Contents

Chapter 1

Vector Analysis

1.1 Vector Algebra

1.1.1 Definition

Scalars are quantities that is determined only by its magnitude (e.g. mass, density or temperature). Real or complex numbers are scalars. In most cases, a scalar means a real number, positive or negative.

Vectors are quantities that has direction as well as magnitude (e.g. force, velocity). Note that we do not need a coordinate system to define a vector.

If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are vectors and m, n are scalars, we have:

$$\begin{aligned} \mathbf{A}+\mathbf{B} &= \mathbf{B}+\mathbf{A} && (1.1) \\ (\mathbf{A}+\mathbf{B})+\mathbf{C} &= \mathbf{A}+(\mathbf{B}+\mathbf{C}) && (1.2) \\ m(n\mathbf{A}) = (mn)\mathbf{A} &= n(m\mathbf{A}) && (1.3) \\ m(\mathbf{A}+\mathbf{B}) &= m\mathbf{A}+m\mathbf{B}\ . && (1.4) \end{aligned}$$

$(-1)\mathbf{A} = -\mathbf{A}$ is the vector with same magnitude with $\mathbf{A}$ but with opposite direction. **Unit vector** is a vector with unit magnitude, $\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|}$, where $|\mathbf{A}|$ or A denote the magnitude of $\mathbf{A}$.

1.1.2 Component Forms

Let $\hat{\imath}$,$\hat{\jmath}$ and $\hat{k}$ be unit vectors parallel to the positive x, y and z axes of a Cartesian coordinate. An arbitrary vector $\mathbf{A}$ can be written as

$$\mathbf{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\ , \tag{1.5}$$

where A_x, A_y and A_z are the components of the vector along the three directions. We also write it as (A_x, A_y, A_z). In this component form, we have

$$\mathbf{A}+\mathbf{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) + (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

$$
\begin{aligned}
&= (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k} \qquad (1.6)\\
m\mathbf{A} &= m(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\\
&= (mA_x)\hat{\imath} + (mA_y)\hat{\jmath} + (mA_z)\hat{k}\ , \qquad (1.7)
\end{aligned}
$$

or

$$
\begin{aligned}
(A_x, A_y, A_z) + (B_x, B_y, B_z) &= (A_x + B_x, A_y + B_y, A_z + B_z) \qquad (1.8)\\
m(A_x, A_y, A_z) &= (mA_x, mA_y, mA_z)\ . \qquad (1.9)
\end{aligned}
$$

Example 1.1 In component form, the unit vectors are

$$
\begin{aligned}
\hat{\imath} &= (1, 0, 0) \qquad (1.10)\\
\hat{\jmath} &= (0, 1, 0) \qquad (1.11)\\
\hat{k} &= (0, 0, 1)\ . \qquad (1.12)
\end{aligned}
$$

1.1.3 Dot Product (Scalar Product)

The **dot product** of two vectors is defined as

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta \qquad (1.13)$$

where θ is the angle between the two vectors. The dot product of two vectors is a scalar. We have

$$
\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= \mathbf{B}\cdot\mathbf{A} \qquad (1.14)\\
\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \qquad (1.15)\\
m\mathbf{A}\cdot\mathbf{B} &= (m\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(m\mathbf{B}) = (\mathbf{A}\cdot\mathbf{B})m \qquad (1.16)\\
\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} &= 1 \qquad (1.17)\\
\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} &= 0 \qquad (1.18)\\
\mathbf{A}\cdot\mathbf{B} &= A_xB_x + A_yB_y + A_zB_z \qquad (1.19)\\
\mathbf{A}\cdot\mathbf{A} &= A_x^2 + A_y^2 + A_z^2 = A^2\ . \qquad (1.20)
\end{aligned}
$$

(How to prove Eq. (1.15)?)

Example 1.2 The cosine law can be easily proved, Fig. 1.1. We have $\mathbf{C} = \mathbf{A} - \mathbf{B}$, hence,

$$
\begin{aligned}
C^2 &= \mathbf{C}\cdot\mathbf{C} = (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})\\
&= \mathbf{A}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B}\\
&= A^2 + B^2 - 2AB\cos\theta\ . \qquad (1.21)
\end{aligned}
$$

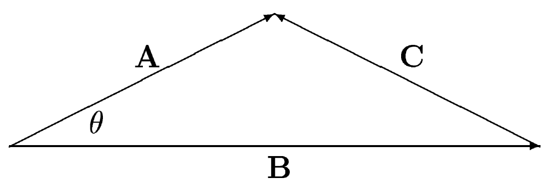

Figure 1.1: The cosine law.

1.1.4 Cross Product (Vector Product)

The **cross product** or **vector product** of two vectors is defined as $\mathbf{A}\times\mathbf{B} \equiv AB\sin\theta\,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is an unit vector perpendicular to the plane defined by $\mathbf{A}$ and $\mathbf{B}$, and we choose the direction of $\hat{\mathbf{n}}$ by the so-called right-hand rule. We have

$$
\begin{aligned}
\mathbf{A}\times\mathbf{B} &= -\mathbf{B}\times\mathbf{A} && (1.22)\\
\mathbf{A}\times(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C} && (1.23)\\
m(\mathbf{A}\times\mathbf{B}) &= (m\mathbf{A})\times\mathbf{B} = \mathbf{A}\times(m\mathbf{B}) = (\mathbf{A}\times\mathbf{B})m && (1.24)\\
\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} &= 0 && (1.25)\\
\hat{\imath}\times\hat{\jmath} &= \hat{k} && (1.26)\\
\hat{\jmath}\times\hat{k} &= \hat{\imath} && (1.27)\\
\hat{k}\times\hat{\imath} &= \hat{\jmath}\ . && (1.28)
\end{aligned}
$$

In component form, in general,

$$
\begin{aligned}
\mathbf{A}\times\mathbf{B} &= (A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k})\times(B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k})\\
&= (A_yB_z-A_zB_y)\hat{\imath}+(A_zB_x-A_xB_z)\hat{\jmath}+(A_xB_y-A_yB_x)\hat{k}\\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z \end{vmatrix}\ . \qquad (1.29)
\end{aligned}
$$

Example 1.3

$$
\begin{aligned}
& (5,8,4)\times(6,-2,9)\\
=\ & (8\cdot 9-4\cdot(-2), 4\cdot 6-5\cdot 9, 5\cdot(-2)-8\cdot 6)\\
=\ & (80,-21,-58)\ . \qquad (1.30)
\end{aligned}
$$

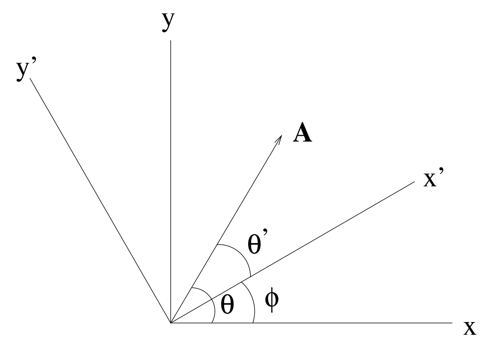

Figure 1.2: The transformation of coordinate systems.

1.1.5 Triple Product

For the product of three vectors, in general,

$$(\mathbf{A}\cdot\mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) \tag{1.31}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}\ . \tag{1.32}$$

For other triple products,

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \\
&= \mathbf{A}\cdot\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\
&= A_x\begin{vmatrix} B_y & B_z \\ C_y & C_z \end{vmatrix} + A_y\begin{vmatrix} B_z & B_x \\ C_z & C_x \end{vmatrix} + A_z\begin{vmatrix} B_x & B_y \\ C_x & C_y \end{vmatrix} \\
&= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})\ .
\end{aligned} \tag{1.33}$$

1.1.6 Vector Transformation

Although one may think that there is no direct application of the transformation property of vectors, in fact, the precise definition of vector is given by the transformation property.

In this subsection, we will talk about passive transformation, which means that the vector itself does not move at all. Only the coordinate system is rotated. For two dimensional vectors, let consider that the coordinate system is rotated by an angle ϕ in Fig. 1.2.

$$A_x = A\cos\theta \tag{1.34}$$

$$A_y = A\sin\theta \tag{1.35}$$

$$\begin{aligned} A_{x'} &= A\cos\theta' = A\cos(\theta-\phi) \\ &= A(\cos\theta\cos\phi+\sin\theta\sin\phi) \\ &= A_x\cos\phi + A_y\sin\phi \qquad (1.36) \\ A_{y'} &= A\sin\theta' = A\sin(\theta-\phi) \\ &= A(\sin\theta\cos\phi-\cos\theta\sin\phi) \\ &= -A_x\sin\phi + A_y\cos\phi \qquad (1.37) \\ \begin{pmatrix} A_{x'} \\ A_{y'} \end{pmatrix} &= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} . \qquad (1.38) \end{aligned}$$

The two by two matrix is the rotational matrix, which does not depend on the vector at all. This is the transformation property of vectors. In three dimension, the situation is similar. We have

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} R_{x'x} & R_{x'y} & R_{x'z} \\ R_{y'x} & R_{y'y} & R_{y'z} \\ R_{z'x} & R_{z'y} & R_{z'z} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} . \qquad (1.39)$$

The three by three rotational matrix does not depend on the vector.

1.2 Vector Calculus

1.2.1 Vector Functions

In many cases, a vector depends on some parameter, for example, the time. We would like to consider the change of the vector with respect to the change of this parameter. If the vector $\mathbf{A}$ depends on u, the simple differentiation is defined as

$$\frac{\partial \mathbf{A}}{\partial u} = \lim_{\Delta u \to 0} \frac{\mathbf{A}(u+\Delta u) - \mathbf{A}(u)}{\Delta u} . \qquad (1.40)$$

This is still a vector (depended on the same parameter). If $\mathbf{A}(u) = A_x(u)\hat{\imath} + A_y(u)\hat{\jmath} + A_z(u)\hat{k}$,

$$\frac{\partial \mathbf{A}}{\partial u} = \frac{\partial A_x}{\partial u}\,\hat{\imath} + \frac{\partial A_y}{\partial u}\,\hat{\jmath} + \frac{\partial A_z}{\partial u}\,\hat{k} , \qquad (1.41)$$

and

$$\frac{\partial^2 \mathbf{A}}{\partial u^2} = \frac{\partial^2 A_x}{\partial u^2}\,\hat{\imath} + \frac{\partial^2 A_y}{\partial u^2}\,\hat{\jmath} + \frac{\partial^2 A_z}{\partial u^2}\,\hat{k} . \qquad (1.42)$$

Also, we have

$$\begin{aligned} \frac{\partial}{\partial u}(\varphi\mathbf{A}) &= \varphi\frac{\partial \mathbf{A}}{\partial u} + \frac{\partial \varphi}{\partial u}\mathbf{A} \qquad (1.43) \\ \frac{\partial}{\partial u}(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial u} + \frac{\partial \mathbf{A}}{\partial u}\cdot\mathbf{B} \qquad (1.44) \\ \frac{\partial}{\partial u}(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}\times\frac{\partial \mathbf{B}}{\partial u} + \frac{\partial \mathbf{A}}{\partial u}\times\mathbf{B} . \qquad (1.45) \end{aligned}$$

Example 1.4 For $\mathbf{A}(t) = (\sin\omega t, e^{-i\omega t}, at^2)$ where ω and a are constants, then

$$\frac{\mathrm{d}\mathbf{A}(t)}{\mathrm{d}t} = (\omega\cos\omega t, -i\omega e^{-i\omega t}, 2at) \ . \tag{1.46}$$

If also $\mathbf{B}(t) = (ut, vt, (u-v)t)$, then $\mathbf{A}\cdot\mathbf{B} = ut\sin\omega t + vte^{-i\omega t} + a(u-v)t^3$ and

$$\frac{\mathrm{d}\mathbf{B}(t)}{\mathrm{d}t} = (u, v, u-v) \ . \tag{1.47}$$

One can easily verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\cdot\mathbf{B} \ . \tag{1.48}$$

1.2.2 Gradient

A quantity that depends on the position and time is called a **field**. If it takes scalar value, it is a **scalar field**. If it takes vector value, it is a **vector field**. For example, the temperature at each point of a room is a scalar field. The velocity of air particle at point in space in a vector field.

Let $T(x, y, z, t)$ be a scalar field, the **gradient** of T is defined as

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k} \ . \tag{1.49}$$

In some books, gradient is denoted by **grad**T. Note that the gradient of a scalar field is a vector field. The value of the vector field at a point is the direction in which the rate of change of $T(x, y, z)$ is maximum.

Example 1.5 If $T = x^2$, $\nabla T = 2x\hat{\imath}$. The gradient points to positive x-axis if $x > 0$, and to negative x-axis if $x < 0$. These are the directions in which T is increasing.

If c is any value, then $T(x, y, z) = c$ is an equation of the surface in which T takes the value c at all points. ∇T is the normal vector of this surface, pointing to the increasing side, if it is not zero.

Example 1.6 Let $T = x^2 + y^2 + z^2$ be a scalar field. For a fixed $c > 0$, $T(x, y, z) = c$ defines a surface. In fact, it is a sphere with radius $\sqrt{c}$. $\nabla T = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$. This is a radial pointing vector. It is normal to the surface (the sphere). And it is pointing in the direction of increasing value of T, which means, if a point is a bit off the surface along ∇T, the value of T is larger.

Sometimes, we call ∇ a vector operator "del",

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k} \ . \tag{1.50}$$

1.2.3 Divergence

Let $\mathbf{v}$ be a vector field, the **divergence** of $\mathbf{v}$ is defined as

$$\begin{aligned}\nabla\cdot\mathbf{v} &= (\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k})\cdot(v_x\hat{\imath}+v_y\hat{\jmath}+v_z\hat{k})\\ &= \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\ . \end{aligned}\tag{1.51}$$

Note that $\nabla\cdot\mathbf{v}$ is a scalar field and it is also denoted as $\mathbf{div}\,\mathbf{v}$.

1.2.4 Curl

For a vector field $\mathbf{v}$, the **curl** of $\mathbf{v}$ is defined as

$$\begin{aligned}\nabla\times\mathbf{v} &= \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}\\ &= (\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z})\hat{\imath}+(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x})\hat{\jmath}+(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y})\hat{k}\ . \end{aligned}\tag{1.52}$$

Note that $\nabla\times\mathbf{v}$ is a vector field and it is also denoted as $\mathbf{curl}\,\mathbf{v}$.

We can prove that

$$\nabla\cdot(\nabla\times\mathbf{v})=0\tag{1.53}$$

by commuting the partial differentiations,

$$\begin{aligned}&\nabla\cdot(\nabla\times\mathbf{v})\\ =\ &\frac{\partial}{\partial x}(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z})+\frac{\partial}{\partial y}(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x})+\frac{\partial}{\partial z}(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y})\\ =\ &0\ . \end{aligned}\tag{1.54}$$

Example 1.7 We verify that $\nabla\cdot(\nabla\times\mathbf{v})=0$ for $\mathbf{v}=x^2yz\hat{\imath}+\sin(xy^3)\hat{\jmath}+e^{-y^2}\hat{k}$.

$$\begin{aligned}\nabla\times\mathbf{v} &= (\frac{\partial e^{-y^2}}{\partial y}-\frac{\partial\sin(xy^3)}{\partial z})\hat{\imath}+(\frac{\partial(x^2yz)}{\partial z}-\frac{\partial e^{-y^2}}{\partial x})\hat{\jmath}\\ &\quad+(\frac{\partial\sin(xy^3)}{\partial x}-\frac{\partial(x^2yz)}{\partial y})\hat{k}\\ &= -2ye^{-y^2}\hat{\imath}+x^2y\hat{\jmath}+(y^3\cos(xy^3)-x^2z)\hat{k}\ . \end{aligned}\tag{1.55}$$

Hence,

$$\begin{aligned}\nabla\cdot(\nabla\times\mathbf{v}) &= \nabla\cdot(-2ye^{-y^2}\hat{\imath}+x^2y\hat{\jmath}+(y^3\cos(xy^3)-x^2z)\hat{k})\\ &= \frac{\partial(x^2y)}{\partial y}+\frac{\partial(y^3\cos(xy^3)-x^2z)}{\partial z}\\ &= x^2-x^2=0\ . \end{aligned}\tag{1.56}$$

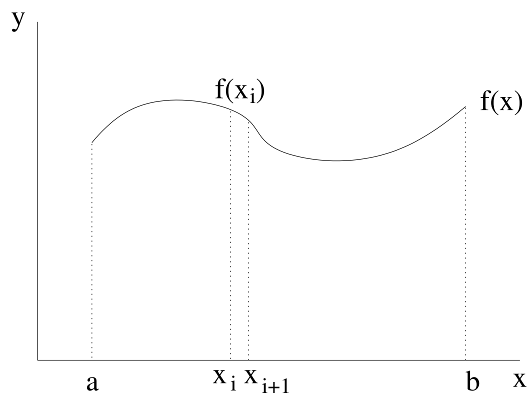

Figure 1.3: Integral as the area under the curve.

1.2.5 Laplacian

For scalar field, the **Laplacian** is defined as

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \ . \tag{1.57}$$

Note that $\nabla^2 T = \nabla \cdot (\nabla T)$. There are non-trivial functions whose Laplacian is zero. For example, if $T = \sin \omega x \, e^{\omega y}$, then

$$\begin{aligned} \nabla^2 T &= \frac{\partial^2 \sin \omega x \, e^{\omega y}}{\partial x^2} + \frac{\partial^2 \sin \omega x \, e^{\omega y}}{\partial y^2} \\ &= -\omega^2 \sin \omega x \, e^{\omega y} + \omega^2 \sin \omega x \, e^{\omega y} \\ &= 0 \ . \end{aligned} \tag{1.58}$$

For vector field,

$$\nabla^2 \mathbf{v} = \frac{\partial^2 \mathbf{v}}{\partial x^2} + \frac{\partial^2 \mathbf{v}}{\partial y^2} + \frac{\partial^2 \mathbf{v}}{\partial z^2} \ . \tag{1.59}$$

But be careful, $\nabla^2 \mathbf{v} \neq \nabla(\nabla \cdot \mathbf{v})$.

1.3 Integral Calculus

Recall what is the definition of a definite integral over an interval of real line. If $f(x)$ is a continuous function for $a \leq x \leq b$, let $a = x_0 < x_1 < \cdots < x_n = b$ be a set of real numbers between a and b, then

$$\int_a^b f(x)\, \mathrm{d}x = \lim_{\Delta x \to 0} \sum_{i=0}^{n-1} f(x_i)\, \Delta x_i \tag{1.60}$$

where $\Delta x_i \equiv x_{i+1} - x_i$ and Δx is the maximum of those Δx_i, Fig. 1.3. The integral is essentially the sum of the values of the function times the length of the infinitesimal intervals.

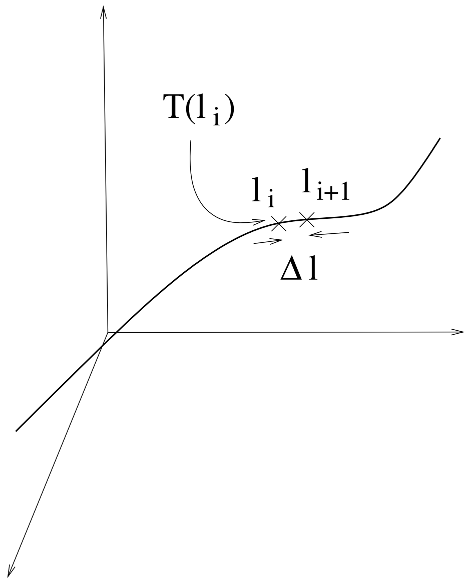

Figure 1.4: Line integral of a scalar field.

1.3.1 Line Integrals

Let $l(t)$ for $a \leq t \leq b$ be a line in space. We can think of it as the path of a particle from time $t = a$ to $t = b$. Here l is in fact three functions of the coordinates of the particle, $l(t) = (x(t), y(t), z(t))$. The infinitesimal path length is $\mathrm{d}l = \sqrt{(\mathrm{d}x/\mathrm{d}t)^2 + (\mathrm{d}y/\mathrm{d}t)^2 + (\mathrm{d}z/\mathrm{d}t)^2}\mathrm{d}t$.

The line integral of a scalar field is defined as

$$\int_{\text{path}} T\,\mathrm{d}l = \lim_{\Delta l \to 0} \sum_{i=0}^{n-1} T(l_i)\,\Delta l_i \tag{1.61}$$

where l_i are points on the path and Δl_i is the distance between two adjacent points, Fig. 1.4. This definition parallels the definite integral. If the path is parametrized as above, we have

$$\int_{\text{path}} T\,\mathrm{d}l = \int_a^b T(l(t)) \sqrt{(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + (\frac{\mathrm{d}y}{\mathrm{d}t})^2 + (\frac{\mathrm{d}z}{\mathrm{d}t})^2}\,\mathrm{d}t\ . \tag{1.62}$$

The line integral is independent of the parametrization.

Example 1.8 Suppose $T = xye^{-z}$ and the path is (t, t, t) for $0 \leq t \leq 1$. (The path is the diagonal of the unit cube.) We have $\mathrm{d}l = \sqrt{3}\,\mathrm{d}t$,

$$\int_{\text{path}} T\,\mathrm{d}l = \int_0^1 t^2 e^{-t}\sqrt{3}\,\mathrm{d}t = \sqrt{3}(2 - 5/e)\ . \tag{1.63}$$

Example 1.9 If the path is now the unit circle on the xy-plane with the counterclockwise direction, then $l(t) = (\cos t, \sin t, 0)$ for $0 \leq t \leq 2\pi$. We

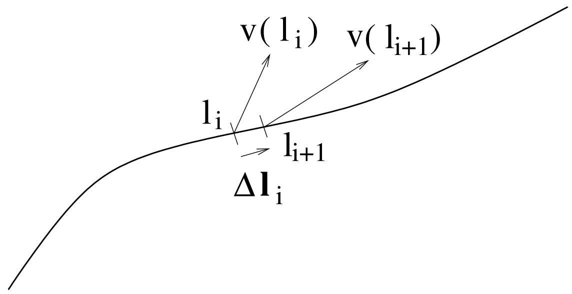

Figure 1.5: Line integral of a vector field.

have $\mathrm{d}l = \mathrm{d}t$,

$$\int_{\text{path}} T\,\mathrm{d}l = \int_0^{2\pi} \sin t \cos t\,\mathrm{d}t = 0\ . \tag{1.64}$$

Example 1.10 The integral still makes sense even if the path is not smooth. If the path is

$$(x, y, z) = \begin{cases} (t, t, 0) & \text{if } 0 \le t \le 1 \\ (1, 1, t-1) & \text{if } 1 \le t \le 2 \end{cases}\ , \tag{1.65}$$

it is the diagonal on the xy-plane and then goes vertical upward. The integral is

$$\int_{\text{path}} T\,\mathrm{d}l = \int_0^1 t^2\sqrt{2}\mathrm{d}t + \int_1^2 e^{-(t-1)}\,\mathrm{d}t = \frac{\sqrt{2}}{3} + 1 - \frac{1}{e}\ . \tag{1.66}$$

The line integral of a vector field is defined as

$$\int_{\text{path}} \mathbf{v}\cdot\mathrm{d}\mathbf{l} = \lim_{\Delta l\to 0}\sum_{i=0}^{n-1}\mathbf{v}(l_i)\cdot\Delta\mathbf{l}_i \tag{1.67}$$

where l_i are points on the path and $\Delta\mathbf{l}_i$ is the infinitesimal vector $\mathbf{l}_{i+1} - \mathbf{l}_i$, Fig. 1.5. This definition is similar to the scalar field case, but with major differences. We have to compute the dot product of the value of the vector field and the direction of the path. For parametrized path, $\mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\imath} + \mathrm{d}y\,\hat{\jmath} + \mathrm{d}z\,\hat{k}$, or

$$\mathrm{d}\mathbf{l} = \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\hat{\imath} + \frac{\mathrm{d}y(t)}{\mathrm{d}t}\hat{\jmath} + \frac{\mathrm{d}z(t)}{\mathrm{d}t}\hat{k}\right)\mathrm{d}t\ . \tag{1.68}$$

In summary,

$$\int_{\text{path}} \mathbf{v}\cdot\mathrm{d}\mathbf{l} = \int_a^b \left(v_x\frac{\mathrm{d}x(t)}{\mathrm{d}t} + v_y\frac{\mathrm{d}y(t)}{\mathrm{d}t} + v_z\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right)\,\mathrm{d}t\ . \tag{1.69}$$

Example 1.11 Suppose $\mathbf{v} = 5(y-3)z^2\hat{\imath} + (x^2 - y^2)\hat{\jmath} + xz^3\hat{k}$ and the path is (t, t, t) for $0 \le t \le 1$, then

$$\int_{\text{path}} \mathbf{v}\cdot\mathrm{d}\mathbf{l} = \int_0^1 (5(t-3)t^2 + t^4)\,\mathrm{d}t = -\frac{71}{20}\ . \tag{1.70}$$

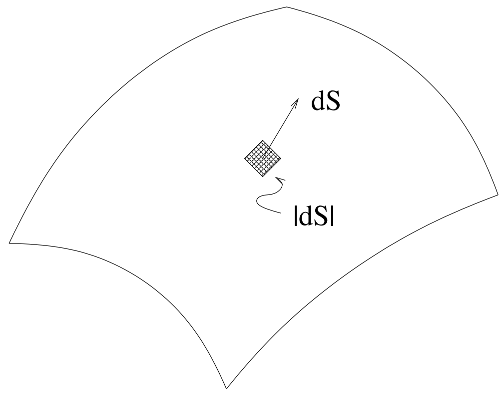

Figure 1.6: The infinitesimal surface element, with or with direction.

