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SE - A (A3)

DM TUTORIAL – 4

Q2 Find the generating function for the following sequence 1, 2, 3, 4, 5, 6...

ANS 2. Given sequence :- {1, 2, 3, 4, 5, 6 ...}
∴ ordinary generating function is obtained by multiplying each member of the sequence successively by $x^0, x^1, x^2, x^3 \ldots$ and taking the sum.

$$\therefore A(x) = 1x^0 + 2x^1 + 3x^2 + 4x^3 + \ldots$$
$$= 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$\therefore \text{Let } S = 1 + 2x + 3x^2 + 4x^3 + \ldots$$
$$xS = x + 2x^2 + 3x^3 + \ldots$$
$$(-) \quad (-)$$

$$S(1-x) = 1 + x + x^2 + x^3 + \ldots$$
$$S(1-x) = \frac{1}{1-x}$$

$$\therefore S = (1-x)^{-2}$$

$$\boxed{\therefore A(x) = (1-x)^{-2}} \longrightarrow (1)$$

∴ Exponential generating function is obtained by multiplying each number of the sequence successively by $\frac{x^0}{0!}, \frac{x^1}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!} \ldots$ & taking the sum.

$$\therefore B(x) = 1.\frac{x^0}{0!} + 2\frac{x^1}{1!} + 3\frac{x^2}{2!} + 4\frac{x^3}{3!} + \ldots$$

$\therefore B(x) = 1 + 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \dots$

$$\boxed{\therefore B(x) = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}} \longrightarrow (2)$$

Q4 Solve the recurrence relation $a_{x+2} - a_{x+1} - 6a_x = 4$

ANS 4. Since this is a non-homogeneous equation, its solution consists of 2 parts:

(i) The solution of the corresponding homogeneous equation

(ii) The particular solution we shall first obtain the solution of the corresponding homogeneous equation.

$$a_{x+2} - a_{x+1} - 6a_x = 0$$

Let $a^{x+2} = n^x$ then the characteristic equation is x.

$$n^2 - n - 6 = 0$$

$$(n-3)(n+2) = 0$$

$$\therefore n = 3 \mid n = -2$$

The roots are real, rational and distinct. let the solution of the corresponding homogeneous equation be

$$a_x^{(h)} = A(-2)^x + B(3)^x$$

$\therefore$ Since $f(x)$ = a constant, we assume the particular solution to be a constant i.e. we assume $\boxed{a_x^{(p)} = c}$, a constant.

$$\therefore \boxed{a_{x+1} = a_{x+2} = c}$$

Putting these values in the given recurrence relation

$$a_x^{(h)} = A(-2)^n + B(3)^n$$

$$\therefore c - c - 6c = 4$$

$$\therefore c = \frac{-2}{3}$$

$\therefore$ The particular solution is $\boxed{a_x^{(p)} = -2/3}$

Hence, the solution of the given recurrence relation is

$$a_x = a_x^{(h)} + a_x^{(p)}$$

$$= A(-2)^x + B(3)^x + (-2/3)$$

$$\boxed{a_x = A(-2)^x + B(3)^x - \frac{2}{3}}$$

Q6 Find the generating functions for the following sequence.
i] 0, 0, 0, 1, 2, 3, 4, 5, 6, 7...
ii] 6, -6, 6, -6, 6, -6, 6, -6...

ANS 6 i) Given : 0, 0, 0, 1, 2, 3, 4, 5, 6, 7...
$\therefore$ Ordinary generating function is obtained by multiplying each member of the sequence successively by $x^0, x^1, x^2, x^3, \ldots$ and taking the ~~solution~~ sum.

$\therefore A(x) = 0x^0 + 0x^1 + 0x^2 + 1x^3 + 2x^4 + \ldots$

$= 0 + 0 + 0 + x^3 + 2x^4 + \ldots$

$= x^3 + 2x^4 + 3x^5 + \ldots$

$= x^3 \left[1 + 2x + 3x^2 + \ldots\right]$

$$\boxed{A(x) = x^3 (1-x)^{-2}} \longrightarrow 1$$

$\therefore$ Exponential generating function is obtained by multiplying each number of the sequence successfuly by $\frac{x^0}{0!}, \frac{x^1}{1!}, \frac{x^2}{2!}, \ldots$ and taking the sum.

$$\therefore B(x) = 0\frac{x^0}{0!} + 0\frac{x^1}{1!} + 0\frac{x^2}{2!} + 1\frac{x^3}{3!} + 2\frac{x^4}{4!} + 3\frac{x^5}{5!} + \ldots$$

$$\boxed{\therefore B(x) = \sum_{n=3}^{\infty} \frac{(n-2)\,x^n}{n!}}$$

ii) Given sequence : 6, -6, 6, -6, 6, -6, ...

