Equations of Motion for a Translating Compound Pendulum

CMU 15-462 (Fall 2015)

November 18, 2015

In this note we will derive the equations of motion for a compound pendulum being driven by external motion at the center of rotation. A *compound pendulum* is a pendulum consisting of a single rigid body rotating around a fixed axis. In contrast to the simple pendulum we studied in class, a compound pendulum can have an arbitrary distribution of mass, which means that it better models real-world objects. In the context of our animation tool, it means that the shape and size of each joint will have an effect on the dynamics of our character. The fact that the pendulum is moving means that we can have pendulums hanging from our character (e.g., feathers, scales, or other gadgets).

1 Lagrange's Equations of Motion

Let's first review our procedure for deriving equations of motion using Lagrangian mechanics. For any system described by a configuration q and velocity $\dot{q}$ in generalized coordinates, we can take the following approach:

- Write down the kinetic energy K.
- Write down the potential energy U.
- The Lagrangian is just the difference $\mathcal{L} := K - U$.
- The equations of motion can then be found by plugging $\mathcal{L}$ into the *Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}.$$

2 Basic Pendulum

Consider a pendulum of length L with mass m concentrated at its endpoint, whose configuration is completely determined by the angle θ made with the vertical, and whose velocity is the corresponding angular velocity $\dot{\theta}$.

2.1 Kinetic Energy

Recall that the *kinetic energy* of any system is one-half the mass times the velocity squared. Suppose we use y to denote the endpoint of the pendulum, and let $\dot{y}$ denote the corresponding *linear* velocity. Then the kinetic energy is

$$K = \tfrac{1}{2}m|\dot{y}|^2,$$

where $|\cdot|$ denotes the usual Euclidean norm. We would now like to express the kinetic energy in terms of our original angular velocity $\dot{\theta}$. What is the relationship between linear and angular velocity? To answer this question, we can first ask ourselves,

How far does the endpoint move if the pendulum makes one full revolution?

The answer, of course, is that it moves around the circumference of a circle of radius L, i.e., it moves along a linear distance $2\pi L$. So then what's the amount of angular distance traveled relative to the amount of linear distance? I.e., how far does y move for a given change in θ? Well, we move a distance $2\pi L$ for every 2π change in angle, which means the *amount of linear motion per angle* is

$$2\pi L/2\pi = L.$$

In other words, if we double the length of our pendulum, the endpoint will move twice as fast (for the same speed of rotation). This makes sense, because the circumference of the circle is *linear* in the length of the bar. Thus, the relationship between linear and angular speed is

$$|\dot{y}| = |L\dot{\theta}|.$$

Returning to our kinetic energy, that means we have

$$K = \tfrac{1}{2}m|L\dot{\theta}|^2 = \tfrac{1}{2}mL^2\dot{\theta}^2.$$

This formula may look familiar: the quantity $I := mL^2$ is sometimes referred to as the *moment of inertia* for a point mass, and if we use the variable $\omega := \dot{\theta}$ to denote the angular velocity, then we get the usual expression for the kinetic energy of a rotating point mass: $K = \frac{1}{2}I\omega^2$. (Note, however, that we are not making an "appeal to authority" here—we really just derived this formula for ourselves, from scratch! Not too hard, right?) This line of thinking will come in handy later, when we derive the equations of motion for a compound pendulum.

2.2 Potential Energy

There are many different sources of potential energy, but for our pendulum we will be concerned only with the influence of *gravity*. The gravitational potential energy for any point mass is

$$U = mgh,$$

where the constant g is the acceleration due to gravity, and h is the height relative to some fixed "ground," though it does not matter where this "ground" is—as we will see in a moment, we are going to take the derivative of this quantity, hence any constant we add or subtract will make absolutely no difference.

So, what is the height h for the endpoint of our pendulum? If we assume that the center of rotation is at the origin ($h = 0$), then the height of the endpoint is

$$h = -L\cos(\theta).$$

Let's do some quick sanity checks to make sure we got this expression right: if $\theta = 0$, i.e., the pendulum is hanging straight down, then $-L\cos(\theta) = -L\cos(0) = -L(1) - L$, i.e., the endpoint is at a distance L below the ground. Great. And if $\theta = \pi$, i.e., the pendulum is pointing straight up, then $-L\cos(\theta) = -L\cos(\pi) = -L(-1) = L$, i.e., the endpoint is at a distance L above the ground. Finally, if $\theta = \pi/2$, i.e., the pendulum is sticking out to the right, then $-L\cos(\theta) = -L\cos(\pi/2) = -L(0) = 0$, i.e., the endpoint is also sitting right along the ground. Perfect. So, we can write our gravitational potential energy as

$$U = mgh = -mgL\cos(\theta).$$

This line of thinking will also come in handy when we consider a general compound pendulum.

