形式化方法导引

第 4 章 逻辑问题求解——一种通用求解方法:SAT/SMT 求解
4.1 应用

黄文超
http://staff.ustc.edu.cn/~huangwc/fm.html

课程回顾

- 第 1 章: 自动机、可计算性、复杂度理论
    - 问题是什么? 可以解么? 有多难?
- 第 2 章: 怎样用逻辑来定义一个验证器问题?
    - Propositional logic, first-order logic, higher-order logic
- 第 3 章: 怎样进一步定义一种演算规则rules 来降低求解难度?
    - The proof calculus of natural deduction
- 本章: 如何针对上述的 rules求解?

课程回顾

- 第 1 章：自动机、可计算性、复杂度理论
  - 问题是什么？可以解么？有多难？
- 第 2 章：怎样用逻辑来定义一个验证器问题？
  - Propositional logic, first-order logic, higher-order logic
- 第 3 章：怎样进一步定义一种演算规则rules 来降低求解难度？
  - The proof calculus of natural deduction
- 本章：如何针对上述的 rules求解？

课程回顾

- 第 1 章：自动机、可计算性、复杂度理论
  - 问题是什么？可以解么？有多难？
- 第 2 章：怎样用逻辑来定义一个验证器问题？
  - Propositional logic, first-order logic, higher-order logic
- 第 3 章：怎样进一步定义一种演算规则rules 来降低求解难度？
  - The proof calculus of natural deduction
- 本章：如何针对上述的 rules求解？

课程回顾

- 第 1 章: 自动机、可计算性、复杂度理论
  - 问题是什么? 可以解么? 有多难?
- 第 2 章: 怎样用逻辑来定义一个验证器问题?
  - Propositional logic, first-order logic, higher-order logic
- 第 3 章: 怎样进一步定义一种演算规则rules 来降低求解难度?
  - The proof calculus of natural deduction
- 本章: 如何针对上述的 rules求解?

本章内容

1. 应用：如何用工具解决经典逻辑相关的问题?
   - SAT,SMT 问题
   - 问题求解工具 Z3
   - 案例实现
     - Satisfiability
     - Validity
     - Numbers and inequalities
     - Eight Queens problem
     - Binary Arithmetic
     - Rectangle fitting
     - Solving Sudoku
   - 其它应用: Symbolic execution
2. 理论：这些工具的核心算法?

本章内容

1. 应用：如何用工具解决经典逻辑相关的问题?
   - SAT,SMT 问题
   - 问题求解工具 Z3
   - 案例实现
     - Satisfiability
     - Validity
     - Numbers and inequalities
     - Eight Queens problem
     - Binary Arithmetic
     - Rectangle fitting
     - Solving Sudoku
   - 其它应用: Symbolic execution
2. 理论：这些工具的核心算法?

定义: Propositional Logic in *BNF*

$$\phi ::= p \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$$

where p stands for any atomic proposition and each occurrence of ϕ to the right of $::=$ stands for any already constructed formula.

定义: Verification in Logics

Most logics used in the design, specification and verification of computer systems fundamentally deal with a *satisfaction relation*:

$$\mathcal{M} \vDash \phi$$

问题: $\mathcal{M} \vDash \phi$ 在命题逻辑中更简单的表达是什么?

定义: Propositional Logic in *BNF*

$$\phi ::= p \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$$

where p stands for any atomic proposition and each occurrence of ϕ to the right of $::=$ stands for any already constructed formula.

定义: Verification in Logics

Most logics used in the design, specification and verification of computer systems fundamentally deal with a *satisfaction relation*:

$$\mathcal{M} \vDash \phi$$

问题: $\mathcal{M} \vDash \phi$ 在命题逻辑中更简单的表达是什么?

问题：$\mathcal{M} \vDash \phi$ 在命题逻辑中更简单的表达是什么?

引理: (去除 $\mathcal{M}$)

Given formulas $\phi_1, \phi_2, \ldots, \phi_n$ and ψ of propositional logic, $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds iff $\vDash \phi_1 \to (\phi_2 \to (\phi_3 \to \cdots \to (\phi_n \to \psi)))$ holds.

答：去除 $\mathcal{M}$ 后，求解 validity (见如下定义)

定义: Validity

We call ϕ *valid*, if $\vDash \phi$ holds.

- We also call ϕ as a *tautology* (重言式), if ϕ is valid.

下一个问题：怎么求解 validity?
答：换一个问题求解: Satisfiability (见下页)

问题：$\mathcal{M} \vDash \phi$ 在命题逻辑中更简单的表达是什么?

引理: (去除 $\mathcal{M}$)

Given formulas $\phi_1, \phi_2, \ldots, \phi_n$ and ψ of propositional logic, $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds iff $\vDash \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow \cdots \rightarrow (\phi_n \rightarrow \psi)))$ holds.

答：去除 $\mathcal{M}$ 后，求解 validity (见如下定义)

定义: Validity

We call ϕ *valid*, if $\vDash \phi$ holds.

- We also call ϕ as a *tautology* (重言式), if ϕ is valid.

下一个问题：怎么求解 validity?

答：换一个问题求解: Satisfiability (见下页)

问题：$\mathcal{M} \vDash \phi$ 在命题逻辑中更简单的表达是什么?

引理: (去除 $\mathcal{M}$)

Given formulas $\phi_1, \phi_2, \ldots, \phi_n$ and ψ of propositional logic, $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds iff $\vDash \phi_1 \to (\phi_2 \to (\phi_3 \to \cdots \to (\phi_n \to \psi)))$ holds.

答：去除 $\mathcal{M}$ 后，求解 validity (见如下定义)

定义: Validity

We call ϕ *valid*, if $\vDash \phi$ holds.

- We also call ϕ as a *tautology* (重言式), if ϕ is valid.

下一个问题：怎么求解 validity?
答：换一个问题求解: Satisfiability (见下页)

问题：$\mathcal{M} \vDash \phi$ 在命题逻辑中更简单的表达是什么?

引理: (去除 $\mathcal{M}$)

Given formulas $\phi_1, \phi_2, \ldots, \phi_n$ and ψ of propositional logic, $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds iff $\vDash \phi_1 \to (\phi_2 \to (\phi_3 \to \cdots \to (\phi_n \to \psi)))$ holds.

答：去除 $\mathcal{M}$ 后，求解 validity (见如下定义)

定义: Validity

We call ϕ *valid*, if $\vDash \phi$ holds.

- We also call ϕ as a *tautology* (重言式), if ϕ is valid.

下一个问题：怎么求解 validity?

答：换一个问题求解: Satisfiability (见下页)

问题：$\mathcal{M} \vDash \phi$ 在命题逻辑中更简单的表达是什么?

引理: (去除 $\mathcal{M}$)

Given formulas $\phi_1, \phi_2, \ldots, \phi_n$ and ψ of propositional logic, $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds iff $\vDash \phi_1 \to (\phi_2 \to (\phi_3 \to \cdots \to (\phi_n \to \psi)))$ holds.

答：去除 $\mathcal{M}$ 后，求解 validity (见如下定义)

定义: Validity

We call ϕ *valid*, if $\vDash \phi$ holds.

- We also call ϕ as a *tautology* (重言式), if ϕ is valid.

下一个问题：怎么求解 validity?
答：换一个问题求解: Satisfiability (见下页)

定义: Satisfiability

Given a formula ϕ in propositional logic, we say that ϕ is *satisfiable* if it has a valuation in which is evaluates to **T**.

例子:

$p \vee q \to p$ is *satisfiable*, since it computes **T** if we assign **T** to p.
Note that $p \vee q \to p$ is *not valid*.

定理:

Let ϕ be a formula of propositional logic. Then ϕ is *satisfiable* iff $\neg\phi$ is *not valid*.
In other words, ϕ is valid iff $\neg\phi$ is not satisfiable.

总结: 求解 $\mathcal{M} \vDash \phi$ ——求解 Validity ——求解 Satisfiability.

定义: Satisfiability

Given a formula ϕ in propositional logic, we say that ϕ is *satisfiable* if it has a valuation in which is evaluates to **T**.

例子:

$p \vee q \to p$ is *satisfiable*, since it computes **T** if we assign **T** to p.
Note that $p \vee q \to p$ is *not valid*.

定理:

Let ϕ be a formula of propositional logic. Then ϕ is *satisfiable* iff $\neg\phi$ is *not valid*.
In other words, ϕ is valid iff $\neg\phi$ is not satisfiable.

总结: 求解 $\mathcal{M} \vDash \phi$ ——求解 Validity ——求解 Satisfiability.

定义: Satisfiability

Given a formula ϕ in propositional logic, we say that ϕ is *satisfiable* if it has a valuation in which is evaluates to **T**.

例子:

$p \vee q \to p$ is *satisfiable*, since it computes **T** if we assign **T** to p.
Note that $p \vee q \to p$ is *not valid*.

定理:

Let ϕ be a formula of propositional logic. Then ϕ is *satisfiable* iff $\neg\phi$ is *not valid*.

In other words, ϕ is valid iff $\neg\phi$ is not satisfiable.

总结: 求解 $\mathcal{M} \vDash \phi$ ——求解 Validity ——求解 Satisfiability.

定义: Satisfiability

Given a formula ϕ in propositional logic, we say that ϕ is *satisfiable* if it has a valuation in which is evaluates to **T**.

例子:

$p \vee q \to p$ is *satisfiable*, since it computes **T** if we assign **T** to p.
Note that $p \vee q \to p$ is *not valid*.

定理:

Let ϕ be a formula of propositional logic. Then ϕ is *satisfiable* iff $\neg\phi$ is *not valid*.
In other words, ϕ is valid iff $\neg\phi$ is not satisfiable.

总结: 求解 $\mathcal{M} \vDash \phi$ ——求解 Validity ——求解 Satisfiability.

定义: Satisfiability

Given a formula ϕ in propositional logic, we say that ϕ is *satisfiable* if it has a valuation in which is evaluates to **T**.

例子:

$p \vee q \to p$ is *satisfiable*, since it computes **T** if we assign **T** to p.
Note that $p \vee q \to p$ is *not valid*.

定理:

Let ϕ be a formula of propositional logic. Then ϕ is *satisfiable* iff $\neg\phi$ is *not valid*.
In other words, ϕ is valid iff $\neg\phi$ is not satisfiable.

总结: 求解 $\mathcal{M} \models \phi$ ——求解 Validity ——求解 Satisfiability.

定义: Satisfiability

Given a formula ϕ in propositional logic, we say that ϕ is *satisfiable* if it has a valuation in which is evaluates to **T**.

例子:

$p \vee q \to p$ is *satisfiable*, since it computes **T** if we assign **T** to p.
Note that $p \vee q \to p$ is *not valid*.

定理:

Let ϕ be a formula of propositional logic. Then ϕ is *satisfiable* iff $\neg\phi$ is *not valid*.
In other words, ϕ is valid iff $\neg\phi$ is not satisfiable.

