CHAPTER 4

Fast Fourier Transform

Syllabus :

Need of FFT, Radix-2 DIT-FFT algorithm, DIT-FFT Flow graph for N = 4 and 8, Inverse FFT algorithm.

4.1 Introduction

We studied the Discrete Fourier Transform (DFT) in the Chapter 3.

The DFT is used to identify the frequency components present in a discrete time signal. The DFT converts a time domain signal into a frequency domain signal.

A N–point DFT is given by the equation

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \ ; \quad k = 0, 1, \ldots, N-1 \qquad \ldots(4.1.1)$$

In the matrix form it is given by the equation

$$X = [W]x \qquad \ldots(4.1.2)$$

4.2 Computational Complexity of DFT

Now, the question that we need to ask ourselves is How many multiplications and additions are required to compute a N-point DFT ?

We will solve an example to answer this question.

Solved Example

Ex. 4.2.1 : Compute the DFT of the sequence

$x(n) = \{0, 1, 2, 1\}$.

Soln. : Since the length of the signal is 4,

$$N = 4 \rightarrow \text{4-Point DFT}$$

We generate a 4×4 DFT matrix and use Equation (4.1.2).

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & +1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\begin{array}{ccccc} & & & \text{Number of multiplication} & \text{Number of addition} \\ \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} & \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} & \begin{matrix} \rightarrow 4\times \\ \rightarrow 4\times \\ \rightarrow 4\times \\ \rightarrow 4\times \end{matrix} & \begin{matrix} 3+ \\ 3+ \\ 3+ \\ 3+ \end{matrix} \\ & & & \overline{16\times} & \overline{12+} \end{array}$$

We have also written the number of multiplication and addition required to calculate each X(k)

$\therefore X(0) = 1\cdot x(0) + 1\cdot x(1) + 1\cdot x(2) + 1\cdot x(3)$ $\rightarrow$4 multi. & 3 add

$X(1) = 1\cdot x(0) + (-j)\cdot x(1) + (-1)\cdot x(2) + j\cdot x(3)$ $\rightarrow$4 multi. & 3 add

$X(2) = 1\cdot x(0) + (-1)x(1) + (1)\cdot x(2) + -(1)x(3)$ $\rightarrow$4 multi. & 3 add

$X(3) = 1\cdot x(0) + j\,x\cdot(1) + (-1)\cdot x(2) + -(j)\cdot x(3)$ $\rightarrow$4 multi. & 3 add

16multi. & 12add

$$\therefore X(k) = \{4, -2, 0, -2\}$$

As is shown each X(k) requires 4 multiplications and 3 additions.

Hence a 4-point DFT requires a total of 16 multiplications and 12 additions. i.e.$(4)^2$ multiplications and 4×3additions.

We can generalize this by stating the A N-point DFT requires N^2 multiplications and N (N – 1) additions.

Therefore computing the DFT is computationally very expensive.

It has been mentioned in earlier chapters that because of high sampling rates in the real world, the length of x(n) is very large and runs into thousands of samples.

Computing the DFT of such signals would require a machine which has a lot of computation power. It would also take a lot of time.

To cut a long story short, the DFT is computationally very demanding and we require something that would reduce the number of computations.

This is where the Fast Fourier Transform comes in.

Fast Fourier Transform (FFT)

- In 1965, Cooley and Tukey proposed a method of computing the DFT which required only N $\log_2$ N calculations.
- This method came to be known as the Fast Fourier Transform (F.F.T)
- The FFT refers to a class of algorithms for efficiently computing the DFT.
- Hence FFT is not a new transform but a set of algorithms used for computing the DFT efficiently.
- The results obtained from the DFT and the FFT are the same.
- The computational efficiency of the FFT becomes clear when the values of N^2 and N $\log_2$ N are compared for several values of N.

Number of Samples	Number of computations (DFT) N^2	Number of computations (FFT) $N \log_2 N$
32	1024	160
128	16,384	896
2048	4,194,304	22,528

- We notice that as N increases, the FFT becomes more efficient.
- As mentioned earlier, FFT is an algorithm. The commonly used technique is called the RADIX-2 FFT algorithm.
- We have studied the Twiddle factor in the previous chapter. There are two important properties of the Twiddle factor.

$$W_N^{k+\frac{N}{2}} = -W_N^k \qquad \text{Symmetric Property}$$

$$W_N^{k+N} = W_N^k \qquad \text{Periodic property}$$

- The FFT exploits these two properties to reduce the number of complex multiplications.
- The basic principle of the FFT algorithm is to break (decompose) a N-point DFT into smaller DFTS. As mentioned earlier, FFT is an algorithm and is known as Radix-2 which simply means that the number of output -points is expressed an a power of 2 i.e. $N=2^m$, where m is an integer.
- Hence when the FFT algorithm breaks a DFT into smaller DFTS' the output of the smaller DFTS' are always powers of 2.
- The two commonly used algorithms are
  1. Decimation in Time FFT (DIT-FFT)
  2. Decimation in Frequency FFT (DIF-FFT)
- We shall derive and discuss each one in detail but before doing that let us try to visualize a N-point DFT formula.

We know a N-point DFT is given by the equation

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}\ ;\ k = 0, 1, \ldots, N-1$$

- Here x(n) is the input and X(k) the output. This equation can be pictorially represented as shown in Fig. 4.2.1

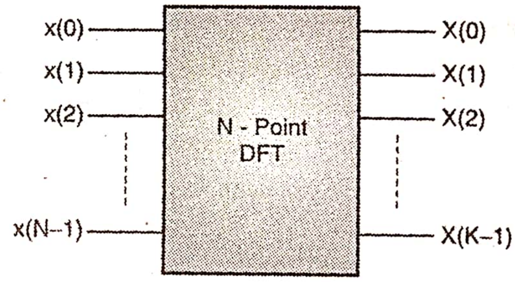

Fig. 4.2.1

- The DFT block computes the DFT of the input x(n) and gives us the output X(k).
- Both, DIF-FFT and DIT-FFT, are based on the basic principle of breaking this N-Point DFT into smaller DFT Blocks.

4.3 Decimation In Time FFT (DIT-FFT)

- To decimate means to kill. Decimation in time means killing (re-arranging) the time signal.
- Here we exploit the symmetric and periodic property of the twiddle factor W_N

$$W_N^{k+N/2} = -W_N^k \rightarrow \text{Symmetric}$$

$$W_N^{k+N} = W_N^k \rightarrow \text{Periodic}$$

- These two important properties reduce the number of computations.

A N-point-DFT is given by the equation,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \qquad k = 0, 1, 2, \ldots N-1$$

- In decimation-in-time technique, we split the input x(n) into odd and even parts.

$$X(k) = \sum_{n \to \text{even}} x(n)\, W_N^{nk} + \sum_{n \to \text{odd}} x(n)\, W_N^{nk}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{(2n+1)k}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{2nk}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{2nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{2nk}$$

Now, $W_N^2 = W_{N/2}$

$$X(k) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_{N/2}^{nk}}_{\text{N/2 point DFT}} + \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_{N/2}^{nk}}_{\text{N/2 point DFT}} \quad \ldots(4.3.1)$$

Let $\sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_{N/2}^{nk} = F_1(k)$

and $\sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_{N/2}^{nk} = F_2(k)$; $k = 0, 1, \ldots \frac{N}{2}$

$$\therefore\ X(k) = F_1(k) + W_N^k F_2(k) \quad \ldots(4.3.2)$$

This equation gives us the first $\frac{N}{2}$ values

- Since $F_1(k)$ and $F_2(k)$ are periodic with period N/2 we can write

$$F_1(k + N/2) = F_1(k) \text{ and}$$
$$F_2(k + N/2) = F_2(k)$$

also $W_N^{k+N/2} = -W_N^k$

$\therefore$ Equation (4.3.2) can be written as,

$$\therefore\quad X(k) = F_1(k) + W_N^k F_2(k);\quad k = 0, 1, \ldots \frac{N}{2} - 1$$

$$\therefore\ X\left(k + \frac{N}{2}\right) = F_1(k) - W_N^k F_2(k);\ k = 0, 1, \ldots \frac{N}{2} - 1 \quad \ldots(4.3.3)$$

- We observe that $F_1(k)$ requires $(N/2)^2$ complex multiplications and the same applies to $F_2(k)$. Furthermore, there are N/2 additional complex multiplications required to compute $W_N^k F_2(k)$. Hence the computation of X(k) requires complex multiplications which is almost half the computations required by the conventional method.

$$2 \times \left(\frac{N}{2}\right)^2 + \frac{N}{2} = \frac{N^2}{2} + \frac{N}{2}$$

Let us take an example where N = 8,

- Equation (4.3.3) can be shown using a signal flow graph.

Since in DIT - FFT, we have split the input x(n) into even and odd parts, we have

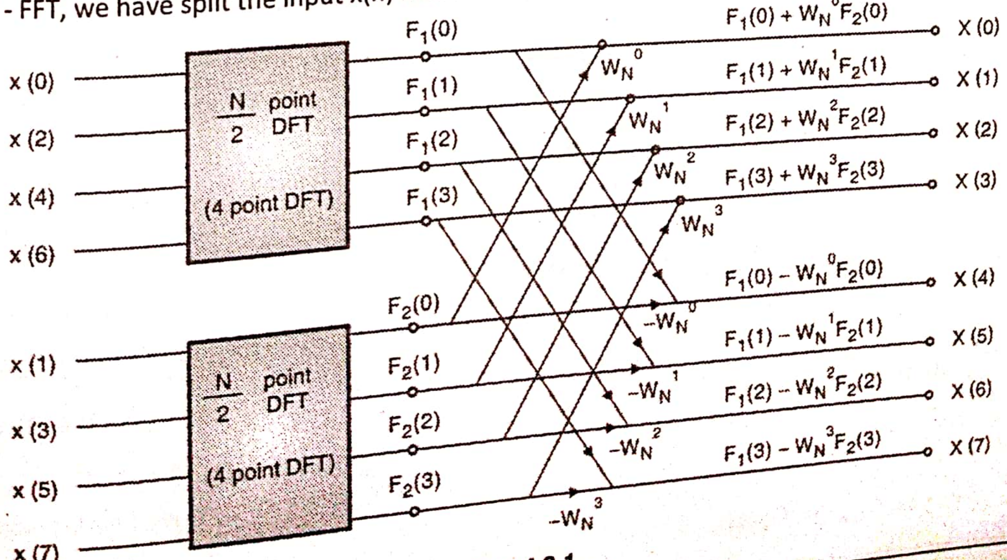

Fig. 4.3.1

– As stated earlier, $F_1(k)$ and $F_2(k)$ are N/2 point DFT's (4 point DFT's if N = 8). These can be further split up into odd and even parts. We rewrite Equation 4.3.1.

$$\text{i.e.}\quad X(k) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_{N/2}^{nk}}_{F_1(k)} + W_N^k \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_{N/2}^{nk}}_{F_2(k)}$$

– Splitting up $F_1(k)$ and $F_2(k)$ into odd and even parts, we get,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_{N/2}^{nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_{N/2}^{nk}$$

2 parts of N/4 each 2 parts of N/4 each

(A) (B)

$$\text{(A)}\quad F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_{N/2}^{nk} = \sum_{n=0}^{\frac{N}{4}-1} x(4n)\, W_{N/2}^{2nk} + \sum_{n=0}^{\frac{N}{4}-1} x(4n+2)\, W_{N/2}^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{u}-1} x(un)\, W_{N/L}^{2nk} + \sum_{n=0}^{\frac{N}{u}-1} x(4n+2)\, W_{N/2}^{2nk} . W_{N/2}^{k}$$

$$= \sum x(un)\, W_{N/2}^{2nk} + W_{N/2}^{k} \sum x(un+2)\, W_{N/2}^{2nk}$$

$$\text{Now}\quad W_{N/2}^{2} = W_{N/4}^{1}$$

$$\therefore\quad F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_{N/2}^{nk} = \underbrace{\sum_{n=0}^{\frac{N}{4}-1} x(4n)\, W_{N/4}^{nk}}_{\text{N/4 point DFT}} + W_{N/2}^{k} \underbrace{\sum_{n=0}^{\frac{N}{4}-1} x(4n+2)\, W_{N/4}^{nk}}_{\text{N/4 point DFT}}$$

$$\therefore\quad F_1(k) = \sum_{n=0}^{\frac{N}{4}-1} x(4n)\, W_{N/4}^{nk} + W_{N/2}^{k} \sum_{n=0}^{\frac{N}{4}-1} x(4n+2)\, W_{N/4}^{nk}$$

$$\text{Let}\quad \sum_{n=0}^{\frac{N}{4}-1} x(4n)\, W_{N/4}^{nk} = G_1(k)$$

$$\text{and}\quad \sum_{n=0}^{\frac{N}{4}-1} x(4n+2)\, W_{N/4}^{nk} = G_2(k)$$

$$\therefore F_1(k) = G_1(k) + W_{N/2}^{k}\ G_2(k)$$

This equation will give us the first $\frac{N}{4}$ values

In a similar manner we can write for (B).

(B) $$F_2(k) = H_1(k) + W_{N/2}^{k}\ H_2(k)$$

Now, $G_1(k)$, $G_2(k)$, $H_1(k)$ and $H_2(k)$ are all N/4 point DFT's and are periodic with period N/4. Using the symmetric and periodic properties of the twiddle factor we get

$$\left.\begin{aligned} F_1(k) &= G_1(k) + W_{N/2}^{k}\ G_2(k) \\ \therefore\quad F_1(k+N/4) &= G_1(k) - W_{N/2}^{k}\ G_2(k) \\ F_2(k) &= H_1(k) + W_{N/2}^{k}\ H_2(k) \end{aligned}\right\} k = 0, 1, 2, \ldots, \frac{N}{4} - 1 \qquad \ldots (4.3.4)$$

$$\therefore\quad F_2(k+N/4) = H_1(k) - W_{N/2}^{k}\ H_2(k)$$

For N = 8, k will have values 0, 1.

We have split $F_1(k)$ and $F_2(k)$ which were N/2 point DFT's into two N/4 point DFT's each.

We draw the signal flow graph using Equation (4.3.4) and the earlier Fig. 4.3.1. Let N = 8

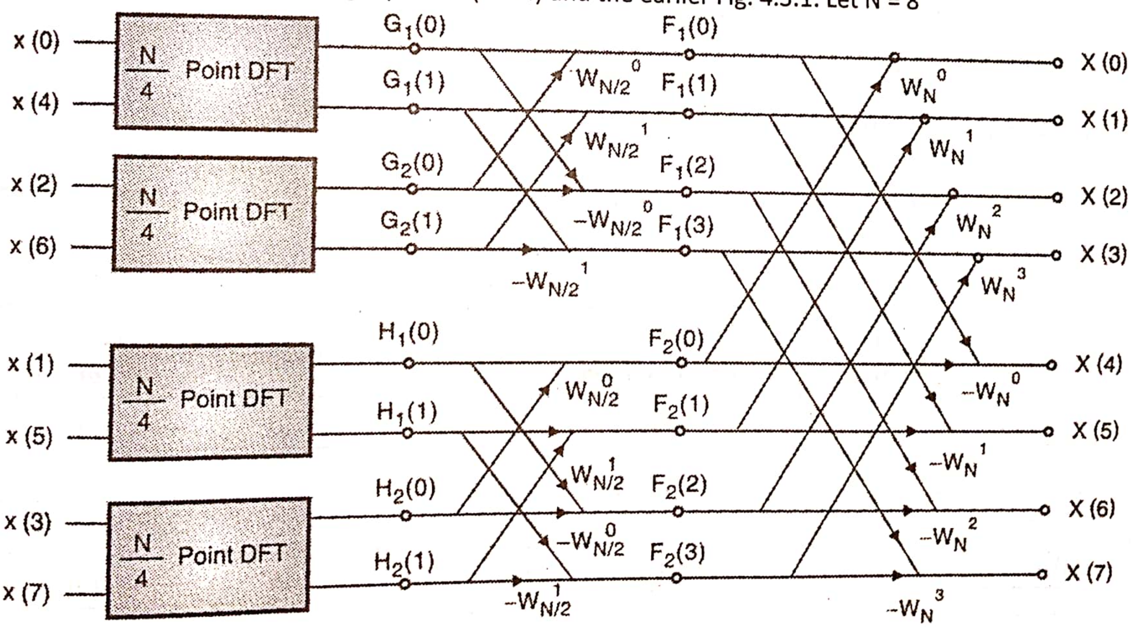

Fig. 4.3.2

A 2-Point DFT is drawn as,

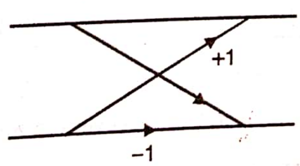

Fig. 4.3.3

Hence the final signal flow graph is shown in Fig. 4.3.4.

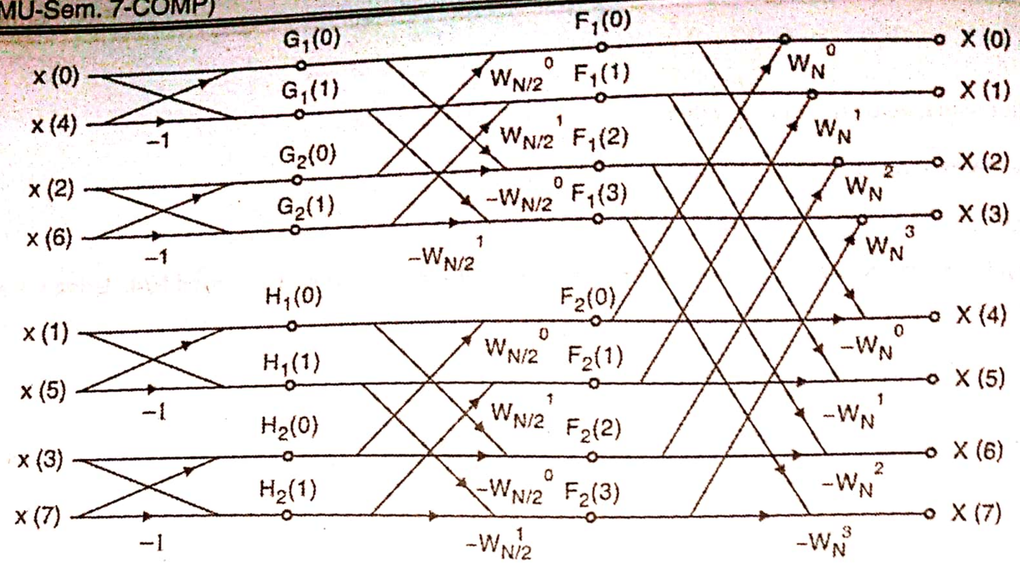

Fig. 4.3.4 : 8-point DIT-FFT Butterfly diagram

This diagram is also called an 8-point butterfly diagram.

The DIT-FFT refers to the method in which the input samples are broken into odd and even parts. This time decimation leads to the scrambled order of the input.

The first decimation gives x(0), x(2), x(4), x(6), x(1)....

The second decimation gives x(0), x(4), x(2), x(6)....

4.4 Bit Reversal Format

The shuffling of the input data is known as bit-reversal. This is because the order of the input data can be obtained by reversing the bits of the binary representation of the normal input data index order

$x(4) = 100 \rightarrow BR \rightarrow 001 \rightarrow x(1)$

	Original Samples	Binary Representation	Bit reversed	Decimal	New order of input
x(0) ⟶	0	000	000	0	⟶ x(0)
x(1) ⟶	1	001	100	4	⟶ x(4)
x(2) ⟶	2	010	010	2	⟶ x(2)
x(3) ⟶	3	011	110	6	⟶ x(6)
x(4) ⟶	4	100	001	1	⟶ x(1)
x(5) ⟶	5	101	101	5	⟶ x(5)
x(6) ⟶	6	110	011	3	⟶ x(3)
x(7) ⟶	7	111	111	7	⟶ x(7)

Hence the input that is fed to the algorithm is in the bit reversed format.

4.4.1 Solved Examples on Bit Reversal Format

Ex. 4.4.1 : An 8-point sequence x(n) is given by , {0, 1, 2, 3, 2, 1, 5, 2, 1}

Find the DFT of the sequence using DIT-FFT.

Soln. : Since N = 8, we draw a 8-point DIT-FFT Butterfly diagram remember, the input to the network is in the bit reversed order.

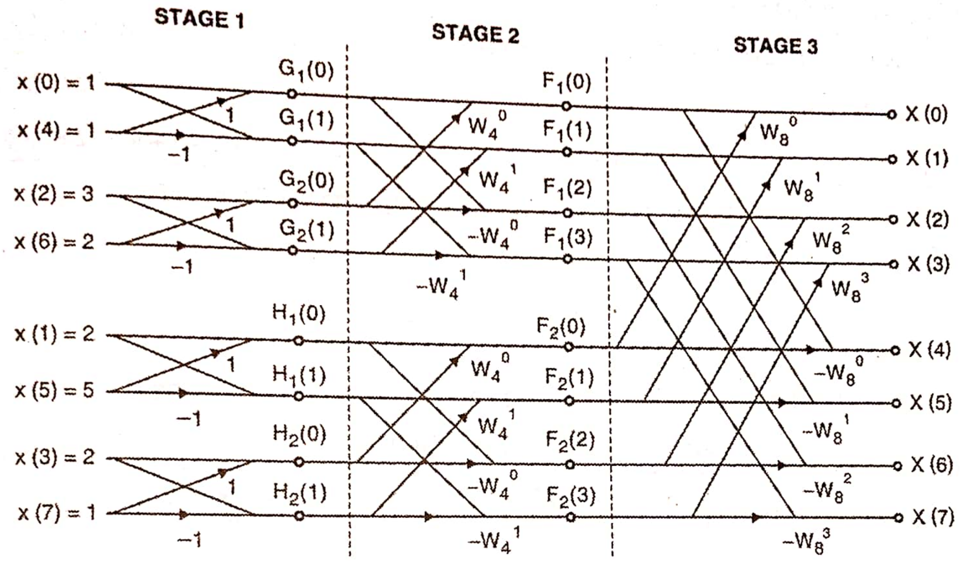

Fig. P.4.4.1

$$x(n) = \{1, 2, 3, 2, 1, 5, 2, 1\}$$

x(0) x(1) x(2) x(3) x(4) x(5) x(6) x(7)

Here $W_4^0 = 1$

$$W_4^1 = e^{\frac{-j2\pi 1}{4}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Similarly,

$$W_8^0 = 1$$

$$W_8^1 = e^{\frac{-j2\pi \cdot 1}{8}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j0.707$$

$$W_8^2 = e^{\frac{-j2\pi \cdot 2}{8}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi \cdot 3}{8}} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j0.707$$

We write down the equations at each stage and calculate the outputs.

