Stochastic Volatility Modeling Quant Research Ideas III

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Preface

- This is a literature review notes on quantitative finance, a self-learning slides by Junfan Zhu. Feel free to share with others.
- No confidential information of projects, or outspread of course materials are written in the slides. References are listed in the end, thanks to their insightful publications, I learned a great deal.
- Due to limitations of author's capability, if you find mistakes or have comments, please contact me at `junfanzhu@uchicago.edu`.
- This is the third slides of my series. You may find the previous two through link:

Previous Relevant Slides

- **Stochastic Volatility Quant Research Ideas II** (Feb 29, 2020 , **Click me**)
- **UChicago Project Lab Presentation: Ideas I** (Feb 19, 2020, **Click me**)

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Distribution Metrics

Kullback-Leibler (KL) divergence: $D_{KL}(p||q) = \int_x p(x) ln(\frac{p(x)}{q(x)}) dx$

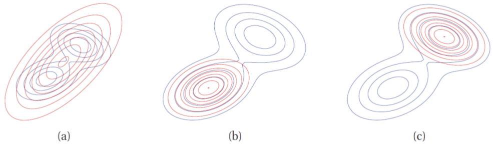

Figure 21.1 Illustrating forwards vs reverse KL on a bimodal distribution. The blue curves are the contours of the true distribution p. The red curves are the contours of the unimodal approximation q. (a) Minimizing forwards KL: q tends to "cover" p. (b-c) Minimizing reverse KL: q locks on to one of the two modes. Based on Figure 10.3 of (Bishop 2006b). Figure generated by `KLfwdReverseMixGauss`. [a]

[a] Source: Machine Learning: A Probabilistic Perspective, p.734

f-divergence: $D_f(p||q) = \int_x q(x) f(\frac{p(x)}{q(x)}) dx$

- KL divergence is a type of f-divergence, but it's unbounded.
- Take $f(t) = -\log t$, we get KL-divergence. Other types of f-divergence: Hellinger distance, total variation distance.

Wasserstein Distance

Wasserstein Distance

p^{th} **Wasserstein distance** between probability measures μ and ν:

$$W_p(\mu,\nu) := \left(\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{M \times M} d(x,y)^p \mathrm{d}\gamma(x,y) \right)^{1/p}$$

$\Gamma(\mu,\nu)$ is the set of all **couplings** of μ and ν.

- WGAN in Machine Learning, W stands for Wasserstein Distance.
- Wasserstein metric is good to describe **the reason** how 2 distributions differ in **geometry** by **continuous** transformation.
- It comes from **Optimal Transport**, and relates to stochastic control in Nonlinear PDE option pricing with Hamilton-Jacobi-Bellman type.

How does Wasserstein fits with stochastic differential equations?

Wasserstein metric is related to model-independent bounds on multi-asset European option prices.

Financial meaning of Wasserstein distance: Copula

We can find a probability measure τ (copula) on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals $\mathbb{P}_1$ on the first coordinates and $\mathbb{P}_2$ on the second coordinates that attains the infimum of the expcectation of $d(\bullet, \bullet)^p$.

$$d_{\mathrm{MK}}\left(\mathbb{P}_1, \mathbb{P}_2\right)^p = \inf_{\tau \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}^n) \text{ with marginals } \mathbb{P}_1 \text{ and } \mathbb{P}_2} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} d(x, y)^p \tau(dx, dy) \right) \tag{1}$$

Multi-asset example: European Options

Euro option price depends on 2 assets, $X = S_T^1$ and $Y = S_T^2$, maturity T. Payoff $= d(X, Y)^p$. Our model is calibrated to vanilla smiles at T for both assets $\rightarrow$ marginal risk-neutral distributions $\mathbb{P}_1$, $\mathbb{P}_2$ of X, Y at T.

Price $C(T, K)$ of T-vanilla options with strike K, the risk-neutral density at T is $\mathbb{P}(K) = \partial_K^2 C(T, K)$. Option price: choose a joint distribution τ of (X, Y) with marginals $\mathbb{P}_1$ and $\mathbb{P}_2$ that minimizes the option fair value $\mathbb{E}^\tau[d(X, Y)^p]$. This is exactly Wasserstein distance.

