PHYS2155 Methods in physics II

Tutorial 3

1. Evaluate the line integrals along the curve C.

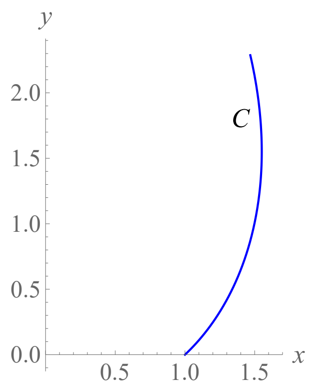
(a) $\displaystyle\int_C (1+xy^2)\,\mathrm{d}s, \quad C\colon \mathbf{r}(t) = (1-t)\,\hat{\imath} + 2(1-t)\,\hat{\jmath},\ 0 \le t \le 1.$

(b) $\displaystyle\int_C \left(\frac{1}{1+x}\right)\mathrm{d}s, \quad C\colon \mathbf{r}(t) = t\,\hat{\imath} + \tfrac{2}{3}t^{3/2}\,\hat{\jmath},\ 0 \le t \le 3.$

(c) $\displaystyle\int_C (3x-y)\,\mathrm{d}s,$ C consists of the line segment C_1 from $(1,2)$ to $(3,3)$ followed by the circular path C_2 given by $x^2+y^2=18$ from $(3,3)$ to $(3,-3)$.

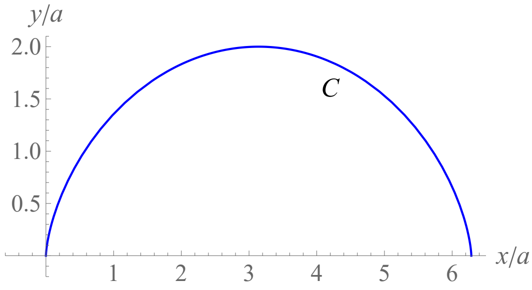
(d) $\displaystyle\int_C xy\,\mathrm{d}s,$ C consists of the parabolic path C_1 given by $y = 2 - 2x^2/9$ from $(0,2)$ to $(3,0)$ followed by the line segment C_2 from $(3,0)$ to $(0,2)$.

2. (a) A thin wire in the shape of the curve C defined parametrically by $x = e^t\cos t$, $y = e^t \sin t$, $0 \le t \le 1$, has linear mass density λ that is proportional to the distance from the origin. Find the mass of the wire.



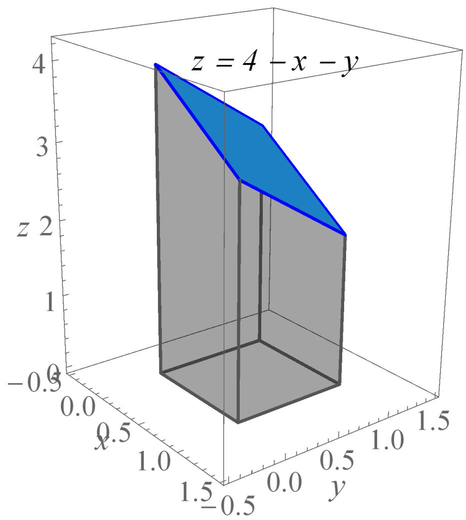
(b) A thin wire in the shape of the curve C given by $y = x^2$, $0 \le x \le 3$, has linear mass density $\lambda(x,y) = x$. Find the mass and the location of the center of mass of the wire.

(c) A thin wire in the shape of an arch of the cycloid C defined parametrically by $x = a(t-\sin t)$, $y = a(1-\cos t)$, $0 \le t \le 2\pi$, has constant linear mass density k where a and k are positive constants. Find the mass and the location of the center of mass of the wire.

