The shallow water equations for two dimensional waves in three dimensional shallow water involve variables $h(x, y, t)$ (the height of the water surface over the bottom), $v_x(x, y, t)$ (the mean x velocity, depth averaged), and $v_y(x, y, t)$ (the y velocity, depth averaged). Depth averaged means the average of the three dimensional velocity V over the "water column":

$$v_x(x, y, t) = \frac{1}{\text{depth}} \int_{z=bottom}^{z=h(x,y,t)} V_x(x, y, z, t)\, dz$$

The water is "shallow" when the x, y wavelength is large compared to the depth. The more "shallow" the water, the more accurate the equations. Tsunamis in the deep ocean are an example. The wavelength is several tens of kilometers and the depth is a few kilometers. The equations given below assume that the fluid density is $\rho = 1$, which is nearly true for sea water in CGS units. g is the gravitational acceleration constant, which is about 9.8 m/sec^2.

$$\partial_t\, h + \partial_x\,(hv_x) + \partial_y\,(hv_y) = 0$$
$$\partial_t\,(hv_x) + \partial_x\left(hv_x^2 + \frac{1}{2}gh^2\right) + \partial_y\,(hv_xv_y) = 0$$
$$\partial_t\,(hv_y) + \partial_x\,(hv_xv_y) + \partial_y\left(hv_y^2 + \frac{1}{2}gh^2\right) = 0$$

The *primitive variables* are the depth and velocity: h, v_x, and v_y. The *conserved variables* are depth (mass, because water is not compressed in this approximation) and momentum variables $m_x = hv_x$ and $m_y = hv_y$. These equations resemble the simple compressible gas equations given in class, but with "pressure" given by $\frac{1}{2}h^2$, and h taking the place of density.

The $h \sim$ density is supported by the idea that the amount of water above a two dimensional region of ocean, A, is (with unit density, as before)

$$\int_A h(x, y, t)\, dxdy$$

In conserved variables, the equations are

$$\partial_t\, h + \partial_x\,(m_x) + \partial_y\,(m_y) = 0$$
$$\partial_t\,(m_x) + \partial_x\left(h^{-1}m_x^2 + \frac{1}{2}gh^2\right) + \partial_y\,(h^{-1}m_xm_y) = 0$$
$$\partial_t\,(m_y) + \partial_x\,(h^{-1}m_xm_y) + \partial_y\left(h^{-1}m_y^2 + \frac{1}{2}gh^2\right) = 0$$

Bathymetry refers to measurements of the depth of the ocean. The equations above are for ocean with a flat bottom, "constant bathymetry". Suppose $b(x, y)$ is the height of the bottom (probably negative, measured from where the surface would be if the water were flat. The "variable depth" shallow water equations, in conserved variables, are

$$\partial_t\, h + \partial_x\,(m_x) + \partial_y\,(m_y) = 0$$
$$\partial_t\,(m_x) + \partial_x\left(h^{-1}m_x^2 + \frac{1}{2}gh^2\right) + \partial_y\,(h^{-1}m_xm_y) = gh\partial_x\, b$$
$$\partial_t\,(m_y) + \partial_x\,(h^{-1}m_xm_y) + \partial_y\left(h^{-1}m_y^2 + \frac{1}{2}gh^2\right) = gh\partial_y\, b$$

The variable $h(x, y, t)$ still represents the height (depth) of the water above the bottom. This means that the water surface is flat if $b(x, y) + h(x, y, t) = $ const. You can check that flat water with no movement is a solution; if $h + b = $ const and $v_x = v_y = 0$ then all three equations are satisfied.

1. Find the linearized equations assuming flat bathymetry and base constant solution $\bar{h} = h_0$, $\bar{v}_x = \bar{v}_y = 0$. Write them as a first order system with 3×3 matrices A_x and A_y. Show that the linearized equations have plane wave solutions that propagate in any direction at a wave speed $s_0 = \sqrt{gh_0}$. Do this by finding the eigenvalues and eigenvectors of the matrix $A_\omega = \omega_x A_x + \omega_y A_y$. Show that, from this analysis, the water moves in the same direction the wave is moving.

