Homework Assignment
on

The Method of Regularized Stokeslets

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A work submitted on
the 20^{th} anniversary of the method

1. The original Regularized Stokeslet paper uses the 2D blob

$$\phi_\epsilon(\mathbf{x}) = \frac{3\epsilon^3}{2\pi(r^2+\epsilon^2)^{5/2}}$$

The corresponding functions G_ϵ and B'_ϵ are given in section 2.1.1 of the paper. Nicer formulas are obtained if we use a different blob, namely

$$\psi_\epsilon(\mathbf{x}) = \frac{2\epsilon^4}{\pi(r^2+\epsilon^2)^3}$$

Verify that $\int_{\mathbb{R}^2} \psi_\epsilon(\mathbf{x})d\mathbf{x} = 1$.

Recall that $\Delta G_\epsilon = \psi_\epsilon$ and that for radially-symmetric functions (using polar coordinates), this reduces to $(rG'_\epsilon(r))' = r\psi_\epsilon(r)$. Similarly, $\Delta B_\epsilon = G_\epsilon$ reduces to $(rB'_\epsilon(r))' = rG_\epsilon(r)$.

Derive the corresponding functions G_ϵ and B'_ϵ for this blob. Then find the expression for the regularized Stokeslet (a force $\mathbf{f}_k$ applied at $\mathbf{x}_k$)

$$\mathbf{u}(\mathbf{x}) = \mathbf{f}_k H_1(r) + [\mathbf{f}_k \cdot (\mathbf{x}-\mathbf{x}_k)](\mathbf{x}-\mathbf{x}_k)H_2(r)$$

where $r = \|\mathbf{x}-\mathbf{x}_k\|$, $H_1(r) = B'_\epsilon(r)/r - G_\epsilon(r)$ and $H_2(r) = (rB''_\epsilon(r) - B'_\epsilon(r))/r^3$.

i.

$$\begin{aligned}\int_{\mathbb{R}^2} \psi_\epsilon(\mathbf{x})d\mathbf{x} &= \int_0^{2\pi}\int_0^\infty \frac{2\epsilon^4}{\pi(r^2+\epsilon^2)^3} r\,dr\,d\theta \\ &= 2\pi\frac{\epsilon^4}{\pi}\left[\frac{(r^2+\epsilon^2)^{-2}}{-2}\right]_{r=0}^{r=\infty} = \epsilon^4(\epsilon^2)^{-2} = 1\end{aligned}$$