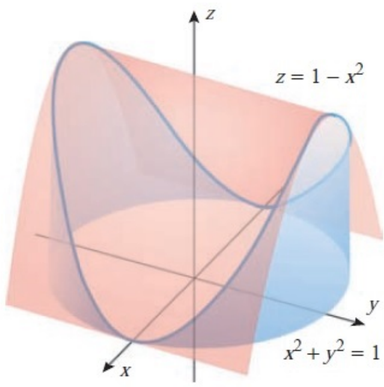


3. If C is a curve in the xy-plane and $f(x,y)$ is a nonnegative continuous function defined on C, then $\int_C f\,\mathrm{d}s$ can be interpreted as the area A of the "sheet" that is swept out by a vertical line segment that extends upward from the point (x,y) to a height of $f(x,y)$ and moves along C from one endpoint to the other. Use such interpretation to the area of the following surfaces.

   (a) The surface extending upward from the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ in the xy-plane to the plane $z = 4 - x - y$.



   (b) The surface extending upward from the circle $x^2 + y^2 = 1$ in the xy-plane to the parabolic cylinder $z = 1 - x^2$.

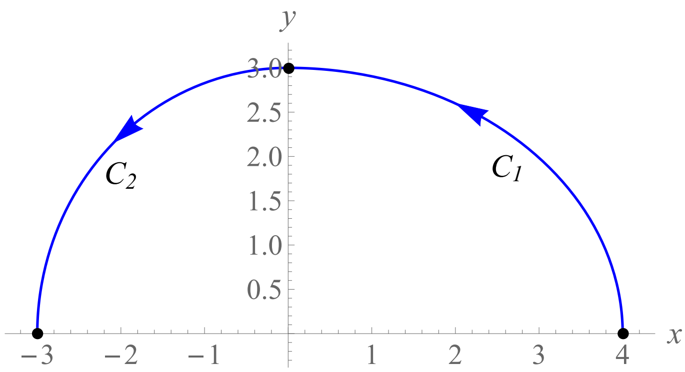


4. Evaluate the line integrals along the curve C.

(a) $\int_C 2xy\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y$, $C\colon \mathbf{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$, $0 \le t \le \pi/2$.

(b) $\int_C y\,\mathrm{d}x + x^2\,\mathrm{d}y$, C is the parabolic arc given by $y = 4x - x^2$ from $(4, 0)$ to $(1, 3)$.

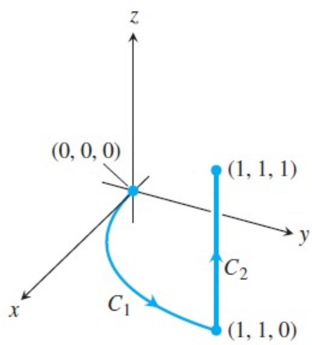
(c) $\int_C (2x + y)\,\mathrm{d}x + 2y\,\mathrm{d}y$, C consists of the elliptical path C_1 given by $9x^2 + 16y^2 = 144$ from $(4, 0)$ to $(0, 3)$ followed by the circular path C_2 given by $x^2 + y^2 = 9$ from $(0, 3)$ to $(-3, 0)$.



5. Evaluate the line integrals along the curve C.

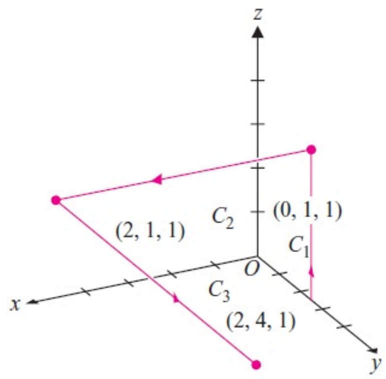
(a) $\int_C (x^2 - y + 3z)\,\mathrm{d}s$, C is the line segment from $(0, 0, 0)$ to $(1, 2, 1)$

(b) $\int_C (x + \sqrt{y} - z^2)\,\mathrm{d}s$, C consists of the parabolic arc given by $y = x^2$ from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the line segment C_2 from $(1, 1, 0)$ to $(1, 1, 1)$.

