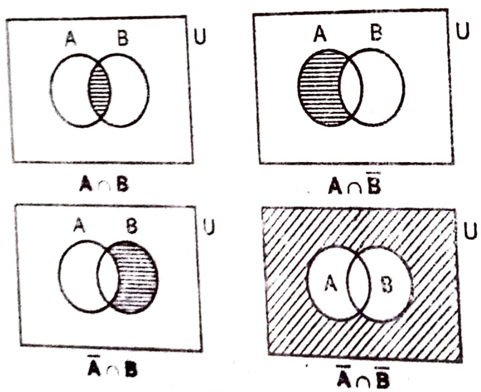



**Fig. 1.20**

1. Note that the union of all fundamental **products** is the universal set $U$.
2. Note also that any two of these fundamental **products are** disjoint.



**Example 2 :** If $A$, $B$, $C$ are three sets, write down all the fundamental products.

**Sol. :** If $x$ is an element of $U$ then $x$ belongs to $A$ or $x$ does not belong to $A$. So is the case for $B$ and $C$. Thus, for each set there are 2 possibilities and since there are 3 sets, there will be $2^3$ fundamental products. They are shown in the Fig. 1.21.

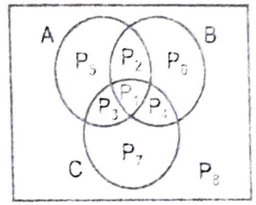



**Fig. 1.21**

The product sets are listed below.

$P_1 = A \cap B \cap C$, $P_2 = A \cap B \cap \overline{C}$

$P_3 = A \cap \overline{B} \cap C$, $P_4 = \overline{A} \cap B \cap C$

$P_5 = A \cap \overline{B} \cap \overline{C}$, $P_6 = \overline{A} \cap B \cap \overline{C}$

$P_7 = \overline{A} \cap \overline{B} \cap C$, $P_8 = \overline{A} \cap \overline{B} \cap \overline{C}$

Note that the union of all these fundamental product sets is the universal set. In general if there are $n$ sets $A_1, A_2, \ldots, A_n$ then (i) there will be $2^n$ fundamental products, (ii) the union of these fundamental products is the universal set and (iii) any two of these fundamental products are disjoint.

**Example 3 :** Write $A \cap (B \cup C)$ as the union of (disjoint) fundamental products.

**Sol. :** In the Venn diagram (Fig. 1.22), we show the set $A \cap (B \cup C)$ by shaded area. This area consists of three disjoint sets $P_1$, $P_2$ and $P_3$. Hence,

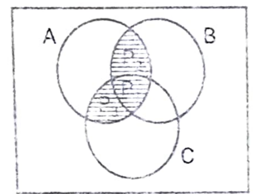



**Fig. 1.22**

$$A \cap (B \cup C) = P_1 \cup P_2 \cup P_3$$
$$= (A \cap B \cap C) \cup (A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C)$$

### EXERCISE - IV

1. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 5, 7\}$, $B = \{2, 3, 4, 6\}$. Write down all fundamental products. Show that the union of these fundamental products is $U$.

[ **Ans. :** There are $2^2 = 4$ fundamental products. They are $A \cap B = \{2, 3\}$

$A \cap \overline{B} = \{1, 5, 7\}$; $\overline{A} \cap B = \{4, 6\}$, $\overline{A} \cap \overline{B} = \{8, 9, 10\}$. Also

$(A \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = U$ ]

2. If $U = \{a, b, c, d, e, f, g, h, i\}$, $A = \{a, b, e, g\}$, $B = \{b, c, d, e, h\}$. (i) Write down all fundamental products. (ii) Show that the union of these fundamental products is $U$.

[ **Ans. :** (i) $A \cap B = \{b, e\}$, $A \cap \overline{B} = \{a, g\}$

$\overline{A} \cap B = \{c, d, h\}$ $\overline{A} \cap \overline{B} = \{f, i\}$ (ii) is left to you. ]

## 22. Class of Sets

We know that a set is a collection of objects called elements. But in some situations we may need to talk about the subsets of a given set as elements. Such a collection of sets is called a **class of sets** or a **family of sets**. Suppose we are given $A = \{a, b, c\}$. From this we can have a set whose elements are subsets of $A$. *e.g.*, $S = \{\{a\}, \{b\}, \{a, b\}\}$. Such a set $S$ is called a class of sets.

**Example :** Let $S = \{a, b, c, d\}$. Find the class of subsets of $S$ which contain (1) exactly two elements, (2) exactly three elements, (3) $c$ and two other elements.

**Sol. :** We have

$$A = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$$

$$B = \{\{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{c, d, a\}\}$$

$$C = \{\{c, a, b\}, \{c, b, d\}, \{c, a, d\}\}$$

(M.U. 2010)

## 23. Power Set

**Definition :** If $S$ is a given set then the **set of all subsets of** $S$ is called power set of $S$ and is denoted by $P(S)$. Clearly $\Phi$ and $S$ are the elements of $P(S)$.

**Example 1 :** If $S = \{\Phi\}$, then $P(S) = \{\Phi, \{\Phi\}\}$ **(M.U. 2000)**

**Example 2 :** If $S = \{a, b\}$, then $P(S) = \{\Phi, \{a\}, \{b\}, \{a, b\}\}$

**Example 3 :** If $S = \{a, b, c\}$ then

$$P(S) = \{\Phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$$

If $S$ has $N$ elements then $P(S) = 2^N$ elements and hence some times the power set is also denoted by $2^N$. See that in Ex. 1 above $S$ has only 1 element and $P(S)$ has $2^1 = 2$ elements. In Ex. 2, $S$ has 2 elements and $P(S)$ has $2^2 = 4$ elements. In Ex. 3, $S$ has 3 elements and hence $P(S)$ has $2^3 = 8$ elements.

### EXERCISE - V

1. If $A = \{1, 2, 3\}$ determine the power set of $A$

 **[ Ans. :** $P(A) = \{\Phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ **]**

2. If $A = \{a, b, c, d\}$ determine the power set of $A$

 **[ Ans. :** $P(A) = \{\Phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\},$
 $\{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\},$
 $\{b, c, d\}, \{c, d, a\}, \{a, b, c, d\}\}$ **]**



## 24. Partition of Sets

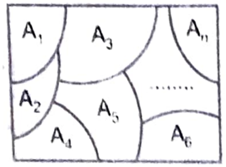



Fig. 1.23

If $S$ is a non-empty set by partition of $S$, we mean the division of $S$ into **disjoint** subsets such that their **union** is $S$. As shown in the Fig. 1.23, if $S$ is the given set then $A_1, A_2, A_3, \ldots\ldots, A_n$ form the partition of $S$.

**Definition :** A collection $\{A_i\}$ of non-empty subsets of $S$ is called a **partition** of $S$ if (i) each element of $S$ belongs to one subset $A_i$ *i.e.* $\cup A_i = S$ and (ii) the subsets $A_i$ are mutually disjoint *i.e.* $A_i \cap A_j = \Phi$.

The subsets in a partition are called **cells**.

**Example 1 :** Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and consider the following collections of subsets of $S$.

(i) $A_1 = \{\{1, 2, 3\}, \{3, 4, 5\}, \{6, 7, 8\}\}$

(ii) $A_2 = \{\{1, 2\}, \{3, 4, 5\}, \{6, 7\}\}$

(iii) $A_3 = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}, \{8\}\}$

Which of them is a partition of $S$ and why ? Also draw diagrams.

**Sol. :** The collection $A_1$, $A_2$, $A_3$ are shown in Fig. 1.24.

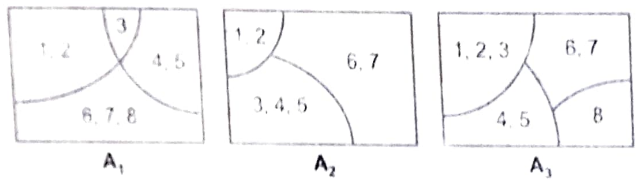



Fig. 1.24

$A_1$ is **not** a partition because subsets $\{1, 2, 3\}$ and $\{3, 4, 5\}$ are **not** disjoint. Their intersection is $\{3\}$.

$A_2$ is **not** a partition because their union is **not** $S$. The element 8 does not belong to any subset.

$A_3$ is a partition because each element of $S$ belongs to one of the subsets *i.e.* the union of the subsets is $S$ and no element belongs to two (or more) subsets *i.e.* all subsets are mutually disjoint.

**Example 2 :** Let $Z_0, Z_1, Z_2, Z_3$ denote the set of integers which leave remainder zero, one, two, three respectively when divided by four *i.e.*

$$Z_0 = \{\ldots\ldots, -8, -4, 0, 4, 8, \ldots\ldots\}$$

$$Z_1 = \{\ldots\ldots, -7, -3, 1, 5, 9, \ldots\ldots\}$$

$$Z_2 = \{\ldots\ldots, -6, -2, 2, 6, 10, \ldots\ldots\}$$

$$Z_3 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$$

Do $Z_i$ partition $Z$ ?

**Sol.** : It is easy to see that the sets $Z_0, Z_1, Z_2, Z_3$ are disjoint and their union is $Z$. Hence, they partition $Z$ as shown in the Fig. 1.25.

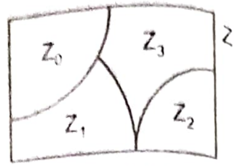
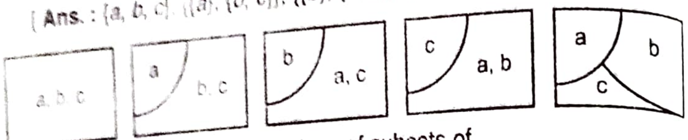



**Fig. 1.25**

## EXERCISE - VI

1. Determine all partitions of $S = \{a, b, c\}$. Draw Venn diagrams.

[ **Ans.** : $\{a, b, c\}$, $\{\{a\}, \{b, c\}\}$, $\{\{b\}, \{c, a\}\}$, $\{\{c\}, \{a, b\}\}$, $\{\{a\}, \{b\}, \{c\}\}$ ]




2. Consider the following collections of subsets of

$$S = \{a, b, c, d, e, f, g, h\}$$

(i) [ $\{a, b, c\}$, $\{d, e, f\}$, $\{f, g, h\}$ ]

(ii) [ $\{a, b, c\}$, $\{d, e, f\}$, $\{g, h\}$ ]

(iii) [ $\{a, b\}$, $\{d, e, f\}$, $\{g, h\}$ ]

Which of these collections is a partition and which are not ? Give reasons. Draw figures.