Example 1.12 If we choose the path as in Example 1.10, then

$$\begin{aligned} &\int_{\text{path}} \mathbf{v}\cdot \mathrm{d}\mathbf{l} \\ = &\int_0^1 (0)\cdot(\hat{\imath}+\hat{\jmath})\mathrm{d}t + \int_1^2 (t-1)^3\hat{k}\cdot\hat{k}\,\mathrm{d}t \\ = &\frac{1}{4}\ . \end{aligned} \tag{1.71}$$

1.3.2 Surface Integrals

If the surface is parametrized by $S(u,v) = (x(u,v), y(u,v), z(u,v))$ for $a \le u \le b$ and $c \le v \le d$, the infinitesimal surface element is, Fig. 1.6,

$$\mathrm{d}\mathbf{S} = \Big((y_u z_v - z_u y_v)\hat{\imath} + (z_u x_v - x_u z_v)\hat{\jmath} + (x_u y_v - y_u x_v)\hat{k}\Big)\,\mathrm{d}u\mathrm{d}v \tag{1.72}$$

where $y_u \equiv \partial y/\partial u$, etc.

Example 1.13 If the surface is defined by $z = f(x,y)$ for a function f, then, we have $S = (x, y, f(x,y))$ and Eq. (1.72) gives us

$$\mathrm{d}\mathbf{S} = (-\frac{\partial f}{\partial x}\hat{\imath} - \frac{\partial f}{\partial y}\hat{\jmath} + \hat{k})\mathrm{d}x\mathrm{d}y\ . \tag{1.73}$$

Example 1.14 The surface defined by $z = c$, where c is a constant, is a plane parallel to the xy-plane. Now, the function $f = c$ is a constant. Hence, the surface element is

$$\mathrm{d}\mathbf{S} = \hat{k}\,\mathrm{d}x\mathrm{d}y\ . \tag{1.74}$$

If the surface is $z = ax + by + c$ with a, b and c constants, then the surface element is

$$\mathrm{d}\mathbf{S} = (-a\hat{\imath} - b\hat{\jmath} + \hat{k})\,\mathrm{d}x\mathrm{d}y\ . \tag{1.75}$$

Make sure you understand the relation between the orientation of plane and the direction of the surface element.

The surface integral of a scalar field T is defined as

$$\int_{\text{surface}} T\,|\mathrm{d}\mathbf{S}| \tag{1.76}$$

where $|\mathrm{d}\mathbf{S}|$ is the magnitude of surface element. For the surface defined by a function f,

$$|\mathrm{d}\mathbf{S}| = \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,\mathrm{d}x\mathrm{d}y \tag{1.77}$$

The surface integral of a vector field $\mathbf{v}$ is defined as

$$\int_{\text{surface}} \mathbf{v}\cdot\mathrm{d}\mathbf{S}\ . \tag{1.78}$$

Note that there are two directions for a surface, corresponding to which parameter is u and which is v in Eq. (1.72). The problem must specify the direction.

Example 1.15 Suppose $T = xye^{-z}$ and $\mathbf{v} = 5(y-3)z^2\hat{\imath}+(x^2-y^2)\hat{\jmath}+xz^3\hat{k}$, and the surface is the surface of a sphere with radius R with outward normal. What are the surface integrals? The surface element is

$$\mathrm{d}\mathbf{S} = R^2\sin\theta\,\hat{\mathbf{r}}\,\mathrm{d}\theta\mathrm{d}\phi \tag{1.79}$$

where $\hat{\mathbf{r}}$ is the radial unit vector field. It is the vector field that at each point, the vector is pointing away from the origin and with unit length. The vector field is undefined at the origin. Hence,

$$\begin{aligned} & \int_{\text{sphere}} T\,|\mathrm{d}\mathbf{S}| \\ = & \int_0^{2\pi}\int_0^{\pi}(R\sin\theta\cos\phi R\sin\theta\sin\phi e^{-R\cos\theta})R^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi \\ = & \ 0\ . \end{aligned} \tag{1.80}$$

In the last step, we notice that the integration of ϕ gives us zero, because only terms with even powers of sine and cosine, $\cos^{2n}\phi\sin^{2m}\phi$, give non-zero result.

$$\begin{aligned} & \int_{\text{surface}} \mathbf{v}\cdot\mathrm{d}\mathbf{S} \\ = & \int_0^{2\pi}\int_0^{\pi}(5(R\sin\theta\sin\phi-3)R^2\cos^2\theta\sin\theta\cos\phi \\ & +R^2(\sin^2\theta\cos^2\phi-\sin^2\theta\sin^2\phi)\sin\theta\sin\phi \\ & +R^4\sin\theta\cos\phi\cos^3\theta\cos\theta)R^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi \\ = & \ 0\ . \end{aligned} \tag{1.81}$$

1.3.3 Volume Integrals

The volume integral of a scalar field is just the triple integral of a function. But be careful about the region of integration. It could be complicated. The volume integral of a vector field is just the vector sum of the three integrals of the components of the vector field.

Example 1.16 The density of a point of a solid sphere depends on its distance from the center: $\rho(r) = \rho_0\, e^{-r/R}$, where ρ_0 is constant and the radius of the sphere is R. The total mass is

$$\begin{aligned} M &= \int_{r=0}^{R} \rho_0\, e^{-r/R}\, \mathrm{d}^3\mathbf{r} \\ &= \int_{r=0}^{R} \rho_0\, e^{-r/R}\, r^2 \sin\theta\, \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\ &= 4\pi\rho_0 \int_{r=0}^{R} r^2 e^{-r/R}\, \mathrm{d}r \\ &= 4\pi\rho_0 R^3 (2 - 5e^{-1}) \ . \end{aligned} \tag{1.82}$$

1.4 Fundamental Theorems

The fundamental theorem of calculus is

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}\, \mathrm{d}x = f(b) - f(a) \ . \tag{1.83}$$

We will state a few more theorems of vector calculus similar to this one, but we will not give the proof.

1.4.1 Fundamental Theorem for Gradient

The fundamental theorem for gradient is

$$\int_{\text{path}} (\nabla T) \cdot \mathrm{d}\mathbf{l} = T(b) - T(a) \tag{1.84}$$

where $T(a)$ and $T(b)$ are the values of the scalar field at the initial and final points of the path. Note that $\int_a^b (\nabla T) \cdot \mathrm{d}\mathbf{l}$ is independent of paths as long as the initial and final points are the same. Hence, for any closed path $(a = b)$,

$$\oint (\nabla T) \cdot \mathrm{d}\mathbf{l} = T(b) - T(a) = 0 \ . \tag{1.85}$$

1.4.2 Green's Theorem

$$\int_{\text{volume}} (\nabla \cdot \mathbf{v}) \mathrm{d}^3\mathbf{r} = \oint_{\text{surface}} \mathbf{v} \cdot \mathrm{d}\mathbf{S} \tag{1.86}$$

where the closed surface on the right is the boundary of the volume on the left, with the outward normal.

1.4.3 Stokes' Theorem

$$\int_{\text{surface}} (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{S} = \oint_{\text{curve}} \mathbf{v} \cdot \mathrm{d}\mathbf{l} \tag{1.87}$$

where the closed curve on the right is the boundary of the surface on the left, with the sense given by the right-hand rule.

Example 1.17 Suppose $\mathbf{v} = 5(y-3)z^2\hat{\imath} + (x^2 - y^2)\hat{\jmath} + xz^3\hat{k}$, and the surface is the unit square on the xy-plane, $(x, y, 0)$ with $0 \leq x, y \leq 1$ and we choose the $+\hat{k}$ as the normal. Then, the boundary of the surface is

$$(x, y, z) = \begin{cases} (t, 0, 0) & \text{if } 0 \leq t \leq 1 \\ (1, t-1, 0) & \text{if } 1 \leq t \leq 2 \\ (3-t, 1, 0) & \text{if } 2 \leq t \leq 3 \\ (0, 4-t, 0) & \text{if } 3 \leq t \leq 4 \end{cases} . \tag{1.88}$$

We have $\nabla \times \mathbf{v} = (10(y-3)z - z^3)\hat{\jmath} + (2x - 5z^2)\hat{k}$, and

$$\int_{\text{surface}} (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{S} = \iint (2x - 5z^2)\, \mathrm{d}x\mathrm{d}y = 1 . \tag{1.89}$$

Also, only the second and the fourth segments of the path contribute

$$\oint_{\text{curve}} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_0^1 (1 - y^2)\, \mathrm{d}y + \int_1^0 -y^2\, \mathrm{d}y = 1 . \tag{1.90}$$

1.5 Curvilinear Coordinates

1.5.1 Spherical Coordinates

The spherical coordinates are defined as, Fig. 1.7,

$$x = r \sin\theta \cos\phi \tag{1.91}$$
$$y = r \sin\theta \sin\phi \tag{1.92}$$
$$z = r \cos\theta . \tag{1.93}$$

In this coordinate system, the line element is

$$\mathrm{d}\mathbf{l} = \mathrm{d}r\, \hat{r} + r\, \mathrm{d}\theta\, \hat{\theta} + r \sin\theta\, \mathrm{d}\phi\, \hat{\phi} , \tag{1.94}$$

while the volume element is

$$\mathrm{d}^3\mathbf{r} = \mathrm{d}l_r\, \mathrm{d}l_\theta\, \mathrm{d}l_\phi = r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi . \tag{1.95}$$

The gradient of a scalar field is

$$\nabla T = \frac{\partial T}{\partial r}\, \hat{r} + \frac{1}{r} \frac{\partial T}{\partial \theta}\, \hat{\theta} + \frac{1}{r \sin\theta} \frac{\partial T}{\partial \phi}\, \hat{\phi} . \tag{1.96}$$

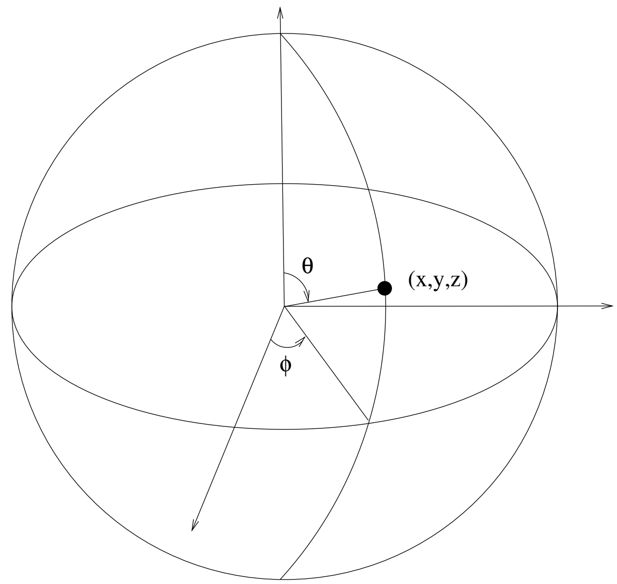

Figure 1.7: Spherical and Cartesian coordinate systems.

If a vector field $\mathbf{v}$ is $v_r\,\hat{r} + v_\theta\,\hat{\theta} + v_\phi\,\hat{\phi}$ in terms of the vector field $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$, then the divergence of the vector field is

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\,\frac{\partial}{\partial r}\,(r^2 v_r) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\,\frac{\partial v_\phi}{\partial\phi}\ . \tag{1.97}$$

The curl of the vector field is

$$\begin{aligned}\nabla \times \mathbf{v} \;&=\; \frac{1}{r\sin\theta}\,[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}]\,\hat{r} + \frac{1}{r}\,[\frac{1}{\sin\theta}\,\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r\, v_\phi)]\,\hat{\theta} \\ &\quad + \frac{1}{r}\,[\frac{\partial}{\partial r}(r\, v_\theta) - \frac{\partial v_r}{\partial\theta}]\,\hat{\phi}\ . \end{aligned} \tag{1.98}$$

The Laplacian is

$$\begin{aligned}\nabla^2 T \;&=\; \frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2\,\frac{\partial T}{\partial r}) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}(\sin\theta\,\frac{\partial T}{\partial\theta}) \\ &\quad + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 T}{\partial\phi^2}\ . \end{aligned} \tag{1.99}$$

1.5.2 Cylindrical Coordinates

The cylindrical coordinates are defined by

$$x \;=\; r\cos\phi \tag{1.100}$$
$$y \;=\; r\sin\phi \tag{1.101}$$
$$z \;=\; z\ . \tag{1.102}$$

Please refer the books for the detailed formula. Here, only the Laplacian is given.

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \ . \tag{1.103}$$

1.6 The Dirac Delta Function

This is not a real function in the mathematical sense. (It is a distribution, if you really want to know.) The definition of the delta function is

$$\int_a^b \delta(x)\, f(x)\, \mathrm{d}x = f(0) \tag{1.104}$$

if f is continuous at 0 and $a < 0 < b$. Similarly, we have:

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, f(x)\, \mathrm{d}x = f(x_0) \ . \tag{1.105}$$

One way to visualize the delta function is

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \ . \tag{1.106}$$

A few ways to realize the delta function is

$$\delta(x) = \lim_{a\to\infty} \sqrt{\frac{a}{\pi}}\, e^{-ax^2} \ , \tag{1.107}$$

$$\delta(x) = \lim_{a\to\infty} \frac{a}{\pi}\frac{\sin(ax)}{ax} \tag{1.108}$$

where we understand that the limit $a \to \infty$ is taken after we do the integration in Eq. (1.104). Now, we prove that

$$\nabla \cdot (\frac{\hat{r}}{r^2}) = 4\pi\delta^3(\mathbf{r}) \ . \tag{1.109}$$

For the vector field $\hat{r}/r^2$ and $r \neq 0$, by Eq. (1.97),

$$\nabla \cdot (\frac{\hat{r}}{r^2}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{1}{r^2}) = 0 \ . \tag{1.110}$$

Hence, $\nabla \cdot (\hat{r}/r^2) = k\delta^3(\mathbf{r})$ for a constant k. (The above derivation is not valid at $r = 0$ because it is singular there.) To determine k (it could be zero), we multiply both sides with $\exp(-r^2)$ and integrate over the whole space. We have

$$\int e^{-r^2}\nabla \cdot (\hat{r}/r^2)\mathrm{d}^3\mathbf{r} = k$$

$$\begin{aligned}
-\int \nabla(e^{-r^2}) \cdot (\hat{r}/r^2)\mathrm{d}^3\mathbf{r} &= k \\
-\int (-2r)e^{-r^2}\hat{r} \cdot (\hat{r}/r^2)\mathrm{d}^3\mathbf{r} &= k \\
\int \frac{2}{r}\, e^{-r^2}\mathrm{d}^3\mathbf{r} &= k \\
\int_0^\infty \frac{2}{r}\, e^{-r^2} 4\pi r^2 \mathrm{d}r &= k \\
8\pi \int_0^\infty e^{-r^2} r\mathrm{d}r &= k \\
4\pi &= k \; . \qquad (1.111)
\end{aligned}$$

1.7 Summary

1.7.1 Differential Vector Identities

$$\begin{aligned}
\nabla \cdot \nabla\, \phi &= \nabla^2\, \phi && (1.112) \\
\nabla \cdot \nabla \times \mathbf{F} &= 0 && (1.113) \\
\nabla \times \nabla\, \phi &= 0 && (1.114) \\
\nabla \times (\nabla \times \mathbf{F}) &= \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\, \mathbf{F} && (1.115) \\
\nabla\, (\phi\psi) &= (\nabla\phi)\psi + \phi(\nabla\psi) && (1.116) \\
\nabla(\mathbf{F} \cdot \mathbf{G}) &= (\mathbf{F} \cdot \nabla)\, \mathbf{G} + \mathbf{F} \times (\nabla \times \mathbf{G}) && \\
&\quad +(\mathbf{G} \cdot \nabla)\, \mathbf{F} + \mathbf{G} \times (\nabla \times \mathbf{F}) && (1.117) \\
\nabla \cdot (\phi\mathbf{F}) &= (\nabla\phi) \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F}) && (1.118) \\
\nabla \cdot (\mathbf{F} \times \mathbf{G}) &= (\nabla \times \mathbf{F}) \cdot \mathbf{G} - (\nabla \times \mathbf{G}) \cdot \mathbf{F} && (1.119) \\
\nabla \times (\phi\mathbf{F}) &= (\nabla\phi) \times \mathbf{F} + \phi\nabla \times \mathbf{F} && (1.120) \\
\nabla \times (\mathbf{F} \times \mathbf{G}) &= (\nabla \cdot \mathbf{G})\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} && \\
&\quad +(\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} && (1.121)
\end{aligned}$$

1.7.2 Vector Integral Theorems

Divergence theorem

$$\int_V \nabla \cdot \mathbf{v}\, \mathrm{d}^3\mathbf{r} = \oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{S} \qquad (1.122)$$

Stokes' Theorem

$$\int_S (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{S} = \oint_l \mathbf{v} \cdot \mathrm{d}\mathbf{l} \qquad (1.123)$$

Chapter 2

Electrostatics

In this chapter, we would try to find out the electric field induced by static electric charge distribution.

2.1 The Electric Field

2.1.1 Coulomb's Law

Discovered by experiments, the force acting on a point charge Q at position $\mathbf{r}_1$ by another point charge q at $\mathbf{r}_2$ is

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r} \tag{2.1}$$

where ε_0 is called the **permittivity** of free space and its value is $8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{Nm}^2$ in SI unit. The distance between them is $r = |\mathbf{r}_1 - \mathbf{r}_2|$ and the unit vector $\hat{r}$ is $(\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2|$.

2.1.2 The Electric Field

According to the principle of superposition, this is also found by experiments, the total force on Q caused by q_1, q_2, q_3, … is:

$$\begin{aligned}
\mathbf{F} &= \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \cdots \\
&= \frac{1}{4\pi\varepsilon_0}(\frac{q_1 Q}{r_1^2}\hat{r_1} + \frac{q_2 Q}{r_2^2}\hat{r_2} + \cdots) \\
&= \frac{Q}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i^2}\hat{r_i} \qquad (2.2)
\end{aligned}$$

where r_i is the distance between Q and q_i and $\hat{r_i}$ is the unit vector from q_i to Q. Thus,

$$\mathbf{F} = Q\mathbf{E} \tag{2.3}$$

where

$$\mathbf{E} \equiv \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{r_i^2} \hat{r}_i \tag{2.4}$$

is called the electric field. It is a function of position. The position of our test charge Q is called the **field point**, while the position of other charge is called the **source point**. In this particular case, there are many source points because there are many point charges.

2.1.3 Continuous Charge Distribution

Apart from point charges, we will also consider continuous charge distribution, including line charge density: λ is the charge per unit length; surface charge density: σ is the charge per unit area; volume charge density: ρ is the charge per unit volume.

Thus, for example, Eq. (2.4) can be written as:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}')\, \mathrm{d}^3\mathbf{r}' \ , \tag{2.5}$$

in which, $\mathbf{r}$ is the field point and $\mathbf{r}'$ is the source point. Explicitly, in Cartesian coordinates:

$$\begin{aligned} & \mathbf{E}(x, y, z) \\ = \ & \frac{1}{4\pi\varepsilon_0} \int_V \frac{(x - x')\hat{\imath} + (y - y')\hat{\jmath} + (z - z')\hat{k}}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \rho(x', y', z')\, \mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \ . \end{aligned} \tag{2.6}$$

Example 2.1 We calculate the electric field of a point charge q at the origin of our coordinate. Suppose a positive test charge Q is located at $\mathbf{r}$. From Eq. (2.1), we have

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{r^2} \hat{r} \tag{2.7}$$

$$\mathbf{E} = \mathbf{F}/Q = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} \ . \tag{2.8}$$

Example 2.2 An **electric dipole** is a pair of point charges $+q$ and $-q$ separated by a fixed distance l. Usually, we will take the limit that $q \to \infty$ and $l \to 0$ such that the electric dipole moment $p \equiv ql$ is a constant. We will calculate the electric field of a dipole for far away field point, $l \ll r$.

We choose the coordinate such that $+q$ is at $l/2\hat{\imath}$ and $-q$ is at $-l/2\hat{\imath}$, Fig. 2.1. Obviously, there is a cylindrical symmetry along the dipole axis.

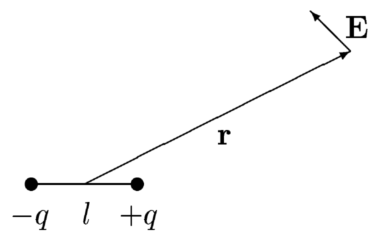

Figure 2.1: The electric field of electric dipole.

We can consider just the field points on the xy-plane. The electric field is

$$\begin{aligned}
&\mathbf{E}(x,y) \\
= &\frac{q}{4\pi\varepsilon_0}\left(\frac{\mathbf{r}-l/2\hat{\imath}}{|\mathbf{r}-l/2\hat{\imath}|^3}-\frac{\mathbf{r}+l/2\hat{\imath}}{|\mathbf{r}+l/2\hat{\imath}|^3}\right) \\
= &\frac{q}{4\pi\varepsilon_0}\left(\frac{\mathbf{r}-l/2\hat{\imath}}{(r^2-rl\cos\theta+l^2/4)^{3/2}}-\frac{\mathbf{r}+l/2\hat{\imath}}{(r^2+rl\cos\theta+l^2/4)^{3/2}}\right)
\end{aligned} \tag{2.9}$$

where in the last equality, polar coordinates are employed. Since we just consider the region $r \gg l$, we have

$$\begin{aligned}
&(r^2 \pm rl\cos\theta + l^2/4)^{-3/2} \\
= &\frac{1}{r^3}(1 \pm l\cos\theta/r + l^2/(4r^2))^{-3/2} \\
= &\frac{1}{r^3}(1 \mp \frac{3\cos\theta}{2}\frac{l}{r} + O\left(\frac{l^2}{r^2}\right)) \; .
\end{aligned} \tag{2.10}$$

Hence, we have

$$\begin{aligned}
&\mathbf{E}(x,y) \\
= &\frac{q}{4\pi\varepsilon_0 r^3}\left((\mathbf{r}-l/2\hat{\imath})(1+\frac{3\cos\theta}{2}\frac{l}{r})-(\mathbf{r}+l/2\hat{\imath})(1-\frac{3\cos\theta}{2}\frac{l}{r})\right) \\
= &\frac{q}{4\pi\varepsilon_0 r^3}\left(-l\hat{\imath}+\frac{3\cos\theta l\mathbf{r}}{r}\right) \\
= &\frac{1}{4\pi\varepsilon_0}\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}}{r^3}
\end{aligned} \tag{2.11}$$

where $\mathbf{p} = ql\hat{\imath}$ is the vector electric dipole moment. Note that the electric field strength falls off as $1/r^3$.

2.2 The Gauss's Law

2.2.1 Field Lines

A system of **field lines** is a qualitative description of the electric field of a system of static charges. It is a set of curves such that the tangent at each point on these curves indicates the direction of the electric field at that location. A few remarks about the field lines:
1. Field lines originate on positive charges and terminate on negative charges;
2. If the system has the same amount of positive and negative charges, all field lines starting from positive charge will terminate on negative charge. They cannot stop in the midair;
3. Field line can never cross each other. It ensures electric field at a point only has only one direction;
4. Field line will not form a closed path;
5. The density of field lines indicates the strength of the field. Density is defined by the number of lines passing through an unit area cross section, $E \propto \Delta N/\Delta S$.

2.2.2 Gauss's Law

The integral form of **Gauss's law** is

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0} \tag{2.12}$$

where S is a closed surface and Q is the amount of net charges enclosed by S. Gauss's law can be deduced from Coulomb's law and the principle of superposition as follows.