$\therefore$ Ordinary generating function is obtained by multiplying each member of the sequence successively by $x^0, x^1, x^2, x^3, \ldots$ and taking the sum.

$\therefore A(x) = 6x^0 - 6x^1 + 6x^2 - 6x^3 + \ldots$

$= 6\left[1 - x + x^2 - x^3 + \ldots\right]$

$$\boxed{\therefore A(x) = 6(1+x)^{-1}} \longrightarrow 1$$

∴ Exponential generating function is obtained by multiplying each number of the sequence successively by $\frac{x^0}{0!}, \frac{x^1}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots$ and taking the sum.

$$\therefore B(x) = 6\left[\frac{x^0}{0!} + \frac{(-x^1)}{1!} + \frac{x^2}{2!} + \frac{(-x^3)}{3!} + \ldots\right]$$

$$= 6\left[1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right]$$

$$\boxed{B(x) = 6e^{-x}}$$

Q8 Find the ordinary generating functions for the given sequences

i] $\{0, 1, 2, 3, 4, 5 \ldots\}$

ii] $\{1, 2, 3, 4, 5 \ldots\}$

iii] $\{0, 3, 32, 33 \ldots\}$

iv] $\{2, 2, 2, 2, 2 \ldots\}$

ANS 8 To find the ordinary generating functions. It is obtained by multiplying each member of the given sequence successively by $x^0, x^1, x^2, \ldots$ and taking the sum.

i) Given sequence : $\{0, 1, 2, 3, 4, 5 \ldots\}$

corresponding ordinary generating function :-

$$A(x) = 0x^0 + 1x^1 + 2x^2 + 3x^3 + \ldots$$

$$= x\left[1 + 2x + 3x^2 + \ldots\right]$$

$$\boxed{\therefore A(x) = x(1-x)^{-2}}$$

ii) Given sequence : $\{1, 2, 3, 4 \ldots\}$

corresponding ordinary generating function.

$$B(x) = 1x^0 + 2x^1 + 3x^2 + 4x^3 + \ldots$$
$$= 1 + 2x + 3x^2 + 4x^3 + \ldots$$
$$\boxed{B(x) = (1-x)^{-2}}$$

iii) Given sequence : $\{0, 3, 32, 33 \ldots\}$

corresponding ordinary generating function.

$$C(x) = 0x^0 + 3x^1 + 32x^2 + 33x^3 + \ldots$$
$$\boxed{C(x) = 3x + 32x^2 + 33x^3 + \ldots}$$

iv) Given sequence : $\{2, 2, 2, 2, 2 \ldots\}$

corresponding ordinary generating function.

$$D(x) = 2\,[x^0 + x^1 + x^2 + x^3 + \ldots]$$
$$= 2\,[1 + x^1 + x^2 + x^3 + \ldots]$$
$$\boxed{\therefore D(x) = 2\,(1-x)^{-1}}$$

Q10 Find the generating function for the following finite segments.

i] 2, 2, 2, 2, 2, 2

ii] 1, 1, 1, 1, 1, 1

ANS 10 To find ordinary generating function just multiply each term of the given sequence successively by $x^0, x^1, x^2, x^3 \ldots$ and taking the sum.

To find exponential generating function, just multiply each term of the given sequence successively by $\frac{x^0}{0!}, \frac{x^1}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots$ and taking the sum.