[ **Ans.** : (i) not a partition as $f$ belongs to two subsets $\{d, e, f\}$ and $\{f, g, h\}$

(ii) is a a partition,

(iii) is not a partition as $c$ does not belong to any subset. ]

3. Let $S$ = {red, green, blue, white}.

Determine which of the following is a partition.

(1) $A$ = {{red}, {green}, {blue}, {red, white}}

(2) $B$ = {{green}, {blue}, {white}}

(3) $C$ = {{red, green}, {blue, white}}

(4) $D$ = {{red, green, blue, white}}

(5) $E$ = {Φ, {red, blue}, {green, white}}

[ **Ans.** : **(1)** No. red belongs to two cells. **(2)** No. red does not belong to any cell. **(3)** Yes. **(4)** Yes. The cell contains all elements of $S$. **(5)** No. Φ cannot be an element of any cell. ]

4. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Determine which of the following is a partition of $S$

**(a)** $\{\{1, 2, 3\}, \{5, 6, 7\}, \{8, 9\}\}$



**(b)** $\{\{1, 3, 6\}, \{2, 4, 5, 9\}, \{7, 8\}\}$

**(c)** $\{\{3, 4, 6\}, \{1, 2, 5\}, \{7, 8\}\}$

**(d)** $\{\{1, 2\}, \{3, 4, 5\}, \{2, 6\}, \{7, 8, 9\}\}$

[ **Ans.** : **(a)** No. 4 is missing. **(b)** Yes. **(c)** No. 9 is missing. **(d)** No. 2 belongs to two cells. ]

5. Let $S = \{1, 2, 3, 4, \ldots, \infty\}$. Determine which of the following is a partition of $S$.

**(a)** $\{\{x \mid x \in S \text{ and } x > 4\} \{1, 2, 3\}\}$

**(b)** $\{\{1, 2, 3\}, \{4, 5, 6\}, \{x \mid x \in S, x > 6\}\}$

**(c)** $\{\{x \mid x \in S, x \text{ is odd}\} \{x \mid x \in S, x \text{ is even}\}\}$

**(d)** $\{\{1, 3, 5, 7, \ldots\} \{2, 4, 6, 8, \ldots\}\}$

**(e)** $\{\{1, 3, 5, 7, \ldots\} \{3, 6, 9, 12, \ldots\}\}$

[ **Ans.** : **(a)** No. 4 does not belong to any cell. **(b)** Yes. **(c)** Yes. **(d)** Yes. **(e)** No. ]

## 25. Ordered Set

A set as we know is a collection of objects with no reference to the order in which the elements of the set are written. For this reason the sets $\{a, b, c\}$, $\{b, c, a\}$, $\{c, a, b\}$ are equal. If we assign a position to each element of a set then the set is called an ordered set.

**Definition** : A set of elements such that each element is assigned a position is called an **ordered set**. If $a_1, a_2, \ldots, a_n$ are the elements of an ordered set in this order, it is denoted by $(a_1, a_2, a_3, \ldots, a_n)$. An ordered set with $n$ elements is called an ***n*-tupple.**

Two $n$-tupples are **equal** if and only if their corresponding elements are equal. The ordered set $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal if and only if $a_i = b_i$ for every $i$.

For example, in the number system a number is an ordered set of digits. The number '2435' is an ordered set of the digits 2, 3, 4 and 5. Similarly, a word is an ordered set of letters. The word 'theory' is an ordered set of the letters t, h, e, o, r, y.

## 26. Cartesian Product

The **Cartesian product** of two sets $A$ and $B$ denoted by $A \times B$ is the set of all ordered pairs $(a, b)$ where $a \in A$ and $b \in B$. Thus,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

**Example 1 :** Write the cartesian product $A \times B$ where $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$.

**Sol. :** By definition the cartesian product $A \times B$ is the set $(a, b)$ where the first element $a \in A$ and the second element $b \in B$. Each element of $A$ is to be paired with each element of $B$.

$\therefore \; A \times B = \{(a, 1), (a, 2), (a, 3),$
$(b, 1), (b, 2), (b, 3),$
$(c, 1), (c, 2), (c, 3)\}$

A cartesian product can be represented pictorially by taking the set $A$ on the horizontal axis and the set $B$ on the vertical as usual. [Fig. 1.26(a)]

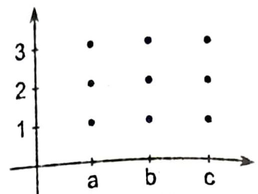


**Fig. 1.26 (a)**

Similarly, we get

$B \times A = \{(1, a), (1, b), (1, c), (2, a), (2, b),$
$(2, c), (3, a), (3, b), (3, c)\}$

Pictorial representation of this product is as shown in Fig. 1.26 (b).

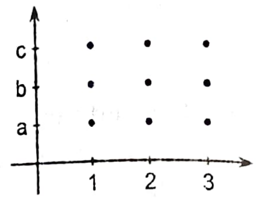


**Fig. 1.26 (b)**

Note that since $(a, 1)$ and $(1, a)$ are ordered pairs, they are not equal. Hence, in this case the sets $A \times B$ and $B \times A$ are not equal.

**Example 2 :** If $A$ and $B$ are the sets as given in Ex. 1, write $A \times A$, $B \times B$.

**Sol. :** Clearly, we have

$A \times A = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

$B \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

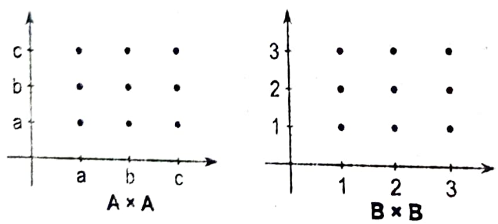


**Fig. 1.27**

**Example 3 :** Find the cartesian product $N \times N$ and $R \times R$.

**Sol. :** $N \times N$ is the ordered pair $(a, b)$ where $a \in N$ and $b \in N$ i.e.

$$N \times N = \{(a, b) \mid a \in N, b \in N\}$$



Diagrammatically it can be represented as shown in Fig. 1.28 (a).

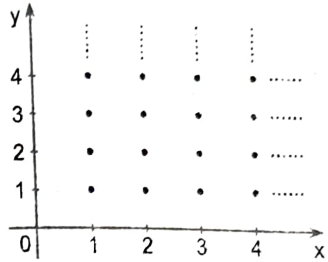


**Fig. 1.28 (a)**

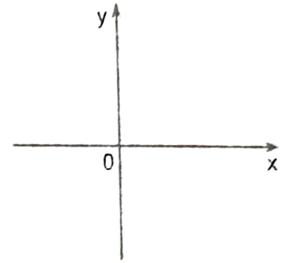


**Fig. 1.28 (b)**

$R \times R$ is the set of all ordered pairs of the type $(x, y)$ where $x \in R$ and $y \in R$. It is represented by our usual $x$-$y$ plane or Cartesian plane. [Fig. 1.28(b)]

The definition of the cartesian product of two sets can be extended to $n$ sets. The cartesian product of $n$ sets $A_1, A_2, \ldots, A_n$ consists of all possible $n$-tupples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$ for every $i$. We denote this cartesian product by $A_1 \times A_2 \times \ldots \times A_n$.

If all $A$'s are requal to $A$, the cartesian product is denoted by $A^n$.

**Example 4 :** If $A = \{a\}$, $B = \{b, c\}$, $C = \{d, e, f\}$, find $A \times B \times C$.

**Sol. :** By definition,

$$A \times B \times C = \{(a_1, a_2, a_3) \mid a_1 \in A, a_2 \in B, a_3 \in C\}$$
$$= \{(a, b, d), (a, b, e), (a, b, f), (a, c, d), (a, c, e), (a, c, f)\}$$

**Example 5 :** If $A = \{1\}$, $B = \{2, 3\}$, $C = \{a, b, c\}$, find $A \times B \times C$.

**Sol. :** As above

$$A \times B \times C = \{(1, 2, a), (1, 2, b), (1, 2, c), (1, 3, a), (1, 3, b), (1, 3, c)\}$$

**Example 6 :** Prove the following relations

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ **(M.U. 2003, 06, 08)**

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ **(M.U. 2009)**

**Sol. :** We have

**(i)** $A \times (B \cup C) = \{(x, y) \mid x \in A \text{ and } y \in B \cup C\}$

$= \{(x, y) \mid x \in A \text{ and } y \in B \text{ or } y \in C\}$

$= \{(x, y) \mid (x \in A \text{ and } y \in B) \textbf{ or } (x \in A \text{ and } y \in C)\}$

$= \{(x, y) \mid x \in A \text{ and } y \in B\} \cup \{(x, y) \mid x \in A \text{ and } y \in C\}$

$= (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = \{(x, y) \mid x \in A \text{ and } y \in (B \cap C)\}$

$= \{(x, y) \mid x \in A \text{ and } (y \in B \text{ and } y \in C)\}$

$= \{(x, y) \mid (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\}$

$= \{(x, y) \mid x \in A \text{ and } y \in B\} \cap \{(x, y) \mid x \in A \text{ and } y \in C\}$

$= (A \times B) \cap (A \times C)$

## EXERCISE - VII

1. If $A = \{a, b\}$, $B = \{x, y\}$, $C = \{x, z\}$, find the following.

(i) $A \times \Phi$, (ii) $A \times B$, (iii) $A \times B \times \Phi$,

(iv) $A \times B \times C$, (v) $A \times (B \cup C)$

[**Ans.** : (i) $\Phi$, (ii) $\{(a, x), (a, y), (b, x), (b, y)\}$, (iii) $\Phi$,

(iv) $\{(a, x, x), (a, x, z), (a, y, x), (a, y, z), (b, x, x), (b, x, z), (b, y, x), (b, y, z)\}$

(v) $\{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$

2. If $A = \{a, b, c\}$, $B = \{b, c, d\}$, $C = \{a, b, e\}$, $D = \{b, e, f\}$, verify that

(i) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

(ii) $(A - B) \times C = (A \times C) - (B \times C)$

(iii) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

(iv) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

3. If $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{a, b\}$, verify that $A \times B \neq B \times A$ and $(A \times B) \times C = A \times (B \times C)$.

