Department of Mechanical & Mechatronics Engineering

MTE 219 – Mechanics of Deformable Solids

Project - Optimum Design of a Truss

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Group #28

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Executive Summary

Purpose

Using the given materials of three $1/8'' \times 3/4'' \times 36''$ balsa wood planks and one $1/8'' \times 36''$ hardwood dowel, the objective is to design a bridge capable of being loaded and holding the given load for greater than 5 seconds and achieving the specified performance value of 200 or greater. The performance value of the bridge is the load to mass ratio.

Methodology

The team initially deliberated on the key factors that had to be considered for the design of the members, including the key failure modes. Then, 3 initial designs were created. A well-defined cost function is developed based on a set of assumptions, and the 3 truss concepts were optimized exhaustively with a computer program and the cheapest truss was selected. Additionally, preliminary designs tailored against specific failure modes were created to further aid in the bridge stability. Tests were then conducted upon prototype versions of the final design, which showed possibility for further improvement due to construction changes. These were taken into account for the final bridge design.

Results

After the final bridge was tested to failure, the bridge did not perform up to the expected load, due to shear forces occurring at the edge of compression members. Even though it performed better than previous iterations due to enhanced geometric stability (reflectional symmetry of the structure), the estimated 20 kg load capacity was not reached.

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4. Introduction

This project has multiple problems and objectives that comprises both theorizing and practical aspects of Mechanics of Deformable Solids subject. In Essense, it is an truss-bridge optimization problem with a given constraint model and a custom cost model.

4.1 Problem Definition

Consider a truss, $T = (\mathcal{J}, \mathcal{M})$ where $\mathcal{J}$ and $\mathcal{M}$ are the set of joints and members. Let $\mathcal{J} = \{j_1, j_2, \dots, j_r\}$, $\mathcal{M} = \{m_1, m_2, \dots, m_k\}$ where $r, k \in \mathbb{Z}^+$.

To maximize load to weight ratio PV, we define custom cost function $C(T)$ that takes in a truss T as parameter and returns the cost of the truss. The *1st problem* is to find a well-defined **Cost Function** for a truss, and the *2nd problem* is to apply an existing optimization algorithm to minimize the cost of the truss (which in turn maximizes the truss's PV value).

4.2 Constraint Model (Design Constraints)

Dimension Constraint: Truss should be 40 ± 1 cm long and 10 ± 0.5 cm tall. There must exist a loading joint at the center of the truss.

Geometric Constraint: The truss must *not* be *statistically indeterminant*, *overly defined*, or contain *zero-force members*.

4.3 Cost Model (Design Criteria)

The cost model is essentially the minimums weight of the truss given a constant loading. When comparing 2 trusses, the truss with less cost is considered more optimal. When a member is under tension or compression, I will consider the minimum weight possible for the given truss that failure won't occur, i.e. the **critical weight.** To achieve so, I will need to find the minimum section area of a member given a axial loading - **critical section area**. If one knows the critical section area and the length of the member, one can calculate the volume. Since volume $\propto$ weight, comparing the sum of the **critical volume** of all members is equivalent to comparing the minimum weight of a truss.

Define function $\boldsymbol{C}(\boldsymbol{T})$ as the **cost function** of the truss that is essentially the **critical volume** given a constant loading. $\boldsymbol{A}(\boldsymbol{F},\boldsymbol{L})$ is the **critical section area function** at the given force F and length L. Let $\boldsymbol{L}(\boldsymbol{m_i})$ denotes the length and $\boldsymbol{F}(\boldsymbol{m_i})$ denotes the internal force of $m_i \in \mathcal{M}$.

$$C(T) = \sum_{i=1}^{r} A\left(F(m_i), L(m_i)\right) \times L(m_i)$$

To determine the critical area function $A(F, L)$, different failure modes must be taken into consideration. Member in *tension* may fail due to **Rupture** and **Member Tearing;** Member in *compression* may fail due to **Buckling** and **Bearing Stress.** Pin shear and bending are not considered here due to its small volume but will be discussed in later sections.

4.3.1 Dimensioning Assumption

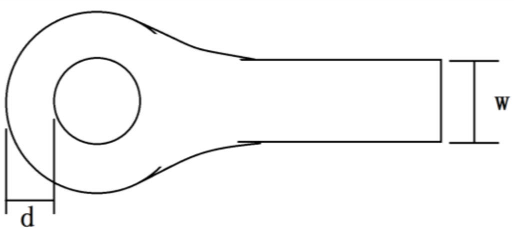


Figure 1: Sample Dimensions

Distance between inner and outer diameter is denoted $\boldsymbol{d}$*; Minimum width for rupture calculation is denoted* w*; Thickness is denoted* $\boldsymbol{t}$*; Length is denoted* $\boldsymbol{L}$*; Load would be* $\boldsymbol{F}$

Note that this theoretical member is the projection of 3D bridge to 2D. Meaning that one 'member' in 2D comprises multiple beams in 3D. The cross-sectional area and the force undertaken by the beam are combined in 2D projection so that it will not cause confusion.

- $\boldsymbol{w} = \boldsymbol{2d}$ to maximize rupture resistance
- $\boldsymbol{w} = \boldsymbol{t}$ to equalize second moment of inertia ($I_x = I_y$)

4.3.2 Tension Failure Mode ($\boldsymbol{F} > \mathbf{0}$)

Rupture failure: $\frac{F}{wt} = \frac{F}{2dt} \geq \sigma_{ut} = 7.7\ MPa$

Tearing failure: $\frac{F}{2dt} \geq \tau_{ut} = 1\ MPa\ (along\ grain)$

Therefore, **Tearing** will happen before **Rupture**, $\therefore\ F_{fail} \geq 2dt\tau_{ut}$4.444

4.3.3 Compression Failure Mode ($F < 0$)

Bearing Stress is not taken into consideration here (as part of cost function) because it does not impact the critical section area of the member. There are also many mechanisms we can employ to minimize risks of bearing failure.

Buckling load, however, is closely linked to the critical section area. We use the **Euler Buckling Load Estimation**: $P_{cr} = \pi^2 EI/(KL)^2$ Since both sides of a member under compression is pin connected, the K value is 1.

$$I_x = 2\int_0^{w/2} y^2 \ t \times dy = 2t\frac{(w/2)^3}{3} = \frac{w^3 t}{12} \qquad I_y = 2 \times \int_0^{t/2} x^2 \ w \times dx = \frac{wt^3}{12}$$

$$\therefore F_{fail} \geq P_{cr} = \min(\frac{\pi^2 E_b w t^3}{12L^2}, \frac{\pi^2 E_b w^3 t}{12L^2}) \quad \Leftarrow K = 1$$

Therefore, cross sectional area $A = wt = 2dt$. Recall that cross sectional area is directly proportional to the cost incurred by the member. Find A with respect to F:

In *tension* ($F > 0$): $\quad F_{fail} = 2dt\tau_{ut} = A\tau_{ut} \quad \Longrightarrow \boldsymbol{A = F/\tau_{ut} = F/1.0MPa}$

In *compression* ($F < 0$): $\quad F_{fail} = \frac{\pi^2 E_b w t^3}{12L^2} = \frac{\pi^2 E_b A^2}{12L^2} \quad \therefore \boldsymbol{A} = \sqrt{\frac{F_{fail} \times 12L^2}{\pi^2 E_b}} = \left[\boldsymbol{\frac{1}{\pi}\sqrt{\frac{12}{E_b}}}\right]\boldsymbol{L\sqrt{F_{fail}}}$

Overall, the **critical area function $\boldsymbol{A(F, L)}$** for the cost function is:

$$A(F, L) = \begin{cases} F \geq 0, & \dfrac{F}{1.0MPa} \\ F < 0, & \left[\dfrac{1}{\pi}\sqrt{\dfrac{12}{E_b}}\right] L\sqrt{F_{fail}} \end{cases}$$

Unit of A is mm^2, of L is mm. Unit of F is N. $E = 3510\ MPa$

4.4 Material Properties

	$\rho\ (kg/m^3)$	$E\ (GPa)$	$\sigma_{ut}\ (MPa)$	$\tau_{ut}\ (MPa)$	
Balsa wood	123 ± 23	3.51 ± 1.05	7.7 ± 2.5	1.0 (along grain); 1.6 (transverse)	
Dowel Pin	650	17	171	23	$M_{max} = 368\ N.mm$

Figure 2: Key Material Properties

We used the data posted on learn. For simplicity, uncertainties are ignored.