1. The **flux** across a surface S, not necessarily closed, is defined by

$$\Phi \equiv \int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} \ . \tag{2.13}$$

If a point charge is located at the origin of a sphere S, the total flux will be

$$\Phi = \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \oint_S \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} \cdot (r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{r}) = \frac{q}{\varepsilon_0} \ . \tag{2.14}$$

Note that the result is independent of the size of the sphere.

2. If a point charge is enclosed by an arbitrary surface, the flux is

$$\Phi = \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{q}{4\pi\varepsilon_0} \oint_S \frac{\mathbf{r} \cdot \mathrm{d}\mathbf{S}}{r^3} \ . \tag{2.15}$$

The quantity $\mathbf{r} \cdot \mathrm{d}\mathbf{S}/r = \mathrm{d}S \cos\theta$ is the projection of $\mathrm{d}S$ on the plane perpendicular to $\mathbf{E}$, Fig. 2.2. When it is divided by r^2, it is the solid angle subtended by $\mathrm{d}S$. The solid angle of a sphere is 4π, hence

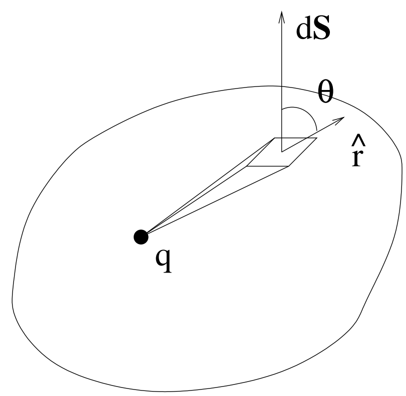

Figure 2.2: An arbitrary surface encloses a point charge.

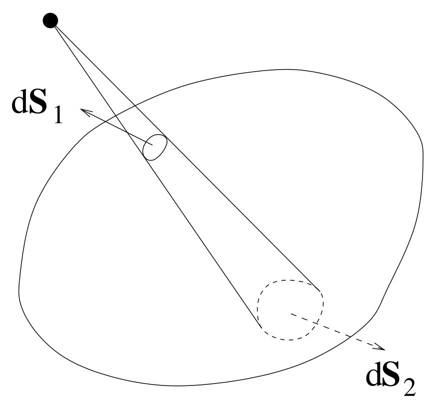

Figure 2.3: An arbitrary surface encloses no charge.

$$\oint_S \frac{\mathbf{r}\cdot \mathrm{d}\mathbf{S}}{r^3} = 4\pi \ , \tag{2.16}$$

and

$$\Phi = \oint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{q}{\varepsilon_0} \ . \tag{2.17}$$

3. If the surface encloses no charge, Fig. 2.3, the solid angle subtended by S_1 and S_2 will be the same but with opposite sign. We have

$$\begin{aligned} \mathrm{d}\Phi_1 &= \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}_1\cdot \mathrm{d}\mathbf{S}_1}{r_1^3} = \frac{q}{4\pi\varepsilon_0}\mathrm{d}\Omega_1 \\ &= \frac{q}{4\pi\varepsilon_0}\mathrm{d}(-\Omega_2) = \frac{q}{4\pi\varepsilon_0}\frac{-\mathbf{r}_2\cdot \mathrm{d}\mathbf{S}_2}{r_2^3} \\ &= -\mathrm{d}\Phi_2 \ . \end{aligned} \tag{2.18}$$

The total integral vanishes

$$\Phi = \oint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} = 0 \ . \tag{2.19}$$

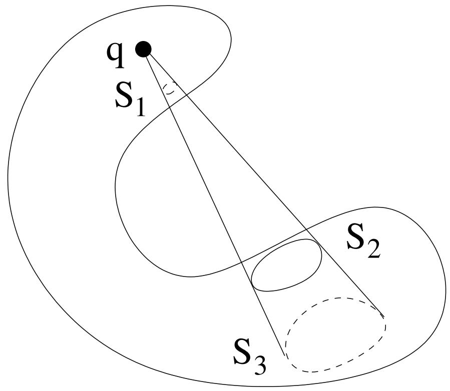

Figure 2.4: An element of solid angle cuts the surface more than once.

4. If an element of solid angle cuts the surface S more than once, Fig. 2.4, then the surface elements with opposite orientations will cancel each other,

$$\begin{aligned} \Phi &= \Phi_1 + \Phi_2 + \Phi_3 + \cdots \\ &= \Phi_1 - \Phi_1 + \Phi_1 + \cdots \ . \end{aligned} \tag{2.20}$$

5. For many point charges, by the principle of superposition, we have

$$\begin{aligned} \Phi &= \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \oint_S (\sum_i \mathbf{E_i}) \cdot \mathrm{d}\mathbf{S} \\ &= \sum_i \oint_S \mathbf{E_i} \cdot \mathrm{d}\mathbf{S} = \sum_i \Phi_i = \sum_i \frac{q_i}{\varepsilon_0} = \frac{1}{\varepsilon_0} \sum_i q_i \end{aligned} \tag{2.21}$$

where the sum is over point charges.

2.2.3 Differential Form of Gauss's Law

If the charge distribution is continuous, we could replace the sum over point charges by an integral of the charge density ρ over the region enclosed by the closed surface

$$\sum_i q_i \to \int_V \rho \, \mathrm{d}V \ . \tag{2.22}$$

Even if there are also some point charges, we could add to the charge density some appropriate delta functions. Hence,

$$\int_V \frac{\rho}{\varepsilon_0} \mathrm{d}V = \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \int_V \nabla \cdot \mathbf{E} \, \mathrm{d}V \ . \tag{2.23}$$

Since the above equation is valid for any volume V, the integrands of two sides must equal,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \ . \tag{2.24}$$

This is the differential form of the Gauss's law.

2.2.4 The Curl of E

We are going to prove that for static charge distribution, the curl of the electric field is zero. The line element in spherical coordinates is given in Eq. (1.94),

$$\mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{r} + r\mathrm{d}\theta\,\hat{\theta} + r\sin\theta\,\mathrm{d}\phi\,\hat{\phi}\ . \tag{2.25}$$

Hence, for a point charge, the line integral of the electric field along a path from point a to point b is

$$\begin{aligned}\int_a^b \mathbf{E}\cdot\mathrm{d}\mathbf{l} &= \frac{1}{4\pi\varepsilon_0}\int_a^b \frac{q}{r^2}\,\mathrm{d}r \\ &= \frac{q}{4\pi\varepsilon_0}\,(\frac{1}{r_a}-\frac{1}{r_b})\end{aligned} \tag{2.26}$$

where r_a and r_b are the distances of the two points from the origin. We have found that the line integral is independent of path. Thus, if the integral is calculated along a closed path ($r_a = r_b$), we have

$$\oint \mathbf{E}\cdot\mathrm{d}\mathbf{l} = 0\ . \tag{2.27}$$

Applying the Stokes' law, we have $\int(\nabla\times\mathbf{E})\cdot\mathrm{d}\mathbf{S} = \oint \mathbf{E}\cdot\mathrm{d}\mathbf{l} = 0$. Since the closed path is arbitrary, the integrand must be zero,

$$\nabla\times\mathbf{E} = 0 \tag{2.28}$$

for point charge. Similar to the proof of the Gauss's law, we can conclude that Eq. (2.28) is valid for any static charge distribution.

Example 2.3 We consider the electric field of an uniformly charged solid sphere with total charge q and of radius R. The charge density is $\rho = q/(4\pi R^3/3)$. By the symmetry of the configuration, the electric field must be along the radial direction. For $r \le R$, apply the Gauss's law,

$$4\pi r^2 E = \oint_{S_r}\mathbf{E}\cdot\mathrm{d}\mathbf{S} = \int_{V_r}\frac{\rho}{\varepsilon_0}\,\mathrm{d}V = \frac{qr^3}{\varepsilon_0 R^3} \tag{2.29}$$

where S_r and V_r is the surface and volume of the sphere with radius r. As a result,

$$\mathbf{E} = E\hat{r} = \frac{q}{4\pi\varepsilon_0}\,\frac{r}{R^3}\,\hat{r}\ . \tag{2.30}$$

For $r > R$, the total charge inside the sphere of radius r is just q. Hence, $\mathbf{E} = q/(4\pi\varepsilon_0 r^2)$. In summary, Fig. 2.5,

$$\mathbf{E} = \begin{cases} \dfrac{q}{4\pi\varepsilon_0}\,\dfrac{r}{R^3}\hat{r} & \text{if } r \le R \\ \dfrac{q}{4\pi\varepsilon_0}\,\dfrac{1}{r^2}\hat{r} & \text{if } r > R \end{cases}\ . \tag{2.31}$$

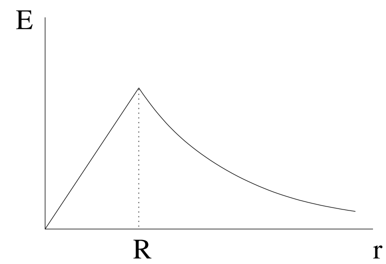

Figure 2.5: The electric field strength of the uniformly charged solid sphere.

2.3 Electric Potential

For static charge distribution, we found that

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{2.32}$$

$$\nabla \times \mathbf{E} = 0 \,. \tag{2.33}$$

In particular, the second equation tells us that the line integral of $\mathbf{E}$ is independent of path. Hence, the following scalar field is well defined

$$V(P) = -\int_O^P \mathbf{E} \cdot d\mathbf{l} \tag{2.34}$$

where O is an arbitrary but fixed point. This is the **electric potential**. We also have

$$\mathbf{E} = -\nabla V \,. \tag{2.35}$$

Eq. (2.33) is automatically satisfied because $\nabla \times (\nabla V) = 0$ for any scalar field. Some comments on the potential:

1. Potential is not the energy.

2. The reason to introduce potential is that to determine the vector field $\mathbf{E}$, we need three equations for the three components. But to find V, one equation is sufficient. This does not mean that less information is obtained. It only means that the three components of the electric field are not independent. The potential is a more efficient way to express it.

3. The potential difference is given by

$$\Delta V = V(P) - V(Q) = -\int_O^P \mathbf{E} \cdot d\mathbf{l} - (-\int_O^Q \mathbf{E} \cdot d\mathbf{l}) = -\int_Q^P \mathbf{E} \cdot d\mathbf{l} \,. \tag{2.36}$$

This is independent of the choice of the reference point.

4. The choice of reference is arbitrary. Changing the reference point just adds a constant to V

$$\begin{aligned} V'(P) &= -\int_{O'}^P \mathbf{E} \cdot d\mathbf{l} \\ &= -\int_O^P \mathbf{E} \cdot d\mathbf{l} - \int_{O'}^O \mathbf{E} \cdot d\mathbf{l} \\ &= V(P) + \text{constant} \,. \end{aligned} \tag{2.37}$$

Usually we choose the infinity to be the reference point and set $V(\infty) = 0$.

5. The potential also obeys the superposition principle because if $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \cdots$, the potential will be

$$\begin{aligned} V(P) &= -\int_{\infty}^{P} \mathbf{E} \cdot \mathrm{d}\mathbf{l} \\ &= -\int_{\infty}^{P} (\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \cdots) \cdot \mathrm{d}\mathbf{l} \\ &= -\int_{\infty}^{P} \mathbf{E}_1 \cdot \mathrm{d}\mathbf{l} - \int_{\infty}^{P} \mathbf{E}_2 \cdot \mathrm{d}\mathbf{l} - \int_{\infty}^{P} \mathbf{E}_3 \cdot \mathrm{d}\mathbf{l} - \cdots \\ &= V_1(P) + V_2(P) + V_3(P) + \cdots \ . \end{aligned} \tag{2.38}$$

6. A **equipotential surface** is a surface which consists of all points with the same potential.

(i) Equipotential surfaces are perpendicular to the electric field because for any two points P and Q on a surface, $V(P) - V(Q) = 0$. This should be equal to the work done to move a charge q on the surface, which is

$$\delta W = q\mathbf{E} \cdot \delta\mathbf{l} = 0 \tag{2.39}$$

where $\delta\mathbf{l}$ is the displacement between the two points. Hence, $\mathbf{E}$ is perpendicular to the displacement between any two points on the surface, i.e. it is perpendicular to the surface.

(ii) A small spacing between equipotential surfaces indicates stronger field strength.

2.3.1 Poisson's Equation and Laplace's Equation

Substitute Eq. (2.35) into the Gauss's law, we have

$$\nabla^2 V = -\nabla \cdot (-\nabla V) = -\nabla \cdot \mathbf{E} = -\frac{\rho}{\varepsilon_0} \ . \tag{2.40}$$

This is the **Poisson's equation**. If there is no charge or if we consider region without charge, the equation is

$$\nabla^2 V = 0 \ . \tag{2.41}$$

This is the **Laplace's equation**.

2.3.2 The Potential of Point Charges

For a point charge, the potential at distance r from it is

$$V(r) = \frac{-1}{4\pi\varepsilon_0} \int_{\infty}^{r} \frac{q}{r'^2}\, \mathrm{d}r' = \frac{q}{4\pi\varepsilon_0 r} \ . \tag{2.42}$$

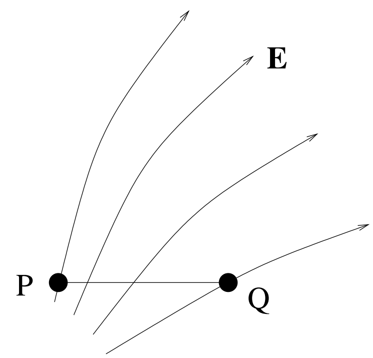

Figure 2.6: An example of electric field lines.

For a collection of point charges,

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|} \tag{2.43}$$

where $\mathbf{r}_i$ is the location of the ith charge. For continuous charge distribution,

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3\mathbf{r}' \ . \tag{2.44}$$

In general, there is no easy way to calculate this integral.

2.3.3 Discussions of Potential

1. If the work done by the Coulomb's force depended on the path, can we introduce the concept of electric potential?

2. If the field $\mathbf{E}$ at a point is stronger, will the potential at this point also be higher?

3. If $V(\mathbf{r})$ is higher, will $\mathbf{E}(\mathbf{r})$ be stronger?

4. If $\mathbf{E} = 0$ at some point, will V also be zero there? If $V = 0$ at some point, will the field $\mathbf{E}$ be zero there?

5. If $\mathbf{E}(P) = \mathbf{E}(Q)$ at two points P and Q, is $V(P) = V(Q)$?

6. Consider the situation indicated by Fig. 2.6. (i) To move a positive charge from P to Q, will the electric field make a positive or a negative work done? Will the potential energy of the charge increase or decrease? Which one of the $V(P)$ and $V(Q)$ is higher?

(ii) If the charge is negative in question (i), what will be the answer?

7. The potential always decreases along a field line. Why?

2.4 Work and Energy in Electrostatics

We calculate the work done needed to move a test charge Q from a point a to another point b.

$$W = \int_a^b \mathbf{F} \cdot \mathrm{d}\mathbf{l} = -\int_a^b Q\mathbf{E} \cdot \mathrm{d}\mathbf{l} = Q(V(b) - V(a)) \ . \tag{2.45}$$

Thus, the potential difference is the work done per unit charge.

For a collection of point charges, we put in the charges one by one. There is no work done needed to put in the first charge, so $W_1 = 0$. If we denote the distance between the ith and jth charge be r_{ij}, the work done to put in the second charge is

$$W_2 = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}} \ . \tag{2.46}$$

The work done to put in the third charge is

$$W_3 = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) \ . \tag{2.47}$$

Hence, the total work is

$$\begin{aligned} W &= W_1 + W_2 + W_3 + \cdots + W_n \\ &= \frac{1}{4\pi\varepsilon_0} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{q_i q_j}{r_{ij}} \ . \end{aligned} \tag{2.48}$$

We can also count each pair twice and then divide it by two to have

$$W = \frac{1}{8\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{r_{ij}} \ . \tag{2.49}$$

Another form of Eq. (2.49) is

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{ij}} \right) = \frac{1}{2} \sum_{i=1}^{n} q_i V(P_i) \tag{2.50}$$

where $V(P_i)$ is the potential caused by all charges but q_i.

For the energy of a continuous charge distribution, we replace the sum by integral in Eq. (2.50)

$$\begin{aligned} W &= \frac{1}{2} \int \rho V \, \mathrm{d}^3\mathbf{r} \\ &= \frac{\varepsilon_0}{2} \int V \nabla \cdot \mathbf{E} \, \mathrm{d}^3\mathbf{r} \\ &= \frac{\varepsilon_0}{2} \int (\nabla \cdot (V\mathbf{E}) - \nabla V \cdot \mathbf{E}) \, \mathrm{d}^3\mathbf{r} \\ &= \frac{\varepsilon_0}{2} \oint_S V\mathbf{E} \cdot \mathrm{d}\mathbf{S} + \frac{\varepsilon_0}{2} \int \mathbf{E}^2 \, \mathrm{d}^3\mathbf{r} \ . \end{aligned} \tag{2.51}$$

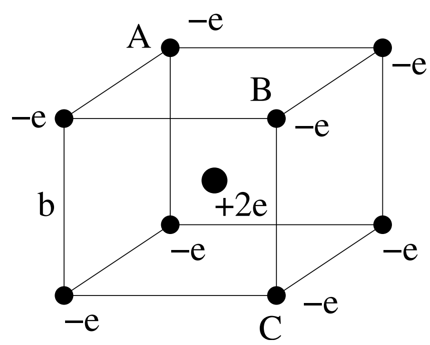

Figure 2.7: An example of point charge system for work done calculation.

As large r, $E \sim r^{-2}$ and $V \sim r^{-1}$. When the integration region increases, the integrand in the first term goes as r^{-3} but the area of the surface goes only as r^2, the surface integral decreases. Take the integration region to be the whole space, $\oint_S$ goes to zero, and we have

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} \mathbf{E}^2 \mathrm{d}^3\mathbf{r} \ . \tag{2.52}$$

Note that the electrostatic energy does not obey the superposition principle since W usually contains cross items

$$\begin{aligned} W &= \frac{\varepsilon_0}{2}\int \mathbf{E}^2 \mathrm{d}^3\mathbf{r} \\ &= \frac{\varepsilon_0}{2}\int (\mathbf{E_1}+\mathbf{E_2})^2 \mathrm{d}^3\mathbf{r} \\ &= \frac{\varepsilon_0}{2}\int (\mathbf{E}_1^2+\mathbf{E}_2^2+2\mathbf{E}_1\cdot\mathbf{E}_2)\mathrm{d}^3\mathbf{r} \\ &= W_1 + W_2 + \varepsilon_0\int \mathbf{E_1}\cdot\mathbf{E_2}\mathrm{d}^3\mathbf{r} \ . \end{aligned} \tag{2.53}$$

The last term is the interaction energy between the two subsystems.

Example 2.4 We calculate the electrostatic energy of the charge system as shown in Fig.2.7. For the 12 pairs of neighboring charges, the energy is

$$12(\frac{e^2}{4\pi\varepsilon_0 b}) \ . \tag{2.54}$$

For the pairs like A and B, the separation is $\sqrt{2}b$ and there are 12 pairs

$$12(\frac{e^2}{4\pi\varepsilon_0\sqrt{2}b}) \ . \tag{2.55}$$

For pairs like A and C, separation is $\sqrt{3}b$. There are 4 pairs

$$4(\frac{e^2}{4\pi\varepsilon_0\sqrt{3}b}) \ . \tag{2.56}$$

There are 8 pairs from the corners to center, with separation $\sqrt{3}b/2$,

$$8(\frac{-2e^2}{4\pi\varepsilon_0\sqrt{3}b/2}) \ . \tag{2.57}$$

The total interaction energy of the system is

$$\begin{aligned} W &= \frac{e^2}{4\pi\varepsilon_0 b}(12+\frac{12}{\sqrt{2}}+\frac{4}{\sqrt{3}}-\frac{32}{\sqrt{3}}) \\ &= 0.344(\frac{e^2}{\varepsilon_0 b}) \ . \end{aligned} \tag{2.58}$$

Example 2.5 We will calculate the electrostatic interaction energy of NaCl crystal. Sodium chloride crystal is a square lattice, with sodium and chlorine atoms at alternate lattice site. Since there is a huge number of ions, we first calculate the interaction energy for one ion and then multiply by N, the total number of ions. Similarly to the calculation of the above example,

$$\begin{aligned} W(\mathrm{Na}) &= \frac{e^2}{4\pi\varepsilon_0 a}(-6+\frac{12}{\sqrt{2}}-\frac{8}{\sqrt{3}}+\cdots) \\ &= -\frac{8.738e^2}{4\pi\varepsilon_0 a} \ . \end{aligned} \tag{2.59}$$

Since the energy for the two kinds of ions is the same, the total energy is

$$W = NW(\mathrm{Na}) = -\frac{8.738Ne^2}{4\pi\varepsilon_0 a} \ . \tag{2.60}$$

Comparing with experiment, the error is about 10%. The main reason is that the ions are not point charges.

Example 2.6 For an uniformly charged shell with radius R and total surface charge q, the potential is $V = q/4\pi\varepsilon_0 R$. Hence the self energy is

$$W_{\mathrm{surface}} = \frac{1}{2}Vq = \frac{q^2}{8\pi\varepsilon_0 R} \ . \tag{2.61}$$

Example 2.7 For an uniformly charged solid sphere, the potential can be calculated from Eq. (2.31) as followings. For $r > R$, it is obvious that $V(r) = q/4\pi\varepsilon_0 r$. For $r \leq R$,

$$\begin{aligned} V(r) &= -\int_R^r \mathbf{E}\cdot d\mathbf{r} + \frac{q}{4\pi\varepsilon_0 R} \\ &= -\int_R^r \frac{qr}{4\pi\varepsilon_0 R^3}\,dr + \frac{q}{4\pi\varepsilon_0 R} \\ &= \frac{q}{8\pi\varepsilon_0}(\frac{3}{R}-\frac{r^2}{R^3}) \ . \end{aligned} \tag{2.62}$$

The self energy of the solid sphere is

$$\begin{aligned} W &= \frac{1}{2}\int V\rho\,\mathrm{d}^3\mathbf{r} \\ &= \frac{q}{16\pi\varepsilon_0}\int_0^R (\frac{3}{R}-\frac{r^2}{R^3})\,\frac{3q}{4\pi R^3}\,4\pi r^2\,\mathrm{d}r \\ &= \frac{3q^2}{16\pi\varepsilon_0 R^3}\int_0^R (\frac{3r^2}{R}-\frac{r^4}{R^3})\,\mathrm{d}r \\ &= \frac{3q^2}{16\pi\varepsilon_0 R^3}\,\frac{4R^2}{5} \\ &= \frac{3q^2}{20\pi\varepsilon_0 R}\ . \end{aligned} \tag{2.63}$$

If we model the electron as a solid charged sphere and equate the self energy with the rest mass, we have the classical radius of the electron as

$$r_e = \frac{3e^2}{20\pi\varepsilon_0\, m_e c^2} = 1.12\times 10^{-15}\mathrm{m}\ . \tag{2.64}$$

2.5 Conductors

Charges can freely move inside a conductor. This makes conductor play a very special role in electrostatic.

1. $\mathbf{E}=0$ inside a conductor, otherwise, the free charges in the conductor will move.

2. The whole conductor is in the same potential. Hence, the surface of a conductor is an equipotential surface.

3. The electric field is always perpendicular to the surface of conductors, because the surface is an equipotential surface.

4. No net charge is inside the conductor. All net charge resides on the surface.

5. If the curvature of a point on the surface is larger, the charge density there is higher.

2.5.1 Electric Field Near a Boundary

If there are localized charges on boundary, we can use a Gaussian "pillbox", Fig. 2.8, to determine the electric field. If the area ΔS is small, we could assume that the surface charge density, σ_e, is uniform. Also assume that the height of the pillbox tends to zero. Then, according to Gauss's law,

$$\begin{aligned} \frac{\sigma_e\Delta S}{\varepsilon_0} &= \oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} \\ &= \int_{\text{above}}\mathbf{E}\cdot\mathrm{d}\mathbf{S} + \int_{\text{below}}\mathbf{E}\cdot\mathrm{d}\mathbf{S} + \int_{\text{side}}\mathbf{E}\cdot\mathrm{d}\mathbf{S} \end{aligned} \tag{2.65}$$

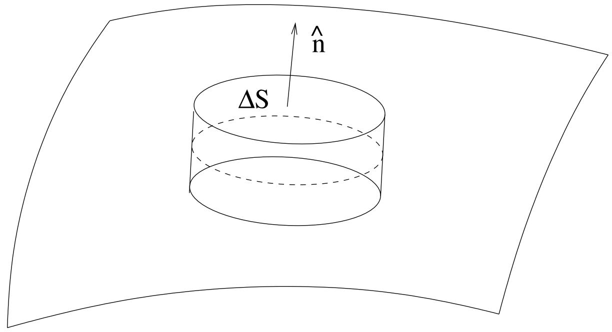

Figure 2.8: A pillbox across the boundary.