### Cardinality of a Set

The **cardinality** of a set $A$, denoted by $|A|$ is the number of elements in the set $A$. Clearly, since $\Phi$ does not have any element, the cardinality of $\Phi$ is 1 *i.e.*, $|\Phi| = 0$. The cardinality of a singleton set *i.e.*, a set containing only one element is 1 *e.g.*, $|\{a\}| = 1$, $|\{\Phi\}| = 1$, similarly, $|\{a, b\}| = 2$, $|\{a, b, c\}| = 3$. The cardinality of a finite set containing $n$ elements is $n$.

### Countably Infinite and Uncountably Infinite Sets

The sets of positive integers $N = \{1, 2, 3, ....\}$ is infinite, the set of integers $Z = \{...., -3, -2, -1, 0, 1, 2, 3, .....\}$ is also infinite and the set of real numbers $R$ is also infinite. The cardinality of the set of positive integers is denoted by $\aleph_0$ (read 'aleph-naught'), aleph ($\aleph$) being the first letter of Hebrew alphabet. This symbol for cardinality of $N$ was by suggested by Cantor. Any set which can be put into one-one correspondence with $N$ has also cardinality $\aleph_0$. Such sets are called **countably infinite** or countable sets.



The set that is not countable is called **uncountably infinite** or **uncountable**. The set of real numbers is uncountable.

A countable infinite set is also called **denumerable** and an uncountably infinite set is called **non-denumerable**.

## 27. The Inclusion-Exclusion Principle (M.U. 2008)

The inclusion-exclusion principle is concerned with the number of elements in the set operations. Since in a set an element is to be counted once the principle states that while counting the elements in a situation if some elements are not counted already they are to be **included** and if some elements are already counted they are to be **excluded**. For example, if we want the number of elements in $A \cup B$ denoted by $n(A \cup B)$ we add $n(A)$ and $n(B)$ and subtract $n(A \cap B)$ *i.e.* in our counting we 'include' $n(A)$ and $n(B)$ and 'exclude' $n(A \cap B)$ to avoid double counting because they are counted in $n(A)$ as well as in $n(B)$. This is inclusion-exclusion principle.

### Inclusion-Exclusion Principle (M.U. 2008)

**Theorem 1 :** If $A$ and $B$ are two finite sets and $n(A)$, $n(B)$ denote the number of elements in $A$ and $B$ then,

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)} \quad .......... (1)$$

**Proof :** Suppose $n(A \cap B) = k$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ we can assume that for some non-negative integers $n_1$ and $n_2$, $n(A) = n_1 + k$ and $n(B) = n_2 + k$.

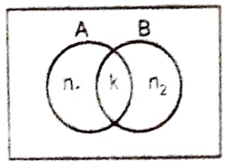



Fig. 1.29

Then, $n(A \cup B) = n_1 + k + n_2$

$= (n_1 + k) + (n_2 + k) - k$

$= n(A) + n(B) - n(A \cap B)$

**Corollary :** If $A$ and $B$ are finite disjoint sets then since $n(A \cap B) = 0$, we have

$$\boxed{n(A \cup B) = n(A) + n(B)} \quad .......... (2)$$

This is also known as **addition principle**.

**Theorem 2 :** If $A$, $B$, $C$ are three finite sets then

$$\boxed{\begin{aligned} n(A \cup B \cup C) = {} & n(A) + n(B) + n(C) - n(A \cap B) \\ & - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C) \end{aligned}} \quad .......... (3)$$

(M.U. 2008)

**Proof :** We have

$$\begin{aligned}
n(A \cup B \cup C) &= n[A \cup (B \cup C)] && [\text{ Theorem 1 }]\\
&= n(A) + n(B \cup C) - n(A \cap (B \cup C))\\
&= n(A) + n(B \cup C) - n[(A \cap B) \cup (A \cap C)] && [\text{ Distributivity }]\\
&= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B)\\
&\quad - n(A \cap C) + n[(A \cap B) \cap (A \cap C)] && [\text{ Theorem 1 }]\\
&= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)\\
&\quad - n(C \cap A) + n(A \cap B \cap C)
\end{aligned}$$

From the Venn-diagram given below or from the algebra of sets, the results (1 to 8) can be easily established.

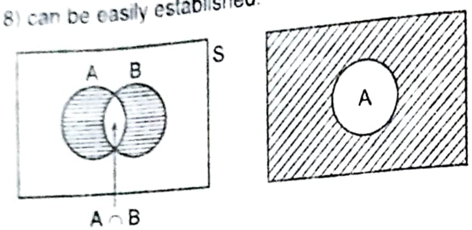



**Fig. 1.30**

1. $n(A - B) = n(A) - n(A \cap B)$
2. $n(B - A) = n(B) - n(A \cap B)$
3. $n(A \Delta B) = n(A - B) + n(B - A)$
   $= n(A) + n(B) - 2n(A \cap B)$
4. $n(\overline{A}) = n(S) - n(A)$
5. $n(\overline{A \cup B}) = n(S) - n(A \cup B)$
6. $n(\overline{A \cap B}) = n(S) - n(A \cap B)$
7. $n(\overline{A \cup B}) = n(\overline{A} \cap \overline{B})$
8. $n(\overline{A \cap B}) = n(\overline{A} \cup \overline{B})$

**Example 1 :** Find the number of integers $\leq 200$ which are divisible 2 or 5.

**Sol. :** Let $A = \{x \in N \mid x \leq 200 \text{ and } x \text{ divisible by } 2\}$

$\therefore \quad A = \{2, 4, 6, 8, \ldots, 200\}$

Let $B = \{x \in N \mid x \leq 200 \text{ and } x \text{ divisible by } 5\}$

$\therefore \quad B = \{5, 10, 15, \ldots, 200\}$

Then $A \cap B$ consists of positive integers $\leq 200$ which are divisible by 2 or 5 *i.e.* divisible by 10.

$\therefore \quad A \cap B = \{10, 20, \ldots, 200\}$

n(S) = 200

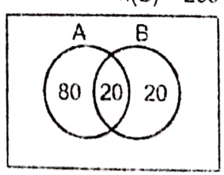



Fig. 1.31



$$n(A) = \frac{200}{2} = 100, \quad n(B) = \frac{200}{5} = 40, \quad n(A \cap B) = \frac{200}{10} = 20$$

$$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 100 + 40 - 20 = 120$$

**Example 2 :** In a group of 300 persons 160 drink tea and 170 drink coffee, 80 of them drink both. How many persons do not drink either ?

**Sol. :** $A = \{x \mid x \text{ drink tea}\}$,

$B = \{x \mid x \text{ drink coffee}\}$

Then, we have

$n(A) = 160, \quad n(B) = 170,$

$n(A \cap B) = 80 \quad \text{and} \quad n(S) = 300$

n(S) = 300

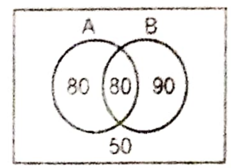



**Fig. 1.32**

We want $n(\overline{A} \cap \overline{B})$.

$$\begin{aligned}
\text{Now, } n(\overline{A} \cap \overline{B}) &= n(\overline{A \cup B}) && [\text{ De Morgans }]\\
&= n(S) - n(A \cup B) && [\text{ By (5), page 1-34 }]\\
&= n(S) - [n(A) + n(B) - n(A \cap B)]\\
&= 300 - [160 + 170 - 80]\\
&= 50
\end{aligned}$$

**Example 3 :** In a town there are 2000 literate persons. Of them 60% read newspaper *A*, 55% read newspaper *B* and 20% read neither *A* nor *B*. How many persons read (i) both the newspapers *A* and *B* ? (ii) only one newspaper ?

**Sol. :** In such problems it is convenient to draw Venn-diagrams. Let us talk in terms of percentages. Let there be 100 literate persons in the town.

We are given that 20 of them do not read any newspaper. This means

100 − 20 = 80 read *A* or *B* or both.

Since 60 read *A*,

$\therefore$ 80 − 60 = 20 read *B* only.

Since 55 read *B*,

$\therefore$ 80 − 55 = 25 read *A* only

$\therefore$ 80 − (20 + 25) = 35 read both.

This is shown in the Fig. 1.33.

S = 100

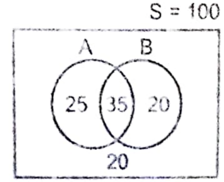



**Fig. 1.33**

$$\therefore \text{ Number of persons reading both newspapers} = 2000 \times \frac{35}{100} = 700.$$

**Aliter : (i) The problem can also be solved analytically as follows.**

We are given that, $n(S) = 100$, $n(A) = 60$, $n(B) = 55$, $n(\overline{A} \cap \overline{B}) = 20$

By De Morgan's Laws.

$$n(\bar{A} \cap \bar{B}) = n(\overline{A \cup B})$$
$$= n(S) - n(A \cup B)$$
$$\therefore \quad 20 = 100 - n(A \cup B)$$
$$\therefore \quad n(A \cup B) = 80$$

But $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$$\therefore \quad n(A \cap B) = n(A) + n(B) - n(A \cup B)$$
$$= 60 + 50 - 80 = 35.$$

**(ii)** From Venn-diagram the percentage of persons who read only one newspaper = 25 + 20 = 45%.

Alternatively, by (3) page 1-34, the percentage,

$$n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$$
$$= 60 + 55 - 2(35) = 45\%$$

$\therefore$ Number of persons who read only one newspaper

$$= 2000 \times \frac{45}{100} = 900$$

**Example 4 :** In a class of 25 students of Economics and Politics, 12 students have taken Economics. Out of these 8 have taken Economics but not Politics. Find the number of students who have taken Economics and Politics and those who have taken Politics but not Economics.