2.3 Lagrangian

Using the kinetic and potential energies derived above, we find that our Lagrangian can be expressed explicitly in terms of the angle θ and the angular velocity $\dot{\theta}$ via

$$\mathcal{L} = K - U = \tfrac{1}{2}mL^2\dot{\theta}^2 + mgL\cos(\theta).$$

The only other variables in this equation are constants: the mass m, the length L, and the gravitational acceleration g. It is important to get our Lagrangian in this form, so that we can easily differentiate it once we plug it into the Euler-Lagrange equations.

It is perhaps worth stopping here to note a remarkable fact: the entire dynamical behavior of our pendulum has been boiled down into a single scalar function, the Lagrangian, which takes as input the current position and velocity, and splits out a number. In fact, almost any conservative system (i.e., system that does not gain or lose energy) can be formulated this way: as a scalar function on positions and velocities, and one can then very easily move to the corresponding equations of motion via the Euler-Lagrange equations. Moreover, by analyzing *symmetries* of the Lagrangian (i.e., changes of coordinates that do not change its value), one can easily deduce properties of the mechanical system like conservation of energy, conservation of momentum, etc. On the whole, these questions are the focus of *Lagrangian mechanics*, and can provide a rather powerful perspective when developing numerical simulation algorithms both from a systems perspective (i.e., many different systems can be formally encoded and manipulated using one unified representation), and from a numerical analysis point of view (in particular, it becomes reasonably easy to develop integrators that conserve energy and momentum, as we saw in class with *symplectic Euler*).

2.4 Euler-Lagrange Equations

Now that we have our Lagrangian $\mathcal{L}$, we can derive the equations of motion via a completely mechanical procedure, i.e., by taking derivatives with respect to the configuration, velocity, and time.[1] The derivative with respect to the configuration θ is

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\tfrac{1}{2}mL^2\dot{\theta}^2 + mgL\cos(\theta)\right) = 0 + mgL\frac{\partial}{\partial \theta}\cos(\theta) = -mgL\sin(\theta).$$

The derivative with respect to the velocity $\dot{\theta}$ is

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\tfrac{1}{2}mL^2\dot{\theta}^2 + mgL\cos(\theta)\right) = \tfrac{1}{2}mL^2\frac{\partial}{\partial \dot{\theta}}(\dot{\theta}^2) = mL^2\dot{\theta},$$

and the time derivative of this expression is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\frac{d}{dt}\dot{\theta} = mL^2\ddot{\theta}.$$

Putting it all together, our Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}.$$

become

$$mL^2\ddot{\theta} = -mgL\sin(\theta),$$

or equivalently,

$$\boxed{\ddot{\theta} = -g\sin(\theta)/L.}$$

(This is the same equation we derived in class.)

[1] Here, for instance, one could imagine designing an animation system that allows a user to simply describe a purely in terms of various potentials, automatically performing symbolic differentiation to get the equations of motion.

3 Moving Pendulum

Suppose now that our pendulum is no longer rotating around a fixed point x, but that this point itself is translating, i.e., x is a function of time $x(t)$, and has an associated velocity $\dot{x}(t) := \frac{d}{dt}x(t)$. We will still let y denote the position of the rotating endpoint *relative to the current center of rotation* x, so that at all times $|x-y| = L$. This system is a bit more fun than the previous one, because we can get the pendulum to move by "shaking it around." In our animation tool, for instance, we can attach pendulums to joints of our character. How exactly do the equations of motion change? Let's find out.