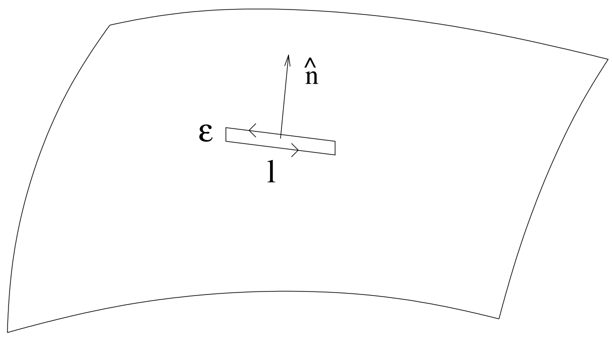

Figure 2.9: A small loop with length l and height ϵ across the boundary.

Since the area of the "side" tends to zero, The third term is zero. We have

$$(\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}) \cdot \hat{n} = \frac{\sigma_e}{\varepsilon_0} . \tag{2.66}$$

To determine the tangential component of the field, consider a small loop across the boundary, Fig. 2.9. By Eq. (2.27),

$$0 = \oint \mathbf{E} \cdot d\mathbf{l} = E^{||}_{\text{above}} l - E^{||}_{\text{below}} l \tag{2.67}$$

where $E^{||}$ is the component of $\mathbf{E}$ parallel the boundary. Hence, the parallel component is continuous across the boundary. Put them together, we have

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma_e}{\varepsilon_0} \hat{n} . \tag{2.68}$$

If one side of the boundary is a conductor, $\mathbf{E} = 0$ inside it. We conclude that the electric field outside of a conductor is $\mathbf{E} = \sigma_e \hat{n}/\varepsilon_0$.

2.5.2 Conducting Shells

For a cavity surrounded by conductor, If there is no charge in the cavity, charge can only be on the outer surface of the conductor. On the inside

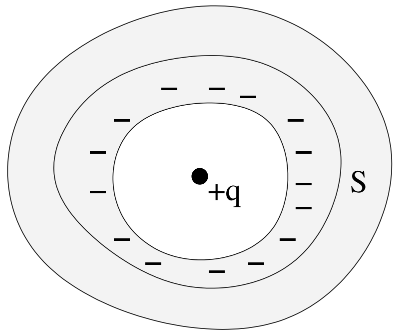

Figure 2.10: A cavity surrounded by conductor.

surface of the cavity, there is no charge. If there are some charges in the cavity, the total induced charge on the cavity wall is equal and opposite to the charge inside, because we can choose a surface totally inside the conductor as the Gaussian surface, S in Fig. 2.10. Then, because $\mathbf{E} = 0$ on S, net charge inside S is zero and the induced charge is $-q$.

2.6 Capacitors

2.6.1 Capacitance

Suppose there are two conductors separated by a distance. If the net charge on one is $+Q$ and on another is $-Q$, then we find that the potential V between them is proportional to Q. (This can be proved by the uniqueness theorems in next chapter.) The constant of proportionality is called the **capacitance** $C \equiv Q/V$.

Capacitance is a purely geometrical quantity, determined by the sizes, shapes and separation of the two conductors. Its physical meaning is the amount of charge needed to increase one unit of potential. In MKS units, capacitance is measured in **farads**.

The single conductor can also have a capacitance. In this case, the second conductor is an imaginary spherical shell of infinite radius.

Example 2.8 To calculate the capacitance of a pair of parallel plates, we assume the square root of the area $\sqrt{A}$ is much larger than the distance d between the plates, and then the electric field E will be independent of position between the plates. We have

$$V = Ed = \frac{\sigma_e d}{\varepsilon_0} = \frac{Qd}{\varepsilon_0 A} \ . \tag{2.69}$$

The capacitance is

$$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d} \ . \tag{2.70}$$

Example 2.9 The electric field between two concentric spherical shells is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \ . \tag{2.71}$$

The potential between them is

$$V = \int_{r_a}^{r_b} E\,\mathrm{d}r = \frac{Q}{4\pi\varepsilon_0}\,(\frac{1}{r_a} - \frac{1}{r_b}) \ . \tag{2.72}$$

The capacitance is

$$C = 4\pi\varepsilon_0\,\frac{r_a r_b}{r_b - r_a} \tag{2.73}$$

where r_a and r_b are the radii of the smaller and larger shells respectively.

Example 2.10 Let the radii of two concentric cylindrical tubes be a and b ($b > a$). If the length of two tubes L is much larger than their radii, we could neglect the edge effect. The electric field is

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 r} \ . \tag{2.74}$$

(Why?) Here λ is the charge per unit length. Thus, the total charge on one tube is $Q = \lambda L$. The potential is

$$V = \int_a^b \frac{\lambda}{2\pi\varepsilon_0\, r}\mathrm{d}r = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a} \ . \tag{2.75}$$

The capacitance is

$$C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)} \ . \tag{2.76}$$

2.6.2 Energy Stored in Capacitor

The infinitesimal work done on a capacitor is

$$\mathrm{d}W = V\,\mathrm{d}q \ . \tag{2.77}$$

Thus, the energy stored in a capacitor is

$$W = \int \mathrm{d}W = \int_0^Q V\,\mathrm{d}q = \int_0^Q \frac{q}{C}\mathrm{d}q = \frac{Q^2}{2C} \tag{2.78}$$

which also equals to

$$W = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV \ . \tag{2.79}$$

Chapter 3

Special Approaches in Problem Solving

In this chapter, we will discuss some special techniques to solve the electrostatic problems. Since the field is independent of time, we could replace the equation about the electric field by the equation of potential. This will be easier to solve as we only have to deal with a scalar field instead of a vector field.

3.1 Laplace's Equation

The potential satisfies the Poisson's equation,

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \, . \tag{3.1}$$

If $\rho = 0$, the equation becomes the Laplace's equation, $\nabla^2 V = 0$. This is a partial differential equation. All kinds of techniques to solve a partial differential equation can be applied here.

Separation of variables is a useful trick to solve P.D.E. We illustrate with a two dimensional case

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \, . \tag{3.2}$$

We will find solutions that are products of functions, each function depends only on one variable, $V(x, y) = X(x)Y(y)$. The general solution will be linear combinations of them. Eq. (3.2) is reduced to

$$\begin{aligned} Y\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + X\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} &= 0 \\ \frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} &= -\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} \, . \end{aligned} \tag{3.3}$$

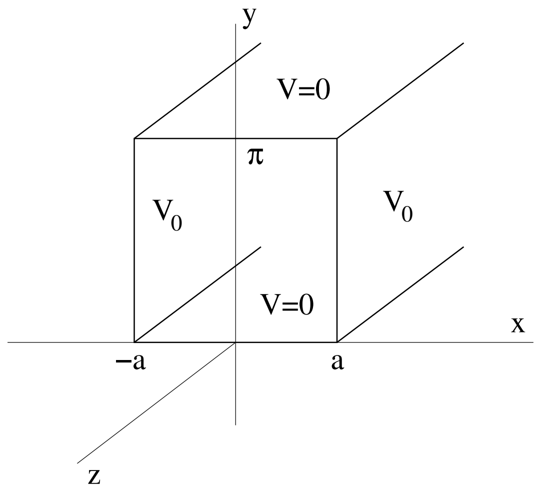

Figure 3.1: A rectangular conducting pipe.

Since the two sides depend on different independent variables, this equation is valid only if both sides equal to a constant, k^2, say. Thus,

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = k^2 = -\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} \ , \tag{3.4}$$

which implies

$$\begin{aligned} X(x) &= Ae^{kx} + Be^{-kx} \\ Y(y) &= C\sin ky + D\cos ky \ . \end{aligned} \tag{3.5}$$

We could fix the constants A, B, C, D and k with the boundary or initial conditions.

Example 3.1 Two infinitely long grounded metal plates are connected at $x = \pm a$ by metal strips kept at potential V_0, Fig. 3.1. We are going to find the potential inside the rectangular pipe. By the symmetry of the problem, the result is independent of z. Thus, the potential satisfies Eq. (3.2). The boundary conditions are

i. $V(x,0) = V(x,\pi) = 0$;

ii. $V(\pm a, y) = V_0$.

From Eq. (3.5), the solution should be in the form $(Ae^{kx}+Be^{-kx})(C\sin ky + D\cos ky)$. The solution should be even with respect to x, so $A = B$ and we have

$$\cosh kx(C\sin ky + D\cos ky) \tag{3.6}$$

where we have redefined the constant C and D. The boundary condition (i) requires that $D = 0$ and k is a positive integer. Thus, the general solution is

$$V(x,y) = \sum_{k=1}^{\infty} C_k \cosh kx \sin ky \ . \tag{3.7}$$

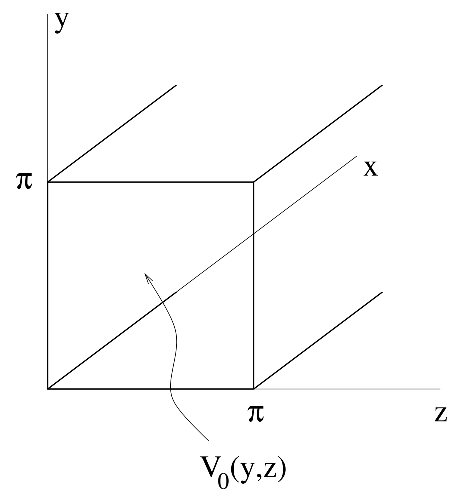

Figure 3.2: A semi-infinitely long square conducting pipe.

The boundary condition (ii) requires that $\sum_{k=1}^{\infty} C_k \cosh ka \sin ky = V_0$. To determine the coefficients C_k, we can multiply both sides by $\sin ny$ and integrating from 0 to π

$$\begin{aligned} \sum_{k=1}^{\infty} C_k \cosh ka \int_0^{\pi} \sin ky \sin ny \, \mathrm{d}y &= \int_0^{\pi} V_0 \, \sin ny \mathrm{d}y \\ C_n \cosh na &= \frac{2V_0}{\pi} \int_0^{\pi} \sin ny \, \mathrm{d}y \\ &= \frac{2V_0}{n\pi}(1 - \cos n\pi) \ . \end{aligned} \tag{3.8}$$

Therefore, the potential in this case is given by

$$V(x,y) = \frac{4V_0}{\pi} \sum_{k=1,3,5\ldots} \frac{1}{k} \frac{\cosh kx}{\cosh ka} \sin ky \ . \tag{3.9}$$

In summary, the method of separation of variables can be described as follows,

1. Find out the boundary conditions from the problem;
2. From the shape and configuration of the system, choose a suitable coordinate system;
3. Use the method of separation of variables, find the general solutions of Laplace's equation.
4. The solution of the problem should be able to be expressed as a linear combination of the general solutions;
5. Use the boundary conditions to determine the coefficients.

Example 3.2 An semi-infinitely long square metal pipe with a size π is grounded, but the potential at the end is given by $V(0,y,z) = V_0(y,z)$. Find the potential in the pipe. The boundary conditions are

$$\begin{cases} V(y=0) = V(y=\pi) = 0 \\ V(z=0) = V(z=\pi) = 0 \\ V(x \to \infty) = 0 \\ V(0,y,z) = V_0(y,z) \end{cases} , \tag{3.10}$$

with the Laplace's equation in the form

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \ . \tag{3.11}$$

Suppose $V = X(x)Y(y)Z(z)$. Substitute into the above equation, we have

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + \frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = 0 \ . \tag{3.12}$$

Similar to the argument in the previous example, the three terms must be equal to constants, with the sum of the three constants zero. Let the constant corresponding to the x term be $k^2 + l^2$, that of the y term be $-k^2$ and z be $-l^2$. The solutions are

$$\begin{cases} X(x) = Ae^{\sqrt{k^2+l^2}x} + Be^{-\sqrt{k^2+l^2}x} \\ Y(y) = C\sin ky + D\cos ky \\ Z(z) = E\sin lz + F\cos lz \end{cases} . \tag{3.13}$$

From the boundary conditions, we have

$$\begin{cases} V(x \to \infty) = 0 & \to \quad A = 0 \\ V(y=0) = 0 & \to \quad D = 0 \\ Z(z=0) = 0 & \to \quad F = 0 \\ V(y=\pi) = 0 & \to \quad k \text{ is a positive integer} \\ V(z=\pi) = 0 & \to \quad l \text{ is a positive integer} \end{cases} \tag{3.14}$$

Hence, the solution is in the form $C_{kl}\, e^{-\sqrt{k^2+l^2}x} \sin ky \sin lz$, and the general solution is a linear combination

$$V = \sum_{k,l=1}^{\infty} C_{kl}\, e^{-\sqrt{k^2+l^2}x} \sin ky \sin lz \ . \tag{3.15}$$

To evaluate C_{kl}, multiply by $\sin ny \sin mz$ and integrate

$$\begin{aligned} & \sum_{k,l=1}^{\infty} C_{kl} \int_0^{\pi} \sin ky \sin ny \,\mathrm{d}y \int_0^{\pi} \sin lz \sin mz \,\mathrm{d}z \\ = \; & \int_0^{\pi}\int_0^{\pi} V_0(y,z) \sin ny \sin mz \,\mathrm{d}y\mathrm{d}z \end{aligned} \tag{3.16}$$

where we have used the boundary condition $V(x=0) = V_0(y,z)$. Note that $\int_0^\pi \sin ky \sin ny \, \mathrm{d}y = \delta_{kn}\pi/2$. Thus,

$$C_{nm} = \left(\frac{2}{\pi}\right)^2 \int_0^\pi \int_0^\pi V_0(x,y) \sin ny \sin mz \, \mathrm{d}y \mathrm{d}z \ . \tag{3.17}$$

We cannot simplify the above equation without further knowledge of V_0. For instance, if $V_0(y,z) = V_0$ is a constant, we can deduce that

$$C_{nm} = \begin{cases} 0 & \text{if } n \text{ or } m \text{ is even} \\ \dfrac{16V_0}{\pi^2 nm} & \text{if both } n \text{ and } m \text{ are odd} \end{cases} , \tag{3.18}$$

and the solution is

$$V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{n,m=1,3,5\ldots}^{\infty} \frac{1}{nm} e^{-\sqrt{n^2+m^2}x} \sin ny \sin mz \ . \tag{3.19}$$

It decreases very rapidly.

3.2 Legendre Functions

We are going to use the Legendre polynomials in the next section. We will briefly discuss them in this section.

The **Legendre equation** is an ordinary differential equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + l(l+1)y = 0 \ . \tag{3.20}$$

Its solution is Legendre functions. If $l = 0, 1, 2, \ldots$ are non-negative integers, its general solution can be written as

$$y = c_1 P_l(x) + c_2 Q_l(x) \tag{3.21}$$

where $P_l(x)$ are **Legendre polynomials** and $Q_l(x)$ is the Legendre function of the second kind. These are not very important to us because Q_l are singular at ± 1, $Q_l(\pm 1) \to \infty$.

The Legendre polynomials are

$$\begin{aligned} P_l(x) &= \frac{(2l-1)(2l-3)\ldots 1}{l!}\left(x^l - \frac{l(l-1)}{2(2l-1)}x^{l-2} + \cdots\right) \\ &= \frac{1}{2^l}\sum_{k=0}^{[l/2]}(-1)^k \frac{(2l-2k)!}{k!(l-k)!(l-2k)!}x^{l-2k} \end{aligned} \tag{3.22}$$

where $[\frac{l}{2}]$ is the maximum integer smaller than or equal to $\frac{l}{2}$, $[\frac{l}{2}] = l/2$ if l is even, $[\frac{l}{2}] = (l-1)/2$ if l is odd.

For small l, we have

$$\begin{aligned} P_0(x) &= 1 && (3.23)\\ P_1(x) &= x && (3.24)\\ P_2(x) &= \frac{1}{2}(3x^2-1) && (3.25)\\ P_3(x) &= \frac{1}{2}(5x^3-3x) && (3.26)\\ P_4(x) &= \frac{1}{8}(35x^4-30x^2+3)\ . && (3.27) \end{aligned}$$

We list without proof some properties of Legendre polynomials.

1. For all values of l, $P_l(1)=1$, $P_l(-1)=(-1)^l$.
2. The **Rodrigues formula** is

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l\ . \tag{3.28}$$

3. The generating function of Legendre polynomials is

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)\, t^l\ . \tag{3.29}$$

4. The recursion relations are, for $l \geq 1$,

$$\begin{aligned} P_{l+1}(x) &= \frac{2l+1}{l+1}xP_l(x) - \frac{l}{l+1}P_{l-1}(x) && (3.30)\\ P'_{l+1}(x) &= P'_{l-1}(x) + (2l+1)P_l(x)\ . && (3.31) \end{aligned}$$

5. The orthogonality relation is

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \frac{2}{2n+1}\,\delta_{n,m}\ . \tag{3.32}$$

We will show that $\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = 0$ for $n \neq m$ as an illustration. As P_m and P_n are solutions of Legendre equation,

$$\begin{aligned} (1-x^2)P''_m - 2xP'_m + m(m+1)P_m = 0 && (3.33)\\ (1-x^2)P''_n - 2xP'_n + n(n+1)P_n = 0\ . && (3.34) \end{aligned}$$

Multiply Eq. (3.34) by P_m and Eq. (3.33) by P_n, the difference is

$$\begin{aligned} & (1-x^2)(P_nP''_m - P_mP''_n) - 2x(P_nP'_m - P_mP'_n)\\ = \ & [n(n+1) - m(m+1)]P_mP_n\ . && (3.35) \end{aligned}$$

It can be written as

$$\begin{aligned} & (1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}(P_nP'_m - P_mP'_n) - 2x(P_nP'_m - P_mP'_n)\\ = \ & [n(n+1) - m(m+1)]P_mP_n\ , && (3.36) \end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)(P_n P_m' - P_m P_n')\right] = [n(n+1) - m(m+1)]P_m P_n \; . \tag{3.37}$$

Integrate both sides from -1 to 1, the left hand side is zero because of the factor $(1-x^2)$. If $n \neq m$, then the pre-factor on the right hand side is non-zero, and we have

$$\int_{-1}^{1} P_m(x) P_n(x)\,\mathrm{d}x = 0 \; . \tag{3.38}$$

Because of the orthogonality relation Eq. (3.32), we have the **Legendre polynomial series**. If $f(x)$ is a continuous function for $-1 < x < 1$, then we can expand it in terms of the Legendre polynomials

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \tag{3.39}$$

where

$$A_l \;=\; \frac{2l+1}{2}\int_{-1}^{1} f(x)\,P_l(x)\,\mathrm{d}x \; . \tag{3.40}$$

3.3 Laplace Equation in Spherical Symmetric Systems

In many spherical symmetric systems, a convenient coordinate system is the spherical coordinate system. The Laplace's equation in this coordinate system is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} \; . \tag{3.41}$$

In many cases, V is independent of ϕ. Thus, we have

$$\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) = 0 \; . \tag{3.42}$$

As before, we look for solutions by separation of variables and assume that $V(r,\theta) = R(r)\Theta(\theta)$. Substitute this in Eq. (3.42), multiply by $1/R\Theta$, we have

$$\frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Theta}{\partial\theta}) = 0 \; . \tag{3.43}$$

The two terms must equal to a constant, let it be λ^2.

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2\frac{\partial R}{\partial r}) + \lambda^2 R \;=\; 0 \tag{3.44}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}) - \lambda^2\Theta\sin\theta \;=\; 0 \; . \tag{3.45}$$

Eq. (3.44) can be written as

$$r^2 R'' + 2rR' + \lambda^2 R = 0 \ . \tag{3.46}$$

Its general solution is

$$R = Ar^l + \frac{B}{r^{l+1}} \tag{3.47}$$

where $l = -\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda^2}$, or $\lambda^2 = -l(l+1)$. If we let $x = \cos\theta$, then $\mathrm{d}\Theta/\mathrm{d}x = -\mathrm{d}\Theta/\mathrm{d}\theta/\sin\theta$. Hence,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta \frac{\mathrm{d}\Theta}{\mathrm{d}\theta}) &= \frac{\mathrm{d}}{\mathrm{d}\theta}[-(1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}] \\ &= \sin\theta \frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}] \ . \end{aligned} \tag{3.48}$$

Eq. (3.45) can then be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}] + l(l+1)\Theta = 0 \ . \tag{3.49}$$

This is the Legendre equation. Thus, the solution of Eq. (3.42) is

$$V(r,\theta) = (Ar^l + \frac{B}{r^{l+1}})[CP_l(\cos\theta) + DQ_l(\cos\theta)] \ . \tag{3.50}$$

As before, the method of separation of variables leads to an infinite set of solutions. If we demand a non-singular solution at $\theta = 0$ and π, we have to set $D = 0$. The general solution is a linear combination

$$V(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta) \ . \tag{3.51}$$

Example 3.3 The potential $V_0(\theta)$ is specified on the surface of a hollow, empty spherical shell of radius R. What is the potential?

Inside the sphere, the potential must be non-singular at $r = 0$. So, we must set $B_l = 0$, and we have

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{3.52}$$

and $V(R,\theta) = V_0$. To find A_l, because the Legendre polynomials are orthogonal to each other, we have

$$A_m R^m \frac{2}{2m+1} = \int_0^{\pi} V_0(\theta) P_m(\cos\theta)\sin\theta \,\mathrm{d}\theta \tag{3.53}$$

and

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta \ . \tag{3.54}$$

For instance, if $V_0(\theta) = k\sin^2(\theta/2)$ where k is a constant,

$$V_0(\theta) = \frac{k}{2}(1 - \cos\theta) = \frac{k}{2}\left(P_0(\cos\theta) - P_1(\cos\theta)\right) \ . \tag{3.55}$$

Thus, $A_0 = k/2$, $A_1 = -k/(2R)$ and all other $A_l = 0$. Therefore, for $r < R$,

$$V(r,\theta) = \frac{k}{2}(P_0(\cos\theta) - \frac{r}{R}P_1(\cos\theta)) = \frac{k}{2}(1 - \frac{r}{R}\cos\theta) \ . \tag{3.56}$$

Outside the sphere, we demand $V(r \to \infty) = 0$. In this case, $A_l = 0$ and

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \ . \tag{3.57}$$

Set $r = R$, multiply by $P_m(\cos\theta)\sin\theta$ and integrate

$$\begin{aligned} \frac{B_m}{R^{m+1}}\frac{2}{2m+1} &= \int_0^\pi V_0(\theta)P_m(\cos\theta)\sin\theta\,\mathrm{d}\theta \\ B_m &= \frac{(2m+1)R^{m+1}}{2}\int_0^\pi V_0(\theta)P_m(\cos\theta)\sin\theta\,\mathrm{d}\theta \ . \end{aligned} \tag{3.58}$$

Continue with the above example, for $r > R$,

$$V(r,\theta) = \frac{k}{2}(\frac{R}{r} - \frac{R^2}{r^2}\cos\theta) \ . \tag{3.59}$$

3.4 Uniqueness Theorems

The Poisson's equation or the Laplace's equation alone cannot determine the potential or electric field of the problem, just like an ordinary differential equation cannot completely determine the solution. We need boundary conditions, or initial conditions, to fully fix the solution. However, the boundary conditions cannot be over-specified. Otherwise, it could lead to inconsistencies. How much boundary condition is just right is the content of the uniqueness theorems. We will state without proof some uniqueness theorems.

Our **first uniqueness theorem** is that the solution of the Laplace's equation of some region is completely determined if the value on the boundary surface of the region is specified.

This theorem was illustrated in the previous three examples. We have found the potentials inside the pipes in the first two examples and potential of whole space in the third. The existence of the solutions are guaranteed by the theorem, and they are the unique solutions.

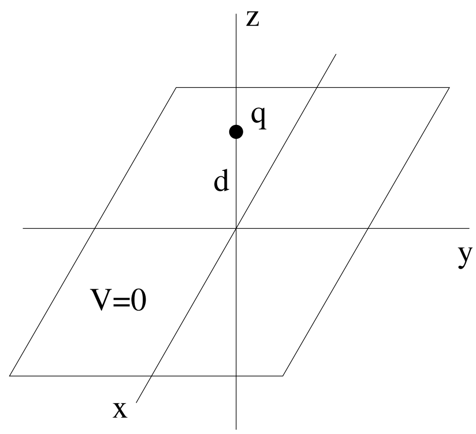

Figure 3.3: A point charge near an infinite conducting plate.