**Sol. :** Let $E$ and $P$ denote these sets. Then we have :-

There are in all 25 students.

Out of these 12 have taken Economics.

Since the class consists of students of Economics and Politics only the remaining 25 – 12 = 13 students have taken Politics.

Out of 12 students 8 have taken only Economics. This means 12 – 8 = 4 students have taken Economics and Policies. This is shown in Fig. 1.34.

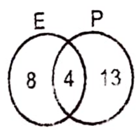



Fig. 1.34

Analytically, we have $n(S) = n(E \cup P) = 25$, $n(E) = 12$, $n(E \cap \bar{P}) = 8$

But $n(E \cap \bar{P}) = n(E - P) = 8$

But $n(E - P) = n(E) - n(E \cap P)$

$\therefore \quad 8 = 12 - n(E \cap P) \qquad \therefore \ n(E \cap P) = 12 - 8 = 4$

$n(E \cup P) = n(E) + n(P) - n(E \cap P)$

$\therefore \quad 25 = 12 + n(P) - 4 \qquad \therefore \ n(P) = 17$

$n(P - E) = n(P) - n(P \cap E) = 17 - 4 = 13$



**Example 5 :** In an examination there were two papers $A$ and $B$. 900 students appeared for the examination. Exactly 740 and 660 passed in papers $A$ and $B$ respectively. 640 passed in both. Find the number of students who failed in both papers.

**Sol. :** Since 640 passed in both $A$ and $B$ and 740 passed in $A$, 740 – 640 = 100 passed only in $A$. Similarly, 660 – 640 = 20 passed only in $B$.

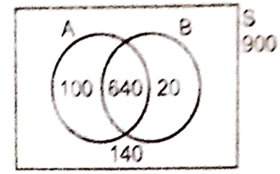



Fig. 1.35

$\therefore$ Number of students who failed

$$= 900 - (100 + 640 + 20)$$
$$= 900 - 760 = 140.$$

**Example 6 :** In a class of 60 students 30 students got first class in semester I examination and 25 got first class in semester II examination. If 20 students did not get first class in either examination, how many got first class in both examinations ?

**Sol. :** We are given

$$n(S) = 60,\ n(A) = 30,\ n(B) = 25,\ n(\overline{A \cup B}) = 20$$
$$\therefore \quad n(A \cup B) = n(S) - n(\overline{A \cup B}) = 60 - 20 = 40$$

We want $n(A \cap B)$.

But $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$$\therefore \quad n(A \cap B) = n(A) + n(B) - n(A \cup B)$$
$$= 30 + 25 - 40 = 15.$$

Using Venn-diagram, since there are 60 students, 20 did not get first class in either semester I or semester II.

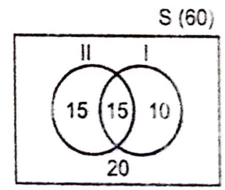



Fig. 1.36

Number of students who got first class in either semester I or semester II = 60 – 20 = 40.

Number of students who got Ist class in semester I only = 40 – 30 = 10

Number of students who got Ist class in semester II only

$$= 40 - 25 = 15$$

$\therefore$ Number of students who got Ist class in both

$$= 40 - (10 + 15) = 15.$$

**Example 7 :** In a survey of 500 students it was found that 300 are locals, 300 use glasses and 275 are vegetarians. 200 are locals who use glasses, 170 are locals who are vegetarian. 200 of them use glasses and are vegetarians and 125 are locals who use glasses and are vegetarians.

(i) Find the number of students who are not local students, who do not use glasses and who are not vegetarians.

(ii) Find the number of students who are locals, who do not use glasses and who are not vegetarians.

**Sol.** : We have $n(S) = 500$, $n(L) = 300$, $n(G) = 300$, $n(V) = 275$,

$n(L \cap G) = 200$, $n(L \cap V) = 170$, $n(G \cap V) = 200$, $n(L \cap G \cap V) = 125$.

$$\therefore \quad n(L \cup G \cup V) = n(L) + n(G) + n(V) - n(L \cap G) - n(G \cap V) - n(V \cap L) + n(L \cap G \cap V)$$

$$\therefore \quad n(L \cup G \cup V) = 300 + 300 + 275 - (200 + 170 + 200) + 125$$

$$= 875 - 570 + 125 = 430$$

$$\therefore \quad n(\bar{L} \cup \bar{G} \cup \bar{V}) = 500 - 430 = 70$$

Alternatively, we can solve the problem by Venn-diagram.

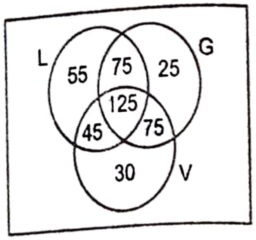



**Fig. 1.37**

There are 125 persons who are locals, who use glasses and who are vegetarians.

Since 200 use glasses and are vegetarians,

$$200 - 125 = 75$$

use glasses and are vegetarians only.

Since 170 are locals who are vegetarians,

$170 - 125 = 45$ are locals and vegetarians only.

Since, 200 are locals who use glasses,

$200 - 125 = 75$ are locals using glasses only.

Since 300 are locals,

$300 - (45 + 75 + 125) = 55$ are only locals.

Since 300 uses glasses,

$300 - (75 + 125 + 75) = 25$ use only glasses.

Since 275 are vegetarians,

$275 - (75 + 125 + 45) = 30$ are only vegetarians.

$\therefore$ **Number of students who are not locals and who do not use glasses and who are not vegetarians**

$$= 500 - (55 + 25 + 30) - (45 + 75 + 75) - 125$$

$$= 500 - (110 - 195 - 125) = 70$$



**Example 8 :** In a class of 75 students, the number of students who passed in Mathematics is equal to the number of students who passed in Electronics. The total number of students who passed exactly in one subject is 60. It was found that 5 students did not pass in either subject. Find the number of students who passed in Mathematics only, who passed in Electronics only and who passed in both the subjects.

**Sol. :** Let $M$ and $E$ denote the set of students who passed in Mathematics and Electronics respectively.

Further, $n(M) = n(E)$. Let $n(M) = n(E) = x$ and also let $n(M \cap E) = y$

$$n(M \cup E) = n(S) - n(\overline{M \cup E})$$

$$= n(S) - n(\bar{M} \cap \bar{E})$$

$$\therefore \quad n(M \cup E) = 75 - 5 = 70$$

Further, $n(M \cup E) = n(M) + n(E) - n(M \cap E)$

$$70 = x + x - y = 2x - y$$

Since, the total number of students who passed in exactly one (of the two) subject is 60, we have

$$n(M) - n(M \cap E) + n(E) - n(M \cap E) = 60$$

$$\therefore \quad x - y + x - y = 60$$

$$\therefore \quad 2x - 2y = 6$$

$$\therefore \quad x - y = 30$$

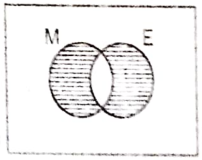



Fig. 1.38

Solving the two equations,

$$2x - y = 70 \ ; \ x - y = 30$$

$$\therefore \quad x = 40 \qquad \therefore \quad y = 10$$

$\therefore$ Number of students who passed in only Mathematics $= x - y = 40 - 10 = 30$

Number of students who passed in only Electronics

$$= x - y = 40 - 10 = 30$$

Number of students who passed on both $= y = 10$.

**Example 9 :** In a class, 42% students passed in Mathematics, 45% passed in Physics. 41% passed in Chemistry, 16% passed in Mathematics and Physics, 19% passed in Physics and Chemistry, 18% passed in Chemistry and Mathematics. Find the number of students who passed in all the three subjects if there were 260 students in the class and 15% students failed in all the subjects.

**Sol. :** We are given the following percentages *i.e.* the number of students passing out of 100.

$n(P) = 45$

$n(M) = 42$, $n(M \cap P) = 16$

$n(C) = 41$, $n(C \cap M) = 18$

$n(P \cap C) = 19$

and we want $n(M \cap P \cap C)$, we have

$n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(P \cap C)$
$\qquad - n(C \cap M) + n(M \cap P \cap C)$

$= 42 + 45 + 41 - (16 + 19 + 18) + n(M \cap P \cap C)$

Since 15% students failed in all the three subjects

$n(M \cup P \cup C) = n(U) - n(M' \cap P' \cap C')$

$= 100 - 15 = 85$

$85 = 128 - 53 + n(M \cap P \cap C)$

$n(M \cap P \cap C) = 85 - 128 + 53 = 10\%$

Number of students who passed in all the three subjects

$= 280 \times \dfrac{10}{100} = 28$

**Alternatively :** We can use Venn diagram as shown in Fig. 1.30.

Let us suppose that x% students passed in all the three subjects. Then by data, we get the following results.

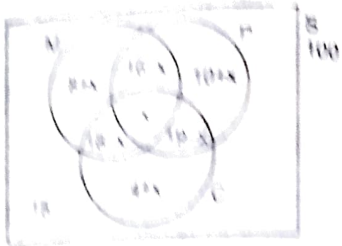



**Fig. 1.30**

% of passing M and P = 16 − x

% of passing in P and C = 19 − x

% of passing in C and M = 18 − x

% of passing in M only = 42 − (16 − x + 18 − x) − x

$= 8 + x$

% of passing in P only = 45 − (16 − x + 19 − x) − x

$= 10 + x$

% of passing in C only = 41 − (19 − x + 18 − x) − x

$= 4 + x$

% of failing in all M, P, C = 15

From the figure we see that

$(8 + x + 10 + x + 4 + x) + (16 - x + 19 - x + 18 - x) + x$

$= 100 - 15 = 85$

$(22 + 3x) + (53 - 3x) + x = 85 \qquad \therefore x = 10$



**Example 10 :** Find the number of integers between 1 and 500 which are divisible by (i) two, three or five, (ii) neither by two nor by three nor by five.

