

Q1

(a) $\ell : \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} k, k \in \mathbb{R}$

Let $v = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

Projection of v to ℓ : $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \cdot \left| \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right|^{-1} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{12}{\sqrt{6}}$

Perpendicular vector of v to ℓ : $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - \frac{12}{\sqrt{6}} \left| \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right|^{-1} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Distance: $\left| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right| = \sqrt{3}$

(b) (i) Let $f(u, v) = \begin{pmatrix} u \\ v \\ u^2 - v^2 + 3 \end{pmatrix}$

$$f_u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}, f_v = \begin{pmatrix} 0 \\ 1 \\ -2v \end{pmatrix}$$

$$f_u(P) = \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix}, f_v(P) = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$$

$$\pi : \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$$

(ii) To find the intersection, we solve the equation: $\begin{pmatrix} u \\ v \\ u^2 - v^2 + 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$

$$u = 3 + \lambda, v = 3 + \mu$$

$$u^2 - v^2 + 3 = (3 + \lambda)^2 - (3 + \mu)^2 + 3 = 3 + 6\lambda - 6\mu$$

$$\therefore \lambda^2 - \mu^2 = 0$$

Hence, the line $\ell_1 : x = y, z = 3$ corresponds to $\lambda = \mu$, which satisfies the equation above. Hence, it lies in the intersection of S and π .

(iii) The other line would be $\lambda = -\mu$.

$$\ell_2 : \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 12 \end{pmatrix}$$

Q2

(a) $f(x, y) = x^3 - y^3 + 2xy$

$$f_x(x, y) = 3x^2 + 3y, f_y(x, y) = -3y^2 + 3x$$

To find critical points, we solve for $f_x = f_y = 0$.

$$3x^2 + 3y = -3y^2 + 3x = 0 \text{ only at } (0, 0) \text{ and } (1, -1)$$

$(0, 0)$ is a saddle point, because if we fix $y = 0$ and vary x , we can see that f increases as x increases and decreases as x decreases.

$$f_{xx} = 6x, f_{yy} = -6y, f_{xy} = 3. D(1, -1) = 36 - 9 > 0.$$

Hence, $(1, -1)$ is a local min.

(b) $g(x) = x^2 + 2xy + 2y^2$

$$\nabla g(x) = \begin{pmatrix} 2x + 2y \\ 2x + 4y \end{pmatrix}, \text{ which is non zero under the condition } g(x) = 5.$$

$$\nabla f(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\text{By Lagrang multiplier, } \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 2x + 2y \\ 2x + 4y \end{pmatrix} \implies 2\lambda(2x + 4y) = \lambda(2x + 2y) \implies x = -3y$$

$$\text{Sub } x = -3y \text{ to } x^2 + 2xy + 2y^2 = 5: 9y^2 + 2(-3y)y + 2y^2 = 5 \implies y^2 = 1$$

$$\therefore (x, y) = (-3, 1) \text{ or } (3, -1).$$

$$f(-3, 1) = -5, f(3, -1) = 5.$$

Min is -5 , max is 5 .

Q3

(a) Let $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x + y \\ y - 2x \end{pmatrix}$

Then A, B, C would have $u - v$ coordinates $(0, 0), (3, -6), (3, 3)$. Let the region be S .

$$\frac{\partial(u, v)}{\partial(x, y)} = \left| \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \right| = \left| \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \right| = 3$$

$$\begin{aligned} \iint_R \sqrt{x+y}(y-2x)^2 dx dy &= \iint_S uv^2 \frac{1}{3} dv du \\ &= \int_0^3 \int_{-2u}^u \frac{1}{3} uv^2 dv du \\ &= \int_0^3 \left[\frac{1}{9} uv^3 \right]_{-2u}^u du \\ &= \int_0^3 u^4 du \\ &= \left[\frac{1}{5} u^5 \right]_0^3 \\ &= \frac{3^5}{5} \end{aligned}$$

(b) $f(x, y, z) = xy \sin z$ is a potential function for F .

$$\int_C F \cdot dr = f\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) - f(0, 0, 0) = \frac{\pi^2}{4}$$

Q4

(a) $P = 7y - e^{\sin x}$, $Q = 9x - \cos(y^3 + 7y)$

$$\begin{aligned}\int P \, dx + Q \, dy &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA \\ &= \iint_D 9 - 7 \, dA \\ &= 2(\pi \times 2^2) \\ &= 8\pi\end{aligned}$$

(b) Cone: $\sqrt{3}z = \sqrt{x^2 + y^2}$

Sphere: $x^2 + y^2 + (z - 1)^2 = 1$

Let $r^2 = x^2 + y^2$, and $x = r \cos \theta$, $y = r \sin \theta$.