5. Preliminary Design

The 3 proposed designs consist of a triangular truss (**Figure 3**), trapezoidal truss (**Figure 4**), and a more complicated hexagonal truss (**Figure 5**). The first thing we noticed with these 3 designs is that the *outer-rim* members of the truss are under compression, whereas the *inner* members (that are linked to the center loading point) are under tension.

The **triangular** truss (**figure 3**), being the simplest of them all, has very little number of joints. This concept is favourable as the dowels are 3 times denser than balsa wood. A sample load of 10kN is shown in **Figure 3**. As can be seen, the 3 members connecting to (20,0) joint have almost equivalent tensilve force as load, being vulnerable to rupture. Furthermore, the other 2 members have slightly more than twice the amount in compressive force. This is highly unfavourable as only few members are taking in all the force.

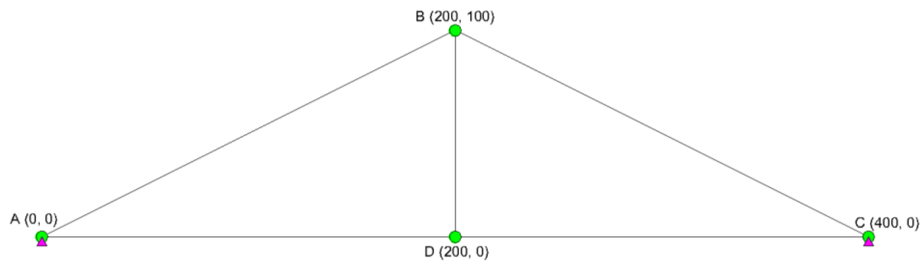


Figure 3: Triangular Truss (mm)

As a potential improvement of the triangular truss, an additional joint along the top would help distribute the load for the inner tension members, potentially allowing the truss to sustain heavier load. It ends up reducing the tension for each tension member significantly, and this leads to the **trapezoidal** design (Figure 4).

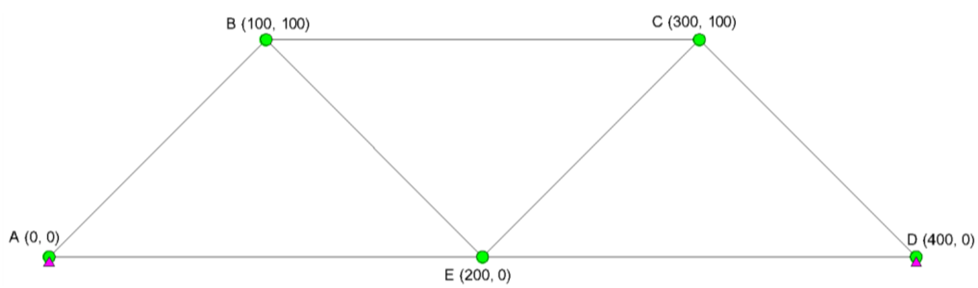


Figure 4: Trapezoidal Truss (mm)

Now we turn our attention to the outer-rim compression members. Since with buckling load estimation, $P_{cr} \propto 1/L^2$, a decrease in length would significantly reduce the risk of buckling. We came up with another potentially improved design – **hexagonal** truss (Figure 5). The hexagonal truss reduces both the compressive force and the compression member length, and therefore minimizes the risk of buckling compared to the other truss designs. Additionally, individual tensile forces for the inner members have lessened, reducing the risk of rupture in these members.

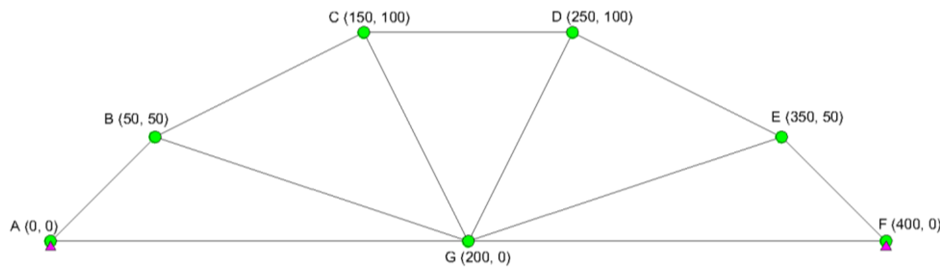


Figure 5: Hexagonal Truss (mm)

To evaluate which truss concept has the *superior* configuration, the 3 designs' cost $C(T)$ (which is our design criteria) are evaluated. However, none of the above trusses above are considered optimal, because moving certain joints around may yield a cheaper truss. In the following section *Design Optimization*, each truss configuration is optimized exhaustively, and then, their costs $C(T)$ are compared against each other. Finally, the cheapest-costing truss will be selected as the superior concept (see ***Final Selection***).

6. Optimization and Design Analysis

6.1 Design Optimization

As the *introduction/Cost Model* section discussed, for a given truss T, the goal is to minimize its cost $C(T)$ by moving its free joints around.

I devised a combinatoric algorithm to find the 'local minimum' cost of a given truss by displacing some of its joints, as explained in detail below. The program is elegantly implemented in C++ with backtracking, and the source code is here: github.com/thomasgaozx/truss-optimizer.

In short, there are 3 major roadblocks:

1. **Truss simulation**: a truss is represented by a weighted undirected graph T, nodal data structure is used instead of matrix for better runtime, the unknown forces are stored in a hashmap with `key` being member object and `value` being the force(`double`).
2. **Truss solving** (internal forces): for all joints, dump its system of equations ($\sum F_x = 0, \sum F_y = 0$) into one huge matrix, solve the matrix using QR decomposition for superior accuracy (compared to Gaussian Elimination). If the rank of the solved matrix equals to the number of unknown forces, then there is no *statistical indeterminacy*.
3. **Combinatoric algorithm** for optimization: see the following subsection.

6.1.1 Combinatoric Algorithm

For a given truss T_0, I used the brute force $O(K^N)$ run-time solution to complete one optimization cycle, where K is the possible displacement, $\mathcal{J}^{Free} \subset \mathcal{J}$ is a set of joints that are allowed to move, and N is the size of $\mathcal{J}^{Free}$. A *displacement* of a joint is defined by a point around the current joint, that the joint could potentially be at. K *displacements* are determined by drawing a circle around a joint with radius ρ, and find $K - 1$ equidistance points that the joint (the current position of the joint is also a displacement).

In one optimization cycle, $\forall j \in \mathcal{J}^{Free}$, j must go through all K displacement. Clearly, there are K^N possible truss configurations to check. For each configuration T_i, tests are performed to ensure the truss T_i meet the constraints, only then are the internal forces solved and the cost of the configuration is evaluated. If the cost is smaller than the minimum tracked cost, the current truss T_i is recorded.

After 1 optimization cycle, a more cost effective truss T' is yielded. I then use this new configuration as the base truss T_0 for the next optimization cycle. The cycles only stop when base T_0 is the most cost effective one of all checked configurations. If I run one truss against with multiple K and ρ values, I am *guaranteed* to have a locally-cheapest truss.

Run time analysis: The program runs decently fast with up to 8 free joints (joints whose displacement is allowed), due to the exponential worst-case run-time of this algorithm. In this project, however, 2-4 free joints are sufficient, so the run-time is not really an issue.

6.1.2 Final Selection

We used the algorithm described above to optimize the 3 selected trusses as discussed in the *Preliminary Design* section, with various K and ρ values.

Under the loading of $200N$, the cost of each optimized design is listed in the table below:

Design	**hexagonal**	**trapezoidal**	**triangular**
$C(T)$ (mm^3)	87160.8	96977.6	125050

Figure 6: Final Optimized Costs

As one can see, the optimized hexagonal is the cheapest truss design and beats the trapezoidal design by a noticeable margin, it is therefore selected as the superior concept. The specific coordinates of the locally-cheapest truss are shown in the figure below.

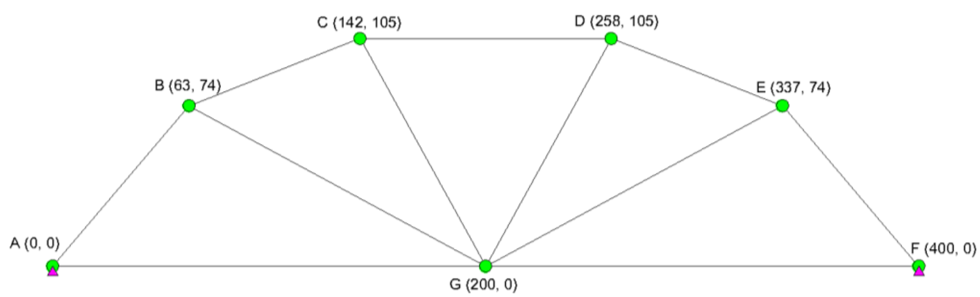


Figure 7: Hexagonal Truss with Optimized Coordinates (mm)

6.2 2D Static Force Analysis

The theoretical P value, 200 N, was used for the calculation of 2D Truss analysis.