   The system of shallow water equations is

$$\begin{bmatrix} h \\ hu \\ hv \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}_x + \begin{bmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix}_y = 0.$$

   Let

$$q(x,y,t) = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \quad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}, \quad \text{and} \quad g(q) = \begin{bmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix}.$$

   Then, for smooth $q(x,y,t)$,

$$\begin{aligned} (f(q))_x = f'(q)q_x &= \begin{bmatrix} q_2 \\ q_2^2/q_1 + \frac{1}{2}gq_1^2 \\ q_2q_3/q_1 \end{bmatrix}_x \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -(q_2/q_1)^2 + gq_1 & 2q_2/q_1 & 0 \\ -q_2q_3/q_1^2 & q_3/q_1 & q_2/q_1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}_x \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -uv & v & u \end{bmatrix} \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}_x \end{aligned}$$

   and similarly,

$$(g(q))_y = g'(q)q_y = \begin{bmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2 + gh & 0 & 2v \end{bmatrix} \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}_y.$$

   To obtain the linearization at $q_0 = \begin{bmatrix} h_0 & h_0u_0 & h_0v_0 \end{bmatrix}^T$, we expand $(q_0 + \Delta q)_t + (f(q_0 + \Delta q))_x + (g(q_0 + \Delta q))_y = 0$ and drop terms of $O(\Delta q^2)$. This gives $(\Delta q)_t + f'(q_0)(\Delta q)_x + g'(q_0)(\Delta q)_y = 0$.

   Since $h = h_0$ and $u_0 = v_0 = 0$ in this question,

$$A_x = f'(q_0) = \begin{bmatrix} 0 & 1 & 0 \\ gh_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_y = g'(q_0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ gh_0 & 0 & 0 \end{bmatrix}.$$

   A_ω represents the Jacobian matrix in direction ω. $\omega = (\omega_x, \omega_y)$ is the unit vector. It follows that

$$A_\omega = (A_x, A_y) \cdot (\omega_x, \omega_y) = \omega_x A_x + \omega_y A_y = \begin{bmatrix} 0 & \omega_x & \omega_y \\ \omega_x gh_0 & 0 & 0 \\ \omega_y gh_0 & 0 & 0 \end{bmatrix}.$$

   The eigenvalues of A_ω are $\lambda_1 = 0$, $\lambda_2 = \sqrt{gh_0}$, and $\lambda_3 = -\sqrt{gh_0}$, with the corresponding eigenvectors

$$r_1 = \begin{bmatrix} 0 \\ \omega_y \\ -\omega_x \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ \omega_x\sqrt{gh_0} \\ \omega_y\sqrt{gh_0} \end{bmatrix}, \quad \text{and} \quad r_3 = \begin{bmatrix} 1 \\ -\omega_x\sqrt{gh_0} \\ -\omega_y\sqrt{gh_0} \end{bmatrix}.$$

Let $R = \begin{bmatrix} r_1 \,|\, & r_2 \,|\, & r_3 \end{bmatrix}$ and $L = R^{-1}$, and denote the rows of the matrix L by $l_1,\ l_2$, and l_3. Then, the solution can be written in terms of the initial data $\mathring{q}$ as

$$q(x_\omega, t) = \sum_{i=1}^{3} \left[l_i \ \mathring{q}(x_\omega - \lambda_i t)\right] r_i.$$

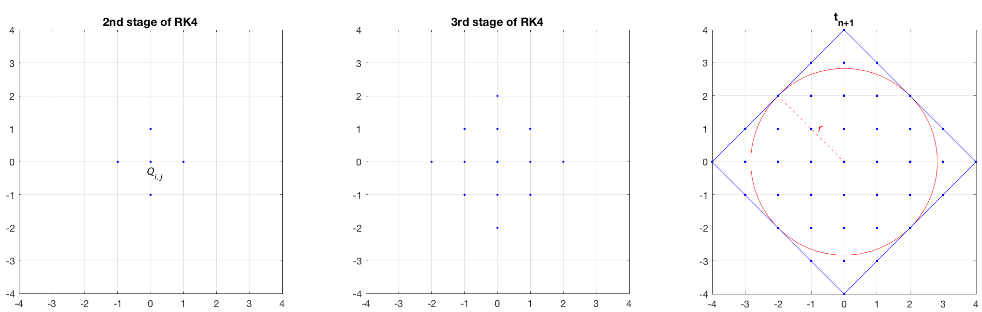
$q(x_\omega, t)$ is a linear combination of $r_1,\ r_2$, and r_3, with each r_i moving in the velocity of λ_i.