**Sol. :** Let $n(A)$ = number of integers divisible by 2

$n(B)$ = number of integers divisible by 3

$n(C)$ = number of integers divisible by 5

(i) Number of integers divisible by 2 or 3 or 5

$= n(A \cup B \cup C)$

$= n(A) + n(B) + n(C) - n(A \cap B)$
$\qquad - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

Now, $n(A) = \dfrac{500}{2} = 250$, $n(B) = \dfrac{500}{3} = 166$, $n(C) = \dfrac{500}{5} = 100$

$n(A \cap B) = \dfrac{500}{6} = 83$, $n(B \cap C) = \dfrac{500}{15} = 33$

$n(C \cap A) = \dfrac{500}{10} = 50$, $n(A \cap B \cap C) = \dfrac{500}{30} = 16$

$n(A \cup B \cup C) = (250 + 166 + 100) - (83 + 33 + 50) + 16$

$= 516 - (166) + 16 = 366$

(ii) Number of integers divisible by neither 2 or 3 or 5 = 500 − 366 = 134

(Problems of this type are more conveniently solved by using Venn diagram as illustrated in the following examples)

**Example 11 :** Find the number of integers between 1 and 60 which are not divisible by 2, nor by 3 nor by 5. **(M.U. 2006, 08)**

**Sol. :** With the notation of the above example.

Number of integers divisible by 2 or 3 or 5

$= n(A \cup B \cup C)$

$= n(A) + n(B) + n(C)$

$\qquad - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

$n(A) = \dfrac{60}{2} = 30$, $n(B) = \dfrac{60}{3} = 20$, $n(C) = \dfrac{60}{5} = 12$

$n(A \cap B) = \dfrac{60}{6} = 10$, $\quad n(B \cap C) = \dfrac{60}{15} = 4$,

$n(C \cap A) = \dfrac{60}{10} = 6$, $\quad n(A \cap B \cap C) = \dfrac{60}{30} = 2$

$n(A \cup B \cup C) = 30 + 20 + 12 - 10 - 4 - 6 + 2$

$= 44$

Number of numbers divisible neither by 2 nor 3 nor 5 = 60 − 44 = 16

$n(M) = 42$, $\quad n(P) = 45$

$n(C) = 41$, $\quad n(M \cap P) = 16$

$n(P \cap C) = 19$, $\quad n(C \cap M) = 18$

and we want $n(P \cup M \cup C)$, we have

$$n(M \cup P \cup C) = n(M) + n(P) + n(c) - n(M \cap P) - n(P \cap C) - n(C \cap M) + n(M \cap P \cap C)$$

$$= 42 + 45 + 41 - (16 + 19 + 18) + n(M \cap P \cap C)$$

Since 15% students failed in all the three subjects

$$n(M \cup P \cup C) = n(S) - n(\overline{M} \cap \overline{P} \cap \overline{C})$$

$$= 100 - 15 = 85$$

$$85 = 128 - 53 + n(M \cap P \cap C)$$

$$n(M \cap P \cap C) = 85 - 128 + 53 = 10\%$$

Number of students who passed in all the three subjects

$$= 260 \times \frac{10}{100} = 26$$

**Alternatively :** We can use Venn-diagram as shown in Fig. 1.39.

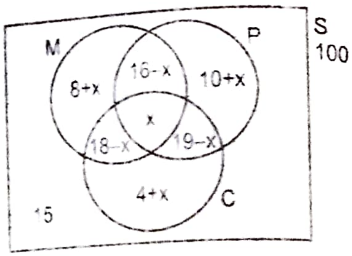



Fig. 1.39

Let us suppose that $x$% students passed in all the three subjects. Then by data, we get the following results.

% of passing $M$ and $P = 16 - x$

% of passing in $P$ and $C = 19 - x$

% of passing in $C$ and $M = 18 - x$

% of passing in $M$ only $= 42 - (16 - x + 18 - x) - x$

$= 8 + x$

% of passing in $P$ only $= 45 (16 - x + 19 - x) - x$

$= 10 + x$

% of passing in $C$ only $= 41 - (19 - x + 18 - x) - x$

$= 4 - x$

% of failing in all $M, P, C = 15$.

From the figure we see that,

$$(8 + x + 10 + x + 4 + x) + (16 - x + 19 - x + 18 - x) + x = 100 - 15 = 85$$

$$(22 + 3x) + (53 - 3x) + x = 85 \quad \therefore x = 10$$



**Example 10 :** Find the number of integers between 1 and 500 which are divisible by (i) two, three or five, (ii) neither by two nor by three nor by five.

**Sol. :** Let $n(A)$ = number of integers divisible by 2.

$n(B)$ = number of integers divisible by 3.

$n(C)$ = number of integers divisible by 5

(i) Number of integers divisible by 2 or 3 or 5

$$= n(A \cup B \cup C)$$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

Now, $\quad n(A) = \dfrac{500}{2} = 250$, $\quad n(B) = \dfrac{500}{3} = 166$, $\quad n(C) = \dfrac{500}{5} = 100$

$n(A \cap B) = \dfrac{500}{6} = 83$, $\quad n(B \cap C) = \dfrac{500}{15} = 33$,

$n(C \cap A) = \dfrac{500}{10} = 50$, $\quad n(A \cap B \cap C) = \dfrac{500}{30} = 16$.

$$\therefore \quad n(A \cup B \cup C) = (250 + 166 + 100) - (83 + 33 + 50) + 16$$

$$= 516 - (166) + 16 = 366$$

(ii) Number of integers divisible by neither 2 or 3 or 5 = 500 − 366 = 134

(Problems of this type are more conveniently solved by using Venn-diagram as illustrated in the following examples)

**Example 11 :** Find the number of integers between 1 and 60 which are not divisible by 2, nor by 3 nor by 5. **(M.U. 2006, 08)**

**Sol. :** With the notation of the above example,

Number of integers divisible by 2 or 3 or 5

$$= n(A \cup B \cup C)$$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

$n(A) = \dfrac{60}{2} = 30$, $\quad n(B) = \dfrac{60}{3} = 20$, $\quad n(C) = \dfrac{60}{5} = 12$.

$n(A \cap B) = \dfrac{60}{6} = 10$, $\quad n(B \cap C) = \dfrac{60}{15} = 4$,

$n(C \cap A) = \dfrac{60}{10} = 6$, $\quad n(A \cap B \cap C) = \dfrac{60}{30} = 2$.

$$n(A \cup B \cup C) = 30 + 20 + 12 - 10 - 4 - 6 + 2$$

$$= 44$$

Number of numbers divisible neither by 2 nor 3 nor 5 = 60 − 44 = 16

**Example 12** : Find the number of positive integers < 500 which are (i) divisible by 5 or 7 but not by 3, (ii) divisible by 3 nor by 5 nor by 7, (iii) not divisible by 3 nor by 5 nor by 7 ?

**Sol.** : We have

$n(A)$ = number of integers divisible by 3 = $\frac{500}{3} = 166$

$n(B)$ = number of integers divisible by 5 = $\frac{500}{5} = 100$

$n(C)$ = number of integers divisible by 7 = $\frac{500}{7} = 71$

$n(A \cap B)$ = number of integers divisible by 3 and 5 *i.e.* by 15 = $\frac{500}{15} = 33$

$n(B \cap C)$ = number of integers divisible by 5 and 7 *i.e.* by 35 = $\frac{500}{35} = 14$

$n(C \cap A)$ = number of integers divisible by 7 and 3 *i.e.* by 21 = $\frac{500}{21} = 23$

$n(A \cap B \cap C)$ = number of integers divisible by 3, 5 and 7 *i.e.* by 105

$= \frac{500}{105} = 4$

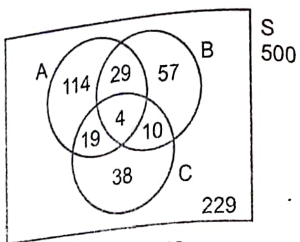



Fig. 1.40

∴ Number of integers divisible by 3 and 5 but not by 7 = 33 − 4 = 29

Number of integers divisible by 5 and 7 but not by 3 = 14 − 4 = 10

∴ Number of integers divisible by 7 and 3 but not by 5 = 23 − 4 = 19

∴ Number of integers divisible by 3 only = 166 − 29 − 19 − 4 = 114

Number of integers divisible by only 5 = 100 − 29 − 10 − 4 = 57

Number of integers divisible by only 7 = 71 − 19 − 10 − 4 = 38.

We can now answer any question about the number of integers in any subset from the figure.

Number of integers divisible 5 or 7 but not by 3

$= n(\bar{A} \cap B \cap C) = 38 + 10 + 57 = 105$

Number of integers divisible of by 3 or by 5 or by 7

$= n(A \cup B \cup C)$

$= 114 + 57 + 38 + 29 + 19 + 10 + 4$

$= 271$



Number of integers not divisible by 3 nor by 5 nor by 7

$= n(\bar{A} \cap \bar{B} \cap \bar{C}) = 500 - 271 = 229$

**Example 13** : If was found that in a class 80 students passed in English, 60 passed in Science and 50 passed in Mathematics. It was also known that 30 students passed in both English and Science, 15 passed in English and Mathematics and 20 passed in Science and Mathematics. 10 students passed in all the three subjects. If there were 150 students in the class find,

(i) how many students passed in at least one subject

(ii) how many passed in English only

(iii) how many passed in at least two subjects.

(iv) how many failed in all the subjects.

**Sol. :** As seen above, it is more convenient to take the help of Venn-diagram and answer all the relevant question rather than use formulae.

We are given that number of students who passed in all the three subjects = 10.

