Q1

(a) $\ell : \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} k, k \in \mathbb{R}$

Let $v = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

Projection of v to ℓ: $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \cdot \left| \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right|^{-1} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{12}{\sqrt{6}}$

Perpendicular vector of v to ℓ: $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - \frac{12}{\sqrt{6}} \left| \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right|^{-1} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Distance: $\left| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right| = \sqrt{3}$

(b) (i) Let $f(u, v) = \begin{pmatrix} u \\ v \\ u^2 - v^2 + 3 \end{pmatrix}$

$f_u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}, f_v = \begin{pmatrix} 0 \\ 1 \\ -2v \end{pmatrix}$

$f_u(P) = \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix}, f_v(P) = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$

$\pi : \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$

(ii) To find the intersection, we solve the equation: $\begin{pmatrix} u \\ v \\ u^2 - v^2 + 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$

$u = 3 + \lambda, v = 3 + \mu$
$u^2 - v^2 + 3 = (3 + \lambda)^2 - (3 + \mu)^2 + 3 = 3 + 6\lambda - 6\mu$
$\therefore \lambda^2 - \mu^2 = 0$
Hence, the line $\ell_1 : x = y, z = 3$ corresponds to $\lambda = \mu$, which satisfies the equation above. Hence, it lies in the intersection of S and π.

(iii) The other line would be $\lambda = -\mu$.

$\ell_2 : \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 12 \end{pmatrix}$

Q2

(a) $f(x, y) = x^3 - y^3 + 2xy$

$f_x(x, y) = 3x^2 + 3y, f_y(x, y) = -3y^2 + 3x$

To find critical points, we solve for $f_x = f_y = 0$.

$3x^2 + 3y = -3y^2 + 3x = 0$ only at $(0, 0)$ and $(1, -1)$

$(0,0)$ is a saddle point, because if we fix $y = 0$ and vary x, we can see that f increases as x increases and decreases as x decreases.

$f_{xx} = 6x, f_{yy} = -6y, f_{xy} = 3$. $D(1,-1) = 36 - 9 > 0$.

Hence, $(1,-1)$ is a local min.

(b) $g(x) = x^2 + 2xy + 2y^2$

$\nabla g(x) = \begin{pmatrix} 2x+2y \\ 2x+4y \end{pmatrix}$, which is non zero under the condition $g(x) = 5$.

$\nabla f(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

By Lagranch multiplier, $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 2x+2y \\ 2x+4y \end{pmatrix} \implies 2\lambda(2x+4y) = \lambda(2x+2y) \implies x = -3y$

Sub $x = -3y$ to $x^2 + 2xy + 2y^2 = 5$: $9y^2 + 2(-3y)y + 2y^2 = 5 \implies y^2 = 1$

$\therefore (x,y) = (-3,1)$ or $(3,-1)$.

$f(-3,1) = -5, f(3,-1) = 5$.

Min is -5, max is 5.

Q3

(a) Let $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x+y \\ y-2x \end{pmatrix}$

Then A, B, C would have $u-v$ coordinates $(0,0), (3,-6), (3,3)$. Let the region be S.

$\frac{\partial(u,v)}{\partial(x,y)} = \left| \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \right| = \left| \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \right| = 3$

$$\begin{aligned}
\iint_R \sqrt{x+y}(y-2x)^2 \, dx \, dy &= \iint_S uv^2 \frac{1}{3} \, dv \, du \\
&= \int_0^3 \int_{-2u}^{u} \frac{1}{3} uv^2 \, dv \, du \\
&= \int_0^3 \left[\frac{1}{9} uv^3 \right]_{-2u}^{u} \, du \\
&= \int_0^3 u^4 \, du \\
&= \left[\frac{1}{5} u^5 \right]_0^3 \\
&= \frac{3^5}{5}
\end{aligned}$$

(b) $f(x,y,z) = xy \sin z$ is a potential function for F.

$$\int_C F \cdot dr = f\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) - f(0,0,0) = \frac{\pi^2}{4}$$

Q4

(a) $P = 7y - e^{\sin x}$, $Q = 9x - \cos(y^3 + 7y)$

$$\begin{aligned}\int P\ dx + Q\ dy &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA \\ &= \iint_D 9 - 7\ dA \\ &= 2(\pi \times 2^2) \\ &= 8\pi\end{aligned}$$

(b) Cone: $\sqrt{3}z = \sqrt{x^2 + y^2}$

Sphere: $x^2 + y^2 + (z-1)^2 = 1$

Let $r^2 = x^2 + y^2$, and $x = r\cos\theta, y = r\sin\theta$.