This also give us another way to find solutions. If we have come up with a function which satisfies the Laplace's equation and the boundary conditions, we know for sure that it is the unique solution. How we get the function is completely not important. We are going to do something like this in the next section.

A **corollary** is that if the charge distribution in a region is specified and the potential on the boundary surface of the region is also specified, then there is an unique solution for the potential in the region.

The **second uniqueness theorem** is that if a region is surrounded by conductors, the charge density in the region is specified and the total charges carried by the conductors are specified, then the potential in the region is uniquely determined.

This theorem provides the theoretical basis for the capacitors, because we can deduce that the potential is proportional to the amount of charge. (How?)

3.5 The Method of Images

For some highly symmetric situations, the potential in the concerned region can be described by the effect of one or more imaginary charges out of the region. These imaginary charges are called the **image charges** or images. This is the basic steps of the method of images.

Example 3.4 Consider a point charge q at a distance d from an infinite conducting plate, Fig. 3.3, where the conducting plate is set to be $V = 0$. What is the potential above the plane? Note q will induce some negative charge on the surface of the conductor. The boundary conditions are

1. $V(z = 0) = 0$;
2. $V \to 0$ if $x^2 + y^2 + z^2 \gg d^2$.

To solve this problem, consider another problem in which not only is there a charge q at $(0,0,d)$, but also another $-q$ at $(0,0,-d)$. The potential for this is

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right) . \tag{3.60}$$

This solution satisfies the Laplace's equation and the above boundary conditions. From the uniqueness theorem, we are sure that this is the solution for $z > 0$. For $z < 0$, the charge distribution is different, thus the solution will be different. However we don't care. The charge at $(0,0,-d)$ is the image.

Since the fields around q are the same for the two problems, the force is

$$\mathbf{F} = \frac{-1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{k} . \tag{3.61}$$

However, the energy is different! For $+q$ and $-q$ without conducting plane,

$$W = \frac{-1}{4\pi\varepsilon_0}\frac{q^2}{2d} . \tag{3.62}$$

For $+q$ and the conducting plane,

$$W = \int_\infty^d \mathbf{F}\cdot \mathrm{d}\mathbf{l} = \frac{1}{4\pi\varepsilon_0}\int_\infty^d \frac{q^2}{4z^2}\,\mathrm{d}z = \frac{1}{4\pi\varepsilon_0}\left.(-\frac{q^2}{4z})\right|_\infty^d = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d} . \tag{3.63}$$

Example 3.5 A charge q is at a distance d to a very large conducting plane. At $t = 0$ the charge is released. When will the charge reach the plane?

According to Eq. (3.61), if the charge is of a distance x to the plane, the equation of motion is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4x^2} . \tag{3.64}$$

Multiply both sides by $\mathrm{d}x/\mathrm{d}t$

$$\begin{aligned} m(\frac{\mathrm{d}x}{\mathrm{d}t})(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}) &= -\frac{1}{4\pi\varepsilon_0}(\frac{q^2}{4x^2})(\frac{\mathrm{d}x}{\mathrm{d}t}) \\ \frac{\mathrm{d}}{\mathrm{d}t}[\frac{1}{2}m(\frac{\mathrm{d}x}{\mathrm{d}t})^2] &= \frac{q^2}{16\pi\varepsilon_0}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{x} \\ \frac{1}{2}m(\frac{\mathrm{d}x}{\mathrm{d}t})^2 &= \frac{q^2}{16\pi\varepsilon_0}(\frac{1}{x}-\frac{1}{d}) . \end{aligned} \tag{3.65}$$

Here we have used the initial conditions $x(0) = d$ and $\mathrm{d}x/\mathrm{d}t = 0$ at $t = 0$. Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sqrt{\frac{q^2}{8m\pi\varepsilon_0}\frac{d-x}{xd}} \tag{3.66}$$

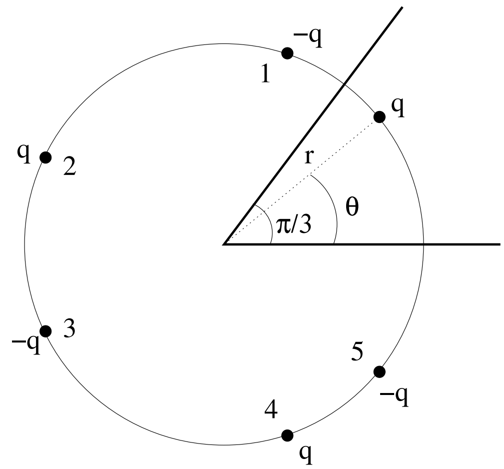

Figure 3.4: A point charge near a $\pi/3$ wedge of conducting plates.

and the time taken is

$$\begin{aligned} T &= \int_0^T \mathrm{d}t \\ &= -\int_d^0 \frac{\sqrt{8m\pi\varepsilon_0}}{q}\sqrt{\frac{xd}{d-x}}\,\mathrm{d}x \\ &= \frac{\sqrt{8m\pi\varepsilon_0\,d^3}}{q}\int_0^{\pi/2}\sqrt{\frac{\sin^2\theta}{\cos^2\theta}}\,2\sin\theta\cos\theta\,\mathrm{d}\theta \\ &= \frac{\sqrt{8m\pi\varepsilon_0\,d^3}}{q}\,\frac{\pi}{2} \\ &= \frac{\sqrt{2m\pi^3\varepsilon_0\,d^3}}{q} \end{aligned} \tag{3.67}$$

where we have let $x = d\sin^2\theta$.

Example 3.6 If there are two conducting plates making an angle of $\pi/3$ with each other and a charge q is in between them, Fig. 3.4, then the image charges are at

$$\begin{aligned} &q_1 = -q(r, 2\pi/3 - \theta), \quad q_2 = q(r, 2\pi/3 + \theta), \\ &q_3 = -q(r, 4\pi/3 - \theta), \quad q_4 = q(r, 4\pi/3 + \theta), \\ &q_5 = -q(r, 6\pi/3 - \theta)\ . \end{aligned} \tag{3.68}$$

Potential at a point P is

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3} + \frac{1}{r_4} - \frac{1}{r_5}\right) \tag{3.69}$$

where r_i is the distance of P from q_i. This is exactly like the case of two mirrors.

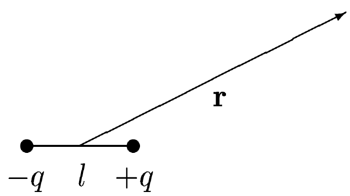

Figure 3.5: An electric dipole.

3.6 Multipole Expansion

For a localized charge distribution, if we observe from a very large distance, the potential will tend to $Q/(4\pi\varepsilon_0\, r)$ where Q is the net charge.

For an electric dipole,

$$V(P) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right) \tag{3.70}$$

where r_+ is the distance between $+q$ and the field point, similarly for r_-. From the law of cosines,

$$\begin{aligned} r_\pm^2 &= r^2 \mp rl\cos\theta + \left(\frac{l}{2}\right)^2 = r^2\left(1 \mp \frac{l}{r}\cos\theta + \frac{l^2}{4r^2}\right) \\ \frac{1}{r_\pm} &= \frac{1}{r}\left(1 \mp \frac{l}{r}\cos\theta\right)^{-1/2} \approx \frac{1}{r}\left(1 \pm \frac{l}{2r}\cos\theta\right) \\ \frac{1}{r_+} - \frac{1}{r_-} &\approx \frac{l}{r^2}\cos\theta \,. \end{aligned} \tag{3.71}$$

Therefore,

$$V(P) = \frac{1}{4\pi\varepsilon_0}\frac{ql}{r^2}\cos\theta \,. \tag{3.72}$$

We conclude that the potential of a dipole goes as $1/r^2$. Similarly, potential of a quadrupole goes as $1/r^3$, for an octopole, it goes as $1/r^4$. (Can you integrate Eq. (2.11) to get Eq. (3.72)?)

We can, in fact, obtain an expansion for the potential of an arbitrary localized charge distribution in powers of $1/r$. Assume that the charge distribution is cylindrical symmetric, ρ does not depend on ϕ. (If it does, it is only technically more difficult.) In general,

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}' \,. \tag{3.73}$$

Then, using the law of cosines, let $s = |\mathbf{r}-\mathbf{r}'|$, we have

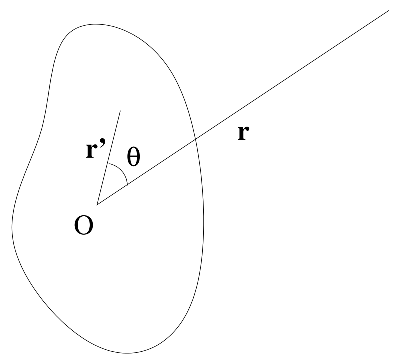

Figure 3.6: The configuration for the expansion for the potential.

$$s^2 = r^2 + r'^2 - 2rr'\cos\theta = r^2(1 + \frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta) \ . \tag{3.74}$$

We can rewrite it as $s = r\sqrt{1+\delta}$ where $\delta = (\frac{r'}{r})(\frac{r'}{r} - 2\cos\theta)$. By binomial expansion,

$$\begin{aligned}
\frac{1}{s} &= \frac{1}{r}(1+\delta)^{-1/2} \\
&= \frac{1}{r}(1 - \frac{1}{2}\delta + \frac{3}{8}\delta^2 - \frac{5}{16}\delta^3 + \ldots) \\
&= \frac{1}{r}[1 - \frac{1}{2}(\frac{r'}{r})(\frac{r'}{r} - 2\cos\theta) + \frac{3}{8}(\frac{r'}{r})^2(\frac{r'}{r} - 2\cos\theta)^2 \\
&\quad -\frac{5}{16}(\frac{r'}{r})^3(\frac{r'}{r} - 2\cos\theta)^3 + \cdots] \\
&= \frac{1}{r}[1 + (\frac{r'}{r})\cos\theta + (\frac{r'}{r})^2(\frac{3}{2}\cos^2\theta - \frac{1}{2}) \\
&\quad +(\frac{r'}{r})^3(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta) + \cdots] \\
&= \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^n P_n(\cos\theta) \qquad (3.75)
\end{aligned}$$

where P_n are the Legendre polynomials. The last equality can directly be obtained by Eq.(3.29). Thus, the multipole expansion of V in powers of $1/r$ can be written as

$$\begin{aligned}
V(P) &= \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int \rho\, r'^n P_n(\cos\theta)\,\mathrm{d}^3\mathbf{r}' \\
&= \frac{1}{4\pi\varepsilon_0}[\frac{1}{r}\int \rho\,\mathrm{d}^3\mathbf{r}' + \frac{1}{r^2}\int \rho\, r'\cos\theta\,\mathrm{d}^3\mathbf{r}' \\
&\quad +\frac{1}{r^3}\int \rho\, r'^2(\frac{3}{2}\cos^2\theta - \frac{1}{2})\,\mathrm{d}^3\mathbf{r}' + \cdots] \ . \qquad (3.76)
\end{aligned}$$

The first term ($n = 0$) is the monopole term. It goes as $1/r$. It provided the approximated V at large r. The second term is the dipole term, $1/r^2$.

The third term is the quadrupole term, $1/r^3$. The forth term is the octopole term, $1/r^4$. We can include more terms if greater precision is required.

Chapter 4

Electrostatic Fields in Dielectrics

4.1 Polarization

Common materials can be classified into two broad categories: conductors and **insulators** or **dielectrics**. Charges inside a conductor can go freely. All charges in insulators are attached to specific atoms or molecules. With an applied electric field $\mathbf{E}$, the charges within an atom will be displaced from their respective equilibrium positions. We say that the atoms are polarized. The polarized atoms are similar to dipoles which have induced dipole moment $\mathbf{p} = \alpha\mathbf{E}$, where α is called the **atomic polarizability**.

For some matters, their molecules have permanent dipole moments. They are called **polar molecules**. Water, H_2O, is a common material with permanent dipole moment. Refer to Fig. 4.1, the force on a dipole is $\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q\,(\mathbf{E}_+ - \mathbf{E}_-) = q\,\delta\mathbf{E}$ where $\delta\mathbf{E} = (\mathbf{l}\cdot\nabla)\mathbf{E}$, or

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}\ . \tag{4.1}$$

If the electric field is uniform in the size of the dipole, the dipole will experience no net force. However, it will still experiences a torque

$$\mathbf{N} = \mathbf{p}\times\mathbf{E} \tag{4.2}$$

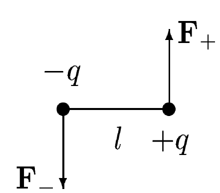

Figure 4.1: An electric dipole under the influence of an external electric field.

where $\mathbf{E}$ is the average electric field, or the electric field at the center of the dipole.

In summary, when a field is applied, non-polar material will have displacement and become polarized. For polar material, the dipoles will also response by rotating. Hence, all insulators or dielectrics becomes polarized. To measure this effect, we can use the **polarization**

$$\mathbf{P} = \lim_{\Delta V \to 0} \frac{\sum \mathbf{p}}{\Delta V} \ . \tag{4.3}$$

This is the density of dipoles. The volume is taken to be small compared to macroscopic measures and large compared to the size of atoms.

The alignment of the polar molecules has a much stronger response to electric field than the displacement of non-polar molecules. To an oscillating fields with a high frequency, as the mass of the nuclei is larger than mass of electrons, only displacement of electrons will be significant.

4.2 Bound Charges

Since the charges induced by the polarization are bounded to the atoms, they are called the **bound charges**. What is the potential created by the polarization? By Eq. (3.72),

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\text{volume}} \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, \mathrm{d}^3\mathbf{r}' \tag{4.4}$$

where the integration region is the volume of the dielectrics. Because

$$\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{4.5}$$

(The differentiation is with respect to $\mathbf{r}'$.),

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0} \int_{\text{volume}} (\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}) \cdot \mathbf{P}(\mathbf{r}')\, \mathrm{d}^3\mathbf{r}' \\ &= \frac{1}{4\pi\varepsilon_0} \int_{\text{volume}} \left(\nabla' \cdot \frac{\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|} - \frac{\nabla'\cdot\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|} \right) \mathrm{d}^3\mathbf{r}' \\ &= \frac{1}{4\pi\varepsilon_0} \int_{\text{surface}} \frac{\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|} \cdot \mathrm{d}\mathbf{S} - \frac{1}{4\pi\varepsilon_0} \int_{\text{volume}} \frac{\nabla'\cdot\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|}\, \mathrm{d}^3\mathbf{r}' \ . \end{aligned} \tag{4.6}$$

The first term is the contribution from some surface charge. We interpret $\mathbf{P}\cdot\hat{\mathbf{n}}$ as the surface bound charge induced by the polarization, where $\hat{\mathbf{n}}$ is the outward pointing normal of the surface. The second term is the contribution from some charge density. Hence, $-\nabla\cdot\mathbf{P}$ is the charge density of the polarization.

We can have a rough physical understanding of why $\mathbf{P}\cdot\hat{\mathbf{n}}$ is the surface charge. For uniform dielectric, the induced charges in the bulk will cancel each other since the positive charge of one dipole will cancel the effect of the negative charge of another nearby dipole. Thus, the net charge are localized on the surface, in the thin volume segment near the surface of thickness of l. For a small surface area δS and the dipole density n, the charge in this layer is $Q = nql\delta S\cos\theta = \mathbf{P}\cdot \mathrm{d}\mathbf{S}$, where $\cos\theta$ is the angle between the dipole moment and the normal of the surface. Therefore, the surface charge density is $\sigma = Q/\delta S = \mathbf{P}\cdot\hat{\mathbf{n}}$.

4.3 Electric Displacement

The total net charge in dielectrics consists of bound charges and free charges. The charge density is given by $\rho = \rho_b + \rho_f$. By Gauss's law, we have

$$\varepsilon_0 \nabla\cdot\mathbf{E} = \rho = \rho_b + \rho_f = -\nabla\cdot\mathbf{P} + \rho_f \ . \tag{4.7}$$

Put the divergence on the right hand side to the left, we have $\nabla\cdot(\varepsilon_0\mathbf{E}+\mathbf{P}) = \rho_f$. Define $\mathbf{D} = \varepsilon_0\mathbf{E}+\mathbf{P}$. It is the **dielectric displacement**. Then, Gauss's law in dielectrics becomes

$$\nabla\cdot\mathbf{D} = \rho_f \tag{4.8}$$

or in integral form, $\oint_S \mathbf{D}\cdot \mathrm{d}\mathbf{S} = Q_f$, where Q_f is the total free charge enclosed by S.

Note that there is no Coulomb's law for $\mathbf{D}$, i.e. $\mathbf{D} \neq \frac{1}{4\pi}\int \rho_f \hat{r}/r^2\, \mathrm{d}^3\mathbf{r}$. The reason is that unlike the electric field, $\nabla\times\mathbf{D} = \varepsilon_0(\nabla\times\mathbf{E})+\nabla\times\mathbf{P} = \nabla\times\mathbf{P} \neq 0$ in general. And only knowing the divergence is not sufficient to determine a vector. By the same reason, there is in general no "potential" for $\mathbf{D}$ because the curl of it is non-zero.

4.3.1 Boundary Conditions of Dielectrics

Similar to Subsection 2.5.1, we can derive the boundary conditions across dielectrics. Now, we have $\nabla\cdot\mathbf{D} = \rho_f$ instead of $\nabla\cdot\mathbf{E} = \rho_f/\varepsilon_0$. If the surface free charge density is σ_f, the conditions are

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f \tag{4.9}$$

$$E^{||}_{\text{above}} - E^{||}_{\text{below}} = 0 \ . \tag{4.10}$$

4.4 Linear Dielectrics

If the electric field is not too strong, the polarization is proportional to $\mathbf{E}$ for an isotropic dielectrics

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \tag{4.11}$$

where χ_e is called the **electric susceptibility** and such dielectrics is called linear dielectrics. We have

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = \varepsilon_0 (1 + \chi_e) \mathbf{E} \equiv \varepsilon \mathbf{E} \tag{4.12}$$

where ε is the **permittivity** of the material, and $\kappa \equiv 1 + \chi_e$ is the **dielectric constant** of the material. The bound charge is given by

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot (\varepsilon_0 \frac{\chi_e}{\varepsilon} \mathbf{D}) = -\frac{\chi_e}{1+\chi_e} \rho_f \ . \tag{4.13}$$

For anisotropic material, usually a crystal, the polarization could be in a different direction from the electric field, although the response could be still linear. In this case, χ_e is a tensor in the sense that

$$P_x = \varepsilon_0 (\chi_{exx} \mathbf{E}_x + \chi_{exy} \mathbf{E}_y + \chi_{exz} \mathbf{E}_z) \tag{4.14}$$

$$P_y = \varepsilon_0 (\chi_{eyx} \mathbf{E}_x + \chi_{eyy} \mathbf{E}_y + \chi_{eyz} \mathbf{E}_z) \tag{4.15}$$

$$P_z = \varepsilon_0 (\chi_{exx} \mathbf{E}_x + \chi_{exy} \mathbf{E}_y + \chi_{exz} \mathbf{E}_z) \ . \tag{4.16}$$

We will not talk much about anisotropic materials.

Example 4.1 If the whole space is filled with dielectric which has permittivity ε, and there is a point charge q_0 at point O, what is the electric field? Consider a spherical surface S with center at O.

$$q_0 = \oint_{\mathrm{S}} \mathbf{D} \cdot \mathrm{d}\mathbf{S} = 4\pi r^2 D \ , \tag{4.17}$$

and hence,

$$E = \frac{D}{\varepsilon} = \frac{q_0}{4\pi\varepsilon r^2} \ . \tag{4.18}$$

In comparison, in free space, $E_0 = \frac{q_0}{4\pi\varepsilon_0 r^2}$ and we have $E = \frac{E_0}{1+\chi_e}$. E is reduced because the charge q_0 is surrounded by some bound charges. The bound charges will screen off the free charge and produce a shielding effect to q_0.

Example 4.2 A metal ball with a charge Q and radius a is surrounded by a shell of linear dielectric material of ε with outer radius b. What is the potential at the center?

By Gauss's law, for $r > a$, $\mathbf{D} = (Q\hat{r})/(4\pi r^2)$. Because $\mathbf{D} = \varepsilon \mathbf{E}$,

$$E = \begin{cases} 0 & r < a \\ Q/4\pi\varepsilon r^2 & a < r < b \\ Q/4\pi\varepsilon_0 r^2 & r > b \end{cases} \ . \tag{4.19}$$

Hence,

$$\begin{aligned} V &= -\int_\infty^0 \mathbf{E} \cdot \mathrm{d}\mathbf{l} \\ &= -\int_\infty^b \frac{Q}{4\pi\varepsilon_0 r^2} \,\mathrm{d}r - \int_b^a \frac{Q}{4\pi\varepsilon r^2} \,\mathrm{d}r - \int_a^0 0 \,\mathrm{d}r \\ &= \frac{Q}{4\pi} \left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right) \ . \end{aligned} \tag{4.20}$$

If $a = b$ or $\varepsilon = \varepsilon_0$, $V = Q/4\pi\varepsilon_0 a$ because both conditions mean that there is no dielectric shell.

4.5 Energy in Dielectric Systems

The energy of an electrostatic system of point charges in vacuum is

$$W = \frac{\varepsilon_0}{2}\int E^2\,\mathrm{d}^3\mathbf{r}\ . \tag{4.21}$$

What is the energy if the space is filled with dielectric? If we increase the amount of free charge by $\delta\rho_f$, the change of the energy will be

$$\delta W = \int (\delta\rho_f)V\,\mathrm{d}^3\mathbf{r}\ . \tag{4.22}$$

By Gauss's law, $\nabla\cdot\mathbf{D} = \rho_f$ and $\delta\rho_f = \nabla\cdot(\delta\mathbf{D})$. The change in energy becomes

$$\begin{aligned}
\delta W &= \int [\nabla\cdot(\delta\mathbf{D})]V\mathrm{d}^3\mathbf{r} \\
&= \int \nabla\cdot(\delta\mathbf{D}V)\,\mathrm{d}^3\mathbf{r} - \int \delta\mathbf{D}\cdot\nabla V\,\mathrm{d}^3\mathbf{r} \\
&= \int \nabla\cdot(\delta\mathbf{D}V)\,\mathrm{d}^3\mathbf{r} + \int \delta\mathbf{D}\cdot\mathbf{E}\,\mathrm{d}^3\mathbf{r}\ .
\end{aligned} \tag{4.23}$$

If we integrate over the whole space, the first term will tend to 0. We also have

$$(\delta\mathbf{D})\cdot\mathbf{E} = \varepsilon(\delta\mathbf{E})\cdot\mathbf{E} = \frac{1}{2}\delta(\varepsilon E^2) = \frac{1}{2}\delta(\mathbf{D}\cdot\mathbf{E})\ . \tag{4.24}$$

Put everything together, we have

$$\delta W = \delta(\frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,\mathrm{d}^3\mathbf{r}) \tag{4.25}$$

And we interpret that $W_e \equiv \frac{1}{2}\mathbf{D}\cdot\mathbf{E}$ is the energy density.

Example 4.3 Find the energy of an uniformly charged dielectric ball with a radius R and total charge q. The outside of the ball is vacuum. The electric field is easily found to be

$$E = \left\{\begin{array}{ll} qr/4\pi\varepsilon R^3 & r < R \\ q/4\pi\varepsilon_0 r^2 & r > R \end{array}\right. . \tag{4.26}$$

The energy is then

$$\begin{aligned}
W &= \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,\mathrm{d}^3\mathbf{r} \\
&= \frac{\varepsilon}{2}\int_0^R (\frac{qr}{4\pi\varepsilon R^3})^2 4\pi r^2\mathrm{d}r + \frac{\varepsilon_0}{2}\int_R^\infty (\frac{q}{4\pi\varepsilon_0 r^2})^2 4\pi r^2\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^2}{8\pi\varepsilon R^6}\int_0^R r^4 \mathrm{d}r + \frac{q^2}{8\pi\varepsilon_0}\int_R^\infty \frac{1}{r^2}\,\mathrm{d}r \\
&= \frac{q^2}{40\pi\varepsilon R} + \frac{q^2}{8\pi\varepsilon_0 R}\ . \qquad (4.27)
\end{aligned}
$$

In comparison, if the sphere is a conductor, the electric field is

$$
E = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 r^2} & r > R \\ 0 & r < R \end{cases}\ , \qquad (4.28)
$$

and the energy is

$$
\begin{aligned}
W &= \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,\mathrm{d}^3\mathbf{r} \\
&= \frac{1}{2\varepsilon_0}\int_R^\infty (\frac{q}{4\pi r^2})^2 4\pi r^2 \mathrm{d}r \\
&= \frac{q^2}{8\pi\varepsilon_0 R}\ . \qquad (4.29)
\end{aligned}
$$

The energy for a conductor is lower than that of a dielectric.