Dual Problem: Portfolio of assets

Dual Problem

The previous minimization formula is an infinite-dimensional linear programming problem, we can dualize it with Lagrange's multipliers w.r.t. **marginal constraints**.

$$d_{\mathrm{MK}}\left(\mathbb{P}_1,\mathbb{P}_2\right)^p = \inf_{\mathbb{P}\in M_+} \sup_{u_1,u_2} \left\{\mathbb{E}^{\mathbb{P}}\left[d^p(X,Y)-u_1(X)-u_2(Y)\right] + \mathbb{E}^{\mathbb{P}_1}\left[u_1(X)\right]+\mathbb{E}^{\mathbb{P}_2}\left[u_2(X)\right]\right\}$$

Interpretations

Maximize objective function $u_1(X)+u_2(Y)$: portfolio consisting 2 options X and Y with market prices $\mathbb{E}^{\mathbb{P}_1}[u_1(X)]$ and $\mathbb{E}^{\mathbb{P}_2}[u_2(X)]$.

Constraints: the intrinsic value of portfolio $<$ payoff $c(X,Y)=d(X,Y)^p$.

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Local Stochastic Volatility Models (LSVM): Intuition

- How to build a stochastic volatility model[a] that calibrates exactly to a full surface of implied vol? We embed a local vol $\sigma(t, f)$ into the stochastic vol model. LSVM is an extension of Dupire's LV model.
- $df_t = a_t f_t \sigma(t, f_t) dW_t$
  - a_t: (possibly multi-factor) stochastic process, an extension of Dupire local vol/stochastic vol model (In stochastic vol models we can't perfectly calibrate to the whole implied vol surface). But here we calibrate market smiles exactly by 'decorating' vol of forward with a local vol $\sigma(t, f)$.
- SIR-LSVM (LSVM + stochastic interest rate): $\frac{dS_t}{S_t} = r_t dt + \sigma(t, S_t) a_t dW_t$

[a] Sometimes people abbreviate 'Stochastic Volatility Models' as SVM, but it seems like Support Vector Machine, so I incline to write the full name

Stochastic Volatility Models

Table 11.1: Examples of SVMs ($\sigma(t,f)=1$).

Name	SDE
Stein-Stein	$\frac{df_t}{f_t} = a_t\,dW_t$ $da_t = \lambda(a_t - \bar{a})\,dt + \zeta\,dZ_t, \quad d\langle W,Z\rangle_t = \rho\,dt$
Geometric	$\frac{df_t}{f_t} = a_t\,dW_t$ $da_t = \lambda a_t\,dt + \zeta a_t\,dZ_t, \quad d\langle W,Z\rangle_t = \rho\,dt$
3/2-model	$\frac{df_t}{f_t} = a_t\,dW_t$ $da_t^2 = \lambda(a_t^2 - \bar{v}a_t^4)\,dt + \zeta a_t^3\,dZ_t, \quad d\langle W,Z\rangle_t = \rho\,dt$
SABR	$\frac{df_t}{f_t} = a_t f_t^{\beta-1}\,dW_t$ $da_t = \nu a_t\,dZ_t, \quad d\langle W,Z\rangle_t = \rho\,dt$
Scott-Chesney	$\frac{df_t}{f_t} = e^{y_t}\,dW_t$ $dy_t = \lambda\,(\bar{y} - y_t)\,dt + \zeta\,dZ_t, \quad d\langle W,Z\rangle_t = 0$
Heston	$\frac{df_t}{f_t} = a_t\,dW_t$ $da_t^2 = \lambda(\bar{v} - a_t^2)\,dt + \zeta a_t\,dZ_t, \quad d\langle W,Z\rangle_t = \rho\,dt$

[1]

[1] Source: Julien Guyon, Pierre Henry-Labordere, Nonlinear Option Pricing, p.275

Local Stochastic Volatility Model (LSVM)

Extensions of LSVM

- How to calibrate? Gyongy's Thm: LSVM is calibrated to European call prices with positive maturities and strikes, if local vol function = ratio of Dupire local vol function over root conditional mean square of the stochastic volatility factor, given the spot value.
- LSVM leads to nonlinear SDE in the sense of McKean. How to calibrate efficiently? Particle methods based on a kernel approximation of the conditional expectation is a good idea.
- **Regime Switching local volatility**: stochastic volatility factor is a jump process taking finitely many values and with jump intensities depending on the spot level. Can be used for pricing VIX options.
- Suppose we have m securities $S_i(t)$, with drift velocities $\mu_i(t)$ driven by n white noise $R_j(t)$ and volatilities $\sigma_j^i(t)$, we have m coupled Langevin equations (discussed last time, **Click Here**):
- $\frac{dS_i(t)}{dt} = \mu_i(t)S_i(t) + S_i(t)\sum_{j=1}^{n} \sigma_j^i(t)R_j(t)$