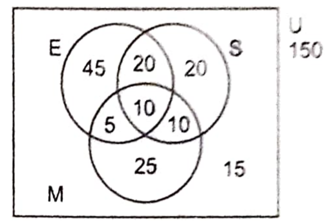



**Fig. 1.41**

∴ Number of students passed in English and Science but not in Mathematics

= 30 − 10 = 20

Number of students passed in English and Mathematics but not in Science

= 15 − 10 = 5

Number of students passed in Mathematics and Science but not in English

= 20 − 10 = 10

Number of students passed in English only

= 80 − 20 − 5 − 10 = 45

Number of students passed in Science only

= 60 − 20 − 10 − 10 = 20

Number of students passed in Mathematics only

= 50 − 5 − 10 − 10 = 25

We shall answer the first part by using analytical method as well as Venn diagram.

***By analytical method i.e. by formulae***

Number of student passing in at least one subject

$= n(E \cup S \cup M)$

$= n(E) + n(S) + n(M) - n(E \cup S) - n(E \cup M)$
$- n(S \cup M) + n(E \cup S \cup M)$

$= (80 + 60 + 50) - (30 + 15 + 20) + 10$

$= 135$

**Alternatively by Venn diagram**, number of students passing in at least one subject

= (passing in exactly three) + (passing in exactly two) + (passing in exactly one)

$= 10 + (5 + 10 + 20) + (45 + 20 + 25)$

$= 135$

(ii) Number of students passing in English only = 45

(iii) Number of students passing in at least two subjects

= number of students passing in exactly two subjects + number of students passing in three subjects

$= (20 + 5 + 10) + 10 = 45$

(iv) Number of students failed in all subjects

$= 150 - (45 + 20 + 25) - (20 + 5 + 10) - 10$

$= 15$

**Example 14 :** A group of 100 students went to a hotel where three dishes $A$, $B$, $C$ were available, each dish costing Rs. 10. It is known that 30 students had taken all the three dishes and 60 had taken at least two of the three dishes. The total expenses were Rs. 1600. Find the number of students who did not order any dish.

**Sol. :** Total number of students = 100.

Total expenditure = Rs. 1600

$\therefore$ Total dishes ordered $= \dfrac{1600}{10} = 160$

Since 30 students had taken all the three dishes and 60 had taken at least two dishes;

$60 - 30 = 30$ students had taken exactly 2 dishes

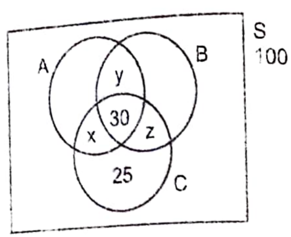



**Fig. 1.42**

Suppose $x, y, z$ denote the number of students taking exactly two dishes i.e. the dishes $A$ and $B$; $B$ and $C$, $C$ and $A$, we have

$x + y + z = 30$



Number of children who had taken exactly one dish

= total number of dishes ordered
– (number of students ordering exactly two dishes) × 2
– (number of students ordering exactly three dishes) × 3

$= 160 - (x + y + z) \cdot 2 - 30 \times 3$

$= 160 - (30) \times 2 - 30 \times 3 = 10$

$\therefore$ Children who did not order any dish $= 100 - (30 + 30 + 10) = 30$.

**Example 15 :** In a survey of 100 cars it was found that 48 cars had air-conditioning ($A$), 45 cars had C.D. players ($C$) and 47 cars had power steering ($P$), 22 cars had air-conditioning and C.D. players, 24 had C.D. players and power-steering, 25 had air-conditioning and power steering, 10 had all the three - air-conditioning, C.D. players and power steering.

Find the number of cars having (i) only air-conditioning, (ii) only C.D players, (iii) only power steering, (iv) air-conditioning and C.D. player but not power steering, (v) C.D. player and power steering but not air-conditioning, (vi) power steering and air conditioning but not C.D. player, (vii) only one of the three facilities, (viii) at least one of these facilities, (ix) at least two of these facilities, (x) none of these facilities.

**Sol. :** As discussed in above problems, we calculate the number of cars in each subset.

Number of cars having air-conditioning and C.D. player only

$= 22 - 10 = 12$

Number of cars having C.D. player and power steering only

$= 24 - 10 = 12$

Number of cars having air-conditioning and power steering only

$= 25 - 10 = 15$

Number of cars having air-conditioning only

$= 48 - (12 + 15 + 10) = 11$

Number of cars having C.D. player only

$= 45 - (12 + 14 + 10) = 9$

Number of cars having power steering only

$= 45 - (15 + 14 + 10) = 8$

These figures are shown in the Fig. 1.40 on the next page.

Now we can answer any question about the number of elements in any subset from the figure

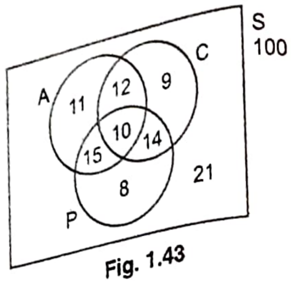



Fig. 1.43

(1) Number of cars having air-conditioning only = 11

(2) Number of cars having C.D. player only = 9

(3) Number of cars having power-steering only = 8

(4) Number of cars having air-conditioning and C.D. players but not power-steering = 12

(5) Number of cars having C.D. players and power steering but not air-conditioning = 14

(6) Number of cars having power-stearing, air-conditioning but not C.D. player = 15

(7) Number of cars having only one of the three facilities
$= 11 + 9 + 8 = 28$

(8) Number of cars having at least one facility
$= (11 + 9 + 8) + (15 + 12 + 14) + (10)$
$= 79$

(9) Number of cars having at least two facilities
$= (15 + 12 + 14) + (10) = 51$

(10) Number of cars having none of these facilities
$= 100 - 79 = 21$

**Example 16 :** Find the number of positive integers $n$ where $1 \le n \le 100$ and $n$ is not divisible by 2, 3 or 5. **(M.U. 2005, 10)**

**Sol. :** Let us denote the sets by $A$, $B$, $C$ respectively. Then we have

$n(A) = 50,\ n(B) = 33,\ n(C) = 20$

$n(A \cap B) = 16,\ n(A \cap C) = 10,$

$n(B \cap C) = 6,\ n(A \cap B \cap C) = 3.$

Number of integers divisible by 2 or 3 or 5 is

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$
$$= 50 + 33 + 20 - 16 - 6 - 10 + 3$$
$$= 74$$



Numbers which are not divisible by 2 or 3 or 5

$$= n(S) - n(A \cup B \cup C)$$
$$= 100 - 74 = 26.$$

**Example 17 :** Among the integers 1 and 300,

(i) how many are divisible by 3, 5 and 7 and not divisible by 3 nor by 5 nor by 7 ?

(ii) how many of them are divisible by 3 but not by 5 nor by 7. **(M.U. 2009)**

**Sol. :** Let us note the sets by $A$, $B$, $C$. Then we have

$$n(A) = \left|\frac{300}{3}\right| = 100,\ n(B) = \left|\frac{300}{5}\right| = 60,\ n(C) = \left|\frac{300}{7}\right| = 42,$$

where $\left|\frac{a}{b}\right|$ denotes the integral part of the quotient.

$$n(A \cap B) = \left|\frac{300}{3 \times 5}\right| = 20, \qquad n(B \cap C) = \left|\frac{300}{5 \times 7}\right| = 8,$$

$$n(C \cap A) = \left|\frac{300}{3 \times 7}\right| = 14, \qquad n(A \cap B \cap C) = \left|\frac{300}{3 \times 5 \times 7}\right| = 2.$$

Now, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

$= 100 + 60 + 42 - 20 - 8 - 14 + 2$

$= 162$

There are 162 numbers which are divisible by 3 or 5 or 7.

$\therefore$ The numbers which are not divisible by 3 nor by 5 nor by 7

$= 300 - 162 = 138$

(ii) Now, the numbers divisible by 3 but not by 5 nor by 7

$= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

$= 100 - 20 - 14 + 2$

$= 68$

**Example 18 :** Find the number of integers between 1 and 2000 which are divisible by 2, 3, 5 or 7. **(M.U. 1997)**

**Sol. :** Let the four sets be denoted by $A$, $B$, $C$, $D$.

$\therefore\ n(A) = 1000,\ n(B) = 666,\ n(C) = 400,\ n(D) = 285$

$n(A \cap B) = 333, \qquad n(B \cap C) = 133,$

$n(C \cap D) = 57, \qquad n(A \cap C) = 200,$

$n(A \cap D) = 142, \qquad n(B \cap D) = 95.$

$n(A \cap B \cap C) = 66, \quad n(A \cap B \cap D) = 47,$

$n(A \cap C \cap D) = 28 \quad n(B \cap C \cap D) = 19,$

$n(A \cap B \cap C \cap D) = 9$

$$n(A \cup B \cup C \cup D) = n(A) + n(B) + n(C) + n(D)$$
$$- n(A \cap B) - n(A \cap C) - n(A \cap D) - n(B \cap C)$$
$$- n(B \cap D) - n(C \cap D) + n(A \cap B \cap C)$$
$$+ n(A \cap C \cap D)$$
$$+ n(A \cap B \cap D) + n(B \cap C \cap D) - n(A \cap B \cap C \cap D)$$
$$= (1000 + 666 + 400 + 285) - (333 + 200 + 142 + 133 + 95 + 57)$$
$$+ (66 + 47 + 28 + 19) - 9$$
$$= 1542$$

**Example 19 :** In a survey of 60 persons, it was found that 25 read Newsweek, 26 read Time and 26 read Fortune. Further 9 read both Newsweek and Fortune, 11 read both Newsweek and Time, 8 read both time and Fortune and 8 read none of these.

(i) Find the number of people who read all the three magazines.

(ii) Find the number of people who read exactly one magazine.