First, we find the intersection of the sphere and cone:

$$\begin{gathered} z = \frac{\sqrt{x^2+y^2}}{\sqrt{3}} = 1 + \sqrt{1 - (x^2 + y^2)} \\ \frac{r}{\sqrt{3}} = 1 + \sqrt{1 - r^2} \\ 1 - r^2 = \left(\frac{r}{\sqrt{3}} - 1\right)^2 = \frac{r^2}{3} - \frac{2r}{\sqrt{3}} + 1 \\ \frac{4}{3}r^2 - \frac{2}{\sqrt{3}}r = 0 \implies r = \frac{\sqrt{3}}{2} \end{gathered}$$

Hence, the volume is:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\sqrt{3}/2}\int_{r/\sqrt{3}}^{1+\sqrt{1-r^2}} r\ dr\ d\theta &= \int_0^{2\pi}\int_0^{\sqrt{3}/2}\left(1 + \sqrt{1-r^2} - \frac{r}{\sqrt{3}}\right) r\ dr\ d\theta \\ &= 2\pi\left[-\frac{1}{3}(1-r^2)^{3/2} + \frac{1}{2}r^2 - \frac{1}{3\sqrt{3}}r^3\right]_0^{\sqrt{3}/2} \\ &= \frac{13}{12}\pi\end{aligned}$$

Q5

(a) The original conditions given are:

$$\begin{gathered} -1 \le x \le 1 \\ x^2 \le y \le 1 \\ 0 \le z \le 1 - y \end{gathered}$$

Hence,

$$\begin{gathered} 0 \le z \le 1 - y \le 1 - x^2 \le 1 \\ x^2 \le y \le 1 - z \implies -\sqrt{1-z} \le x \le \sqrt{1-z} \\ x^2 \le y \le 1 - z \end{gathered}$$

Hence,

$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f(x,y,z)\ dz\ dy\ dx = \int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} f(x,y,z)\ dy\ dx\ dz$$

(b)

$$r(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ \cos^2\theta - \sin^2\theta \end{pmatrix}, 0 \le \theta \le 2\pi$$

$$\begin{aligned}
\int_C F \cdot dr &= \int_0^{2\pi} F(r(\theta)) \cdot r'(\theta)\ d\theta \\
&= \int_0^{2\pi} \begin{pmatrix} \sin^3\theta \\ \cos\theta \\ \sin^3\theta\cos^3\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \\ \cos^2\theta - \sin^2\theta \end{pmatrix}\ d\theta \\
&= \int_0^{2\pi} -\sin^4\theta + \cos^2\theta + \sin^3\theta\cos^5\theta - \sin^5\theta\cos^3\theta\ d\theta \\
&= \int_0^{2\pi} -\sin^4\theta + \cos^2\theta\ d\theta \\
&= \frac{\pi}{4}
\end{aligned}$$

Q6

(a)

$$\begin{aligned}
\int_C f\nabla g \cdot dr &= \iint_\Sigma \nabla \times (f\nabla g) \cdot d\Sigma \\
&= \iint_\Sigma \nabla \times \begin{pmatrix} fg_x \\ fg_y \\ fg_z \end{pmatrix} \cdot d\Sigma \\
&= \iint_\Sigma \left[\left(\frac{\partial}{\partial y}(fg_z) - \frac{\partial}{\partial z}(fg_y) \right) i - \left(\frac{\partial}{\partial x}(fg_z) - \frac{\partial}{\partial z}(fg_x) \right) j + \left(\frac{\partial}{\partial x}(fg_y) - \frac{\partial}{\partial y}(fg_x) \right) k \right] \cdot d\Sigma \\
&= \iint_\Sigma \left[(f_y g_z - f_z g_y)\, i - (f_x g_z - f_z g_x)\, j + (f_x g_y - f_y g_x)\, k \right] \cdot d\Sigma
\end{aligned}$$

Similarly,

$$\int_C g\nabla f \cdot dr = \iint_\Sigma \left[(g_y f_z - g_z f_y)\, i - (g_x f_z - g_z f_x)\, j + (g_x f_y - g_y f_x)\, k \right] \cdot d\Sigma$$

Hence,

$$\int_C f\nabla g \cdot dr = -\int_C g\nabla f \cdot dr = \int_{-C} g\nabla f \cdot dr$$

(b)

$$\nabla \cdot F = 2z$$

Let E be the solid ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$.
Hence,

$$\begin{aligned}
\int_S F \cdot dS &= \iiint_E (\nabla \cdot F)\ dV \\
&= \iiint_E 2z\ dV \\
&= 0
\end{aligned}$$

Due to symmetry.