Uninsured Idiosyncratic Risk and Aggregate Saving
S. Rao Aiyagari (1994) QJE

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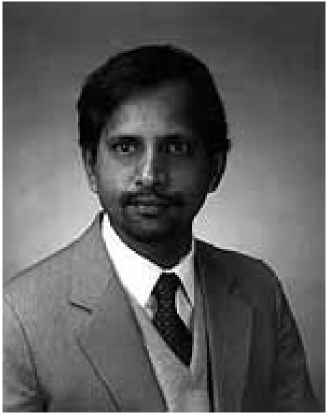


Sudhakar Rao Aiyagari (Nov. 9, 1951 – May 20, 1997)

Between 1986 and 1996, Aiyagari was a research economist and officer at the Minneapolis Fed. Before joining the Fed, he taught economics at the University of Wisconsin, Madison, and Carnegie-Mellon University, Pittsburgh. 1996 fall he fulfilled a dream of returning to academia when he accepted a position as a full professor at the University of Rochester, New York.

PhD in economics, the University of Minnesota, 1981
MA in economics and physics from Jawaharlal Nehru University in New Delhi and the Indian Institute of Technology, Delhi, India.

Introduction

- Goal 1: Provide a framework where aggregate outcomes arise from idiosyncratic shocks and individual saving behavior.
- Goal 2: Quantify how much precautionary saving contributes to aggregate saving.
- Key features: **precautionary motives**, **incomplete markets**, **borrowing constraints**.
- Foundation of **Heterogeneous Agent Models**.

Main features

Since Aiyagari (1994) model is a general equilibrium version of Bewley's model, the model is also called Bewley model or Bewley-Aiyagari model. The main features of the basic Aiyagari model:

1. There are mass of agents. Each agent is atomless and thus a price taker.
2. Agents are ex-ante homogeneous but ex-post heterogeneous, depending on the history of realizations of idiosyncratic shocks (Aiyagari's model has only shocks to labor income).
3. The set of assets which are traded is exogenously determined. In particular, there is only one asset (risk-free asset or capital) allowed to be traded.
4. Therefore, agents cannot fully insure away their idiosyncratic risks. They can only self-insure by saving.
5. Only the steady state equilibrium is studied.
6. Prices (wage and interest rate) are determined competitively.

Model Overview

- Infinitely-lived agents facing i.i.d. or persistent labor income shocks.
- Standard neoclassical production: Cobb-Douglas $Y = K^{\alpha} L^{1-\alpha}$.
- Incomplete markets: agents cannot insure idiosyncratic shocks.
- Borrowing constraint $a_t \geq -\bar{b}$.
- General equilibrium: interest rate r clears the asset market.

The Income Fluctuation Problem

Each agent solves:

$$\max_{\{c_t, a_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \quad U(c) = \frac{c^{1-\mu}}{1-\mu}, \mu > 0$$

subject to:

$$\begin{gathered} c_t + a_{t+1} = w\ell_t + (1+r)a_t, \\ a_t \geq -\bar{b}, \quad c_t \geq 0 \end{gathered}$$

- One asset a_t, earns interest rate r.
- Stochastic labor income $l_t \in \{l_{\text{min}}, l_{\text{max}}\}$, i.i.d. distribution $dF(l)$.
- Idiosyncratic income risk is *not* shared. Markets are incomplete.

- **Fixed limit**: $a_t \geq -b$
- **Borrowing limit** :

$$a_t \geq -\phi, \qquad \phi = \min\left\{b, \frac{wl_{\min}}{r}\right\} (\text{if } r \leq 0,\ \phi = b)$$

- Sequence problem:

$$\max_{\{c_{it}, a_{it}\}} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

$$\text{s.t.} \quad c_{it} + a_{it} = wl_{it} + (1+r)a_{i,t-1}$$

$$a_{it} \geq -\phi$$

FOC:

$$\frac{1}{c_{it}} = \lambda_{it}$$
$$\lambda_{it} = \beta(1+r)\mathbb{E}_t\lambda_{i,t+1} + \mu_{it}.$$

Note: even if $\mu_{it} = 0$ today, if $\mu_{it} > 0$ is some future state with positive probability, then this will affect current consumption: precautionary savings.

Implies that a stationary equilibrium needs $(1+r)\beta < 1$. Why? Integrate over all individuals to obtain

$$\bar{\lambda} = \beta(1+r)\bar{\lambda} + \bar{\mu},$$

where $\bar{\lambda}$ and $\bar{\mu}$ are the population averages (time-invariant). So long as a positive mass of individuals is constrained, $\beta(1+r) < 1$.
Otherwise individuals would accumulate infinitely many assets to perfectly insure against idiosyncratic risk.
(Can formally prove this using that a non-negative supermartingale u_c converges almost surely to a finite limit.)