Stage 1

$$G_1(0) = x(0) + x(4) = 1 + 1 = 2$$

$$G_1(1) = x(0) - x(4) = 1 - 1 = 0$$

$$G_2(0) = x(2) + x(6) = 3 + 2 = 5$$

$$G_2(1) = x(2) - x(6) = 3 - 2 = 1$$

$$H_1(0) = x(1) + x(5) = 2 + 5 = 7$$

$$H_1(1) = x(1) - x(5) = 2 - 5 = -3$$

$$H_2(0) = x(3) + x(7) = 2 + 1 = 3$$

$$H_2(1) = x(3) - x(7) = 2 - 1 = 1$$

Stage 2

$$F_1(0) = G_1(0) + W_4^0 G_2(0) = 2 + 1 \cdot (5) = 7$$

$$F_1(1) = G_1(1) + W_4^1 G_2(1) = 0 + (-j)(1) = -j$$

$$F_1(2) = G_1(0) - W_4^0 G_2(0) = 2 - 1 \cdot (5) = -3$$

$$F_1(3) = G_1(1) - W_4^1 G_2(1) = 0 - (-j)(1) = +j$$

$$F_2(0) = H_1(0) + W_4^0 H_2(0) = 7 + 1 \cdot (3) = 10$$

$$F_2(1) = H_1(1) + W_4^1 H_2(1) = -3 + (-j)(1) = -3 - j$$

$$F_2(2) = H_1(0) - W_4^0 H_2(0) = 7 - 1 \cdot (3) = 4$$

$$F_2(3) = H_1(1) - W_4^1 H_2(1) = -3 - (-j)(1) = -3 + j$$

Stage 3

$$X(0) = F_1(0) + W_8^0 F_2(0) = 7 + 1 \cdot (10) = 17$$

$$X(1) = F_1(1) + W_8^1 F_2(1) = -j + (0.707 - j\,0.707)(-3 - j) = -2.83 + j0.41$$

$$X(2) = F_1(2) + W_8^2 F_2(2) = -3 + (-j)(4) = -3 - j4$$

$$X(3) = F_1(3) + W_8^3 F_2(3) = j + (-0.707 - j0.707)(-3 + j) = 2.83 + j2.41$$

$$X(4) = F_1(0) - W_8^0 F_2(0) = 7 - 1 \cdot (10) = -3$$

$$X(5) = F_1(1) - W_8^1 F_2(1) = -j-(0.707 - j0.707)(-3-j) = 2.83 - j2.41$$

$$X(6) = F_1(2) - W_8^2 F_2(2) = -3-(-j)(4) = -3+j4$$

$$X(7) = F_1(3) - W_8^3 F_2(3) = j-(-0.707-j0.707)(-3+j) = -2.83 - j0.41$$

$$\therefore \quad X(k) = \{17, -2.83+j0.41, -3-j4, 2.83+j2.41, -3, 2.83-j2.41, -3+j4, -2.83-j0.41\}$$

Ex. 4.4.2 Find s-point DIT of the sequence x(n) = {1, 2, 3, 4, 4, 3, 2, 1} using DIT Radix 2 FFT

Soln. : Since N = 8, we draw a 8-point DIT-FFT diagram

$$x(n) = \{\underset{x(0)}{1}, \underset{x(1)}{2}, \underset{x(2)}{3}, \underset{x(3)}{4}, \underset{x(4)}{4}, \underset{x(5)}{3}, \underset{x(6)}{2}, \underset{x(7)}{1}\}$$

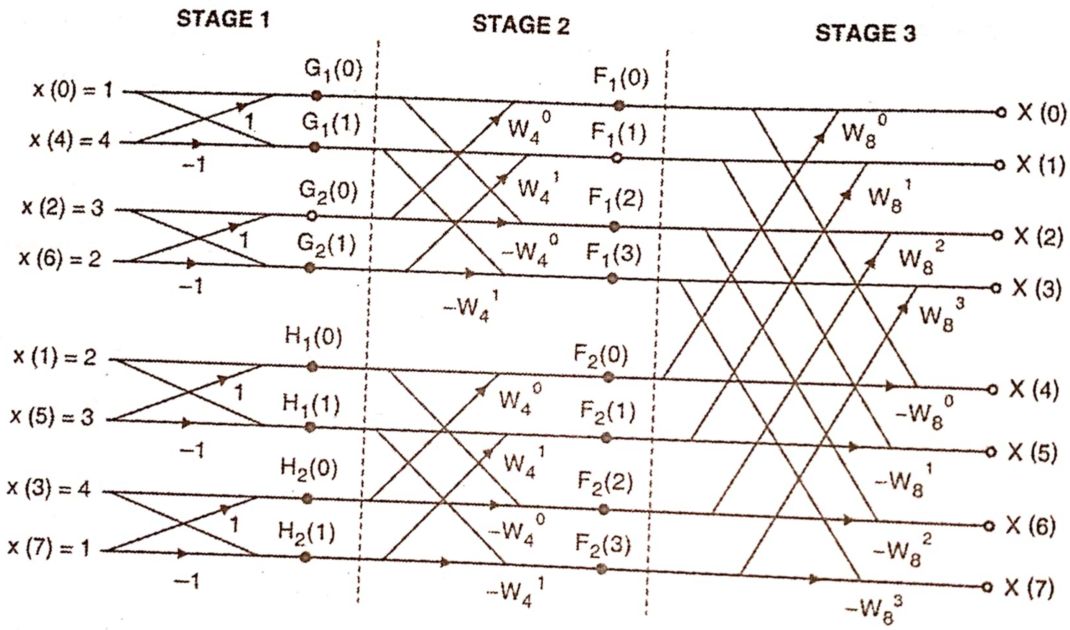

Fig. P. 4.4.2

Here $W_4^0 = 1$

$$W_4^1 = e^{\frac{-j2\pi}{4} \cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Similarly,

$$W_8^0 = 1$$

$$W_8^1 = e^{\frac{-j2\pi}{4} \cdot 1} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j0.707$$

$$W_8^2 = e^{\frac{-j2\pi}{4} \cdot 2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi}{4} \cdot 3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j0.707$$

We write down the equation at each stage and calculate the outputs.

Stage 1

$$G_1(0) = x(0) + x(4) = 1 + 4 = 5$$

$$G_1(1) = x(0) - x(4) = 1 - 4 = -3$$

$$G_2(0) = x(2) + x(6) = 3 + 2 = 5$$

$$G_2(1) = x(2) - x(6) = 3 - 2 = 1$$

$$H_1(0) = x(1) + x(5) = 2 + 3 = 5$$

$$H_1(1) = x(1) - x(5) = 2 - 3 = -1$$

$$H_2(0) = x(3) + x(7) = 4 + 1 = 5$$

$$H_2(1) = x(3) - x(7) = 4 - 1 = 3$$

Stage 2

$$F_1(0) = G_1(0) + W_4^0 G_2(0) = 5 + 1(5) = 10$$

$$F_1(1) = G_1(1) + W_4^1 G_2(1) = -3 + (-j)(1) = -3-j$$

$$F_1(2) = G_1(0) - W_4^0 G_2(0) = 5 - 1(5) = 0$$

$F_1(3) = G_1(1) - W_4^1 G_2(1) = -3-(-j)(1) = -3+j$

$F_2(0) = H_1(0) + W_4^0 H_2(0) = 5+1(5) = 10$

$F_2(1) = H_1(1) + W_4^1 H_2(1) = -1+(-j)(3) = -3-j3$

$F_2(2) = H_1(0) - W_4^0 H_2(0) = 5-1(5) = 0$

$F_2(3) = H_1(1) - W_4^1 H_2(1) = -1-(-j)(3) = -1+j3$

Stage 3

$X(0) = F_1(0) + W_8^0 F_2(0) = 10+(1)(10) = 20$

$X(1) = F_1(1) + W_8^1 F_2(1)$

$= (-3-j) + (0.707 - j\,0.707)(-1-j3)$

$= -5.825 - j\,2.414$

$X(2) = F_1(2) + W_8^2 F_2(2)$

$= 0 + (-j)(0) = 0$

$X(3) = F_1(3) + W_8^3 F_2 + (3)$

$= (-3+j) + (-0.707 - j0.707)(-1+j3)$

$= -0.172 - j\,0.414$

$X(4) = F_1(0) - W_8^0 F_2(0) = 10-(1)(10) = 0$

$X(5) = F_1(1) - W_8^1 F_2(1)$

$= (-3-j) - (0.707 - j0.707)(-1-j3)$

$= -0.172 + j0.414$

$X(6) = F_1(2) - W_8^2 F_2(2) = 0-(-j)(0) = 0$

$X(7) = F_1(3) - W_8^3 F_2(3)$

$= (-3-j) - (-0.707 - j0.707)(-1+j3)$

$= -5.828 + j2.414$

$\therefore \quad X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j\,2.414$

Ex. 4.4.3 : Given $x(n)=2^n$. Find X(k) using DIT-FFT algorithms. Assume x(n) is of length 8.

Soln. : Here N=8we vary n from 0to 7.

$\therefore \; x(n) = 2^n, \quad 0 \le n < 7$

$\therefore \; x(0) = 2^0 = 1$

$\therefore \; x(1) = 2^1 = 2$

$\therefore \; x(2) = 2^2 = 4$

$\therefore \; x(3) = 2^3 = 8$

$\therefore \; x(4) = 2^4 = 16$

$\therefore \; x(5) = 2^5 = 32$

$\therefore \; x(6) = 2^6 = 64$

$\therefore \; x(7) = 2^7 = 128$

$\therefore \; x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$

Since N= 8, we draw a 8 point DIT-FFT. Remember the input is the bit-reversed format.

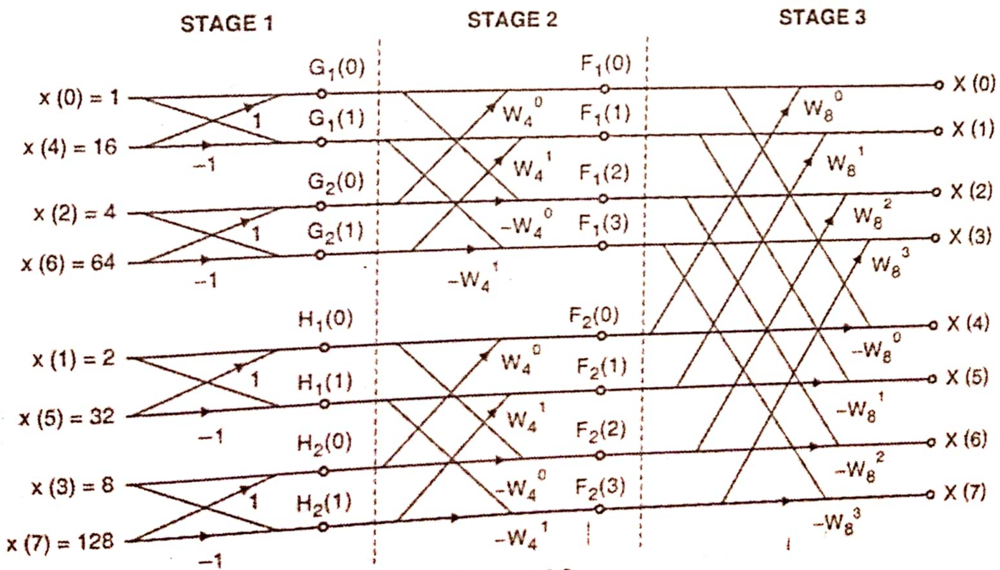

Fig. P.4.4.3

Here $\quad W_4^0 = 1$

$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$

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Similarly, $W_8^0 = 1$

$$W_8^1 = e^{\frac{-j2\pi}{8}1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$

$$= 0.707 - j\,0.707$$

$$W_8^2 = e^{\frac{-j2\pi}{8}2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi}{8}3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}$$

$$= -0.707 - j\,0.707$$

We write down the equations at each stage

Stage 1

$G_1(0) = x(0) + x(4) = 1 + 16 = 17$

$G_1(1) = x(0) - x(4) = 1-16 = -15$

$G_2(0) = x(2) + x(6) = 4 + 64 = 68$

$G_2(1) = x(2) - x(6) = 4 - 64 = -60$

$H_1(0) = x(1) + x(5) = 2 + 32 = 34$

$H_1(1) = x(1) - x(5) = 2 - 32 = -30$

$H_2(0) = x(3) + x(7) = 8 + 128 = 136$

$H_2(1) = x(3) - x(7) = 8-128 = -120$

Stage 2

$F_1(0) = G_1(0) + W_4^0 G_2(0) = 17 + 1 \cdot (68) = 85$

$F_1(1) = G_1(1) + W_4^1 G_2(1)$

$= -15 + (-j)(-60) = -15 + j60$

$F_1(2) = G_1(0) - W_4^0 G_2(0)$

$= 17 - 1(68) = -51$

$F_1(3) = G_1(1) - W_4^1 G_2(1)$

$= -15 - (-j)(-60) = -15 - j\,60$

$F_2(0) = H_1(0) + W_4^0 H_2(0)$

$= 34 + 1 \cdot (136) = 170$

$F_2(1) = H_1(1) + W_4^1 H_2(1)$

$= -30 + (-j)(-120) = -30 + j120$

$F_2(2) = H_1(0) - W_4^0 H_2(0) = 34 - 1 \cdot (136) = -102$

$F_2(3) = H_1(1) - W_4^1 H_2(1)$

$= -30 - (-j)(-120) = -30 - j120$

Stage 3

$X(0) = F_1(0) + W_8^0 F_2(0) = 85 + 1(170) = 255$

$X(1) = F_1(1) + W_8^1 F_2(1)$

$= (-15 + j60) + (0.707 - j0.707)(-30 + j120)$

$= 48.64 + j166.07$

$X(2) = F_1(2) + W_8^2 F_2(2)$

$= -51 + (-j)(-102) = -51 + j102$

$X(3) = F_1(3) + W_8^3 F_2(3)$

$= (-15 - j60) + (-0.707 - j0.707)(-30 - j120)$

$= -78.64 + j46.07$

$X(4) = F_1(0) - W_8^0 F_2(0) = 85 - 1 \cdot (170) = -85$

$X(5) = F_1(1) - W_8^1 F_2(1)$

$= (-15 + j60) - (0.707 - j0.707)(-30 + j\,120)$

$= -78.64 - j46.07$

$X(6) = F_1(2) - W_8^2 F_2(2)$

$= -51 - (-j)(-120) = -51 - j102$

$X(7) = F_1(3) - W_8^3 F_2(3)$

$= (-15 - j60) - (-0.707 - j0.707)(-30 - j120)$

$= 48.6 - j166.07$

$X(k) = \{255, 48.64 + j166.047, -51 + j102,$
$-78.64 + j46.07, -85, -78.64 - j46.07,$
$-51 - j102, 48.6 - j166.07\}$

Ex.4.4.4 : Compute the DFT of x(n)= {1,0,2,0,3,0,4,0} using DIT-FFT.

Soln. : Since N = 8, we drawn a 8-point DIT-FFT Butterfly diagram. The input is in the bit reversed order.

$x(n) = \{1, 0, 2, 0, 3, 0, 4, 0\}$

Here $W_4^0 = 1$

$$W_4^1 = e^{\frac{-j2\pi \cdot 1}{u}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Similarly,

$W_8^0 = 1$

$$W_8^1 = e^{\frac{-j2\pi \cdot 1}{8}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j0.707$$

$$W_8^2 = e^{\frac{-j2\pi \cdot 2}{8}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi \cdot 3}{8}} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j0.707$$

We write down the equation at each stage and calculate the outputs

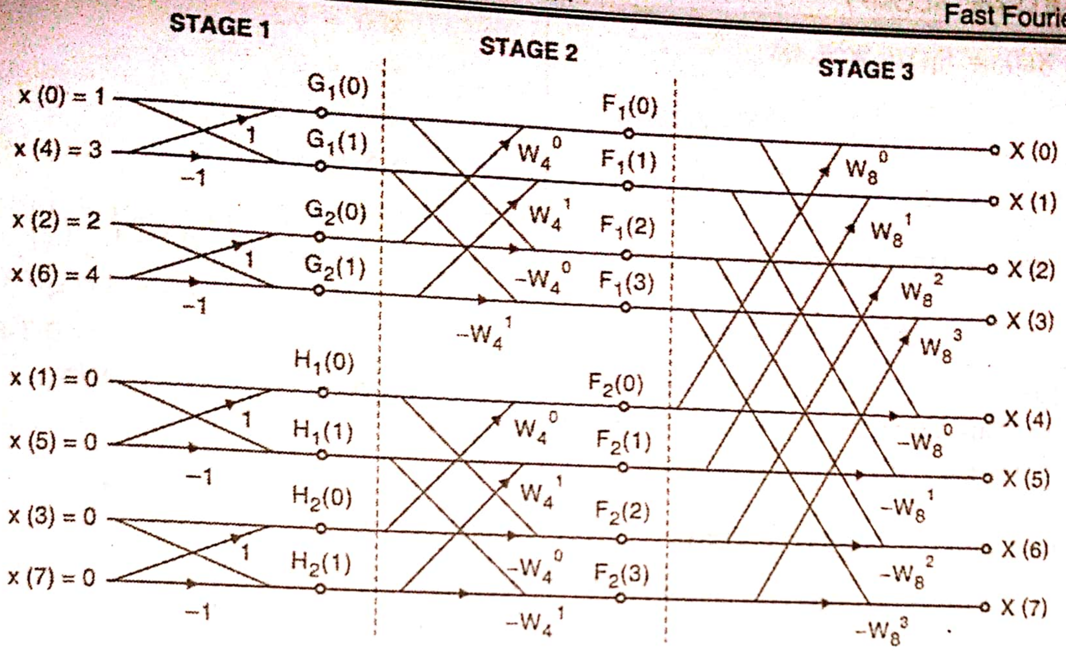

Fig. P. 4.4.4

Stage 1

$$G_1(0) = x(0) + x(4) = 1 + 3 = 4$$

$$G_1(1) = x(0) - x(4) = 1 - 3 = -2$$

$$G_2(0) = x(2) + x(6) = 2 + 4 = 6$$

$$G_2(1) = x(2) - x(6) = 2 - 4 = -2$$

$$H_1(0) = x(1) + x(5) = 0 + 0 = 0$$

$$H_1(1) = x(1) - x(5) = 0 - 0 = 0$$

$$H_2(0) = x(3) + x(7) = 0 + 0 = 0$$

$$H_2(1) = x(3) - x(7) = 0 - 0 = 0$$

Stage 2

$$F_1(0) = G_1(0) + W_4^0 G_2(0) = 4 + 1 \cdot (6) = 10$$

$$F_1(1) = G_1(1) + W_4^1 G_2(1) = -2 + (-j)(-2)$$

$$= -2 + j2$$

$$F_1(2) = G_1(0) - W_4^0 G_2(0) = 4 - 1 \cdot (6) = -2$$

$$F_1(3) = G_1(1) - W_4^1 G_2(1) = -2 - (-j)(-2)$$

$$= -2 - j2$$

$$F_2(0) = H_1(0) + W_4^0 H_2(0) = 0 + 1 \cdot (0) = 0$$

$$F_2(1) = H_1(1) + W_4^1 H_2(1) = 0 + (-j)(0) = 0$$

$$F_2(2) = H_1(0) - W_4^0 H_2(0) = 0 - 1 \cdot (0) = 0$$

$$F_2(3) = H_1(1) - W_4^1 H_2(1) = 0 - (-j)(0) = 0$$

Stage 3

$$X(0) = F_1(0) + W_8^0 F_2(0) = 10 + 1 \cdot (0) = 10$$

$$X(1) = F_1(1) + W_8^1 F_2(1)$$

$$= (-2 + 2j) + (0.707 - j0.707) \cdot 0$$

$$= -2 + j2$$

$$X(2) = F_1(2) + W_8^2 F_2(2) = -2 + (-j)(0) = -2$$

$$X(3) = F_1(3) + W_8^3 F_2(3)$$

$$= (-2 - 2j) + (-0.707 - j\,0.707) \cdot 0$$

$$= -2 - j2$$

$$X(4) = F_1(0) - W_8^0 F_2(0) = 10 - 1 \cdot (0) = 10$$

$$X(5) = F_1(1) - W_8^1 F_2(1)$$

$$= (-2 + 2j) - (0.707 - j\,0.707) \cdot 0$$

$$= -2 + j2$$

$$X(6) = F_1(2) - W_8^2 F_2(2) = -2 - (-j)(0)$$

$$= -2$$

$$X(7) = F_1(3) - W_8^3 F_2(3)$$

$$= (-2-2j) - (-0.707 - j0.707) \cdot 0$$

$$= -2 - j2$$

$$\therefore \quad X(k) = \{10, -2+j2, -2, -2-j2, 10, -2+j2, -2, -2-j2\}$$

Ex. 4.4.5 : Sample the given continuous time signal $x(t) = \sin(2\pi 1000t) + 0.5 \sin(2\pi 1000 t)$ at 8000 sample sec. Find out the Eight point DFT using the DIT-FFT algorithm.

Soln. : $x(t) = \sin(2\pi 1000 t) + 0.5 \sin(2\pi 2000 t)$

To obtain a sampled version of x(t), we replace t by $\frac{n}{F_s}$.

Here F_s = 8000 samples /sec

$$\therefore x(n) = \sin\left(\frac{2\pi 1000 n}{F_s}\right) + 0.5 \sin\left(\frac{2\pi 2000 n}{F_s}\right)$$

$$\therefore x(n) = \sin\left(\frac{2\pi 1000 n}{8000}\right) + 0.5 \sin\left(\frac{2\pi 2000 n}{8000}\right)$$

$$\therefore x(n) = \sin\left(\frac{\pi n}{4}\right) + 0.5 \sin\left(\frac{\pi n}{2}\right) \quad \ldots(1)$$

Since we require a 8 point DFT, we vary n from 0 to 7 in Equation (1),

$$x(0) = \sin\left(\frac{\pi}{4} \cdot 0\right) + 0.5 \text{ Sin}\left(\frac{\pi}{2} \cdot 0\right) = 0$$

Similarly, $x(1) = 1.2$

$x(2) = 1$

$x(3) = 0.2$

$x(4) = 0$

$x(5) = -0.2$

$x(6) = -1$

$x(7) = -1.2$

$$x(n) = \{0, 1.2, 1, 0.2, 0, -0.2, -1, -1.2\}$$

x(0) x(1) x(2) x(3) x(4) x(5) x(6) x(7)

Since N = 8, we draw a 8 – point DIT-FFT Butterfly diagram. The input is in bit reversed order.