- Members under compression: member AB (131.33 N), member BC (168.16 N), member CD (190.48 N), member DE (168.16 N), member EF (131.33 N)
- Members under tension: member AG (85.14 N), member BG (81.16N), member CG (70.19 N), member DG (70.19 N), member EG (81.16N), member FG (85.14 N)
- Maximum compressive force: member BC (168.16 N) and member DE (168.16 N)
- Maximum tensile force: member FG (85.14 N), member AG (85.14 N)

Due to symmetry, the force distribution on the left side of the bridge is the mirror image of the right side of the bridge. The detailed calculation for each member is attached in the appendix.

6.3 Shear Bending Analysis

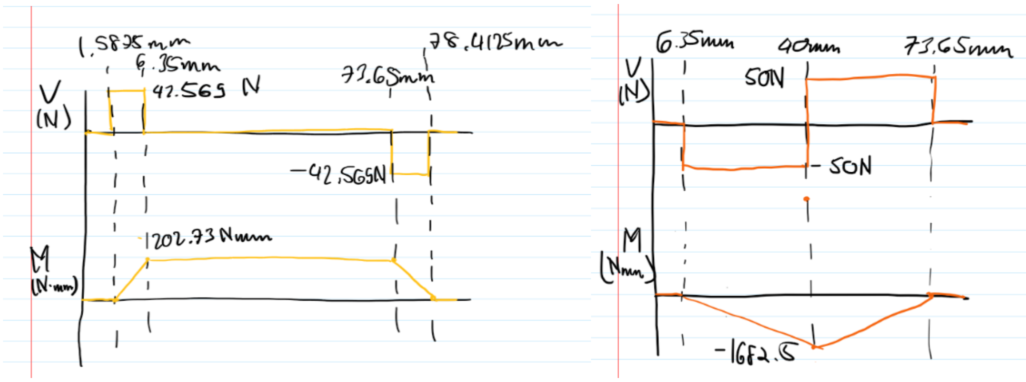


Figure 8: Shear and Bending Moment Diagram in X and Y - Directions for Pin A

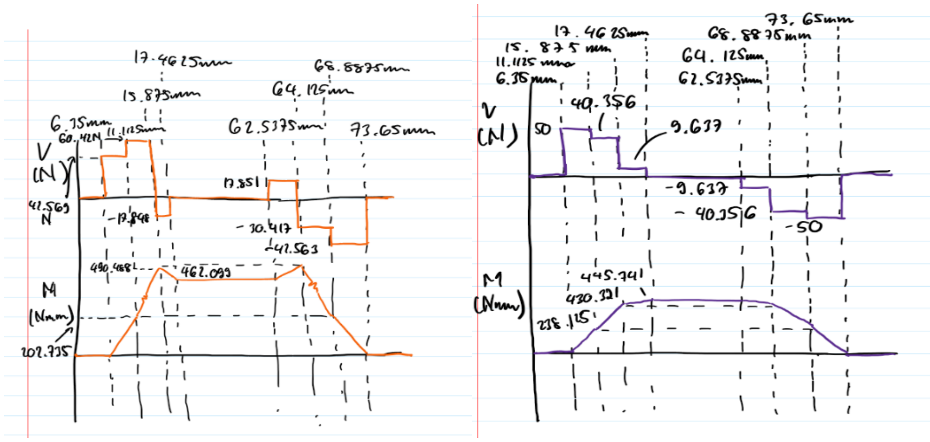


Figure 9: Shear and Bending Moment Diagram in X and Y - Directions for Pin B

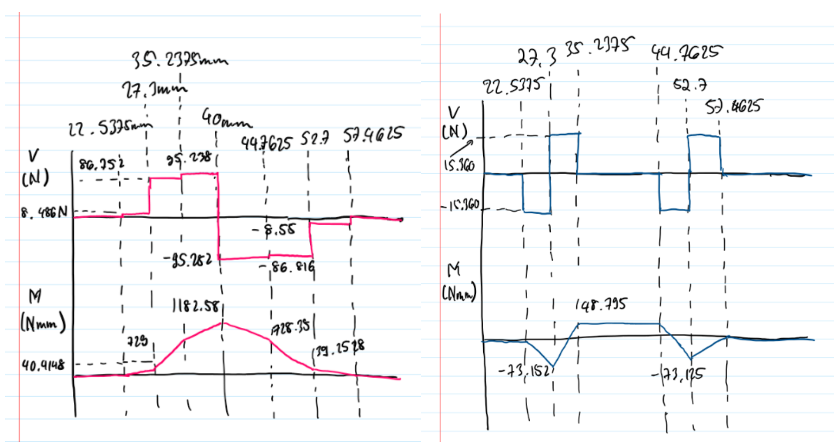


Figure 9: Shear and Bending Moment Diagram in X and Y - Directions for Pin C

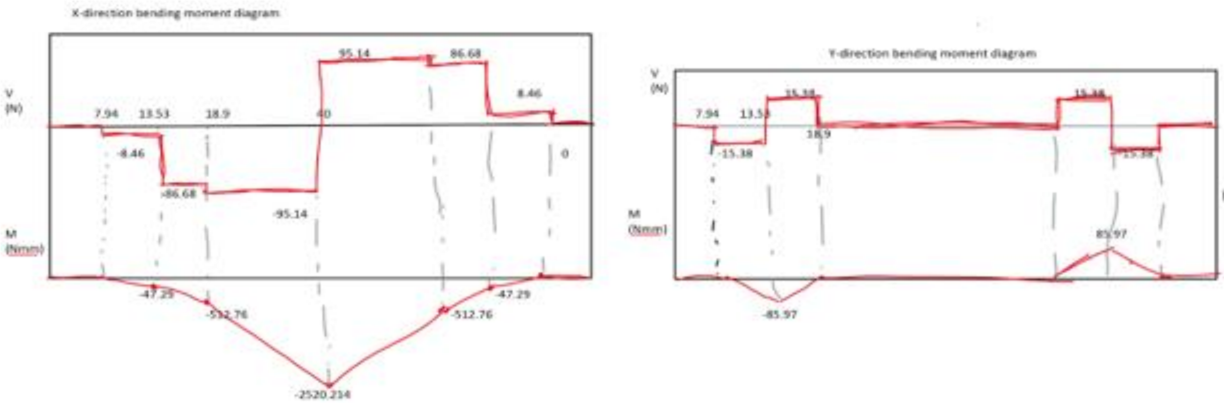


Figure 11: Shear and Bending Moment Diagram in X and Y - Directions for Pin D

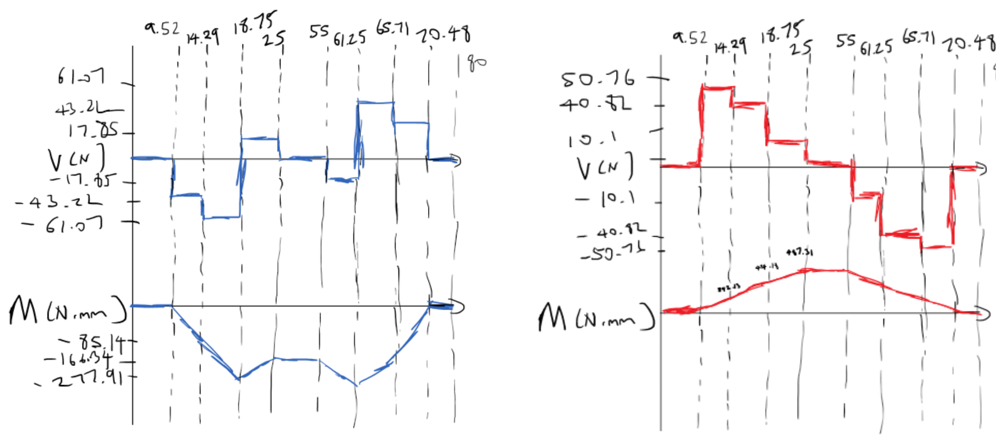


Figure 11: Shear and Bending Moment Diagram in X and Y - Directions for Pin E

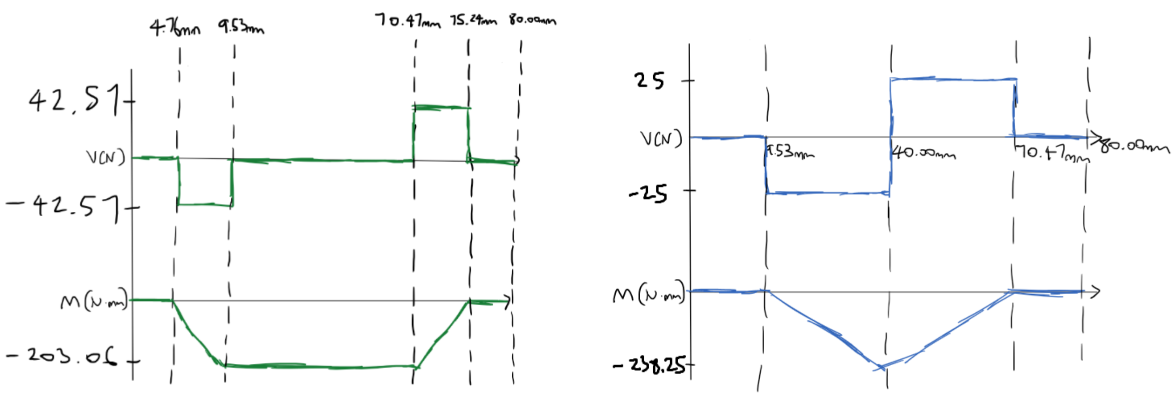


Figure 13: Shear and Bending Moment Diagram in X and Y - Directions for Pin F

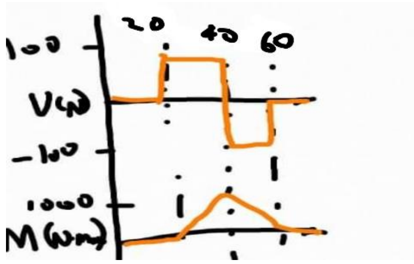


Figure 14: Shear and Bending Moment Diagram in X and Y - Directions for Pin G

6.4 Failure Analysis

The modes of failure are extensively discussed in the *Cost Model* section of the introduction. Nonetheless, the manual failure analysis verification is attached in the appendix I.

6.5 Final Design Drawings and Theoretical PV

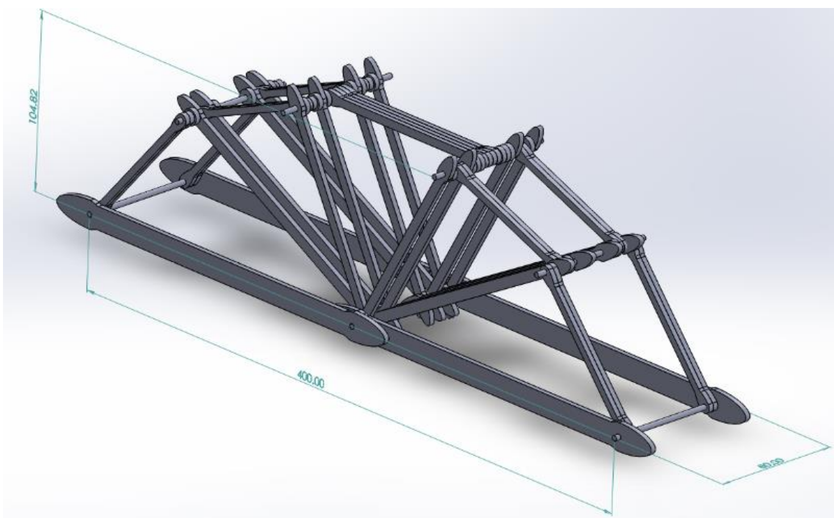


Figure 15: 3D View of Final Design

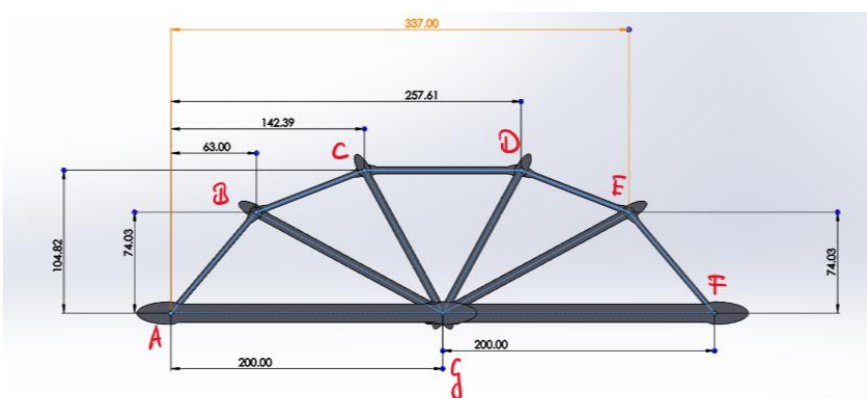


Figure 16: Dimensions for each Key Joint

Using the 3D Solidworks model and the dimensions given, the mass of the bridge was estimated to be 28g. Considering the load this bridge was designed for was 20 kg or 20000g, the theoretical PV would be 714.29.

7. Construction, Testing and Refinement

After the design finalization, the bridge members were drawn on AutoCAD. We also drawn the 3D representation of each member and performed virtual assembly on SolidWorks. We then did laser-cutting, and assembled the bridge. The compression member is made such that its cross section is as *square* as possible (glueing 2 pieces together). each tension member is separated to 4 beams because we wanted to maximize geometric stability.

7.1 Testing History and Reflection

Due to lack of material, we only performed 2 testings: **1)** testing to failure (first bridge); **2)** testing with minimum PV value (second bridge)

The first bridge was assembled in a way that on the top view, it has 180^o rotational symmetry. Under a load (5 kg) around the minimum PV value, its structure appears to experience torsion and quickly failed. We deduce it was due to the lack of reflectional symmetry on the top view that leads to geometric instability under heavy load.

