1 Limitations of Standard FDTD Algorithm

Standard FDTD algorithm cannot cater for negative values of permittivity or permeability. This is because of the Courant stability criterion. As soon as the permeability or permittivity becomes less than unity the algorithm will not remain stable. A metamaterial object can be modelled as a dispersive substance using either the Lorentz or Drude dispersive models. These models can yield negative values of permittivity (or permeability) for certain frequency ranges [1]. Using these dispersive models, FDTD update equations are modified and permittivity and permeabilities are replaced with terms dependent on frequency of operation (ω).

2 Drude Dispersive Model

In ideal conditions the permittivity (and permeability) of a material remain constant for any frequency and throughout the structure of that material. Speed of electromagnetic waves in such a medium remain constant if frequency changes. Additionally, there is no loss in energy as the waves pass through the medium.

In reality, such a material does not exist. Speed of EM waves varies with frequency of operation. Also, there is a loss associated with the material. A material is dispersive if its permittivity or permeability is dependent on frequency [2, Ch. 10].

This section follows the treatment of [2, Ch. 10, 289–290] for deriving permittivity relationship given by Drude model. The Drude model is described by the following second order differential equation that relates the net force on charges moving under the influence of an electric field and facing an impeding force due to collisions with material.

$$M\frac{d^2\mathbf{x}}{dt^2} = Q\mathbf{E}(t) - Mg\frac{d\mathbf{x}}{dt} \tag{1}$$

The left side of this equation gives mass times acceleration or the net force on charge. Here M is the mass of charge, Q the amount of charge, $\mathbf{E}(t)$ is electric field that may vary with time and g is the damping coefficient. Equation 1 can be converted to frequency domain using the relationships $\partial^2/\partial t^2 \rightarrow (j\omega)^2$ and $\partial/\partial t \rightarrow j\omega$, obtaining

$$M(j\omega)^2\hat{\mathbf{x}}(\omega) + Mg(j\omega)\hat{\mathbf{x}}(\omega) = Q\hat{\mathbf{E}}(\omega). \tag{2}$$

The polarization vector $\mathbf{P}$ is given by

$$\hat{\mathbf{P}} = NQ\hat{\mathbf{x}}. \tag{3}$$

Here, N is the number of dipoles per unit volume. By eliminating $\hat{\mathbf{x}}$ from equations 2 and 3 and rearranging terms polarization can be expressed as

$$\hat{\mathbf{P}}(\omega) = -\epsilon_0\frac{\frac{NQ^2}{M\epsilon_0}}{\omega^2 - jg\omega}\hat{\mathbf{E}}(\omega). \tag{4}$$

By letting $\omega_p^2 = NQ^2/M\epsilon_0$, the electric susceptibility is obtained as

$$\hat{\chi_e}(\omega) = -\frac{\omega_p^2}{\omega^2 - jg\omega}. \tag{5}$$

The relative permittivity in Drude model is then

$$\hat{\epsilon_r}(\omega) = \epsilon_\infty - \frac{\omega_p^2}{\omega^2 - jg\omega}. \tag{6}$$

Setting $g = 0$ and $\epsilon_\infty = 1$, relative permittivity comes out to be negative for $\omega/\omega_p > 1$ (figure 1). Thus, Drude model can be effectively used to model metamaterials with permittivity or permeability less than one by incorporating it into FDTD update equations.

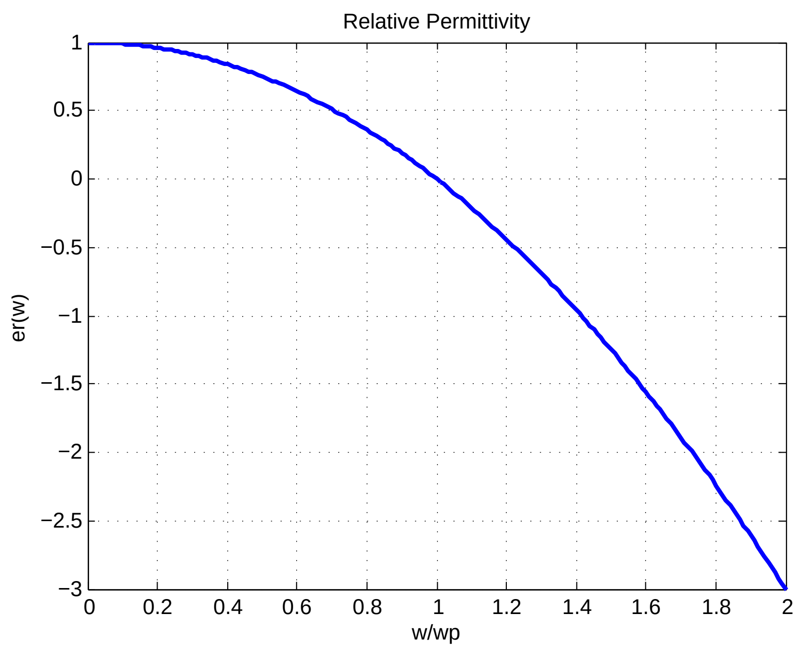

Figure 1: ϵ_r plotted against ω/ω_p for $\epsilon_\infty = 1$ and $g = 0$

3 FDTD Equations in 1D

Here, the approach outlined in [3] for derivation of FDTD auxiliary update equations is followed. Magnetic field **H** and magnetic flux density **B** are related by the constitutive parameter μ, expressed as