2. Write a code to solve the linearized shallow water equations, assuming $h_0 = 3$ km. *Warning*: g changes when you express length in kilometers. Apply periodic boundary conditions in space with $h(x+L_x, y, t) = h(x, y+L_y, t)$ (and for the other variables). Use grid points $x_j = j\Delta x$, and $y_k = k\Delta y$ (same spacing in x and y). Choose $\Delta t = \lambda s_0 \Delta x$. Compute the space derivatives by second order or fourth order centered differences to get a semi-discrete scheme, then evolve in time using the four stage fourth order Runge Kutta method. Verify the code by computing plane waves in various directions (not just x or y or 45 degrees). Take an initial disturbance to be mode in direction ω with a Gaussian $e^{-u^2/2l^2}$ profile with a length $l = 30$ km and compute the propagation for a time that lets the waves move a distance of $3l = 90$ km. Do a convergence study to see that the scheme is second order accurate with second order space derivatives and fourth order with fourth order derivatives in space. Show that the scheme is stable for $\lambda < 2\sqrt{2}$ and unstable for $\lambda > 2\sqrt{2}$.

(a) Stability

i. The CFL condition

Consider the scheme of the four stage fourth order Runge Kutta method with second order space derivatives. The figure below shows the stencils of this scheme. The radius of the circular base of the characteristic cone enclosed by the pyramid is $2\sqrt{2}\Delta x$. Hence, $\Delta t \leq 2\sqrt{2}\Delta x / s_0$ is a necessary condition for the scheme to be stable.



ii. Absolute stability

Because of periodic boundary conditions in space, the second order centered finite difference operator matrix in use is circulant. This allows us to find its eigenvalues by setting $\frac{1}{2\Delta x}\left[e^{2\pi i k(j+1)\Delta x} - e^{2\pi i k(j-1)\Delta x}\right] = \lambda_k e^{2\pi i k j \Delta x}$, $k \in \mathbb{Z}$. λ_k is purely imaginary.

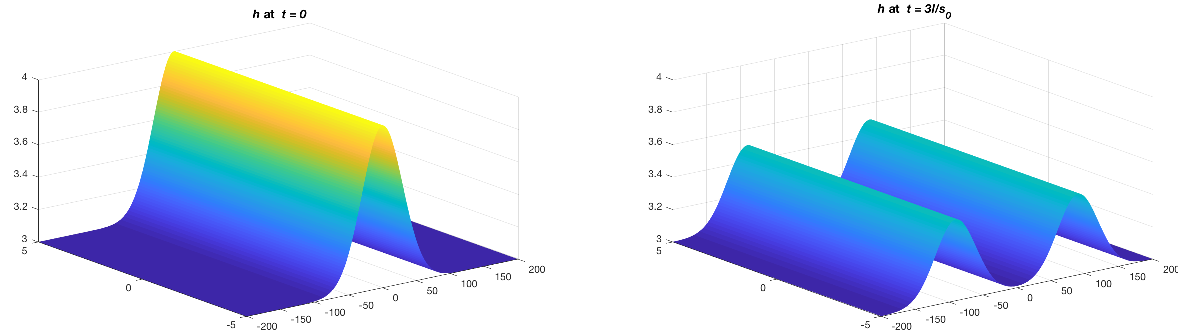
The stability region for the four stage fourth order Runge Kutta method intersect the imaginary axis at $\pm 2\sqrt{2}i$. Therefore, $(s_0/\Delta x)\, \Delta t \leq 2\sqrt{2}$ is also a sufficient condition for the scheme to be stable.

(b) Convergence study

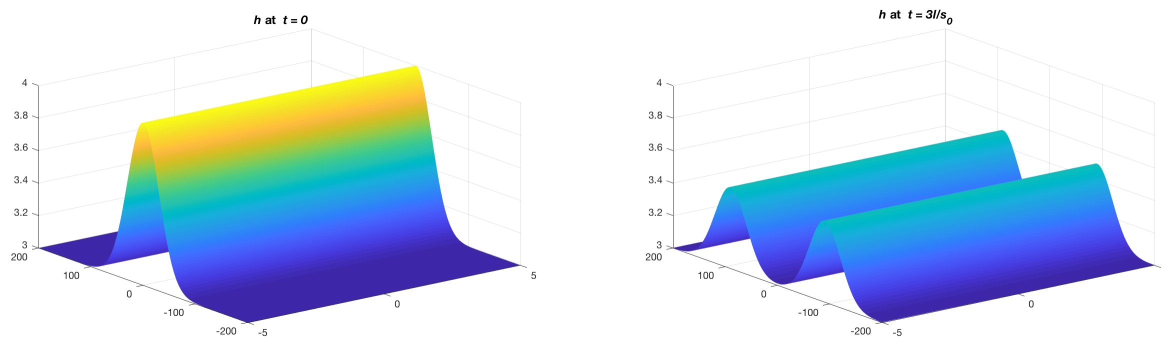
See the output of the code next page.