总结: 求解 $\mathcal{M} \vDash \phi$ ——求解 Validity ——求解 Satisfiability.

定义: Satisfiability

Given a formula ϕ in propositional logic, we say that ϕ is *satisfiable* if it has a valuation in which is evaluates to **T**.

例子:

$p \vee q \to p$ is *satisfiable*, since it computes **T** if we assign **T** to p.
Note that $p \vee q \to p$ is *not valid*.

定理:

Let ϕ be a formula of propositional logic. Then ϕ is *satisfiable* iff $\neg\phi$ is *not valid*.
In other words, ϕ is valid iff $\neg\phi$ is not satisfiable.

总结: 求解 $\mathcal{M} \vDash \phi$ ——求解 Validity ——求解 Satisfiability.

定义: SAT 问题

SAT is the *decision* problem: given a propositional formula, is it *satisfiable*?

SAT 问题可用于模型的验证!

回顾: 问题可以解么? – 问题 4

Given a set $A \subseteq S$, and $x \in S$, *whether there is a machine* that can compute whether $x \in A$.

- Define a new machine, named *Turing machine, 图灵机*.
- If yes, i.e., there is a Turing machine M for A, language A is *decidable*.
- If no, but there is a Turing machine M that can only accept s, if $s \in A$, language A is *still* *Turing-recognizable*.

SAT 是可计算的 (见下页)

定义: SAT 问题

SAT is the *decision* problem: given a propositional formula, is it *satisfiable*?

SAT 问题可用于模型的验证!

回顾：问题可以解么? – 问题 4

Given a set $A \subseteq S$, and $x \in S$, *whether there is a machine* that can compute whether $x \in A$.

- Define a new machine, named *Turing machine, 图灵机*.
- If yes, i.e., there is a Turing machine M for A, language A is *decidable*.
- If no, but there is a Turing machine M that can only accept s, if $s \in A$, language A is *still* *Turing-recognizable*.

SAT 是可计算的 (见下页)

定义: SAT 问题

SAT is the *decision* problem: given a propositional formula, is it *satisfiable*?

SAT 是可计算的:

- Essentially, this consists of computing the values of the formula for all 2^n ways to choose **T** or **F** for the n variables.

问：SAT 属于哪一类问题? (复杂度)

答：SAT 属于经典的 NP-Complete 问题 (1970 年开始研究)(Bad news)。

回顾：定义: NP-complete and NP-hard

A language B is *NP-complete* if it satisfies two conditions:

1. B is in NP, and
2. every A in NP is polynomial time *reducible* to B.

Here, B is *NP-hard* if it satisfies condition 2.

Classical *NP-complete* problem: *SAT* (形式化方法的重要问题之一).

定义: SAT 问题

SAT is the *decision* problem: given a propositional formula, is it *satisfiable*?

SAT 是可计算的:

- Essentially, this consists of computing the values of the formula for all 2^n ways to choose **T** or **F** for the n variables.

问：SAT 属于哪一类问题? (复杂度)

答：SAT 属于经典的 NP-Complete 问题 (1970 年开始研究)(Bad news)。

回顾：定义: NP-complete and NP-hard

A language B is *NP-complete* if it satisfies two conditions:

1. B is in NP, and
2. every A in NP is polynomial time *reducible* to B.

Here, B is *NP-hard* if it satisfies condition 2.

Classical *NP-complete* problem: *SAT* (形式化方法的重要问题之一).

定义: SAT 问题

SAT is the *decision* problem: given a propositional formula, is it *satisfiable*?

SAT 是可计算的:

- Essentially, this consists of computing the values of the formula for all 2^n ways to choose **T** or **F** for the n variables.

问: SAT 属于哪一类问题? (复杂度)

答: SAT 属于经典的 NP-Complete 问题 (1970 年开始研究)(Bad news)。

回顾: 定义: NP-complete and NP-hard

A language B is *NP-complete* if it satisfies two conditions:

1. B is in NP, and
2. every A in NP is polynomial time *reducible* to B.

Here, B is *NP-hard* if it satisfies condition 2.

Classical *NP-complete* problem: *SAT* (形式化方法的重要问题之一).

定义: SAT 问题

SAT is the *decision* problem: given a propositional formula, is it *satisfiable*?

SAT 是可计算的:

- Essentially, this consists of computing the values of the formula for all 2^n ways to choose **T** or **F** for the n variables.

问：SAT 属于哪一类问题? (复杂度)
答：SAT 属于经典的 NP-Complete 问题 (1970 年开始研究)(Bad news)。

回顾：定义: NP-complete and NP-hard

A language B is *NP-complete* if it satisfies two conditions:

1. B is in NP, and
2. every A in NP is polynomial time *reducible* to B.

Here, B is *NP-hard* if it satisfies condition 2.
Classical *NP-complete* problem: *SAT* (形式化方法的重要问题之一).

Good news: Current SAT solvers are successful for several big formulas.

- 例子: solving the n-queens problem for n=100 yields a 50Mb formula over 10000 variables, but is solved in 10 seconds by the SAT solver *Z3*.

定义: SMT problem

Extension of SAT, to deal with *numbers* and *inequalities*.

SAT, SMT 求解工具:

- Z3, YICES, CVC4
- For non-commercial use they are free to download and to use

Good news: Current SAT solvers are successful for several big formulas.

- 例子: solving the n-queens problem for n=100 yields a 50Mb formula over 10000 variables, but is solved in 10 seconds by the SAT solver *Z3*.

定义: SMT problem

Extension of SAT, to deal with *numbers* and *inequalities*.

SAT, SMT 求解工具:

- Z3, YICES, CVC4
- For non-commercial use they are free to download and to use

Good news: Current SAT solvers are successful for several big formulas.

- 例子: solving the n-queens problem for n=100 yields a 50Mb formula over 10000 variables, but is solved in 10 seconds by the SAT solver *Z3*.

定义: SMT problem

Extension of SAT, to deal with *numbers* and *inequalities*.

SAT, SMT 求解工具:

- Z3, YICES, CVC4
- For non-commercial use they are free to download and to use

Z3

- Z3 is a theorem prover from Microsoft Research.

Z3 interfaces

- Default input format is SMTLIB2
- Other native foreign function interfaces:
  - C++ API
  - .NET API
  - Java API
  - Python API
  - C
  - OCaml
  - Julia

参考阅读

- SAT/SMT by Example

Z3

- Z3 is a theorem prover from Microsoft Research.

Z3 interfaces

- Default input format is SMTLIB2
- Other native foreign function interfaces:
  - C++ API
  - .NET API
  - Java API
  - Python API
  - C
  - OCaml
  - Julia

参考阅读

- SAT/SMT by Example

Z3

- Z3 is a theorem prover from Microsoft Research.

Z3 interfaces

- Default input format is SMTLIB2
- Other native foreign function interfaces:
  - C++ API
  - .NET API
  - Java API
  - Python API
  - C
  - OCaml
  - Julia

参考阅读

- SAT/SMT by Example

问题: Is ϕ satisfiable?

$$\phi = (p \to q) \land (r \leftrightarrow \neg q) \land (\neg p \lor r)$$

```
from z3 import *
p = Bool('p')
q = Bool('q')
r = Bool('r')
solve(Implies(p, q), r == Not(q), Or(Not(p), r))
```

运行结果:

$python3 z3-1-sat.py

[q = True, p = False, r = False]

解析:

存在一个解，满足 $\vDash \phi$。解为 $q = \mathbf{T} \land p = \mathbf{F} \land r = \mathbf{F}$

问题: Is ϕ satisfiable?

$\phi = (p \rightarrow q) \wedge (r \leftrightarrow \neg q) \wedge (\neg p \vee r)$

```
from z3 import *
p = Bool('p')
q = Bool('q')
r = Bool('r')
solve(Implies(p, q), r == Not(q), Or(Not(p), r))
```

运行结果:

$python3 z3-1-sat.py

[q = True, p = False, r = False]

解析:

存在一个解，满足 $\models \phi$。解为 $q = \mathbf{T} \wedge p = \mathbf{F} \wedge r = \mathbf{F}$

问题: Is ϕ satisfiable?

$$\phi = (p \rightarrow q) \wedge (r \leftrightarrow \neg q) \wedge (\neg p \vee r)$$

```python
from z3 import *
p = Bool('p')
q = Bool('q')
r = Bool('r')
solve(Implies(p, q), r == Not(q), Or(Not(p), r))
```

运行结果:

$python3 z3-1-sat.py

[q = True, p = False, r = False]

解析:

存在一个解，满足 $\vDash \phi$。解为 $q = \mathbf{T} \wedge p = \mathbf{F} \wedge r = \mathbf{F}$

问题: Is ϕ satisfiable?

$\phi = (p \to q) \land (r \leftrightarrow \neg q) \land (\neg p \lor r)$

```python
from z3 import *
p = Bool('p')
q = Bool('q')
r = Bool('r')
solve(Implies(p, q), r == Not(q), Or(Not(p), r))
```

运行结果:

$python3 z3-1-sat.py

[q = True, p = False, r = False]

解析:

存在一个解，满足 $\models \phi$。解为 $q = \mathbf{T} \land p = \mathbf{F} \land r = \mathbf{F}$

```python
from z3 import *
p, q = Bools('p q')
demorgan = \
    And(p, q) == \
        Not(Or(Not(p), Not(q)))
def prove(f):
    s = Solver()
    s.add(Not(f))
    if s.check() == unsat:
        print("proved")
    else:
        print("failed to prove")
prove(demorgan)
```

问题: Is ϕ valid ? (i.e., $\models \phi$?)

$\phi = (p \wedge q) \leftrightarrow \neg(\neg p \vee \neg q)$

运行结果:

$python3 z3-2-valid.py

proved

解析:

ϕ is valid,

iff $\neg\phi$ is not satisfiable.

So, ϕ is valid.

```python
from z3 import *
p, q = Bools('p q')
demorgan = \
    And(p, q) == \
        Not(Or(Not(p), Not(q)))
def prove(f):
    s = Solver()
    s.add(Not(f))
    if s.check() == unsat:
        print("proved")
    else:
        print("failed to prove")
prove(demorgan)
```

问题: Is ϕ valid ? (i.e., $\models \phi$?)

$\phi = (p \wedge q) \leftrightarrow \neg(\neg p \vee \neg q)$

运行结果:

$python3 z3-2-valid.py

proved

解析:

ϕ is valid,

iff $\neg\phi$ is not satisfiable.

So, ϕ is valid.

```python
from z3 import *
p, q = Bools('p q')
demorgan = \
    And(p, q) == \
        Not(Or(Not(p), Not(q)))
def prove(f):
    s = Solver()
    s.add(Not(f))
    if s.check() == unsat:
        print("proved")
    else:
        print("failed to prove")
prove(demorgan)
```

问题: Is ϕ valid ? (i.e., $\models \phi$?)