4 Kinetic Energy

As before, kinetic energy is one-half the mass times the (linear) velocity squared. If the whole pendulum is moving, *and* its endpoint is swinging, what's the velocity v at the endpoint? It's nothing more than

$$v = \dot{x} + \dot{y}.$$

In other words, it's the sum of the motion at the endpoint with the overall motion. Then kinetic energy then becomes

$$K = \tfrac{1}{2}m|v|^2 = \tfrac{1}{2}m|\dot{x}+\dot{y}|^2.$$

Now here's a very useful fact: the squared norm of a vector is the same as its inner product with itself. I.e., if u is any vector, then

$$|u|^2 = \langle u, u\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product. (Do you believe it? If not, try writing this relationship out in components $u = (u_1, u_2)$, using the standard Euclidean dot product.) Therefore, we can expand the final term in the kinetic energy as

$$|\dot{x}+\dot{y}|^2 = \langle \dot{x}+\dot{y}, \dot{x}+\dot{y}\rangle = \langle \dot{x},\dot{x}\rangle + \langle \dot{x},\dot{y}\rangle + \langle \dot{y},\dot{x}\rangle + \langle \dot{y},\dot{y}\rangle = |\dot{x}|^2 + 2\langle \dot{x},\dot{y}\rangle + |\dot{y}|^2.$$

(Can you explain to yourself why the inner product of two vectors is symmetric, i.e., why we're allowed to swap the order of $\dot{x}$ and $\dot{y}$? You could write it out in coordinates; you could also just think about what an inner product means geometrically, in terms of the lengths and angles of the vectors. Likewise, why does the inner product distribute over addition? Can you explain this fact geometrically, or can you only do by expanding the expression into individual components?)

From here, our expression for kinetic energy becomes

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(|\dot{x}|^2 + 2\langle \dot{x},\dot{y}\rangle + |\dot{y}|^2\right) = \tfrac{1}{2}m\left(|\dot{x}|^2 + 2\langle \dot{x},\dot{y}\rangle + L^2\dot{\theta}^2\right)$$

where in the final step we used a result from our analysis of the stationary pendulum ($|\dot{y}|^2 = L^2\dot{\theta}^2$).

Earlier, we investigated the relationship between linear and angular *speed*, and discovered that $|\dot{y}| = L|\dot{\theta}|$, i.e., that linear speed is just angular speed times the length of the bar (or equivalently, the radius of the circle). Here, it will also be useful to understand the relationship between linear and angular *velocity*, i.e., in what *direction* does y move as a result of changing θ? Geometrically, the relationship is actually pretty clear: y is moving around in a circle, so the velocity $\dot{y}$ must be *tangent* to the circle; otherwise, y would necessarily fly off of the circle! And for any point y on a circle, the tangent direction is just the *orthogonal* direction, which we can denote by $y^\perp$; here we have to be careful about orientation: by convention, we will say that $y^\perp$ denotes a counter-clockwise rotation by a quarter-turn ($\pi/2$ radians, or 90 degrees). Hence, the vector

$$\frac{y^\perp}{|y^\perp|} = \frac{y^\perp}{|y|} = \frac{y^\perp}{L}$$

is a *unit* vector tangent to the circle, i.e., it points in the *direction* of the linear velocity, but may not have the right *magnitude*. Of course, we already know the magnitude: it's the speed $|\dot{y}| = L|\dot{\theta}|$. So, being careful to note that $y^\perp$ points in the counter-clockwise direction, we get

$$\dot{y} = L\dot{\theta}\frac{y^\perp}{|y^\perp|} = L\dot{\theta}\frac{y^\perp}{L} = \dot{\theta}y^\perp.$$

Let's just summarize that, because it turned out to be so darn simple:

$$\dot{y} = \dot{\theta}y^\perp,$$

the linear velocity vector is just the position rotated by 90 degrees, times the angular velocity. Great[2].

Plugging this relationship back into our expression for kinetic energy, we get

$$K = \tfrac{1}{2}m\left(|\dot{x}|^2 + 2L\dot{\theta}\langle\dot{x}, y^\perp\rangle + L^2\dot{\theta}^2\right) = \frac{1}{2}m\left(|\dot{x}|^2 + 2L\dot{\theta}\langle\dot{x}, (\cos(\theta), \sin(\theta))\rangle + L^2\dot{\theta}^2\right).$$

