# PYP Answer - MA3269 AY1718Sem1

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| | $(0, K_1]$ | $(K_1, K_2]$ | $(K_2, K_3]$ | $(K_3, \infty)$ |
|---|---|---|---|---|
| Long 1 $K_1$ call | 0 | $S_T - K_1$ | $S_T - K_1$ | $S_T - K_1$ |
| Short 2 $K_2$ call | 0 | 0 | $-2S_T + 2K_2$ | $-2S_T + 2K_2$ |
| Long 1 $K_3$ call | 0 | 0 | 0 | $S_T - K_3$ |
| Sum | 0 | $S_T - K_1$ | $-S_T + K_3$ | $S_T - K_3$ |

1. (a) i. Since all the sums are greater or equal to 0 yet the investment is initially costing 0, it is an arbitrage opportunity.

   ii. We have

$$\begin{aligned} C_i + Ke^{-rT} &= P_i + S_0, i = 1, 2, 3 \\ \Rightarrow C_2 - \frac{1}{2}(C_1 + C_3) &= P_2 - \frac{1}{2}(P_1 + P_3) \leq 0 \end{aligned}$$

   (b) i. We employ the two-period binomial model. Here, $q = 0.613636$ and let $a := e^{-r\delta t} = 0.970446$. $F_1^u = a(q \times 0 + (1 - q)(42 - 44 \times 0.92)) = 0.569916$ and $F_1^d = a(q(42 - 40.48) + (1 - q)(42 - 33.856)) = 3.95871$ and finally $F_0 = a(qF_1^u + (1 - q)F_1^d) = 1.82$.

   ii. Here $q$ and $a$ remains the same. $F_1^u = a(q \times 8.4 + (1 - q) \times 0.48) = 5.18218$ and $F_1^d = a(q \times 0.48) = 0.28584$ and $F_0 = 3.19$.

2. (a) By definition of certainty equivalent,

$$U(c) = 0.2U(0.8) + 0.6U(1) + 0.2U(1.25) \Rightarrow c = 1.00$$

   (b) Solving $U(1) > pU(0.8) + 3pU(1) + (1 - 4p)U(1.25)$ gives $p > 0.201$.

   (c) We calculate ARA of $U^2(x)$ to be $-\frac{-(x+1)^{-2}}{(x+1)^{-1}} = (1 + x)^{-1}$. By definition of ARA, we have $-\frac{R_1''}{R_1'} = (1 + x)^{-1}$. Therefore,

$$\begin{aligned} \ln(R')' &= -(x + 1)^{-1} \\ \ln(R') &= c_1 - \ln(x + 1) \\ R' &= A_1(x + 1)^{-1} \\ R(x) &= A_1 \ln(x + 1) + A_2 \text{ where } A_1 > 0, A_2 \in \mathbb{R} \end{aligned}$$

(d) $W = 1 - \frac{1}{V} = 1 - V^{-1}$. Differentiating once gives $W' = V^{-2}V'$ and twice gives $W'' = -2V^{-3}V' + V^{-2}V'' < 0$ since $V > 0, V' > 0$ and $V'' < 0$. Therefore, investor C is risk averse.

(e) $W_{\text{ARA}} = -\frac{W''}{W'} = 2V^{-1} - \frac{V''}{V'}$. Since $V_{\text{ARA}} = -\frac{V''}{V'}$, we have $W_{\text{ARA}} = V_{\text{ARA}} + 2V^{-1} > V_{\text{ARA}}$ for all $x > 0$. Therefore, C is globally more risk averse than B.

(f) We have $Z = W \circ U^{-1}$. Therefore, $Z' = W'(U^{-1})\frac{1}{U'(U^{-1})}$ and $Z'' = W''(U^{-1})\frac{1}{U'(U^{-1})} - W'(U^{-1})(U'(U^{-1}))^{-2}U''(U^{-1})\frac{1}{U'(U^{-1})} < 0$.

3. (a) ($\Rightarrow$) By two-fund theorem, all frontier portfolios are spanned by $\frac{\mathbf{C}^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{1}}$ and $\frac{\mathbf{C}^{-1}\boldsymbol{\mu}}{\mathbf{1}^{\mathrm{T}}\mathbf{C}^{-1}\boldsymbol{\mu}}$. Since $\mathbf{u}$ is uncorrelated with all frontier portfolio, we have

$$\mathbf{w}_u^{\mathrm{T}}\mathbf{C}\mathbf{C}^{-1}\mathbf{1} = 0 \text{ and}$$
$$\mathbf{w}_u^{\mathrm{T}}\mathbf{C}\mathbf{C}^{-1}\boldsymbol{\mu} = 0$$

The first equation shows $\mathbf{u}$ is hedge and second equation show $\mathbf{u}$ is zero-mean.
($\Leftarrow$) From the condition zero-mean and hedge we can arrive at the above pair of equations. Then any portfolio $x$'s correlation with this portfolio is

$$\mathbf{w}_u^{\mathrm{T}}\mathbf{C}\mathbf{w}_x = c_1\mathbf{w}_u^{\mathrm{T}}\mathbf{C}\mathbf{C}^{-1}\mathbf{1} + c_2\mathbf{w}_u^{\mathrm{T}}\mathbf{C}\mathbf{C}^{-1}\boldsymbol{\mu} = 0$$

by two fund theorem.