$\phi = (p \wedge q) \leftrightarrow \neg(\neg p \vee \neg q)$

运行结果:

$python3 z3-2-valid.py

proved

解析:

ϕ is valid,

iff $\neg\phi$ is not satisfiable.

So, ϕ is valid.

问题: Solve the following system of constraints

$x > 2 \wedge y < 10 \wedge x + 2y = 7$, where x, y are integers.

```
from z3 import *
x = Int('x')
y = Int('y')
solve(x > 2, y < 10, x + 2*y == 7)
```

运行结果:

$python3 z3-3-inequalities.py

[y = 0, x = 7]

解析:

该问题为 SMT 问题

问题: Solve the following system of constraints

$x > 2 \wedge y < 10 \wedge x + 2y = 7$, where x, y are integers.

```
from z3 import *
x = Int('x')
y = Int('y')
solve(x > 2, y < 10, x + 2*y == 7)
```

运行结果:

$python3 z3-3-inequalities.py

[y = 0, x = 7]

解析:

该问题为 SMT 问题

问题: Solve the following system of constraints

$x > 2 \wedge y < 10 \wedge x + 2y = 7$, where x, y are integers.

```
from z3 import *
x = Int('x')
y = Int('y')
solve(x > 2, y < 10, x + 2*y == 7)
```

运行结果:

$python3 z3-3-inequalities.py

[y = 0, x = 7]

解析:

该问题为 SMT 问题

问题: Solve the following system of constraints

$x > 2 \wedge y < 10 \wedge x + 2y = 7$, where x, y are integers.

```python
from z3 import *
x = Int('x')
y = Int('y')
solve(x > 2, y < 10, x + 2*y == 7)
```

运行结果:

$python3 z3-3-inequalities.py

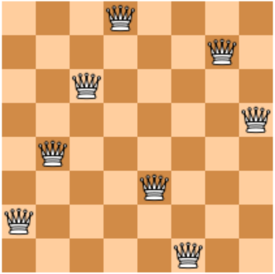
[y = 0, x = 7]

解析:

该问题为 SMT 问题

问题: Eight-Queens

The eight queens puzzle is the problem of placing eight chess queens on an 8×8 chessboard so that *no two queens attack each other*. Thus, a solution requires that *no two queens* share the *same row, column, or diagonal*.

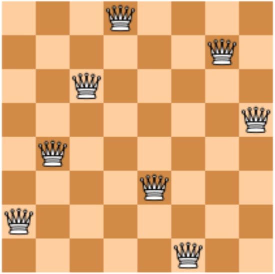


As usual in SAT/SMT, don't think about how to solve it, but *only specify the problem*.

- Pure SAT: only boolean variables, no numbers, no inequalities.
- For every position (i, j) on the board: boolean variable p_{ij} expresses whether there is a queen or not.

问题: Eight-Queens

The eight queens puzzle is the problem of placing eight chess queens on an 8×8 chessboard so that *no two queens attack each other*. Thus, a solution requires that *no two queens* share the *same row, column, or diagonal*.

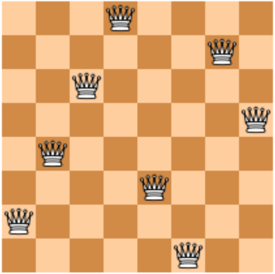


As usual in SAT/SMT, don't think about how to solve it, but *only specify the problem*.

- Pure SAT: only boolean variables, no numbers, no inequalities.
- For every position (i, j) on the board: boolean variable p_{ij} expresses whether there is a queen or not.

问题: Eight-Queens

The eight queens puzzle is the problem of placing eight chess queens on an 8×8 chessboard so that *no two queens attack each other*. Thus, a solution requires that *no two queens* share the *same row, column, or diagonal*.

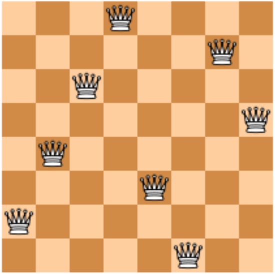


As usual in SAT/SMT, don't think about how to solve it, but *only specify the problem*.

- Pure SAT: only boolean variables, no numbers, no inequalities.
- For every position (i, j) on the board: boolean variable p_{ij} expresses whether there is a queen or not.

问题: Eight-Queens

The eight queens puzzle is the problem of placing eight chess queens on an 8×8 chessboard so that *no two queens attack each other*. Thus, a solution requires that *no two queens* share the *same row, column, or diagonal*.



As usual in SAT/SMT, don't think about how to solve it, but *only specify the problem*.

- Pure SAT: only boolean variables, no numbers, no inequalities.
- For every position (i, j) on the board: boolean variable p_{ij} expresses whether there is a queen or not.

p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}
p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}
p_{41}	p_{42}	p_{43}	p_{44}	p_{45}	p_{46}	p_{47}	p_{48}
p_{51}	p_{52}	p_{53}	p_{54}	p_{55}	p_{56}	p_{57}	p_{58}
p_{61}	p_{62}	p_{63}	p_{64}	p_{65}	p_{66}	p_{67}	p_{68}
p_{71}	p_{72}	p_{73}	p_{74}	p_{75}	p_{76}	p_{77}	p_{78}
p_{81}	p_{82}	p_{83}	p_{84}	p_{85}	p_{86}	p_{87}	p_{88}

(1) At least one queen on row i:

- $p_{i1} \vee p_{i2} \vee p_{i3} \vee p_{i4} \vee p_{i5} \vee p_{i6} \vee p_{i7} \vee p_{i8}$

$$\bigvee_{j=1}^{8} p_{ij}$$

(2) At most one queen on row i:

- For every $j < k$ not both p_{ij} and p_{ik} are true

$$\bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

Requirements until now (on *every* row):

$$\bigwedge_{i=1}^{8} \bigvee_{j=1}^{8} p_{ij} \quad \wedge \quad \bigwedge_{i=1}^{8} \bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}
p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}
p_{41}	p_{42}	p_{43}	p_{44}	p_{45}	p_{46}	p_{47}	p_{48}
p_{51}	p_{52}	p_{53}	p_{54}	p_{55}	p_{56}	p_{57}	p_{58}
p_{61}	p_{62}	p_{63}	p_{64}	p_{65}	p_{66}	p_{67}	p_{68}
p_{71}	p_{72}	p_{73}	p_{74}	p_{75}	p_{76}	p_{77}	p_{78}
p_{81}	p_{82}	p_{83}	p_{84}	p_{85}	p_{86}	p_{87}	p_{88}

(1) At least one queen on row i:

- $p_{i1} \vee p_{i2} \vee p_{i3} \vee p_{i4} \vee p_{i5} \vee p_{i6} \vee p_{i7} \vee p_{i8}$

$$\bigvee_{j=1}^{8} p_{ij}$$

(2) At most one queen on row i:

- For every $j < k$ not both p_{ij} and p_{ik} are true

$$\bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

Requirements until now (on *every* row):

$$\bigwedge_{i=1}^{8} \bigvee_{j=1}^{8} p_{ij} \quad \wedge \quad \bigwedge_{i=1}^{8} \bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}
p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}
p_{41}	p_{42}	p_{43}	p_{44}	p_{45}	p_{46}	p_{47}	p_{48}
p_{51}	p_{52}	p_{53}	p_{54}	p_{55}	p_{56}	p_{57}	p_{58}
p_{61}	p_{62}	p_{63}	p_{64}	p_{65}	p_{66}	p_{67}	p_{68}
p_{71}	p_{72}	p_{73}	p_{74}	p_{75}	p_{76}	p_{77}	p_{78}
p_{81}	p_{82}	p_{83}	p_{84}	p_{85}	p_{86}	p_{87}	p_{88}

(1) At least one queen on row i:

- $p_{i1} \vee p_{i2} \vee p_{i3} \vee p_{i4} \vee p_{i5} \vee p_{i6} \vee p_{i7} \vee p_{i8}$

$$\bigvee_{j=1}^{8} p_{ij}$$

(2) At most one queen on row i:

- For every $j < k$ not both p_{ij} and p_{ik} are true

$$\bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

Requirements until now (on *every* row):

$$\bigwedge_{i=1}^{8} \bigvee_{j=1}^{8} p_{ij} \quad \wedge \quad \bigwedge_{i=1}^{8} \bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}
p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}
p_{41}	p_{42}	p_{43}	p_{44}	p_{45}	p_{46}	p_{47}	p_{48}
p_{51}	p_{52}	p_{53}	p_{54}	p_{55}	p_{56}	p_{57}	p_{58}
p_{61}	p_{62}	p_{63}	p_{64}	p_{65}	p_{66}	p_{67}	p_{68}
p_{71}	p_{72}	p_{73}	p_{74}	p_{75}	p_{76}	p_{77}	p_{78}
p_{81}	p_{82}	p_{83}	p_{84}	p_{85}	p_{86}	p_{87}	p_{88}

(1) At least one queen on row i:

- $p_{i1} \vee p_{i2} \vee p_{i3} \vee p_{i4} \vee p_{i5} \vee p_{i6} \vee p_{i7} \vee p_{i8}$

$$\bigvee_{j=1}^{8} p_{ij}$$

(2) At most one queen on row i:

- For every $j < k$ not both p_{ij} and p_{ik} are true

$$\bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

Requirements until now (on *every* row):

$$\bigwedge_{i=1}^{8} \bigvee_{j=1}^{8} p_{ij} \quad \wedge \quad \bigwedge_{i=1}^{8} \bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}
p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}
p_{41}	p_{42}	p_{43}	p_{44}	p_{45}	p_{46}	p_{47}	p_{48}
p_{51}	p_{52}	p_{53}	p_{54}	p_{55}	p_{56}	p_{57}	p_{58}
p_{61}	p_{62}	p_{63}	p_{64}	p_{65}	p_{66}	p_{67}	p_{68}
p_{71}	p_{72}	p_{73}	p_{74}	p_{75}	p_{76}	p_{77}	p_{78}
p_{81}	p_{82}	p_{83}	p_{84}	p_{85}	p_{86}	p_{87}	p_{88}

(1) At least one queen on row i:

- $p_{i1} \vee p_{i2} \vee p_{i3} \vee p_{i4} \vee p_{i5} \vee p_{i6} \vee p_{i7} \vee p_{i8}$

$$\bigvee_{j=1}^{8} p_{ij}$$

(2) At most one queen on row i:

- For every $j < k$ not both p_{ij} and p_{ik} are true

$$\bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

Requirements until now (on *every* row):

$$\bigwedge_{i=1}^{8} \bigvee_{j=1}^{8} p_{ij} \quad \wedge \quad \bigwedge_{i=1}^{8} \bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}
p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}
p_{41}	p_{42}	p_{43}	p_{44}	p_{45}	p_{46}	p_{47}	p_{48}
p_{51}	p_{52}	p_{53}	p_{54}	p_{55}	p_{56}	p_{57}	p_{58}
p_{61}	p_{62}	p_{63}	p_{64}	p_{65}	p_{66}	p_{67}	p_{68}
p_{71}	p_{72}	p_{73}	p_{74}	p_{75}	p_{76}	p_{77}	p_{78}
p_{81}	p_{82}	p_{83}	p_{84}	p_{85}	p_{86}	p_{87}	p_{88}