Value function

To write down the recursive problem, define cash-on-hand as

$$z_{it} = wl_{it} + (1+r)a_{i,t-1},$$

$$V(z) = \max_{a \in [-\phi, z]} \left\{ u(z-a) + \beta \int V[wl' + (1+r)a] \, dF(l') \right\}.$$

Solve using your favorite value-function iteration (alternative: policy-function iteration).

Characteristics of the solution:

- Value function $V(z)$ is continuous and strictly concave (see Stokey–Lucas).
- Consumption is strictly increasing in z: From the envelope condition,

$$V_z(z) = u_c[c(z)] \quad \Rightarrow \quad c_z(z) = \frac{V_{zz}(z)}{u_{cc}[c(z)]} > 0.$$

Solution

Whenever the borrowing constraint does not bind, then $a_z(z) > 0$:

$$u_c[c(z)] = \beta(1+r)\int V_z[(1+r)a(z) + wl']\,dF(l')$$

$$\Rightarrow u_{cc}[c(z)]c_z(z) = \beta(1+r)\int V_{zz}(z')\,dF(l')\,a_z(z)$$

$$\Rightarrow \ a_z(z) = \frac{u_{cc}[c(z)]c_z(z)}{\beta(1+r)\int V_{zz}(z')\,dF(l')} > 0.$$

There exists $\bar{z}$, s.t. $a(z) = -\phi$ for all $z \leq \bar{z}$, and $a_z(z) = 0$. The existence follows from optimality and a finite borrowing limit. If you are at the borrowing constraint, then MU is high, so rather consume than save.

- Next, suppose that $a(\tilde{z}) > -\phi$ for some $\tilde{z} < \bar{z}$. We get

$$u_c[c(\bar{z})] \geq \beta(1+r)\int V_z[(1+r)a(\bar{z}) + wl']dF(l')$$

$$u_c[c(\tilde{z})] = \beta(1+r)\int V_z[(1+r)a(\tilde{z}) + wl']dF(l')$$

$$\Rightarrow u_c[c(\bar{z})] \geq \beta(1+r)\int V_z[-(1+r)\phi + wl']dF(l') > \beta(1+r)\int V_z[(1+r)a(\tilde{z}) + wl']dF(l')$$

$$= u_c[c(\tilde{z})]$$

$$\Rightarrow c(\bar{z}) < c(\tilde{z})$$

which is a contradiction.

Cash-on-hand function in Aiyaga

Finally, from the budget constraint,

$$c_z(z) + a_z(z) = 1,$$

so the MPC is 1 when constrained and less than 1 otherwise.

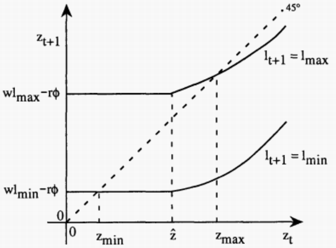


FIGURE Ib
Evolution of Total Resources

- Stationary distribution $\Phi(z)$ will only have positive mass over $[z_{\min}, z_{\max}]$.
- At $z_{\max} = wl_{\max} + (1 + r)a_{\max}$, individuals choose $a' = a_{\max}$ and $c_{\max} = wl_{\max} + ra_{\max}$. The MPC out of wealth is r, similar to a complete-markets model. In that sense, individuals with high wealth are fully self-insured against hitting the borrowing constraint.

Technology

- Output is produced using Cobb–Douglas production,

$$y_t = k_{t-1}^{\alpha} l_t^{1-\alpha}.$$

- Capital follows a standard accumulation equation

$$k_t = k_{t-1}(1-\delta) + i_t.$$

- Firms are perfectly competitive, so

$$r = \alpha K^{\alpha-1} L^{1-\alpha} - \delta \Rightarrow K = \left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} L$$

$$w = (1-\alpha) K^{\alpha} L^{1-\alpha} = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(r+\delta)^{\frac{\alpha}{\alpha-1}} L$$

- Normalize total labor supply to

$$l = p l_{\min} + (1-p) l_{\max} = 1$$

and use stationarity.

Steady-State Equilibrium

By Walras' law, we can forget about goods market and we only need to check input market clearing. Define asset market clearing condition:

$$k = \int a(z, r, w, \phi)\, d\Phi(z),$$

where $d\Phi$ is the distribution of assets.

- A steady-state equilibrium consists of policy function $a(z, r, w, \phi)$, a steady-state distribution $d\Phi(z)$, a capital stock k, and prices r, w such that
  1. The policy function is optimal given w, r.
  2. The steady-state distribution is consistent with the policy functions.
  3. Capital, labor, and asset markets clear.
- Important property: Aggregates are deterministic, but individual allocations are not.