Here, $W_4^0 = 1$

$$W_4^1 = e^{\frac{-j2\pi \cdot 1}{4}} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2} = -j$$

Similarly,

$$W_8^0 = 1$$

$$W_8^1 = e^{\frac{-j2\pi \cdot 1}{8}} = \cos\frac{\pi}{4} - j \sin\frac{\pi}{4} = 0.707 - j\,0.707$$

$$W_8^2 = e^{\frac{-j2\pi \cdot 2}{8}} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi \cdot 3}{8}} = \cos\frac{3\pi}{4} - j \sin\frac{3\pi}{4}$$

$$= -0.707 - j\,0.707$$

We write down the equations at each stage and calculate the outputs.

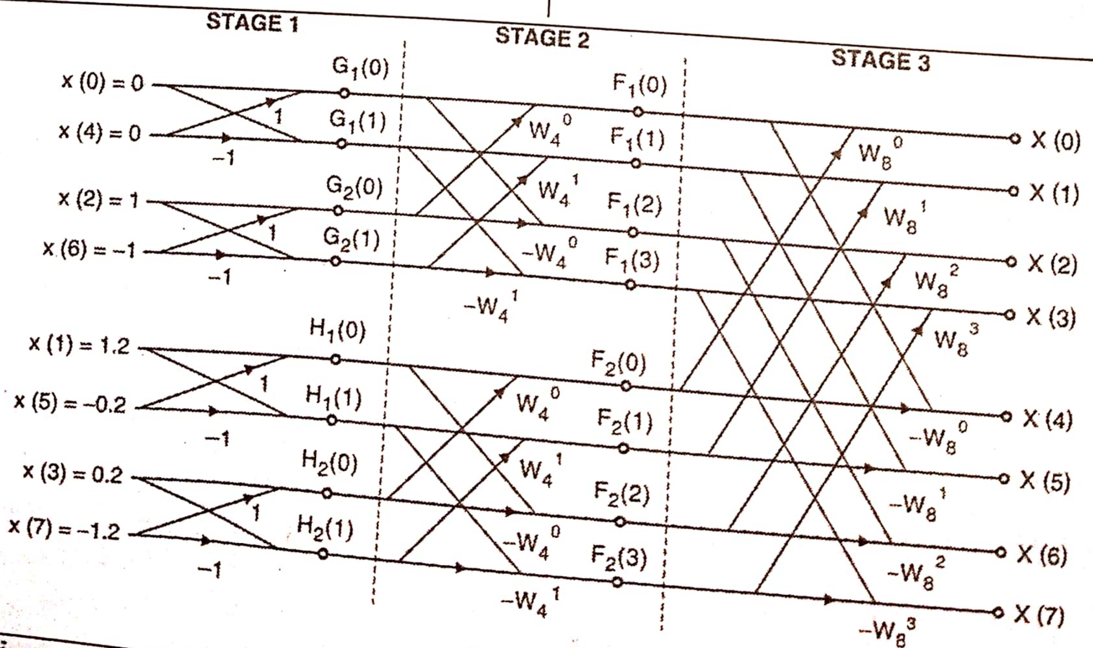

Fig. P.4.4.5

Stage 1

$$G_1(0) = x(0) + x(4) = 0 + 0 = 0$$
$$G_1(1) = x(0) - x(4) = 0 - 0 = 0$$
$$G_2(0) = x(2) + x(6) = 1 + (-1) = 0$$
$$G_2(1) = x(2) - x(6) = 1 - (-1) = 2$$
$$H_1(0) = x(1) + x(5) = 1.2 + (-0.2) = 1$$
$$H_1(1) = x(1) - x(5) = 1.2 - (-0.2) = 1.4$$
$$H_2(0) = x(3) + x(7) = 0.2 + (-1.2) = -1$$
$$H_2(1) = x(3) - x(7) = 0.2 - (-1.2) = 1.4$$

Stage 2

$$F_1(0) = G_1(0) + W_4^0 G_2(0) = 0 + 1 \cdot (0) = 0$$
$$F_1(1) = G_1(1) + W_4^1 G_2(1) = 0 + (-j)(2) = -j2$$
$$F_1(2) = G_1(0) - W_4^0 G_2(0) = 0 - 1 \cdot (0) = 0$$
$$F_1(3) = G_1(1) - W_4^1 G_2(1) = 0 - (-j)(2) = +j2$$
$$F_2(0) = H_1(0) + W_4^0 H_2(0) = 1 + 1(-1) = 0$$
$$F_2(1) = H_1(1) + W_4^1 H_2(1)$$
$$= 1.4 + (-j)(1.4) = 1.4 - j1.4$$
$$F_2(2) = H_1(0) - W_4^0 H_2(0)$$
$$= 1 - 1(-1) = 2$$
$$F_2(3) = H_1(1) - W_4^1 H_2(1)$$
$$= 1.4 - (-j)(1.4) = 1.4 + j1.4$$

Stage 3

$$X(0) = F_1(0) + W_8^0 F_2(0) = 0 + 1 \cdot (0) = 0$$
$$X(1) = F_1(1) + W_8^1 F_2(1)$$
$$= -j2 + (0.707 - j0.707)(1.4 - j1.4) = -j3.98$$
$$X(2) = F_1(2) + W_8^2 F_2(2) = 0 + (-j)(2) = -j2$$
$$X(3) = F_1(3) + W_8^3 F_2(3)$$
$$= j2 + (-0.707 - j0.707)(1.4 + j1.4)$$
$$= j0.02$$
$$X(4) = F_1(0) - W_8^0 F_2(0) = 0 - 1(0) = 0$$
$$X(5) = F_1(1) - W_8^1 F_2(1)$$
$$= -j2 - (0.707 - j0.707)(1.4 - j1.4) = -j0.02$$
$$X(6) = F_1(2) - W_8^2 F_2(2) = 0 - (-j)(2) = j2$$
$$X(7) = F_1(3) - W_8^3 F_2(3)$$
$$= j2 - (-0.707 - j0.707)(1.4 + j1.4)$$
$$= j0.398$$
$$\therefore \quad X(k) = \{0, -j3.98, -2j, j0.02, 0 - j0.02, j2, j0.398\}$$

Ex. 4.4.6 : An eight point sequence x(n)is given by $x_1(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

(i) Find DFT of $x_1(n)$ using any of the FFT technique

(ii) Let $x_2(n) = \{5, 6, 7, 8, 1, 2, 3, 4\}$

Using appropriate DFT property and answer of earlier part determine $X_2(k)$

(iii) Again we DFT property and Find $x_3(n) = x_1(n) + x_2(n)$

Soln. :

(i) Computing the DFT of $X_1(n)$:

We use the DIT-FFT. Since N = 8, we draw the 8-point DIT-FFT Butterfly diagram. The input is in the bit reversed order.

$$x_1(n) = \{\underset{x(0)}{1}\ \underset{x(1)}{2}\ \underset{x(2)}{3}\ \underset{x(3)}{4}\ \underset{x(4)}{5}\ \underset{x(5)}{6}\ \underset{x(6)}{7}\ \underset{x(7)}{8}\}$$

Here, $W_4^0 = 1$

$$W_4^1 = e^{\frac{-j2\pi 1}{4}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Similarly,

$$W_8^0 = 1$$
$$W_8^1 = e^{\frac{-j2\pi \cdot 1}{8}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j\,0.707$$
$$W_8^2 = e^{\frac{-j2\pi \cdot 2}{8}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$
$$W_8^3 = e^{\frac{-j2\pi \cdot 3}{8}} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}$$
$$= -0.707 - j\,0.707$$

We write down the equations at each stage and calculate the outputs.

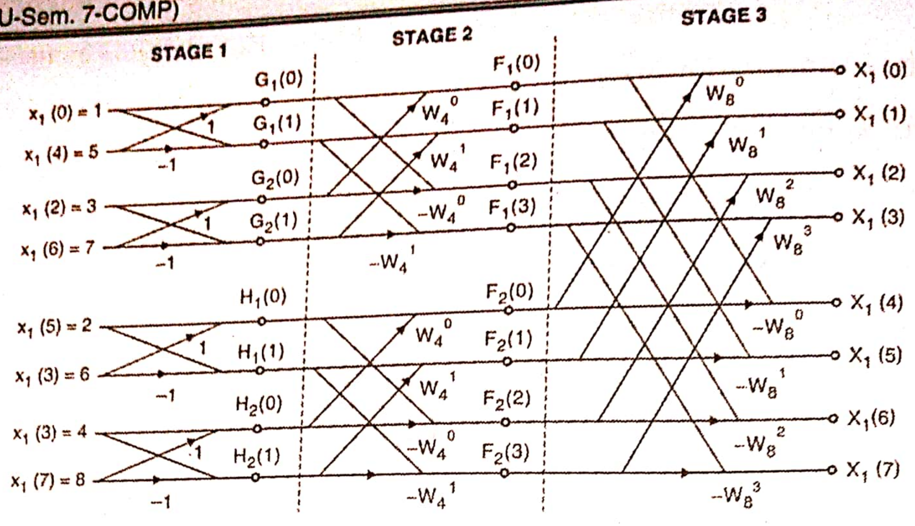

Fig. P. 4.4.6

Stage 1

$$G_1(0) = x_1(0)+x_1(4) = 1+5 = 6$$

$$G_1(1) = x_1(0)-x_1(4) = 1-5 = -4$$

$$G_2(0) = x_1(2)+x_1(6) = 3+7 = 10$$

$$G_2(1) = x_1(2)-x_1(6) = 3-7 = -4$$

$$H_1(0) = x_1(1)+x_1(5) = 2+6 = 8$$

$$H_1(1) = x_1(1)-x_1(5) = 2-6 = -4$$

$$H_2(0) = x_1(3)+x_1(7) = 4+8 = 12$$

$$H_2(1) = x_1(3)-x_1(7) = 4-8 = -12$$

Stage 2

$$F_1(0) = G_1(0)+W_4^0G_2(0) = 6+1\cdot(10) = 16$$

$$F_1(1) = G_1(1)+W_4^1G_2(1) = -4+(-j)(-4) = -4+j4$$

$$F_1(2) = G_1(0)-W_4^0G_2(0) = 6-1\cdot(10) = -4$$

$$F_1(3) = G_1(1)-W_4^1G_2(1) = -4-(-j)(-4) = -4-j4$$

$$F_2(0) = H_1(0)+W_4^0H_2(0) = 8+1\cdot(12) = 20$$

$$F_2(1) = H_1(1)+W_4^1H_2(1) = -4+(-j)(-12) = -4+j12$$

$$F_2(2) = H_1(0)-W_4^0H_2(0) = 8-1\cdot(12) = -4$$

$$F_2(3) = H_1(1)-W_4^1H_2(1) = -4-(-j)(-12) = -4-j12$$

Stage 3

$$X_1(0) = F_1(0)+W_8^0F_2(0) = 16+1\cdot(20) = 36$$

$$X_1(1) = F_1(1)+W_8^1F_2(1)$$

$$= (-4+j4)+(0.707-j0.707)(-4+j12)$$

$$X_1(2) = F_1(2)+W_8^2F_2(2) = -4+(-j)(-4) = -4+j4$$

$$X_1(3) = F_1(3)+W_8^3F_2(3)$$

$$= (-4+j4)+(-0.707-j0.707)(-4+j12)$$

$$X_1(4) = F_1(0)-W_8^0F_2(0) = 16-1\cdot(20) = -4$$

$$X_1(5) = F_1(1)-W_8^1F_2(1)$$

$$= (-4+j4)-(0.707-j0.707)(-4+j12)$$

$$X_1(6) = F_1(2)-W_8^2F_2(2) = -4-(-j)(-4) = -4-j4$$

$$X_1(7) = F_1(3)-W_8^3F_2(3)$$

$$= (-4-j12)-(-0.707-j0.707)(-4-j12)$$

$$X(k) = \{36, -4+j9.65, -4+j4, -4+j1.65, -4, -4-j1.65, -4-j4, -4-j9.656\}$$

(ii) Compute the DFT of $x_2(n)$ using appropriate DFT property.

Here, $x_1(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

and $x_2(n) = \{5, 6, 7, 8, 1, 2, 3, 4\}$

Imagine $x_1(n)$ and $x_2(n)$ to lie on 2 circles.

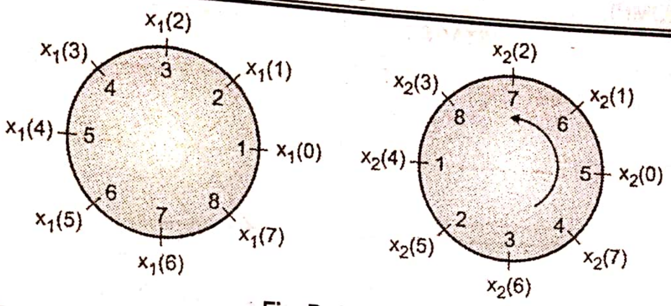

Fig. P. 4.4.6(a)

If we rotate the $x_2(n)$ circle in the clockwise direction by 4 steps, we would get the $x_1(n)$.circle.

Hence, $x_2(n)$ is a delayed version of $x_1(n)$.

$$\therefore \quad x_2(n) = x_1((n-4))$$

From the time shifting property of the DFT we know,

$$\text{If } x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$\text{Then } x((n-m))_N \xleftrightarrow[N]{DFT} e^{\frac{-j2\pi km}{N}} X(k)$$

i.e. $X_2(k) = \text{DFT}\{x_2(n)\} = \text{DFT}\{x_1((n-4))\}$

$$= e^{\frac{-j2\pi km}{N}} X_1(k)$$

Since the delay is of 4 units, $m = 4$, $N = 8$.

$\therefore$ We can write

$$X_2(k) = e^{\frac{-j2\pi 4 \cdot k}{8}} X_1(k)$$

$$\therefore \quad X_2(k) = X_1(k)\, e^{-j\pi k}$$

We have calculated $X_1(k)$ in the previous part of the example.

We vary k from 0 to 7

$$X_2(0) = e^{-j\pi \cdot 0} X_1(0) = (1) \cdot X_1(0) = (1).36 = 36$$

$$X_2(1) = e^{-j\pi \cdot 1} X_1(1) = (-1) \cdot X_1(1)$$

$$= (-1)(-4+j9.65) = 4-j9.65$$

$$X_2(2) = e^{-j\pi \cdot 2} X_1(2) = (1)\, X_1(2) = (1)(-4+j4)$$

$$= -4+j4$$

$$X_2(3) = e^{-j\pi \cdot 3} X_1(3) = (-1)\, X_1(3)$$

$$= (-1)(-4+j1.65) = 4-j1.65$$

$$X_2(4) = e^{-j\pi \cdot 4} X_1(4) = (1)\, X_1(4) = (1)(-4) = -4$$

$$X_2(5) = e^{-j\pi \cdot 5} X_1(5) = (1)\, X_1(5) = (-1)(-4-j\,1.65)$$

$$= 4+j1.65$$

$$X_2(6) = e^{-j\pi \cdot 6} X_1(6) = (1)\, X_1(6) = (1)(-4-j4)$$

$$= -4-j4$$

$$X_2(7) = e^{-j\pi \cdot 7} X_1(7) = (-1)\, X_1(7) = (-1)(-4-j9.65)$$

$$= 4+j9.65$$

$$\therefore \quad X_2(k) = \{36, 4-j9.65, -4+j4, 4-j1.65, -4, 4+j1.65, -4-j4, 4+j\,9.65\}$$

(iii) Compute $X_3(k)$ if $x_3(n) = x_1(n) + x_2(n)$.

We use the linearity property which is as follows then

$$\text{If } \quad x(n) \xleftrightarrow{DFT} X(k)$$

$$a x_1(n) + b\, x_2(n) \xleftrightarrow{DFT} a\, X_1(k) + b\, X_2(k)$$

Since, $x_3(n) = x_1(n) + x_2(n)$

$$X_3(k) = \text{DFT}\{x_3(n)\} = \text{DFT}\{x_1(n) + x_2(n)\}$$

$$\therefore \quad X_3(k) = X_1(k) + X_2(k)$$

$$X_3(0) = X_1(0) + X_2(0) = 36 + 36 = 72$$

$$X_3(1) = X_1(1) + X_2(1) = (-4+j9.65) + (-4-j\,9.65)$$

$$= 0$$

$$X_3(2) = X_1(2) + X_2(2) = (-4+j4) + (-4-j\,4)$$

$$= -8+j8$$

$$X_3(3) = X_1(3) + X_2(3) = (-4+j\,1.65) + (4-j1.65) = 0$$

$$X_3(4) = X_1(4) + X_2(4) = (-4) + (-4) = -8$$

$$X_3(5) = X_1(5) + X_2(5) = (-4-j\,1.65) + (4+j\,1.65) = 0$$

$$X_3(6) = X_1(6) + X_2(6) = (-4-j4) + (-4-j4) = -8-j8$$

$$X_3(7) = X_1(7) + X_2(7) = (-4-j9.65) + (4+j9.65) = 0$$

$$\therefore \quad X_3(k) = \{72, 0, -8+j8, 0, -8, 0, -8-j8, 0\}$$

Ex. 4.4.7 : Compute the DFT of the sequence $X(0) = \{1, 2, 3, 4\}$. Use DIT-FFT algorithm.

Soln. : Since the length of the sequence is 4, N = 4. We use the 4 - point DIT-FFT Butterfly diagram. The input is in the bit reversed order,

$$W_4^0 = e^{\frac{j2\pi}{4} \cdot 0} = 1$$

$$W_4^1 = e^{\frac{-j\pi}{4} \cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

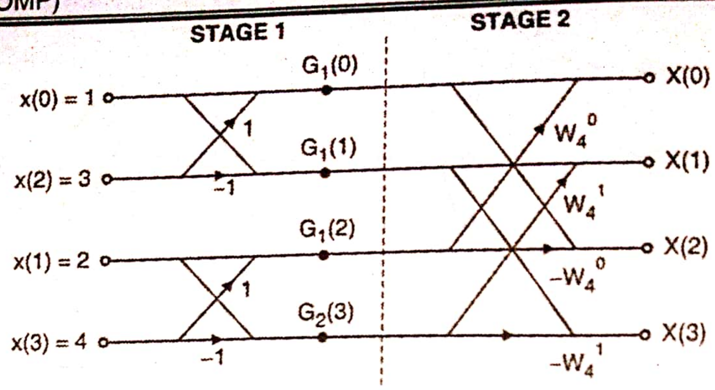

Fig. P. 4.4.7

Stage 1

$$G_1(0) = x(0) + x(2) = 1 + 3 = 4$$

$$G_1(1) = x(0) - x(2) = 1 - 3 = -2$$

$$G_1(2) = x(1) + x(3) = 2 + 4 = 6$$

$$G_1(3) = x(1) - x(3) = 2 - 4 = -2$$

Stage 2

$$X(0) = G_1(0) + W_4^0 G_1(2) = 4 + 1(6) = 10$$

$$X(1) = G_1(1) + W_4^1 G_1(3) = -2 + (-j)(-2)$$

$$= -2 + j2$$

$$X(2) = G_1(0) - W_4^0 G_1(2) = 4 - 1(6) = -2$$

$$X(3) = G_1(1) - W_4^1 G_1(3) = -2 - (-j)(-2)$$

$$= -2 - j2$$

$$X(k) = \{10, -2 + j2, -2, -2 - j2\}$$

All along we have observed that in DIT-FFT, the input gets reordered and the output is in order.

Ex. 4.4.8 : Compute 4-point DFT of the sequence given by $x(n) = (-1)^n$ using DIT-FFT algorithms.

Soln. Since N = 4, we vary n from 0 to 3

$$\therefore\ x(n) = (-1)^n$$

$$\therefore\ x(0) = (-1)^0 = 1$$

$$\therefore\ x(1) = (-1)^1 = -1$$

$$\therefore\ x(2) = (-1)^2 = 1$$

$$\therefore\ x(3) = (-1)^3 = 1$$

$$\therefore\ x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$$

$$x(n) = \{\underset{x(0)}{1}, \underset{x(1)}{-1}, \underset{x(2)}{1}, \underset{x(3)}{-1}\}$$

The twiddle factors required are

$$W_4^0 = e^{\frac{-j2\pi}{4}\cdot 0} = 1$$

$$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

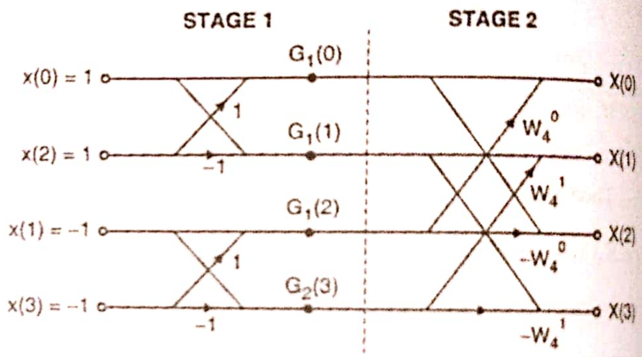

Fig. P. 4.4.8

Stage 1

$$G_1(0) = x(0) + x(2) = 1 + 1 = 2$$

$$G_1(1) = x(0) - x(2) = 1 - 1 = 0$$

$$G_1(2) = x(1) + x(3) = -1 - 1 = -2$$

$$G_1(3) = x(1) - x(3) = -1 - (-1) = 0$$

Stage 2

$$X(0) = G_1(0) + W_4^0 G_1(2) = 2 + (1)(-2) = 0$$

$$X(1) = G_1(1) + W_4^1 G_1(3) = 0 + (-j)(0) = 0$$

$$X(2) = G_1(0) + W_4^0 G_1(2) = 2 - (1)(-2) = 4$$

$$X(3) = G_1(0) + W_4^1 G_1(3) = 2 - (-j)(0) = 0$$

$$\therefore X(k) = \{0, 0, 4, 0\}$$

Ex. 4.4.9 : Compute 4-point DFT of the sequence $x(n) = \{1, 2, 3, 1\}$ using DIT – FFt Radix-2 algorithms.