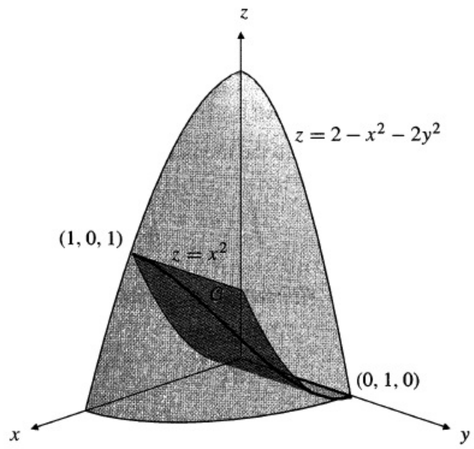


(c) $\int_C yz\,\mathrm{d}x - xz\,\mathrm{d}y + xy\,\mathrm{d}z$, $C\colon \mathbf{r}(t) = e^t\,\hat{\imath} + e^{3t}\,\hat{\jmath} + e^{-t}\,\hat{k}$, $0 \le t \le 1$.

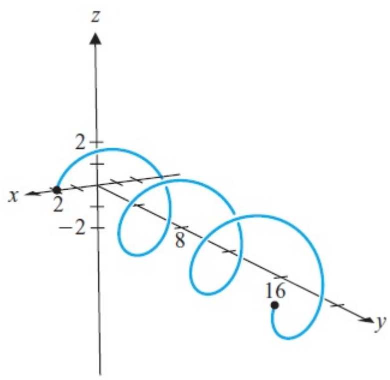
(d) $\int_C 4x\,\mathrm{d}y + 2y\,\mathrm{d}z$, C consists of the line segment C_1 from $(0, 1, 0)$ to $(0, 1, 1)$, followed by the line segment C_2 from $(0, 1, 1)$ to $(2, 1, 1)$, and followed by the line segment C_3 from $(2, 1, 1)$ to $(2, 4, 1)$.



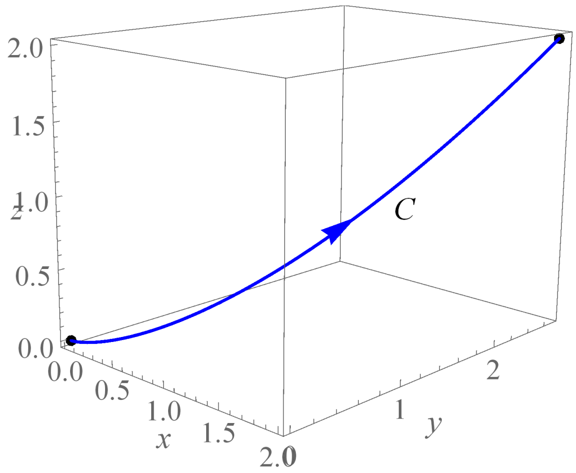
6. (a) A wire lies along the curve C that is the first octant part of the curve of the intersection of the elliptic paraboloid $z = 2 - x^2 - 2y^2$ and the parabolic cylinder $z = x^2$ between $(0, 1, 0)$ and $(1, 0, 1)$. Find the mass of the wire if it has linear mass density $\lambda(x, y, z) = xy$.



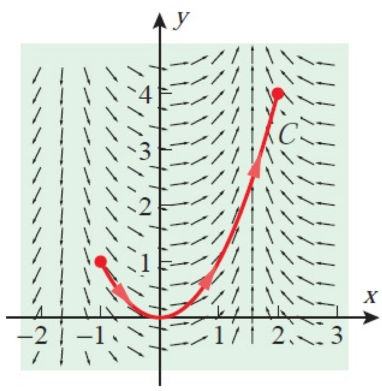
(b) A spring in the shape of a helix C defined parametrically by $x = 2\cos t, y = t, z = 2\sin t, 0 \leq t \leq 6\pi$, has linear mass density $\lambda(x, y, z) = 2y$. Find the mass and the location of the center of mass of the spring.