Unfortunately, the expected ratio of convergence for the scheme that uses fourth order space derivatives has not been achieved. The correct ratio of convergence is achieved when applying the scheme to the system of 2D linearized shallow water equations. See Appendix.

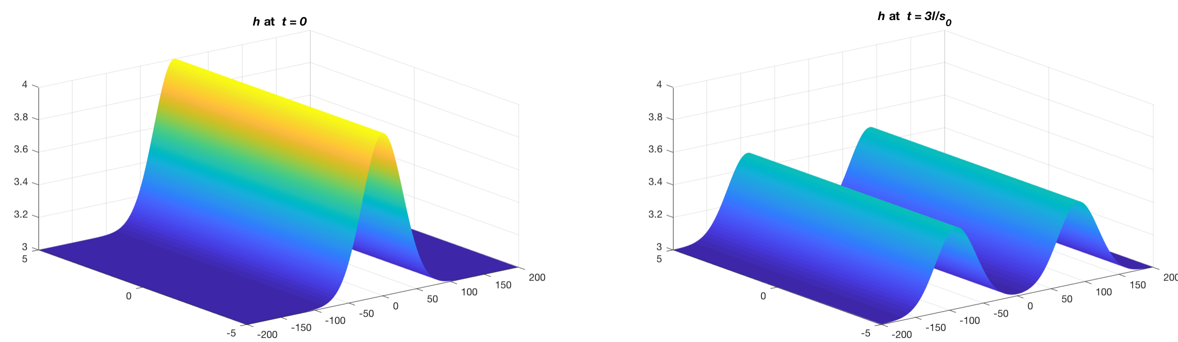
```
>> ErrorAnalysis
w = 0
X-range = [-200 200]
Y-range = [-5 5]
Order of Space Derivative = 2nd
dx        L1-error     L2-error     L_inf-error  ratio-1   ratio-2   ratio-3
1.00000   1.87780591   0.03747447   0.00115409
0.50000   0.46824049   0.00936490   0.00028773   4.01035   4.00159   4.01097
0.25000   0.11733680   0.00234153   0.00007188   3.99057   3.99948   4.00269
0.12500   0.02973053   0.00058780   0.00001797   3.94668   3.98352   4.00031
>> ErrorAnalysis
w = pi/2
X-range = [-5 5]
Y-range = [-200 200]
Order of Space Derivative = 2nd
dx        L1-error     L2-error     L_inf-error  ratio-1   ratio-2   ratio-3
1.00000   1.87780591   0.03747447   0.00115409
0.50000   0.46824049   0.00936490   0.00028773   4.01035   4.00159   4.01097
0.25000   0.11733680   0.00234153   0.00007188   3.99057   3.99948   4.00269
0.12500   0.02973053   0.00058780   0.00001797   3.94668   3.98352   4.00031
>> ErrorAnalysis
w = 0
X-range = [-200 200]
Y-range = [-5 5]
Order of Space Derivative = 4th
dx        L1-error     L2-error     L_inf-error  ratio-1   ratio-2   ratio-3
1.00000   0.01962346   0.00061548   0.00006262
0.50000   0.00605462   0.00031145   0.00003382   3.24107   1.97617   1.85142
0.25000   0.00290579   0.00016212   0.00001763   2.08364   1.92111   1.91851
0.12500   0.00147250   0.00008282   0.00000900   1.97337   1.95741   1.95818
>> ErrorAnalysis
w = pi/2
X-range = [-5 5]
Y-range = [-200 200]
Order of Space Derivative = 4th
dx        L1-error     L2-error     L_inf-error  ratio-1   ratio-2   ratio-3
1.00000   0.01962346   0.00061548   0.00006262
0.50000   0.00605462   0.00031145   0.00003382   3.24107   1.97617   1.85142
0.25000   0.00290579   0.00016212   0.00001763   2.08364   1.92111   1.91851
0.12500   0.00147250   0.00008282   0.00000900   1.97337   1.95741   1.95818
>>
```