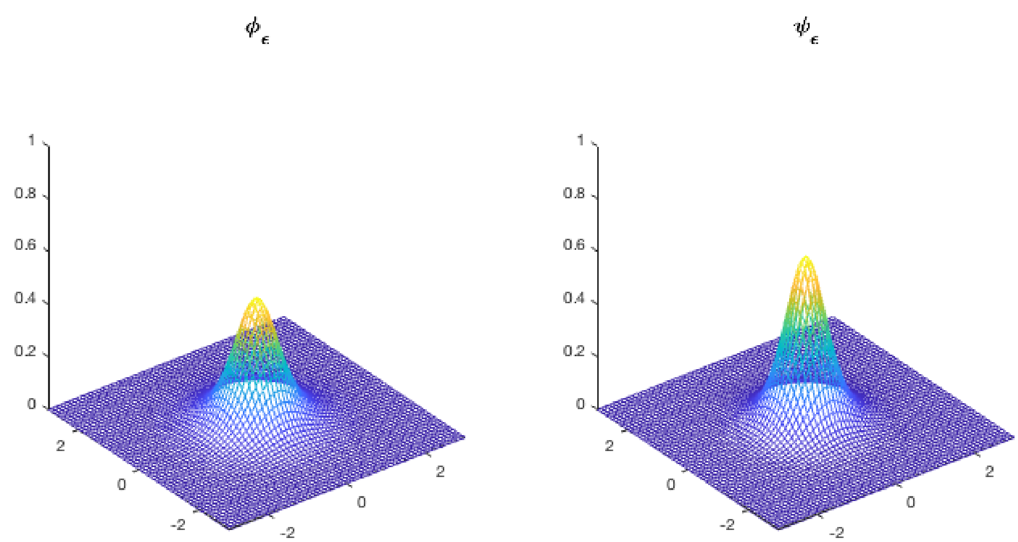


Figure 1: $\epsilon = 1$

ii. Since $(rG'_\epsilon(r))' = 2r\epsilon^4/(\pi(r^2+\epsilon^2)^3)$,

$$\begin{aligned} rG'_\epsilon(r) &= \int_0^r \frac{2s\epsilon^4}{\pi(s^2+\epsilon^2)^3} ds \\ &= \frac{\epsilon^4}{\pi}\left[\frac{(s^2+\epsilon^2)^{-2}}{-2}\right]_{s=0}^{s=r} = -\frac{\epsilon^4}{2\pi}\left(\frac{1}{(r^2+\epsilon^2)^2} - \frac{1}{(\epsilon^2)^2}\right) \\ &= \frac{r^2(r^2+2\epsilon^2)}{2\pi(r^2+\epsilon^2)^2} \end{aligned}$$

Thus, $G'_\epsilon(r) = r(r^2+2\epsilon^2)/(2\pi(r^2+\epsilon^2)^2)$ and

$$\begin{aligned} G_\epsilon(r) &= \int_a^r \frac{s(s^2+2\epsilon^2)}{2\pi(s^2+\epsilon^2)^2} ds \\ &= \frac{1}{4\pi}\int_a^r 2s\left(\frac{1}{s^2+\epsilon^2} + \frac{\epsilon^2}{(s^2+\epsilon^2)^2}\right) ds \\ &= \frac{1}{4\pi}\left(\ln(r^2+\epsilon^2) - \frac{\epsilon^2}{r^2+\epsilon^2}\right) \end{aligned}$$

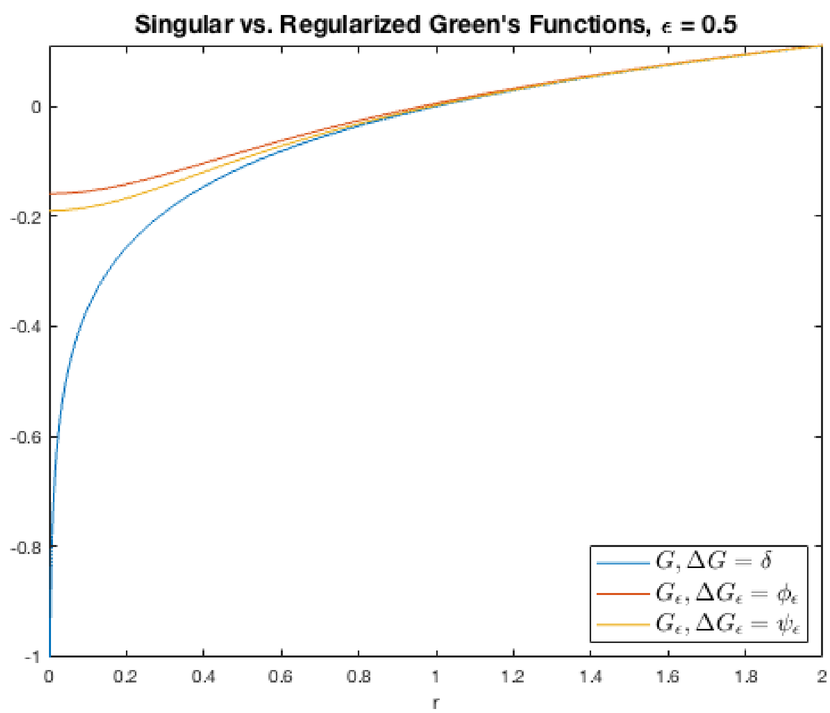


Figure 2: $G_\epsilon(r)$ is an approximation to the Green's function. [1]