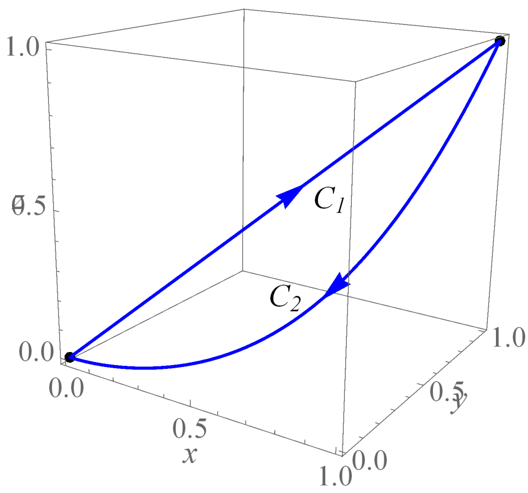
(c) A wire lying along the curve C defined by $\mathbf{r}(t) = t\,\hat{\imath} + (2\sqrt{2}t^{3/2}/3)\,\hat{\jmath} + (t^2/2)\,\hat{k}$, $0 \leq t \leq 2$, has linear mass density $\lambda(x, y, z) = 1/(x + 1)$. Find the location of the center of mass and the moment of inertia about the y-axis of the wire.



7. Find the work done by the following force fields $\mathbf{F}$ on a particle that moves along the curves C.
(a) $\mathbf{F}(x, y) = \cos x\,\hat{\imath} + \sin x\,\hat{\jmath}$, $C\colon \mathbf{r}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath}$, $-1 \leq t \leq 2$.



(b) $\mathbf{F}(x, y) = xe^y\,\hat{\imath} + y\,\hat{\jmath}$, C consists of the parabolic arc C_1 given by $y = x^2$ from $(-1, 1)$ to $(2, 4)$ followed by the line segment C_2 from $(2, 4)$ to $(4, 6)$.
(c) $\mathbf{F}(x, y, z) = 4y\,\hat{\imath} + 2xz\,\hat{\jmath} + 3y\,\hat{k}$, C is the helix $x = 2\cos t$, $y = 2\sin t$, $z = 3t$, $0 \leq t \leq \pi$.
(d) $\mathbf{F}(x, y, z) = \sqrt{z}\,\hat{\imath} - 2x\,\hat{\jmath} + \sqrt{y}\,\hat{k}$, C consists of the line segment C_1 from $(0, 0, 0)$ to $(1, 1, 1)$, followed by the curve C_2 defined parametrically by $x = (1-t)$, $y = (1-t)^2$, $z = (1-t)^4$, $0 \leq t \leq 1$.



8. In electromagnetics theory, the electric potential difference between two points, A and B, is given by the line integral

$$V_B - V_A = -\int_C \mathbf{E} \cdot \mathrm{d}\mathbf{r} \tag{1}$$

where $\mathbf{E}$ is the electric field and C is *any* path from A to B (with both $\mathbf{E}$ and V in appropriate units). Find the electric potential differences between the points A and B for the following electric fields $\mathbf{E}$.
(a) $\mathbf{E}(x, y, z) = 2xy^2z\,\hat{\imath} + 2x^2yz\,\hat{\jmath} + x^2y^2\,\hat{k}$; $A(0, 0, 0)$, $B(1, 1, 6)$.
(b) $\mathbf{E}(x, y, z) = (y^2 - x)\,\hat{\imath} + (2xy + \sin z)\,\hat{\jmath} + (y \cos z)\,\hat{k}$; $A(1, 3, 1)$, $B(2, -1, 4)$.
(Hint: Consider the line segment from A to B.)

9. (a) Show that $\mathbf{F}(x, y) = (2xy - 3)\,\hat{\imath} + (x^2 + 4y^3 + 5)\,\hat{\jmath}$ is conservative by showing that it is the gradient of the potential function $\phi(x, y) = x^2y - 3x + y^4 + 5y$. Hence, evaluate the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is any piecewise-smooth curve from $(-1, 2)$ to $(2, 3)$.
(b) Show that $\mathbf{F}(x, y, z) = (2x \cos z - x^2)\,\hat{\imath} + (z - 2y)\,\hat{\jmath} + (y - x^2 \sin z)\,\hat{k}$ is conservative by showing that it is the gradient of the potential function $\phi(x, y, z) = -x^3/3 + x^2 \cos z - y^2 + yz$. Hence, evaluate the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is any piecewise-smooth curve from $(3, -2, 1)$ to $(1, 0, \pi)$.