(1) At least one queen on row i:

- $p_{i1} \vee p_{i2} \vee p_{i3} \vee p_{i4} \vee p_{i5} \vee p_{i6} \vee p_{i7} \vee p_{i8}$

$$\bigvee_{j=1}^{8} p_{ij}$$

(2) At most one queen on row i:

- For every $j < k$ not both p_{ij} and p_{ik} are true

$$\bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

Requirements until now (on *every* row):

$$\bigwedge_{i=1}^{8} \bigvee_{j=1}^{8} p_{ij} \quad \wedge \quad \bigwedge_{i=1}^{8} \bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}
p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}
p_{41}	p_{42}	p_{43}	p_{44}	p_{45}	p_{46}	p_{47}	p_{48}
p_{51}	p_{52}	p_{53}	p_{54}	p_{55}	p_{56}	p_{57}	p_{58}
p_{61}	p_{62}	p_{63}	p_{64}	p_{65}	p_{66}	p_{67}	p_{68}
p_{71}	p_{72}	p_{73}	p_{74}	p_{75}	p_{76}	p_{77}	p_{78}
p_{81}	p_{82}	p_{83}	p_{84}	p_{85}	p_{86}	p_{87}	p_{88}

(1) At least one queen on row i:

- $p_{i1} \vee p_{i2} \vee p_{i3} \vee p_{i4} \vee p_{i5} \vee p_{i6} \vee p_{i7} \vee p_{i8}$

$$\bigvee_{j=1}^{8} p_{ij}$$

(2) At most one queen on row i:

- For every $j < k$ not both p_{ij} and p_{ik} are true

$$\bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

Requirements until now (on *every* row):

$$\bigwedge_{i=1}^{8} \bigvee_{j=1}^{8} p_{ij} \quad \wedge \quad \bigwedge_{i=1}^{8} \bigwedge_{0<j<k\leq 8} (\neg p_{ij} \vee \neg p_{ik})$$

p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}
p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}
p_{41}	p_{42}	p_{43}	p_{44}	p_{45}	p_{46}	p_{47}	p_{48}
p_{51}	p_{52}	p_{53}	p_{54}	p_{55}	p_{56}	p_{57}	p_{58}
p_{61}	p_{62}	p_{63}	p_{64}	p_{65}	p_{66}	p_{67}	p_{68}
p_{71}	p_{72}	p_{73}	p_{74}	p_{75}	p_{76}	p_{77}	p_{78}
p_{81}	p_{82}	p_{83}	p_{84}	p_{85}	p_{86}	p_{87}	p_{88}

DIY

Similarly, on every *column*:

$$\bigwedge_{j=1}^{8} \bigvee_{i=1}^{8} p_{ij} \quad \wedge \quad \bigwedge_{j=1}^{8} \bigwedge_{0<i<k\leq 8} (\neg p_{ij} \vee \neg p_{kj})$$

p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}
p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}
p_{41}	p_{42}	p_{43}	p_{44}	p_{45}	p_{46}	p_{47}	p_{48}
p_{51}	p_{52}	p_{53}	p_{54}	p_{55}	p_{56}	p_{57}	p_{58}
p_{61}	p_{62}	p_{63}	p_{64}	p_{65}	p_{66}	p_{67}	p_{68}
p_{71}	p_{72}	p_{73}	p_{74}	p_{75}	p_{76}	p_{77}	p_{78}
p_{81}	p_{82}	p_{83}	p_{84}	p_{85}	p_{86}	p_{87}	p_{88}

p_{ij} and $p_{i'j'}$ on such a diagnoal

$$i - j = i' - j'$$

1. 应用

1.3 案例实现 | Eight-Queens

p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}
p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}
p_{41}	p_{42}	p_{43}	p_{44}	p_{45}	p_{46}	p_{47}	p_{48}
p_{51}	p_{52}	p_{53}	p_{54}	p_{55}	p_{56}	p_{57}	p_{58}
p_{61}	p_{62}	p_{63}	p_{64}	p_{65}	p_{66}	p_{67}	p_{68}
p_{71}	p_{72}	p_{73}	p_{74}	p_{75}	p_{76}	p_{77}	p_{78}
p_{81}	p_{82}	p_{83}	p_{84}	p_{85}	p_{86}	p_{87}	p_{88}

p_{ij} and $p_{i'j'}$ on such a diagonal

$$i + j = i' + j'$$

So far all i, j, i', j' with $(i,j) \neq (i',j')$ satisfying $i+j = i'+j'$ or $i-j = i'-j'$:

$$\neg p_{ij} \vee \neg p_{i'j'}$$

stating that on (i,j) and (i',j') being two distinct positions on a diagonal, no two queens are allowed.

We may restrict to $i < i'$, yielding

$$\bigwedge_{0<i<i'\leq 8} \Big(\bigwedge_{j,j': i+j=i'+j' \vee i-j=i'-j'} \neg p_{ij} \vee \neg p_{i'j'} \Big)$$

So far all i, j, i', j' with $(i, j) \neq (i', j')$ satisfying $i + j = i' + j'$ or $i - j = i' - j'$:

$$\neg p_{ij} \vee \neg p_{i'j'}$$

stating that on (i, j) and (i', j') being two distinct positions on a diagonal, no two queens are allowed.

We may restrict to $i < i'$, yielding

$$\bigwedge_{0<i<i'\leq 8} \Big(\bigwedge_{j,j':i+j=i'+j' \vee i-j=i'-j'} \neg p_{ij} \vee \neg p_{i'j'} \Big)$$

$$\bigwedge_{i=1}^{8}\bigvee_{j=1}^{8} p_{ij} \wedge \bigwedge_{i=1}^{8}\bigwedge_{0<j<k\leq 8}(\neg p_{ij} \vee \neg p_{ik}) \wedge \bigwedge_{j=1}^{8}\bigvee_{i=1}^{8} p_{ij} \wedge \bigwedge_{j=1}^{8}\bigwedge_{0<i<k\leq 8}(\neg p_{ij} \vee \neg p_{kj})$$

$$\bigwedge_{0<i<i'\leq 8}\Big(\bigwedge_{j,j':i+j=i'+j' \vee i-j=i'-j'} \neg p_{ij} \vee \neg p_{i'j'}\Big)$$

```python
from z3 import *
Q = [ Int('Q_%i' % (i + 1)) for i in range(8) ]
val_c = [ And(1 <= Q[i], Q[i] <= 8) for i in range(8) ]
col_c = [ Distinct(Q) ]
diag_c = [ If(i == j, True,
              And(i+Q[i]!=j+Q[j], i+Q[j]!=j+Q[i]))
           for i in range(8) for j in range(i) ]
solve(val_c + col_c + diag_c)
```

$$\bigwedge_{i=1}^{8}\bigvee_{j=1}^{8}p_{ij}\wedge\bigwedge_{i=1}^{8}\bigwedge_{0<j<k\leq 8}(\neg p_{ij}\vee\neg p_{ik})\wedge\bigwedge_{j=1}^{8}\bigvee_{i=1}^{8}p_{ij}\wedge\bigwedge_{j=1}^{8}\bigwedge_{0<i<k\leq 8}(\neg p_{ij}\vee\neg p_{kj})$$

$$\bigwedge_{0<i<i'\leq 8}\Big(\bigwedge_{j,j':i+j=i'+j'\vee i-j=i'-j'}\neg p_{ij}\vee\neg p_{i'j'}\Big)$$

```python
from z3 import *
Q = [ Int('Q_%i' % (i + 1)) for i in range(8) ]
val_c = [ And(1 <= Q[i], Q[i] <= 8) for i in range(8) ]
col_c = [ Distinct(Q) ]
diag_c = [ If(i == j, True,
              And(i+Q[i]!=j+Q[j], i+Q[j]!=j+Q[i]))
           for i in range(8) for j in range(i) ]
solve(val_c + col_c + diag_c)
```

			o				
	o						
							o
					o		
o							
		o					
				o			
						o	

运行结果:

- $python3 z3-4-queens.py
  - [Q_5 = 1, Q_8 = 7, Q_3 = 8, Q_2 = 2, Q_6 = 3, Q_4 = 6, Q_7 = 5, Q_1 = 4]

解析:

- From Eight-Queens to N-Queens?
  - 26s for 40 queens

思考: 不使用 SMT, 而是换之前解释的方法（pure SAT）, 效率会不会更高?

- 看下面讲解

			o				
	o						
							o
					o		
o							
		o					
				o			
						o	

运行结果:

$python3 z3-4-queens.py

[Q_5 = 1, Q_8 = 7, Q_3 = 8, Q_2 = 2,
Q_6 = 3, Q_4 = 6, Q_7 = 5, Q_1 = 4]

解析:

From Eight-Queens to N-Queens?

26s for 40 queens

思考: 不使用 SMT, 而是换之前解释的方法（pure SAT），效率会不会更高?

看下面讲解

			o				
	o						
							o
					o		
o							
		o					
				o			
						o	

运行结果:

$python3 z3-4-queens.py

[Q_5 = 1, Q_8 = 7, Q_3 = 8, Q_2 = 2,
Q_6 = 3, Q_4 = 6, Q_7 = 5, Q_1 = 4]

解析:

From Eight-Queens to N-Queens?

26s for 40 queens

思考: 不使用 SMT, 而是换之前解释的方法（pure SAT）, 效率会不会更高?

看下面讲解

			o				
	o						
							o
					o		
o							
		o					
				o			
						o	

运行结果:

$python3 z3-4-queens.py

[Q_5 = 1, Q_8 = 7, Q_3 = 8, Q_2 = 2,
Q_6 = 3, Q_4 = 6, Q_7 = 5, Q_1 = 4]

解析:

From Eight-Queens to N-Queens?

26s for 40 queens

思考: 不使用 SMT, 而是换之前解释的方法 (pure SAT), 效率会不会更高?

看下面讲解

Arithmetic: addition, subtraction, multiplication of integers

问题

Compute 13+7?

解析: Covered by SMT, why do it in pure SAT?

- Interesting how to express a non-SAT looking problem in SAT
- Introduces flavor of *bounded model checking*
  - BMC, 有界模型检测, 见第 5 章
  - one of the most important applications of SAT/SMT
- Often (e.g., in hardware verification), SAT encoding of arithmetic *outperforms* SMT

Arithmetic: addition, subtraction, multiplication of integers

问题

Compute 13+7?

解析: Covered by SMT, why do it in pure SAT?

- Interesting how to express a non-SAT looking problem in SAT
- Introduces flavor of *bounded model checking*
  - BMC, 有界模型检测, 见第 5 章
  - one of the most important applications of SAT/SMT
- Often (e.g., in hardware verification), SAT encoding of arithmetic *outperforms* SMT

Arithmetic: addition, subtraction, multiplication of integers

问题

Compute 13+7?