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i) Given sequence : 2, 2, 2, 2, 2, 2 [$\because$ Finite sequence]

corresponding ordinary generating function :

$$A(x) = 2\left[x^0 + x^1 + x^2 + x^3 + x^4 + x^5\right]$$
$$= 2\left[1 + x^1 + x^2 + x^3 + x^4 + x^5\right]$$
$$= 2\left[\frac{x^6 - 1}{x - 1}\right]$$

$$\boxed{\therefore A(x) = \frac{2(x^6 - 1)}{(x-1)}}$$

Corresponding Exponential generating functions :

$$B(x) = 2\left[\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right]$$
$$= 2\left[1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right]$$

$$\boxed{\therefore B(x) = 2\sum_{n=0}^{5} \frac{x^n}{n!}}$$

ii) Given sequence : 1, 1, 1, 1, 1, 1 [$\because$ Finite sequence]

corresponding ordinary generating functions.

$$C(x) = x^0 + x^1 + x^2 + x^3 + x^4 + x^5$$
$$= 1 + x^1 + x^2 + x^3 + x^4 + x^5$$

$$\boxed{\therefore C(x) = \frac{x^6 - 1}{x - 1}}$$

corresponding exponential generating function :

$$D(x) = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$\boxed{D(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}}$$

Q.12 Solve the recurrance relation $a_n = -3[a_{n-1} + a_{n-2}] - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$ and $a_2 = 15$

ANS 12 The given recurrance relation is $\boxed{a_n + 3[a_{n-1} + a_{n-2}] + a_{n-3} = 0}$ — (1)

let $a_n = x^n$ be a solution of (1)

The characteristic equation of (1) is, $x^3 + 3x^2 + 3x + 1 = 0$

$\therefore (x+1)^3 = 0$

$\boxed{\therefore x = -1, -1, -1}$

The roots are real, rational and repeated thrice.

Hence, let the general solution be

$$a_n = b_1 \alpha^n + b_2 n^1 \alpha^n + b_3 n^2 \alpha^n$$

$\boxed{\text{Here } \alpha = -1}$

$\therefore$ $\boxed{a_n = b_1 (-1)^n + b_2 n (-1)^n + b_3 n^2 (-1)^n}$ — (2)

We now use initial conditions to find b_1, b_2, b_3

putting $n=0$ in (2), we get $a_0 = b_1(-1)^0 + b_2(0)(-1)^0 + b_3(0)^2(-1)^0$

$\boxed{\therefore b_1 = 5}$ — (3)

Putting $n = 1$, in (2) we get

$a_1 = b_1(-1)^1 + b_2(1)(-1)^1 + b_3(1)^2(-1)^1$

$-9 = 5(-1) + (-b_2) - b_3$

$\boxed{\therefore b_2 + b_3 = 4}$ — (4)

Putting $n = 2$, in (2) we get

$a_2 = b_1(-1)^2 + b_2(2)(-1)^2 + b_3(2)^2(-1)^2$

$15 = 5 + 2b_2 + 4b_3$

$\boxed{\therefore b_2 + 2b_3 = 5}$ — (5)

From (4), (5) we get $b_3 = 1$, $b_2 = 3$

$\therefore$ The required recurrence relation is

$\boxed{a_n = 5(-1)^n + 3n(-1)^n + n^2(-1)^n}$

Q14 Find the generating function for the following finite sequences.

i] 1, 2, 3, 4

ii] 1, 1, 1, 1, 1, 1

ANS 14 To find ordinary generating function, just multiply each term of the given sequence successively by $x^0, x^1, x^2, \ldots$ and taking the sum.

To find emponential generating function, just multiply each term of the given sequence with successively by $\frac{x^0}{0!}, \frac{x^1}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!} \ldots$ and taking the sum.

i] 1, 2, 3, 4 → Finite given sequence.

corresponding ordinary generating function:

$$A(x) = 1x^0 + 2x^1 + 3x^2 + 4x^3$$

$$A(x) = 1 + 2x + 3x^2 + 4x^3$$

$$\boxed{\therefore A(x) = -\frac{(x^4-1)}{(x-1)^2} = \frac{1-x^4}{(x-1)^2}}$$

corresponding emponential generating function:

$$B(x) = \frac{1x^0}{0!} + \frac{2x^1}{1!} + \frac{3x^2}{2!} + \frac{4x^3}{3!}$$

$$B(x) = 1 + \frac{2x}{1!} + \frac{3x^2}{2!} + \frac{4x^3}{3!}$$

$$\boxed{B(x) = \sum_{n=0}^{3} (n+1)\frac{x^n}{n!}}$$

ii] 1, 1, 1, 1, 1, 1 → Finite given sequence.