$$\mathbf{B} = \mu\mathbf{H}. \tag{7}$$

Where, $\mu = \mu_0\mu_r$. For a lossless, homogeneous, anisotropic and dispersion–less material, relative permeability or μ_r is a scalar constant. For dispersive material μ_r is a function of frequency. Similar to equation 6, relative permeability for Drude model is given by

$$\begin{aligned}\hat{\mu_r}(\omega) &= \mu_\infty - \frac{\omega_p^2}{\omega^2 - jg\omega} \\ &= \frac{\mu_\infty(\omega^2 - jg\omega) - \omega_p^2}{\omega^2 - jg\omega}.\end{aligned} \tag{8}$$

From equation 7 and 8,

$$\mathbf{B}\left(\frac{\omega^2 - jg\omega}{\mu_\infty(\omega^2 - jg\omega) - \omega_p^2}\right) = \mathbf{H}$$

and

$$\omega^2\mathbf{B} - g(j\omega)\mathbf{B} = \mu_\infty\omega^2\mathbf{H} - \omega_p^2\mathbf{H} - \mu_\infty g(j\omega)\mathbf{H}. \tag{9}$$

Frequency domain quantities can be converted to time–domain using the relationships $j\omega \to \partial/\partial t$ and $\omega^2 \to -\partial^2/\partial t^2$. Moreover, fields multiplying with ω_p^2 are averaged in time. Second–order difference scheme is used, both for single and double derivatives to keep all the terms in accordance with the second–order nature of whole expression. That would result in easier implementation. Assuming anisotropic nature of material, only H_y and B_y can be considered. Also, ω_{pm} and ω_{pe} are used for magnetic and electric field expressions instead of ω_p, whereas, γ_m and γ_e are used for magnetic and electric collision frequencies, respectively.

$$\begin{aligned}\left(\frac{B_y^{n+1} - 2B_y^n + B_y^{n-1}}{(\Delta t)^2}\right) + \gamma_m\left(\frac{B_y^{n+1} - B_y^{n-1}}{2\Delta t}\right) = \\ \mu_0\mu_\infty\left(\frac{H_y^{n+1} - 2H_y^n + H_y^{n-1}}{(\Delta t)^2}\right) \\ + \mu_0\omega_{pm}^2\left(\frac{H_y^{n+1} + 2H_y^n + H_y^{n-1}}{4}\right) \\ + \mu_0\mu_\infty\gamma_m\left(\frac{H_y^{n+1} - H_y^{n-1}}{2\Delta t}\right).\end{aligned} \tag{10}$$

By separating H_y^{n+1}, the final form of equation is

$$\begin{aligned}H_y^{n+1} &= a_m\left(B_y^{n+1} - 2B_y^n + B_y^{n-1}\right) + b_m\left(B_y^{n+1} - B_y^{n-1}\right) \\ &\quad + c_m\left(2H_y^n - H_y^{n-1}\right) + d_m\left(2H_y^n + H_y^{n-1}\right) + e_m H_y^{n-1}.\end{aligned} \tag{11}$$

Where,

$$a_m = \frac{4}{4\mu_0\mu_\infty + \mu_0\omega_{pm}^2 \left(\Delta t\right)^2 + \mu_0\mu_\infty\gamma_m \left(2\Delta t\right)},$$
$$b_m = \frac{\gamma_m \left(2\Delta t\right)}{4\mu_0\mu_\infty + \mu_0\omega_{pm}^2 \left(\Delta t\right)^2 + \mu_0\mu_\infty\gamma_m \left(2\Delta t\right)},$$
$$c_m = \frac{4\mu_0\mu_\infty}{4\mu_0\mu_\infty + \mu_0\omega_{pm}^2 \left(\Delta t\right)^2 + \mu_0\mu_\infty\gamma_m \left(2\Delta t\right)},$$
$$d_m = \frac{-\mu_0\omega_{pm}^2 \left(\Delta t\right)^2}{4\mu_0\mu_\infty + \mu_0\omega_{pm}^2 \left(\Delta t\right)^2 + \mu_0\mu_\infty\gamma_m \left(2\Delta t\right)},$$
$$e_m = \frac{\mu_0\mu_\infty\gamma_m \left(2\Delta t\right)}{4\mu_0\mu_\infty + \mu_0\omega_{pm}^2 \left(\Delta t\right)^2 + \mu_0\mu_\infty\gamma_m \left(2\Delta t\right)}.$$