解析: Covered by SMT, why do it in pure SAT?

- Interesting how to express a non-SAT looking problem in SAT
- Introduces flavor of *bounded model checking*
  - BMC, 有界模型检测, 见第 5 章
  - one of the most important applications of SAT/SMT
- Often (e.g., in hardware verification), SAT encoding of arithmetic *outperforms* SMT

Arithmetic: addition, subtraction, multiplication of integers

问题

Compute 13+7?

解析: Covered by SMT, why do it in pure SAT?

- Interesting how to express a non-SAT looking problem in SAT
- Introduces flavor of *bounded model checking*
  - BMC, 有界模型检测, 见第 5 章
  - one of the most important applications of SAT/SMT
- Often (e.g., in hardware verification), SAT encoding of arithmetic *outperforms* SMT

Arithmetic: addition, subtraction, multiplication of integers

问题

Compute 13+7?

解析: Covered by SMT, why do it in pure SAT?

- Interesting how to express a non-SAT looking problem in SAT
- Introduces flavor of *bounded model checking*
  - BMC, 有界模型检测, 见第 5 章
  - one of the most important applications of SAT/SMT
- Often (e.g., in hardware verification), SAT encoding of arithmetic *outperforms* SMT

Arithmetic: addition, subtraction, multiplication of integers

问题

Compute 13+7?

解析: Covered by SMT, why do it in pure SAT?

- Interesting how to express a non-SAT looking problem in SAT
- Introduces flavor of *bounded model checking*
  - BMC, 有界模型检测, 见第 5 章
  - one of the most important applications of SAT/SMT
- Often (e.g., in hardware verification), SAT encoding of arithmetic *outperforms* SMT

In SAT we only have *Boolean variables*.

问题: Binary representation

How to Express a number by a sequence of Boolean values?

解: Binary representation

$$a_1 a_2 \cdots a_n$$

of number a:

$$a_i \in \{0,1\} \text{ and}$$

$$a = a_n + 2a_{n-1} + 4a_{n-2} + \cdots = \sum_{i=1}^{n} a_i * 2^{n-i}$$

例:

- 01101 represents 8+4+1=13
- 00111 represents 4+2+1=7

In SAT we only have *Boolean variables*.

问题: Binary representation

How to Express a number by a sequence of Boolean values?

解: Binary representation

$$a_1 a_2 \cdots a_n$$

of number a:

$$a_i \in \{0,1\} \text{ and}$$

$$a = a_n + 2a_{n-1} + 4a_{n-2} + \cdots = \sum_{i=1}^{n} a_i * 2^{n-i}$$

例:

- 01101 represents 8+4+1=13
- 00111 represents 4+2+1=7

In SAT we only have *Boolean variables*.

问题: Binary representation

How to Express a number by a sequence of Boolean values?

解: Binary representation

$$a_1 a_2 \cdots a_n$$

of number a:

$$a_i \in \{0, 1\} \text{ and}$$

$$a = a_n + 2a_{n-1} + 4a_{n-2} + \cdots = \sum_{i=1}^{n} a_i * 2^{n-i}$$

例:

- 01101 represents 8+4+1=13
- 00111 represents 4+2+1=7

In SAT we only have *Boolean variables*.

问题: Binary representation

How to Express a number by a sequence of Boolean values?

解: Binary representation

$$a_1 a_2 \cdots a_n$$

of number a:

$$a_i \in \{0, 1\} \text{ and}$$

$$a = a_n + 2a_{n-1} + 4a_{n-2} + \cdots = \sum_{i=1}^{n} a_i * 2^{n-i}$$

例:

- 01101 represents 8+4+1=13
- 00111 represents 4+2+1=7

问题: Add

How to compute $d = a + b$?

- Basic rules: Take care of *carry* c

- $0 + 0 + 0 = 0$, carry $= 0$
- $0 + 0 + 1 = 1$, carry $= 0$
- $0 + 1 + 1 = 0$, carry $= 1$
- $1 + 1 + 1 = 1$, carry $= 1$

Start by rightmost carry $= 0$, compute from right to left

carries c:	0	1	1	1	1	0
number a=13:		0	1	1	0	1
number b=7:		0	0	1	1	1
result d		1	0	1	0	0

indeed yielding 10100 representing 20

问题: Add

How to compute $d = a + b$?

- Basic rules: Take care of *carry* c

- $0+0+0=0$, carry $=0$
- $0+0+1=1$, carry $=0$
- $0+1+1=0$, carry $=1$
- $1+1+1=1$, carry $=1$

Start by rightmost carry $=0$, compute from right to left

carries c:	0	1	1	1	1	0
number a=13:		0	1	1	0	1
number b=7:		0	0	1	1	1
result d		1	0	1	0	0

indeed yielding 10100 representing 20

问题: Add

How to compute $d = a + b$?

- Basic rules: Take care of *carry* c

- $0 + 0 + 0 = 0$, carry $= 0$
- $0 + 0 + 1 = 1$, carry $= 0$
- $0 + 1 + 1 = 0$, carry $= 1$
- $1 + 1 + 1 = 1$, carry $= 1$

Start by rightmost carry $= 0$, compute from right to left

carries c:	0	1	1	1	1	0
number a=13:		0	1	1	0	1
number b=7:		0	0	1	1	1
result d		1	0	1	0	0

indeed yielding 10100 representing 20

问题: $d = a + b$?

Result d_i in a formula for $i = 1, \ldots, n$:

$$d_i \leftrightarrow (a_i \leftrightarrow (b_i \leftrightarrow c_i)) \quad (1)$$

correct, since $(a_i \leftrightarrow (b_i \leftrightarrow c_i)$ yields true if and only if 1 or 3 among $\{a_i, b_i, c_i\}$ yield true

Carry c_{i-1} in a formula for $i = 1, \ldots, n$:

$$c_{i-1} \leftrightarrow ((a_i \wedge b_i) \vee (a_i \wedge c_i) \vee (b_i \wedge c_i)) \quad (2)$$

correct, since $(a_i \wedge b_i) \vee (a_i \wedge c_i) \vee (b_i \wedge c_i)$ yields true if and only at least 2 among $\{a_i, b_i, c_i\}$ yield true.

问题: $d = a + b$?
Result d_i in a formula for $i = 1, \ldots, n$:

$$d_i \leftrightarrow (a_i \leftrightarrow (b_i \leftrightarrow c_i)) \quad (1)$$

correct, since $(a_i \leftrightarrow (b_i \leftrightarrow c_i)$ yields true if and only if 1 or 3 among $\{a_i, b_i, c_i\}$ yield true

Carry c_{i-1} in a formula for $i = 1, \ldots, n$:

$$c_{i-1} \leftrightarrow ((a_i \wedge b_i) \vee (a_i \wedge c_i) \vee (b_i \wedge c_i)) \quad (2)$$

correct, since $(a_i \wedge b_i) \vee (a_i \wedge c_i) \vee (b_i \wedge c_i)$ yields true if and only at least 2 among $\{a_i, b_i, c_i\}$ yield true.

问题: $d = a + b$?
Result d_i in a formula for $i = 1, \ldots, n$:

$$d_i \leftrightarrow (a_i \leftrightarrow (b_i \leftrightarrow c_i)) \tag{1}$$

correct, since $(a_i \leftrightarrow (b_i \leftrightarrow c_i)$ yields true if and only if 1 or 3 among $\{a_i, b_i, c_i\}$ yield true

Carry c_{i-1} in a formula for $i = 1, \ldots, n$:

$$c_{i-1} \leftrightarrow ((a_i \wedge b_i) \vee (a_i \wedge c_i) \vee (b_i \wedge c_i)) \tag{2}$$

correct, since $(a_i \wedge b_i) \vee (a_i \wedge c_i) \vee (b_i \wedge c_i)$ yields true if and only at least 2 among $\{a_i, b_i, c_i\}$ yield true.

问题: $d = a + b$?
Result d_i in a formula for $i = 1, \ldots, n$:

$$d_i \leftrightarrow (a_i \leftrightarrow (b_i \leftrightarrow c_i)) \quad (1)$$

correct, since $(a_i \leftrightarrow (b_i \leftrightarrow c_i)$ yields true if and only if 1 or 3 among $\{a_i, b_i, c_i\}$ yield true

Carry c_{i-1} in a formula for $i = 1, \ldots, n$:

$$c_{i-1} \leftrightarrow ((a_i \wedge b_i) \vee (a_i \wedge c_i) \vee (b_i \wedge c_i)) \quad (2)$$

correct, since $(a_i \wedge b_i) \vee (a_i \wedge c_i) \vee (b_i \wedge c_i)$ yields true if and only at least 2 among $\{a_i, b_i, c_i\}$ yield true.

问题: $d = a + b$?
Result d_i in a formula for $i = 1, \ldots, n$:

$$d_i \leftrightarrow (a_i \leftrightarrow (b_i \leftrightarrow c_i)) \tag{1}$$

correct, since $(a_i \leftrightarrow (b_i \leftrightarrow c_i)$ yields true if and only if 1 or 3 among $\{a_i, b_i, c_i\}$ yield true

Carry c_{i-1} in a formula for $i = 1, \ldots, n$:

$$c_{i-1} \leftrightarrow ((a_i \wedge b_i) \vee (a_i \wedge c_i) \vee (b_i \wedge c_i)) \tag{2}$$

correct, since $(a_i \wedge b_i) \vee (a_i \wedge c_i) \vee (b_i \wedge c_i)$ yields true if and only at least 2 among $\{a_i, b_i, c_i\}$ yield true.