Soln. :

Here, N = 4

$$x(n) = \{\underset{x(0)}{1}, \underset{x(1)}{2}, \underset{x(2)}{3}, \underset{x(3)}{1}\}$$

The twiddle factors required are

$$W_4^0 = e^{\frac{-j2\pi}{4}\cdot 0} = 1$$

$$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

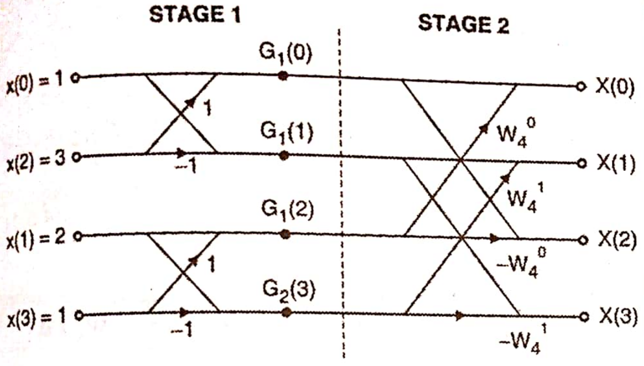

Fig. P. 4.4.9

Stage 1

$$G_1(0) = x(0) + x(2) = 1 + 3 = 4$$

$$G_1(1) = x(0) - x(2) = 1 - 3 = -2$$

$$G_1(2) = x(1) + x(3) = 2 + 1 = 3$$

$$G_1(3) = x(1) - x(3) = 2 - 1 = 1$$

Stage 2

$$X(0) = G_1(0) + W_4^0\,G_1(2) = 4 + (1)\,(3) = 7$$

$$X(1) = G_1(1) + W_4^1\,G_1(3) = -2 + (-j)\,(1) = -2 - j$$

$$X(2) = G_1(0) - W_4^0\,G_1(2) = 4 - (1)\,(3) = 1$$

$$X(3) = G_1(1) - W_4^1\,G_1(3) = -2 - (-j)\,(1) = -2 + j$$

$$\therefore X(k) = \{7,\ -2-j,\ 1,\ -2+j\}$$

Ex. 4.4.10 : Given x(n) = [0, 1, 2, 3].

Compute X(k) using DIT – FFT.

Soln. :

Here, N = 4

$$x(n) = \{\underset{x(0)}{0},\ \underset{x(1)}{1},\ \underset{x(2)}{2},\ \underset{x(3)}{3}\}$$

The twiddle factors required are

$$W_4^0 = e^{\frac{-j2\pi}{4}\cdot 0} = 1$$

$$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

We draw a 4-points DIT – FFT Butterfly

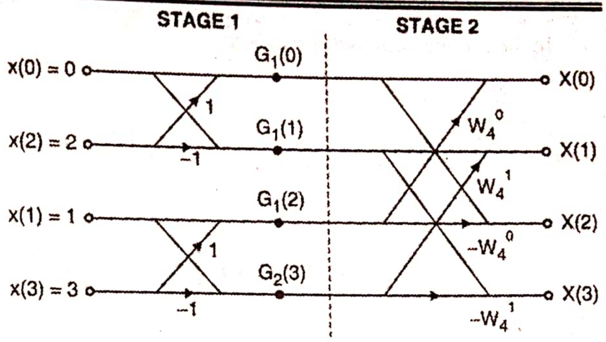

Fig. P. 4.4.10

Stage 1

$$G_1(0) = x(0) + x(2) = 0 + 2 = 2$$

$$G_1(1) = x(0) - x(2) = 0 - 2 = -2$$

$$G_1(2) = x(2) + x(3) = 1 + 3 = 4$$

$$G_1(3) = x(1) - x(3) = 1 - 3 = -2$$

Stage 2

$$X(0) = G_1(0) + W_4^0\,G_1(2) = 2 + 1\,(4) = 6$$

$$X(1) = G_1(1) + W_4^1\,G_1(3)$$

$$= -2 + (-j)\,(-2) = -2 + j2$$

$$X(2) = G_1(0) + W_4^0\,G_1(2) = 2 - 1\,(4) = -2$$

$$X(3) = G_1(1) + W_4^1\,G_1(3) = -2 - (-j)\,(-2) = -2 - j2$$

$$\therefore X(k) = \{6, -2 + j2, -2, -2 - j2\}$$

4.5 Decimation in Frequency FFT (DIT-FFT)

- DIF-FFT is another important FFT-algorithm. Decimation in frequency implies retaining the order of the input sequence but getting an output which is shuffled up.
- In DIF-FFT, we split up the input sequence into the first N/2 data points and the second N/2 data points (instead of odd and even as was the case in DIT-FFT).

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{nk} + \sum_{n'=\frac{N}{2}}^{N-1} x(n')\, W_N^{n'k}$$

In the second sum, put $n' = n + \frac{N}{2}$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{(n+N/2)\,k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{nk}\, W_N^{nk}\, W_N^{kN/2}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{nk} + W_N^{kN/2} \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{nk}$$

Now

$$W_N^{kN/2} = (-1)^k$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{nk} + (-1)^k \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{nk}$$

$$\therefore X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k\, x\left(n + \frac{N}{2}\right)\right] W_N^{nk}$$

We now split X(k) into odd and even parts,

$$(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n + \frac{N}{2}\right)\right] W_N^{2nk} \qquad (\because\ (-1)^{2k} = +1)$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^{(2k+1)n} \qquad (\because\ (-1)^{2k+1} = -1)$$

$$\text{Now } W_N^2 = W_{N/2}^1$$

$$\therefore X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n + \frac{N}{2}\right)\right] W_{N/2}^{nk};$$

$$k = 0, 1, \ldots \frac{N}{2} - 1 \quad \ldots(4.5.1)$$

$$\therefore X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left\{\left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^n\right\} W_{N/2}^{nk}$$

$$;\ k = 0, 1, 2, \ldots \frac{N}{2} - 1 \quad \ldots(4.5.2)$$

Let $g_1(n) = \left[x(n) + x\left(n + \frac{N}{2}\right)\right]$

and $g_2(n) = \left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^n$

$$\therefore \quad X(2k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n)\, W_{N/2}^{nk} \quad \ldots(4.5.3)$$

$$\text{and} \quad X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} g_2(n)\, W_{N/2}^{nk} \quad \ldots(4.5.4)$$

We now draw the signal flow graph using Equations (4.5.1), (4.5.2), (4.5.3) and (4.5.4) using N = 8.

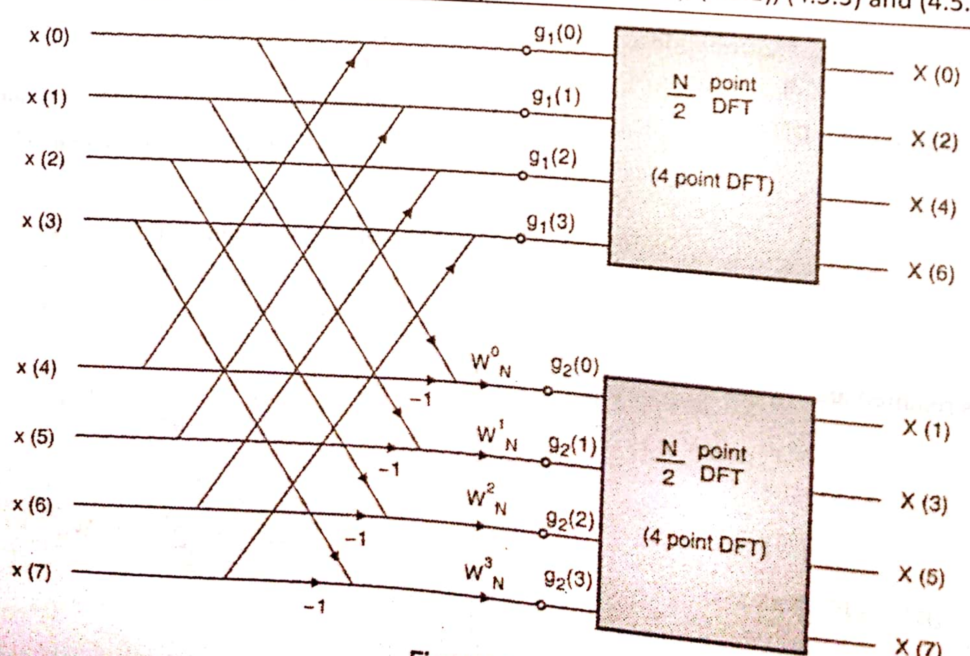

Fig. 4.5.1

As is evident, the order of the input is retained but the output is shuffled.

Equations (4.5.3) and (4.5.4) represent two N/2 point DFT's. Since N = 8, these two equations represent two 4-point DFT's.

Each of these can be further split up into two - 2 point DFT's. We will do just that,

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n)\, W_{N/2}^{nk}$$

Splitting X(2k) we get,

$$X(2k) = \sum_{n=0}^{\frac{N}{4}-1} g_1(n)\, W_{N/2}^{nk} + \sum_{n=0}^{\frac{N}{4}-1} g_1\left(n+\frac{N}{4}\right) W_{N/2}^{k\,(n+N/4)}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g_1(n)\, W_{N/2}^{nk} + W_{N/2}^{kN/4} \sum_{n=0}^{\frac{N}{4}-1} g_1\left(n+\frac{N}{4}\right) W_{N/2}^{nk}$$

Now, $\quad W_{N/2}^{kN/4} = (-1)^k$

$$X(2k) = \sum_{n=0}^{\frac{N}{4}-1} g_1(n)\, W_{N/2}^{nk} + (-1)^k \sum_{n=0}^{\frac{N}{4}-1} g_1\left(n+\frac{N}{4}\right) W_{N/2}^{nk}$$

$$X(2k) = \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + (-1)^k g_1\left(n+\frac{N}{2}\right)\right] W_{N/2}^{nk}$$

Splitting X(2k) into odd and even parts we get,

$$X(4k) = \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + g_1\left(n+\frac{N}{4}\right)\right] W_{N/4}^{nk};$$

$$k = 0, 1, \ldots \frac{N}{4}-1 \qquad \ldots(4.5.5)$$

$$X(4k+2) = \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) - g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{n}\, W_{N/4}^{nk};$$

$$k = 0, 1, \ldots \frac{N}{4}-1 \qquad \ldots(4.5.6)$$

We also split X(2K + 1) into two N/4 point DFTs. Using Equations (4.5.5) and (4.5.6), we draw the signal flow graph.

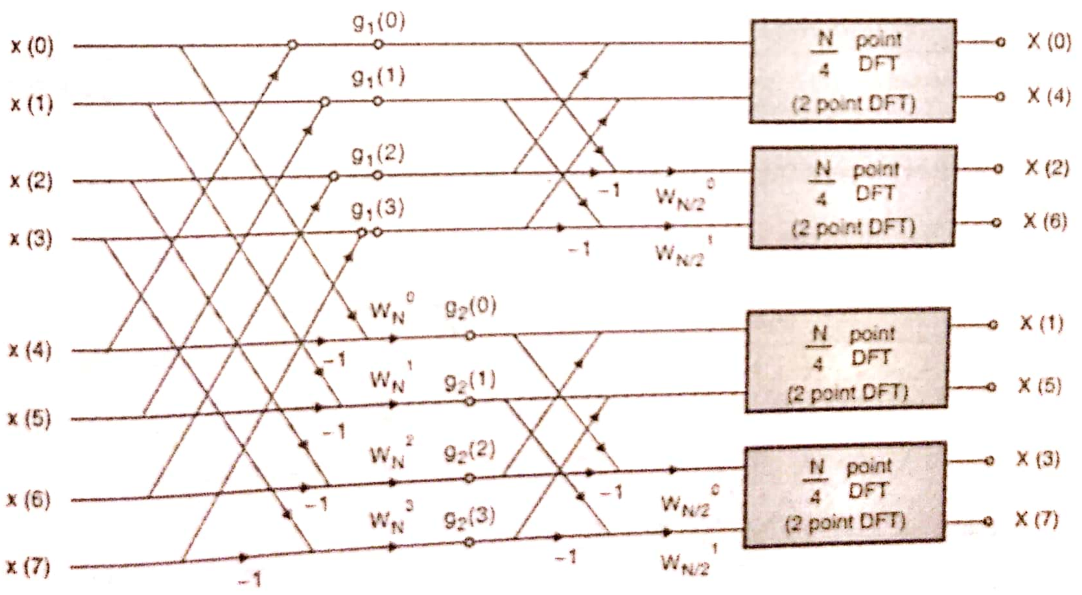

Fig. 4.5.2

A 2-point DFT can be represented as shown in Fig. 4.5.3.

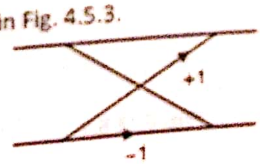

Fig. 4.5.3

The final 8-point DIF-FFT signal flow graph is shown in Fig. 4.5.4.

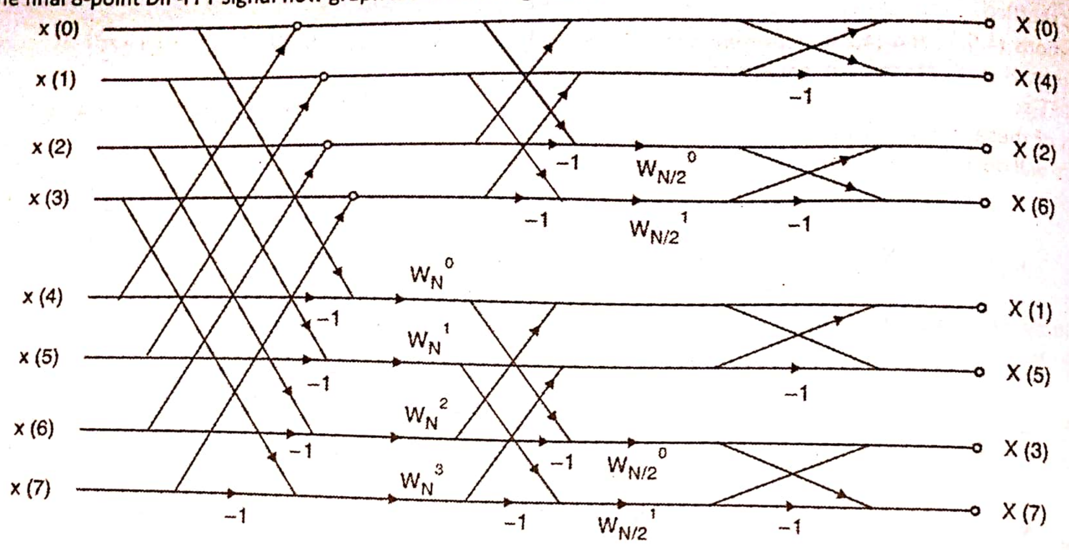

Fig. 4.5.4 : 8-point DIF-FFT Butterfly diagram

In the DIF-FFT, the output is in the bit reversed format while the input is unaltered.

The number of computations required by the DIT-FFT and the DIF-FFT are the same.

As stated earlier, FFT refers to a class of algorithms for efficiently computing the DFT. Hence FFT is not an approximation of the DFT. It is the DFT with a reduced number of computations.

4.5.1 Solved Examples on DIT-FFT

Ex. 4.5.1 : A 8-Point sequence x(n) is given by {1, 2, 3, 2, 1, 5, 2, 1} find the DFT of the sequence using DIF-FFT.

Soln. : $x(n)=\{1, 2, 3, 2, 1, 5, 2, 1\}$

Since N = 8 we draw a 8-point DIF-FFT Butterfly diagram. Remember the input is ordered but the output obtained is in bit reversed order.

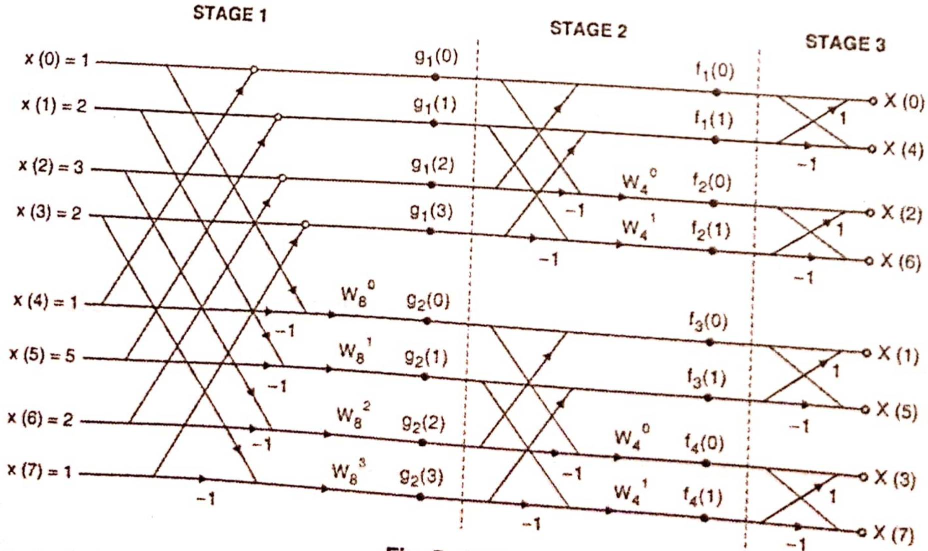

Fig. P. 4.5.1

Here $W_4^0 = e^{\frac{-j2\pi 0}{4}} = \cos 0 - j \sin 0 = 1$

$W_4^1 = e^{\frac{-j2\pi 1}{4}} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2} = -j$

Similarly,

$W_8^0 = e^{\frac{-j2\pi}{8} \cdot 0} = \cos 0 - j \sin 0 = 1$

$W_8^1 = e^{\frac{-j2\pi}{8} \cdot 1} = \cos\frac{\pi}{4} - j \sin\frac{\pi}{4} = 0.707 - j\,0.707$

$W_8^2 = e^{\frac{-j2\pi}{8} \cdot 2} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2} = -j$

$W_8^3 = e^{\frac{-j2\pi}{8} \cdot 3} = \cos\frac{3\pi}{4} - j \sin\frac{3\pi}{4} = -0707 - j0.707$

Stage 1

$g_1(0) = x(0) + x(4) = 1 + 1 = 2$

$g_1(1) = x(1) + x(5) = 2 + 5 = 7$

$g_1(2) = x(2) + x(6) = 3 + 2 = 5$

$g_1(3) = x(3) + x(7) = 2 + 1 = 3$

$g_2(0) = [x(0) - x(4)]\, W_8^0 = [1 - 1]\,(1) = 0$

$g_2(1) = [x(1) - x(5)]\, W_8^1$

$= [2 - 5]\,(0.707 - j0.707) = -2.12 + j2.12$

$g_2(2) = [x(2) - x(6)]\, W_8^2 = [3 - 2]\,(-j) = -j$

$g_2(3) = [x(3) - x(7)]\, W_8^3$

$= [2 - 1]\,(-0.707 - j0.707) = -0.707 - j0.707$

Stage 2

$f_1(0) = g_1(0) + g_1(2) = 2 + 5 = 7$

$f_1(1) = g_1(1) + g_1(3) = 7 + 3 = 10$

$f_2(0) = [g_1(0) - g_1(2)]\, W_4^0 = [2 - 5]\,(1) = -3$

$f_2(1) = [g_1(1) - g_1(3)]\, W_4^1 = [7 - 3]\,(-j) = -4j$

$f_3(0) = g_2(0) + g_2(2) = 0 + (-j) = -j$

$f_3(1) = g_2(1) + g_2(3)$

$= (-2.12 + j2.12) + (-0.707 - j0.707)$

$= -2.827 + j1.413$

$f_4(0) = [g_2(0) - g_2(2)]W_4^0 = 0 - (-j)$

$= 0 - (-j)\,(1) = +j$

$f_4(1) = [g_2(1) - g_2(3)]W_4^1$

$= [(-2.12 + j2.12) - (-0.707 - j0.707)]\,(-j)$

$= 2.827 + j1.413$

Stage 3

$X(0) = f_1(0) + f_1(1) = 7 + 10 = 17$

$X(4) = f_1(0) - f_1(1) = 7 - 10 = -3$

$X(2) = f_2(0) + f_2(1) = -3 + (-4j) = -3 - 4j$

$X(6) = f_2(0) - f_2(1) = -3 - (-4j) = -3 + 4j$

$X(1) = f_3(0) + f_3(1) = (-j) + (-2.827 + j1.413) = -2.827 + j0.413$

$X(5) = f_3(0) - f_3(1) = (-j) - (-2.827 + j\,1.413) = 2.827 - j\,2.413$

$X(3) = f_4(0) + f_4(1) = j + (2.827 + j1.413)$

$= 2.827 + 2.413$

$X(7) = f_4(0) - f_4(1) = j - (2827 + j1413)$

$= 2.827 - j0.413$

We now rearrange the output in the proper order

$\therefore \quad X(k) = \{17, -2.827 + j\,0.413, -3 - 4j, 2.827 + j2.413, -3, 2.827 - j2.413, -3 + 4j, -2.827 - j0.413\}$

Ex. 4.5.2 : Find 8-point DFT using Radix – 2 – DIF FFT algorithms for a given sequences

$x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$

Soln. :

$$x(n) = \{\underset{x(0)}{-1}, \underset{x(1)}{0}, \underset{x(2)}{2}, \underset{x(3)}{0}, \underset{x(4)}{-4}, \underset{x(5)}{0}, \underset{x(6)}{2}, \underset{x(7)}{0}\}$$