[1] Note that $\lim_{\epsilon\to 0} G_\epsilon(r) = \frac{1}{2\pi}\ln(r)$ implies $K=0$ in $G_\epsilon(r) = \frac{1}{4\pi}\left(\ln(r^2+\epsilon^2) - \frac{\epsilon^2}{r^2+\epsilon^2}\right) + K$.

iii. Since $(rB'_\epsilon(r))' = r/(4\pi)\,[\ln(r^2+\epsilon^2) - \epsilon^2/(r^2+\epsilon^2)]$,

$$\begin{aligned}
rB'_\epsilon(r) &= \frac{1}{4\pi}\int_0^r s\left(\ln(s^2+\epsilon^2) - \frac{\epsilon^2}{s^2+\epsilon^2}\right)ds \\
&= \frac{1}{4\pi}\left(\int_0^r s\ln(s^2+\epsilon^2)ds - \int_0^r \frac{\epsilon^2 s}{s^2+\epsilon^2}ds\right) \\
&= \frac{1}{4\pi}\left(\frac{r^2}{2}\ln(r^2+\epsilon^2) - \int_0^r \frac{s^3}{s^2+\epsilon^2}ds - \int_0^r \frac{\epsilon^2 s}{s^2+\epsilon^2}ds\right) \\
&= \frac{1}{4\pi}\left(\frac{r^2}{2}\ln(r^2+\epsilon^2) - \int_0^r sds\right) \\
&= \frac{r^2}{8\pi}\left(\ln(r^2+\epsilon^2) - 1\right)
\end{aligned}$$

Therefore, $B'_\epsilon(r) = r/(8\pi)\,[\ln(r^2+\epsilon^2) - 1]$.

iv.

$$\begin{aligned}
H_1(r) &= -B''_\epsilon(r) = \frac{1}{8\pi}\left(\frac{2\epsilon^2}{r^2+\epsilon^2} - \ln\left(r^2+\epsilon^2\right)\right) - \frac{1}{8\pi} \\
H_2(r) &= \frac{rB''_\epsilon(r) - B'_\epsilon(r)}{r^3} = \frac{1}{4\pi(r^2+\epsilon^2)}
\end{aligned}$$

2. Suppose $C(a)$ is the boundary of an ellipse $x = a\cos\theta$ and $y = b\sin\theta$. An ellipse translating at a constant velocity $(1, 0)$ can be modeled by computing

$$\mathbf{u}(\mathbf{x}) = a\int_0^{2\pi} [\mathbf{f}(\theta)H_1(r) + [\mathbf{f}(\theta)\cdot(\mathbf{x}-\mathbf{x}(\theta))](\mathbf{x}-\mathbf{x}(\theta))H_2(r)]\,\|\mathbf{x}'(\theta)\|\,d\theta$$

where $r = \|\mathbf{x}-\mathbf{x}(\theta)\|$, $\mathbf{x}(\theta) = (a\cos\theta, b\sin\theta)$. Approximate this integral by discretizing the ellipse with N points and writing the equations for $j = 1, 2, \ldots, N$

$$\mathbf{u}(\mathbf{x}_j) = \sum_{k=1}^{N} \mathbf{f}_k H_1(r) + [\mathbf{f}_k\cdot(\mathbf{x}_j-\mathbf{x}_k)](\mathbf{x}_j-\mathbf{x}_k)H_2(r)$$

(the factors $\|\mathbf{x}'(\theta)\|\,d\theta$ can be absorbed into the forces).

Solve this linear system (with $(u, v) = (1, 0)$ and $a = 0.25, b = 0.4$) for the forces. Show the velocity field in the reference frame of the ellipse by subtracting $(1, 0)$ from the computed velocities at all points.