First, we find the intersection of the sphere and cone:

$$\begin{aligned}z &= \frac{\sqrt{x^2 + y^2}}{\sqrt{3}} = 1 + \sqrt{1 - (x^2 + y^2)} \\ \frac{r}{\sqrt{3}} &= 1 + \sqrt{1 - r^2} \\ 1 - r^2 &= \left(\frac{r}{\sqrt{3}} - 1 \right)^2 = \frac{r^2}{3} - \frac{2r}{\sqrt{3}} + 1 \\ \frac{4}{3}r^2 - \frac{2}{\sqrt{3}}r &= 0 \implies r = \frac{\sqrt{3}}{2}\end{aligned}$$

Hence, the volume is:

$$\begin{aligned}\int_0^{2\pi} \int_0^{\sqrt{3}/2} \int_{r/\sqrt{3}}^{1+\sqrt{1-r^2}} r \, dr \, d\theta &= \int_0^{2\pi} \int_0^{\sqrt{3}/2} \left(1 + \sqrt{1 - r^2} - \frac{r}{\sqrt{3}} \right) r \, dr \, d\theta \\ &= 2\pi \left[-\frac{1}{3}(1 - r^2)^{3/2} + \frac{1}{2}r^2 - \frac{1}{3\sqrt{3}}r^3 \right]_0^{\sqrt{3}/2} \\ &= \frac{13}{12}\pi\end{aligned}$$

Q5

(a) The original conditions given are:

$$\begin{aligned}-1 &\leq x \leq 1 \\ x^2 &\leq y \leq 1 \\ 0 &\leq z \leq 1 - y\end{aligned}$$

Hence,

$$\begin{aligned}0 &\leq z \leq 1 - y \leq 1 - x^2 \leq 1 \\ x^2 \leq y \leq 1 - z &\implies -\sqrt{1 - z} \leq x \leq \sqrt{1 - z} \\ x^2 &\leq y \leq 1 - z\end{aligned}$$

Hence,

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx = \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} f(x, y, z) \, dy \, dx \, dz$$

(b)

$$r(\theta) = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ \cos^2 \theta - \sin^2 \theta \end{pmatrix}, 0 \leq \theta \leq 2\pi$$

$$\begin{aligned} \int_C F \cdot dr &= \int_0^{2\pi} F(r(\theta)) \cdot r'(\theta) d\theta \\ &= \int_0^{2\pi} \begin{pmatrix} \sin^3 \theta \\ \cos \theta \\ \sin^3 \theta \cos^3 \theta \end{pmatrix} \cdot \begin{pmatrix} -\sin \theta \\ \cos \theta \\ \cos^2 \theta - \sin^2 \theta \end{pmatrix} d\theta \\ &= \int_0^{2\pi} -\sin^4 \theta + \cos^2 \theta + \sin^3 \theta \cos^5 \theta - \sin^5 \theta \cos^3 \theta d\theta \\ &= \int_0^{2\pi} -\sin^4 \theta + \cos^2 \theta d\theta \\ &= \frac{\pi}{4} \end{aligned}$$

Q6

(a)

$$\begin{aligned} \int_C f \nabla g \cdot dr &= \iint_{\Sigma} \nabla \times (f \nabla g) \cdot d\Sigma \\ &= \iint_{\Sigma} \nabla \times \begin{pmatrix} f g_x \\ f g_y \\ f g_z \end{pmatrix} \cdot d\Sigma \\ &= \iint_{\Sigma} \left[\left(\frac{\partial}{\partial y}(f g_z) - \frac{\partial}{\partial z}(f g_y) \right) i - \left(\frac{\partial}{\partial x}(f g_z) - \frac{\partial}{\partial z}(f g_x) \right) j + \left(\frac{\partial}{\partial x}(f g_y) - \frac{\partial}{\partial y}(f g_x) \right) k \right] \cdot d\Sigma \\ &= \iint_{\Sigma} [(f_y g_z - f_z g_y) i - (f_x g_z - f_z g_x) j + (f_x g_y - f_y g_x) k] \cdot d\Sigma \end{aligned}$$

Similarly,

$$\int_C g \nabla f \cdot dr = \iint_{\Sigma} [(g_y f_z - g_z f_y) i - (g_x f_z - g_z f_x) j + (g_x f_y - g_y f_x) k] \cdot d\Sigma$$

Hence,

$$\int_C f \nabla g \cdot dr = - \int_C g \nabla f \cdot dr = \int_{-C} g \nabla f \cdot dr$$

(b) Let

$$G(x, y, z) = \left\langle \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle = \frac{r}{|r|^3}$$
$$H(x, y, z) = \langle 0, 0, z^2 \rangle$$

Then, $F(x, y, z) = G(x, y, z) + H(x, y, z)$

Note that

$$\nabla \cdot G = 0 \text{ for } r \neq 0$$

$$\nabla \cdot H = 2z$$

Let E be the solid ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$.

$$\begin{aligned}
\int_S H \cdot dS &= \iiint_E (\nabla \cdot H) \, dV \\
&= \iiint_E 2z \, dV \\
&= 0
\end{aligned}$$

Due to symmetry

Let D be the solid ellipsoid minus the sphere centered at $(0, 0, 0)$ with radius $r_0 = 1/2$, and let C be the surface of that sphere.

$$\begin{aligned}
\int_S G \cdot dS - \int_C G \cdot dS &= \iiint_D (\nabla \cdot G) \, dV = 0 \\
\therefore \int_S G \cdot dS &= \int_C G \cdot dS \\
\int_C G \cdot dS &= \int_C (G \cdot n) \, dS = \int_C \left(\frac{r}{|r|^3} \cdot n \right) \, dS = \int_C \frac{1}{|r|^2} \, dS = 4\pi r_0^2 \frac{1}{r_0^2} = 4\pi
\end{aligned}$$

Hence,

$$\int_S F \cdot dS = \int_S H \cdot dS + \int_S G \cdot dS = 4\pi$$