**(M.U. 1998, 2002, 06, 07, 12, 13)**

**Sol. :** We are given $n(S) = 60$, $n(N) = 25$, $n(T) = 26$, $n(F) = 26$, $n(N \cap F) = 9$, $n(N \cap T) = 11$, $n(T \cap F) = 8$, $n(\bar{N} \cap \bar{T} \cap \bar{F}) = 8$

Now, $n(N \cup T \cup F) = n(N) + n(T) + n(F) - n(N \cap T) - n(T \cap F) - n(N \cap F) + n(N \cap T \cap F)$

Now, $n(N \cup T \cup F) = n(S) - n(\bar{N} \cap \bar{T} \cap \bar{F})$
$= 60 - 8 = 52$

$\therefore \quad 52 = 25 + 26 + 26 - 11 - 8 - 9 + n(N \cap T \cap F)$

$\therefore \quad n(N \cap T \cap F) = 3$

**Alternatively :**

Number of persons who read $N$ and $F$ only $= 9 - 3 = 6$

Number of persons who read $N$ and $T$ only $= 11 - 3 = 8$

Number of persons who read $T$ and $F$ only $= 8 - 3 = 5$

Enter these figure in the diagram. Then,

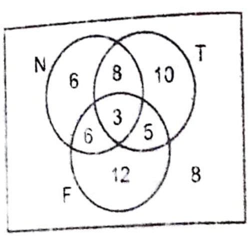



**Fig. 1.44**



Number of persons who read $T$ only $= 26 - 8 - 3 - 5 = 10$

Number of persons who read $N$ only $= 25 - 8 - 3 - 6 = 8$

Number of persons who read $F$ only $= 26 - 6 - 3 - 5 = 12$

**Example 20 :** A survey of 500 T.V. watchers gave the following results

285 watch football.

195 watch hockey.

115 watch basketball.

45 watch football and basketball.

70 watch football and hockey.

50 watch hockey and basketball

50 do not watch any of the three.

How many people watch all the three games ?

How many people watch exactly one of the three games ?

**(M.U. 2000, 11)**

**Sol. :** We are not given the number of persons who watch all the three games. Let this number be $x$. Then we can enter all the numbers in the subsets. The sum of all elements in the subsets is 450.

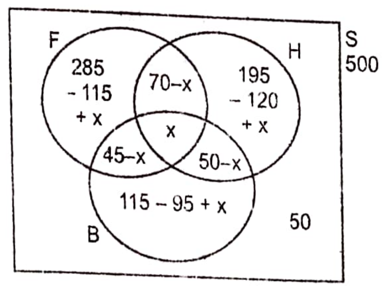



**Fig. 1.45**

$\therefore \quad (170 + x) + (70 - x) + (75 + x) + (50 - x) + (20 + x) + (45 - x) + x = 450$

$\therefore \quad x = 20$

Number of persons who watch all the three games $x = 20$

Number of persons who watch only football $= 170 + x = 190$

Number of persons who watch only Hockey $= 75 + x = 95$

Number of persons who watch only Basketball $= 20 + x = 40$

$\therefore$ Number of persons who watch only one game
$= 190 + 95 + 40 = 325$

**Example 21 :** Out of 250 candidates who failed in either in Mathematics or Physics or in Aggregate, it was revealed that 128 failed in Mathematics, 87 in Physics and 134 in Aggregate, 31 failed in Mathematics and in Physics, 54 failed in Aggregate and Mathematics, 30 failed in Aggregate and Physics. Find how many candidates failed

(i) in all the three subjects,

(ii) in Mathematics but not in Physics

(iii) in Aggregate but not in Mathematics.

(iv) in Physics but not in Aggregate or Mathematics **(M.U. 2012)**

**Sol. :** Let $n(M)$ = number of students who failed in Mathematics,

$n(P)$ = number of students who failed in Physics,

$n(A)$ = number of students who failed in Aggregate.

We are given

$n(M \cup P \cup A) = 250, \quad n(M) = 128, \quad n(P) = 87, \quad n(A) = 134,$

$n(M \cap P) = 31, \quad n(A \cap M) = 54, \quad n(P \cap A) = 30.$

We have

(i) $n(M \cup P \cup A) = n(M) + n(P) + n(A) - n(M \cap P) - n(A \cap M) - n(P \cap A) + n(M \cap P \cap A)$

$\therefore \quad 250 = 128 + 87 + 134 - 31 - 54 - 30 + x$

$\therefore \quad x = 16$

$\therefore$ The number of students who failed in all the three = 16.

(ii) Number of students who failed in Mathematics and Physics both

$= 31 - 16 = 15$

Number of students who failed in Mathematics and Aggregate both

$= 54 - 16 = 38$

Number of students who failed in Aggregate and Physics both

$= 30 - 16 = 14$

$\therefore$ Number of students who failed in Mathematics but not in Physics

$$n(M \cup \bar{P}) = n(M) - n(M \cap P) - n(M \cap P \cap A) = 128 - 15 - 16 = 97$$

(iii) Number of students who failed in aggregate but not in Mathematics

$$n(A \cup \bar{M}) = n(A) - n(A \cap M) - n(A \cap M \cap P) = 134 - 34 - 16 = 80$$



(iv) Number of students who failed in Physics but not in aggregate

$$n(P \cup \bar{A}) = n(P) - n(P \cap A) - n(P \cap A \cap P) = 87 - 14 - 16 = 57$$

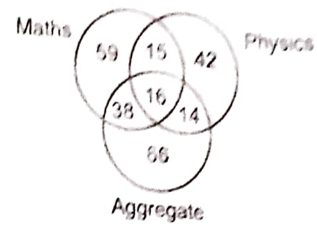



**Fig. 1.46**

## EXERCISE-VIII

1. A construction company has to appoint 30 Civil engineers on road construction and 25 Civil engineers on building construction. Of these 15 will be asked to work on both types of construction. How many Civil engineers the company must appoint ? **[ Ans. : 40 ]**

2. In a class of 38 students, 23 students have taken Mathematics, 13 have taken Mathematics but not Economics. Find the number of students who have taken Mathematics and Economics and the number of students who have taken Economics but not Mathematics. **[ Ans. : (i)** 10, **(ii)** 15 **]**

3. In a group of 500 persons 40% drink tea, 50% drink coffee and 20% drink both. Find the number of persons who drink (i) only tea, (ii) only coffee, (iii) neither tea nor coffee. **[ Ans. : (i)** 150, **(ii)** 100, **(iii)** 150 **]**

4. In a group of 500 persons 100 speak English only and 350 speak Marathi only. Find the number of persons who speak (i) Marathi, (ii) English, (iii) both ? **[ Ans. : (i)** 400, **(ii)** 150, **(iii)** 150 **]**

5. In an examination out of 140 students 60 passed in paper I, 90 passed in paper II and 40 passed in both papers. Find the number of students who (i) failed in both papers, (ii) passed in exactly one paper. **[ Ans. : (i)** 30, **(ii)** 70 **]**

6. Find the number of positive integers $\leq 200$ which are divisible by either 2 or 3. **[ Ans. : 133 ]**

7. In a group of 100 students it was observed that 53 like blue colour, 47 like yellow colour and 41 like red colour, 23 like blue and yellow, 18 like yellow and red and 20 like blue and red and 8 like all they three colours.

Find the number of students who like (i) blue but not red, (ii) blue and red but not yellow, (iii) blue or red but not yellow. **[ Ans. : (i)** 33, **(ii)** 41, **(iii)** 29 **]**

8. Find the number of positive integers less the 500 divisible by two, three or five. **[ Ans. : 366 ]**

9. Find the number of positive integers less than or equal to five hundred which are divisible (i) by 2 or 3, or 5 but not 30, (ii) by 2 or 3 but not 5, (iii) by 3 or 5 but not 2. **[ Ans. : (i)** 350, **(ii)** 266 **(iii)** 116 **]**

**10.** *In a class of 200 students 98 take a course in Marathi, 75 take a course in Hindi, 70 take a course in Sanskrit, 35 take a course in Marathi and Hindi, 40 take a course in Hindi and Sanskrit, 42 take a course in Sanskrit and Marathi and 25 take a course in all the three languages.*

*Find the number of students who take a course in (i) Marathi, Hindi or Sanskrit, (ii) none of these languages, (iii) Marathi or Hindi but not in Sanskrit, (iv) exactly one language, (v) exactly two languages, (vi) only Sanskrit, (vii) at least two languages*

[ **Ans.** : **(i)** 151, **(ii)** 49, **(iii)** 81, **(iv)** 84, **(v)** 42, **(vi)** 13, **(vii)** 67 ]

## MISCELLANEOUS EXERCISE

1. Find the power set of $\{\alpha, \beta, \gamma\}$. How many elements are there in the power set ? **(M.U. 1999)**

[ **Ans.** : $\{\Phi, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\alpha, \beta\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \{\alpha, \beta, \gamma\}; \ 2^3 = 8\}$ ]

2. Find the power set of $\{\Phi\}$. **(M.U. 2004)** [ **Ans.** : $\{\Phi, \{\Phi\}\}$ ]

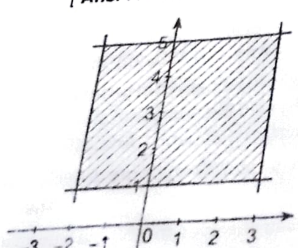
3. Find $A \times B$ if $A = \{a, b\}$ and $B = \{1, 2, 3\}$. **(M.U. 1999)**

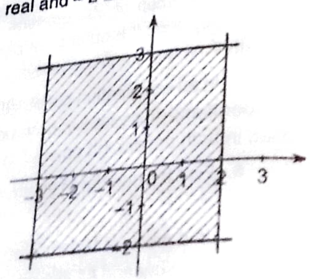
[ **Ans.** : $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$ ]

4. Find $A \times B$ if $A = \{x \mid x$ is real and $-2 \le x \le 3\}$, **(M.U. 1998)**

$B = \{y \mid y$ is real and $1 \le y \le 5\}$.