7.2 Improvements and Final Contest

Therefore, the second time we assembled the bridge, we made sure it is reflectionally symmetric on the top view, it does show much better PV value, as it was able to take minimum weight with ease (5 kg for our bridge).

Before the final contest, we applied glue on the compression members (merging 2 sub-beams into 1 beam). We did not anticipate, however, that the wood glue is of poor quality. It was extremely heavy, and performs poorly. Often the glued members fall apart, and we had to glue it again, adding more weight. In total, the glue adds up to 5g, which is quite a deadly blow to our design. Since the members we are glueing are over-estimated compression members, our bridge is not really more performant. This was undesirable. Had we tested our bridge earlier, we would have known and used the much lighter super glue.

The final design mas was $\mathbf{27.73g}$, and the max load was $\mathbf{12kg}$, and the $PV = \mathbf{432.74}$.

7.3 Deviation from Theoretical PV Values

There are many explanations for the deviation from the theoretical PV Values.

1. The posted properties for the wood and pin may deviates from our wood and pin.
2. 3D geometric stability: since the truss analysis is all done in 2D, when transforming into 3D, there are many potential geometric stability issues due to shear, bending and slight inclination of members on the truss side view during the assembly. Our truss failed essentially due to the geometric instability on the roller-pin support ends. If we hold the support members steady, the truss will be able to reach our theoretical PV value.
3. There are lots of **assumptions** while we were theorizing. The cost function assumed the relation of dimension for width, thickness, and cross section area.

8. Conclusions, Recommendations

8.1 Objective Analysis

The project's objective of reaching 200 PV value is *successfully* met. However, the maximum loading was not able to reach the 20 kg as theorized. Therefore, despite the completeness of our theorization, the practical objectives were not completely met.

8.2 Recommendations

There are a few recommendations that we could think of:

1. We did not consider the weight of pins in our cost model, which had devastating consequences, since pins are much heavier than the balsa wood. Had we incorporated weight of pins into the cost function, Design 2 might have beaten Design 1.
2. We spent most of the time theorizing and very little time testing, and we realized that in engineering, it should be the other way around. Many groups that got good results started testing 2 weeks before the demo day. An ideal combination is to theorizing while testing. This will easily make sure we stay away from the sloppy mistakes such as using wood glue and assemble the bridge in an unsymmetric fashion.