We draw a 8-point DIF-FFT Butterfly diagram,

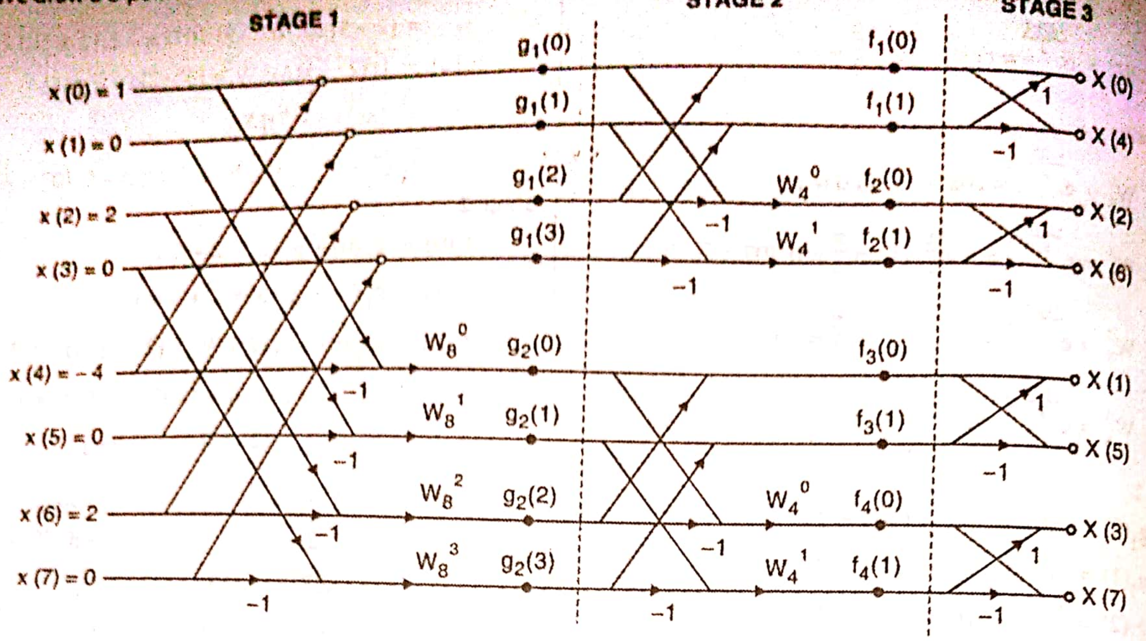

Fig. P. 4.5.2

Here,

$$W_4^0 = e^{\frac{-j2\pi}{4}\cdot 0} = \cos 0 - j \sin 0 = 1$$

$$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2} = -j$$

Similarly,

$$W_8^0 = e^{\frac{-j2\pi}{8}\cdot 0} = \cos 0 - j \sin 0 = 1$$

$$W_8^1 = e^{\frac{-j2\pi}{8}\cdot 1} = \cos\frac{\pi}{4} - j \sin\frac{\pi}{4} = 0.707 - j0.707$$

$$W_8^2 = e^{\frac{-j2\pi}{8}\cdot 2} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi}{8}\cdot 3} = \cos\frac{3\pi}{4} - j \sin\frac{3\pi}{4} = -0.707 - j\,0.707$$

Stage 1

$$g_1(0) = x(0) + x(4) = -1 + (-4) = -5$$

$$g_1(1) = x(1) + x(5) = 0 + 0 = 0$$

$$g_1(2) = x(2) + x(6) = 2 + 2 = 4$$

$$g_1(3) = x(3) + x(7) = 0 + 0 = 0$$

$$g_2(0) = [x(0) - x(4)]\, W_8^0 = [-1 - (-4)]\,(1) = 3$$

$$g_2(1) = [x(1) - x(5)]\, W_8^1$$

$$= [0 - 0]\,(0.707 - j0.707) = 0$$

$$g_2(2) = [x(2) - x(6)]\, W_8^2 = [2 - 2]\,(-j) = 0$$

$$g_2(3) = [x(3) - x(7)]\, W_8^3 = [0 - 0]\,(-0.707 - j0.707) = 0$$

Stage 2

$$f_1(0) = g_1(0) + g_1(2) = -5 + 4 = -1$$

$$f_1(1) = g_1(1) + g_1(3) = 0 + 0 = 0$$

$$f_2(0) = [g_1(0) - g_1(2)]\, W_4^0 = [-5 - 4]\,(1) = -9$$

$$f_2(1) = [g_1(1) - g_1(3)]\, W_4^1 = [0 - 0]\,(-j) = 0$$

$$f_3(0) = g_2(0) + g_2(2) = 3 + 0 = 3$$

$$f_3(1) = g_2(1) + g_2(3) = 0 + 0 = 0$$

$$f_4(0) = [g_2(0) - g_2(2)]\, W_4^0 = [3 - 0]\,(1) = 3$$

$$f_4(1) = [g_2(1) - g_2(3)]\, W_4^1 = [0 - 0]\,(-j) = 0$$

Stage 3

$$X(0) = f_1(0) + f_1(1) = -1 + 0 = -1$$

$$X(4) = f_1(0) - f_1(1) = -1 - 0 = -1$$

$$X(2) = f_2(0) + f_2(1) = -9 + 0 = -9$$

$$X(6) = f_2(0) - f_2(1) = -9 - 0 = -9$$

$$X(1) = f_3(0) + f_3(1) = 3 + 0 = 3$$

$$X(5) = f_3(0) - f_3(1) = 3 - 0 = 3$$

$$X(3) = f_4(0) + f_4(1) = 3 + 0 = 3$$

$$X(7) = f_4(0) - f_4(1) = 3 - 0 = 3$$

We now arrange the output in the proper order

$$\therefore \quad X(k) = \{-1, 3, -9, 3, -1, 3, -9, 3\}$$

Ex. 4.5.3 : Given x(n) = n + 1 and N = 8, find DFT X(k) using DIF-FFT algorithm.

Soln. :

Given :

$$x(n) = n+1 \ ; \ 0 \le n \le 7 \quad (\because N = 8)$$

$$x(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

We draw a 8 – point DIF-FFT Butterfly diagram. In DIF-FFT, the input is ordered but the output obtained is in bit reversed order.

Here

$$W_4^0 = e^{\frac{-j2\pi}{4}\cdot 0} = \cos 0 - j \sin 0 = 1$$

$$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Similarly,

$$W_8^0 = e^{\frac{-j2\pi}{8}\cdot 0} = \cos 0 - j\sin 0 = 1$$

$$W_8^1 = e^{\frac{-j2\pi}{8}\cdot 1} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j\,0.707$$

$$W_8^2 = e^{\frac{-j2\pi}{8}\cdot 2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi}{8}\cdot 3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j\,0.707$$

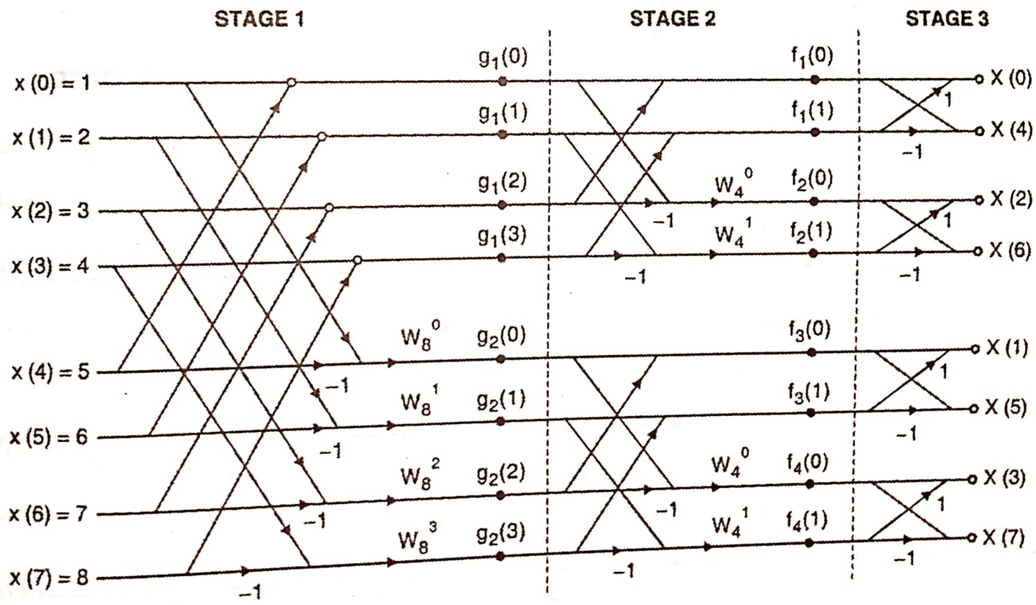

Fig. P. 4.5.3

Stage 1:

$$g_1(0) = x(0) + x(4) = 1 + 5 = 6$$

$$g_1(1) = x(1) + x(5) = 2 + 6 = 8$$

$$g_1(2) = x(2) + x(6) = 3 + 7 = 10$$

$$g_1(3) = x(3) + x(7) = 4 + 8 = 12$$

$$g_2(0) = [x(0) - x(4)]\, W_8^0 = [1 - 5]\,(1) = -4$$

$$g_2(1) = [x(1) - x(5)]\, W_8^1 = [2 - 6]\,(0.707 - j0.707)$$

$$= -2.83 + j2.83$$

$$g_2(0) = [x(2) - x(6)]\, W_8^2 = [3 - 7]\,(-j) = +4j$$

$$g_2(3) = [x(3) - x(7)]\, W_8^3 = [4 - 8]\,(-0.707 - j0.707)$$

$$= 2.83 + j2.83$$

Stage 2 :

$f_1(0) = g_1(0) + g_1(2) = 6 + 10 = 16$

$f_1(1) = g_1(1) + g_1(3) = 8 + 12 = 20$

$f_2(0) = [g_1(0) - g_1(2)]\ W_4^0 = [6 - 10]\ (1) = -4$

$f_2(1) = [g_1(1) - g_1(3)]\ W_4^1 = [8 - 12]\ (-j) = +4j$

$f_3(0) = g_2(0) + g_2(2) = -4 + 4j$

$f_3(1) = g_2(1) + g_2(3)$

$= (-2.83 + j2.83) + (2.83 + j2.83)$

$= j5.66$

$f_4(0) = [g_2(0) - g_2(2)]W_4^0 = [-4 - (+4j)]$

$= -4 - 4j$

$f_4(1) = [g_2(1) - g_2(3)]W_4^1$

$= (-2.83 + j2.83)\ (2.83 + j2.83)\ (-j) = j5.66$

Stage 3

$X(0) = f_1(0) + f_1(1) = 16 + 20 = 36$

$X(4) = f_1(0) - f_1(1) = 16 - 20 = -4$

$X(2) = f_2(0) + f_2(1) = -4 + 4j$

$X(6) = f_2(0) - f_2(1) = -4 - 4j$

$X(1) = f_3(0) + f_3(1) = (-4 + 4j) + (j5.66)$

$= -4 + j9.66$

$X(5) = f_3(0) - f_3(1) = (-4 + 4j) - (j5.66)$

$= -4 - j1.66$

$X(3) = f_4(0) + f_4(1) = (-4 - 4j) + (j5.66)$

$= -4 + j1.66$

$X(7) = f_4(0) - f_4(1) = (-4 - 4j) - (j5.66)$

$= -4 - j9.66$

We now rearrange the output in the proper order

$\therefore \quad X(k) = \{ 36, -4 + j9.66, -4 + 4j, -4 + 1.66j, -4, -4 - j1.66, -4 - 4j, -4 - j9.66\}$

Ex. 4.5.4 : Compute DFT of sequence $x(n) = \{1, 2, 2, 2, 1, 0, 0, 0\}$ using DIF-FFT algorithm. Sketch its magnitude spectrum.

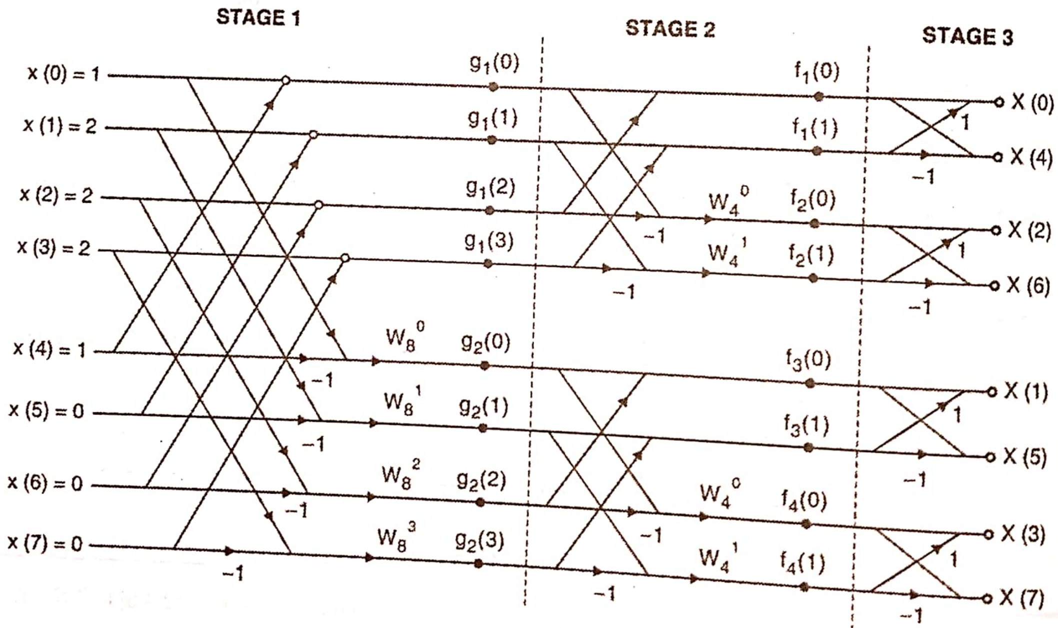

Fig. P. 4.5.4

Soln. :

$$x(n) = \{1, 2, 2, 2, 1, 0, 0, 0\}$$

We draw a 8-point DIF-FFT Butterfly diagram in DIF-FFT, the input is ordered but the output obtained in bit reversed order.

Here,

$$W_4^0 = e^{\frac{-j2\pi}{4}\cdot 0} = \cos 0 - j\sin 0 = 1$$

$$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Similarly,

$$W_8^0 = e^{\frac{-j2\pi}{8}\cdot 0} = \cos 0 - j\sin 0 = 1$$

$$W_8^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j\,0.707$$

$$W_8^2 = e^{\frac{-j2\pi}{4}\cdot 2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi}{4}\cdot 3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j0.707$$

Stage 1

$$g_1(0) = x(0) + x(4) = 1 + 1 = 2$$

$$g_1(1) = x(1) + x(5) = 2 + 0 = 2$$

$$g_1(2) = x(2) + x(6) = 2 + 0 = 2$$

$$g_1(3) = x(3) + x(7) = 2 + 0 = 2$$

$$g_2(0) = [x(0) - x(4)]\, W_8^0 = [1 - 1]\,(1) = 0$$

$$g_2(1) = [x(1) - x(5)]\, W_8^1 = [2 - 0]\,(0.707 - j0.707)$$

$$= 1.414 - j1.414$$

$$g_2(2) = [x(2) - x(6)]\, W_8^2 = [2 - 0]\,(-j) = -2j$$

$$g_2(3) = [x(3) - x(7)]\, W_8^3 = [2 - 0]\,(-0.707 - j0.707)$$

$$= 1.414 - j1.414$$

Stage 2

$$f_1(0) = g_1(0) + g_1(2) = 2 + 2 = 4$$

$$f_1(1) = g_1(1) + g_1(3) = 2 + 2 = 4$$

$$f_2(0) = [g_1(0) - g_1(2)]\, W_4^0 = [2 - 2]\,(1) = 0$$

$$f_2(1) = [g_1(1) - g_1(3)]\, W_4^1 = [2 - 2]\,(-j) = 0$$

$$f_3(0) = g_2(0) + g_2(2) = 0 + (-j2) = -j2$$

$$f_3(1) = g_2(1) + g_2(3)$$

$$= (1.414 - j1.414) + (-1.414 - j1.414)$$

$$= -j2.83$$

$$f_4(0) = [g_2(0) - g_2(2)]W_4^0 = 0 - (-j2) = +j2$$

$$f_4(1) = [g_2(1) - g_2(3)]W_4^1$$

$$= (1.414 - j1.414) - (-1.414 - j1.414)\,(-j)$$

$$= -j2.83$$

Stage 3

$$X(0) = f_1(0) + f_1(1) = 4 + 4 = 8$$

$$X(4) = f_1(0) - f_1(1) = 4 - 4 = 0$$

$$X(2) = f_2(0) + f_2(1) = 0 + 0 = 0$$

$$X(6) = f_2(0) - f_2(1) = 0 - 0 = 0$$

$$X(1) = f_3(0) + f_3(1) = -j2 + (-j2.83) = -j4.83$$

$$X(5) = f_3(0) - f_3(1) = -j2 - (-j2.83) = +j0.83$$

$$X(3) = f_4(0) + f_4(1) = j2 + (-j2.83) = -j0.83$$

$$X(7) = f_4(0) - f_4(1) = j2 - (-j2.83) = j4.83$$

We now rearrange the output in the proper order.

$$X(k) = \{8, -j4.83, 0, -j0.83, 0, +j0.83, 0, j4.83\}$$

To sketch the magnitude spectrum. We compute the magnitude of the Fourier coefficient X(k)

$$\therefore\ |X(k)| = \{8, 4.83, 0, 0.83, 0, 0.83, 0, 4.83\}$$

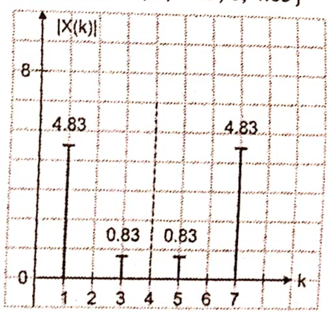

Fig. P. 4.5.4(a)

We observer that the magnitude spectrum is symmetric about the $\frac{N}{2}$ point, in this case 4.

Ex. 4.5.5 : N = 8 point DFT using Radix FFT for the given sequence x(n) = {1, 1, 1, 1, 0, 0, 0, 0}

Soln. :

$$x(n) = \{\underset{x(0)}{1},\ \underset{x(1)}{1},\ \underset{x(2)}{1},\ \underset{x(3)}{1},\ \underset{x(4)}{0},\ \underset{x(5)}{0},\ \underset{x(6)}{0},\ \underset{x(7)}{0}\}$$

We draw a 8-point DIF-FFT Butterfly diagram

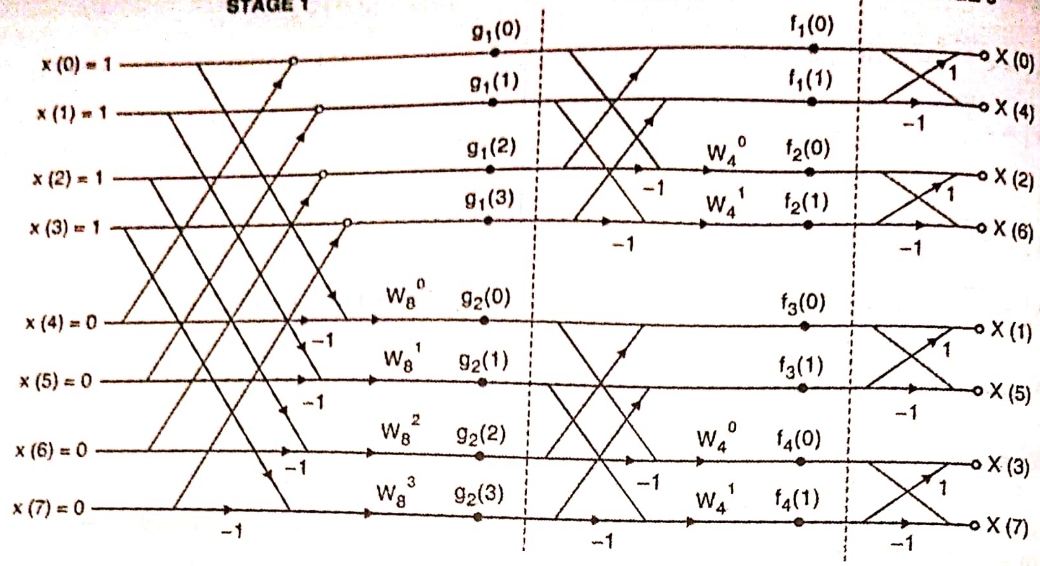

Fig. P. 4.5.5

Here,

$$W_4^0 = e^{\frac{-j2\pi}{4}\cdot 0} = \cos 0 - j\sin 0 = 1$$

$$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Similarly,

$$W_8^0 = e^{\frac{-j2\pi}{8}\cdot 0} = \cos 0 - j\sin 0 = 1$$

$$W_8^1 = e^{\frac{-j2\pi}{8}\cdot 1} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j\,0.707$$

$$W_8^2 = e^{\frac{-j2\pi}{8}\cdot 2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi}{8}\cdot 3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j\,0.707$$

Stage 1

$$g_1(0) = x(0) + x(4) = 1 + 0 = 1$$

$$g_1(1) = x(1) + x(5) = 1 + 0 = 1$$

$$g_1(2) = x(2) + x(6) = 1 + 0 = 1$$

$$g_1(3) = x(3) + x(7) = 1 + 0 = 1$$

$$g_2(0) = [x(0) - x(4)]\,W_8^0 = [1 - 0]\,(0) = 1$$

$$g_2(1) = [x(1) - x(5)]\,W_8^1$$

$$= [1 - 0]\,(0.707 - j0.707)$$

$$= 0.707 - j0.707$$

$$g_2(2) = [x(2) - x(6)]\,W_8^2 = [1 - 0]\,(-j) = -j$$

$$g_2(3) = [x(3) - x(7)]\,W_8^3$$

$$= [1 - 0]\,(-0.707 - j0.707)$$

$$= -0.707 - j0.707$$

Stage 2

$$f_1(0) = g_1(0) + g_1(2) = 1 + 1 = 2$$

$$f_1(1) = g_1(1) + g_1(3) = 1 + 1 = 2$$

$$f_2(0) = [g_1(0) - g_1(2)]\,W_4^0 = [1 - 1]\,(1) = 0$$

$$f_2(1) = [g_1(1) - g_1(3)]\,W_4^1 = [1 - 1]\,(-j) = 0$$

$$f_3(0) = g_2(0) + g_2(2) = 1 + (-j) = 1 - j$$

$$f_3(1) = g_2(1) + g_2(3)$$

$$= (0.707 - j0.707) + (-0.707 - j0.707)$$

$$= -j1.414$$

$$f_4(0) = [g_2(0) - g_2(2)]\,W_4^0 = [1 - (-j)]\,(1) = 1 + j$$

$$f_4(1) = [g_2(1) - g_2(3)]\,W_4^1$$

$$= [(0.707 - j0.707) - (-0.707 - j0.707)]\,(-j)$$

$$= -j1.414$$

Stage 3

$$X(0) = f_1(0) + f_1(1) = 2 + 2 = 4$$

$$X(4) = f_1(0) - f_1(1) = 2 - 2 = 0$$

$X(2) = f_2(0) + f_2(1) = 0 + 0 = 0$

$X(6) = f_2(0) - f_2(1) = 0 - 0 = 0$

$X(1) = f_3(0) + f_3(1) = (1 - j) + (-j1.414) = 1 - j2.414$

$X(5) = f_3(0) - f_3(1) = (1 - j) - (-j0.414) = 1 + j0.414$

$X(3) = f_4(0) + f_4(1) = (1 + j) + (-j0.414) = 1 - j0.414$

$X(7) = f_4(0) - f_4(1) = (1 + j) - (1 - j1.414) = 1 + j2.414$

We now rearrange the output in the proper order

$\therefore \quad X(k) = \{4, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$

Ex. 4.5.6 : Using DIF - FFT Find a

$x(n) = \{1, 2, 1, 2, 4, 2, 1, 2\}$

If $x_1(n) = x(-n)$, find $x_1(k)$ using earlier solution and not otherwise.