The only reason for writing out the final expression in terms of $\cos(\theta)$ and $\sin(\theta)$ is that it makes it extremely clear where and how K depends on the configuration θ and the velocity $\dot{\theta}$. But if you are geometrically minded (and careful), you can often avoid writing everything out long-hand like this. Alternatively, you might use the complex expressions $y = Le^{\imath\theta}$ and $y^\perp = \imath y$, which are a bit more compact. In summary, we have

$$K = \tfrac{1}{2}m\left(|\dot{x}|^2 + 2\dot{\theta}\langle\dot{x}, y^\perp\rangle + L^2\dot{\theta}^2\right),$$

or equivalently,

$$K = \frac{1}{2}m\left(|\dot{x}|^2 + 2L\dot{\theta}\langle\dot{x}, (\cos(\theta), \sin(\theta))\rangle + L^2\dot{\theta}^2\right),$$

or equivalently,

$$K = \frac{1}{2}m\left(|\dot{x}|^2 + 2L\dot{\theta}\langle\dot{x}, e^{\imath(\theta+\pi/2)})\rangle + L^2\dot{\theta}^2\right),$$

depending on how you like to work with vectors in the plane (as vectors, in terms of components, or using complex numbers, respectively). We will keep working with the first expression, which is perhaps the most succinct.

4.1 Potential Energy

How about the potential energy? Our height above the plane is no longer determined just by the angle θ, but also by the height associated with the center of rotation x, which we will denote as simply $h(x)$. (Again, if you prefer coordinates you could say that $x = (x_1, x_2)$, in which case $h(x) = x_2$. But in general, we'll encourage you to do your calculations without resorting to coordinates wherever possible.) Overall, then, the gravitational potential energy is

$$U = mgh = mg(h(x) + h(y)) = mg(h(x) - L\cos(\theta)).$$

This time, really not much different from the stationary case: the potential energy just increases by a term $mgh(x)$, describing how far the center of the rotation is "off the ground." As we'll see in a moment, this extra term won't affect our equations of motion at all because when we differentiate the Lagrangian in terms of the configuration θ and the velocity $\dot{\theta}$, this term will disappear entirely.

[2] If you really like working with explicit coordinates, you can also work this relationship out by writing $y = L(\cos(\theta), \sin(\theta))$ and taking a time derivative. Not surprisingly, you get the same result: $\frac{d}{dt}y = L\dot{\theta}(-\sin(\theta), \cos(\theta)) = \dot{\theta}y^\perp$. You could also write it out in terms of complex numbers: if $y = Le^{\imath\theta}$, then $\frac{d}{dt}y = L\dot{\theta}\imath e^{\imath\theta} = \dot{\theta}y^\perp$, since the complex unit $\imath$ is just a rotation by a quarter-turn in the counter-clockwise direction. As you can see, there are often many different ways to do it—though if you can reason geometrically you can often avoid grinding through a lot of extra algebra, trigonometry, etc.

4.2 Lagrangian

Taking the difference between kinetic and potential energy yields the Lagrangian for our moving pendulum

$$\mathcal{L} = K - U = \tfrac{1}{2}m\left(|\dot{x}|^2 + 2\dot{\theta}\langle\dot{x}, y^\perp\rangle + L^2\dot{\theta}^2\right) - mg(h(x) - L\cos(\theta)).$$

A bit more complicated than for our stationary pendulum, but not unmanageable. Let's see how these changes to the Lagrangian affect our equations of motion.

4.3 Euler-Lagrange Equations

What's the derivative of our Lagrangian with respect to the configuration θ? Well, $\mathcal{L}$ has two terms that depend on θ: one coming from the kinetic energy ($m\dot{\theta}\langle\dot{x}, y^\perp\rangle$), and one coming from the potential energy ($mgL\cos(\theta)$). Remember that $y^\perp$ depends on θ: it is the position of y, rotated by 90 degrees. And what's the derivative of $y^\perp$ with respect to θ? No big surprise: we rotate it again by 90 degrees, yielding

$$\tfrac{\partial}{\partial\theta} y^\perp = -y.$$

The derivative of $\mathcal{L}$ with respect to θ is therefore

$$\frac{\partial\mathcal{L}}{\partial\theta} = -m(\dot{\theta}\langle\dot{x}, y\rangle + gL\sin(\theta)).$$

Likewise, there are two terms that depend on $\dot{\theta}$, this time both from the kinetic energy: $m\dot{\theta}\langle\dot{x}, y^\perp\rangle + \frac{1}{2}mL^2\dot{\theta}^2$. The derivative with respect to $\dot{\theta}$ is therefore

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m\langle\dot{x}, y^\perp\rangle + mL^2\dot{\theta},$$

To compute the time derivative, we should remember that the inner product follows a product rule (which arises from the usual product rule on each scalar component), i.e., for any two functions $f(t), g(t)$ that depend on time,

$$\tfrac{d}{dt}\langle f, g\rangle = \langle\dot{f}, g\rangle + \langle f, \dot{g}\rangle.$$

Hence, noting that $\frac{d}{dt}y^\perp = -\dot{\theta}y$, we get

$$\tfrac{d}{dt}\langle\dot{x}, y^\perp\rangle = \langle\ddot{x}, y^\perp\rangle - \dot{\theta}\langle\dot{x}, y\rangle,$$

which means that

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m\left(\langle\ddot{x}, y^\perp\rangle - \dot{\theta}\langle\dot{x}, y\rangle + L^2\ddot{\theta}\right)$$

Putting it all together, we have

$$\begin{array}{rrcl} & \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} & = & \frac{\partial\mathcal{L}}{\partial\theta} \\ \iff & m\left(\langle\ddot{x}, y^\perp\rangle - \dot{\theta}\langle\dot{x}, y\rangle + L^2\ddot{\theta}\right) & = & -m(\dot{\theta}\langle\dot{x}, y\rangle + gL\sin(\theta)) \\ \iff & \langle\ddot{x}, y^\perp\rangle + L^2\ddot{\theta} & = & -gL\sin(\theta), \end{array}$$

or equivalently,

$$\boxed{\ddot{\theta} = -\langle\ddot{x}, y^\perp\rangle/L^2 - g\sin(\theta)/L.}$$

Here we can do a sanity check to make sure we got the right answer. First, if the pendulum isn't moving then we have no acceleration ($\ddot{x} = 0$) and recover the same equations of motion as we did for the stationary pendulum. Second, we can check the units on the new term: $\ddot{x}$ has units of meters per second squared; $y^\perp$ has units of meters, and L^2 has units of meters squared. Overall, then we get units of $1/s^2$, which is exactly what we should have for angular acceleration since angles are unitless.

5 Compound Pendulum

So far our pendulum takes into account the effects of gravity and motion, but we do not have a particularly realistic model of mass or inertia, i.e., we haven't taken into account the specific shape or size of the swinging object. For the moment, let's again assume that the pendulum is just swinging around the origin (rather than a moving point $x(t)$). Let $\Omega \subset \mathbb{R}^2$ denote the shape of the swinging body, and let $\rho : \Omega \to \mathbb{R}$ denote its mass density. Then the total mass m of the body is just the integral of its mass density:

$$m = \int_\Omega \rho(p)\, dp.$$

The *center of mass* c is the mass-weighted average *position* of the body:

$$c = \tfrac{1}{m} \int_\Omega \rho(p) p\, dp.$$

Finally, the *moment of inertia* I is the total moment of inertia of all point masses comprising the body (which we considered for our original pendulum equation):

$$I = \int_\Omega \rho(p) L(p)^2\, dp = \int_\Omega \rho(p)|p|^2\, dp,$$

i.e., the "length" L from our point mass moment of inertia gets replaced by the distance to the origin $|p|$. In general, the moment of inertia for a general rigid body can be very tricky to compute—but at the end of the day, it still boils down to a single constant (in 2D). For your animation tool, we have already taken the trouble to compute the moments of inertia for basic shapes like rectangles, ellipses, polygons, etc., assuming a constant mass density $\rho = 1$. Note that you do *not* have to derive these expressions yourself! On the other hand, these formulas are not too hard to derive: just take the expression above and plug in the specific geometry of your shape. If the shape is simple enough, you can usually find a nice closed-form expression. Otherwise, you might have to approximate the moment of inertia using numerical quadrature (as we discussed in our lecture on Monte Carlo integration).

5.1 Kinetic Energy

For a compound pendulum, we can get the total kinetic energy by integrating the kinetic energy of each point mass over the entire body:

$$K = \tfrac{1}{2} \int_\Omega \rho |\dot{y}|^2\, dp = \tfrac{1}{2} \int_\Omega \rho |p|^2 \dot{\theta}^2\, dp.$$

Of course, since the body is *rigid*, the angular velocity $\dot{\theta}$ is the same at every point and we can simply pull it out of the integral:

$$K = \tfrac{1}{2}\dot{\theta}^2 \int_\Omega \rho |p|^2\, dp = \tfrac{1}{2} I \dot{\theta}^2.$$

(Once again, the famous "$K = \frac{1}{2} I \omega^2$" appears. Starting to see where this formula comes from?)

5.2 Potential Energy

The gravitational potential energy is likewise the integral of the gravitational potential of each particle:

$$U = \int_\Omega \rho g h(p)\, dp = g \int_\Omega \rho h(p)\, dp.$$

Notice, then, that U is actually just the height h of the center of mass, up to a constant factor mg:

$$U = mgh(c).$$

Also, since rotating the body by an angle θ simply rotates its center of mass by θ (why?), the gravitational potential of the rotated body is just g times the height of the rotated center:

$$U = mgh(e^{\imath\theta}c).$$

5.3 Lagrangian

The Lagrangian for our compound pendulum is then

$$\mathcal{L} = K - U = \tfrac{1}{2}I\dot{\theta}^2 - mgh(e^{\imath\theta}c).$$

Compare this expression to our Lagrangian for the simple pendulum. Do they agree? (In some sense, this general expression is even simpler than our expression for the "simple" pendulum!)

5.4 Euler-Lagrange Equations

The derivative of the Lagrangian with respect to our configuration variable θ is

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgh(\imath e^{\imath\theta}c).$$

What exactly did we do here? Ignoring the constants mg, we basically said "the derivative of the height is the height of the derivative." Then noted that the derivative of a rotation with respect to the rotation angle is the direction orthogonal to the vector being rotated, times the length of the vector being rotated. But that's just the same as rotating the vector by 90 degrees, which we can do by multiplying by the imaginary unit $\imath$. (If this complex notation is uncomfortable for you, try writing it out again your favorite way: either using components, or using the "$\perp$" notation.) The derivative of the Lagrangian with respect to $\dot{\theta}$ is

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\dot{\theta},$$

whose time derivative is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\ddot{\theta}.$$

Hence, the Euler-Lagrange equations are

$$\boxed{\ddot{\theta} = -mgh(e^{\imath(\theta+\pi/2)}c)/I,}$$

where again $e^{\imath(\theta+\pi/2)}$ means "rotate the center of mass by $\theta + \pi/2$ radians in the counter-clockwise direction," and $h(\cdot)$ just extracts the vertical coordinate of the rotated vector (the "height").

We can do a sanity check here by making sure this result agrees with our result for the simple pendulum. There, the center of mass was at $c = L(\sin(\theta), -\cos(\theta))$, and a rotation by 90 degrees yields $L(\cos(\theta), \sin(\theta))$. The moment of inertia was $I = mL^2$. Hence, our general Euler-Lagrange equations simplifies to just $\ddot{\theta} = -g\sin(\theta)/L$, as before.

One interesting thing to consider here is an object that does not have mirror symmetry across the vertical axis. In this case the original ("un-rotated") center of mass will not have a horizontal component of zero, and as a result the height $h(e^{\imath(\theta+\pi/2)}c)$ also will not be zero. In other words, the object will accelerate even when $\theta = 0$. But this makes perfect sense: if the pendulum is heavier on one side than the other, we expect it to swing a little bit before coming to rest.

6 Moving Compound Pendulum

The whole enchilada would be to finally write the equations of motion for a *moving* compound pendulum, combining the ideas and techniques above. This derivation can certainly be done, but is strictly harder than the previous two; in particular, it demands clever application of *Stokes' theorem* to resolve a cross term in the computation of kinetic energy—ambitious students who are well-versed in vector calculus are invited to attack this derivation as extra credit on Assignment 4. Otherwise, in your assignment you may use the approximate equations of motion

$$\boxed{\ddot{\theta} = -\langle \ddot{x}, y^{\perp} \rangle / L^2 - mgh(e^{\imath(\theta+\pi/2)}c)/I.}$$

Here the first term comes from our derivation for a simple moving pendulum; the second term comes from our analysis of a stationary compound pendulum. The only approximation we're making here is that the acceleration due to "shaking" the object will not be quite right, because it doesn't properly account for the shape and size of the object (instead treating it as a point mass). This approximation is unlikely to be noticed in cartoon animation, but is unacceptable for engineering applications—in those situations, one should at very least use a computational differentiation technique (like numerical, automatic, or symbolic differentiation) to get results that quantitatively approximate the correct answer. These techniques are particularly important in situations where derivatives cannot be practically computed by hand.