Chapter 5

Magnetostatics

A stationary charge will produce electric field, while a moving charge will produce magnetic field together with the electric field. In this chapter, we will consider cases where the net charge is zero, hence the electric field is zero, and the current is steady, which means the magnetic field produced does not change with time.

5.1 Biot-Savart Law

5.1.1 Continuity Equation

Current is the motion of charges. In general, both positive and negative charges will contribute, although usually, only the negative charges, the electrons, move in conductors. The current in a wire is the total charge passing through a cross section per unit time.

The **current density** is defined as the total charge passing through a surface per unit time per unit area. For a small area with normal direction $\Delta\mathbf{S}$, the current density $\mathbf{J}$ tells us that the amount of charges passing through the area per unit time is $\mathbf{J} \cdot \Delta\mathbf{S}$. If the current is given by the velocity $\mathbf{v}$ of some charge density ρ, then $\mathbf{J} = \rho\mathbf{v}$.

By the definition of current density, the total charge leaving a volume V per unit time is given by

$$\int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \int_V \nabla \cdot \mathbf{J}\, \mathrm{d}^3\mathbf{r} \tag{5.1}$$

where S is the boundary surface of V. Another expression of total charge leaving the volume per unit time is of course

$$-\frac{\partial}{\partial t}\int_V \rho\, \mathrm{d}^3\mathbf{r} = \int_V -\frac{\partial \rho}{\partial t}\, \mathrm{d}^3\mathbf{r}\ . \tag{5.2}$$

The two must be equal, and we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0\ . \tag{5.3}$$

This is called the **continuity equation**. There are similar equations for the surface current density or line current density.

5.1.2 Biot-Savart Law

We found by experiments that there is an attractive force between two parallel wires with current in the same direction. Since the wire is neutral, it is not the effect of the electric force. The force is proportional to the product of the current and inversely proportional to the distance between them. The force per unit length is

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d} , \tag{5.4}$$

where μ_0 is a constant called **permeability of free space**

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A}^2 . \tag{5.5}$$

This is an exact number because we use this formula to define the unit of current, and then the unit of charge.

We try to generalize the formula in Eq. (5.4). If the first current is formed by some current density J_1, then the current itself is $I_1 = J_1 \Delta S_1$ for some cross sectional area ΔS_1. Thus, the force on a current segment of length Δl_1 is given by

$$F = (J_1 \Delta^3 r_1)\,\frac{\mu_0 I_2}{2\pi d} \tag{5.6}$$

where $\Delta^3 r_1 \equiv \Delta S_1 \Delta l_1$ is the small volume around the current segment.

If the second current is also formed by current density J_2, we have to find a formula such that after integrating over infinitely long straight line, it gives $\mu_0 J_2 \Delta S_2/(2\pi d)$. The correct one is given by

$$\frac{\mu_0}{4\pi}\,\frac{J_2 \Delta^3 r_2 \sin\theta}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \tag{5.7}$$

where $\Delta^3 r_2 \equiv \Delta S_2 \Delta l_2$ is the small volume around the second current segment, and θ is the angle between the current density and the vector $\mathbf{r}_1 - \mathbf{r}_2$. Hence, if we put back the direction in terms of vectors, a current segment $\mathbf{J}_1 \mathrm{d}^3\mathbf{r}_1$ will experiment a force from another current segment $\mathbf{J}_2 \mathrm{d}^3\mathbf{r}_2$ nearby

$$\mathrm{d}\mathbf{F}_{21} = \mathbf{J_1}\,\mathrm{d}^3\mathbf{r}_1 \times \left(\frac{\mu_0}{4\pi}\frac{\mathbf{J_2}\,\mathrm{d}^3\mathbf{r}_2 \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}\right) . \tag{5.8}$$

The above argument is not a proof. But we could verify the above equation by experiments, for example, by measuring the force on wires with different orientation and length. The force on a current segment by a wire is thus

$$\mathrm{d}\mathbf{F}_{21} = \mathbf{J_1}\mathrm{d}^3\mathbf{r}_1 \times \left(\frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_2\,\mathrm{d}^3\mathbf{r}_2 \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}\right) \equiv \mathbf{J_1}\mathrm{d}^3\mathbf{r}_1 \times \mathbf{B} \tag{5.9}$$

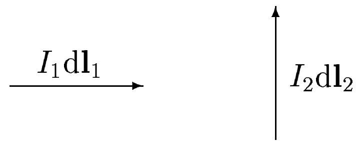

Figure 5.1: Two current segments that are perpendicular to each other.

where $\mathbf{B}$ is called the magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \, \mathrm{d}^3\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{5.10}$$

This is called the **Biot-Savart law**. If the current is on a wire, then the magnetic field is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint \frac{I \, \mathrm{d}\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{5.11}$$

where $\mathrm{d}\mathbf{l}$ is an element of the wire with the direction of the current.

The unit of magnetic field is Tesla, T = NA^{-1}m^{-1}. Another common unit is gauss (CGS unit), where 1T = 10^4 gauss.

Unlike the electric field $\mathbf{E}$, the magnetic field of a moving point charge cannot be easily written down, in particular,

$$\mathbf{B} \neq \frac{\mu_0 \, q}{4\pi} \frac{\mathbf{v} \times \hat{r}}{r^2} \, . \tag{5.12}$$

It is because a moving point charge cannot provide steady current.

Also notice that the interaction between the two current segments may differ from the Newton's third law. However, if we integrate along a closed path, the interaction will obey the Newton's third law. For example, consider the two current segments as shown in Fig. 5.1. We have

$$\begin{aligned} \mathbf{r}_1 - \mathbf{r}_2 &\propto -\hat{\imath} \\ \mathrm{d}\mathbf{l}_1 &\propto \hat{\imath} \\ \mathrm{d}\mathbf{l}_2 &\propto \hat{\jmath} \, . \end{aligned} \tag{5.13}$$

Hence, $\mathbf{F}_{21} \propto \hat{\imath} \times (\hat{\jmath} \times (-\hat{\imath})) = -\hat{\jmath} \neq 0$ and $\mathbf{F}_{12} \propto \hat{\jmath} \times (\hat{\imath} \times \hat{\imath}) = 0$.

5.2 The Lorentz Force Law

Eq. (5.9) gives us the magnetic force on current density as

$$\mathbf{F} = \int \mathbf{J} \times \mathbf{B} \, \mathrm{d}^3\mathbf{r} \, . \tag{5.14}$$

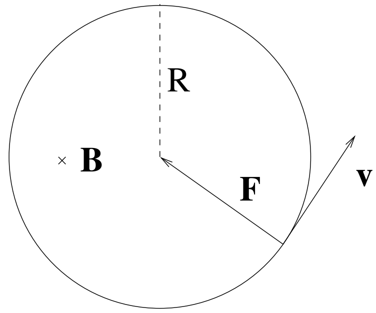

Figure 5.2: The charged particle will be in circular motion in an uniform magnetic field. The magnetic field is point into the paper, as indicated by the cross.

If the current density is given by the motion of a point charge, then $\mathbf{J} = Q\delta^3(\mathbf{r}_1 - \mathbf{r})\mathbf{v}$, where $\mathbf{r}$ is the position of the charge and $\mathbf{v}$ is its velocity. The magnetic force is

$$\mathbf{F} = Q(\mathbf{v} \times \mathbf{B}) \ . \tag{5.15}$$

Because $\mathbf{v} \times \mathbf{B} \cdot \mathbf{v} = 0$, the magnetic force does not do work. The total electromagnetic force is

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ . \tag{5.16}$$

This is the **Lorentz force law**.

Example 5.1 We will investigate the motion of a charged particle in uniform magnetic field **B**.

i. If the initial velocity **v** is perpendicular to **B**, since **F** is also perpendicular to **B**, it implies that the particle is kept in the plane perpendicular to **B**. Thus, if the magnetic field is pointing into the paper and the charge is positive, then we have Fig. 5.2. (If the charge is negative, the particle goes in the opposite direction.) The magnitude of the force is qvB, hence

$$\begin{aligned} qvB &= \frac{mv^2}{R} \\ R &= \frac{mv}{qB} \ . \end{aligned} \tag{5.17}$$

The period is

$$T = \frac{2\pi R}{v} = \frac{2\pi m}{qB} \ . \tag{5.18}$$

Note that the frequency $f = 1/T$ is independent of v and R.

Suppose that the charged particle is moving inside a short conducting cylindrical container. With the magnetic field in place, the particle will circle around forever. Now we cut the container into two D-shape halves, and separate them by a small gap, and apply an oscillating potential on the two

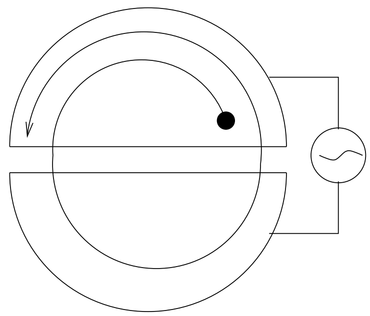

Figure 5.3: Schematic diagram of a cyclotron.

halves, Fig. 5.3. When the particle is in one half, because the electric field is zero inside the half, the particle does not feel the changing of potential. However, when it passes through the gap, it will be accelerated by the difference of potential between the two halves. If we apply an A.C. voltage with frequency of $f = 1/T = qB/2\pi m$, the particle will be accelerated every time it passes through the gap. For the radius R, the maximum speed obtained can be

$$v_{\max} = \frac{qBR}{m} \ . \tag{5.19}$$

For example, if B is about 10^4gauss, R is about 1m, the energy obtained is about 10MeV for proton. This is the basic principle of **cyclotron**, a very useful tool to accelerate charged particle to high energy.

ii. If $\mathbf{v}$ is not perpendicular to $\mathbf{B}$, we decompose $\mathbf{v}$ into components $\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp}$ parallel and perpendicular to $\mathbf{B}$. Since the force is always perpendicular to $\mathbf{B}$, $\mathbf{v}_{\perp}$ does not change at all. The particle will move along a helical path with the same frequency $qB/2\pi m$.

5.3 Calculation of Magnetic Field

In this section, we will calculate the magnetic field of several common situations.

5.3.1 The Straight Wire

For a straight wire carrying a steady current I, refer to Fig. 5.4. Now, $\mathrm{d}\mathbf{l}\times\mathbf{r} = r\sin\phi\,\mathrm{d}l$ and the direction is pointing out of the paper. By Eq. (5.11),

$$\begin{aligned} B(z) &= \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\frac{Ir\sin\phi\,\mathrm{d}l}{r^3} \\ &= \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{\cos\theta}{r^2}\,\mathrm{d}l \end{aligned}$$

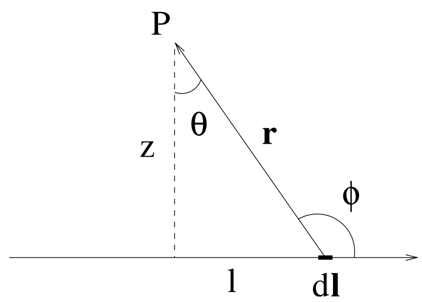

Figure 5.4: The geometry of calculation of the magnetic field at point P of a straight wire.

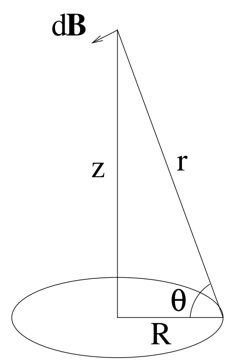

Figure 5.5: The geometry of calculation of the magnetic field at point P of a circular loop.

$$
\begin{aligned}
&= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\cos\theta}{z^2/\cos^2\theta}\,\mathrm{d}(z\tan\theta) \\
&= \frac{\mu_0 I}{4\pi z} \int_{-\pi/2}^{\pi/2} \cos^3\theta\,\frac{1}{\cos^2\theta}\,\mathrm{d}\theta \\
&= \frac{\mu_0 I}{2\pi z}\ . \qquad (5.20)
\end{aligned}
$$

This is exactly what we need in the paragraph above Eq. (5.7).

5.3.2 Circular Loop

We calculate the field at a distance z above the center of a circular loop of radius R carrying a steady current I. By symmetry, only the component $B_z \neq 0$. Since $\mathrm{d}\mathbf{l} \perp \mathbf{r}$, the contribution of the current segment to the total magnetic field is

$$
\mathrm{d}B = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}l}{r^2}\ . \qquad (5.21)
$$

The z component of the magnetic field is $\mathrm{d}B_z = \cos\theta\,\mathrm{d}B$, hence

$$
B_z \;=\; \oint \mathrm{d}B\cos\theta
$$

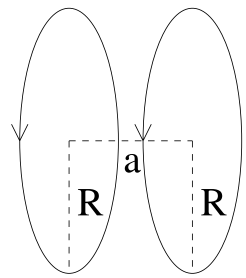

Figure 5.6: The Helmholtz Coil.

$$
\begin{aligned}
&= \frac{\mu_0}{4\pi}\frac{I}{z^2}\sin^2\theta\cos\theta\oint \mathrm{d}l \\
&= \frac{\mu_0}{4\pi}\frac{I}{z^2}\frac{z^2}{R^2+z^2}\frac{R}{\sqrt{R^2+z^2}}2\pi R \\
&= \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\ .
\end{aligned}
\tag{5.22}
$$

At the center, $z=0$, $B_z = \mu_0 I/(2R)$. If $z \gg R$, $B_z = \mu_o I R^2/(2z^3)$.

5.3.3 Helmholtz Coil

Helmholtz coil is a pair of coils parallel to each other, separated by by distance a, Fig. 5.6. The field of individual coil has be obtained in Eq. (5.22). The field near the center of the coil is given by

$$
B(x) = \frac{\mu_0 I}{2}\left(\frac{R^2}{(R^2+(x+a/2)^2)^{3/2}} + \frac{R^2}{(R^2+(x-a/2)^2)^{3/2}}\right)\ . \tag{5.23}
$$

We can expand $B(x)$ into Taylor series at x=0

$$
B(x) = B(0) + x(\frac{\partial B}{\partial x})_{x=0} + \frac{x^2}{2!}(\frac{\partial^2 B}{\partial x^2}) + \frac{x^3}{3!}(\frac{\partial^3 B}{\partial x^3})_{x=0} + \frac{x^4}{4!}(\frac{\partial^4 B}{\partial x^4})_{x=0} + \cdots\ . \tag{5.24}
$$

As $B(x) = B(-x)$, the odd terms in the expansion should be zero, in particular,$B'(0) = B^{(3)}(0) = 0$.

If $B''(0)=0$, then $B(0)+O(x^4)$, where $O(x^4)$ means x^4 or higher power terms, and the magnetic field is quite uniform near the center. Now, $B''(0) = 0$ implies that $a = R$. (Check this!) Thus, to have uniform magnetic field, we can arrange two coils separated by a distance equal to their radius.

5.3.4 Helical Coil

For a helical coil, we assume that the number of turns per unit length is n, Fig. 5.7. The total current in a small section of the coil $\mathrm{d}l$ is $In\,\mathrm{d}l$. By

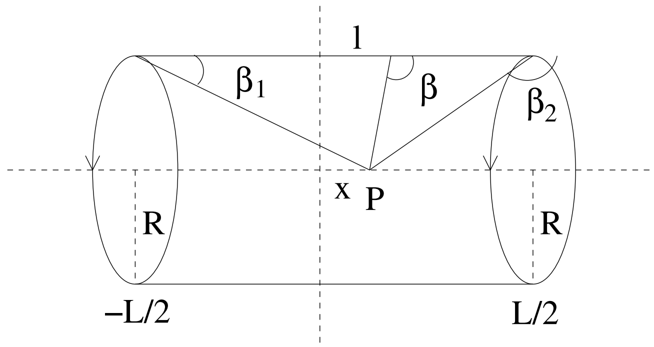

Figure 5.7: The cross sectional view of a helical coil.

Eq. (5.22), the magnetic field at the point P by that small section is

$$\mathrm{d}B = \frac{\mu_0}{2} \frac{R^2 In\, \mathrm{d}l}{(R^2 + (x-l)^2)^{3/2}} \; . \tag{5.25}$$

From the figure, we have $-\tan\beta = R/(l-x)$ or $l - x = -R\cot\beta$, hence $\mathrm{d}l = R\,\mathrm{d}\beta/\sin^2\beta$. For the whole helical coil,

$$\begin{aligned} B &= \frac{\mu_0}{2} \int_{-L/2}^{L/2} \frac{R^2 In\, \mathrm{d}l}{[R^2 + (x-l)^2]^{3/2}} \\ &= \frac{\mu_0 In}{2} \int_{\beta_1}^{\beta_2} \frac{R^3\, \mathrm{d}\beta}{\sin^2\beta [R^2 + R^2\cot^2\beta]^{3/2}} \\ &= \frac{\mu_0 In}{2} \int_{\beta_1}^{\beta_2} \sin\beta\, \mathrm{d}\beta \\ &= \frac{\mu_0 In}{2} (\cos\beta_1 - \cos\beta_2) \; . \end{aligned} \tag{5.26}$$

where

$$\cos\beta_1 = \frac{x + \frac{L}{2}}{\sqrt{R^2 + (x + \frac{L}{2})^2}} \tag{5.27}$$

$$\cos\beta_2 = \frac{x - \frac{L}{2}}{\sqrt{R^2 + (x - \frac{L}{2})^2}} \; . \tag{5.28}$$

By some more detailed calculation, we found that if $L \gg R$, the magnetic field is quite uniform in the coil. If we take the limit $L \to \infty$, $\beta_1 = 0$, $\beta_2 = \pi$ and

$$B = \mu_0 nI \; . \tag{5.29}$$

In fact, $B = \mu_0 nI$ not only along the axis, but also for the whole region inside the coil. For the field at the end of the coil, $\beta_1 = 0$ and $\beta_2 = \frac{\pi}{2}$, and we have $B = \frac{\mu_0 nI}{2}$. For any real coil, as long as $L \gg R$, the above formula is very accurate.

5.4 Ampere's Law

5.4.1 The Divergence

We have Eq. (5.10)

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}^3\mathbf{r}' \ . \tag{5.30}$$

Thus,

$$\begin{aligned}\nabla\cdot\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int \nabla\cdot(\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3})\,\mathrm{d}^3\mathbf{r}' \\ &= \frac{-\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\cdot\left(\nabla\times\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)\mathrm{d}^3\mathbf{r}' \\ &= 0 \ . \end{aligned}\tag{5.31}$$

We have the second equality because $\mathbf{J}$ does not depend on $\mathbf{r}$ and Eq. (1.119), and the third equality because $\nabla\times(\mathbf{r}/r^3) = 0$. (There is not even a delta function.)

Similar to the electric field lines, we can draw the magnetic field lines. Because of Eq. (5.31), the magnetic field lines are closed.

5.4.2 Ampere's Law

Using Biot-Savart law,

$$\nabla\times\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \nabla\times(\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3})\,\mathrm{d}^3\mathbf{r}' \ . \tag{5.32}$$

Now,

$$\begin{aligned}&\nabla\times(\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}) \\ =\ &\mathbf{J}(\nabla\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}) - (\mathbf{J}\cdot\nabla)\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \ .\end{aligned}\tag{5.33}$$

In the second term, because $(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|^3$ is an odd function of $\mathbf{r}-\mathbf{r}'$, we can replace $(-\nabla)$ by ∇'. Consider one component

$$(\mathbf{J}\cdot\nabla')(\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3}) = \nabla'\cdot(\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3}\mathbf{J}) - \frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3}\nabla'\cdot\mathbf{J} \ . \tag{5.34}$$

Since $\mathbf{J}$ is a steady current density, $\nabla'\cdot\mathbf{J} = 0$. Also,

$$\int_V \nabla'\cdot\left(\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3}\mathbf{J}(\mathbf{r}')\right)\mathrm{d}^3\mathbf{r}' = \oint_S \frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3}\mathbf{J}(\mathbf{r}')\cdot\mathrm{d}\mathbf{S}' = 0 \tag{5.35}$$

because the integration region is very large and includes all the current density, so there is no current on the boundary surface and the result of the integration is zero.

What is left is only the first term in Eq. (5.33). But we know that $\nabla \cdot ((\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|^3) = 4\pi\delta^3(\mathbf{r}-\mathbf{r}')$. Finally, we have

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\, 4\pi\delta^3(\mathbf{r}-\mathbf{r}')\, \mathrm{d}^3\mathbf{r}' = \mu_0\, \mathbf{J}(\mathbf{r}) \ . \tag{5.36}$$

This is the **Ampere's law** in differential form. If we integrate along a closed path,

$$\oint_{\text{path}} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0\, I \tag{5.37}$$

where I is total current enclosed by the path, and the directions of the enclosed current and the path obey the right-hand rule. Notice that the field B in Eq. (5.36) is the total field caused by all currents in the space, including those currents which are not enclosed in L.

Ampere's law in integral form can be used to find the magnetic field in some highly symmetrical systems.

Example 5.2 For the field of a long straight wire, consider a circular path of radius r with center at the wire, we have

$$\mu_0 I = \oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B \oint \mathrm{d}\mathbf{l} = 2\pi r B \tag{5.38}$$

or

$$B = \frac{\mu_0 I}{2\pi r} \ . \tag{5.39}$$

5.5 Magnetic Vector Potential

Since $\nabla \cdot \mathbf{B} = 0$, we can introduce a **vector potential A** as

$$\mathbf{B} = \nabla \times \mathbf{A} \ . \tag{5.40}$$

We can add to $\mathbf{A}$ the gradient of any scalar field $\nabla\lambda$ and still have the same magnetic field

$$\nabla \times (\mathbf{A} + \nabla\lambda) = \nabla \times \mathbf{A} + \nabla \times \nabla\lambda = \nabla \times \mathbf{A} \ . \tag{5.41}$$

We utilize this freedom to constraint the vector potential to also satisfy

$$\nabla \cdot \mathbf{A} = 0 \ . \tag{5.42}$$

(See the textbook for the proof.) Then, we have

$$\mu_0\mathbf{J} = \nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = -\nabla^2\mathbf{A} \ . \tag{5.43}$$

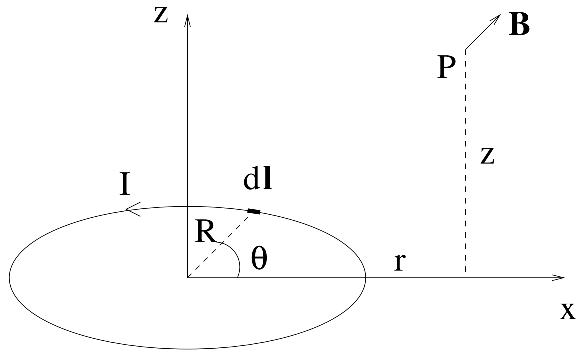

Figure 5.8: The magnetic field of a current loop.

This is just the Poisson's equation, one for each component. The solution is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}' \ . \tag{5.44}$$

Since $\mathbf{A}$ is still a vector, it is not as useful as V in the electrostatic case.

