Stochastic Volatility Surface Quant Research Ideas II

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Overall Thinking

We construct **SABR** model to describe stochastic volatility.
Use **Quasi Monte Carlo** (QMC) with Brownian bridge for option pricing.

Quasi Monte Carlo (QMC)

QMC: choose points spread out uniformly through $[0, 1]^d$ with minimal clumps and voids. QMC is stratification taken to extremes.

Koksma-Hlawka Inequality: Heuristic Explanation

- Extremely important theorem in QMC. It says that QMC is much better than Monte Carlo when n large enough, f variation bounded.
- Koksma-Hlawka inequality isn't probabilistic, it holds with 100% confidence. It holds for finite n, while CLT only holds as $n \to \infty$
- Use a low discrepancy sequence, we achieve $\mid \mu^d - \mu \mid = O\left(n^{-1+\epsilon}\right)$

Derivatives of Quasi Monte Carlo (QMC)

1. QMC has 3 families
   - **Digital Set** (Haar wavelet); **Lattice rule**; Randomized QMC (**RQMC**)
2. QMC vs MCMC
   - **QMC**: low discrepancy sequance. **MCMC**: ergodicity of sequence.
3. QMC Advantages
   - QMC holds strong advantage for high dimensional function dominated by smooth low dimensional ANOVA components

How to improve (reduce effective dimensions)? RQMC.

- RQMC has good equidistribution in all projections onto coordinates.
- Array-RQMC uses RQMC methods to sample a large number n of Markov chains through T time steps each.
- Array-RQMC's empirical result so far outstrips theoretical proofs.
  - Gerber (2015): scrambled nets for RQMC points has variance $O(n^{-1})$.

Example: Price an Asian Option w/ QMC

1. QMC is based on low discrepancy sequence. One of a good candidate of low discrepancy sequences is the **Sobol sequence**
2. Use Sobol points with a nested uniform scramble in $[0, 1]^d$
3. RQMC outperforms MC and the principal components construction

Stochastic Vol Algo example w/ RQMC (Puchhammer, 2019)

- Independent points, which corresponds to crude Monte Carlo (MC);
- Stratified sampling over the unit hypercube (Stratif);
- Sobol points with a random linear matrix scrambling and a digital random shift (Sobol+LMS);
- Sobol points with nested uniform scrambling (Sobol+NUS);
- A rank-1 lattice rule with a random shift modulo 1 followed by a baker's transformation (Lattice+baker).

More Stochastic Simulation Methods, and CEV

1. τ-**leaping**: a speed up version of Gillespie simulation
2. **Multilevel Monte Carlo**: O(ϵ^{-2}) cost rate holds for path-dependent exotic options (lookback, Asian, digital, barrier)
3. Simulated Annealing, Quantum Annealing (thrust, CUDA, OpenMP).
4. Hamiltonian MCMC (will talk about in Chapter 4)

Constant Elasticity of Variance (CEV)

- $dX_t = \delta X_t dt + \sigma X_t^{\beta+1} dB_t$
- This SDE has a strong solution if $\beta \geqslant$ -1/2. If $\beta < 0$, then the volatility $\sigma X_t^{\beta+1}/X_t = \sigma X_t^{\beta}$ increases as X_t increases.
- This property is **leverage effect**, which geometric BM didn't consider.
  - Carr (2017) separates leverage effect from vol feedback, self-exciting jumps
- If $-1/2 \leqslant \beta < 0$, this process can reach 0 in infinite time and remain there. Thus CEV model includes the possibility of bankruptcy.
- SABR is an extension of CEV: volatility follows stochastic process.

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Stochastic Volatility Inspired (SVI): Ideas

SVI: Logical Steps

- Solve multivariate optimization problem numerically: find right a (vol), b (wing slope), ρ (rotation), m (smile right), σ (ATM curvature) to fit SVI.
- We obtain the interpolating call price curves for each maturity.

Advantages of SVI

- Parameterized model for cross-strike interpolation of option price.
- Smooth implied volatility; price curves using only few parameters.
- Each parameter has a financial interpretation, helps develop intuition.

How to construct better interpolation algorithm?

- SVI is bad in fitting an arbitrary combination of option prices (Gatheral 2014): interpolated implied vol occasionally crosses the bid and ask values.
- Let's also take a look at stochastic volatility models.

Why not local volatility model?

1. Local vol models don't fit well the interest rate options prices, fail to reflect market dynamics, eg. **wing effect** (implied vol tends to ↑ for high strikes forming the familiar smile shape)

How to take this into account?

- **Stochastic volatility models**. Add a new stochastic factor to the dynamics, assume volatility parameter itself follows stochastic process.
- **Jump diffusion models**. Itô semimartingale for volvol and vol jumps. Blumenthal–Getoor index (i.e., the activity) to price jumps. But jump is difficult to identify if data sampled sparsely. Also excludes long-memory vol.
- **Fractional Brownian motion**. Long-memory volatility model. Log-volatility behaves as a fractional BM with Hurst exponent H of order 0.1.

2. SABR model is market standard for quoting cap and swaption volatilities. Nowadays also used in FX and equity markets.
3. SABR shows term structure, eg. α shape starts out high for short-dated options, declines monotonically as option expiration increases.

SABR model

Stochastic $\alpha\beta\rho$ model (SABR)

- $dF_t = \sigma_t F_t^{\beta} dW_t$
- $d\sigma_t = \alpha \sigma_t dZ_t$
- Two Brownian Motions are correlated: $\mathbb{E}\left[dW_t dZ_t\right] = \rho dt$, where ρ is a const
- Parameter explanation:
  1. σ_t: stochastic volatility parameter
  2. α: volatility of σ_t (volvol)
  3. F_t: forward rate, eg. LIBOR forward / forward swap rate; it's a martingale
  4. $\beta \in [0, 1]$: elasticity of futures spot and ATM volatility.