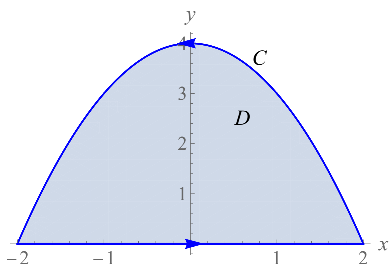
10. Determine whether the vector fields $\mathbf{F}$ are conservative. If so, find the potential functions ϕ such that $\mathbf{F} = \nabla\phi$.
(a) $\mathbf{F}(x, y) = y^2 \cos x\,\hat{\imath} + (2y \sin x + 3)\,\hat{\jmath}$
(b) $\mathbf{F}(x, y) = ye^{xy}\,\hat{\imath} + (xe^{xy} + \cos y)\,\hat{\jmath}$
(c) $\mathbf{F}(x, y, z) = ze^{xz}\,\hat{\imath} + \ln z\,\hat{\jmath} + \left(xe^{xz} + \dfrac{y}{z}\right)\hat{k} \quad$ for $z > 0$
(d) $\mathbf{F}(x, y, z) = y \ln z\,\hat{\imath} - x \ln z\,\hat{\jmath} + \dfrac{xy}{z}\,\hat{k}$.

11. In electrostatic, all electric field $\mathbf{E}$ are conservative ($\nabla \times \mathbf{E} = 0$ and can be written as the negative gradient of a scalar potential V

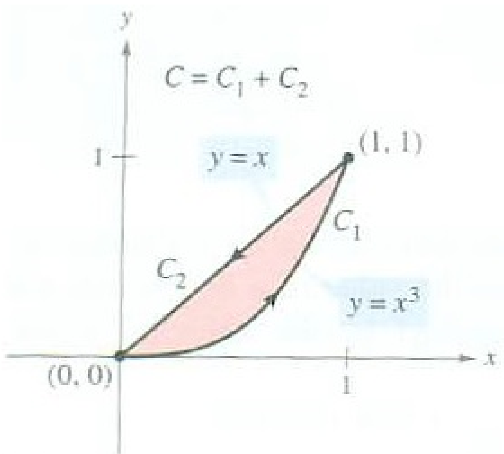
$$\mathbf{E} = -\nabla V\ . \tag{2}$$

(a) Show that the electric field $\mathbf{E}(x, y, z) = yz\,\hat{\imath} + xz\,\hat{\jmath} + xy\,\hat{k}$ is conservative by finding a potential V such that $\mathbf{E} = -\nabla V$.
(b) Determine the values of the constants A and B such that the electric field $\mathbf{E}(x, y, z) = Ax \sin \pi y\,\hat{\imath} + (x^2 \cos \pi y + Bye^{-z})\,\hat{\jmath} + y^2 e^{-z}\,\hat{k}$ is conservative. Find a potential V such that $\mathbf{E} = -\nabla V$.

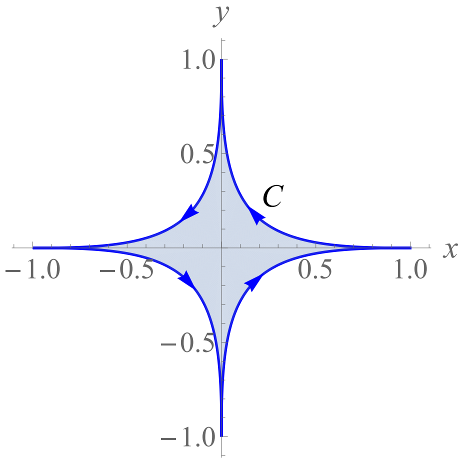
12. Use Green's Theorem to evaluate the line integrals along the curves C which are positively oriented.
(a) $\displaystyle\oint_C (7y - e^{\sin x})\,dx + [15x - \sin(y^3 + 8y)]\,dy$, C is the circle of radius 3 centered at the point $(5, -7)$.
(b) $\displaystyle\oint_C (y^2 + \cos x)\,dx + (x - \tan^{-1} y)\,dy$, C is the boundary of the region bounded by the curve $y = 4 - x^2$ and the x-axis.