Example 5.3 We will find the magnetic field of a loop carrying current I. To get the field of an arbitrary point P, we choose the cylindrical coordinate such that the field point P is on the xz-plane, Fig. 5.8. By symmetry, only the θ component of the vector potential is non-zero, $\mathbf{A} = A_\theta\hat{\theta}$. (For point P, $\hat{\theta} = \hat{\jmath}$.) The component of the current segment that will contribute to the vector potential is $I\cos\theta\,\mathrm{d}l = IR\cos\theta\,\mathrm{d}\theta$. We have

$$A_\theta = \frac{\mu_0}{4\pi}\oint \frac{IR\cos\theta\,\mathrm{d}\theta}{(R^2+r^2+z^2-2Rr\cos\theta)^{1/2}} \ . \tag{5.45}$$

To evaluate this integral, let $\alpha = \frac{\theta-\pi}{2}$. Thus, $\frac{1}{2}\cos\theta\,\mathrm{d}\theta = (2\sin^2\alpha - 1)\mathrm{d}\alpha$. Also, let $k^2 = 4Rr/((R+r)^2+z^2)$. Then,

$$\begin{aligned}
A_\theta &= \frac{\mu_0}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{IR(2\sin^2\alpha-1)\,\mathrm{d}\alpha}{(R^2+r^2+z^2-4Rr\sin^2\alpha+2Rr)^{1/2}} \\
&= \frac{\mu_0 IR}{\pi}\frac{1}{[(R+r)^2+z^2]^{1/2}}\int_0^{\pi/2}\frac{(2\sin^2\alpha-1)\mathrm{d}\alpha}{(1-k^2\sin^2\alpha)^{1/2}} \\
&= \frac{\mu_0 IR}{\pi[(R+r)+z^2]^{1/2}}\int_0^{\pi/2}\frac{2\sin^2\alpha-\frac{2}{k^2}+(\frac{2}{k^2}-1)}{(1-k^2\sin^2\alpha)^{1/2}}\mathrm{d}\alpha \\
&= \frac{\mu_0 Ik}{2\pi}(\frac{R}{r})^{1/2}[-\frac{2}{k^2}\int_0^{\pi/2}(1-k^2\sin^2\alpha)^{1/2}\mathrm{d}\alpha \\
&\quad +(\frac{2}{k^2}-1)\int_0^{\pi/2}\frac{\mathrm{d}\alpha}{(1-k^2\sin^2\alpha)^{1/2}}] \\
&= \frac{\mu_0 Ik}{2\pi}(\frac{R}{r})^{1/2}[-\frac{2}{k^2}E+(\frac{2}{k^2}-1)K]
\end{aligned} \tag{5.46}$$

where

$$K \equiv \int_0^{\pi/2} \frac{\mathrm{d}\alpha}{(1-k^2\sin^2\alpha)^{1/2}} \tag{5.47}$$

$$E \equiv \int_0^{\pi/2} (1-k^2\sin^2\alpha)^{1/2}\mathrm{d}\alpha \tag{5.48}$$

are called the first type and second type of elliptic integral respectively. For small k, we have the expansions

$$K = \frac{\pi}{2}[1+(\frac{1}{2})^2k^2+(\frac{1\times 3}{2\times 4})^2k^4+(\frac{1\times 3\times 5}{2\times 4\times 6})^2k^6+\cdots] \tag{5.49}$$

$$E = \frac{\pi}{2}[1-(\frac{1}{2})^2k^2-(\frac{1\times 3}{2\times 4})^2\frac{k^4}{3}-(\frac{1\times 3\times 5}{2\times 4\times 6})^2\frac{k^6}{5}+\cdots] \ . \tag{5.50}$$

Thus, A_θ can be written as

$$A_\theta = \frac{\mu_0 I k^3}{32}(\frac{R}{r})^{1/2}(1+\frac{3}{4}k^2+\frac{75}{128}k^4+\cdots) \ . \tag{5.51}$$

This is the general solution for a loop. Using $\mathbf{B} = \nabla\times\mathbf{A}$, we can find out the magnetic field, but the result is not very illuminating. Instead, we just consider the case on the axis. For small r, $k = (2R^{1/2}/(R^2+z^2)^{1/2})r^{1/2}+O(r)$ and

$$A_\theta = \frac{\mu_0 I}{4}\frac{R^2}{(R^2+z^2)^{3/2}}\,r+O(r^2) \ . \tag{5.52}$$

The only non-zero component of the magnetic field is

$$B_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)|_{r=0} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}} \ . \tag{5.53}$$

This result exactly equals what we have found in Eq. (5.22).

5.6 Multipole Expansion

Similar to Section 3.6, we can obtain the multipole expansion of a localized current distribution, Fig. 5.9. By Eq. (3.29) or Eq. (3.75),

$$\frac{1}{s} = \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^n P_n(\cos\theta) \ . \tag{5.54}$$

Hence, we have

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0 I}{4\pi}\oint\frac{1}{s}\,\mathrm{d}\mathbf{l} \\ &= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint(r')^n P_n(\cos\theta)\,\mathrm{d}\mathbf{l} \\ &= \frac{\mu_0 I}{4\pi}[\frac{1}{r}\oint\mathrm{d}\mathbf{l}+\frac{1}{r^2}\oint r'\cos\theta\,\mathrm{d}\mathbf{l} \\ &\quad +\frac{1}{r^3}\oint(r')^2(\frac{3}{2}\cos^2\theta-\frac{1}{2})\,\mathrm{d}\mathbf{l}+\cdots]\end{aligned} \tag{5.55}$$

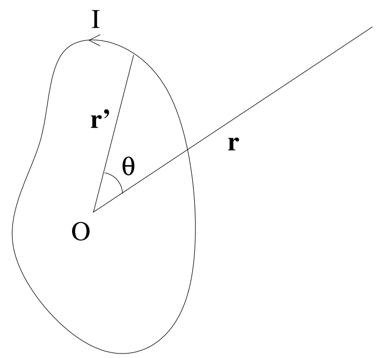

Figure 5.9: The configuration for the expansion for the vector potential.

The first term is called the monopole term. It is always zero because we integrate over closed loop. The second term is the dipole term. The third term is quadrupole and so on.

The **magnetic dipole moment** is defined as

$$\mathbf{m} \equiv I \int \mathrm{d}\mathbf{S} \; . \tag{5.56}$$

For a plane loop, $\int \mathrm{d}\mathbf{S}$ is just the area of the loop with direction given by the right-hand rule. Then, the magnetic dipole moment is just the product of current with area. Another form of this **vector area** is

$$\int \mathrm{d}\mathbf{S} = \frac{1}{2} \oint \mathbf{r}' \times \mathrm{d}\mathbf{l} \; . \tag{5.57}$$

To prove this, consider the Stokes's theorem with the vector field $\mathbf{c} \times \mathbf{r}'$ where $\mathbf{c}$ is an arbitrary constant vector. Then,

$$\begin{aligned} \nabla' \times (\mathbf{c} \times \mathbf{r}') &= -(\mathbf{c} \cdot \nabla')\mathbf{r}' + \mathbf{c}\nabla' \cdot \mathbf{r}' \\ &= -\mathbf{c} + \mathbf{c}3 \\ &= 2\mathbf{c} \; , \end{aligned} \tag{5.58}$$

and

$$\begin{aligned} \int \nabla' \times (\mathbf{c} \times \mathbf{r}') \cdot \mathrm{d}\mathbf{S} &= \oint \mathbf{c} \times \mathbf{r}' \cdot \mathrm{d}\mathbf{l} \\ \mathbf{c} \cdot \int 2\,\mathrm{d}\mathbf{S} &= \mathbf{c} \cdot \oint \mathbf{r}' \times \mathrm{d}\mathbf{l} \; . \end{aligned} \tag{5.59}$$

Since the vector $\mathbf{c}$ is arbitrary, we have Eq. (5.57).

Now, we provide another formula for the dipole term of the vector potential which is similar to corresponding formula of electrostatic, Eq. (3.72). Let $\hat{r} = \mathbf{r}/|\mathbf{r}|$ be the unit vector along the $\mathbf{r}$ direction. This does not depend on $\mathbf{r}'$. Consider again the Stokes' theorem with vector field $(\mathbf{r}' \cdot \hat{r})\mathbf{c}$. We have

$$\begin{aligned}
\int \nabla' \times (\mathbf{r}' \cdot \hat{r}\mathbf{c}) \cdot \mathrm{d}\mathbf{S} &= \oint \mathbf{r}' \cdot \hat{r}\mathbf{c} \cdot \mathrm{d}\mathbf{l} \\
\int (\nabla'(\mathbf{r}' \cdot \hat{r}) \times \mathbf{c}) \cdot \mathrm{d}\mathbf{S} &= \mathbf{c} \cdot \oint (\mathbf{r}' \cdot \hat{r})\, \mathrm{d}\mathbf{l} \\
\int (\hat{r} \times \mathbf{c}) \cdot \mathrm{d}\mathbf{S} &= \mathbf{c} \cdot \oint (r' \cos\theta)\, \mathrm{d}\mathbf{l} \\
-\mathbf{c} \cdot \int \hat{r} \times \mathrm{d}\mathbf{S} &= \mathbf{c} \cdot \oint (r' \cos\theta)\, \mathrm{d}\mathbf{l} \ .
\end{aligned} \tag{5.60}$$

Hence, the dipole term can be written as

$$\begin{aligned}
\mathbf{A} &= \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta\, \mathrm{d}\mathbf{l} \\
&= \frac{\mu_0 I}{4\pi r^2} \int \mathrm{d}\mathbf{S} \times \hat{r} \\
&= \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{r}}{r^2} \ .
\end{aligned} \tag{5.61}$$

Chapter 6

Magnetic Fields in Matter

There are tiny current loops in all matters. They are mainly the spinning of electrons and the orbiting of the electrons around the nuclei. When we apply external magnetic field to the matter, the tiny current loops will align with each other. We say that the material is **magnetized**. It could increase or decrease the field strength, depending on the material. Some will even "memorize" the direction of the magnetic field after the field is removed.

6.1 Magnetization

6.1.1 Effects on Magnetic Dipoles

Consider a rectangular current loop with sides a and b, as shown in Fig. 6.1. The magnitude of the force on each segment of the loop is $F = IbB$. Hence, the torque on the loop is given by

$$\mathbf{N} = Fa \sin\theta \hat{\imath} = Iab\, B\, \sin\theta \hat{\imath} = \mathbf{m} \times \mathbf{B} \tag{6.1}$$

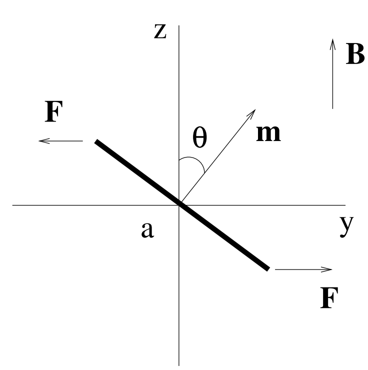

Figure 6.1: A rectangular current loop in uniform magnetic field in the z axis, viewed from the positive x axis. The current I goes into the page on the right hand segment.

where $\mathbf{m}$ is the magnetic moment of the loop. Notice that the torque will try to align the magnetic moment to the magnetic field, hence increasing it. This is **paramagnetism**. Atoms or molecules with an odd number of electrons will have this property.

If the magnetic field in not uniform, there will be net force on the magnetic moment, given by

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \ . \tag{6.2}$$

(See, for example, Problem 6.22 of the textbook for the proof.)

6.1.2 Effects on Atomic Orbits

In the last section, we have assume that the magnetic moment itself does not change by the magnetic field. This is true for the magnetic moment of electrons, for example. But the magnetic moment of the orbiting electron around a nucleus does change under the influence of a magnetic field.

Consider the simplified case where the radius of the orbit of the electron is R and the speed is v. Then, the current is $I = -e/(2\pi R/v) = -ev/2\pi R$. The orbital dipole moment is

$$m = I\pi R^2 = -\frac{1}{2}\, evR \ . \tag{6.3}$$

The speed is given by the balance the centripetal force and the electrical force

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} = m_e\frac{v^2}{R} \ . \tag{6.4}$$

If there is a magnetic field, perpendicular to the orbital plane for example, the force by the magnetic field will contribute to the centripetal force,

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} + ev'B = m_e\frac{v'^2}{R} \ . \tag{6.5}$$

(See Problem 7.49 in p.337 of the textbook for why we could assume that R does not change.) We have

$$ev'B = m_e\frac{v'^2}{R} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} = \frac{m_e}{R}(v'^2 - v^2) = \frac{m_e}{R}(v' - v)(v' + v) \ . \tag{6.6}$$

For small changes, $v \approx v'$,

$$\Delta v \equiv v' - v = \frac{eRB}{2m_e} \ . \tag{6.7}$$

Hence, the change in magnetic moment is

$$\Delta m = -\frac{1}{2}\, e\Delta vR = -\frac{e^2R^2}{4m_e}B \ . \tag{6.8}$$

The change is opposite to the magnetic field. This is **diamagnetism**. This effect is weaker than the paramagnetism. The above derivation is, of course, not entirely correct. But it shows the main idea that orbital magnetic moment will decrease the magnetic field.

6.1.3 Magnetization

The **magnetization** is defined as the magnetic dipole moment per unit volume. It is usually denoted as $\mathbf{M}$.

6.2 The Magnetic Field in Matters

For a single dipole $\mathbf{m}$, the vector potential is

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{r}}{r^2}\ . \tag{6.9}$$

For magnetized object, of course it is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}^3\mathbf{r}'\ . \tag{6.10}$$

Since $\nabla\frac{1}{r} = -\frac{\hat{r}}{r^2}$,

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int(\mathbf{M}\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|})\,\mathrm{d}^3\mathbf{r}' \\ &= \frac{\mu_0}{4\pi}\left[\int\frac{\nabla'\times\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}' - \int\nabla'\times(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|})\,\mathrm{d}^3\mathbf{r}'\right]\ .\end{aligned} \tag{6.11}$$

To simplify the expression, put the vector field $\mathbf{u}\times\mathbf{c}$ where $\mathbf{c}$ is a constant vector in the divergence theorem, we have $\nabla\cdot(\mathbf{u}\times\mathbf{c}) = \mathbf{c}\cdot(\nabla\times\mathbf{u})$ and

$$\begin{aligned}\int\nabla\cdot(\mathbf{u}\times\mathbf{c})\,\mathrm{d}^3\mathbf{r} &= \int(\mathbf{u}\times\mathbf{c})\cdot\mathrm{d}\mathbf{S} \\ \mathbf{c}\cdot\int\nabla\times\mathbf{u}\,\mathrm{d}^3\mathbf{r} &= -\mathbf{c}\cdot\int\mathbf{u}\times\mathrm{d}\mathbf{S} \\ \int\nabla\times\mathbf{u}\,\mathrm{d}^3\mathbf{r} &= -\int\mathbf{u}\times\mathrm{d}\mathbf{S}\end{aligned} \tag{6.12}$$

because $\mathbf{c}$ is arbitrary. Then, Eq. (6.11) can be written as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int\frac{\nabla'\times\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}' + \int\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\times\mathrm{d}\mathbf{S}\right]\ . \tag{6.13}$$

$\mathbf{J}_b \equiv \nabla\times\mathbf{M}$ is the **volume bound current density** and $\mathbf{K}_b \equiv \mathbf{M}\times\hat{n}$ is the **surface bound current density**. Therefore,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int\frac{\mathbf{J}_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}' + \int\frac{\mathbf{K}_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}S\right]\ . \tag{6.14}$$

6.3 Auxiliary Field H

Current can either be bound or free

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \ . \tag{6.15}$$

We can control the free current, while the bound current is induced by the magnetic field. Since the bound current is given by $\mathbf{J}_b = \nabla \times \mathbf{M}$, Ampere's law then can be written as

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J}_b + \mathbf{J}_f = (\nabla \times \mathbf{M}) + \mathbf{J}_f \tag{6.16}$$

or if we let $\mathbf{H} = \mathbf{B}/\mu_\mathbf{0} - \mathbf{M}$, we have

$$\nabla \times \mathbf{H} = \mathbf{J}_f \tag{6.17}$$

$$\oint \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I_f \ . \tag{6.18}$$

In some books, the field $\mathbf{H}$ is called the magnetic field, but not in this lecture note. We will just call it auxiliary field.

Example 6.1 What is $\mathbf{H}$ near a long straight wire with circular cross section of radius R and with current I uniformly distributed in it? The solution is easy with the application of the integral form of the Ampere's law Eq. (6.18). For distance $s < R$ from the axis,

$$2\pi s H = I_f = I\,\frac{\pi s^2}{\pi R^2} \tag{6.19}$$

or $H = Is/(2\pi R^2)$. The direction of $\mathbf{H}$ is given by the right hand rule. For $s > R$, we have

$$H = \frac{I}{2\pi s} \ . \tag{6.20}$$

Since $M = 0$ for $s > R$, $B = \mu_0 H = \mu_0 I/(2\pi s)$.

Similar to the electric displacement, $\nabla \cdot \mathbf{H} \neq 0$. Thus, be careful.

6.4 Linear Media

In diamagnetic and paramagnetic materials, the magnetization is proportional to the magnetic field, provided that the field is not too strong. We have

$$\mathbf{M} = \chi_m \mathbf{H} \tag{6.21}$$

where the constant χ_m is called the **magnetic susceptibility**. $\mathbf{B}$ is also proportional to $\mathbf{H}$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H} \tag{6.22}$$

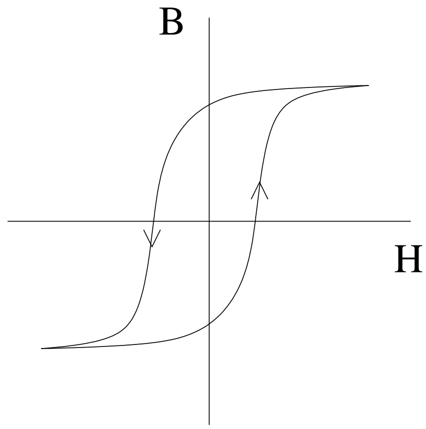

Figure 6.2: The magnetization of ferromagnetic material depends on history.

where $\mu \equiv \mu_0(1+\chi_m)$ is called the **permeability** of the material. In these homogeneous linear material, the bound current is also proportional to the free current

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times (\chi_m \mathbf{H}) = \chi_m \mathbf{J}_f \ . \tag{6.23}$$

6.5 Ferromagnetism

Unlike the linear media, ferromagnetic material can retain the magnetization after the external magnetic field is removed. The reason behind this is that the magnetic dipoles in it like to point to the same direction, that is, their potential energy is much lower if they point to the same direction as their neighbor. This occurs only in material with unpaired electrons in the atoms.

A region in the material in which the magnetic moment of all atoms point in the same direction is called a **domain**. A macroscopic sample of ferromagnetic material may not have non-zero magnetic field because the magnetic moment of different domain may point in different direction.

If we apply an external magnetic field, domains with moment near the direction of the external field will grow in size. If the field is stronger, the size of the domains will be larger, up to a point where almost all magnetic moments point in the direction of the external field. The magnetization is said to be saturated.

Saturated or not, if we reduce the external magnetic field to zero, ferromagnetic material can retain the magnetization. To remove the magnetization, we have to apply an external field in the opposite direction. Thus, the magnetization depends on the history, and we have the **hysteresis loop**, Fig. 6.2. Note that very high temperature can completely remove the magnetization.

Chapter 7

Electrodynamics

7.1 Ohm's Law

We know that electric currents are made up of moving charges. It is found by experiments that for most materials, the current is proportional to the force on the charges. The charges are not accelerated indefinitely because scatterings with the nucleus will stop the motions of the charges. In fact, when we just apply the electric field on a conductor, say, the charges are accelerated and then in a short time, reach equilibrium. For practical purpose, this occurs instantaneously. We have

$$\mathbf{J} = \sigma \mathbf{f} \tag{7.1}$$

where $\mathbf{f}$ is the force per unit charge and σ is the **conductivity** of the material. The reciprocal of conductivity is the **resistivity**

$$\rho = \frac{1}{\sigma} \ . \tag{7.2}$$

There can be various kinds of forces to drive the charges. Here we only consider electromagnetic force

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ . \tag{7.3}$$

Usually, the speed of the charges is small, we can neglect the effect of magnetic force

$$\mathbf{J} \approx \sigma \mathbf{E} = \frac{1}{\rho} \mathbf{E} \ . \tag{7.4}$$

This is called the **Ohm's law**.

For an uniform conducting wire of length l and cross sectional area A, the potential across the two ends is $V = El$, while the current is given by $I = AJ$. Hence,

$$V = El = \rho l J = \frac{\rho l}{A} I = RI \tag{7.5}$$

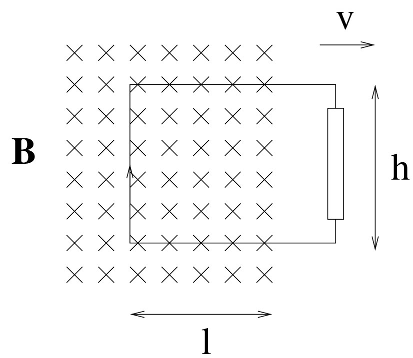

Figure 7.1: A closed loop of circuit moving to the right in a magnetic field **B**.

where $R \equiv= \rho l/A$ is the resistance of the wire. This is the familiar form of Ohm's law.

Notice that Ohm's law is not an universal law. There are many materials with non-linear response to the applied electric or magnetic field, and Eq. (7.3) is valid for those materials.

The power dissipated on the wire is given by

$$VI = I^2R = \frac{V^2}{R} \ . \tag{7.6}$$

This is called the **Joule heating law**.

7.1.1 The Electromotive Force (Emf)

The force in Eq. (7.1), $\mathbf{J} = \sigma\mathbf{f}$, can have many source, including the chemical force of a battery and the electric field generated by the potential difference of the battery, etc. No matter what is the source, the net effect around a loop of circuit is called the **electromotive force (emf)**

$$\mathcal{E} = \oint \mathbf{f} \cdot \mathrm{d}\mathbf{l} = \oint \mathbf{f}_s \cdot \mathrm{d}\mathbf{l} \ . \tag{7.7}$$

Notice that we do not have to include the electric field because the integral of it along a closed path is zero.

7.1.2 Motional Emf

A special kind of emf is illustrated in Fig. 7.1. A rectangular loop is moving to the right with speed v. The magnetic field is pointing into the paper. By Lorentz force law, electrons in the left hand vertical segment of the loop will experience a force pushing downward, hence produced a current upward. The magnitude of the force per unit charge is vB. The emf is $\mathcal{E} = vBh$.

If we let Φ be the flux through the loop, then

$$\Phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{S} = Bhl \tag{7.8}$$

and

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = Bh\frac{\mathrm{d}l}{\mathrm{d}t} = -Bhv = -\mathcal{E}\ . \tag{7.9}$$

7.2 Faraday's Law

It is found by experiments that the electromotive force can be generated in several ways: by moving the loop with stationary magnetic field; or by moving the magnetic field with stationary loop; or by changing the magnetic field strength. In the second and third cases, the charges are not moving. Thus, it is a surprise that there is a current at all. To explain these, Faraday suggested that a changing magnetic field induces an electric field. To find the induced electric field,

$$\begin{aligned}
\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} &= -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
\int \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{S} &= -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathbf{B} \cdot \mathrm{d}\mathbf{S} \\
&= -\int \frac{\partial \mathbf{B}}{\partial \mathbf{t}} \cdot \mathrm{d}\mathbf{S} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}\ . \qquad (7.10)
\end{aligned}$$

This is the **Faraday's law**. For static field, we get back $\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0$.

To determine the direction of the induced current in the loop, we can use **Lenz's law**, which says that the induced current will be in such a direction that the magnetic field it produces tends to counteract the change in flux that induced the emf. In other words, the induced current will yield a field to resist the change of the flux.

7.2.1 Inductance

Suppose there are two conducting loops near each other and there is a current I_1 on the first loop. What is the flux through the second loop? By Biot-Savart law, the magnetic field of the first loop is

$$\mathbf{B}_1(\mathbf{r}) = \frac{\mu_0}{4\pi} I_1 \oint \frac{\mathrm{d}\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\ . \tag{7.11}$$

Hence, the flux through the second loop is

$$\Phi_2 = \int \mathbf{B}_1 \cdot \mathrm{d}\mathbf{S}_2\ . \tag{7.12}$$

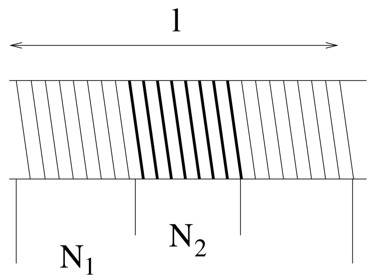

Figure 7.2: Two coils with the same cross sectional area. The longer coil has N_1 turns of total length l. The shorter coil has N_2 turns.

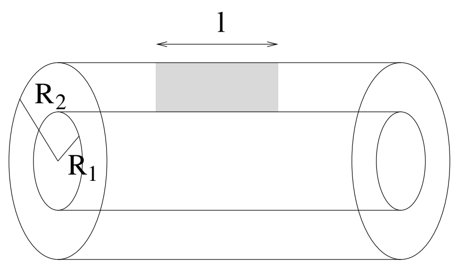

Figure 7.3: A coaxial cable with inner radius R_1 and outer radius R_2.

This is proportional to I_1, and we define the proportionality constant as M_{21}

$$\Phi_2 = M_{21} I_1 \tag{7.13}$$

M_{21} is known as the **mutual inductance** of the two loops.