A similar derivation can be carried out to obtain the auxiliary update equation for electric field given by

$$\begin{aligned} E_x^{n+1} &= a_e \left(D_x^{n+1} - 2D_x^n + D_x^{n-1}\right) + b_e \left(D_x^{n+1} - D_x^{n-1}\right) \\ &\quad + c_e \left(2E_x^n - E_x^{n-1}\right) + d_e \left(2E_x^n + E_x^{n-1}\right) + e_e E_x^{n-1}. \end{aligned} \tag{12}$$

Where

$$a_e = \frac{4}{4\epsilon_0\epsilon_\infty + \epsilon_0\omega_{pe}^2 \left(\Delta t\right)^2 + \epsilon_0\epsilon_\infty\gamma_e \left(2\Delta t\right)},$$
$$b_e = \frac{\gamma_e \left(2\Delta t\right)}{4\epsilon_0\epsilon_\infty + \epsilon_0\omega_{pe}^2 \left(\Delta t\right)^2 + \epsilon_0\epsilon_\infty\gamma_e \left(2\Delta t\right)},$$
$$c_e = \frac{4\epsilon_0\epsilon_\infty}{4\epsilon_0\epsilon_\infty + \epsilon_0\omega_{pe}^2 \left(\Delta t\right)^2 + \epsilon_0\epsilon_\infty\gamma_e \left(2\Delta t\right)},$$
$$d_e = \frac{-\epsilon_0\omega_{pe}^2 \left(\Delta t\right)^2}{4\epsilon_0\epsilon_\infty + \epsilon_0\omega_{pe}^2 \left(\Delta t\right)^2 + \epsilon_0\epsilon_\infty\gamma_e \left(2\Delta t\right)},$$
$$e_e = \frac{\epsilon_0\epsilon_\infty\gamma_e \left(2\Delta t\right)}{4\epsilon_0\epsilon_\infty + \epsilon_0\omega_{pe}^2 \left(\Delta t\right)^2 + \epsilon_0\epsilon_\infty\gamma_e \left(2\Delta t\right)}.$$

For a wave propagating in z direction, FDTD update equations are

$$B_y^{n+1}(k) = B_y^n(k) + \frac{\Delta t}{\Delta z} \left(E_x^n(k) - E_x^n(k+1)\right) \tag{13}$$

and

$$D_x^{n+1}(k) = D_x^n(k) + \frac{\Delta t}{\Delta z} \left(H_y^{n+1}(k-1) - H_y^{n+1}(k)\right). \tag{14}$$

Equations 13 and 14 drive the FDTD algorithm which give future values of By and Dx from past fields. Equations 11 and 12 are auxiliary equations which give future fields H_y and E_x at $n+1$.

4 Simulation of 1D DNG Slab

4.1 Problem Specification

An electromagnetic wave travelling in z direction is incident on a slab with negative values of permittivity and permeability (DNG) at the frequency of operation [4]. Sinusoidal wave, Gaussian pulse and Ricker wavelet are used as sources. Transmission and reflection coefficients are calculated at the air–slab interface. Refractive index of slab for a range of frequencies is also computed. By varying parameters, the simulation can be scaled to any desired frequency or wavelength. Matlab code of this simulation is given in appendix **??**.

4.2 Implementation Details

There are three main phases in the implementation:

1. Initialisation
2. Simulation
3. Post-processing

Figure 2 depicts the flow of program.

4.3 Initialisation

In this phase, simulation parameters are specified. Simulation parameters are total number of spatial steps, time steps, wavelength and frequency of operation etc. Field arrays are then allocated and initialised. Any additional fields that are required for post–processing are also initialised. Arrays containing γ_m, γ_e, ω_{pm}^2 and ω_{pe}^2 must be initialised with values specified for each location in the domain.

4.4 Simulation

A dry run must be carried out without any obstacle if reflection coefficient is to be calculated in order to record the incident field where obstacle is to be placed. The actual simulation consists of five steps:

1. Compute future value of B_y from past values of B_y and E_x (equation 13).
2. Compute future value of H_y from B_y calculated in step 1 (equation 11).
3. Compute future value of D_x from past value of D_x and H_y in step 2 (equation 14).
4. Compute future value of E_x from D_x calculated in step 3 (equation 12).
5. Update additive source at specified source location for current time step.

Electric field snapshots are saved at specified intervals during simulation and played–back as a movie after simulation ends.