9. References

[1] H. Jahed, MTE219 Project Outline. 2019 [Online]. Available: https://learn.uwaterloo.ca/d2l/le/content/441962/viewContent/2478423/View. [Accessed: 19- Mar- 2019]

[2] "Euler's critical load", En.wikipedia.org, 2019. [Online]. Available: https://en.wikipedia.org/wiki/Euler%27s_critical_load?fbclid=IwAR0CWFXNBctDrdJ3ms3kKyUPTu28HLtYIO07hOfwMoN-45S5ttDZagN6f24. [Accessed: 19- Mar- 2019]

Appendix I: 2D Truss Static Analysis Calculations

Link AB/BC/DE/EF Rupture (4 Members)

$$\sigma_{ut} = \frac{P_{max}}{A} = \frac{P_{max}}{(4)(w)(t)}$$

$$P_{max} = (4)(\sigma_{ut})(w)(t) = (4)(7.7MPa)(4.00mm)(3.175mm) = 391.16N$$

Link CD Rupture (4 Members)

$$\sigma_{ut} = \frac{P_{max}}{A} = \frac{P_{max}}{(4)(w)(t)}$$

$$P_{max} = (4)(\sigma_{ut})(w)(t) = (4)(7.7MPa)(5.00mm)(3.175mm) = 488.95N$$

Link AG/FG Rupture (2 Members)

$$\sigma_{ut} = \frac{P_{max}}{A} = \frac{P_{max}}{(2)(w - d_{pin})(t)}$$

$$P_{max} = (2)(\sigma_{ut})\left(w - d_{pin}\right)(t) = (2)(7.7MPa)(16.63mm - 3.18mm)(3.175mm) = 657.64N$$

Link BG/EG Rupture (4 Members)

$$\sigma_{ut} = \frac{P_{max}}{A} = \frac{P_{max}}{(4)(w - d_{pin})(t)}$$

$$P_{max} = (4)(\sigma_{ut})\left(w - d_{pin}\right)(t) = (4)(7.7MPa)(9.65mm - 3.18mm)(3.175mm) = 632.70N$$

Link CG/DG Rupture (4 Members)

$$\sigma_{ut} = \frac{P_{max}}{A} = \frac{P_{max}}{(4)(w - d_{pin})(t)}$$

$$P_{max} = (4)(\sigma_{ut})\left(w - d_{pin}\right)(t) = (4)(7.7MPa)(8.68mm - 3.18mm)(3.175mm) = 537.85N$$

Bearing Failure (4 Members)

$$\sigma_{ut} = \frac{P_{max}}{A} = \frac{P_{max}}{(4)(d_{pin})(t)}$$

$$P_{max} = (4)(\sigma_{ut})\left(d_{pin}\right)(t) = (4)(7.7MPa)(3.18mm)(3.175mm) = 310.97N$$

Buckling in Link AB/EF (2 $\times$ 2 Glued Members)

$$\mathrm{P_{cr}} = \frac{2\pi^2 EI}{(KL)^2} = \frac{2\pi^2 E}{(KL)^2} \times \frac{w^3 t}{12} = \frac{2\pi^2 \times 3510MPa}{(1 \times 97.19mm)^2} \times \frac{(4.00mm)^3(6.25mm)}{12} = 244.50N$$

Buckling in Link BC/DE (2 × 2 Glued Members)

$$\mathrm{P_{cr}} = \frac{2\pi^2 EI}{(KL)^2} = \frac{2\pi^2 E}{(KL)^2} \times \frac{w^3 t}{12} = \frac{2\pi^2 \times 3510MPa}{(1 \times 84.86mm)^2} \times \frac{(4.00mm)^3(6.25mm)}{12} = 320.71N$$

Buckling in Link CD (2 × 2 Glued Members)

$$\mathrm{P_{cr}} = \frac{2\pi^2 EI}{(KL)^2} = \frac{2\pi^2 E}{(KL)^2} \times \frac{w^3 t}{12} = \frac{2\pi^2 \times 3510MPa}{(1 \times 116.00mm)^2} \times \frac{(5.00mm)^3(6.25mm)}{12} = 335.22N$$

Tearing in Link AG/FG (2 Members)

$$\mathrm{\tau_{ut}} = \frac{P_{max}}{(2)(b)(t)}$$

$$P_{max} = (2)(\tau_{ut})(b)(t) = (2)(1.0MPa)(24.85mm)(3.175mm) = 157.80N$$

Tearing in Link BG/EG (4 Members)

$$\mathrm{\tau_{ut}} = \frac{P_{max}}{(4)(b)(t)}$$

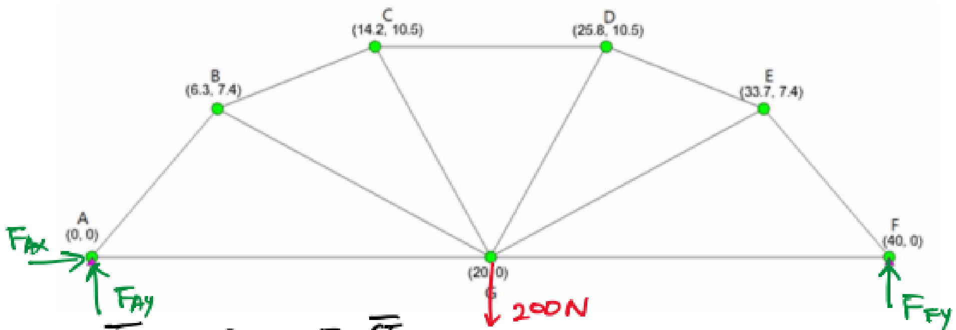
$$P_{max} = (4)(\tau_{ut})(b)(t) = (4)(1.0MPa)(14.48mm)(3.175mm) = 183.90N$$

Tearing in Link CG/DG (4 Members)

$$\mathrm{\tau_{ut}} = \frac{P_{max}}{(4)(b)(t)}$$

$$P_{max} = (4)(\tau_{ut})(b)(t) = (4)(1.0MPa)(13.01mm)(3.175mm) = 165.23N$$

Detail Calculation for 2D Static Force Analysis



$$\overline{AG} = 200\ mm = \overline{GF}$$

$$\overline{AB} = \sqrt{(63-0)^2 + (74-0)^2} = 97.185\ mm = \overline{EF}$$

$$\overline{BC} = \sqrt{(142-63)^2 + (105-74)^2} = 84.864\ mm = \overline{DE}$$

$$\overline{CD} = 258 - 142 = 116\ mm$$

$$\overline{BG} = \sqrt{(63-200)^2 + (74-0)^2} = 155.708 = \overline{EG}$$

$$\overline{CG} = \sqrt{(142-200)^2 + (105-0)^2} = 119.954 = \overline{DG}$$

$$\Sigma F_x = 0 : F_{Ax} = 0$$

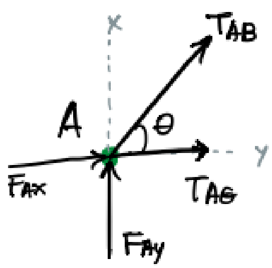
$$\Sigma M_A = 0 \ (+) : -200\ N \cdot 200\ mm + F_{Fy} \cdot 400\ mm = 0$$

$$\therefore F_{Fy} = \frac{200N \cdot 200mm}{400mm} = 100N$$

$$\Sigma F_y = 0 : F_{Ay} - 200N + F_{Fy} = 0$$

$$\therefore F_{Ay} = 200N - 100N = 100\ N$$

Joint A:



$$\theta = \cos^{-1}\left(\frac{\overline{AB}^2 + \overline{AG}^2 - \overline{BG}^2}{2\overline{AB}\,\overline{AG}}\right) \quad \text{(Cosine Law)}$$

$$= \cos^{-1}\left(\frac{(97.185)^2 + (200)^2 - (155.708)^2}{2(97.185)(200)}\right)$$

$$= 49.59^\circ$$

$$\Sigma F_x = 0 : \quad T_{AB}\cos(49.59^\circ) + T_{AG} + F_{Ax} = 0$$

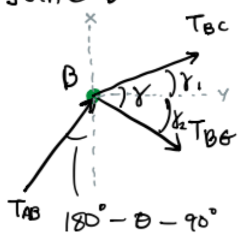
$$\Sigma F_y = 0 : \quad T_{AB}\sin(49.59^\circ) + F_{Ay} = 0$$

$$\therefore T_{AB} = \frac{-100N}{\sin(49.59^\circ)} = -131.333N \Rightarrow 131.333N$$

$$(-131.333N)\cos(49.59^\circ) + T_{AG} + 0N = 0$$

$$\therefore T_{AG} = (131.333N)\cos(49.59^\circ) = 85.137N \text{ (Tension)} \quad ②$$