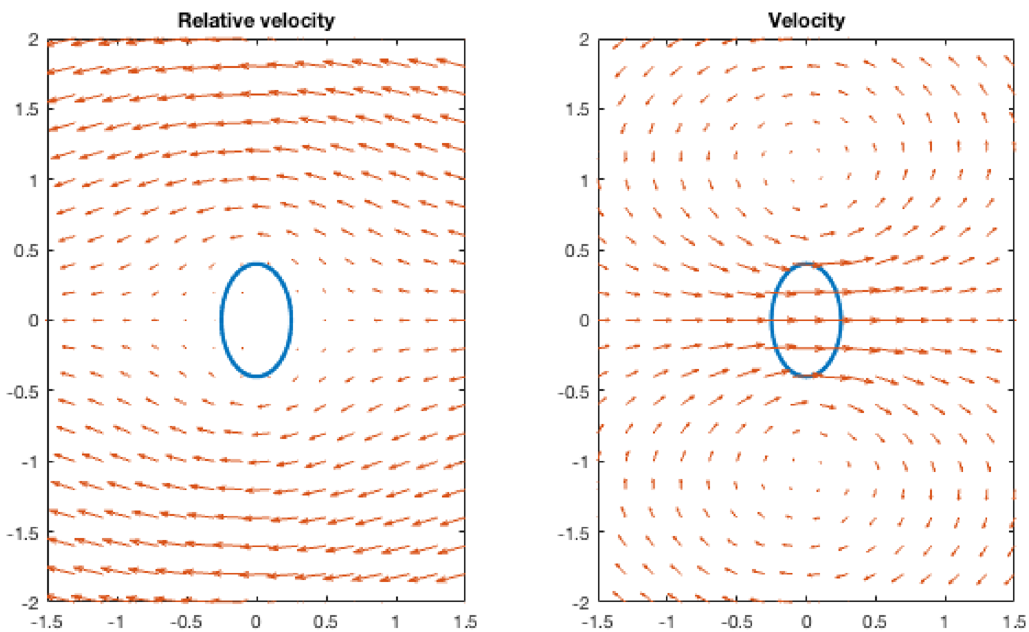


Figure 3: The relative velocities (left) are obtained by subtracting $(1, 0)$ from the computed velocities (right) at all points. The parameter used in computing these velocities are $N = 160$ and $\epsilon = \frac{2\pi a}{4N}$.

3. Now suppose $C(a)$ is the boundary of an "organism" represented by an oscillating finite sine curve $x(s) = s$ and $y(s,t) = A\sin(2\pi s - 2\pi t)$. The goal is to visualize the instantaneous velocity field around it. Note that the organism has velocity given by $u(s) = 0$ and $v(s,t) = -2\pi A\cos(2\pi s - 2\pi t)$.

a. Let $A = 0.2$ and $N = 48$ and discretize the organism at time $t = 0$ with $(x_k, y_k) = (s_k, A\sin(2\pi s_k))$, where $s_k = (k-1)/(N-1)$ for $k = 1, 2, \ldots, N$.

b. Let the velocity be $(u_k, v_k) = (0, -2\pi A\cos(2\pi s_k))$.

c. Then write the equations for $j = 1, 2, \ldots, N$

$$\mathbf{u}(\mathbf{x}_j) = \sum_{k=1}^{N} \mathbf{f}_k H_1(r) + [\mathbf{f}_k \cdot (\mathbf{x}_j - \mathbf{x}_k)](\mathbf{x}_j - \mathbf{x}_k) H_2(r)$$