Soln. : We draw a 8-point DIF-FFT Butterfly diagram. In DIF-FFT, the input is ordered but the output obtained is in bit reversed order.

Here $W_4^0 = e^{\frac{-j2\pi}{4} 0} = \cos 0 - j \sin 0 = 1$

$W_4^1 = e^{\frac{-j2\pi}{4} 1} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2} = -j$

Similarly,

$W_8^0 = e^{\frac{-j2\pi}{8} 0} = \cos 0 - j \sin 0 = 1$

$W_8^1 = e^{\frac{-j2\pi}{8} 1} = \cos\frac{\pi}{4} - j \sin\frac{\pi}{4} = -0.707 - j\,0.707$

$W_8^2 = e^{\frac{-j2\pi}{8} 2} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2} = -j$

$W_8^3 = e^{\frac{-j2\pi}{8} 3} = \cos\frac{3\pi}{4} - j \sin\frac{3\pi}{4} = -0.707 - j0.707$

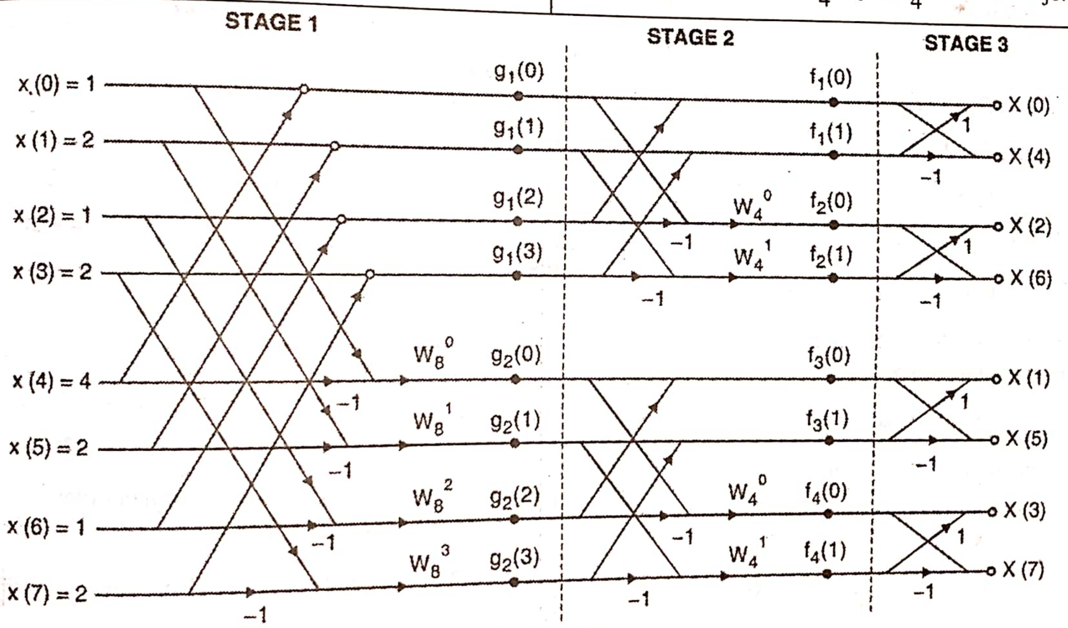

Fig. P. 4.5.6

Stage 1

$$g_1(0) = x(0) + x(4) = 1 + 4 = 5$$

$$g_1(1) = x(1) + x(5) = 2 + 2 = 4$$

$$g_1(2) = x(2) + x(6) = 1 + 1 = 2$$

$$g_1(3) = x(3) + x(7) = 2 + 2 = 4$$

$$g_2(0) = [x(0) - x(4)]\, W_8^0 = [1 - 4]\,(1) = -3$$

$$g_2(1) = [x(1) - x(5)]\, W_8^1 = [2 - 2]\,(0.707 - j0.707) = 0$$

$$g_2(2) = [x(2) - x(6)]\, W_8^2 = [1 - 1]\,(-j) = 0$$

$$g_2(3) = [x(3) - x(7)]\, W_8^3 = [2 - 2]\,(-0.707 - j0.707) = 0$$

Stage 2

$$f_1(0) = g_1(0) + g_1(2) = 5 + 2 = 7$$

$$f_1(1) = g_1(1) + g_1(3) = 4 + 4 = 8$$

$$f_2(0) = [g_1(0) - g_1(2)]\, W_4^0 = [5 - 2]\,(1) = 3$$

$$f_2(1) = [g_1(1) - g_1(3)]\, W_4^1 = [4 - 4]\,(-j) = 0$$

$$f_3(0) = g_2(0) + g_2(2) = -[-3 + 0] = -3$$

$$f_3(1) = g_2(1) + g_2(3) = [0 + 0] = 0$$

$$f_4(0) = [g_2(0) - g_2(2)] W_4^0 = [-3 - 0]\,(1) = -3$$

$$f_4(1) = [g_2(1) - g_2(3)] W_4^1 = [0 - 0]\,(-j) = 0$$

Stage 3

$$X(0) = f_1(0) + f_1(1) = 7 + 8 = 15$$

$$X(4) = f_1(0) - f_1(1) = 7 - 8 = -1$$

$$X(2) = f_2(0) + f_2(1) = 3 + 0 = 3$$

$$X(6) = f_2(0) - f_2(1) = 3 - 0 = 3$$

$$X(1) = f_3(0) + f_3(1) = -3 + 0 = -3$$

$$X(5) = f_3(0) - f_3(1) = -3 - 0 = 3$$

$$X(3) = f_4(0) + f_4(1) = -3 + 0 = -3$$

$$X(7) = f_4(0) - f_4(1) = -3 - 0 = -3$$

We now rearrange the output in the proper order.

$$X(k) = \{15, -3, 3, -3, -1, -3, -3, 3, -3\}$$

We now need to find $X_1(k)$.

Given $\quad x_1(n) = x(-n)$

According to the time reversal properly of the DFT we have

$$x_1((-n))_N \xleftrightarrow{DFT} X((-k))_n$$

$$\therefore \quad x_1(n) = x((-n))_N \xleftrightarrow{DFT} X((-k))_N = X_1(-k)$$

Hence $X_1(k)$ is obtained by circularly folding $X_1(k)$

$$\therefore X_1(k) = \{15, -3, 3, -3, -1, -3, 3, -3\}$$

$\therefore$ $X_1(k)$ is the same as $X(k)$.

Ex. 4.5.7 : Compute the DFT of

$x(n) = 2\delta(n) + 3\delta(n-1) + 4\delta(n-2) + 5\delta(n-3)$

Use DIF-FFT

Soln. : Here $\delta(n)$ is a impulse functions while $\delta(n-1)$, $\delta(n-2)$ and $\delta(n-3)$ are impulse shifted to the right.

$$\therefore \quad x(n) = \{2, 3, 4, 5\}$$

We draw a 4 – point DIF–FFT butterfly diagram. In DIF–FFT, the input is ordered but the output obtained is in the bit reversed order.

Here $\quad W_4^0 = e^{\frac{-j2\pi}{4} \cdot 0} = \cos 0 - j \sin 0 = 0$

$$W_4^1 = e^{\frac{-j2\pi}{4} \cdot 1} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2} = -j$$

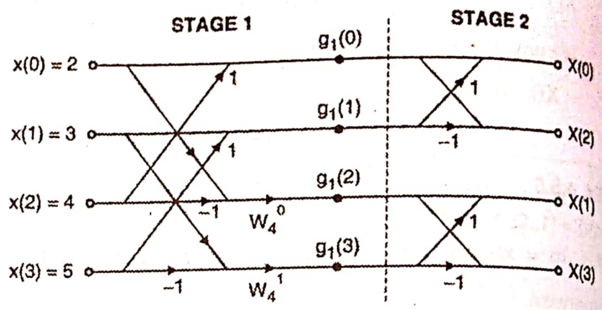

Fig. P. 4.5.7

Stage 1

$$g_1(0) = x(0) + x(2) = 2 + 4 = 6$$

$$g_1(1) = x(1) + x(3) = 3 + 5 = 8$$

$$g_1(2) = [x(0) - x(2)]\, W_4^0 = [2 - 4]\,(1) = -2$$

$$g_1(3) = [x(1) - x(3)]\, W_4^1 = [3 - 5]\,(-j) = +2j$$

Stage 2

$$X(0) = g_1(0) + g_1(1) = 6 + 8 = 14$$

$$X(2) = g_1(0) - g_1(1) = 6 - 8 = -2$$

$$X(1) = g_1(2) + g_1(3) = -2 + (+2j) = -2 + 2j$$

$$X(3) = g_1(2) - g_1(3) = -2 - (+2j) = -2 - 2j$$

We rearrange the output in the proper order

$$\therefore \quad X(k) = \{14, -2 + 2j, -2, -2 - 2j\}$$

Ex. 4.5.8 : Compute the DFT of $x(n) = \cos\frac{n\pi}{2}$ using DIF-FFT. Assume N = 4.

Soln. : Since N = 4, we have $x(n) = \cos\frac{n\pi}{2}$; $0 \le n \le 3$

$$x(0) = \cos(0) = 1$$

$$x(1) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$x(2) = \cos(\pi) = -1$$

$$x(3) = \cos\left(\frac{3\pi}{2}\right) = 0$$

$$\therefore \quad x(n) = \{1, 0, -1, 0\}$$

We draw a 4 - point DIF-FFT Butterfly diagram

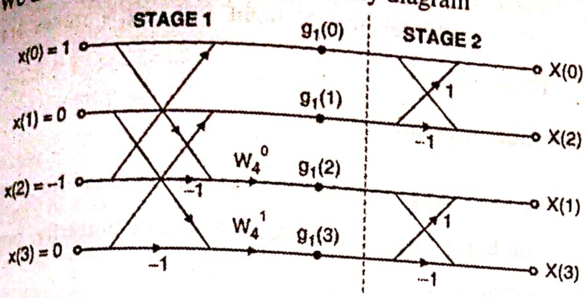

Fig. P. 4.5.8

Here
$$W_4^0 = e^{\frac{-j2\pi}{4}\cdot 0} = 1$$
$$W_4^1 = e^{\frac{-j2\pi}{4}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Stage 1

$$g_1(0) = x(0) + x(2) = 1 - 1 = 0$$
$$g_1(1) = x(1) + x(3) = 0 + 0 = 0$$
$$g_1(2) = [x(0) - x(2)]\, W_4^0 = [1 - (-1)]\,(1) = 2$$
$$g_1(3) = [x_1(1) - x(3)]\, W_4^1 = [0 - 0]\,(-j) = 0$$

Stage 2

$$X(0) = g_1(0) + g_1(1) = 0 + 0 = 0$$
$$X(2) = g_1(0) - g_1(1) = 0 - 0 = 0$$
$$X(1) = g_1(2) + g_1(3) = 2 + 0 = 2$$
$$X(2) = g_1(2) - g_1(3) = 2 - 0 = 2$$

We rearrange the output in the proper order

$$\therefore \quad X(k) = \{0, 2, 0, 2\}$$

Ex. 4.5.9 : If x(n) = {1 + 2j, 3 + 4j, 5 + 6j, 7 + 8j}. Find DFT X(k) using DIF-FFT.

Soln.: Since N = 4, we draw a 4-point DIF-FFT Bufferfly diagram in DIF-FFT, the input is ordered but the output is in bit reversed order.

Here
$$W_4^0 = e^{\frac{-j2\pi}{4}\cdot 0} = 1$$
$$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Stage 1

$$g_1(0) = x(0) + x(2) = (1 + 2j) + (5 + 6j) = 6 + j8$$
$$g_1(1) = x(1) + x(3) = (3 + j4) + (7 + j8) = 10 + j12$$
$$g_2(2) = [x(0) - x(2)]\, W_4^0 = [(1 + 2j) - (5 + 6j)]\,(1)$$
$$= -4 - j4$$
$$g_1(3) = [x(1) - x(3)]\, W_8^1$$
$$= [(3 + j4) - (7 + j8)]\,(-j) = -4 + j4$$

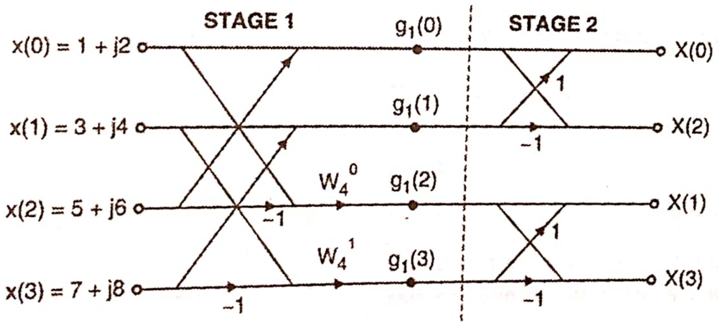

Fig. P. 4.5.9

Stage 2

$$X(0) = g_1(0) + g_1(1) = (6 + j8) + (10 + j12) = 16 + j20$$
$$X(2) = g_1(0) - g_1(1) - (6 + j8) - (10 - j12) = -4 - j4$$
$$X(1) = g_1(2) + g_1(3) = (-4 - j4) + (-4 + j4) = -8$$
$$X(3) = g_1(2) - g_1(3) = (-4 - j4) + (-4 + j4) = -j8$$

We rearrange the output in the proper order

$$\therefore \quad X(k) = \{16 + j20, -8, -4 - j4, -j8\}$$

4.6 Computational Complexity Comparison between DFT and FFT

- We now compare the number of computations required for DFT and FFT.
- It was mentioned earlier that FFT is merely a faster way of computing the DFT. We will now show how much is the reduction in computation when a FFT algorithm is used.

1. Computations required for DFT algorithm

We have shown at the beginning of the chapter that a N-point DFT required N^2 multiplication and $N(N-1)$ additions.

2. Computations required using FFT algorithm

- We begin by calculating the computation complexity required for one butterfly. Consider the general structure of butterfly as shown in Fig. 4.6.1.

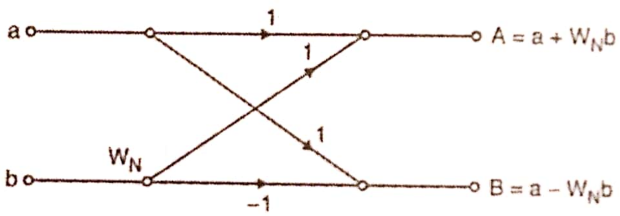

Fig. 4.6.1 : General structure of butterfly

- Here 'a' and 'b' are inputs and A and B are outputs of butterfly. The outputs are given by,

$$A = a + W_N b \quad \text{...(4.6.1)}$$

$$\text{and } B = a - W_N b \quad \text{...(4.6.2)}$$

- To calculate any output (A or B), we need to multiply each input by the twiddle factor W_N. So **one complex multiplication** is required for one butterfly.
- From Equations (4.6.1) and (4.6.2) we note that to calculate output 'A', one complex addition is required, while to calculate output 'B' one complex subtraction is required. But the computational complexity of addition and subtraction is same. So we can say that for one butterfly **two complex additions** are required.
- For a 8 point DFT, 4 butterflies are required at each stage. So for 'N' point DFT, at each stage, $\frac{'N'}{2}$ butterflies are required.
- Three such stages are required to compute 8-point DFT. In general, for 'N' point DFT, $\log_2 N$ stages are required.

Complex multiplications

- At each stage there are $\frac{N}{2}$ butterflies and the total number of stages are $\log_2 N$. For each butterfly, one complex multiplication is required.
- Hence total complex multiplications = $\frac{N}{2}\log_2 N$ ...(4.6.3)

Complex additions

- Total number of stages are $\log_2 N$ and at each stage, $\frac{N}{2}$ butterflies are required. For each butterfly, '2' complex additions are required.

$$\therefore \quad \text{Total complex additions} = 2 \times \frac{N}{2} \log_2 N$$

$$\therefore \quad \text{Total complex additions} = N \log_2 N$$

- Table 4.6.1 shows comparison of direct DFT computation and computation using FFT algorithms.

Table 4.6.1

Number of points 'N'	Direct Computation		Using FFT	
	Complex Multiplication N^2	Complex Additions $(N^2 - N)$	Complex Multiplication $\left(\frac{N}{2}\log_2 N\right)$	Complex Addition $(N \log_2 N)$
4	16	12	2	4
8	64	56	12	24
16	256	240	32	64
32	1024	992	40	80
64	4096	4032	96	192
128	16,384	16,256	224	448

The Table 4.6.1 shows that, by the use of FFT algorithms the number of complex multiplications and complex additions are reduced. So there is tremendous improvement in the speed

4.6.1 Solved Examples on Computation of Inverse DFT using FFT Algorithms

Ex. 4.6.1 : How many computations are required to compute 16 point DFT using DFT and FFT algorithms

Soln. :

Here, N = 16

Using DFT

Number of multiplications = $N^2 = 16^2 = 256$

Number of addition = $N(N-1) = 16 \times 15 = 240$

Using FFT

Number of Multiplication = $\frac{N}{2} \log_2^N = 32$

Number of Addition = $N \log_2^N = 64$

Ex. 4.6.2 : How to find inverse one dimensional DFT using forward DIT FFT flow graph. Derive necessary formula.

Soln. : The 1-D-DFT formula is

$$X(k) = \sum_{k=0}^{N-1} x(n)\, W_N^{nk} \qquad \text{...(1)}$$

The 1-D-IDFT formula is

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{*\,nk} \qquad \text{...(2)}$$

The DIT-FFT algorithm is computed using Equation (1). We now try to make Equation (2) appear like Equation (1)

We take the complex conjugate on both sides of Equation (2).

$$\therefore \quad x^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X^*(k)\, W_N^{**\,nk}$$

$$\therefore \quad x^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X^*(k)\, W_N^{nk} \qquad \text{...(3)}$$

Equation (3) is identical to Equation (1) except for the complex sign and the $\frac{1}{N}$ term.

Hence in DIT-FFT, if we replace the input x(n) by $X^*(k)$ and divide it by $\frac{1}{N}$, we get $x^*(n)$. Taking the conjugate of this gives us x(n). Steps involved in computing DIT-IFFT.

1. Replace x(n) by $X^*(k)$ in the DIT FFT.
2. On obtaining $x^*(n)$, take the conjugate.

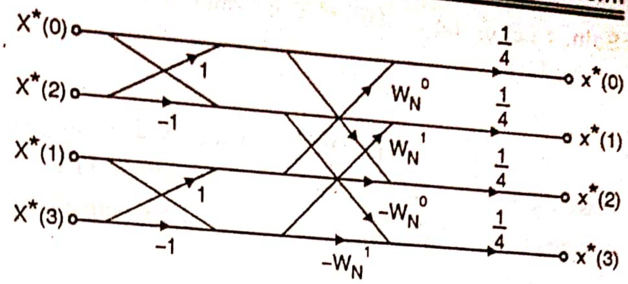

Fig. P. 4.6.2

Ex. 4.6.3 : Find the I-DFT using DIT-FFT of the given example.

$$X(k) = \{10, -2+2j, -2, -2-2j\}$$

Soln. :

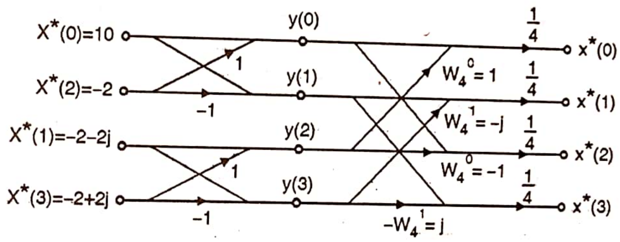

Fig. P. 4.6.3

$$y(0) = X^*(0) + X^*(2) = 8$$

$$y(1) = X^*(0) - X^*(2) = 12$$

$$y(2) = X^*(1) + X^*(3) = -4$$

$$y(3) = X^*(1) - X^*(3) = -4j$$

$$x^*(0) = \frac{1}{4}[Y(0) + Y(2) \cdot (1)] = 1$$

$$x^*(1) = \frac{1}{4}[Y(1) + Y(3) \cdot (-j)] = 2$$

$$x^*(2) = \frac{1}{4}[Y(0) + Y(2)(-1)] = 3$$

$$x^*(3) = \frac{1}{4}[Y(1) + Y(3) \cdot (j)] = 4$$

$$\therefore \quad x(n) = \{1, 2, 3, 4\}$$

We finally take the complex conjugate

$$x^*(n) = \{1, 2, 3, 4\}$$

We have thus found out the inverse DFT using the DIT-FFT.

Ex. 4.6.4 : Given :

$X(k) = \{36, -4 + j9.656, -4 + j4, -4 + j1.656, -4, -4 - j1.656, -4 - j4, -4 - j9.656\}$

Find x(n) using any IFFT algorithm.

Soln. : Let us use the DIF-FFT algorithm to compute the inverse FFT.

Here, N = 8

We draw a 8 point DIF-FFT butterfly diagram and make the required for computation of the inverse FFT. In required DIF-FFT, the input is in ordered but the output is in bit reversed order.

Here,

$$W_4^0 = e^{\frac{-j2\pi}{4}0} = \cos 0 - j \sin 0 = 1$$

$$W_4^1 = e^{\frac{-j2\pi}{4}1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Similarly,

$$W_8^0 = e^{\frac{-j2\pi}{8}0} = \cos 0 - j\sin 0 = 1$$

$$W_8^1 = e^{\frac{-j2\pi}{8}1} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j\,0.707$$

$$W_8^2 = e^{\frac{-j2\pi}{8}2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi}{8}3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}$$

$$= -0.707 - j\,0.707$$

Note: The input given to the network is the complex conjugate of the input and the output obtained needs to be divided by N.