Example 7.1 We calculate the inductance of two helical coils as shown in Fig. 7.2. The magnetic field produced by the first coil is

$$B = \mu_0 \frac{N_1 I_1}{l} \ . \tag{7.14}$$

The magnetic flux in the second coil is then

$$\Phi_2 = N_2 B A = \mu_0 \frac{N_1 N_2 A}{l} I_1 \tag{7.15}$$

where A is the cross section area of the second coil. Thus the inductance is

$$M = \mu_0 \frac{N_1 N_2 A}{l} \ . \tag{7.16}$$

Example 7.2 We will find the self inductance of a coaxial cable, which has current I flowing in it and with inner radius R_1 and outer radius R_2. Use Ampere's law, the field between the inner wire and outer shell is

$$B = \frac{\mu_0}{2\pi} \frac{I}{r} \ . \tag{7.17}$$

The flux passing the shaded area is

$$\Phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_{R_1}^{R_2} Bl\,\mathrm{d}r = \frac{\mu_0 Il}{2\pi} \ln \frac{R_2}{R_1} \ . \tag{7.18}$$

The self inductance is then

$$L = \frac{\Phi}{I} = \frac{\mu_0\, l}{2\pi} \ln \frac{R_2}{R_1} \ . \tag{7.19}$$

We are going to derive another formula for the mutual inductance. Since the vector potential of the first coil is

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{\mathrm{d}\mathbf{l}_1}{r} \tag{7.20}$$

where r is the distance between the source point and the field point. We have

$$\begin{aligned} \Phi_2 &= \int \mathbf{B}_1 \cdot \mathrm{d}\mathbf{S}_2 \\ &= \int (\nabla \times \mathbf{A}_1) \cdot \mathrm{d}\mathbf{S}_2 \\ &= \oint \mathbf{A}_1 \cdot \mathrm{d}\mathbf{l}_2 \\ &= \frac{\mu_0 I_1}{4\pi} \oint (\oint \frac{\mathrm{d}\mathbf{l}_1}{r}) \cdot \mathrm{d}\mathbf{l}_2 \ . \end{aligned} \tag{7.21}$$

Hence,

$$M_{21} = \frac{\mu_0}{4\pi} \oint_1 \oint_2 \frac{\mathrm{d}\mathbf{l}_1 \cdot \mathrm{d}\mathbf{l}_2}{r} = M_{12} \ . \tag{7.22}$$

This is the **Neumann formula**. We see that M_{21} is a purely geometrical quantity, depending on the size, shapes and relative positions of the two loops. Moreover, $M_{21} = M_{12}$. Thus, we don't have to worry about the order of the mutual inductance.

The changing current can also induce an emf in the source loop itself

$$\Phi = LI \tag{7.23}$$

where L is the **self-inductance**. Notice that inductance is a "resistance" to the changing of current, no matter it is mutual or self-inductance.

7.3 Energy in Magnetic Field

When we consider the self-inductance of a loop, the emf induced in the loop is given by Faraday's law

$$\mathcal{E} = -L\frac{\mathrm{d}I}{\mathrm{d}t} \ . \tag{7.24}$$

The power is $\frac{\mathrm{d}W}{\mathrm{d}t} = -\mathcal{E}I$, and we have

$$\begin{aligned} \frac{\mathrm{d}W}{\mathrm{d}t} &= IL\frac{\mathrm{d}I}{\mathrm{d}t} \\ W &= \frac{1}{2}LI^2 \ . \end{aligned} \tag{7.25}$$

This is the energy stored in the coil. Another form may be more illuminating. Since

$$LI = \Phi = \int \mathbf{B}\cdot \mathrm{d}\mathbf{S} = \int \nabla\times\mathbf{A}\cdot \mathrm{d}\mathbf{S} = \oint \mathbf{A}\cdot \mathrm{d}\mathbf{l} \ , \tag{7.26}$$

we have

$$\begin{aligned} W &= \frac{1}{2}I\oint \mathbf{A}\cdot \mathrm{d}\mathbf{l} \\ &= \frac{1}{2}\oint(\mathbf{A}\cdot\mathbf{I})\mathrm{d}l \\ &= \frac{1}{2}\int(\mathbf{A}\cdot\mathbf{J})\,\mathrm{d}^3\mathbf{r} \\ &= \frac{1}{2\mu_0}\int \mathbf{A}\cdot(\nabla\times\mathbf{B})\mathrm{d}^3\mathbf{r} \end{aligned} \tag{7.27}$$

where we have used the Ampere's law. To simplify this, note that

$$\begin{aligned} \nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \\ &= B^2 - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \ . \end{aligned} \tag{7.28}$$

Therefore, we have

$$\begin{aligned} W &= \frac{1}{2\mu_0}\left[\int B^2\mathrm{d}^3\mathbf{r} - \int \nabla\cdot(\mathbf{A}\times\mathbf{B})\,\mathrm{d}^3\mathbf{r}\right] \\ &= \frac{1}{2\mu_0}\left[\int B^2\mathrm{d}^3\mathbf{r} - \oint(\mathbf{A}\times\mathbf{B})\cdot \mathrm{d}\mathbf{S}\right] \ . \end{aligned} \tag{7.29}$$

The integration region is the region where includes all coils. We could take the integration over the whole space, and the second term will be zero

$$W = \frac{1}{2\mu_0}\int B^2\mathrm{d}^3\mathbf{r} \ . \tag{7.30}$$

We could interpret the result as the energy is stored in the magnetic field and the energy per unit volume is $B^2/2\mu_0$. To summarize, the energy stored in the electric and magnetic fields are

$$W_e = \frac{\varepsilon_0}{2}\int E^2\mathrm{d}^3\mathbf{r} \tag{7.31}$$

$$W_m = \frac{1}{2\mu_0}\int B^2\mathrm{d}^3\mathbf{r} \ . \tag{7.32}$$

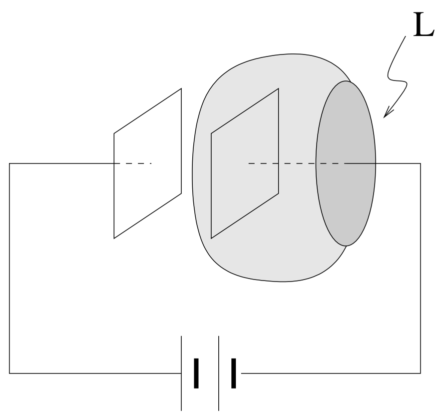

Figure 7.4: A setup to demonstrate of the inconsistency of the original Ampere's law.

7.4 Maxwell Equations

Now, we have a set of four equations

$$\left\{\begin{array}{lcll} \nabla\cdot\mathbf{E} &=& \dfrac{\rho}{\varepsilon_0} & \text{Gauss's law} \\ \nabla\cdot\mathbf{B} &=& 0 & \text{no name} \\ \nabla\times\mathbf{E} &=& -\dfrac{\partial\mathbf{B}}{\partial t} & \text{Faraday's law} \\ \nabla\times\mathbf{B} &=& \mu_0\mathbf{J} & \text{Ampere's law} \end{array}\right. . \tag{7.33}$$

However, these equations are not consistent with each other. To demonstrate the contradiction, consider

$$0 \equiv \nabla\cdot(\nabla\times\mathbf{B}) = \mu_0\nabla\cdot\mathbf{J} \neq 0 \ . \tag{7.34}$$

Another way to see the inconsistency is to consider the setup in Fig. 7.4. When the capacitor is charging up, there is a current flowing around the circuit. Ampere's law tells us

$$\oint_L \mathbf{B}\cdot d\mathbf{l} = \mu_0 I \tag{7.35}$$

where I is the current enclosed by the loop L, but what is it? If we choose the darker shaded surface, I is of course the current of the circuit. If we choose the lighter shaded surface, I is zero. What's wrong?

For a steady current, $\nabla\cdot\mathbf{J}=0$, but in general, $\nabla\cdot\mathbf{J}=-\partial\rho/\partial t$. Maxwell suggested the following correction

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0\nabla\cdot\mathbf{E}) = -\varepsilon_0\nabla\cdot\frac{\partial\mathbf{E}}{\partial t} \tag{7.36}$$

Thus,

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{7.37}$$

and we have obtained a set of equations

$$\begin{cases} \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases} . \tag{7.38}$$

The term $\varepsilon_0 \partial \mathbf{E}/\partial t$ is called the **displacement current**. These are the four **Maxwell's equations**. This is the beginning of electrodynamics.

You might notice the symmetry between the electric field and magnetic field of the Maxwell's equations. If there is magnetic charge or magnetic monopole, the Maxwell's equations can be rewritten in a more symmetric form. If η had represented the density of magnetic charge, then we should have

$$\nabla \cdot \mathbf{B} = \mu_0 \eta \tag{7.39}$$

$$\nabla \times \mathbf{E} = -\mu_0 \mathbf{K} - \frac{\partial \mathbf{B}}{\partial t} \tag{7.40}$$

$$\tag{7.41}$$

where $\mathbf{K}$ is the magnetic current density. The continuity equation for magnetic charge is

$$\nabla \cdot \mathbf{K} = -\frac{\partial \eta}{\partial t} . \tag{7.42}$$

We expect from the equations the existence of magnetic charge. It would fit in so nicely. Dirac pointed in 1931 that if magnetic charge exists, electric and magnetic charge must be quantized,

$$\frac{\mu_0 e g}{4\pi} = n \frac{\hbar}{2} \tag{7.43}$$

where g is the magnetic charge. So far, we have not found any magnetic monopole.

7.5 Maxwell's Equations in Matters

The bound charge and bound current in matters are given by

$$\rho_b = -\nabla \cdot \mathbf{P} \tag{7.44}$$

$$\mathbf{J}_b = \nabla \times \mathbf{M} . \tag{7.45}$$

At the surface, the surface charge density is $\sigma_b = \mathbf{P} \cdot \hat{n}$. Now, if the polarization increases a bit, $\mathbf{P} \to \mathbf{P} + \delta \mathbf{P}$, the bound charge at a small surface of area δS also increases, by $\delta q_b = \delta\sigma_b\delta S = \delta\mathbf{P}\cdot\delta\mathbf{S}$. The increase in current is then

$$\delta I = \frac{\mathrm{d}\delta q_b}{\mathrm{d}t} = \frac{\mathrm{d}\delta\sigma_b}{\mathrm{d}t}\delta S = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}\cdot\delta\mathbf{S}\ . \tag{7.46}$$

The current density induced by the changing of polarization is

$$\mathbf{J}_{\mathrm{p}} = \frac{\partial\mathbf{P}}{\partial t} \tag{7.47}$$

This polarization current has nothing to do with $\mathbf{J}_b$. It is the result of linear motion of charges when $\mathbf{P}$ changes. The continuity equation of the polarization current is

$$\nabla\cdot\mathbf{J}_p = \nabla\cdot\frac{\partial\mathbf{P}}{\partial t} = \frac{\partial}{\partial t}(\nabla\cdot\mathbf{P}) = -\frac{\partial\rho_b}{\partial t}\ . \tag{7.48}$$

So the total charge and current can be written as

$$\rho = \rho_f + \rho_b = \rho_f - \nabla\cdot\mathbf{P} \tag{7.49}$$

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla\times\mathbf{M} + \frac{\partial\mathbf{P}}{\partial t}\ . \tag{7.50}$$

Gauss's law becomes

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}(\rho_f - \nabla\cdot\mathbf{P}) \tag{7.51}$$

or

$$\nabla\cdot\mathbf{D} = \rho_f \tag{7.52}$$

where $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$. Ampere's law becomes

$$\nabla\times\mathbf{B} = \mu_0(\mathbf{J}_f + \nabla\times\mathbf{M} + \frac{\partial\mathbf{P}}{\partial t}) + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{7.53}$$

or

$$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t} \tag{7.54}$$

where $\mathbf{H} = \mathbf{B}/\boldsymbol{\mu_0} - \mathbf{M}$. To summarize, Maxwell's' equations in matter are

$$\nabla\cdot\mathbf{D} = \rho_f \tag{7.55}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{7.56}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{7.57}$$

$$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t} \tag{7.58}$$

More equations are needed to specify $\mathbf{P}$ and $\mathbf{M}$, depending on the nature of the materials. For linear medium,

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \tag{7.59}$$

$$\mathbf{M} = \chi_m\mathbf{H} \tag{7.60}$$

and

$$\begin{aligned} \mathbf{D} &= \varepsilon \mathbf{E} && (7.61) \\ \mathbf{H} &= \frac{1}{\mu}\mathbf{B} \ . && (7.62) \end{aligned}$$

7.6 Gauge Transformations

In electrostatics, we have $\nabla \times \mathbf{E} = 0$. Thus, we could express the electric field in terms of the scalar potential

$$\mathbf{E} = -\nabla V \ . \tag{7.63}$$

For electrodynamics, $\nabla \times \mathbf{E} \neq 0$, but $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$. As $\mathbf{B}$ remains divergenceless, we still have $\mathbf{B} = \nabla \times \mathbf{A}$, and

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) && (7.64) \\ \nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) &= 0 \ . && (7.65) \end{aligned}$$

Thus, we can express $\mathbf{E}$ as

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \ . \tag{7.66}$$

Gauss's law can then be written as

$$\begin{aligned} \nabla \cdot \mathbf{E} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) &= \frac{\rho}{\varepsilon_0} && (7.67) \\ \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) &= -\frac{\rho}{\varepsilon_0} \ . && (7.68) \end{aligned}$$

This replaces the previous Poisson's equation for the electric potential. For the fourth Maxwell's equations, we have

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \times (\nabla \times \mathbf{A}) &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}) \ . && (7.69) \end{aligned}$$

Using the identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, we obtain

$$(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}) - \nabla(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}) = -\mu_0 \mathbf{J} \ . \tag{7.70}$$

The two equations, Eq.(7.68) and Eq.(7.70), contain all the information of the Maxwell's equations. We can simplify this equation by exploring the gauge degree of freedom. The defining equations of the scalar and vector potentials are

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{7.71}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \; . \tag{7.72}$$

For any scalar field λ, consider a new set of potentials

$$V' = V - \frac{\partial \lambda}{\partial t} \tag{7.73}$$

$$\mathbf{A}' = \mathbf{A} + \nabla \lambda \; . \tag{7.74}$$

This is called the **gauge transformation** of the potentials. The corresponding electric and magnetic fields are

$$\begin{aligned}
\mathbf{E}' &= -\nabla V' - \frac{\partial \mathbf{A}'}{\partial t} \\
&= -\nabla V + \nabla \frac{\partial \lambda}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial \nabla \lambda}{\partial t} \\
&= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \\
&= \mathbf{E} \qquad (7.75) \\
\mathbf{B}' &= \nabla \times \mathbf{A}' \\
&= \nabla \times \mathbf{A} + \nabla \times \nabla \lambda \\
&= \mathbf{B} \; . \qquad (7.76)
\end{aligned}$$

Thus, they are invariant under the gauge transformation. The freedom to add a constant to the electric potential in electrostatics is a special case of this invariance.

With the gauge degree of freedom, the functional form of the potentials is not determined by the electric and magnetic field alone. To fully determine the functional form, we could choose an extra relation. This is called choosing a **gauge**.

If we choose the **Coulomb gauge**,

$$\nabla \cdot \mathbf{A} = 0 \; , \tag{7.77}$$

Eq.(7.68) becomes

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \; . \tag{7.78}$$

This is the ordinary Poisson's equation, with solution

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{r} \, \mathrm{d}^3\mathbf{r} \; . \tag{7.79}$$

But then Eq.(7.70) becomes

$$(\nabla^2 \mathbf{A} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}) - \mu_0\varepsilon_0 \nabla(\frac{\partial V}{\partial t}) = -\mu_0 \mathbf{J} \ , \tag{7.80}$$

and it is be very difficult to solve. We can also choose the **Lorentz gauge**

$$\nabla \cdot \mathbf{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t} = 0 \ . \tag{7.81}$$

Eq.(7.68) and Eq.(7.70) becomes

$$\nabla^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \tag{7.82}$$

$$\nabla^2 \mathbf{A} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \ . \tag{7.83}$$

We define the operator $\Box^2 \equiv \nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}$. It is called **d'Alembertian**. (d'Alembertian is always the square. We never need the "square root" of d'Alembertian.) Thus, we can express these two equations as

$$\Box^2 V = -\frac{\rho}{\varepsilon_0} \tag{7.84}$$

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J} \ . \tag{7.85}$$

Under the Lorentz gauge, the equations for V and $\mathbf{A}$ are the same form. Actually, Eq.(7.84) and Eq.(7.85) are the wave equations with sources. Their particular solutions can be written as

$$V(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3\mathbf{r}' \tag{7.86}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3\mathbf{r}' \tag{7.87}$$

where $t' = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}$ and

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{7.88}$$

is the speed of the electromagnetic wave. The value of this speed is very closed to the speed of light and led Maxwell to comment that

> we have strong reason to conclude that light itself is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws.

Although this form of the solution is explicit, it is very difficult to calculate in practice.

7.7 Energy in Electrodynamics

We have found that the total energy stored in electromagnetic fields is

$$W_{EM} = \frac{1}{2}\int (\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2)\,\mathrm{d}^3\mathbf{r}\ . \tag{7.89}$$

On the other hand, the work done on charge $\mathrm{d}q$ is

$$\mathrm{d}W = \mathbf{F}\cdot\mathrm{d}\mathbf{l} = \mathrm{d}q\,(\mathbf{E} + (\mathbf{v}\times\mathbf{B}))\cdot\mathbf{v}\mathrm{d}t = \mathbf{E}\cdot\mathbf{v}\,\mathrm{d}q\,\mathrm{d}t\ . \tag{7.90}$$

Note that $\mathrm{d}q = \rho\,\mathrm{d}^3\mathbf{r}$ and $\mathbf{J} = \rho\mathbf{v}$, we have

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int \mathbf{E}\cdot\mathbf{J}\,\mathrm{d}^3\mathbf{r}\ . \tag{7.91}$$

The product $\mathbf{E}\cdot\mathbf{J}$ is the work per unit time per unit volume, namely the power per unit volume. We convert this formula to a form that only involves the fields. First,

$$\mathbf{E}\cdot\mathbf{J} = \frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \varepsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\ . \tag{7.92}$$

As

$$\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot(\nabla\times\mathbf{B}) \tag{7.93}$$

and

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{7.94}$$

we have

$$\mathbf{E}\cdot(\nabla\times\mathbf{B}) = -\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{B})\ . \tag{7.95}$$

Thus, we have

$$\mathbf{E}\cdot\mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2) - \frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B}) \tag{7.96}$$

Eq.(7.91) becomes

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{2}(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2)\,\mathrm{d}^3\mathbf{r} - \frac{1}{\mu_0}\int(\mathbf{E}\times\mathbf{B})\cdot\mathrm{d}\mathbf{S}\ . \tag{7.97}$$

The meaning of this equation is that the work done on the charges in a region is equal to the decrease of energy stored in the electromagnetic field, the second term, and the energy passing through the surface, the third term. This is called **Poynting's theorem**. The energy passing through a small surface per unit time, per unit area is called the **Poynting vector**

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})\ . \tag{7.98}$$

Combine Eq.(7.91), Eq.(7.97) and Eq.(7.98), we have

$$\begin{aligned}\int \mathbf{E}\cdot\mathbf{J}\,\mathrm{d}^3\mathbf{r} &= -\int \frac{1}{2}\frac{\partial}{\partial t}(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2)\,\mathrm{d}^3\mathbf{r} - \int \mathbf{S}\cdot\mathrm{d}\mathbf{S} \\ &= -\int \frac{1}{2}\frac{\partial}{\partial t}(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2)\,\mathrm{d}^3\mathbf{r} - \int \nabla\cdot\mathbf{S}\,\mathrm{d}^3\mathbf{r} \qquad (7.99)\end{aligned}$$

(Sorry for the notation in the first line of this equation.) Hence,

$$\nabla\cdot\mathbf{S} + \frac{\partial}{\partial t}(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2) = -\mathbf{E}\cdot\mathbf{J}\ . \qquad (7.100)$$

This is just the local version of the Poynting's theorem.

7.8 Boundary Conditions

We summarize the boundary conditions for the electromagnetic field across two media. Apply the first Maxwell's equations to a Gaussian pillbox at the interface, we have

$$\oint \mathbf{D}\cdot\mathrm{d}\mathbf{S} = Q_f \qquad (7.101)$$

or

$$D_{1\perp} - D_{2\perp} = \sigma_f \qquad (7.102)$$

where σ_f is the surface free charge density. Similarly,

$$B_{1\perp} - B_{2\perp} = 0\ . \qquad (7.103)$$

We conclude that $D_\perp$ is discontinuous at the interface while $B_\perp$ is continuous.

For the third Maxwell's equations, we can choose the integral path as shown in Fig. 2.9,

$$\oint \mathbf{E}\cdot\mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathbf{B}\cdot\mathrm{d}\mathbf{S}\ . \qquad (7.104)$$

In the limit very small loop, the surface bounded by it tends to zero, the second term will be zero. We have

$$\oint_{\mathrm{L}} \mathbf{E}\cdot\mathrm{d}\mathbf{l} = 0 \qquad (7.105)$$

or

$$\mathbf{E}_{1||} - \mathbf{E}_{2||} = 0\ . \qquad (7.106)$$

Similarly, from $\oint_{\mathrm{L}} \mathbf{H}\cdot\mathrm{d}\mathbf{l} = I_f + \frac{\mathrm{d}}{\mathrm{d}t}\int \mathbf{D}\cdot\mathrm{d}\mathbf{S}$, we have

$$\mathbf{H}_{1||} - \mathbf{H}_{2||} = \mathbf{K}_f \times \hat{n} \qquad (7.107)$$

where $\mathbf{K}_f$ is the surface free current density. If the media are linear, we obtain

$$\begin{aligned}\varepsilon_1 E_{1\perp} - \varepsilon_2 E_{2\perp} &= \sigma_f &(7.108)\\ B_{1\perp} - B_{2\perp} &= 0 &(7.109)\\ \mathbf{E}_{1||} - \mathbf{E}_{2||} &= 0 &(7.110)\\ \frac{1}{\mu_1}\mathbf{B}_{1||} - \frac{1}{\mu_2}\mathbf{B}_{2||} &= \mathbf{K}_f \times \hat{n} &(7.111)\end{aligned}$$

or

$$\begin{aligned}\hat{n}\cdot(\varepsilon_1\mathbf{E}_1 - \varepsilon_2\mathbf{E}_2) &= \sigma_f &(7.112)\\ \hat{n}\cdot(\mathbf{B}_1 - \mathbf{B}_2) &= 0 &(7.113)\\ \hat{n}\times(\mathbf{E}_1 - \mathbf{E}_2) &= 0 &(7.114)\\ \hat{n}\times(\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{K}_f \times \hat{n}\ . &(7.115)\end{aligned}$$

7.9 Summary of Maxwell's Equations

This chapter contains the foundation of classical electromagnetic theory. The basis are Maxwell's equations

$$\begin{aligned}\nabla\cdot\mathbf{D} &= \rho_f &(7.116)\\ \nabla\cdot\mathbf{B} &= 0 &(7.117)\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} &(7.118)\\ \nabla\times\mathbf{H} &= \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}\ . &(7.119)\end{aligned}$$

The $\mathbf{E}$ and $\mathbf{B}$ fields are operationally defined by Lorentz force

$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v}\times\mathbf{B})] \tag{7.120}$$

The Maxwell's equations have the following consequences:

1. Electric charge is conserved

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0\ . \tag{7.121}$$

2. Energy is conserved

$$\nabla\cdot\mathbf{S} + \frac{\partial v}{\partial t} = -\mathbf{J}\cdot\mathbf{E} \tag{7.122}$$

where the field energy density is $v = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$ and the energy flux per unit area is given by the Poynting vector

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\ . \tag{7.123}$$

3. **B** and **E** can be expressed by potential functions

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{7.124}$$
$$\mathbf{B} = \nabla \times \mathbf{A} \ . \tag{7.125}$$

4. If we choose the Lorentz gauge,

$$\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0 \ , \tag{7.126}$$

   the potential functions satisfy the inhomogeneous wave equations

$$\Box^2 V = -\frac{\rho}{\varepsilon_0} \tag{7.127}$$
$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{7.128}$$

   where the d'Alembertian is $\Box^2 \equiv \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}$. These equations tell us the generation of electromagnetic waves by describing charge and current distributions.

5. Their particular solutions (in vacuum) are

$$V(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3\mathbf{r}' \tag{7.129}$$
$$A(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3\mathbf{r}' \tag{7.130}$$

   where $t' = t - |\mathbf{r} - \mathbf{r}'|/c$.

6. Boundary conditions at the interface between two media are given in last section.

This is the beginning of the story.

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