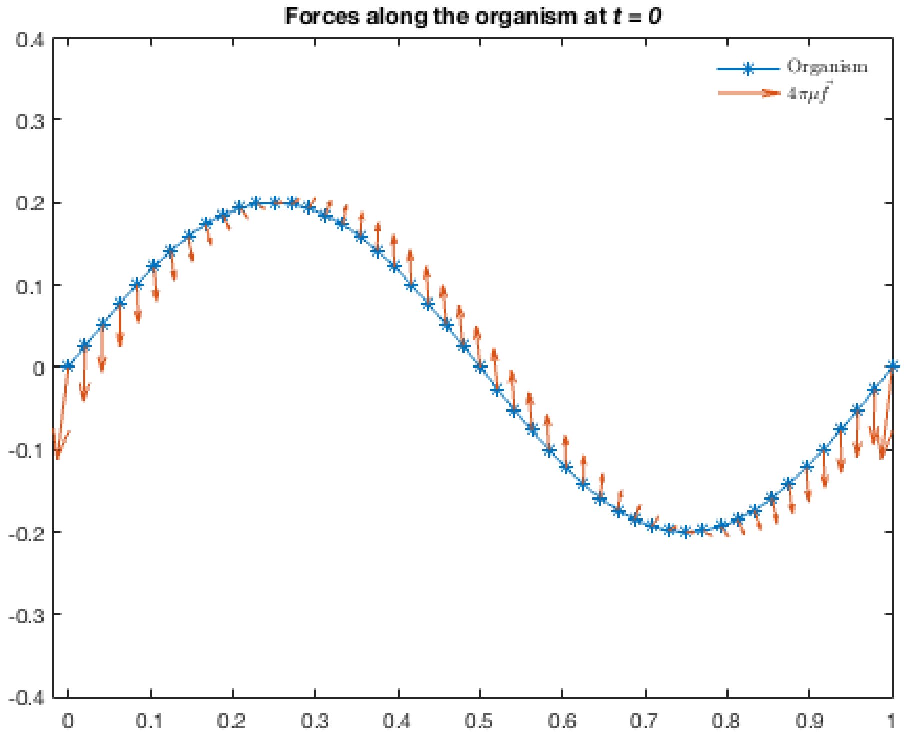
as a linear system of $2N$ equations with $2N$ unknowns (the forces) and solve for the forces.

d. Once you have the forces, use

$$\mathbf{u}(\mathbf{x}) = \sum_{k=1}^{N} \mathbf{f}_k H_1(r) + [\mathbf{f}_k \cdot (\mathbf{x} - \mathbf{x}_k)](\mathbf{x} - \mathbf{x}_k) H_2(r)$$

to evaluate the velocity field in a rectangular region around the organism, $[-0.5, 1.5] \times [-0.5, 0.5]$ and plot the organism, the forces along the organism, and the velocity field (quiver plot).

Go to the animation.



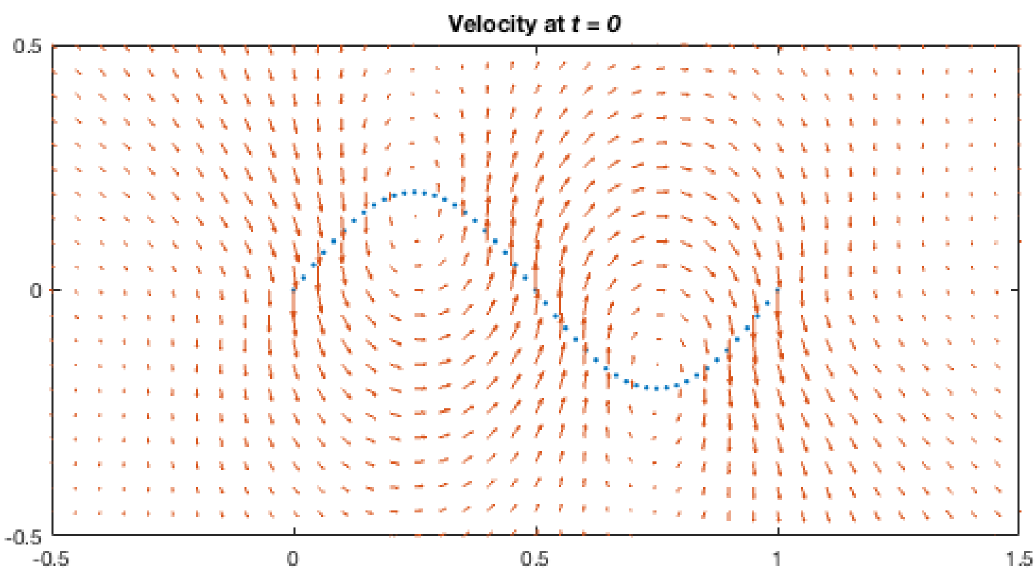


Figure 4: The regularization parameter was set to $\epsilon = \frac{1}{4N}$.

