NATIONAL UNIVERSITY OF SINGAPORE

MA1104 MULTIVARIABLE CALCULUS

(SEMESTER 2: AY 2016-2017)

Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/student number only. Do not write your name.
2. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages.
3. Answer **ALL** questions. This examination carries a total of **100** marks.
4. Please write your answers on the answer book provided.
5. This is a closed book examination. Candidates are allowed to bring ONE (1) A4-sized double-sided help sheet.
6. Candidates may use non-programmable calculators. However, they should lay out systematically the various steps in the calculations.

Question 1. [15 marks]

Consider the surface S given by the equation $xz + 2x^2y + y^2z^3 = 11$.

(i) Find an equation of the tangent plane (in the form $ax + by + cz = d$) to the surface S at the point $(2, 1, 1)$.

(ii) Let C be the intersection curve of the surface S and the plane $x + y + z = 4$. Find *all unit* vectors which are parallel to the tangent line to the curve C at $(2, 1, 1)$.

(iii) Find the coordinates of the point of intersection of the tangent line found in (ii) with the plane $z = 0$.

Question 2. [15 marks]

(a) Suppose that $z = x^2 + xy^3$, where $x = uv^2 + w^3$, $y = u + ve^w$. Find

$$\frac{\partial z}{\partial u}, \quad \frac{\partial z}{\partial v}, \quad \text{and} \quad \frac{\partial z}{\partial w}$$

when $u = 2$, $v = 1$, and $w = 0$.

(b) Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ where $z \geq 1$.

(i) Compute the area of S.

(ii) Suppose that $f(x, y, z) = g(\sqrt{x^2 + y^2 + z^2})$, where g is a function of one variable such that $g(2) = -3$. Evaluate

$$\iint_S f(x, y, z)\, dS.$$

Question 3. [15 marks]

Let $f(x, y, z) = 3xy + 6z$ and $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + 2z^2 \leq 6\}$.

(i) Find the maximum and minimum values of the function f subject to the constraint $(x, y, z) \in D$ and list all the points (x, y, z) where the maximum and minimum values are attained respectively. Justify your answer.

(ii) Find all the points $(x, y, z) \in D$ where $\|\nabla f\|$ achieves its maximum value.

Question 4. [15 marks]

(a) Evaluate the following iterated integral (give the exact answer in simplest form):

$$\int_0^1 \int_{x^2}^1 2x \sin(y^2)\, dy\, dx.$$

(b) By using a change of variables or otherwise, find

$$\iint_D e^{2y-x}\, dA,$$

where D is the region on the xy-plane bounded by the lines $x + y = 4$, $x + y = 7$, $y - 2x = 1$ and $y - 2x = -3$ (give your answer in exact form).

(c) Rewrite the following iterated integral in the order $dz\, dx\, dy$:

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_{3-x}^3 f(x, y, z)\, dy\, dz\, dx.$$

Question 5. [20 marks]

(a) Let D be the region in the plane which lies outside the circle $r = 1$ and inside the cardioid $r = 1 + \cos\theta$, where (r, θ) are polar coordinates. Find the area of D (give your answer in exact form).

(b) Consider the vector field $\mathbf{F}(x, y, z) = \langle 5y + \sin x,\ z^2 + \cos y,\ x^3 \rangle$.

(i) Find $\operatorname{curl}(\mathbf{F})$.

(ii) *Use Stokes' Theorem* to evaluate the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the curve $\mathbf{r}(t) = \langle \sin t,\ \cos t,\ 2 \sin t \cos t \rangle$, $0 \le t \le 2\pi$.

Question 6. [20 marks]

(a) Consider the vector field $\mathbf{G}(x, y) = \langle y + ye^x + \sin y,\ xy + e^x + x\cos y\rangle, \quad (x, y) \in \mathbb{R}^2$.

(i) Determine if $\mathbf{G}$ is conservative. Justify your answer.

(ii) Evaluate the line integral

$$\int_C \mathbf{G} \cdot d\mathbf{r},$$

where C is the curve with parametrization $x = t$, $y = \cos t$, $0 \le t \le \pi/2$.

(b) Let $\mathbf{F}(x, y, z) = \langle x^2 + y^2 - 1,\ x^2 + y^2 - 1,\ -2z(x + y - 1)\rangle$ be a vector field defined in $\mathbb{R}^3$. Suppose that E is the solid bounded by the following surfaces: $z = -(x^2 + y^2)$, $x^2 + y^2 = 1$ and $z = 7$. Let S be the boundary surface of E with the positive (outward-pointing) orientation $\mathbf{n}$.

(i) Compute the flux $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$, where S_1 is the part of the plane $z = 7$ which belongs to S.

(ii) Compute $\iiint_E \operatorname{div} \mathbf{F}\ dV$.

(iii) Compute the flux $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$, where S_2 is the part of the surface $z = -(x^2 + y^2)$ which belongs to S.

END OF PAPER