Stage 1

$$g_1(0) = X^*(0) + X^*(4) = 36 + (-4) = 32$$

$$g_1(1) = X^*(1) + X^*(5)$$
$$= (-4 - j9.656) + (-4 + j1.651)$$
$$= -8 - j8.005$$

$$g_1(2) = X^*(2) + X^*(6) = (-4 - j4) + (-4 + j4)$$
$$= -8$$

$$g_1(3) = X^*(3) + X^*(7)$$
$$= (-4 - j1.656) + (-4 + j9.656)]$$
$$= -8 + j8$$

$$g_2(0) = X^*[(0) - X^*(4)]\ W_8^0 = 36 - (-4)\ (1) = 40$$

$$g_2(1) = X^*[(1) - X^*(5)]\ W_8^1$$
$$= [(-4 - j9.656) - (-4 + j\,1.651)]\ (0.707 - j0.707)$$
$$= -8 - j\,8$$

$$g_2(2) = [\ X^*[(2) - X^*(6)]\ W_8^2$$
$$= [(-4 - j4) - (-4 + j\,4)]\ (-j)] = -8$$

$$g_2(3) = X^*[(3) - X^*(7)]\ W_8^3$$
$$= [(-4 - j1.656) - (-4 + j9.656)]\ \ (-0.707 - 0.707)$$
$$= -8 + j8$$

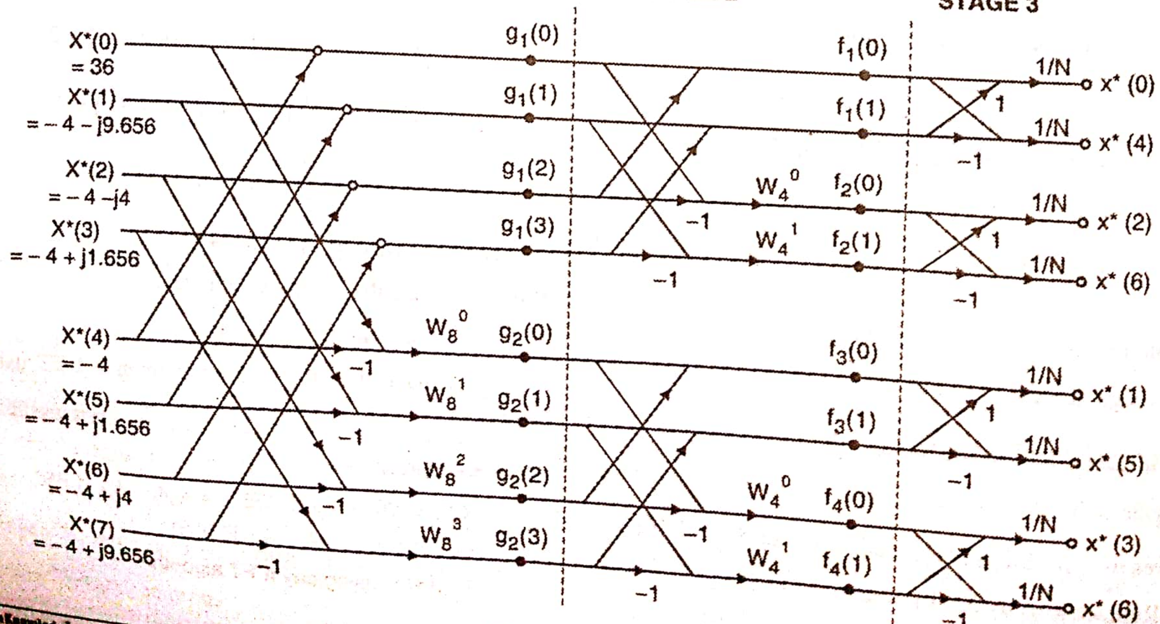

Fig. P. 4.6.4

Stage 2

$f_1(0) = g_1(0) + g_1(2) = 32 + (-8) = 24$

$f_1(1) = g_1(1) + g_1(3) = (-8 - j8) + (-8 + j8)$

$= -16$

$f_2(0) = [g_1(0) + g_1(2)]\, W_4^0 = [32 - (-8)](1) = 40$

$f_2(1) = [g_1(1) - g_1(3)]\, W_4^1$

$= [(-8 - j8) - (-8 + j8)](-j) = -16$

$f_3(0) = g_2(0) + g_2(2) = 40 + (-8) = 32$

$f_3(1) = g_2(1) + g_2(3) = (-8 - j8) + (-8 + j8)$

$= -16$

$f_4(0) = [g_2(0) - g_2(2)]W_4^0 = [40 - (-8)]\,(1) = 48$

$f_4(1) = [g_2(1) - g_2(3)]W_4^1$

$= [(-8 - j8) - (-8 + 8)](-j) = -16$

Stage 3

$x^*(0) = [f_1(0) + f_1(1)]/\,8 = [24 + (-16)]/8 = 1$

$x^*(4) = [f_1(0) - f_1(1)]/8 = [24 - (-16)]\,/\,8 = 5$

$x^*(2) = [f_2(0) + f_2(1)]/8 = [40 + (-16)]/\,8 = 3$

$x^*(6) = [f_2(0) - f_2(1)]/8 = [40 - (-16)]/8 = 7$

$x^*(1) = [f_3(0) + f_3(1)]/8 = [32 + (-16)]/8 = 2$

$x^*(5) = [f_3(0) - f_3(1)]/8 = [32 - (-16)]/8 = 6$

$x^*(3) = [f_4(0) + f_4(1)]/8 = [48 + (-16)]/8 = 4$

$x^*(7) = [f_4(0) - f_4(1)]/8 = [48 - (-16)]/8 = 8$

We rearrange the output in the proper order. Since $x^*(n)$ values are all real

$x(n) = x^*(n)$

$x(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Ex. 4.6.5 : Let $x(n)$ be 8-point sequence. Its corresponding DFT $X(k)$ is,

$X(k) = \{0.5,\ 2 + j,\ 3 + j2,\ j,\ 3,\ -j,\ 3 - j2,\ 2 - j\}$

Soln. :

We will use the DIT – FFT algorithms since N = 8, we use the DIT – FFT algorithms we give $X^*(k)$ as the input and divide the output obtained by N. We also take the complex conjugate of the result obtained.

$x(n) = \{0.5,\ 2 + j,\ 3 + j2,\ j,\ 3,\ -j,\ 3 - j2,\ 2 - j\}$

x(0) x(1) x(2) x(3) x(4) x(5) x(6) x(7)

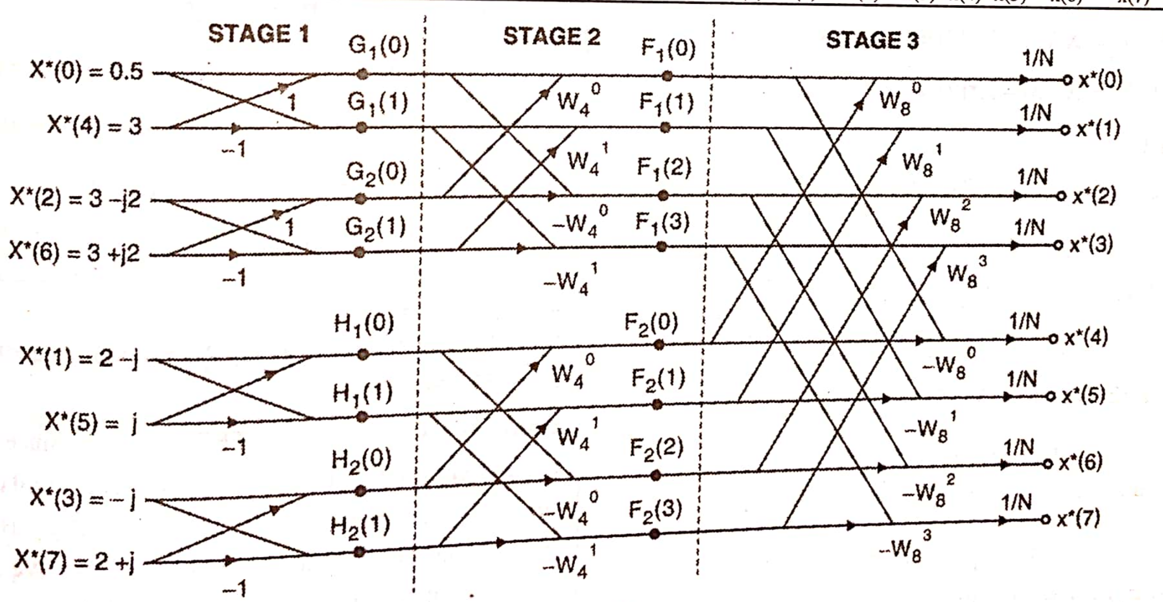

Fig. P. 4.6.5

Here,

$$W_4^0 = 1$$

$$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Similarly,

$$W_8^0 = 1$$

$$W_8^1 = e^{\frac{-j2\pi}{8}\cdot 1} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j0.707$$

$$W_8^2 = e^{\frac{-j2\pi}{8}\cdot 2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi}{8}\cdot 3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j\,0.707$$

We write down the equation at each stage and calculate the outputs

Stage 1

$$G_1(0) = X^*(0) + X^*(4) = 0.5 + 3 = 3.5$$

$$G_1(1) = X^*(0) - X^*(4) = 0.5 - 3 = -2.5$$

$$G_2(0) = X^*(2) + X^*(6) = (3 - j2) + (3 + j2) = 6$$

$$G_2(1) = X^*(2) - X^*(6) = (3 - j2) - (3 + j2) - j4$$

$$H_1(0) = X^*(1) + X^*(5) = (2 - j) + (j) = 2$$

$$H_1(1) = X^*(1) - X^*(5) = (2 - j) - (j) = 2 - j2$$

$$H_2(0) = X^*(3) + X^*(7) = -j + (2 + j) = 2$$

$$H_2(1) = X^*(3) - X^*(7) = -j - (2 + j) = -2 - j2$$

Stage 2

$$F_1(0) = G_1(0) + W_4^0 G_2(0) = 3.5 + (1)(6) = 9.5$$

$$F_1(1) = G_1(1) + W_4^1 G_2(1) = -2.5 + (-j)(-j4) = -6.5$$

$$F_1(2) = G_1(0) - W_4^0 G_2(0) = 3.5 - (1)(6) = -2.5$$

$$F_1(3) = G_1(1) - W_4^1 G_2(1) = 2.5 - (-j)(-j4) = 1.5$$

$$F_2(0) = H_1(0) + W_4^0 H_2(0) = 2 + (1)(2) = 4$$

$$F_2(1) = H_1(1) + W_4^1 H_2(1) = (2 - j2) + (-j)(-2 - j2) = 0$$

$$F_2(2) = H_1(0) - W_4^0 H_2(0) = 2 - (1)(2) = 0$$

$$F_2(3) = H_1(1) - W_4^1 H_2(1) = (2 - j2) - (-j)(-2 - j2) = 4 - j4$$

Stage 3

$$x^*(0) = [F_1(0) + W_8^0 F_2(0)]/8 = [9.5 + (1)\,4]/8 = 1.69$$

$$x^*(1) = [F_1(1) + W_8^1 F_2(1)]/8 = [-6.5 + (0.707 - j0.707\,(0)]/8 = -0.81$$

$$x^*(2) = [F_1(2) + W_8^2 F_2(2)]/8 = [-2.5 + (-j)(0)]/8 = -0.31$$

$$x^*(3) = [F_1(3) + W_8^3 F_2(3)]/8 = [1.5 + (-0.707 - j0.707)(4 - j4)]/8 = -0.52$$

$$x^*(4) = [F_1(0) - W_8^0 F_2(0)]/8 = [9.5 - (1)\,4]/8 = 0.69$$

$$x^*(5) = [F_1(1) - W_8^1 F_2(1)]/8 = [-6.5 - (0.707 - j0.707)(0)]/8 = -0.81$$

$$x^*(6) = [F_1(2) - W_8^2 F_2(2)]/8 = [-2.5 - (-j)(0)]/8 = -0.31$$

$$x^*(7) = [F_1(3) - W_8^3 F_2(3)]/8 = [1.5 - (-0.707 - j0.707)(4 - j4)]/8 = 0.89$$

We finally take the complex conjugate of the result.

$$\therefore\ x(n) = \{1.69, -0.81, -0.31, -0.52, 0.69, -0.81, -0.31, 0.89\}$$

Ex. 4.6.6 : Given $X(k) = \{2, -6j, 2 - 8j, 6j, 2 - 6j, 2 + 8j, 6j, 0\}$ Find x(n) using any IFFT algorithm.

Soln. : We will use the DIT-FFT algorithm. Since N = 8, we use a 8-point DIT-FFT Butterfly diagram. In DIT-FFT, the input is in bit reversed order. We give $X^*(k)$ as the input and divide the output - obtained by N. We also take complex conjugate of the result obtained.

$$X(k) = \{2, -j6, 2 - j8, j6, 2 - j6, 2 + j8, j6, 0\}$$

$$\therefore \quad \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

Here

$$W_4^0 = 1$$

$$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

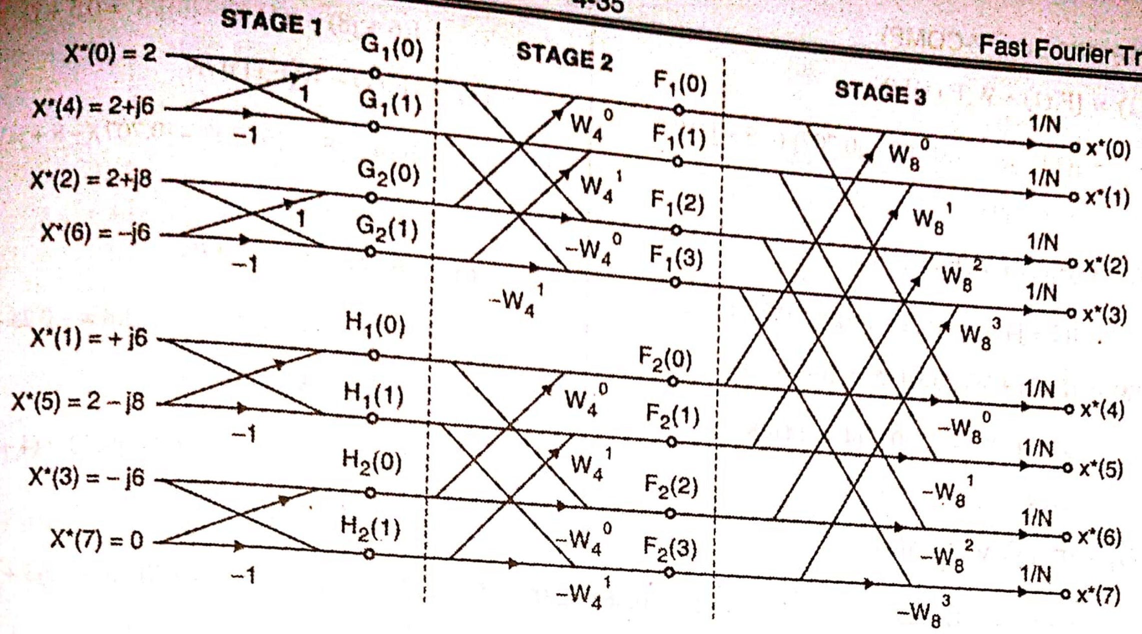

Fig. P. 4.6.6

Similarly,

$$W_8^0 = 1$$

$$W_8^1 = e^{\frac{-j2\pi}{8}\cdot 1} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$$

$$= 0.707 - j\,0.707$$

$$W_8^2 = e^{\frac{-j2\pi}{8}\cdot 2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi}{8}\cdot 3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}$$

$$= -0.707 - j\,0.707$$

We write down the equation at each stage and calculate the outputs.

Stage 1

$$G_1(0) = X^*(0) + X^*(4) = 4 + j6$$

$$G_1(1) = X^*(0) - X^*(4) = -j6$$

$$G_2(0) = X^*(2) + X^*(6) = 2 + j2$$

$$G_2(1) = X^*(2) - X^*(6) = 2 + j14$$

$$H_1(0) = X^*(1) + X^*(5) = 2 - j2$$

$$H_1(1) = X^*(1) - X^*(5) = -2 + j14$$

$$H_2(0) = X^*(3) + X^*(7) = -j6$$

$$H_2(1) = X^*(3) - X^*(7) = -j6$$

Stage 2

$$F_1(0) = G_1(0) + W_4^0 G_2(0) = (4 + j6) + (1)(2 + j2)$$

$$= 6 + j8$$

$$F_1(1) = G_1(1) + W_4^1 G_2(1)$$

$$= -j6 + [(-j)(2 + j14)] = 14 - j8$$

$$F_1(2) = G_1(0) - W_4^0 G_2(0)$$

$$= (4 + j6) - [(1)(2 + j2)] = 2 + j4$$

$$F_1(3) = G_1(1) - W_4^1 G_2(1)$$

$$= -j6 - [(-j)(2 + j14)] = -14 - j4$$

$$F_2(0) = H_1(0) + W_4^0 H_2(0)$$

$$= (2 - j2) + [(1)(-j6)] = 2 - j8$$

$$F_2(1) = H_1(1) + W_4^1 H_2(1)$$

$$= (-2 + j14) + [(-j)(-j6)]$$

$$= -8 + j14$$

$$F_2(2) = H_1(0) - W_4^0 H_2(0)$$

$$= (-2 - j2) - [(1)(-j6)]$$

$$= 2 + j14$$

$$F_2(3) = H_1(1) - W_4^1 H_2(1) = (-2 + j14) - [(-j)(-j6)]$$

$$= 4 + j14$$

Stage 3

$$x^*(0) = [F_1(0) + W_8^0 F_2(0)]/N$$

$$= [(6 + j8)] + [(1)(2 - j8)]/8 = 1$$

$$x^*(1) = [F_1(1) + W_8^1F_2(1)] /N$$
$$= [(14 - j8) + [(0.707 - j0.707)(-8 + j4)]/ 8$$
$$= 2.28 + j0.95$$
$$x^*(2) = [F_1(2) + W_8^2F_2(2)] /N$$
$$= [(2 + j4) + (-j)(2 + j4)]/8 = 0.75 + j0.25$$
$$x^*(3) = [F_1(3) + W_8^3F_2(3)] /N = [(-14 - j4)$$
$$+ [(-0.707 - j0.707)(4 + j14)]/8$$
$$= -0.86 - j2.09$$
$$x^*(4) = [F_1(0) - W_8^0F_2(0)] /N$$
$$= [(6 + j8) - [(1)(2 - j8)]/8 = 0.5 + j2$$
$$x^*(5) = [F_1(1) - W_8^1F_2(1)] /N$$
$$= [(14 - j8) - [(0.707 - j0.707)(-8 + j4)]/8$$
$$= 1.22 - j2.94$$
$$x^*(6) = [F_1(2) - W_8^2F_2(2)] /N$$
$$= [(2 + j4) - [(-j)(2 + j4)]]/8 = -0.25 + j0.75$$
$$x^*(7) = [F_1(3) - W_8^3F_2(3)] /N$$
$$= [(-14 - j4) - [(-0.707 - j0.707)(4 + j14)]/8$$
$$= -2.63 + j1.09$$

$$\therefore \quad x^*(n) = \{1, 2.28 + j0.95, 0.75 + j0.25, -0.86 - j2.09, 0.5 + j2, 1.22 - j2.94, -0.25 + j0.75, -2.63 + j1.09\}$$

What we have obtained is $x^*(n)$

$$\therefore \quad x(n) = \{1, 2.28 - j0.95, 0.75 - j0.25, -0.866 + j2.09, 0.5 - j2, 1.22 + j2.94, -0.25 - j0.75, -2.63 - j1.09\}$$

Ex. 4.6.7 : Find IDFT of X(k) = {10, – 2 + 2j, – 2, – 2 – 2j} using DIT-FFT algorithm.

Soln. :

$$X(k) = \{10, -2 + 2j, -2, -2 - 2j\}$$
$$\{X(0),\ X(1),\ X(2),\ X(3)\}$$

Since N = 4, we use a 4 – point DIT-FFT butterfly diagram.

In DIT-FFT, the input is in bit reversed order. We give $X^*(k)$ as input and divide the output obtained by N. We also take the complex conjugate of the result obtained.

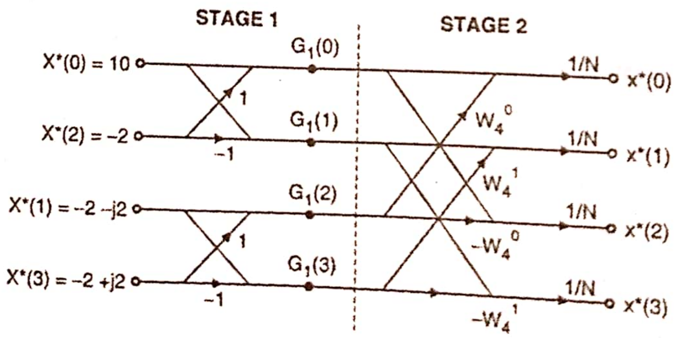

Fig. P. 4.6.7

Here,
$$W_4^0 = 1$$
$$W_4^1 = -j$$

Stage 1

$$G_1(0) = X^*(0) + X^*(2) = 10 + (-2) = 8$$
$$G_1(1) = X^*(1) - X^*(2) = 10 - (-2) = 12$$
$$G_1(2) = X^*(1) + X^*(3) = (-2 - j2) + (-2 + j2) = -4$$
$$G_1(3) = X^*(1) - X^*(3) = (-2 - j2) - (-2 + j2) = -j4$$

Stage 2

$$x^*(0) = [G_1(0) + W_4^0 G_1(2)]/4$$
$$= [8 + (1)(-4)]/4 = 1$$
$$x^*(1) = [G_1(1) + W_4^1 G_1(3)]/4$$
$$= [12 + (-j)(-j4)]/4 = 2$$
$$x^*(2) = [G_1(0) - W_4^0 G_1(2)]/4 = [8 - (1)(-4)]/4 = 3$$
$$x^*(3) = [G_1(1) - W_4^1 G_1(3)]/4$$
$$= [12 - (-j)(-j4)]/4 = 4$$
$$\therefore \quad x^*(n) = \{1, 2, 3, 4\}$$
$$\therefore \quad x(n) = \{1, 2, 3, 4\}$$

4.6.2 Filtering using FFT Algorithms

Ex. 4.6.8 : Determine circular convolution of $x(n) = \{1, 2, 1, 4\}$ and $h(n) = \{1, 2, 3, 2\}$ using Radix-2 - FFT.