To express that we start by rightmost carry = 0, we state

$$\neg c_n \tag{3}$$

To express that the result should fit in n bits, at the end we should not have a carry left, and we state

$$\neg c_0 \tag{4}$$

Let ϕ be the conjunction of all these requirements; this expresses the *correctness* of the corresponding binary addition

To compute $a+b$ for $a=13, b=7$, we apply a SAT solver to

$$\phi \wedge \underbrace{\neg a_1 \wedge a_2 \wedge a_3 \wedge \neg a_4 \wedge a_5}_{a=13=01101} \wedge \underbrace{\neg b_1 \wedge \neg b_2 \wedge b_3 \wedge b_4 \wedge b_5}_{b=7=00111}$$

The resulting satisfying assignment will contain $d_1, \neg d_2, d_3, \neg d_4, \neg d_5$ representing the result $d=20$

To express that we start by rightmost carry $= 0$, we state

$$\neg c_n \tag{3}$$

To express that the result should fit in n bits, at the end we should not have a carry left, and we state

$$\neg c_0 \tag{4}$$

Let ϕ be the conjunction of all these requirements; this expresses the *correctness* of the corresponding binary addition

To compute $a + b$ for $a = 13, b = 7$, we apply a SAT solver to

$$\phi \wedge \underbrace{\neg a_1 \wedge a_2 \wedge a_3 \wedge \neg a_4 \wedge a_5}_{a=13=01101} \wedge \underbrace{\neg b_1 \wedge \neg b_2 \wedge b_3 \wedge b_4 \wedge b_5}_{b=7=00111}$$

The resulting satisfying assignment will contain $d_1, \neg d_2, d_3, \neg d_4, \neg d_5$ representing the result $d = 20$

To express that we start by rightmost carry $= 0$, we state

$$\neg c_n \tag{3}$$

To express that the result should fit in n bits, at the end we should not have a carry left, and we state

$$\neg c_0 \tag{4}$$

Let ϕ be the conjunction of all these requirements; this expresses the *correctness* of the corresponding binary addition

To compute $a + b$ for $a = 13, b = 7$, we apply a SAT solver to

$$\phi \wedge \underbrace{\neg a_1 \wedge a_2 \wedge a_3 \wedge \neg a_4 \wedge a_5}_{a=13=01101} \wedge \underbrace{\neg b_1 \wedge \neg b_2 \wedge b_3 \wedge b_4 \wedge b_5}_{b=7=00111}$$

The resulting satisfying assignment will contain $d_1, \neg d_2, d_3, \neg d_4, \neg d_5$ representing the result $d = 20$

To express that we start by rightmost carry = 0, we state

$$\neg c_n \tag{3}$$

To express that the result should fit in n bits, at the end we should not have a carry left, and we state

$$\neg c_0 \tag{4}$$

Let ϕ be the conjunction of all these requirements; this expresses the *correctness* of the corresponding binary addition

To compute $a + b$ for $a = 13, b = 7$, we apply a SAT solver to

$$\phi \wedge \underbrace{\neg a_1 \wedge a_2 \wedge a_3 \wedge \neg a_4 \wedge a_5}_{a=13=01101} \wedge \underbrace{\neg b_1 \wedge \neg b_2 \wedge b_3 \wedge b_4 \wedge b_5}_{b=7=00111}$$

The resulting satisfying assignment will contain $d_1, \neg d_2, d_3, \neg d_4, \neg d_5$ representing the result $d = 20$

To express that we start by rightmost carry $= 0$, we state

$$\neg c_n \tag{3}$$

To express that the result should fit in n bits, at the end we should not have a carry left, and we state

$$\neg c_0 \tag{4}$$

Let ϕ be the conjunction of all these requirements; this expresses the *correctness* of the corresponding binary addition

To compute $a + b$ for $a = 13, b = 7$, we apply a SAT solver to

$$\phi \wedge \underbrace{\neg a_1 \wedge a_2 \wedge a_3 \wedge \neg a_4 \wedge a_5}_{a=13=01101} \wedge \underbrace{\neg b_1 \wedge \neg b_2 \wedge b_3 \wedge b_4 \wedge b_5}_{b=7=00111}$$

The resulting satisfying assignment will contain $d_1, \neg d_2, d_3, \neg d_4, \neg d_5$ representing the result $d = 20$

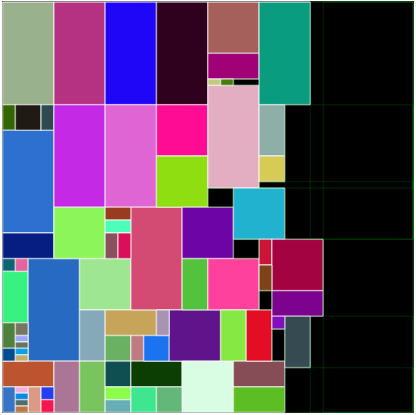
Concluding,

In this way computing $d = a + b$ can be done by SAT solving for any binary numbers a, b.

By adding given values for a_i, d_i to the formula ϕ expressing $d = a + b$, and reading b_i from resulting satisfying assignment, we can compute $b = d - a$ by exploiting the same formula.

问题: Rectangle fitting

Given *a big rectangle* and *a number of small rectangles*, can you fit the small rectangles in the big one such that *no two overlap*.

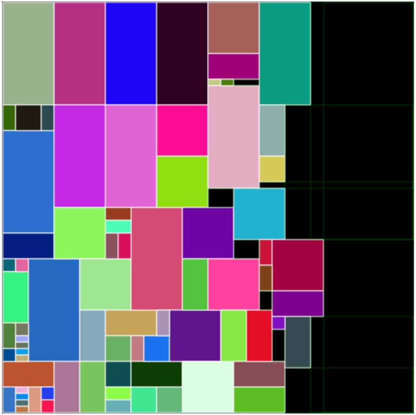


How to specify this problem?

- Number rectangles from 1 to n
- for $i = 1 \ldots n$ introduce variables:
  - w_i is the width of rectangle i
  - h_i is the height of rectangle i
  - x_i is the x-coordinate of the left lower corner of rectangle i
  - y_i is the y-coordinate of the left lower corner of rectangle i

问题: Rectangle fitting

Given *a big rectangle* and *a number of small rectangles*, can you fit the small rectangles in the big one such that *no two overlap*.

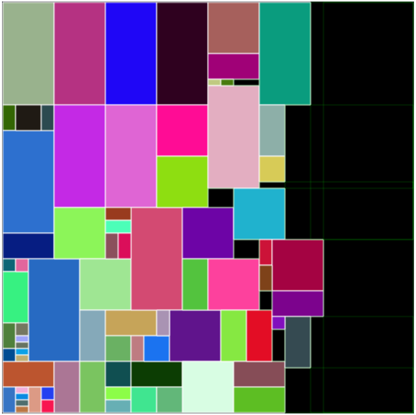


How to specify this problem?

- Number rectangles from 1 to n
- for $i = 1 \ldots n$ introduce variables:
  - w_i is the width of rectangle i
  - h_i is the height of rectangle i
  - x_i is the x-coordinate of the left lower corner of rectangle i
  - y_i is the y-coordinate of the left lower corner of rectangle i

问题: Rectangle fitting

Given *a big rectangle* and *a number of small rectangles*, can you fit the small rectangles in the big one such that *no two overlap*.

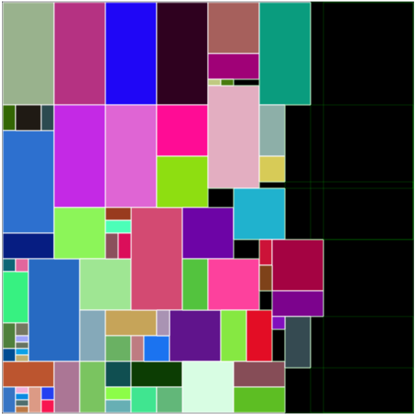


How to specify this problem?

- Number rectangles from 1 to n
- for $i = 1 \dots n$ introduce variables:
  - w_i is the width of rectangle i
  - h_i is the height of rectangle i
  - x_i is the x-coordinate of the left lower corner of rectangle i
  - y_i is the y-coordinate of the left lower corner of rectangle i

问题: Rectangle fitting

Given *a big rectangle* and *a number of small rectangles*, can you fit the small rectangles in the big one such that *no two overlap*.



How to specify this problem?

- Number rectangles from 1 to n
- for $i = 1 \ldots n$ introduce variables:
  - w_i is the width of rectangle i
  - h_i is the height of rectangle i
  - x_i is the x-coordinate of the left lower corner of rectangle i
  - y_i is the y-coordinate of the left lower corner of rectangle i

How to specify this problem?

- Configuration of small rectangles:
  - 例子: First rectangle has width 4 and height 6:
  - $(w_1 = 4 \wedge h_1 = 6) \vee (w_1 = 6 \wedge h_1 = 4)$
- Configuration of the big rectangle
  - (0,0) = lower left corner of big rectangle.
  - W = width of big rectangle.
  - H = height of big rectangle.
- Requirements:

$$x_i \geq 0 \wedge x_i + w_i \leq W$$

$$y_i \geq 0 \wedge y_i + h_i \leq H$$

for all $i = 1, \ldots, n$

How to specify this problem?

- Configuration of small rectangles:
  - 例子: First rectangle has width 4 and height 6:
  - $(w_1 = 4 \wedge h_1 = 6) \vee (w_1 = 6 \wedge h_1 = 4)$
- Configuration of the big rectangle
  - (0,0) = lower left corner of big rectangle.
  - W = width of big rectangle.
  - H = height of big rectangle.
- Requirements:

$$x_i \geq 0 \wedge x_i + w_i \leq W$$

$$y_i \geq 0 \wedge y_i + h_i \leq H$$

for all $i = 1, \ldots, n$

How to specify this problem?

- Configuration of small rectangles:
  - 例子: First rectangle has width 4 and height 6:
  - $(w_1 = 4 \wedge h_1 = 6) \vee (w_1 = 6 \wedge h_1 = 4)$
- Configuration of the big rectangle
  - (0,0) $=$ lower left corner of big rectangle.
  - W $=$ width of big rectangle.
  - H $=$ height of big rectangle.
- Requirements:

$$x_i \geq 0 \wedge x_i + w_i \leq W$$

$$y_i \geq 0 \wedge y_i + h_i \leq H$$

for all $i = 1, \ldots, n$

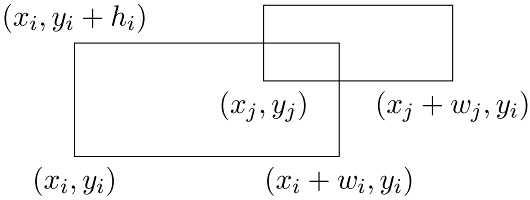
How to specify this problem?