[ **Ans.** : $A \times B = \{(x, y) \mid x, y$ are real and $-2 \le x \le 3, 1 \le y \le 5\}$ ]





5. Find $A \times B$ if $A = \{x \mid x$ is real and $-3 \le x \le 2\}$

and $B = \{y \mid y$ is real and $-2 \le y \le 3\}$

[ **Ans.** : $A \times B = \{(x, y) \mid x, y$ are real and $-3 \le x \le 2, -2 \le y \le 3\}$ ]

6. Determine the number of integers between 1 and 250 that are divisible by 2, 3, 5 or 7. **(M.U. 1997, 2003)** [ **Ans.** : 193 ]

7. An investigator interviewed 100 students to determine their preferences for three drinks - Milk ($M$), Coffee ($C$) and Tea ($T$). He reported the following : 10 students like all the three drinks; 20 like $M$ and $C$, 30 like $C$ and $T$, 25 like $M$ and $T$, 12 like $M$ only, 5 like $C$ only and 8 like $T$ only.

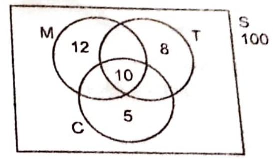


(i) How many did not like any of the three drinks ?

(ii) How many like milk but not coffee ?

(iii) How many like tea and coffee but not milk ? **(M.U. 1997)**

[ **Ans.** : **(i)** 20, **(ii)** 27, **(iii)** 20 ]




( **Hint** : Enter the given figures and find the remaining. )

**8.** Let $A$ denote the set of students who study data structures, $B$ denote the set of students who study discrete structure, $C$ denote the set of students who study assembly language programming, $D$ denote the set of students who study Theory of Computer Science, $E$ denote the set of students who are staying in Hostel, $C$ and $F$ denote the set of students who went to watch a cricket match last Monday.

Express the following statements in set theoretic notation.

(i) All hostelites who study neither data structure no discrete structure went to watch cricket match last Monday.

(ii) The students who went to watch cricket match are only those who study assembly language programming or data structure.

(iii) No student who is studying data structures went to watch the cricket match.

(iv) Those and only those students who are studying theory of computer science and discrete structure went to watch a cricket match. **(M.U. 1996)**

[ **Ans.** : **(i)** $E \cap \overline{A} \cap \overline{B} \subseteq F$, **(ii)** $F = C \cup A$, **(iii)** $A \cap F = \Phi$, **(iv)** $D \cap B = F$ ]

**9.** At one of the colleges in Mumbai it was found that out of 150 students 109 take at least one of the following computer languages; VB, VC$^{++}$, Jawa.

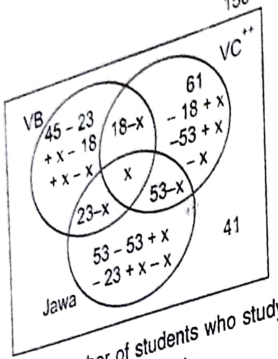
Further it was found that 45 study VB, 61 study VC$^{++}$, 53 study Jawa, 18 study VB and VC$^{++}$, 53 study VC$^{++}$ and Jawa and 23 study VB and Jawa.

Find (i) how many study all the three languages.

(ii) how many study only VC$^{++}$

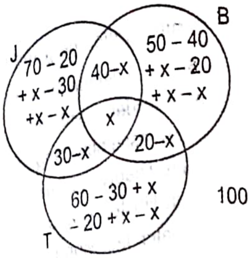
(iii) how many do not study any of the languages **(M.U. 2007, 08)**

[ **Ans.** :




( **Hint** : Let $x$ be the number of students who study all the languages. Then working out backwards, we get $x = 44$. )

**10.** In an university 60% professors play tennis, 50% play bridge, 70% jog, 20% play tennis and bridge, 30% play tennis and jog, 40% play bridge and jog. If someone claimed that 20% jog, play bridge and play tennis, would you believe that claim ? Why ? **(M.U. 2000)**



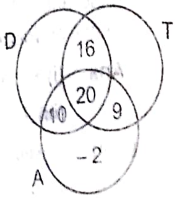

[ **Ans.** : See adjoining figure. ]

( **Hint** : Let $x$ be the percentage of professors who jog, play bridge and play tennis. Then working backwards, we get $x = -10$. Absurd. No. )

**11.** A college record has registered the following information : 119 students enrolled an Introductory Computer Science Course. Of these 96 took Data Structures, 53 took Foundations, 39 took Assembly Language, 31 took both Foundations and Assembly Languages, 32 took both Data structures and Assembly Languages, 38 took Data Structures and Foundations and 22 took all the three courses.

Is the information correct ? Why ?

**(M.U. 1993)**




( **Hint** : See adjoining figure. If we enter the figures starting from the last, in one of the subsets we get a negative figure. Hence, wrong. )

**12.** In a survey of 250 students of a college the following data were obtained :

(i) 64 had taken Mathematics Course.

(ii) 94 had taken Computer Science Course.

(iii) 58 had taken Business Course.



(iv) 28 had taken both Mathematics and Business Course.

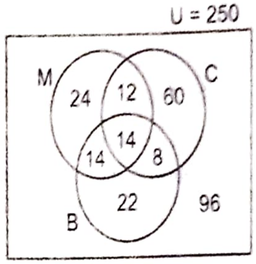
(v) 26 had taken both Mathematics and Computer Science Courses.

(vi) 22 had taken Computer Science and Business Courses.

(vii) 14 had taken all the three Courses.

How many students had taken none of these Courses ?

How many had taken only Computer Science Course ? **(M.U. 2007)**




[ **Ans. :** See adjoining figure.

**(i)** 96, **(ii)** 60 ]

## EXERCISE - IX

### Theory

**1.** Define the following terms.

**(1)** Null set **(2)** Infinite set **(M.U. 2007)** **(3)** Universal set

**(4)** Subset **(5)** Union of two sets

**(6)** Intersection of two sets

**(7)** Complement of a set **(8)** Disjoint sets **(M.U. 2010)**

**(9)** Difference between two sets

**(10)** Symmetric difference between two sets **(M.U. 2010)**

**(11)** Principle of duality in sets.

**(12)** Power Set **(M.U. 2010)**

**2.** State the following laws and show them by Venn-diagrams.

**(1)** Indempotent laws **(2)** Identity laws

**(3)** Inverse laws **(4)** Domination laws

**(5)** Commutative laws **(6)** Associative laws

**(7)** Distributive laws **(8)** De-Morgan's laws

**(9)** Absorption law **(10)** Involution

**3.** Explain the following terms with illustrations.

**(1)** Partitioning of sets **(2)** Cartesian Product

**(3)** The Inclusion-Exclusion Principle **(M.U. 2008)**

**4.** Define symmetric difference of two sets $A$ and $B$ and explain it with Venn-diagrams. **(M.U. 2007)**

5. If $A$, $B$, $C$ are subsets of the universal set $U$ and $A \cap B = A \cap C$, $\bar{A} \cap B = \bar{A} \cap C$ is it necessary that $B = C$? Justify you answer **(M.U. 1995)**

(Hint : Yes $B = B \cap U = B \cap (A \cup \bar{A})$

$= (B \cap A) \cup (B \cap \bar{A}) = (A \cap C) \cup (\bar{A} \cap C)$

$= (A \cup \bar{A}) \cap C = U \cap C = C$ )

6. Prove using the laws of set-theory

$(A \cap B) \cup [B \cap (C \cap D) \cup (C \cap \bar{D})] = B \cap (A \cup C)$ **(M.U. 2006)**

(Hint : l.h.s. $= (A \cap B) \cup [B \cap C \cap (D \cup \bar{D})]$

$= (A \cap B) \cup [B \cap C \cap U]$

$= (A \cap B) \cup (B \cap C) = (B \cap A) \cup (B \cap C)$

$= B \cap (A \cup C)$ )

7. Define (i) Power set, (ii) Partition of a set. Give examples. **(M.U. 2000, 01, 02)**

8. Define disjoint sets. **(M.U. 2004)**

6. Define power set. Write power set of $A = \{a, b, c\}$.

8. Define the following terms :

**(1)** Countable set **(2)** Symmetric difference of two sets.

9. Determine the power sets of the following :

**(i)** $\{a\}$, **(ii)** $\{\{a\}\}$, **(iii)** $\{\Phi\}$, **(iv)** $\{\Phi, \{\Phi\}\}$

[ **Ans. :** **(i)** $\{\Phi, \{a\}\}$, **(ii)** $\{\Phi, \{\{a\}\}\}$, **(iii)** $\{\Phi, \{\Phi\}\}$,

**(iv)** $\{\Phi, \{\Phi\}, \{\{\Phi\}\}, \{\Phi, \{\Phi\}\}\}$ ]

10. If $A = \{a, b, c, d\}$, $B = \{c, b\}$, find $A - B$, $B - A$, $A \cup B$, $A \cap B$, $A \times B$.

11. Let $A = \{a, b, c\}$, $B = \{1, 2\}$. Find $A \times B$, $B \times A$.

12. Prove the following identities

**(1)** $(A \cap B) \cup (A \cap \bar{B}) = A$ **(2)** $(A - C) \cap (C - A) = \Phi$

**(3)** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ **(4)** $(A - B) \cup (A \cap B) = A$

13. Prove the following relations analytically.

**(1)** $A - B \cup (A \cap B) = A$ **(2)** $A \cap B \subset B \subset A \cup B$

**(3)** $A \cup (A \cap B) = A$ **(4)** $\bar{A} - \bar{B} = \bar{A} \cap B$

**(5)** $(A - B) - C = A - (B \cup C)$

✣ ✣ ✣

CHAPTER 2

## 1. Introduction

Human being is a r... separates man from other... From the word logos, the w... of thought.

Logic was very mu... learn today, has Greek c...

In Mathematics a... come across expressio... and $p_2$ then $q_1$ or $q_2$" e...

We want to kno... false. This is studied i...

The "Mathemati... of Aristotelian logic. It... Boole, Augustus de M... others.

## 2. Statemen...

In grammar w...

1. Declarati...
2. Imperativ...
3. Interroga...
4. Exclama...

In logic we... sentences we sh... true or false. Su...

The basic... difference betw... later. Till then v...