Joint B:



$$180^\circ - \theta - 90^\circ = 40.41^\circ$$

$$\gamma = \cos^{-1}\left(\frac{\overline{BC}^2 + \overline{BG}^2 - \overline{CG}^2}{2\,\overline{BC}\,\overline{BG}}\right)$$

$$= \cos^{-1}\left(\frac{(84.864)^2 + (155.708)^2 - (119.954)^2}{2(84.864)(155.708)}\right)$$

$$= 49.81^\circ$$

$$\gamma_1 = \tan^{-1}\left(\frac{\Delta Y_1}{\Delta X_1}\right) = \tan^{-1}\left(\frac{|105 - 741|}{|142 - 631|}\right) = 21.43^\circ \qquad 21.4253$$

$$\gamma_2 = \tan^{-1}\left(\frac{\Delta Y_2}{\Delta X_2}\right) = \tan^{-1}\left(\frac{10 - 741}{1200 - 631}\right) = 28.38^\circ$$

$$\gamma = \gamma_1 + \gamma_2 \Rightarrow 49.81^\circ = 21.43^\circ + 28.38^\circ \checkmark$$

$$\Sigma F_x = 0: \quad T_{BC}\cos(21.43^\circ) + T_{BG}\cos(28.38^\circ) + T_{AB}\sin(40.41^\circ) = 0$$

$$\Rightarrow 0.9309\,T_{BC} + 0.8798\,T_{BG} = -85.137N \quad \cdots (1)$$

$$\Sigma F_y = 0: \quad T_{BC}\sin(21.43^\circ) - T_{BG}\sin(28.38^\circ) + T_{AB}\cos(40.41^\circ) = 0$$

$$\Rightarrow 0.3653\,T_{BC} - 0.4753\,T_{BG} = +100.000N \quad \cdots (2)$$

$$(1) \div 0.9309 \Rightarrow T_{BC} + 0.9451\,T_{BG} = -91.4567N$$

$$(2) \div 0.3653 \Rightarrow T_{BC} - 1.3011\,T_{BG} = -273.7483N$$

$$\therefore +91.4567 - 0.9451\,T_{BG} = +273.7483 + 1.3011\,T_{BG}$$

$$\Rightarrow 2.2462\,T_{BG} = -182.2916N$$

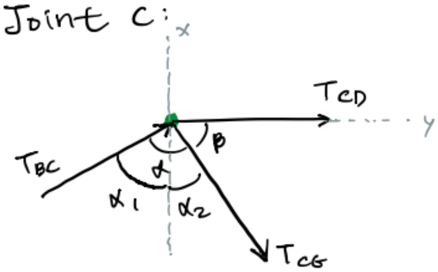
$$\therefore T_{BG} = -81.156N \quad ③$$

$$(1) \div 0.9309 \Rightarrow T_{BC} + 0.9451\,T_{BG} = -91.4567N$$

$$\therefore T_{BC} = -91.4567N - 0.9451(81.156N)$$

$$= -168.157N \Rightarrow 168.157N \text{ (compression)} \quad ④$$

Joint C:



$$\beta = \cos^{-1}\left(\frac{\overline{CD}^2 + \cancel{\overline{CG}^2} - \cancel{\overline{DG}^2}}{2\,\overline{CD}\,\overline{CG}}\right) \qquad \overline{CG} = \overline{DG}$$

$$= \cos^{-1}\left(\frac{(116)^2}{2(116)(119.954)}\right)$$

$$= 61.08^\circ$$

$$\alpha_2 = 90^\circ - \beta = 90^\circ - 61.08^\circ = 28.92^\circ$$

$$\alpha = \cos^{-1}\left(\frac{\overline{BC}^2 + \overline{CG}^2 - \overline{BG}^2}{2\,\overline{BC}\,\overline{CG}}\right)$$

$$= \cos^{-1}\left(\frac{(119.954)^2 + (84.864)^2 - (155.708)^2}{2(119.954)(84.864)}\right)$$

$$= 97.49^\circ$$

$$\alpha_1 = \alpha - \alpha_2 = 97.49^\circ - 28.92^\circ = 68.57^\circ$$

$$\Sigma F_x = 0: \quad T_{CD} - T_{CG}\sin(28.92^\circ) + T_{BC}\sin(68.57^\circ) = 0$$

$$\Sigma F_y = 0: \quad -T_{CG}\cos(28.92^\circ) + T_{BC}\cos(68.57^\circ) = 0$$

$$\therefore T_{CG} = \frac{-(-168.157\,N)\cos(68.57^\circ)}{\cos(28.92^\circ)} = 70.192\,N \quad (5)$$

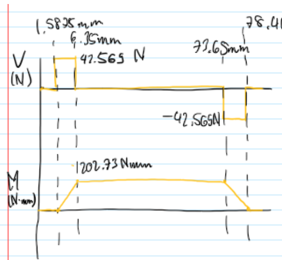
$$T_{CD} = T_{CG}\sin(28.92^\circ) - T_{BC}\sin(68.57^\circ)$$

$$= (70.192)\sin(28.92^\circ) - (-168.157)\sin(68.57^\circ)$$

$$= 190.48\,N \quad (6)$$

Appendix II: Shear Bending Analysis Calculations

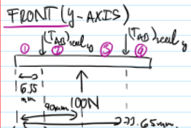
Joint A



$\Sigma M_O = 0 = 42.569(1.5875) - 42.569(6.35) - 42.659(73.65) + 42.659(78.4125) - V_4 x + M_4$

$M_4 = 0$

FRONT (y-AXIS)

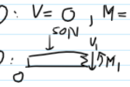


$(T_{AB})_{real_y} = 65.667 \cdot \sin 49.59$

$= 50N$

①: $V = 0$, $M = 0$

②:



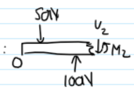
$\Sigma F_y = 0 = -50 - V_1$

$V_1 = -50N$

$\Sigma M_O = 0 = -(50)(6.35) - V_1 x + M_1$

$M_1 = -50x + 317.5$

③:



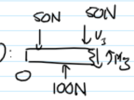
$\Sigma F_y = 0 = -50 + 100 - V_2$

$V_2 = 50N$

$\Sigma M_O = 0 = -(50)(6.35) + (100)(40) - V_2 x + M_2$

$M_2 = 50x - 3682.5$

④:



$\Sigma F_y = 0 = (-50)(2) + 100 - V_3$

$V_3 = 0$

$\Sigma M_O = 0 = -(50)(6.35) + 100(40) - 50(73.65) - V_3 x + M_3$

$M_3 = 0$

Pin A:

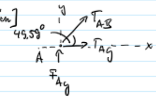
→ due to symmetry of layout + same size members used between joints
↳ assume even distribution of force

→ all pins are 80mm long max

→ due to no forces along axis through centers of pins, assume no force applied

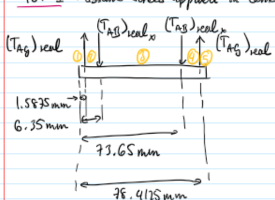
$(T_{AB})_{real} = \frac{T_{AB}}{2} = \frac{131.333}{2} = 65.667$ N [compression]

$(T_{AG})_{real} = \frac{T_{AG}}{2} = \frac{85.137}{2} = 42.569$ N [Tension]



$F_{A_y} = 100$ N

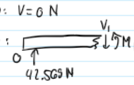
TOP (x-AXIS): → assume forces applied in center of each member (focused!)