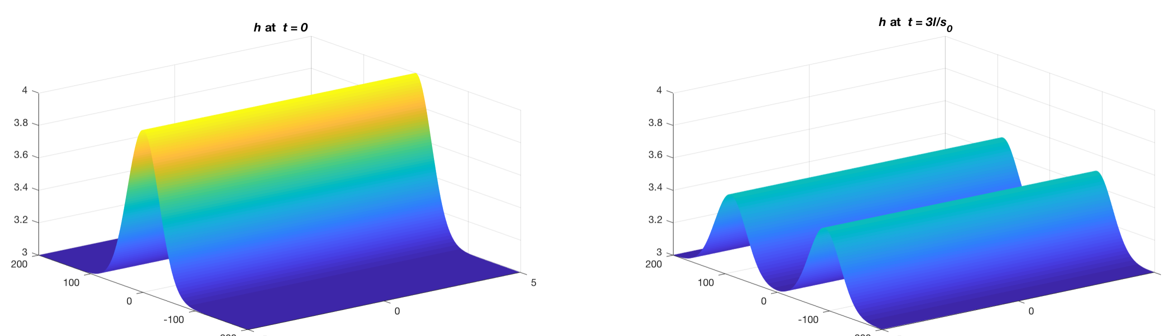
LSWE(0.125, 0, [-200 200], [-5 5], '2nd')



LSWE(0.125, $\pi/2$, [-5 5], [-200 200], '2nd')



LSWE(0.125, 0, [-200 200], [-5 5], '4th')



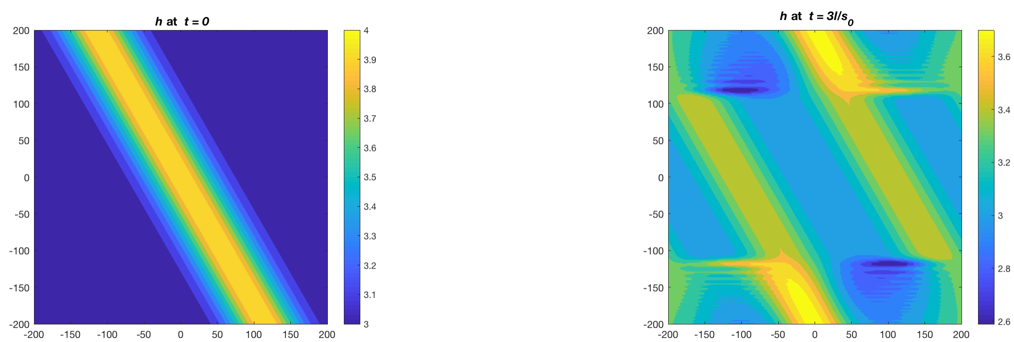
LSWE(0.125, $\pi/2$, [-5 5], [-200 200], '4th')

3. Take bathymetry to represent an undersea island

$$b(x,y) = \frac{1}{2} h_0 e^{-r^2/2l^2}, \quad r^2 = (x - x_0)^2 + y^2$$

Start with an initial plane wave moving in the $+x$ direction starting near $x = 0$ and choose $x_0 = 4l$. Make a movie of the solution. For example, each frame can be a contour plot of $h(\cdot, \cdot, t)$. Choose L_x and L_y large enough to see the scattered wave well.