Soln. : From the circular convolution property of the DFT, it is known than,

$$y(n) = x(n) \circledast h(n) \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}} X(k)\, H(k) = Y(k)$$

Hence to determine circular convolution using DFT, we follow the given steps

1. Compute DFT of x(n) to obtain X(k)
2. Compute DFT of h(n) to obtain H(k)
3. Obtain $\quad Y(k) = X(k) \cdot H(k)$
4. Compute IDFT of Y(k) to obtain y(n)

We will use the DIT-FFT to compute the DFT of x(n) and h(n). Since both are of length 4, we will use a 4-point DIF-FFT butterfly diagram. In DIF-FFT the output is ordered and input is in bit reversed form.

Step 1 : Compute DFT of x(n)

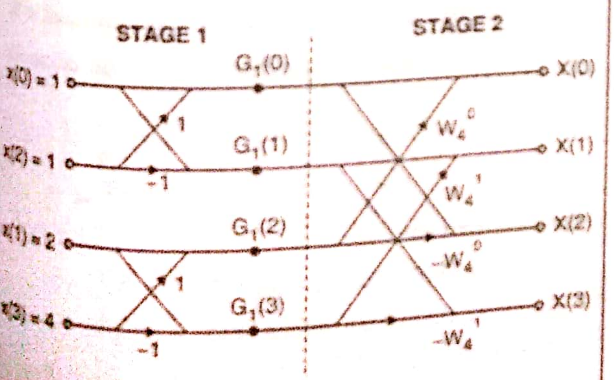

Fig. P. 4.6.8

Here, $W_4^0 = e^{\frac{-j2\pi}{4} \cdot 0} = \cos 0 - j \sin 0 = 1.$

$$W_4^1 = e^{\frac{-j2\pi}{4} \cdot 1} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2} = -j$$

Stage 1

$$G_1(0) = x(0) + x(2) = 1 + 1 = 2$$
$$G_1(1) = x(0) - x(2) = 1 - 1 = 0$$
$$G_1(2) = x(1) + x(3) = 2 + 4 = 6$$
$$G_1(3) = x(1) - x(3) = 2 - 4 = -2$$

Stage 2

$$X(0) = G_1(0) + W_4^0 G_1(2) = 2 + (1)(6) = 8$$
$$X(1) = G_1(1) + W_4^1 G_1(3) = 0 + (-j)(-2) = j2$$
$$X(2) = G_1(0) - W_4^0 G_1(2) = 2 - (1)(6) = -4$$
$$X(3) = G_1(1) - W_4^1 G_1(3) = 0 - (-j)(-2) = -j2$$
$$\therefore \quad X(k) = \{8, j2, -4, -j2\}$$

Step 2 : In a similar manner we compute the DFT of h(n)

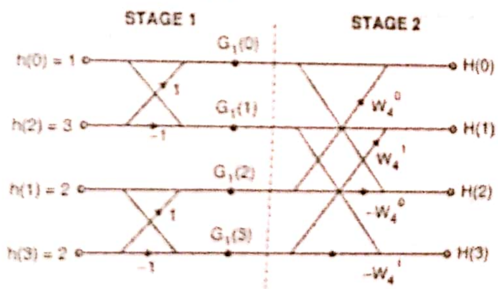

Fig. P. 4.6.8(a)

Stage 1

$$G_1(0) = h(0) + h(2) = 1 + 3 = 4$$
$$G_1(1) = h(0) - h(2) = 1 - 3 = -2$$
$$G_1(2) = h(1) + h(3) = 2 + 2 = 4$$
$$G_1(3) = h(1) - h(3) = 2 - 2 = 0$$

Stage 2

$$H_1(0) = G_1(0) + W_4^0 G_1(2) = 4 + (1)(4) = 8$$
$$H_1(1) = G_1(1) + W_4^1 G_1(3) = -2 + (-j)(0)$$
$$= -2$$
$$H_1(2) = G_1(0) - W_4^0 G_1(2) = 4 - (1)(4) = 0$$
$$H_1(3) = G_1(1) - W_4^1 G_1(3) = -2 - (-j)(0)$$
$$= -2$$
$$\therefore \quad H(k) = \{8, -2, 0, -2\}$$

Step 3 :

$$Y(k) = X(k) \cdot H(k)$$
$$Y(0) = X(0) \cdot H(0) = (8) \cdot (8) = 64$$
$$Y(1) = X(1) \cdot H(1) = (j2) \cdot (-2) = -j4$$
$$Y(2) = X(2) \cdot H(2) = (-4) \cdot (0) = 0$$
$$Y(3) = X(3) \cdot H(3) = (-j2)(-2) = j4$$
$$\therefore \quad Y(k) = \{64, -j4, 0, j4\}$$

Step 4 :

The final step is to obtain y(n). This is done by computing the IDFT of Y(k). We will use the DIT-FFT butterfly diagram with the required modification.

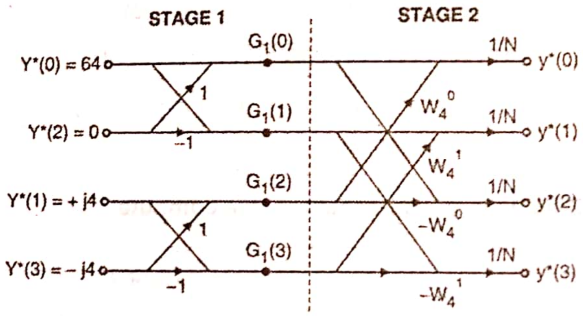

Fig. P. 4.6.8(b)

Stage 1

$$G_1(0) = y^*(0) + y^*(2) = 64 + 0 = 64$$
$$G_1(1) = y^*(0) - y^*(2) = 64 - 0 = 64$$
$$G_1(2) = y^*(1) + y^*(3) = j4 + (-j4) = 0$$
$$G_1(3) = y^*(1) - y^*(3) = j4 - (-j4) = j8$$

Stage 2

$$y^*(0) = [G_1(0) + W_4^0 G_1(2)]/4 = [64 + (1)(0)]/4 = 16$$
$$y^*(1) = [G_1(1) + W_4^1 G_1(3)]/4$$
$$= [64 + (-j)(j8)]/4 = 18$$
$$y^*(2) = [G_1(0) - W_4^0 G_1(2)/4 = [64 - (1)(0)]/4 = 16$$
$$y^*(3) = [G_1(1) - W_4^1 G_1(3)]/4$$
$$= [64 - (-j)(j8)/4 = 14$$
$$y^*(n) = \{16, 18, 16, 14\}$$

Since $y^*(n)$ has all real values

$$y(n) = y^*(n)$$
$$\therefore \quad y(n) = \{16, 18, 16, 14\}$$

We would have obtained the same result had we performed time domain circular convolution. Let us check the result.

$$y(n) = x(n) * h(0)$$

We generate a circular matrix of x(n)

$$\begin{array}{cc} x(n) & h(n) \\ \begin{bmatrix} 1 & 4 & 1 & 2 \\ 2 & 1 & 4 & 1 \\ 1 & 2 & 1 & 4 \\ 4 & 1 & 2 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \end{array} = \begin{bmatrix} 16 \\ 18 \\ 16 \\ 14 \end{bmatrix}$$

$$\therefore \quad y(n) = \{16, 18, 16, 14\}$$

Hence it is seen the results obtained are the same.

Ex. 4.6.9 : By means of FFT-IFFT technique, compute the linear convolution of x(n) = {2, 1, 2, 1}, h(n) = {1, 2, 3, 4}

Soln. : We are supposed to compute linear convolution using FFT-IFFT.

We know from the circular convolution property that,

$$y(n) = x(n) \circledast h(n) \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}} X(k) \cdot H(k) = y(k)$$

We also know that to obtain linear convolution from circular convolution, the lengths of each of signals should be $N_1 + N_2 - 1$.

In this case length x(n) and h(n) = 4.

$\therefore$ If x(n) and h(n) are made equal to (4 + 4 − 1) = 7, then circular convolution and linear convolution will give us the same result. Then we can use the above mentioned property to obtain linear convolution using FFT-IFFT. We increase the size of x(n) and h(n) by Zero padding.

$$\therefore \quad x(n) = \{2, 1, 2, 1, 0, 0, 0\}_{1 \times 7}$$
$$h(n) = \{1, 2, 3, 4, 0, 0, 0\}_{1 \times 7}$$

In order to use the radix-2 algorithm, the lengths of the signal must be powers of 2. We therefore pad one more zero to both x(n) and h(n) and make their lengths equal to 8.

Once the final result is obtained, we shall delete the last value.

$$x(n) = \{2, 1, 2, 1, 0, 0, 0, 0\}_{1 \times 8}$$
$$h(n) = \{1, 2, 3, 4, 0, 0, 0, 0\}_{1 \times 8}$$

Steps involved are:

1. Compute DFT of x(n) to obtain X(k)
2. Compute DFT of h(n) obtain H(k)
3. Obtain Y(k) = X(k) · H(k)
4. Compute IDFT of y(k) to obtain y(n)

We shall use the DIT-FFT algorithm to compute the DFT of x(n) and h(n).

Step 1 : Computing X(k) from x(n)

$$X(n) = \{2, 1, 2, 1, 0, 0, 0, 0\} \quad x(0)\ x(1)\ x(2)\ x(3)\ x(4)\ x(5)\ x(6)\ x(7)$$

Here $W_4^0 = 1$

$$W_4^1 = e^{\frac{-j2\pi}{4}\cdot 1} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Similarly, $W_8^0 = 1$

$$W_8^1 = e^{\frac{-j2\pi}{8}\cdot 1} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j\,0.707$$

$$W_8^2 = e^{\frac{-j2\pi}{8}\cdot 2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi}{8}\cdot 3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j\,0.707$$

We write down the equation at each stage and calculate the outputs.

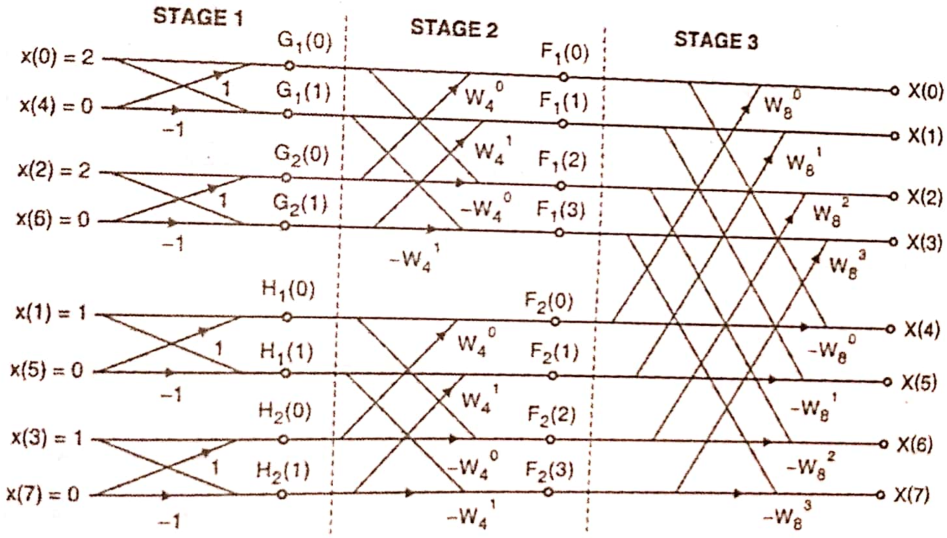

Fig. P. 4.6.9

Stage 1

$$G_1(0) = x(0) + x(4) = 2 + 0 = 2$$

$$G_1(1) = x(0) - x(4) = 2 - 0 = 2$$

$$G_2(0) = x(2) + x(6) = 2 + 0 = 2$$

$$G_2(1) = x(2) - x(6) = 2 - 0 = 2$$

$$H_1(0) = x(1) + x(5) = 1 + 0 = 1$$

$$H_1(1) = x(1) - x(5) = 1 - 0 = 1$$

$$H_2(0) = x(3) + x(7) = 1 + 0 = 1$$

$$H_2(1) = x(3) - x(7) = 1 - 0 = 1$$

Stage 2

$$F_1(0) = G_1(0) + W_4^0\, G_2(0) = 2 + (1)(2) = 4$$

$$F_1(1) = G_1(1) + W_4^1\, G_2(1) = 2 + (-j)(2) = 2 - j2$$

$$F_1(2) = G_1(0) - W_4^0\, G_2(0) = 2 - (1)(2) = 0$$

$$F_1(3) = G_1(1) - W_4^1\, G_2(1) = 2 - (-j)(2) = 2 + j2$$

$$F_2(0) = H_1(0) + W_4^0\, H_2(0) = 1 + (1)(1) = 2$$

$$F_2(1) = H_1(1) + W_4^1\, H_2(1) = 1 + (-j)(1) = 1 - j1$$

$$F_2(2) = H_1(0) - W_4^0\, H_2(0) = 1 - (1)(1) = 0$$

$$F_2(3) = H_1(1) - W_4^1\, H_2(1) = 1 - (-j)(1) = 1 + j1$$

Stage 3

$$X(0) = F_1(0) + W_8^0 F_2(0) = 4 + (1)(2) = 6$$

$$X(1) = F_1(1) + W_8^1 F_2(1)$$

$= (2 - j2) + [(0.0707 - j0.707)(1 - j1)]$

$= 2 - j3.41$

$X(2) = F_1(2) + W_8^2 F_2(2) = 0 + (-j)(0) = 0$

$X(3) = F_1(3) + W_8^3 F_2(3)$

$= (2 + j2) + [(-0.707 - j0.707(1 + j1)]$

$= 2 + j0.58$

$X(4) = F_1(0) - W_8^0 F_2(0) = 4 - (1)(2) = 2$

$X(5) = F_1(1) - W_8^1 F_2(1)$

$= (2 - j2) - [(0.707 - j0.707)(1 - j1)]$

$= 2 - j0.58$

$X(6) = F_1(2) - W_8^2 F_2(2) = 0 - (j)(0) = 0$

$X(7) = F_1(3) - W_8^3 F_2(3)$

$= (2 + j2) - [(-0.707 - j0.707)(1 + j1)]$

$= 2 + j3.41$

$\therefore \quad X(k) = \{6, 2 - j\,3.41, 0, 2 + j0.58, 2, 2 - j0.58, 0, 2 + j3.41\}$

Step 2 : Computing H (k) from h (n)

$h(n) = \{1, 2, 3, 4, 0, 0, 0, 0\}$

$\{h(0), h(1), h(2), h(3), h(4), h(5), h(6), h(7)\}$

We write down the equation at each stage and calculate the outputs

Stage 1

$G_1(0) = h(0) + h(4) = 1 + 0 = 1$

$G_1(1) = h(0) - h(4) = 1 - 0 = 1$

$G_2(0) = h(2) + h(6) = 3 + 0 = 3$

$G_2(1) = h(2) - h(6) = 3 - 0 = 3$

$H_1(0) = h(1) + h(5) = 2 + 0 = 2$

$H_1(1) = h(1) - h(5) = 2 - 0 = 2$

$H_2(0) = h(3) + h(7) = 4 + 0 = 4$

$H_2(1) = h(3) - h(7) = 4 - 0 = 4$

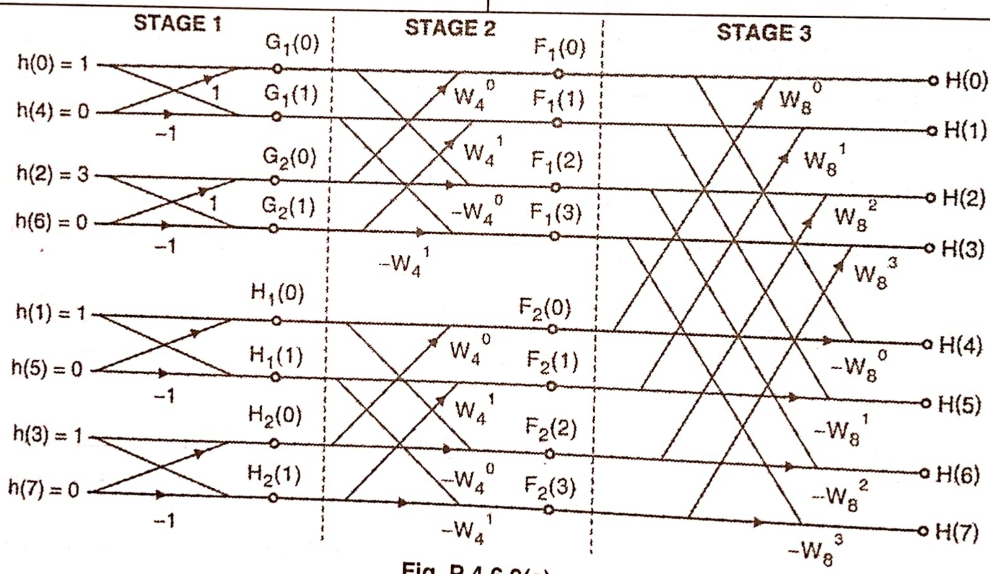

Fig. P.4.6.9(a)

Stage 2

$F_1(0) = G_1(0) + W_4^0 G_2(0) = 1 + (1)(3) = 4$

$F_1(1) = G_1(1) + W_4^1 G_2(1) = 1 + (-j)(3) = 1 - j3$

$F_1(2) = G_1(0) - W_4^0 G_2(0) = 1 - (1)(3) = -2$

$F_1(3) = G_1(1) - W_4^1 G_2(1) = 1 - (-j)(3) = 1 + j3$

$F_2(0) = H_1(0) + W_4^0 H_2(0) = 2 + (1)(4) = 6$

$F_2(1) = H_1(1) + W_4^1 H_2(1) = 2 + (-j)(4) = 2 - j4$

$F_2(2) = H_1(0) - W_4^0 H_2(0) = 2 - (1)(4) = -2$

$F_2(3) = H_1(1) - W_4^1 H_2(1) = 2 - (-j)(-4) = 2 + j4$

Stage 3

$H(0) = F_1(0) + W_8^0 F_2(0) = 4 + (1)(6) = 10$

$H(1) = F_1(1) + W_8^1 F_2(1)$

$= (1 - j3) + [(0.707 - j0.707)(2 - j4)]$

$= -0.41 - j7.24$

$H(2) = F_1(2) + W_8^2 F_2(2) = -2 + [(-j)(-2)] = -2 + j2$

$H(3) = F_1(3) + W_8^3 F_2(3)$

$= (1 + j3) + [(-0.707 - j0.707)\, 2 + j4]$

$= 2.41 - j1.24$

$H(4) = F_1(0) - W_8^0 F_2(0) = 4 - (1)(6) = -2$

$H(5) = F_1(1) - W_8^1 F_2(1)$

$= (1 - j3) - [(0.707 - j0.707)(2 - j4)]$

$= 2.41 + j1.24$

$H(6) = F_1(2) - W_8^2 F_2(2) = -2 - [(-j)(-2)]$

$= -2 - j2$

$H(7) = F_1(3) - W_8^3 F_2(3)$

$= (1 + j3) - [(-0.707 - j0.707)(2 + j4)]$

$= -0.41 + j7.24$

$H(k) = \{10, -0.41 - j7.24, -2 + j2, 2.41 - j1.24, -2, 2.41 + j1.24, -2 - j2, -0.41 + j7.24\}$

Step3: Obtain $Y(k) = X(k) \cdot H(k)$

$Y(0) = X(0) \cdot H(0) = 60$

$Y(1) = X(1) \cdot H(1) = -25.55 - j13.07$

$Y(2) = X(2) \cdot H(2) = 0$

$Y(3) = X(3) \cdot H(3) = 5.55 - j1.07$

$Y(4) = X(4) \cdot H(4) = -4$

$Y(5) = X(5) \cdot H(5) = 5.55 + j1.07$

$Y(6) = X(6) \cdot H(6) = 0$

$Y(7) = X(7) \cdot H(7) = -25.55 + j13.07$

$\therefore \quad Y(k) = \{60, -25.55 - j13.07, 0, 5.55 - j1.07, -4, 5.55 + j\,1.07, 0, -25.55 + j13.07\}$

Step 4: Compute IDFT of y(k) to obtain y(n) we use the DIT–FFT algorithm to compute the IDFT. The final output obtained after computing the IDFT is,

$y(n) = \{2, 5, 10, 16, 12, 11, 4, 0\}$

Since an extra zero was added to x(n) and h(n) to increase their lengths to 8, we remove the last zero of y(n).

$\therefore \quad y(n) = \{2, 5, 10, 16, 12, 11, 4\}$

This is the same result that would be obtained of we perform linear convolutions in the time domain. Let us check the result,

$y(n) = x(n) * h(n)$

We use the matrix method

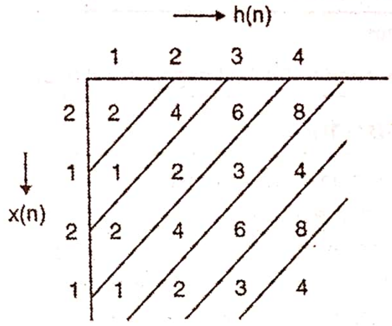

Fig. P.4.6.9(b)

$y(n) = \{2, 5, 10, 16, 12, 11, 4\}$

Hence it is seen that the results obtained one the same.

Summary

In this chapter we study the algorithms proposed by Cooley and Tukey. This method came to be known as the Fast Fourier Transform (F.F.T). The FFT requires only $N \log_2 N$ calculations as opposed to N^2 used by the DFT. Two important algorithms viz. Decimation in Time and decimation in Frequency for Radix 2 were derived. We also learnt how to use the butterfly diagram to compute the Inverse DFT. Many examples were solved to understand how to compute the DFT using the FFT. Filtering using the FFT algorithms were also carried out.

Review Questions

Q. 1 Draw basic butterfly structure for DIT- FFT.

Q. 2 Draw total signal flow graph for DIT-FFT.

Q. 3 Obtain computational complexity of FFT algorithm.

Q. 4 Explain in place computation of FFT.

Q. 5 Derive the first stage of DIF-FFT algorithm. Draw the basic butterfly structure for the same.

Q. 6 Compare DIT-FFT and DIF-FFT algorithms.

Q. 7 Write a short note on bit reversal in FFT.