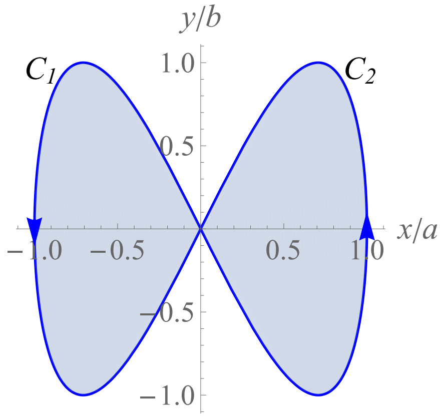
(c) $\displaystyle\oint_C y^3\,dx + (x^3 + 3xy^2)\,dy$, C consists of the curve C_1 given by $y = x^3$ from $(0, 0)$ to $(1, 1)$ followed by the line segment C_2 from $(1, 1)$ to $(0, 0)$.



13. Use a line integral to find the area of the given region.
(a) The region enclosed by the curve C given by $x^{2/5} + y^{2/5} = 1$.



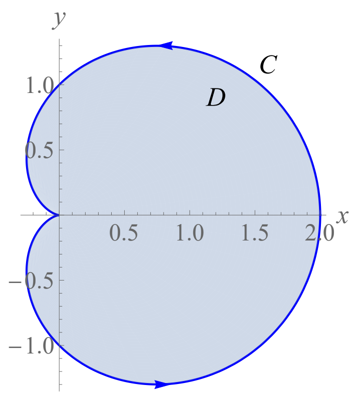
(b) The region enclosed by the curve C defined parametrically by $x = a \sin t$ and $y = b \sin 2t$ where $0 \leq t \leq 2\pi$.



(c) The region enclosed by the line C_1 given by $y = 5x - 3$ and the curve C_2 given by $y = x^2 + 1$.

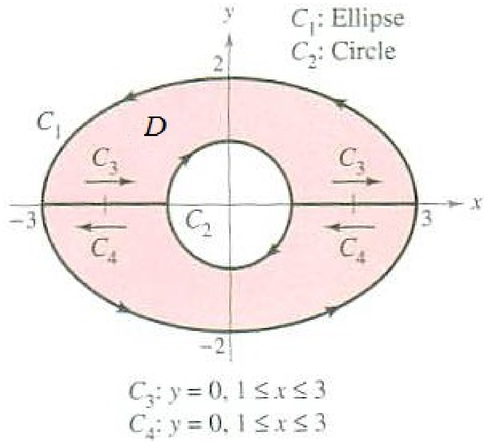
14. Use Green's Theorem to evaluate the line integrals along the curves C which are positive oriented.

(a) $\displaystyle\oint_C 6xy\,\mathrm{d}x + [3x^2 + \ln(1+y)]\,\mathrm{d}y$, C is the cardioid $r = 1 + \cos\theta$, $0 \leq \theta \leq 2\pi$.

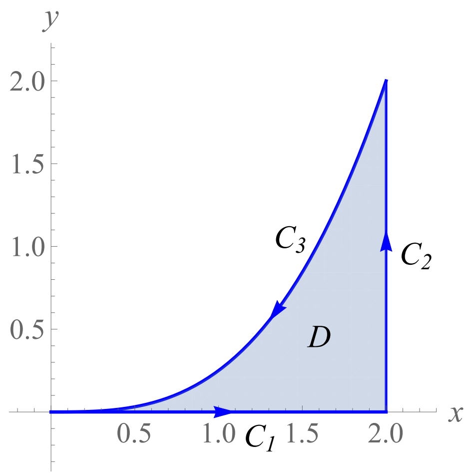


(b) $\oint_C \left(\frac{x^3}{x^4+y^4}\right) \mathrm{d}x + \left(\frac{y^3}{x^4+y^4}\right) \mathrm{d}y$, C is any simple closed curve bounding a region containing the origin.

(c) $\oint_C 2xy\,\mathrm{d}x + (x^2+2x)\,\mathrm{d}y$, C is the boundary of the region lying between the ellipse $4x^2+9y^2=36$ and the circle $x^2+y^2=1$.



15. (a) Use Green's Theorem to find the work done by the force field $\mathbf{F}(x,y) = \sqrt{y}\,\hat{\imath} + \sqrt{x}\,\hat{\jmath}$ on a particle that moves counterclockwise one time around the closed curve C given by the equations $y=0$, $x=2$, and $y=x^3/4$.