Bergomi Model

As a next example, we consider Bergomi's LSV model [58, 128]:

$$
\begin{aligned}
df_t &= f_t \sigma(t, f_t) \sqrt{\xi_t^t}\, dW_t \\
\xi_t^T &= \xi_0^T f^T(t, x_t^T) \\
f^T(t, x) &= \exp(2\sigma x - 2\sigma^2 h(t, T)) \\
x_t^T &= \alpha_\theta \left((1-\theta) e^{-k_X (T-t)} X_t + \theta e^{-k_Y (T-t)} Y_t \right) \\
\alpha_\theta &= \left((1-\theta)^2 + \theta^2 + 2\rho_{XY} \theta (1-\theta) \right)^{-1/2} \\
dX_t &= -k_X X_t\, dt + dW_t^X \\
dY_t &= -k_Y Y_t\, dt + dW_t^Y
\end{aligned}
$$

where

$$
\begin{aligned}
h(t, T) &= (1-\theta)^2 e^{-2k_X (T-t)} \mathbb{E}\left[X_t^2\right] + \theta^2 e^{-2k_Y (T-t)} \mathbb{E}\left[Y_t^2\right] \\
&\quad + 2\theta (1-\theta) e^{-(k_X + k_Y)(T-t)} \mathbb{E}\left[X_t Y_t\right] \\
\mathbb{E}\left[X_t^2\right] &= \frac{1 - e^{-2k_X t}}{2k_X} \\
\mathbb{E}\left[Y_t^2\right] &= \frac{1 - e^{-2k_Y t}}{2k_Y} \\
\mathbb{E}\left[X_t Y_t\right] &= \rho_{XY} \frac{1 - e^{-(k_X + k_Y) t}}{k_X + k_Y}
\end{aligned}
$$

[2]

[2] Source: Julien Guyon, Pierre Henry-Labordere, Nonlinear Option Pricing, p.299

Overall Comments on these Stochastic Vol models

Comments on Bergomi model

- CEV model we discussed last time **Click here** generalizes '3/2' model
- Heston model: vol & volvol move in opposite directions $\rightarrow$ we can use exogenous factor in vol process to drive both the mean level of vol and volvol (the correlation can be illustrated by VVIX)
- Bergomi: After calibrating Heston parameters, daily variations of the calibrated instantaneous variance and volvol showed an impressive correlation of almost 60%. Thus, the market was already pricing a feature of the behaviour of volatility that Heston's model misprices by construction $\rightarrow$ We misprice derivatives that are highly sensitive to the dynamics of implied volatility.

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Particle Method in McKean SDE Option Pricing

- **Particle method** is an elegant stochastic simulation of McKean SDEs. Efficient for calibration. Approximate a flow of prob measures.
- Particle simulation algorithms = sequential Monte Carlo (nonlinear filtering) in Bayesian statistics = particle filters in signal processing
- Related prob models: Feynman-Kac-Schrödinger model, Boltzmann-Gibbs measures, conditional distribution of stochastic process in critical regimes
- Dynamical evolution of calibrated LSVM by McKean SDE: $df_t = f_t \sigma(t, f_t, \mathbb{Q}_T) a_t dW_t$
- where $\mathbb{Q}_t$ denotes the distribution of (f_t, a_t) under $\mathbb{Q}$, with $\sigma(t, f_t, \mathbb{Q}_t) = \frac{\sigma_{\mathrm{Dup}}(t, f_t)}{\sqrt{\mathbb{E}^{\mathbb{Q}}[a_t^2 | f_t]}}$

Related Numerical Methods

- Longstaff-Schwartz style functional regression (Functional regression discussed last time, **Click here**) and Malliavin Monte Carlo for American option.
  - First, improve functional regression approximation using an original adaptive local basis approach.
  - Second, reduce complexity of Malliavin approximate backward dynamics.
- Diffusion approximation techniques, can be used to replace small jump component by a small Brownian motion, and to model microstructure of financial market.
- Last time **(Click here)** we discussed Koksma-Hhlawka inequality: better evenness results in a more accurate integration. (Random sampling is bad.) Sobol's sequence effectively avoids the holes and clusters and fills the hypercube with a better 'evenness'.

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Thanks!

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