- Configuration of small rectangles:
  - 例子: First rectangle has width 4 and height 6:
  - $(w_1 = 4 \wedge h_1 = 6) \vee (w_1 = 6 \wedge h_1 = 4)$
- Configuration of the big rectangle
  - (0,0) = lower left corner of big rectangle.
  - W = width of big rectangle.
  - H = height of big rectangle.
- Requirements:

$$x_i \geq 0 \wedge x_i + w_i \leq W$$

$$y_i \geq 0 \wedge y_i + h_i \leq H$$

for all $i = 1, \ldots, n$



Rectangles i and j overlap if

- $x_j < x_i + w_i$
- and $x_i < x_j + w_j$
- and $y_j < y_i + h_i$
- and $y_i < y_j + h_j$

So for all $i, j = 1, \ldots, n, i < j$, we should add the negation of this overlappingness:

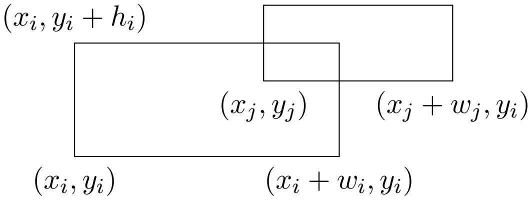
$$\neg(x_j < x_i + w_i \wedge x_i < x_j + w_j \wedge y_j < y_i + h_i \wedge y_i < y_j + h_j)$$

or, equivalently

$$x_j \geq x_i + w_i \vee x_i \geq x_j + w_j \vee y_j \geq y_i + h_i \vee y_i \geq y_j + h_j$$

1. 应用

1.3 案例实现 | Rectangle fitting



Rectangles i and j overlap if

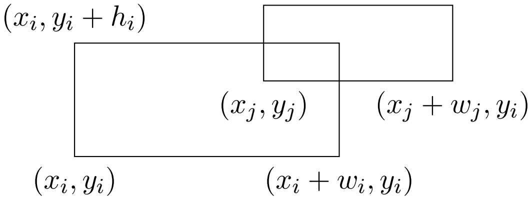
- $x_j < x_i + w_i$
- and $x_i < x_j + w_j$
- and $y_j < y_i + h_i$
- and $y_i < y_j + h_j$

So for all $i, j = 1, \ldots, n, i < j$, we should add the negation of this overlappingness:

$$\neg(x_j < x_i + w_i \wedge x_i < x_j + w_j \wedge y_j < y_i + h_i \wedge y_i < y_j + h_j)$$

or, equivalently

$$x_j \geq x_i + w_i \vee x_i \geq x_j + w_j \vee y_j \geq y_i + h_i \vee y_i \geq y_j + h_j$$



Rectangles i and j overlap if

- $x_j < x_i + w_i$
- and $x_i < x_j + w_j$
- and $y_j < y_i + h_i$
- and $y_i < y_j + h_j$

So for all $i, j = 1, \ldots, n, i < j$, we should add the negation of this overlappingness:

$$\neg(x_j < x_i + w_i \wedge x_i < x_j + w_j \wedge y_j < y_i + h_i \wedge y_i < y_j + h_j)$$

or, equivalently

$$x_j \geq x_i + w_i \vee x_i \geq x_j + w_j \vee y_j \geq y_i + h_i \vee y_i \geq y_j + h_j$$

The following formula is *satisfiable* iff the fitting problem *has a solution*

$$\bigwedge_{i=1}^{n}((w_i = W_i \wedge h_i = H_i) \vee (w_i = H_i \wedge h_i = W_i))$$

$$\wedge \bigwedge_{i=1}^{n}(x_i \geq 0 \wedge x_i + w_i \leq W \wedge y_i \geq 0 \wedge y_i + h_i \leq H)$$

$$\wedge \bigwedge_{1 \leq i < j \leq n}(x_j \geq x_i + w_i \vee x_i \geq x_j + w_j \vee y_j \geq y_i + h_i \vee y_i \geq y_j + h_j)$$

If the formula is satisfiable, then the SMT solver yields a satisfying assignment, that is, the corresponding values of x_i, y_i, w_i, h_i

Applying a standard SMT solver like Z3, Yices, or CVC4: feasible for rectangle fitting problems up to *20 or 25 rectangles*

The following formula is *satisfiable* iff the fitting problem *has a solution*

$$\bigwedge_{i=1}^{n}((w_i = W_i \wedge h_i = H_i) \vee (w_i = H_i \wedge h_i = W_i))$$

$$\wedge \bigwedge_{i=1}^{n}(x_i \geq 0 \wedge x_i + w_i \leq W \wedge y_i \geq 0 \wedge y_i + h_i \leq H)$$

$$\wedge \bigwedge_{1 \leq i < j \leq n}(x_j \geq x_i + w_i \vee x_i \geq x_j + w_j \vee y_j \geq y_i + h_i \vee y_i \geq y_j + h_j)$$

If the formula is satisfiable, then the SMT solver yields a satisfying assignment, that is, the corresponding values of x_i, y_i, w_i, h_i

Applying a standard SMT solver like Z3, Yices, or CVC4: feasible for rectangle fitting problems up to *20 or 25 rectangles*

The following formula is *satisfiable* iff the fitting problem *has a solution*

$$\bigwedge_{i=1}^{n}((w_i = W_i \wedge h_i = H_i) \vee (w_i = H_i \wedge h_i = W_i))$$

$$\wedge \bigwedge_{i=1}^{n}(x_i \geq 0 \wedge x_i + w_i \leq W \wedge y_i \geq 0 \wedge y_i + h_i \leq H)$$

$$\wedge \bigwedge_{1 \leq i < j \leq n}(x_j \geq x_i + w_i \vee x_i \geq x_j + w_j \vee y_j \geq y_i + h_i \vee y_i \geq y_j + h_j)$$

If the formula is satisfiable, then the SMT solver yields a satisfying assignment, that is, the corresponding values of x_i, y_i, w_i, h_i

Applying a standard SMT solver like Z3, Yices, or CVC4: feasible for rectangle fitting problems up to *20 or 25 rectangles*

问题: Sudoku（数独游戏）

Fill the blank cells in such a way that

- every row, and
- every column, and
- every fat 3 block

contains the numbers 1 to 9, all occurring exactly once

				9	4		3	
			5	1				7
	8	9					4	
						2		8
	6		2		1		5	
1		2						
	7					5	2	
9				6	5			
	4		9	7				

最强大脑? : The puzzle may by very hard, and *backtracking* and/or *advanced solving techniques* are required.

For SAT/SMT it is peanuts: *just* specify the problem.

How?

问题: Sudoku（数独游戏）

Fill the blank cells in such a way that

- every row, and
- every column, and
- every fat 3 block

contains the numbers 1 to 9, all occurring exactly once

				9	4		3	
			5	1				7
	8	9					4	
						2		8
	6		2		1		5	
1		2						
	7					5	2	
9				6	5			
	4		9	7				

最强大脑? : The puzzle may by very hard, and *backtracking* and/or *advanced solving techniques* are required.

For SAT/SMT it is peanuts: *just* specify the problem.

How?

问题: Sudoku（数独游戏）

Fill the blank cells in such a way that

- every row, and
- every column, and
- every fat 3 block

contains the numbers 1 to 9, all occurring exactly once

				9	4		3	
			5	1				7
	8	9					4	
						2		8
	6		2		1		5	
1		2						
	7					5	2	
9				6	5			
	4		9	7				

最强大脑? : The puzzle may by very hard, and *backtracking* and/or *advanced solving techniques* are required.
For SAT/SMT it is peanuts: *just* specify the problem.
How?

问题: Sudoku（数独游戏）

Fill the blank cells in such a way that

- every row, and
- every column, and
- every fat 3 block

contains the numbers 1 to 9, all occurring exactly once

				9	4		3	
			5	1				7
	8	9					4	
						2		8
	6		2		1		5	
1		2						
	7					5	2	
9				6	5			
	4		9	7				

最强大脑? : The puzzle may by very hard, and *backtracking* and/or *advanced solving techniques* are required.
For SAT/SMT it is peanuts: *just* specify the problem.
How?

Several approaches, all working well

- Pure SAT: for every cell and every number 1 to 9, introduce boolean variable describing whether that number is on that position, so $9^3 = 729$ boolean variables.
- SMT: for every cell define an integer variable for the corresponding number.

We elaborate the latter.
问题: How to specify that every row (and column, and 3×3 block) contains the numbers 1 to 9, all occurring once?

Several approaches, all working well

- Pure SAT: for every cell and every number 1 to 9, introduce boolean variable describing whether that number is on that position, so $9^3 = 729$ boolean variables.
- SMT: for every cell define an integer variable for the corresponding number.

We elaborate the latter.
问题: How to specify that every row (and column, and 3×3 block) contains the numbers 1 to 9, all occurring once?

Several approaches, all working well

- Pure SAT: for every cell and every number 1 to 9, introduce boolean variable describing whether that number is on that position, so $9^3 = 729$ boolean variables.
- SMT: for every cell define an integer variable for the corresponding number.

We elaborate the latter.

问题: How to specify that every row (and column, and 3×3 block) contains the numbers 1 to 9, all occurring once?

Several approaches, all working well

- Pure SAT: for every cell and every number 1 to 9, introduce boolean variable describing whether that number is on that position, so $9^3 = 729$ boolean variables.
- SMT: for every cell define an integer variable for the corresponding number.

We elaborate the latter.

问题: How to specify that every row (and column, and 3×3 block) contains the numbers 1 to 9, all occurring once?

Several approaches, all working well

- Pure SAT: for every cell and every number 1 to 9, introduce boolean variable describing whether that number is on that position, so $9^3 = 729$ boolean variables.
- SMT: for every cell define an integer variable for the corresponding number.

We elaborate the latter.
问题: How to specify that every row (and column, and 3×3 block) contains the numbers 1 to 9, all occurring once?

Define 9×9 matrix of integer variables.

- Each cell contains a value in 1, ..., 9

```
X = [ [ Int("x_%s_%s" % (i+1, j+1)) for j in range(9) ]
      for i in range(9) ]
cells_c  = [ And(1 <= X[i][j], X[i][j] <= 9)
             for i in range(9) for j in range(9) ]
```

Each row/column/fat 3 block contains a number at most once.

```
rows_c=[ Distinct(X[i]) for i in range(9) ]
cols_c=[ Distinct([ X[i][j] for i in range(9) ])
             for j in range(9) ]
sq_c  =[ Distinct([ X[3*i0 + i][3*j0 + j]
                  for i in range(3) for j in range(3) ])
             for i0 in range(3) for j0 in range(3) ]
```

Define 9×9 matrix of integer variables.

- Each cell contains a value in 1, ..., 9

```
X = [ [ Int("x_%s_%s" % (i+1, j+1)) for j in range(9) ]
      for i in range(9) ]
cells_c  = [ And(1 <= X[i][j], X[i][j] <= 9)
             for i in range(9) for j in range(9) ]
```

Each row/column/fat 3 block contains a number at most once.

```
rows_c=[ Distinct(X[i]) for i in range(9) ]
cols_c=[ Distinct([ X[i][j] for i in range(9) ])
             for j in range(9) ]
sq_c  =[ Distinct([ X[3*i0 + i][3*j0 + j]
                  for i in range(3) for j in range(3) ])
             for i0 in range(3) for j0 in range(3) ]
```

```python
sudoku_c = cells_c + rows_c + cols_c + sq_c
instance = ((0,0,0,0,9,4,0,3,0),
            (0,0,0,5,1,0,0,0,7),
            (0,8,9,0,0,0,0,4,0),
            (0,0,0,0,0,0,2,0,8),
            (0,6,0,2,0,1,0,5,0),
            (1,0,2,0,0,0,0,0,0),
            (0,7,0,0,0,0,5,2,0),
            (9,0,0,0,6,5,0,0,0),
            (0,4,0,9,7,0,0,0,0))
instance_c = [ If(instance[i][j] == 0,
                  True,
                  X[i][j] == instance[i][j])
               for i in range(9) for j in range(9) ]
```

```python
s = Solver()
s.add(sudoku_c + instance_c)
if s.check() == sat:
    m=s.model()
    r=[ [m.evaluate(X[i][j])
          for j in range(9)]
          for i in range(9) ]
    print_matrix(r)
else:
    print("failed to solve")
```