corresponding ordinary generating function:

$$C(x) = x^0 + x^1 + x^2 + x^3 + x^4 + x^5$$

$$= 1 + x^1 + x^2 + x^3 + x^4 + x^5$$

$$\boxed{C(x) = \frac{x^6 - 1}{x - 1}}$$

corresponding exponential generating function :

$$D(x) = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$D(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$\boxed{D(x) = \sum_{n=0}^{5} \frac{x^n}{n!}}$$

Q16 Find the ordinary generating functions for the given sequences

i] $\{1, 2, 3, 4, 5\}$

ii] $\{2, 2, 2, 2 \ldots\}$

iii] $\{1, 1, 1, 1, 1 \ldots\}$

ANS 16 To find ordinary generating function, just multiply each term of the sequence successively with $x^0, x^1, x^2 \ldots$ and taking the sum.

i] Given sequence $\longrightarrow$ $\{1, 2, 3, 4, 5, \ldots\}$

corresponding ordinary generating function :

$$A(x) = 1x^0 + 2x^1 + 3x^2 + 4x^3 + 5x^4 + \ldots$$

$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 \ldots$$

$$\boxed{A(x) = (1-x)^{-2}}$$

ii] Given sequence $\longrightarrow$ $\{2, 2, 2, 2 \ldots\}$

Corresponding ordinary generating function :

$$B(x) = 2\left[x^0 + x^1 + x^2 + x^3 \ldots\right]$$

$$= 2\left[1 + x + x^2 + x^3 + \ldots\right]$$

$$\boxed{\therefore B(x) = 2(1-x)^{-1}}$$

iii] Given sequence $\longrightarrow \{1, 1, 1, 1, 1, \dots\}$

Corresponding ordinary generating function :

$$c(x) = x^0 + x^1 + x^2 + x^3 + \dots$$
$$= 1 + x^1 + x^2 + x^3 + \dots$$
$$\boxed{c(x) = (1-x)^{-1}}$$

Q 18 Find the complete solutions of the general recurrance relations

$$a_n + 2a_{n-1} = n+3 \quad \text{for } n \geq 1 \text{ . with } a_0 = 3 .$$

ANS 18 Since this is a non-homogeneous equation, its solution consists of 2 parts :

i) The solution of the corresponding homogeneous equation.

ii) The Particular solution.

We shall first obtain the solution of the corresponding homogeneous equation $a_n + 2a_{n-1} = 0$.

Let $a_n = x^n$, then the characteristic equation is

$$\therefore x^{n-1}(x+2) = 0$$
$$\therefore x + 2 = 0 \Rightarrow \boxed{x = -2}$$

The root is real, rational and distinct. Let the solution of the corresponding homogeneous equation be :

$$\boxed{a_n^{(h)} = B(-2)^n} \quad — ①$$

Since $f(n) = n+3$ is linear, we assume the particular solution to be $\boxed{a_n^{(p)} = an + b}$ — ②

Putting the value of a_n in the given equation.

$$(an + b) + 2[a(n-1) + b] = n + 3$$
$$\therefore 3an + (3b - 2a) = n + 3$$

$\therefore 3a = 1$ | $3b - 2a = 3$

③ — $\boxed{a = 1/3}$ | $3b - 2/3 = 3$

$$\boxed{b = \frac{11}{9}} \quad — ④$$

Hence from ②, ③ and ④

$$\boxed{a_n^{(P)} = \frac{n}{3} + \frac{11}{9}} \quad \text{—(5)}$$

Now to find constant B in $a_n^{(h)}$ we use the given condition $a_0 = 3$ when $n = 0$.

$\therefore a_0 = 3$

$\therefore$ Final solution : $a_n = a_n^{(h)} + a_n^{(P)}$

$$\boxed{a_n = B(-2)^n + \frac{n}{3} + \frac{11}{9}} \quad \text{—(6)}$$

Putting $n=0$ in (6) we get,

$a_0 = B(-2)^0 + 0 + 11/9$

$3 = B + \frac{11}{9}$

$$\boxed{B = \frac{16}{9}} \quad \text{—— (7)}$$

$\therefore$ From (6), (7) the total solution of the given recurrance relation is $\boxed{a_n = \frac{16}{9}(-2)^n + \frac{n}{3} + \frac{11}{9}}$

Q20 Determine the generating functions of the numeric function a_x

i] $a_x = 3^x + 4^{x+1}$, $x \geq 0$

ii] $a_x = 5$, $x \geq 0$

ANS 20 To find ordinary generating function, just multiply each term of the sequence with $x^0, x^1, x^2, \dots$ successively and taking the sum.