(b) According to Coulomb's law, the electric force exerted by a unit charge at the origin on a charge q at the point (x,y,z) is

$$\mathbf{F}(x,y,z) = \frac{q}{4\pi\epsilon_0 r^3}\,\mathbf{r}\,, \tag{3}$$

where ϵ_0 is the permittivity of free space, $\mathbf{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$, and $r = \|\mathbf{r}\| = \sqrt{x^2+y^2+z^2}$. Show that the work done by the force field

$\mathbf{F}(x, y, z)$ in moving the charge q from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is equal to

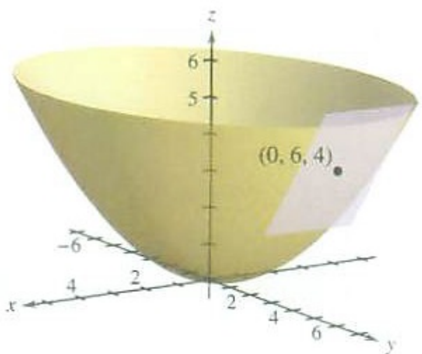
$$W = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \tag{4}$$

where $r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $r_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$.

16. Identify and sketch the parametric surface represented by the given vector functions.
(a) $\mathbf{r}(u, v) = u\,\hat{\imath} + \sqrt{u^2 + v^2}\,\hat{\jmath} + v\,\hat{k}$
(b) $\mathbf{r}(u, v) = u\cos v\,\hat{\imath} + u\sin v\,\hat{\jmath} + u^2\,\hat{k}$
(c) $\mathbf{r}(u, v) = u\,\hat{\imath} + \frac{1}{4}v^3\,\hat{\jmath} + v\,\hat{k}$

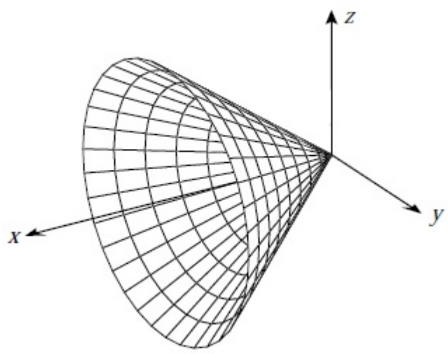
17. Find parametric equations for each of the following surfaces. (There are, of course, many different parametrizations for any surface.)
(a) The surface defined by $z = 1/(x^2 + y^2)$.
(b) The ellipsoid $4x^2 + 9y^2 + 36z^2 = 36$.
(c) The hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$. (Hint: Use the functions $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$.)
(d) The surface generated by rotating the curve $z = y^2 + 1$, $0 \le y \le 2$ about the y-axis.

18. Find an equations of the tangent planes to the surfaces represented by the vector functions at the given points.
(a) $\mathbf{r}(u, v) = uv\,\hat{\imath} + u\,\hat{\jmath} + v^2\,\hat{k}$; $(-2, 2, 1)$
(b) $\mathbf{r}(u, v) = 2u\cos v\,\hat{\imath} + 3u\sin v\,\hat{\mathbf{j}} + u^2\,\hat{k}$; $(0, 6, 4)$

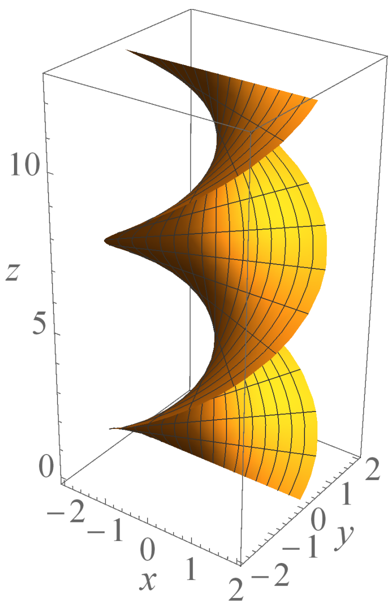


(c) $\mathbf{r}(u, v) = ue^v\,\hat{\imath} + ve^u\,\hat{\jmath} + uv\,\hat{k}$; $(0, \ln 2, 0)$