(b) We want to $\min_{\mathbf{w}} \mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} - \frac{\gamma}{2}(\mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w})$ subject to the constraints $\mathbf{w}^{\mathrm{T}}\mathbf{w}_0 = 0$.
Employ the Lagrange multiplier, we have

$$L = \mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} - \frac{\gamma}{2}(\mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w}) - \lambda(\mathbf{w}^{\mathrm{T}}\mathbf{w}_0)$$

and

$$\frac{\mathrm{d}L}{\mathrm{d}\mathbf{w}} = \boldsymbol{\mu} - \gamma\mathbf{C}\mathbf{w} - \lambda\mathbf{w}_0 = 0$$

gives

$$\mathbf{w} = \frac{1}{\gamma}\mathbf{C}^{-1}(\boldsymbol{\mu} - \lambda\mathbf{w}_0)$$

and substituting it into the constraints, we have

$$\frac{1}{\gamma}\boldsymbol{\mu}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{w}_0 - \frac{\lambda}{\gamma}\mathbf{w}_0\mathbf{C}^{-1}\mathbf{w}_0 = 0$$

so we have $s - \lambda p = 0 \Rightarrow \lambda = \frac{s}{p}$.
Therefore, $\mathbf{w} = \frac{1}{\gamma}\mathbf{C}^{-1}(\boldsymbol{\mu} - \frac{s}{p}\mathbf{w}_0)$.

(c) By definition of beta, $\beta_m = \frac{\mu_m - r_f}{\mu_m - r_f} = 1$.
Since $\sigma_m^2 = \mathbf{w}_m^{\mathrm{T}}\mathbf{C}\mathbf{w}$, equivalently we want to show $\mathbf{w}_m^{\mathrm{T}}\mathbf{C}\mathbf{w} = \frac{1}{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{C}^{-1}\boldsymbol{\beta}}$. Since $\beta_m = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{w}_m$, we evaluate $\mathbf{w}_m^{\mathrm{T}}\mathbf{C}\mathbf{w}\boldsymbol{\beta}^{\mathrm{T}}\mathbf{C}^{-1}\boldsymbol{\beta} = 1$, which proves the claim.

4. (a) From the table, the weight vector is

$$\mathbf{w}_m = \frac{1}{150 \times 2 + 100 \times 2 + 80 \times 2.5 + 100 \times 3} \begin{pmatrix} 150 \times 2 \\ 100 \times 2 \\ 80 \times 2.5 \\ 100 \times 3 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.2 \\ 0.2 \\ 0.3 \end{pmatrix}$$

Portfolio mean is $\mu_m = \boldsymbol{\mu}^{\mathrm{T}} \mathbf{w}_m = 0.103$.

(b) From the asymptote, the minimum-variance frontier has the format of

$$9\sigma^2 = \frac{47}{417}(100\mu - \frac{359}{47})^2 + c$$

where $c$ is some constant. It should also satisfy the market portfolio, so

$$9 \times 0.11 = \frac{47}{417}(100 \times 0.103 - \frac{359}{47})^2 + c$$

so $c = \frac{9}{47}$. Then the required frontier is

$$\sigma^2 = \frac{470000}{3753}(\mu - \frac{359}{4700})^2 + \frac{1}{47}$$

where $x = \frac{470000}{3753}$, $y = \frac{359}{4700}$ and $z = \frac{1}{47}$.

(c) GMVP occurs when $\mu_g = \frac{359}{4700}$ and $\sigma_g^2 = \frac{1}{47}$.

(d) Implicit differentiation on the frontier gives

$$18\sigma \mathrm{d}\sigma = \frac{9400}{417}(100\mu - \frac{359}{47})\mathrm{d}\mu$$

Therefore, at market portfolio, we have

$$\frac{\mathrm{d}\mu}{\mathrm{d}\sigma} = \frac{18 \times \sqrt{0.11}}{\frac{9400}{417}(100 \times 0.103 - \frac{359}{47})} = \frac{3}{10}\sqrt{0.11}$$

Therefore, the CML admits the following equation

$$\mu - 0.103 = \frac{3}{10}\sqrt{0.11}(\sigma - \sqrt{0.11})$$

(e) $r_f = 0.103 + \frac{3}{10}\sqrt{0.11}(0 - \sqrt{0.11}) = 0.07$.

(f) $\beta_3 = \frac{\mu_3 - r_f}{\mu_m - r_f} = \frac{10}{11}$.

(g) Since $\sigma_g^2 = \frac{1}{a}$, we have $a = \frac{1}{\frac{1}{47}} = 47$.
Then $b = a\mu_g = \frac{359}{100}$,
From frontier, we have $\frac{a}{ac - b^2} = \frac{470000}{3753}$, so $c = \frac{1411}{5000}$.