$(T_{AB})_{real_x} = 65.667 \cdot \cos(49.59) = 42.569$ N

①: $V = 0$ N

②:



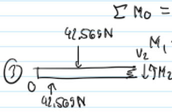
$\Sigma F_x = 0 = 42.569 - V_1$

$V_1 = 42.569$ N

$\Sigma M_O = 0 = (42.569)(1.5875) - V_1 x + M_1$

$M_1 = 42.569x - 67.578$ Nmm

③:



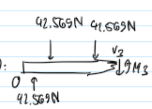
$\Sigma F_x = 0 = 42.569 - 42.569 - V_2$

$V_2 = 0$

$\Sigma M_O = 0 = 42.569(1.5875) - 42.569(6.35) - V_2 x + M_2$

$M_2 = 202.73$ Nmm

④:



$\Sigma F_x = 0 = 42.569 - 2(42.569) - V_3$

$V_3 = -42.569$ N

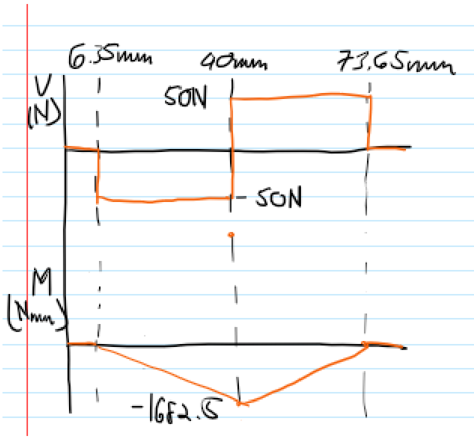
$\Sigma M_O = 0 = 42.569(1.5875) - 42.569(6.35) - 42.569(73.65) - V_3 x + M_3$

$M_3 = -42.569x + 3387.942$

⑤:

$\Sigma F_x = 0 = 42.569(2) - 42.569(2) - V_4$

$V_4 = 0$



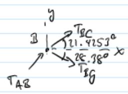
Joint B

<u>Pin B:</u>

$(T_{AB})_{real} = 65.667\ N$ [compression]

$(T_{BC})_{real} = \frac{168.157N}{2} = 84.0785\ N$ [compression]

$(T_{BG})_{real} = \frac{81.156N}{4} = 20.289\ N$ [tension]

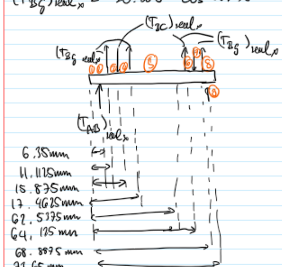


<u>TOP: (X-AXIS)</u>

$(T_{AB})_{real_x} = 42.569\ N$ [C]

$(T_{BC})_{real_x} = 84.0785 \cdot \cos 21.43 = 78.268\ N$ [C]

$(T_{BG})_{real_x} = 20.289 \cdot \cos 28.38 = 17.851\ N$ [T]



① : $V = 0,\ M = 0$

② : $\sum F_x = 0 = 42.569 - V_1$

$V_1 = 42.569N$

$\sum M_0 = 0 = 42.569(6.35) - V_1 x + M_1$

$M_1 = 42.569x - 270.313$

③ : $\sum F_x = 0 = 42.569 + 17.851 - V_2$

$V_2 = 60.42N$

$\sum M_0 = 0 = 42.569(6.35) + 17.851(11.1125) - V_2 x + M_2$

$M_2 = 60.42x - 468.68$

④ : $\sum F_x = 0 = 42.569 + 17.851 - 78.268 - V_3$

$V_3 = -17.848\ N$

$\sum M_0 = 0 = 42.569(6.35) + 17.851(11.1125) - 78.268(15.875) - V_3 x + M_3$

$M_3 = -17.848x + 773.822$

⑤ : $\sum F_x = 0 = 42.569 + 17.851(2) - 78.268 - V_4$

$V_4 = 0$

$\sum M_0 = 0 = 42.569(6.35) + 17.851(11.1125) - 78.268(15.875) + 17.851(17.4625) - V_4 x + M_4$

$M_4 = 462.099$

⑥ : $\sum F_x = 0 = 42.569 + 17.851(3) - 78.268 - V_5$

$V_5 = 17.851\ N$

$M_5 = 17.851x - 654.258$

⑦: $\Sigma F_x = 0 = 42.569 + 17.851(3) - 78.268(2) - V_6$

$V_6 = -30.417\,N$

$\Sigma M_0 = 0 = 42.569(6.35) + 17.851(11.1125) - 78.268(15.875) + 17.851(17.4625) + 17.851(62.5375) - 78.268(64.115) - V_6x + M_6$

$M_6 = -30.417x + 4364.678$

⑧: $\Sigma F_x = 0 = 42.569 + 17.851(4) - 78.268(2) - V_7$

$V_7 = -42.563\,N$

$\Sigma M_0 = 0 = 42.569(6.35) + 17.851(11.1125) - 78.268(15.875) + 17.851(17.4625) + 17.851(62.5375) - 78.268(64.115) + 17.851(68.8825) - V_7x + M_7$

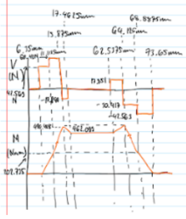
$M_7 = -42.563x + 3134.967$

⑨: $\Sigma F_x = 0 = 42.565(2) + 17.851(4) - 78.268(2) - V_8$

$V_8 = 0$

$\Sigma M_0 = 0 = 42.569(6.35) + 17.851(11.1125) - 78.268(15.875) + 17.851(17.4625) + 17.851(62.5375) - 78.268(64.115) + 17.851(68.8825) + 42.569(73.65) - V_8x + M_8$

$M_8 = 0$

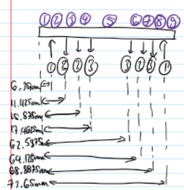


FRONT (y-Direction)

$(T_{AB})_{real_y} = 50\,N$ [C] ①

$(T_{BC})_{real_y} = 84.0785 \cdot \sin 21.43 = 30.719\,N$ [C] ⑥

$(T_{BG})_{real_y} = 20.289 \cdot \sin 28.38 = 9.644\,N$ [T] ⑤



①: $V = 0,\ M = 0$

②: $\Sigma F_y = 0 = 50 - V_1$

$V_1 = 50\,N$

$\Sigma M_0 = 0 = 50(6.35) - V_1x + M_1$

$M_1 = 50x - 317.5$

③: $\Sigma F_y = 0 = 50 - 9.644 - V_2$

$V_2 = 40.356\,N$

$\Sigma M_0 = 0 = 50(6.35) - 9.644(11.1125) - V_2x + M_2$

$M_2 = 40.356x - 210.331$

④: $\Sigma F_y = 0 = 50 - 9.644 - 30.719 - V_3$

$V_3 = 9.637$

$\Sigma M_0 = 0 = 50(6.35) - 9.644(11.1125) - 30.719(15.875) - V_3x + M_3$

$M_3 = 9.637x + 277.333$

⑤: $\Sigma F_y = 0 = 50 - 9.644(2) - 30.719 - V_4$

$V_4 = 0$

$\Sigma M_0 = 0 = 50(6.35) - 9.644(11.1125) - 30.719(15.875) - 9.644(17.4625) - V_4x + M_4$

$M_4 = 445.741$

⑥: $\Sigma F_y = 0 = 50 - 9.644(3) - 30.719 - V_5$

$V_5 = -9.644\,N$

$\Sigma M_0 = 0 = 50(6.35) - 9.644(11.1125) - 30.719(15.875) - 9.644(17.4625) - 9.644(62.5375) - V_5x + M_5$

$M_5 = -9.644x + 561.189$

⑦: $\Sigma F_y = 0 = 50 - 9.644(3) - 30.719(2) - V_6$

$V_6 = -40.363\,N$

$\Sigma M_0 = 0 = 50(6.35) - 9.644(11.1125) - 30.719(15.875) - 9.644(17.4625) - 9.644(62.5375) - 30.719(64.115) - V_6x + M_6$

$M_6 = -40.363x + 3018.709$

⑧: $\Sigma F_y = 0 = 50 - 9.644(4) - 30.719(2) - V_7$

$V_7 = -50\,N$

$\Sigma M_0 = 0 = 50(6.35) - 9.644(11.1125) - 30.719(15.875) - 9.644(17.4625) - 9.644(62.5375) - 30.719(64.115) - 9.644(68.8825) - V_7x + M_7$

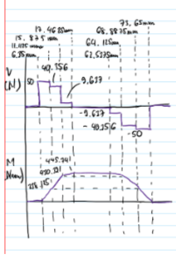
$M_7 = -50x + 3683.06$

⑨: $\Sigma F_y = 50(2) - 9.644(4) - 30.719(2) - V_8$

$V_8 = 0$

$\Sigma M_0 = 0 = 50(6.35) - 9.644(11.1125) - 30.719(15.875) - 9.644(17.4625) - 9.644(62.5375) - 30.719(64.115) - 9.644(68.8825) + 50(73.65) - V_8x + M_8$

$M_8 = 0$



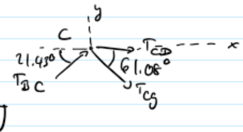
Joint C

Pin C:

$(T_{BC})_{real} = 84.0785\ N\ [C]$

$(T_{CD})_{real} = 190.48\ N\ [C]$

$(T_{CG})_{real} = \frac{T_{CG}}{4} = \frac{70.192}{4} = 17.548\ N\ [T]$



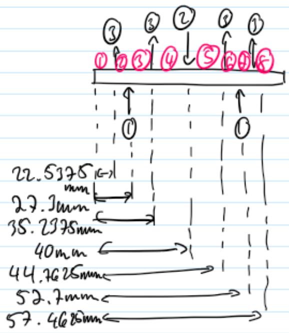
TOP (X-Direction)

