Crecimiento Económico

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Homework Nº3

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OLG Model - Economic Growth

1 Exercise Nº1

Get the value of consumption for an economy with Bonds.

For an economy with the following Utility Function (same as previous examples of class):

$$\mathcal{U} = c_t^h(t) \cdot c_t^h(t+1)^\beta$$

and a Budget Constraint (updated with Bonds):

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w_t(t) - t_t(t) + \frac{w_t(t+1) - t_t(t+1)}{r(t)} - b(t) \cdot \left[p(t) - \frac{1}{r(t)}\right]$$

We can write the following optimization problem:

$$\begin{aligned} \max_{c_t, c_{t+1}} \quad & \mathcal{U} = c_t^h(t) \cdot c_t^h(t+1)^\beta \\ \text{s.a.} \quad & c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w_t(t) - t_t(t) + \frac{w_t(t+1) - t_t(t+1)}{r(t)} - b(t) \cdot \left[p(t) - \frac{1}{r(t)}\right] \end{aligned} \tag{1}$$

We know that First Order Conditions imply:

$$u_1 = r(t) \cdot u_2$$

Solving for u_1 and u_2:

$$\frac{\partial \mathcal{U}}{\partial c_t^h(t)} = \quad u_1 = \quad c_t^h(t+1)^\beta \tag{2}$$

$$\frac{\partial \mathcal{U}}{\partial c_t^h(t+1)} = \quad r(t) \cdot u_2 = \quad r(t) \cdot \beta \cdot c_t^h(t) \cdot c_t^h(t+1)^{\beta-1} \tag{3}$$

For $u_1 = r(t)u_2$ that implies:

$$c_t^h(t+1)^\beta = r(t) \cdot \beta c_t^h(t) \cdot c_t^h(t+1)^{\beta-1} \qquad \Longrightarrow \qquad c_t^h(t+1) = r(t) \cdot \beta \cdot c_t^h(t)$$

Replacing the Budget Constraint (1) as $c_t^h(t+1)$:

$$r(t) \cdot [w_t(t) - t_t(t)] + w_t(t+1) - t_t(t+1) - b(t) \cdot [p(t)r(t) - 1] - c_t^h(t) \cdot r(t) = r(t) \cdot \beta c_t^h(t)$$

With some algebra we get:

$$c_t^h(t) \cdot [r(t) + r(t) \cdot \beta] = r(t) \cdot [w_t(t) - t_t(t)] + w_t(t+1) - t_t(t+1) - b(t) \cdot [p(t)r(t) - 1]$$

$$\boxed{c_t^h(t) = \frac{w_t(t) - t_t(t)}{(1+\beta)} + \frac{w_t(t+1) - t_t(t+1)}{r(t) \cdot (1+\beta)} - \frac{b(t) \cdot [p(t)r(t) - 1]}{r(t) \cdot (1+\beta)}} \tag{4}$$

And that equation (4) for $c_t^h(t)$ is the value of consumption for an economy with Bonds.

2 Exercise Nº2

Replicate the last example of our lesson but with a change in the Utility function.

$$u_t^t = u\left(c_t^h(t); c_t^h(t+1)\right) = \frac{c_t^h(t)^{1-\theta}}{1-\theta} \cdot \frac{\left[c_t^h(t+1)^{1-\theta}\right]^{\beta}}{1-\theta}$$

Taking on mind the new utility function for the economy, we can rewrite our problem as follows:

$$\begin{aligned} \max_{c_t, c_{t+1}} \quad & \mathcal{U} = \frac{c_t^h(t)^{1-\theta}}{1-\theta} \cdot \frac{\left[c_t^h(t+1)^{1-\theta}\right]^{\beta}}{1-\theta} \\ \text{s.a.} \quad & c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w_t(t) \cdot h_t(t) + \frac{w_t(t+1) \cdot h_t(t+1)}{r(t)} \end{aligned} \tag{5}$$

We know, as in the last exercise, that the First Order Conditions imply:

$$u_1 = r(t) \cdot u_2$$

Hence, solving for u_1 and u_2:

$$\frac{\partial \mathcal{U}}{\partial c_t^h(t)} = \quad u_1 = \quad c_t^h(t)^{-\theta} \cdot \frac{c_t^h(t+1)^{\beta(1-\theta)}}{1-\theta} \tag{6}$$

$$\frac{\partial \mathcal{U}}{\partial c_t^h(t+1)} = \quad r(t) \cdot u_2 = \quad r(t) \cdot \frac{c_t^h(t)^{1-\theta}}{1-\theta} \cdot \beta \cdot c_t^h(t+1)^{\beta(1-\theta)-1} \tag{7}$$

