Slides 4: Carr-Wu Method for Volatility Surface

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1. Carr-Wu method: Volatility Surface based on dynamics of implied vol

How to construct Volatility Surface?

- Two ways.
- (1) Assume **dynamics** for the underlying, which can accommodate the **observed** implied vol smiles and skews.
  - Stochastic Volatility models, Levy process, etc.
  - Procedure: Estimate the coefficients of the dynamics **through calibration** from observed option prices.
- (2) Assume entire implied volatility surface has **known** initial value, and **evolves** continuously over time.
  - Explicitly specify dynamics of the implied volatilities, with **market models** of implied vol.
  - Ex: Specify the continuous martingale component of the volatility surface, and then derive the restriction on the risk-neutral drift of the surface imposed by the requirement that all discounted asset prices be martingales.

Carr-Wu method

Idea of Carr-Wu method

- Similar to **(1) market model** approach, it directly models the dynamics of implied volatilities.
- However, both risk-neutral drift and the martingale component of the implied vol are specified, (which means we don't take the initial implied volatility surface as given and infer the risk-neutral drift of the implied volatility dynamics) and then we derive the volatility surface.

Construction of Carr-Wu

- One std Brownian motion drives the whole volatility surface, another partially correlated std Brownian motion drives the underlying security price dynamics.

Carr-Wu method

Construction of Carr-Wu (ctd.)

- Condition: discounted prices of options and their underlying are martingales under the risk-neutral measure
- Then, we obtain a PDE that governs the shape of the implied volatility surface, termed as **Vega-Gamma-Vanna-Volga (VGVV)** methodology. (it links the θ of the options and four Greeks)
- Plug in the analytical solutions for the Greeks, PDE is reduced into an algebraic relation that **links** the shape of the implied volatility surface to its risk-neutral dynamics.
- By parameterizing the implied variance dynamics as a mean-reverting square-root process, the algebraic equation simplifies into a **quadratic equation** of the implied volatility surface as a function of a standardized moneyness measure and time to maturity.

Carr-Wu method

Square Root Variance (SRV) and Lognormal Variance (LNV)

- The coefficients of the quadratic equation are governed by 6 coefficients related to the dynamics of the stock price and implied variance. This model is denoted as the **square root variance (SRV)** model.
- If implied variance dynamics is parametrized as a mean-reverting lognormal process, we obtain **another quadratic equation** of the implied variance: a function of log strike over spot and time to maturity. The implied variance surface is also determined by the 6 coefficients. This model is the **lognormal variance (LNV)** model.
- The computation cost for calibration of **SRV** and **LNV** is very small, because they are quadratic equations.

Mathematical Derivation of Carr-Wu

Carr-Wu Framework

- Stock Price: $dS(t) = S(t)\sqrt{v(t)}dW(t)$
- Implied Vol: $dI(t;K,T) = \mu(t)dt + \omega(t)dZ(t)$ #$Z(t)$ is BM.
- Correlation: $\rho(t)dt = E[dW(t)dZ(t)]$
- Condition: No dynamic arbitrage (NDA) between option $(K;T)$, $(K_0;T_0)$ and stock.
- We derive call/put options with Black-Scholes formula, for a portfolio = put+call to neutralize the exposure on vol risk, and achieve delta neutrality. By Ito's fomula, each option in the portfolio has risk-neutral drift (RND). Based on NDA, both option drifts must vanish, so we derive:
- **(Fundamental PDE)** $-\frac{\partial B}{\partial t} =$
$\mu(t)\frac{\partial B}{\partial \sigma} + \frac{v(t)}{2}S^2(t)\frac{\partial^2 B}{\partial S^2} + \rho(t)\omega(t)\sqrt{v(t)}S(t)\frac{\partial^2 B}{\partial \sigma \partial S} + \frac{\omega^2(t)}{2}\frac{\partial^2 B}{\partial \sigma^2}$

Mathematical Derivation of Carr-Wu

Definition

- The class of implied volatility surfaces defined by the **Fundamental PDE** is the Vega-Gamma-Vanna-Volga (VGVV) model.