General Equilibrium

- In equilibrium:

$$r = f_K(k) - \delta$$
$$w = f_L(k)$$

- $r = f_1(k,1) - \delta \Longrightarrow K(r)$, where $lim_{r \to -\delta} K(r) = \infty$ and $lim_{r \to \infty} K(r) = 0$
  $w = f_2(k,1) \Longrightarrow \omega(r)$, where $lim_{r \to -\delta} \omega(r) = \infty$ and $lim_{r \to \infty} \omega(r) = 0$
- Market clearing:

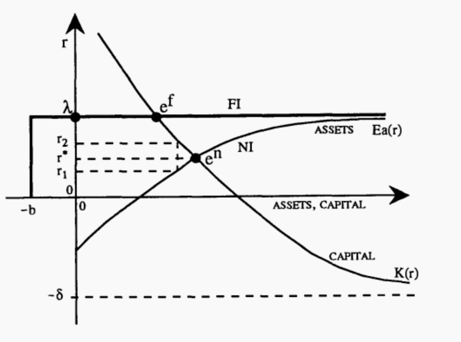
$$\bar{k} = \mathbb{E}[a] = E_a(r)$$

- Steady state r^* satisfies:

$$E_a(r^*) = K(r^*)$$

Characteristics of the Stationary Equilibrium

- Market-clearing picture:
  - Total asset demand converges to infinity as $1 + r \to \beta^{-1}$.
  - Converges to borrowing limit for sufficiently low (negative) interest rate.
  - Downward-sloping factor price r, bounded by $-\delta$.
- (Equilibrium in Aiyagari).



- Inefficiently high capital accumulation relative to full-insurance, because consumers want to self-insure against idiosyncratic income risk.
- General equilibrium works against this: the equilibrium interest rate falls as more capital gets accumulated, making self-insurance more costly.

Solving for the Stationary Equilibrium

1. For given k_0, get r_0; use value-function iteration and compute the distribution of agents $\bar{\Phi}(a_i)$. (Need to approximate the distribution fairly well to avoid Euler-equation errors that accumulate over time.)
2. Market clearing implies value for k_1.
3. Update guess for

$$r_1 = \omega r_0 + (1-\omega)\left(\alpha k_1^{\alpha-1} - \delta\right), \qquad \omega \in (0,1).$$

Comparative Statics on Stationary Equilibrium

- Reduction in borrowing limit. (Figure 2: asset-demand curve shifts out.)
- Increase in income risk (mean-preserving spread). (Figure 2: asset-demand curve shifts out.)
- Provides micro-foundation for changes in the natural rate of interest. Additional ingredients (e.g. sticky prices) then give rise to a recession.
- Key advantages of this model:
    1. Can discipline model using micro-data on income and wealth distribution.
    2. Meaningful welfare differences.

Quantitative Results (Table II)

- Saving rate increases with μ, σ, and ρ.
- With $\mu = 3$, $\sigma = 0.4$, $\rho = 0.9$: saving rate **increased by up to 14 percentage points**.
- For moderate parameters, increase is **only 0–3 pp**.
- Equilibrium interest rate is **below time preference rate** $\rho < \delta$.

TABLE II

A. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.2$)

$\rho \backslash \mu$	1	3	5
0	4.1666/23.67	4.1456/23.71	4.0858/23.83
0.3	4.1365/23.73	4.0432/23.91	3.9054/24.19
0.6	4.0912/23.82	3.8767/24.25	3.5857/24.86
0.9	3.9305/24.14	3.2903/25.51	2.5260/27.36

B. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.4$)

$\rho \backslash \mu$	1	3	5
0	4.0649/23.87	3.7816/24.44	3.4177/25.22
0.3	3.9554/24.09	3.4188/25.22	2.8032/26.66
0.6	3.7567/24.50	2.7835/26.71	1.8070/29.37
0.9	3.3054/25.47	1.2894/31.00	−0.3456/37.63

Welfare and Inequality Implications

- Asset trading significantly reduces consumption volatility:

$$\text{Welfare gain} \approx 14\% \text{ of consumption}$$

- Inequality:
  - Gini (model): Income $\approx 0.12 - 0.4$, Wealth ≈ 0.3
  - Gini (US data): Income ≈ 0.4, Wealth ≈ 0.8
- Distribution: Wealth more skewed than income, which is more skewed than consumption.

Extensions and Interpretations

- Model can be reinterpreted as:
  - Debt neutrality test (Ricardian equivalence)
  - Optimal quantity of money (Bewley-type)
- Results depend crucially on constraint form (fixed vs present value)
- In complete markets: no precautionary saving
- In incomplete markets: aggregate saving **necessarily higher**

Conclusion

- Uninsured idiosyncratic risk + borrowing constraints $\Rightarrow$ higher aggregate saving.
- Precautionary saving modest under moderate parameters.
- Model better matches empirical consumption and wealth dispersion.
- Cannot match extreme wealth inequality.
- Opens path for richer heterogeneity and aggregate shock models.

Discussion

Contribution:

- production economy with capital and labor(heterogeneous agents version of stochastic growth model)
- introduced heterogeneous agents and imperfect market assumptions.

Limition:

- no aggregate risk (which may be more difficult to solve)
- only at the steady state and for the idiosyncratic shocks,without dynamic process

Reference

S. Rao Aiyagari (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics*, Vol. 109, No. 3, pp. 659–684.