Code

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% Exercise 2

N = 160;
a = 0.25;
b = 0.4;
e = 0.25 * 2 * pi * a / N;

U = zeros(2*N, 1);
U(1 : 2 : 2*N) = 1;

k = 1 : N;
theta = 2 * pi / N * (k - 1);
x_k = a * cos(theta); x_k = x_k(:);
y_k = b * sin(theta); y_k = y_k(:);

H1 = @(rsq) 1 / 2 * (2 * e ^ 2 ./ (rsq + e ^ 2) - log(rsq + e ^ 2));

H2 = @(rsq) 1 ./ (rsq + e ^ 2);

M = NaN(2*N, 2*N);

for i = 1 : N

    r = (x_k(i) - x_k) .^ 2 + (y_k(i) - y_k) .^ 2; % indeed, r^2

    M(2 * i - 1, 1 : 2 : 2*N) = H1(r) + (x_k(i) - x_k) .^ 2 .* H2(r);
    M(2 * i - 1, 2 : 2 : 2*N) = (y_k(i) - y_k) .* (x_k(i) - x_k) .* H2(r);

    M(2 * i, 1 : 2 : 2*N) = M(2 * i - 1, 2 : 2 : 2*N);
    M(2 * i, 2 : 2 : 2*N) = H1(r) + (y_k(i) - y_k) .^ 2 .* H2(r);

end

F = M\U;

f = F(1 : 2 : 2*N);
g = F(2 : 2 : 2*N);


u_numerical = @(x, y) 0;
u = cell(N, 1);

for k = 1 : N

    u{k} = @(x, y) f(k) / 2 * (2 * e ^ 2 ./ ((x - x_k(k)) .^ 2 + (y - y_k(k)) .^ 2 + e ^↙
2) - log((x - x_k(k)) .^ 2 + (y - y_k(k)) .^ 2 + e ^ 2)) ...
        + (f(k) * (x - x_k(k)) + g(k) * (y - y_k(k))) .* (x - x_k(k)) ./ ((x - x_k(k)) .^↙
2 + (y - y_k(k)) .^ 2 + e ^ 2);

    u_numerical = @(x, y) u_numerical(x, y) + u{k}(x, y);

end

v_numerical = @(x, y) 0;
v = cell(N, 1);