Improvement Comments

- no closed form; can use asymptotic expansion (SABR formula) to calibrate
- Monte Carlo: quasi Milstein scheme converges faster than Euler scheme.

Calibration of SABR and improvements

1. The goal of calibration is to fit the model parameters to reproduce the market prices or volatilities as close as possible.

Notes on choice of β

- (1) $\beta \approx 1$: trader believe if market fluctuate, at-the money vol won't be affected much (lognormal like). (2) $\beta \ll 1$: if market move, at-the-money vol will move oppositely (normal like). (3) $\beta = 1/2$: interest rates (CIR)
- In a flash crash/crisis, $\beta = 0.5 \Rightarrow$ extreme calibrations of corr ($\rho = \pm 1$). So some practitioners choose high $\beta \to 1$ for short expiry options, and let it decay as option expiry move out.

2. Drawback of static SABR: consider **several** maturities will have error. **Dynamic SABR**: allowing time dependency in some parameters.
3. ZABR (Smile control by Mixing)
   - Antonov (2012): Mix ZABR models to control behaviour of the wings.

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Spectral Analysis in Time Series

Two categories in time series

- (1) Time domain analysis: deal with the observed data directly
- (2) Frequency domain (= Spectral) analysis: apply Fourier transform to data/ACVF first, then proceeds analysis with the transformed data only
- Spectral analysis is an alternative way of viewing a process via decomposing it into a sum of uncorrelated periodic components with different frequencies.

1. α-mixing time series: a sequence of RVs where past and distant future are asymptotically independent.
2. Grigelionis representation: allows for price and volatility jumps, no restriction on dependence of process components $\Rightarrow$ leverage effect
3. chaotic Wiener–Itô spectral representation $\Rightarrow$ delta hedging
4. Nonparametric High Resolution Spectral Estimation (Dahlhaus)
5. functional-coefficient autoregressive (FAR) (nonsaturated nonparam)
6. Marcenko–Pastur Thm: analyze spectral distribution of the integrated covariance matrix for diffusion processes.

Microstructure noise of High Frequency Data volatility

1. Idiosyncrasy of trading process, asynchronous trading (Epps effect)
   - observed transaction $\neq$ quotes implied price? $\Rightarrow$ multi-scale realized vol (MSRV), robust pre-averaging realized volatility (RPRV), etc.
2. High-frequency asymptotic framework
   - Enables nonparametric analysis of Itô semimartingales, dependence, nonstationarity, heteroscedasticity due to stochastic vol and jumps.
   - Don't need strong parametric assumptions required in low-frequency.

Factor GARCH-Itô (Fan 2019), PCA High Frequency Data (Aït 2017)

- Connect discrete time factor models to continuous Itô diffusion process
- Factor GARCH-Itô model: a specific generalized dynamic factor models
- So we can make inferences using high-frequency financial data with more accurate parameter estimators in our nonparametric model.
- Allow for diverging eigenvalue spikes, and characterize limiting distribution of extreme eigenvalues under ultra-high-dimensional regime (dimension can grow faster than sample size).

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Hamiltonian Monte Carlo

1. Hamiltonian $\hat{\mathbb{H}}$: operator in Quantum Mechanics†, means total energy of the system, also known as the Conserved Quantity.

Heuristic Explanation of Hamiltonian Monte Carlo (HMC)

- HMC is a marriage of statistical physics, ML and stochastic process.
- HMC is in a Metropolis-Hastings framework. It enables more efficient exploration of the state space than random walk, i.e. can explore rarely-visited areas and estimate rare-event probability (Barrier option).

Limitations of Hamiltonian Monte Carlo

- Compute gradient of potential energy function to simulate Hamiltonian dynamical system: not applicable for big data or streaming data.

How to marry state space exploration w/ big-data stoch gradients?

- Noisy estimate gradient: minibatch data to scale algorithms (Hoffman 2013)
- Stochastic Gradient Hamilton Monte Carlo (Tianqi et al. 2014, 31st ICML)

Stochastic Gradient Hamilton Monte Carlo (SGHMC)

SGHMC with *friction* (International Conference on Machine Learning)

- **Second-order Langevin dynamics**: maintain desired target distribution as stationary distribution. Counteract effects of noisy gradient, maintain desired target distribution as the invariant distribution in continuous system.
- $d\theta = M^{-1} r dt$
- $dr = -\nabla U(\theta) dt - BM^{-1} r dt + \mathcal{N}(0, 2Bdt)$
- Friction term $BM^{-1}r$ decreases the energy $H(\theta, r)$, thus reducing noise.

Algorithm 2: Stochastic Gradient HMC

for $t = 1, 2 \cdots$ **do**
  optionally, resample momentum r as
  $r^{(t)} \sim \mathcal{N}(0, M)$
  $(\theta_0, r_0) = (\theta^{(t)}, r^{(t)})$
  simulate dynamics in Eq.(13):
  for $i = 1$ ***to*** m **do**
    $\theta_i \leftarrow \theta_{i-1} + \epsilon_t M^{-1} r_{i-1}$
    $r_i \leftarrow r_{i-1} - \epsilon_t \nabla \tilde{U}(\theta_i) - \epsilon_t C M^{-1} r_{i-1}$
      $+\mathcal{N}(0, 2(C - \hat{B})\epsilon_t)$
  end
  $(\theta^{(t+1)}, r^{(t+1)}) = (\theta_m, r_m)$, no M-H step
end

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Thanks!

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