4.5 Post–processing

After the simulation ends, reflection and transmission coefficients are calculated at the air–slab interface. Refractive index is calculated in the slab. A time–domain movie of the simulation is played that shows incident wave passing through the slab and undergoing dispersion.

5 Simulation Results

The simulation is run for both lossless and lossy cases with sinusoidal, Gaussian and Ricker wavelet sources. The slab parameters are set such that at frequency of operation, f_0, the permittivity and permeability of slab are both negative and result in a refractive index $n = -1$.

5.1 Simulation Parameters

The number of spatial steps is set as 4096 and simulation is run for 4×4096 time steps. The slab is located between steps 1365 and 2731. Δz or spatial step is set as 3 mm and time step, Δt, is set as 50 ps. Frequency of operation is $f_0 = 0.1953125$ GHz and Courant number for this configuration comes out to be $S_c = 0.5$. In order to obtain relative permittivity and permeability of -1 at required f_0, plasma frequencies are set as $\omega_{pm}^2 = \omega_{pe}^2 = 2 \times (2\pi f_0)^2$ with $\epsilon_\infty = \mu_\infty = 1$. First order absorbing boundary condition (ABC) are applied on fields at end points.

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[3] Y. H. Argyropoulos C., Yan Zhao, “A radially-dependent dispersive finite-difference time-domain method for the evaluation of electromagnetic cloaks,” *IEEE Transactions on Antennas and Propagation*, vol. 57, pp. 1432–1441, May 2009.

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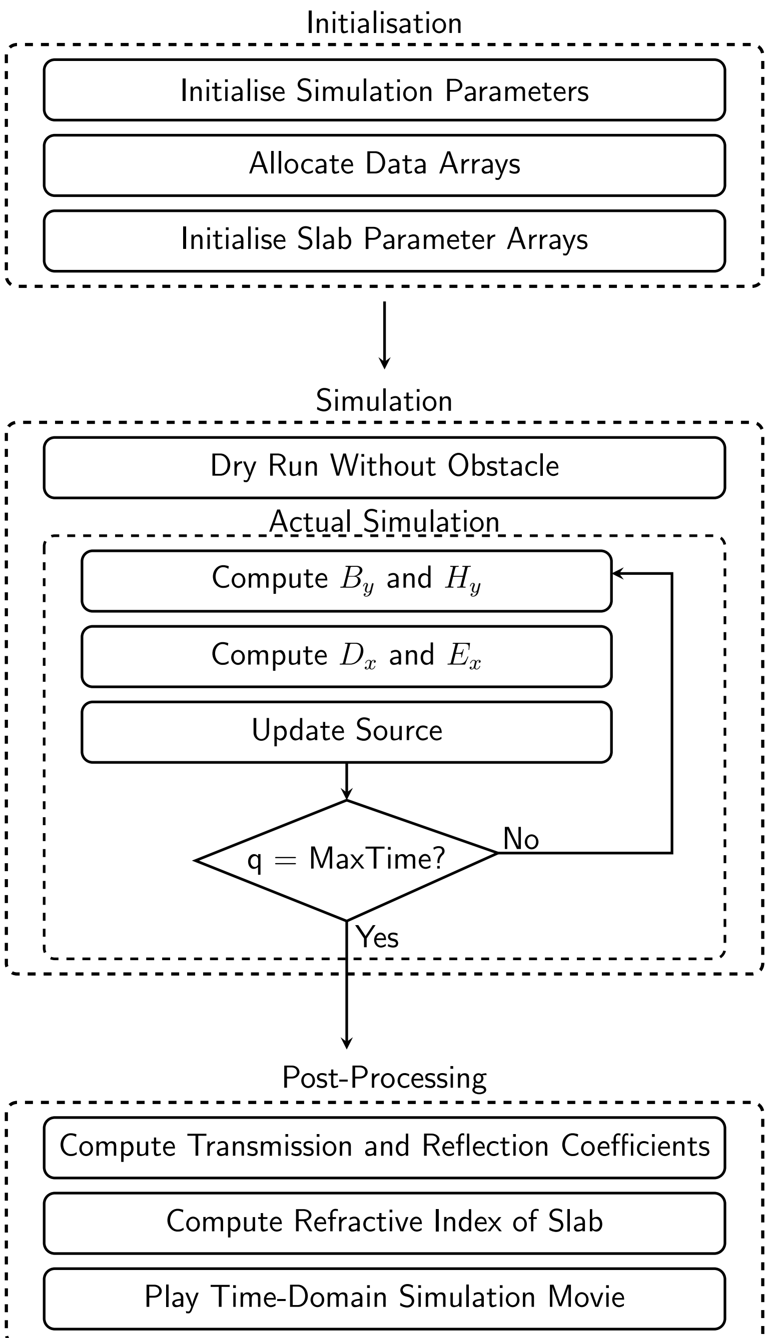

Figure 2: Simulation algorithm