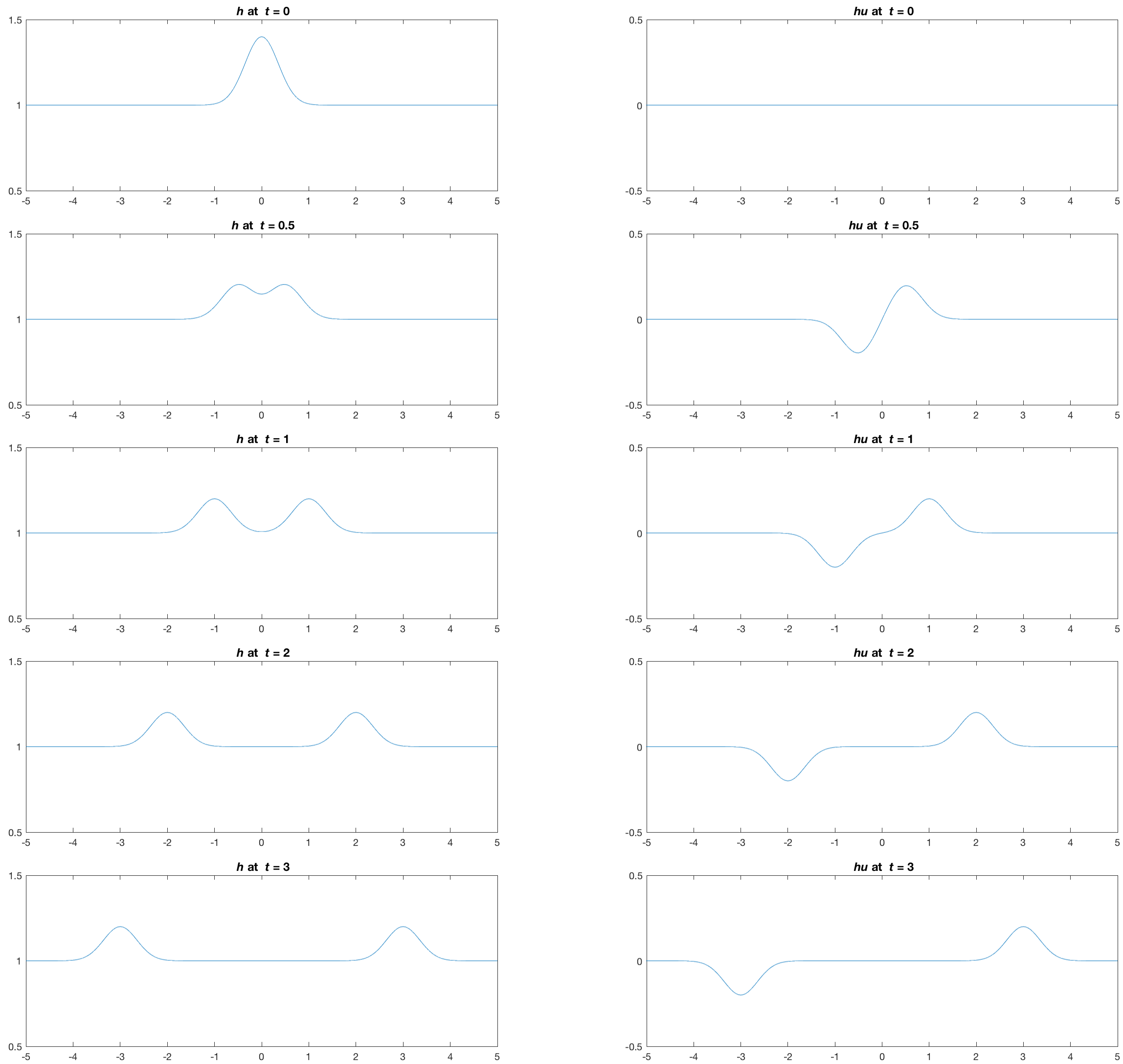
I hope my attempt so far has been on the right track. I am not sure the definition of plane waves in this question. Additionally, I am looking for some articles to learn the assumption of boundary conditions.



LSWE(1, $\pi/6$, [-200 200], [-200 200], '2nd')

Appendix

The system of 2D linearised shallow water equations is solved via diagonalization. The figure below is used to compare with Fig.13.1 in LeVeque-FVMHP, which shows the solution of the full nonlinear shallow water equations under the assumption that $g = 1$.



```
>> ErrorAnalysis2D
T = 3
Order of Space Derivative = 2nd
P  L1-error     L2-error     L_inf-error  ratio-1   ratio-2   ratio-3
1  0.04534578   0.02520764   0.02440522
2  0.01135963   0.00648959   0.00610412   3.99      3.88      4.00
3  0.00283991   0.00162573   0.00148195   4.00      3.99      4.12
4  0.00070960   0.00040648   0.00036713   4.00      4.00      4.04
5  0.00017741   0.00010162   0.00009156   4.00      4.00      4.01
>> ErrorAnalysis2D
T = 3
Order of Space Derivative = 4th
P  L1-error     L2-error     L_inf-error  ratio-1   ratio-2   ratio-3
1  0.00415481   0.00235904   0.00222371
2  0.00027342   0.00015836   0.00014933   15.20      14.90      14.89
3  0.00001729   0.00001004   0.00000954   15.82      15.78      15.65
4  0.00000108   0.00000063   0.00000060   15.94      15.95      15.95
5  0.00000007   0.00000004   0.00000004   15.96      15.99      15.98
>>
```