for k = 1 : N
```

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    v{k} = @(x, y) g(k) / 2 * (2 * e ^ 2 ./ ((x - x_k(k)) .^ 2 + (y - y_k(k)) .^ 2 + e ^↙
2) - log((x - x_k(k)) .^ 2 + (y - y_k(k)) .^ 2 + e ^ 2)) ...
        + (f(k) * (x - x_k(k)) + g(k) * (y - y_k(k))) .* (y - y_k(k)) ./ ((x - x_k(k)) .^↙
2 + (y - y_k(k)) .^ 2 + e ^ 2);

    v_numerical = @(x, y) v_numerical(x, y) + v{k}(x, y);

end

% plot the velocity field
figure

subplot(1, 2, 1)

plot(x_k, y_k, '.')
hold on

[x, y] = meshgrid(-1.5 : 0.2 : 1.5, -2 : 0.2 : 2);
u_diff = u_numerical(x, y) - 1;
v_diff = v_numerical(x, y) - 0;

quiver(x, y, u_diff, v_diff)
axis ([-1.5 1.5 -2 2])
title('Relative velocity')

subplot(1, 2, 2)

plot(x_k, y_k, '.')
hold on

[x, y] = meshgrid(-1.5 : 0.2 : 1.5, -2 : 0.2 : 2);
u = u_numerical(x, y);
v = v_numerical(x, y);

quiver(x, y, u, v)
axis ([-1.5 1.5 -2 2])
title('Velocity')
```

```
% Exercise 3

A = 0.2;

h = 1/48;
N = 1/h + 1;

e = 0.25 * h;

t = 0;

k = 1 : N;
s_k = (k - 1) * h; s_k = s_k(:);

x_k = s_k;
y_k = A * sin(2 * pi * s_k - 2 * pi * t);

U = zeros(2*N, 1);
U(2 : 2 : 2*N) = -2 * pi * A * cos(2 * pi * s_k - 2 * pi * t);

H1 = @(rsq) 1 / 2 * (2 * e ^ 2 ./ (rsq + e ^ 2) - log(rsq + e ^ 2));

H2 = @(rsq) 1 ./ (rsq + e ^ 2);

M = NaN(2*N, 2*N);

for i = 1 : N

    r = (x_k(i) - x_k) .^ 2 + (y_k(i) - y_k) .^ 2; % indeed, r^2

    M(2 * i - 1, 1 : 2 : 2*N) = H1(r) + (x_k(i) - x_k) .^ 2 .* H2(r);
    M(2 * i - 1, 2 : 2 : 2*N) = (y_k(i) - y_k) .* (x_k(i) - x_k) .* H2(r);

    M(2 * i, 1 : 2 : 2*N) = M(2 * i - 1, 2 : 2 : 2*N);
    M(2 * i, 2 : 2 : 2*N) = H1(r) + (y_k(i) - y_k) .^ 2 .* H2(r);

end

F = M\U;

f = F(1 : 2 : 2*N);
g = F(2 : 2 : 2*N);

% plot the velocites and forces along the organism
figure

plot(x_k, y_k, '-*')
hold on

quiver(x_k, y_k, f, g, 0)
axis equal
title(['Forces along the organism at \itt = ', num2str(t)])

leg = legend('Organism', '$4\pi\mu\vec{f}$');
set(leg, 'Interpreter', 'latex')
legend('boxoff')
```

```
u_numerical = @(x, y) 0;
u = cell(N, 1);

for k = 1 : N

    u{k} = @(x, y) f(k) / 2 * (2 * e ^ 2 ./ ((x - x_k(k)) .^ 2 + (y - y_k(k)) .^ 2 + e ^↙
2) - log((x - x_k(k)) .^ 2 + (y - y_k(k)) .^ 2 + e ^ 2)) ...
        + (f(k) * (x - x_k(k)) + g(k) * (y - y_k(k))) .* (x - x_k(k)) ./ ((x - x_k(k)) .^↙
2 + (y - y_k(k)) .^ 2 + e ^ 2);

    u_numerical = @(x, y) u_numerical(x, y) + u{k}(x, y);

end

v_numerical = @(x, y) 0;
v = cell(N, 1);

for k = 1 : N

    v{k} = @(x, y) g(k) / 2 * (2 * e ^ 2 ./ ((x - x_k(k)) .^ 2 + (y - y_k(k)) .^ 2 + e ^↙
2) - log((x - x_k(k)) .^ 2 + (y - y_k(k)) .^ 2 + e ^ 2)) ...
        + (f(k) * (x - x_k(k)) + g(k) * (y - y_k(k))) .* (y - y_k(k)) ./ ((x - x_k(k)) .^↙
2 + (y - y_k(k)) .^ 2 + e ^ 2);

    v_numerical = @(x, y) v_numerical(x, y) + v{k}(x, y);

end

% plot the velocity field
figure

plot(x_k, y_k, '.')
hold on

[x, y] = meshgrid(-0.5 : 0.05 : 1.5, -0.5 : 0.05 : 0.5);
u = u_numerical(x, y);
v = v_numerical(x, y);

quiver(x, y, u, v)
axis ([-0.5 1.5 -0.5 0.5])
title(['Velocity at \itt = ', num2str(t)])
```