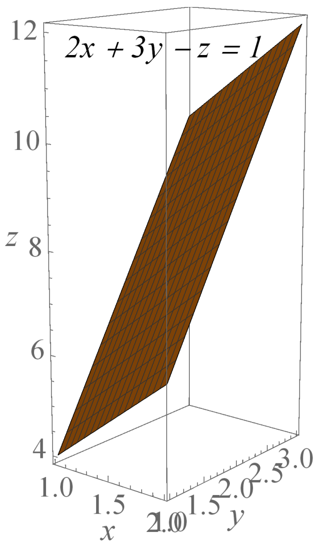
19. Find the surface area of the indicated surface.
(a) The portion of the cone $\mathbf{r}(u, v) = u\,\hat{\imath} + u\cos v\,\hat{\jmath} + u\sin v\,\hat{k}$ with $0 \le u \le 2$ and $0 \le v \le 2\pi$.



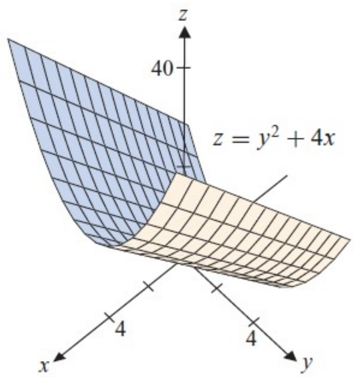
(b) The portion of the spiral ramp $\mathbf{r}(u,v) = u\cos v\,\hat{\imath} + u\sin v\,\hat{\jmath} + 2v\,\hat{k}$ with $0 \le u \le 3$ and $0 \le v \le 2\pi$.

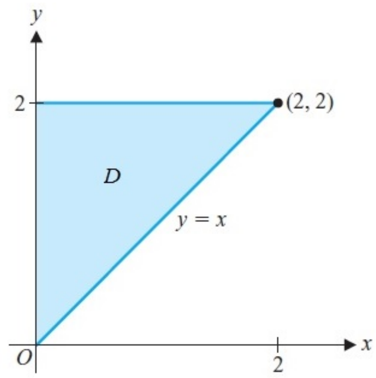


(c) The portion of the plane $2x+3y-z=1$ lying above the rectangular region D with vertices at $(1,1)$, $(1,3)$, $(2,1)$, and $(2,3)$.



(d) The portion of the surface $z = y^2 + 4x$ lying above the triangular region D with vertices at $(0, 0)$, $(0, 2)$ and $(2, 2)$.



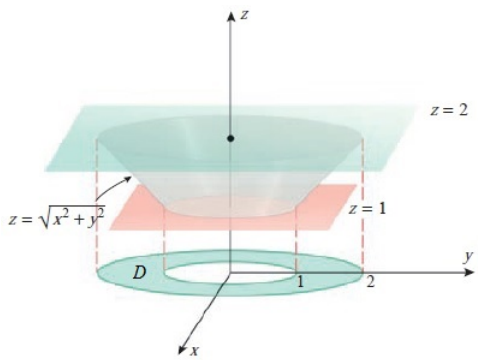


20. Evaluate the surface integrals over the surface S.

(a) $\displaystyle\iint_S y \,\mathrm{d}S$, S is the surface represented by $\mathbf{r}(u, v) = u\cos v\,\hat{\imath} + u\sin v\,\hat{\jmath} + v\,\hat{k}$, where $0 \le u \le 1$ and $0 \le v \le \pi$.

(b) $\displaystyle\iint_S \left(\frac{x-y}{\sqrt{2z+1}}\right) \mathrm{d}S$, S is the surface represented by $\mathbf{r}(u, v) = (u+v)\,\hat{\imath} + (u-v)\,\hat{\jmath} + (u^2+v^2)\,\hat{k}$, $0 \le u \le 1$, $0 \le v \le 2$.

(c) $\displaystyle\iint_S y^2 z^2 \,\mathrm{d}S$, S is the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$.



(d) $\displaystyle\iint_S (y^2 + 2yz) \,\mathrm{d}S$, S is the part of the plane $2x + y + 2z = 6$ in the first octant.