To find exponential generating function, just multiply each term of the sequence successively with $\frac{x^0}{0!}, \frac{x^1}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots$ and taking the sum.

i] $a_r = 3^r + 4^{r+1}$, $r \geq 0$

Putting $r = 0, 1, 2, 3, \ldots$ we get sequence as $\{5, 19, 72, 283, \ldots\}$.

$\therefore$ corresponding ordinary generating function :

$$A(x) = 5x^0 + 19x^1 + 72x^2 + 283x^3 + \ldots$$

$$\boxed{A(x) = 5 + 19x + 72x^2 + 283x^3 + \ldots}$$

$\therefore$ corresponding exponential generating function :

$$B(x) = \frac{5x^0}{0!} + \frac{19x^1}{1!} + \frac{72x^2}{2!} + \frac{283x^3}{3!} + \ldots$$

$$\boxed{B(x) = \sum_{n=0}^{\infty} (3^n + 4^{n+1}) \frac{x^n}{n!}}$$

ii] $a = 5$, $r \geq 0$

Putting $x = 0, 1, 2, 3, \ldots$ we get sequence as $\{5, 5, 5, 5 \ldots\}$

$\therefore$ corresponding ordinary generating function :

$$\therefore c(x) = 5 [x^0 + x^1 + x^2 + x^3 + \ldots]$$

$$= 5 [1 + x + x^2 + x^3 + \ldots]$$

$$\boxed{\therefore c(x) = 5 (1 - x)^{-1}}$$

$\therefore$ corresponding exponential generating function :

$$\therefore D(x) = 5 \left[\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right]$$

$$= 5 \left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right]$$

$$\boxed{D(x) = 5 e^x}$$

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Q22 What is the solution of the recurrance relation.

$a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$ with $a_0 = 8,\ a_1 = 6,\ a_2 = 26$.

ANS 22. The given recurrance relation is:

$$\boxed{a_n + a_{n-1} - 4a_{n-2} - 4a_{n-3} = 0} \quad \text{——(1)}$$

Let $a_n = r^n$ be the solution of (1). The characteristic equation of (1) is $r^3 + r^2 - 4r - 4 = 0$

$\therefore r^2(r+1) = 4(r+1)$

$$\boxed{\therefore r = -2, -1, 2}$$

The roots are real, rational and distinct. Hence, let the general solution be $\boxed{a_n = b_1(-2)^n + b_2(-1)^n + b_3(2)^n}$ —(2)

To find b_1, b_2, b_3 we use the initial conditions given to us i.e. $a_0 = 8,\ a_1 = 6,\ a_2 = 26$

$\therefore$ Putting $n=0$ in (2) we get

$a_0 = b_1(-2)^0 + b_2(-1)^0 + b_3(2)^0$

$$\boxed{8 = b_1 + b_2 + b_3} \quad \text{— (3)}$$

$\therefore$ Putting $n=1$ in (2) we get,

$a_1 = b_1(-2)^1 + b_2(-1)^1 + b_3(2)^1$

$$\boxed{6 = -2b_1 - b_2 + 2b_3} \quad \text{— (4)}$$

Putting $n=2$ in (2) we get,

$\therefore a_2 = b_1(-2)^2 + b_2(-1)^2 + b_3(2)^2$

$$\boxed{\therefore 26 = 4b_1 + b_2 + 4b_3} \quad \text{—(5)}$$

$\therefore$ Adding (3), (4) we get $\boxed{14 = 3b_3 - b_1}$ — (6)

$\therefore$ Adding (4),(5) we get $32 = 2b_1 + 6b_3$

$\boxed{\therefore 16 = 3b_3 + b_1}$ — (7)

$\therefore$ Adding (6),(7) we get $6b_3 = 30$

$\Rightarrow \boxed{b_3 = 5}$ — (8)

$\therefore$ Putting (8) in (6) we get $\boxed{b_1 = 1}$ — (9)

Putting (8),(9) in (3) we get $\boxed{b_2 = 2}$ — (10)

From (2), (8), (9), (10) the solution of the given recurrance relation is

$$\boxed{a_n = (-2)^n + 2(-1)^n + 5(2)^n}$$