运行结果:

```
$python3 z3-5-sudoku.py
    [[7, 1, 5, 8, 9, 4, 6, 3, 2],
    [2, 3, 4, 5, 1, 6, 8, 9, 7],
    [6, 8, 9, 7, 2, 3, 1, 4, 5],
    [4, 9, 3, 6, 5, 7, 2, 1, 8],
    [8, 6, 7, 2, 3, 1, 9, 5, 4],
    [1, 5, 2, 4, 8, 9, 7, 6, 3],
    [3, 7, 6, 1, 4, 8, 5, 2, 9],
    [9, 2, 8, 3, 6, 5, 4, 7, 1],
    [5, 4, 1, 9, 7, 2, 3, 8, 6]]
```

- Solutions of sudoku puzzles are quickly found by just specifying the rules of the game in SMT format, and apply an SMT solver.
- For several other types of puzzles (kakuro, killer sudoku, binario, ...) the SAT/SMT approach to solve or generate them works well too.

```
s = Solver()
s.add(sudoku_c + instance_c)
if s.check() == sat:
    m=s.model()
    r=[ [m.evaluate(X[i][j])
        for j in range(9)]
        for i in range(9) ]
    print_matrix(r)
else:
    print("failed to solve")
```

运行结果:

```
$python3 z3-5-sudoku.py
    [[7, 1, 5, 8, 9, 4, 6, 3, 2],
    [2, 3, 4, 5, 1, 6, 8, 9, 7],
    [6, 8, 9, 7, 2, 3, 1, 4, 5],
    [4, 9, 3, 6, 5, 7, 2, 1, 8],
    [8, 6, 7, 2, 3, 1, 9, 5, 4],
    [1, 5, 2, 4, 8, 9, 7, 6, 3],
    [3, 7, 6, 1, 4, 8, 5, 2, 9],
    [9, 2, 8, 3, 6, 5, 4, 7, 1],
    [5, 4, 1, 9, 7, 2, 3, 8, 6]]
```

- Solutions of sudoku puzzles are quickly found by just specifying the rules of the game in SMT format, and apply an SMT solver.
- For several other types of puzzles (kakuro, killer sudoku, binario, ...) the SAT/SMT approach to solve or generate them works well too.

```
s = Solver()
s.add(sudoku_c + instance_c)
if s.check() == sat:
    m=s.model()
    r=[ [m.evaluate(X[i][j])
          for j in range(9)]
          for i in range(9) ]
    print_matrix(r)
else:
    print("failed to solve")
```

运行结果:

$python3 z3-5-sudoku.py

[[7, 1, 5, 8, 9, 4, 6, 3, 2],
[2, 3, 4, 5, 1, 6, 8, 9, 7],
[6, 8, 9, 7, 2, 3, 1, 4, 5],
[4, 9, 3, 6, 5, 7, 2, 1, 8],
[8, 6, 7, 2, 3, 1, 9, 5, 4],
[1, 5, 2, 4, 8, 9, 7, 6, 3],
[3, 7, 6, 1, 4, 8, 5, 2, 9],
[9, 2, 8, 3, 6, 5, 4, 7, 1],
[5, 4, 1, 9, 7, 2, 3, 8, 6]]

- Solutions of sudoku puzzles are quickly found by just specifying the rules of the game in SMT format, and apply an SMT solver.
- For several other types of puzzles (kakuro, killer sudoku, binario, ...) the SAT/SMT approach to solve or generate them works well too.

SMT 在软件测试中的一个重要应用: 符号执行 (*Symbolic Execution*)

- 文献: *Symbolic Execution for Software Testing: Three Decades Later*

问题: How to explore different program paths and for each path to

- *generate* a set of concrete *input* values exercising that path
- *check* for the presence of *various kinds* of errors

```
int main() {
        x = sym_input();
        y = sym_input();
        testme(x, y);
        return 0;
}
```

```
int twice (int v) {
        return 2*v;
}

void testme (int x, int y) {
        z = twice (y);
        if (z == x) {
                if (x > y+10)
                        ERROR;
        }
}
```

SMT 在软件测试中的一个重要应用: 符号执行 (*Symbolic Execution*)

- 文献: *Symbolic Execution for Software Testing: Three Decades Later*

问题: How to explore different program paths and for each path to

- *generate* a set of concrete *input* values exercising that path
- *check* for the presence of *various kinds* of errors

```
int main() {
        x = sym_input();
        y = sym_input();
        testme(x, y);
        return 0;
}
```

```
int twice (int v) {
        return 2*v;
}

void testme (int x, int y) {
        z = twice (y);
        if (z == x) {
                if (x > y+10)
                        ERROR;
        }
}
```

SMT 在软件测试中的一个重要应用: 符号执行 (*Symbolic Execution*)

- 文献: *Symbolic Execution for Software Testing: Three Decades Later*

问题: How to explore different program paths and for each path to

- *generate* a set of concrete *input* values exercising that path
- *check* for the presence of *various kinds* of errors

```
int main() {
        x = sym_input();
        y = sym_input();
        testme(x, y);
        return 0;
}
```

```
int twice (int v) {
        return 2*v;
}

void testme (int x, int y) {
        z = twice (y);
        if (z == x) {
                if (x > y+10)
                        ERROR;
        }
}
```

SMT 在软件测试中的一个重要应用: 符号执行 (*Symbolic Execution*)

- 文献: *Symbolic Execution for Software Testing: Three Decades Later*

问题: How to explore different program paths and for each path to

- *generate* a set of concrete *input* values exercising that path
- *check* for the presence of *various kinds* of errors

```
int main() {
        x = sym_input();
        y = sym_input();
        testme(x, y);
        return 0;
}
```

```
int twice (int v) {
        return 2*v;
}

void testme (int x, int y) {
        z = twice (y);
        if (z == x) {
                if (x > y+10)
                        ERROR;
        }
}
```

SMT 在软件测试中的一个重要应用: 符号执行 (*Symbolic Execution*)

- 文献: *Symbolic Execution for Software Testing: Three Decades Later*

问题: How to explore different program paths and for each path to

- *generate* a set of concrete *input* values exercising that path
- *check* for the presence of *various kinds* of errors

```
int twice (int v) {
        return 2*v;
}

void testme (int x, int y) {
        z = twice (y);
        if (z == x) {
                if (x > y+10)
                        ERROR;
        }
}

int main() {
        x = sym_input();
        y = sym_input();
        testme(x, y);
        return 0;
}
```

SMT 在软件测试中的一个重要应用: 符号执行 (*Symbolic Execution*)

- 文献: *Symbolic Execution for Software Testing: Three Decades Later*

问题: How to explore different program paths and for each path to

- *generate* a set of concrete *input* values exercising that path
- *check* for the presence of *various kinds* of errors

```
int main() {
        x = sym_input();
        y = sym_input();
        testme(x, y);
        return 0;
}
```

```
int twice (int v) {
        return 2*v;
}

void testme (int x, int y) {
        z = twice (y);
        if (z == x) {
                if (x > y+10)
                        ERROR;
        }
    }
}
```

SMT 在软件测试中的一个重要应用: 符号执行 (*Symbolic Execution*)

- 文献: *Symbolic Execution for Software Testing: Three Decades Later*

问题: How to explore different program paths and for each path to

- *generate* a set of concrete *input* values exercising that path
- *check* for the presence of *various kinds* of errors

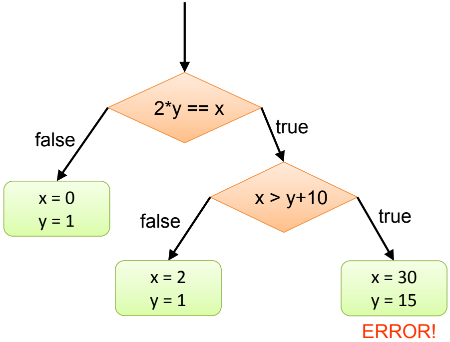
```
int main() {
        x = sym_input();
        y = sym_input();
        testme(x, y);
        return 0;
}
```

```
int twice (int v) {
        return 2*v;
}

void testme (int x, int y) {
        z = twice (y);
        if (z == x) {
                if (x > y+10)
                        ERROR;
                }
        }
}
```

```
int twice (int v) {
        return 2*v;
}

void testme (int x, int y) {
        z = twice (y);
        if (z == x) {
                if (x > y+10)
                        ERROR;
                }
        }
}
```

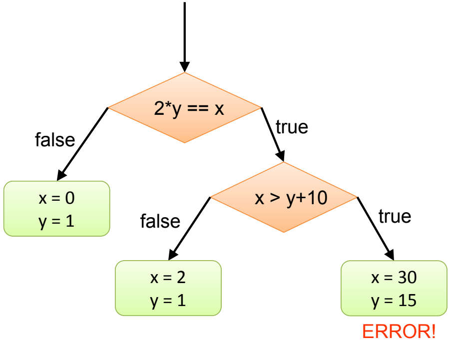


问题: How to reach code ERROR?
SMT problem: Solve x, y satisfying

$$x = 2y \wedge x > y + 10$$

```
int twice (int v) {
        return 2*v;
}

void testme (int x, int y) {
        z = twice (y);
        if (z == x) {
                if (x > y+10)
                        ERROR;
                }
        }
}
```



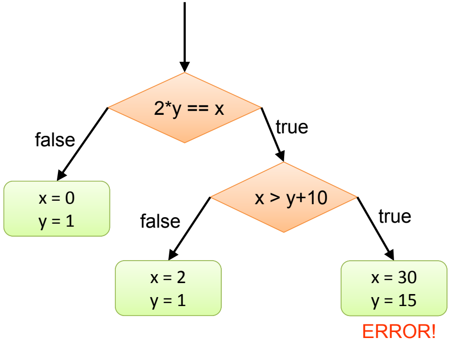
问题: How to reach code ERROR?

SMT problem: Solve x, y satisfying

$$x = 2y \wedge x > y + 10$$

```
int twice (int v) {
        return 2*v;
}

void testme (int x, int y) {
        z = twice (y);
        if (z == x) {
                if (x > y+10)
                        ERROR;
                }
        }
}
```



问题: How to reach code ERROR?
SMT problem: Solve x, y satisfying

$$x = 2y \wedge x > y + 10$$

作业

实验小作业 1: 使用 pure SAT 求解 N-Queen 问题, 并对比 PPT 中 SMT 的实现的效率。要求:

- 代码：使用 SMT 实现代码，和 PureSAT 实现代码
- 文档: 写出实验记录，要求对比 N 取值不同时，两者的效率

实验小作业 2(二选一): 使用 pure SAT 求解 d=a+b 或 d=a-b, 其中,a,b 为正整数。要求:

- 加法和减法问题仅需做一题，减法的实现分数更高
- 代码
- 文档: 简要写出编码思路，代码使用文档和实验结果

实验大作业 (可选): 分别用 Z3 和自己设计的算法求解 rectangle fitting。要求:

- 代码: (1) 使用 Z3 实现 PPT 中的设计 (2) 自己设计算法（使用 C、C++ 语言等)
- 文档: (1) 解释自己的算法思路 (2) 设计测试集，对比两个方法的效率