$(T_{BC})_{real_x} = 78.266\ N\ [C]$ ①

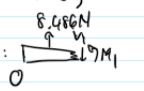
$(T_{CD})_{real_x} = 190.48\ N\ [C]$ ②

$(T_{CG})_{real_x} = 17.548 \cdot \cos(61.08)$

$= 8.486\ N\ [T]$ ③



① $V=0,\ M=0$

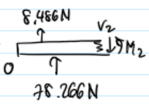
②: 

$\Sigma F_x = 0 = 8.486 - V_1$

$V_1 = 8.486\ N$

$\Sigma M_O = 0 = 8.486(22.5375) - V_1 x + M_1$

$M_1 = 8.486x - 191.253$

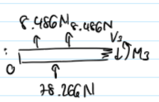
③: 

$\Sigma F_x = 0 = 8.486 + 78.266 - V_2$

$V_2 = 86.752\ N$

$\Sigma M_O = 0 = 8.486(22.5375) + 78.266(27.3) - V_2 x + M_2$

$M_2 = 86.752x - 2327.915$

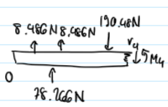
④: 

$\Sigma F_x = 0 = 8.486 + 78.266 + 8.486 - V_3$

$V_3 = 95.238\ N$

$\Sigma M_O = 0 = 8.486(22.5375 + 35.2375) + 78.266(27.3) - V_3 x + M_3$

$M_3 = 95.238x - 2626.940$

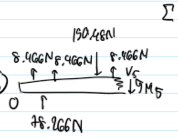
⑤ 

$\Sigma F_x = 0 = 8.466(2) + 78.266 - 190.48 - V_4$

$V_4 = -95.282\ N$

$\Sigma M_O = 0 = 8.466(22.5375 + 35.2375) + 78.266(27.3) - 190.48(40) - V_4 x + M_4$

$M_4 = -95.282x + 4993.415$

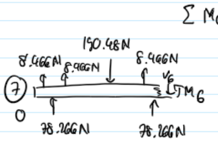
⑥ 

$\Sigma F_x = 0 = 8.466(3) + 78.266 - 190.48 - V_5$

$V_5 = -86.816\ N$

$\Sigma M_O = 0 = 8.466(22.5375 + 35.2375 + 44.7625) + 78.266(27.3) - 190.48(40) - V_5 x + M_5$

$M_5 = -86.816x + 4614.456$

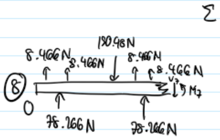
⑦ 

$\Sigma F_x = 0 = 8.466(3) + 78.266(2) - 190.48 - V_6$

$V_6 = -8.55\ N$

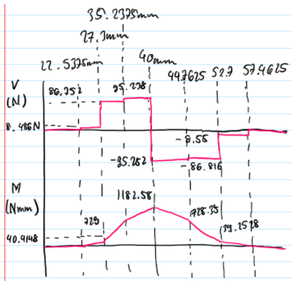
$\Sigma M_O = 0 = 8.466(22.5375 + 35.2375 + 44.7625) + 78.266(27.3 + 52.7) - 190.48(40) - V_6 x + M_6$

$M_6 = -8.55x + 489.838$

⑧ 

$\Sigma F_x = 0 = 8.466(4) + 78.266(2) - 190.48 - V_7$

$V_7 = 0$



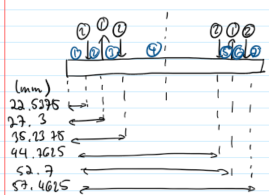
$$\Sigma M_0 = 0 = 8.466(22.5375 + 38.2375 + 44.7625 + 57.4625) + 78.266(27.3 + 52.7) - 180.98(40) - V_7 x + M_7$$

$$M_7 = 0$$

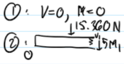
FRONT (y-Direction)

$(T_{BC})_{real_y} = 30.719N\ [C]$ ①

$(T_{CG})_{real_y} = 17.548 \cdot \sin 61.08 = 15.360\ N\ [T]$ ②



① : $V=0,\ M=0$

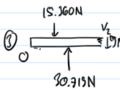
② : 

$$\Sigma F_y = 0 = -15.360 - V_1$$

$$V_1 = -15.360\ N$$

$$\Sigma M_0 = 0 = -(15.360)(22.5375) - V_1 x + M_1$$

$$M_1 = -15.360x + 346.176$$

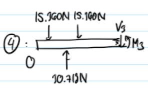
③ 

$$\Sigma F_y = 0 = -15.360 + 30.719 - V_2$$

$$V_2 = 15.359\ N$$

$$\Sigma M_0 = 0 = -(15.360)(22.5375) + (30.719)(27.3) - V_2 x + M_2$$

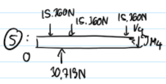
$$M_2 = 15.359x - 492.453$$

④ : 

$$\Sigma F_y = 0 = -15.360(2) + 30.719 - V_3$$

$$V_3 = 0$$

$$M_3 = 48.7953$$

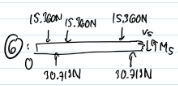
⑤ : 

$$\Sigma F_y = 0 = -15.360(3) + 30.719 - V_4$$

$$V_4 = -15.360\ N$$

$$\Sigma M_0 = 0 = -(15.360)(22.5375 + 35.2375 + 44.7625) + (30.719)(27.3) - V_4 x + M_4$$

$$M_4 = -15.360x + 736.147$$

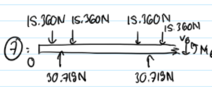
⑥ : 

$$\Sigma F_y = 0 = -(15.360)(3) + 30.719(2) - V_5$$

$$V_5 = 15.358\ N$$

$$\Sigma M_0 = 0 = -(15.360)(22.5375 + 35.2375 + 44.7625) + (30.719)(27.3 + 52.7) - V_5 x + M_5$$

$$M_5 = 15.358x - 882.544$$

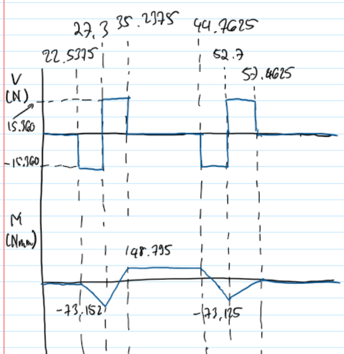
⑦ : 

$$\Sigma F_y = 0 = -(15.360)(4) + 30.719(2) - V_6$$

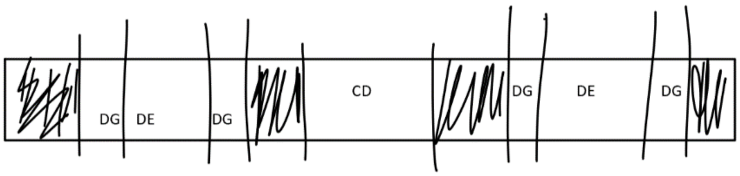
$$V_6 = 0$$

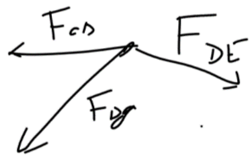
$$\Sigma M_0 = 0 = -(15.360)(22.5375 + 35.2375 + 44.7625 + 57.4625) + (30.719)(27.3 + 52.7) - V_6 x + M_6$$

$$M_6 = 0$$



Joint D





F(DG)=70.2
F(DE)=-168.2
F(CD)=-190.5

$$\hat{u}(DG) = -\frac{[257.61 - 200, 104.82]}{len(DG)} = -\left[\frac{57.61}{119.61}, \frac{104.82}{119.61}\right] = -[0.482, 0.876]$$
$$\hat{u}(CD) = [-1, 0]$$
$$\hat{u}(DE) = \frac{[337 - 257.61, -104.82 + 74.03]}{len(DE)} = \frac{[79.39, -30.79]}{85.15} = [0.93, -0.36]$$
$$\overrightarrow{F_{DG}} = \frac{[-33.84, -61.5]}{4} = [-8.46, -15.38]$$
$$\overrightarrow{F_{DE}} = \frac{[-156.43, 60.55]}{2} = [-78.22, 30.28]$$
$$\overrightarrow{F_{CD}} = [190.5, 0]$$

Joint E

-43.22 , 50.76 EF

-17.85 , -9.64 EG

78.27 , -30.12 ED

Joint F

$T_{EFY} = -100N$ $F_Y = 100N$

$T_{EFX} = 85.14N$ $T_{FG} = -85.14N$

Joint G