More than Fundamental PDE

- Coefficients are **stochastic** in Fundamental PDE.
- Note that this PDE is not meant to solve the value function, because value function $B(S(t), I(t;K,T), t)$ is **known**. It shows that the stochastic values must satisfy this relation to exclude dynamic arbitrage.
- Plugging in Black-Scholes formula, we reduce this PDE to an **algebraic restriction** on the shape of the implied vol surface $I(t;K,T)$. $\frac{I^2(t;K,T)}{2} - \mu(t)I(t;K,T)\tau - \left[\frac{v(t)}{2} - \rho(t)\omega(t)\sqrt{v(t)}\sqrt{\tau}d_2 + \frac{\omega^2(t)}{2}d_1 d_2 \tau\right] = 0$

Mathematical Derivation of Carr-Wu

SRV case

- SRV square-root implied variance dynamics:
  $dI^2(t) = \kappa(t)\left[\theta(t) - I^2(t)\right]dt + 2w(t)e^{-\eta(t)(T-t)}I(t)dZ(t)$
- Assume standard moneyness $z(t) = \frac{ln(K/S(t))+0.5I^2\tau}{I\sqrt{\tau}}$, then $I(z,\tau)$ solves the quadratic equation:
  $$(1+\kappa(t))I^2(z,\tau) + \left(w^2(t)e^{-2\eta(t)\tau}\tau^{1.5}z\right)I(z,\tau)$$
  $$-\left[\left(\kappa(t)\theta(t) - w^2(t)e^{-2\eta(t)\tau}\right)\tau + v(t) + 2\rho(t)\sqrt{v(t)}e^{-\eta(t)\tau}\sqrt{\tau}z\right.$$
  $$\left.+ w^2(t)e^{-2\eta(t)\tau}\tau z^2\right] = 0$$

Mathematical Derivation of Carr-Wu

KNV case

- LNV log-normal implied variance dynamics:
  $dI^2(t) = \kappa(t)\left[\theta(t) - I^2(t)\right]dt + 2w(t)e^{-\eta(t)(T-t)}I(t)dZ(t)$
- Assume log strike $k(t) = ln(K/S(t))$, then $I(k,\tau)$ solves the quadratic equation:

$$\begin{aligned}\frac{w^2(t)}{4}e^{-2\eta(t)\tau}\tau^2 I^4(k,\tau) + [1 &+\kappa(t)\tau + w^2(t)e^{-2\eta(t)\tau}\tau \\ &- \rho(t)\sqrt{v(t)}w(t)e^{-\eta(t)\tau}I^2(k,\tau)] \\ &- \Big[v(t) + \kappa\theta(t)\tau + 2\rho(t)\sqrt{v(t)}e^{-\eta(t)\tau}k \\ &+ w^2(t)e^{-2\eta(t)\tau}k^2] = 0\end{aligned}$$

Final Results for Carr-Wu

- For both SRV and LNV, we have 6 stochastic time-varying coefficients: $\kappa(t), \theta(t), \omega(t), \eta(t), \rho(t), \nu(t)$
- Given time t value of the 6 coefficients, the implied vol surface at time t can be calculated easily as solution to quadratic equations.

Overview of Carr-Wu

- We directly model the implied volatility dynamics, and then **derive** the dynamic-no-arbitrage implication on the shape of the implied volatility surface.
- The two (dynamics and surface shapes) are consistent.
- Two models with extreme simplicity: implied vol surface are solutions to quadratic equations (6th grade math).
- 100 times faster than standard option pricing models, ideal for automated options market making.

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- ***UChicago Project Lab Presentation: Ideas I*** (Feb 19, 2020, ***Click me link***)
- ***Stochastic Volatility Quant Research Ideas III*** (March 9, 2020 , ***Click me link***)

Thanks!

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