That gives:

$$c_t^h(t)^{-\theta} \cdot \frac{c_t^h(t+1)^{\beta(1-\theta)}}{1-\theta} = r(t) \cdot \frac{c_t^h(t)^{1-\theta}}{1-\theta} \cdot \beta \cdot c_t^h(t+1)^{\beta(1-\theta)-1}$$

$$\frac{c_t^h(t+1)^{\beta(1-\theta)}}{c_t^h(t+1)^{\beta(1-\theta)-1}} = \frac{c_t^h(t)^{1-\theta}}{c_t^h(t)^{-\theta}} \cdot \frac{1-\theta}{1-\theta} \cdot \beta \cdot r(t)$$

Operating albraically the exponents:

$$c_t^h(t+1) = c_t^h(t) \cdot \beta \cdot r(t)$$

Replacing the Budget Constraint as $c_t^h(t+1)$:

$$r(t) \cdot w_t(t) \cdot h_t(t) + w_t(t+1) \cdot h_t(t+1) - c_t^h(t) \cdot r(t) = r(t) \cdot \beta c_t^h(t)$$

Solving for $c_t^h(t)$ that gives:

$$\boxed{c_t^h(t) = \frac{r(t) \cdot w_t(t) \cdot h_t(t)}{(1+\beta) r(t)} + \frac{w_t(t+1) \cdot h_t(t+1)}{(1+\beta) r(t)}} \tag{8}$$

The individual savings function is:

$$S_t^h\left(w_t, w_{t+1}, r_t\right) = w_t(t) \cdot h_t(t) - c_t^h(t) = l_t^h(t) + k_t^h(t+1)$$

Replacing 8 in it:

$$S_t^h\left(w_t, w_{t+1}, r_t\right) = w_t(t) \cdot h_t(t) - \frac{r(t) \cdot w_t(t) \cdot h_t(t)}{(1+\beta)r(t)} + \frac{w_t(t+1) \cdot h_t(t+1)}{(1+\beta)r(t)} = l_t^h(t) + k_t^h(t+1)$$

$$\boxed{S_t^h\left(w_t, w_{t+1}, r_t\right) = \frac{\beta \cdot w_t(t) \cdot h_t(t)}{(1+\beta)} - \frac{w_t(t+1) \cdot h_t(t+1)}{(1+\beta)r(t)} = l_t^h(t) + k_t^h(t+1)} \tag{9}$$

Summing over members of generation h, that gives:

$$\boxed{S_t^h\left(w_t, w_{t+1}, r_t\right) = \frac{\beta \cdot w_t(t) \cdot H_t(t)}{(1+\beta)} - \frac{w_t(t+1) \cdot H_t(t+1)}{(1+\beta)r(t)} = K(t+1)} \tag{10}$$

So then, from conditions of the factor markets and the no arbitrage condition, we know that:

$$r(t) = rental_{t+1} \tag{11}$$

$$w_t(t) = (1-\theta) \cdot K(t)^{\theta} \cdot H(t)^{-\theta} \tag{12}$$

$$rental_t = \theta \cdot K(t)^{\theta-1} \cdot H(t)^{1-\theta} \tag{13}$$

See that we can combine 11 with 13 by moving one period forward $(t+1)$.

We can now rewrite equation 10 by replacing the previous conditions (11 , 12 , 13) in it:

$$\boxed{S_t^h\left(w_t, w_{t+1}, r_t\right) = \frac{\beta \cdot \left[(1-\theta) \cdot K(t)^{\theta} \cdot H(t)^{-\theta}\right] \cdot H_t(t)}{(1+\beta)} - \frac{\left[(1-\theta) \cdot K(t+1)^{\theta} \cdot H(t+1)^{-\theta}\right] \cdot H_t(t+1)}{(1+\beta)\left[\theta \cdot K(t+1)^{\theta-1} \cdot H(t+1)^{1-\theta}\right]} = K(t+1)}$$

Our last equation (S_t^h) represents the equilibrium for savings in the economy. After doing some algebra, the expression becomes:

$$\boxed{K(t+1) = \frac{\theta \cdot \beta \cdot \frac{H_t(t)}{H_t(t)^{\theta}}}{\left[\frac{H_t(t+1)}{H_t(t+1)}\right] + \frac{\theta \cdot (1+\beta)}{(1-\theta)}} \cdot K(t)^{\theta} = \kappa \cdot K(t)^{\theta}} \tag{14}$$

Where κ in 14 is a constant equal to:

$$\frac{\theta \cdot \beta \cdot \frac{H_t(t)}{H_t(t)^{\theta}}}{\left[\frac{H_t(t+1)}{H_t(t+1)}\right] + \frac{\theta \cdot (1+\beta)}{(1-\theta)}}$$

Our model dynamics can be seen as follows: