# MA1102R AY1718 Sem 1 Answers

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1. (i) 
$$f(x) = (x^3 + 4x^2 + 11x + 14)e^{-x}$$

$$f'(x) = -(x^3 + 4x^2 + 11x + 14)e^{-x} + (3x^2 + 8x + 11)e^{-x} > 0$$

$$\iff (3x^2 + 8x + 11) - (x^3 + 4x^2 + 11x + 14) > 0$$

$$\iff -x^3 - x^2 - 3x - 3 > 0$$

$$\iff (-x - 1)(x^2 + 3) > 0$$

$$\iff -1 > x$$

f is increasing on  $(-\infty, -1)$  and decreasing on  $(-1, \infty)$ 

(ii) f(-1) = 6e is a local maximum. There is no local minimum.

(iii)

$$f'(x) = (-x^3 - x^2 - 3x - 3)e^{-x}$$

$$f''(x) = (-3x^2 - 2x - 3)e^{-x} - (-x^3 - x^2 - 3x - 3)e^{-x}$$

$$= (x^3 - 2x^2 + x)e^{-x} > 0$$

$$\iff x^3 - 2x^2 + x > 0$$

$$\iff x(x - 1)^2 > 0$$

$$\iff x > 0$$

f is concave up on  $(0,\infty)$  and concave down on  $(\infty,0)$ 

- (iv) f(0) = 14(0, 14)
- 2. (a) For any  $\epsilon > 0$ , choose  $\delta = \min(\epsilon, 1)$ Then for all x such that  $0 < |x - 2| < \delta$

$$\left| \frac{x}{x^2 + 2} - \frac{1}{3} \right| = \left| \frac{3x - (x^2 + 2)}{3(x^2 + 2)} \right| = \left| \frac{(x - 1)(x - 2)}{3(x^2 + 2)} \right|$$

$$< |x + 1| \left| \frac{1}{3(x^2 + 2)} \right| \epsilon$$

$$< 2 \times \frac{1}{3 \times 2} \times \epsilon$$

$$< \epsilon$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^3}{n^2 (n^2 + i^2)} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\left(\frac{i}{n}\right)^3}{1 + \left(\frac{i}{n}\right)^2}$$

$$= \int_0^1 \frac{x^3}{1 + x^2} dx$$

$$= \int_0^1 x - \frac{x}{1 + x^2} dx$$

$$= \left[\frac{1}{2}x - \frac{1}{2}\ln(x^2 + 1)\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{2}\ln 2$$

(c)

$$\lim_{x \to 0^+} \left( \frac{e^2 - 1}{x} \right)^{1/x} = \lim_{x \to 0^+} \exp\left( \frac{1}{x} \ln\left(\frac{e^x - 1}{x}\right) \right)$$

$$= \lim_{x \to 0^+} \exp\left( \left(\frac{x}{e^x - 1}\right) \left(\frac{e^x}{x} - \frac{e^x - 1}{x^2}\right) \right) \quad \text{By L'Hôpital's rule}$$

$$= \lim_{x \to 0^+} \exp\left( \frac{xe^x - e^x + 1}{x(e^x - 1)} \right)$$

$$= \lim_{x \to 0^+} \exp\left( \frac{xe^x}{xe^x + e^x - 1} \right)$$

$$= \lim_{x \to 0^+} \exp\left( \frac{e^x}{e^x + \frac{e^x - 1}{x}} \right)$$

$$= \exp\left( \frac{1}{2} \right)$$

$$= \sqrt{e}$$

#### 3. Let the angle of the sector be $\theta$

$$2r + r\theta = 50 \implies \theta = \frac{50}{r} - 2$$

Area 
$$= \frac{1}{2}r^2\theta$$
$$= \frac{1}{2}r^2\left(\frac{50}{r} - 2\right)$$
$$= 25r - r^2$$
$$= r(25 - r)$$
$$\leq \left(\frac{25}{2}\right)^2$$

By AMGM inequality, with equality at r=12.5  $\,$ 

r = 12.5 m

$$\ln y = (\sec x) \ln(\tan x) + (\tan x) \ln(\sec x)$$

 $\frac{1}{y}\frac{dy}{dx} = \sec^2 x \sin x \ln(\tan x) + \csc x \sec^2 x + \sec^2 x \ln(\sec x) + \sin^2 x \sec^2 x$ 

If 
$$x = \frac{\pi}{4}$$
, then  $y = 1^{\sqrt{2}} \sqrt{2}^1 = \sqrt{2}$ 

Sub  $x = \frac{\pi}{4}$  and  $y = \sqrt{2}$  to the equation

$$\frac{1}{\sqrt{2}}\frac{dy}{dx} = 0 + 2\sqrt{2} + 2\ln\sqrt{2} + 1$$

$$\frac{dy}{dx} = 4 + 2\sqrt{2}\ln\sqrt{2} + \sqrt{2}$$

(b) For  $x \neq 0$ :

$$F'(x) = \frac{d}{dx} \int_0^{x^2} f(t) dt$$
$$= 2xf(x^2)$$
$$= \frac{2\sin(x^2)}{x}$$

For x = 0:

$$F'(x) = 0$$

$$\therefore F'(x) = 0 \text{ for } x = \sqrt{k\pi}, k \in \mathbb{Z}$$

To check if it is a local max or min, we check the concavity For  $x \neq 0$ :

$$F''(x) = 4\cos(x^2) - \frac{2\sin(x^2)}{x^2}$$

For x = 0:

$$F''(x) = 2$$

$$F''(\sqrt{k\pi} = 4(-1)^k \text{ for } k \neq 0 \text{ and } F''(0) = 2$$

f attains local min at  $x = \sqrt{k\pi}$  for even k and local max at  $x = \sqrt{k\pi}$  for odd k.

(c)

$$f''(x) < 0 \implies f'(x)$$
 is decreasing  $\implies f'(x) < 0 \implies f(x)$  is decreasing

Either  $\lim_{x\to\infty} f(x) = -\infty$  or  $\lim_{x\to\infty} f(x) = k$  for some constant k

If  $\lim_{x\to\infty} f(x) = k$ , then  $\lim_{x\to\infty} f'(x) = 0$ ,  $\Longrightarrow \leftarrow$ 

$$\therefore \lim_{x \to \infty} f(x) = -\infty$$

f(x) is decreasing and  $\lim_{x\to\infty} f(x) = -\infty \implies$  exactly 1 root

5.

$$y^2 = 2x = 8 - x^2$$

$$\therefore x = 2, y = \pm 2$$

The curves intersects at (2,2) and (2,-2)

$$x^{2} + y^{2} = 8 \implies x = \sqrt{8 - y^{2}}$$
$$y^{2} = 3x \implies x = \frac{1}{2}y^{2}$$

Area = 
$$\int_{-2}^{2} \sqrt{8 - y^2} - frac12y^2 dy$$
  
=  $\int_{-2}^{2} \sqrt{8 - y^2} dy - \left[ -\frac{1}{6}y^3 \right]_{-2}^{2}$  sub  $y = \sqrt{8} \sin \theta$   
=  $\int_{-\pi/4}^{\pi/4} 8 \cos^2 \theta d\theta - \frac{8}{3}$   
=  $4 \int_{-\pi/4}^{\pi/4} \cos(2\theta) + 1 d\theta - \frac{8}{3}$   
=  $2[\sin(2\theta) + 2\theta]_{-\pi/4}^{\pi/4} - \frac{8}{3}$   
=  $\frac{4}{3} + 2\pi$ 

(ii)

$$x^{2} + y^{2} = 8 \implies y = \sqrt{8 - x^{2}}$$

$$y^{2} = 3x \implies y = \sqrt{2x}$$

$$Volume = 2 \left[ \int_{0}^{2} \sqrt{2x} (2\pi x) \ dx + \int_{2}^{\sqrt{8}} \sqrt{8 - x^{2}} (2\pi x) \ dx \right]$$

$$= 4\pi \left[ \left[ \frac{2}{5} \sqrt{2} x^{5/2} \right]_{0}^{2} + \left[ -\frac{1}{3} (8 - x^{2})^{3/2} \right]_{2}^{\sqrt{8}} \right]$$

$$= 4\pi \left[ \frac{16}{5} + \frac{8}{3} \right]$$

$$= \frac{352}{15} \pi$$

6. (i)

$$\int \frac{x \ln x}{(1+x^2)^2} dx = \left(-\frac{1}{2}\right) \frac{\ln x}{1+x^2} - \int \left(-\frac{1}{2}\right) \frac{1}{x(1+x^2)} dx \quad \text{sub } x = \tan \theta$$

$$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \int \frac{1}{\tan \theta \sec^2 \theta} \sec^2 d\theta$$

$$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \int \frac{\cos \theta}{\sin \theta} d\theta$$

$$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \ln(\sin \theta) + C$$

$$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \ln\left(\frac{x}{\sqrt{1+x^2}}\right) + C$$

$$\begin{split} \lim_{x \to \infty} \left( \frac{1}{2} \ln \frac{x}{\sqrt{1 + x^2}} - \frac{\ln x}{2(1 + x^2)} \right) &= \frac{1}{2} \ln(1) \\ &= 0 \\ \lim_{x \to 0^+} \left( \frac{1}{2} \ln \frac{x}{\sqrt{1 + x^2}} - \frac{\ln x}{2(1 + x^2)} \right) &= \lim_{x \to 0^+} \left( \frac{1}{2} \ln x - \frac{\ln x}{2(1 + x^2)} - \frac{1}{2} \ln \sqrt{1 + x^2} \right) \\ &= \lim_{x \to 0^+} \left( (\ln x) \left( \frac{1}{2} - \frac{1}{2(1 + x^2)} \right) \right) \\ &= \frac{1}{2} \lim_{x \to 0^+} \left( \frac{x^2 \ln x}{1 + x^2} \right) \\ &= \frac{1}{2} \lim_{x \to 0^+} (x^2 \ln x) \\ &= \frac{1}{2} \lim_{x \to 0^+} \frac{1/x}{-2x^{-3}} \qquad \text{By L'Hôpital's rule} \\ &= 0 \end{split}$$

$$\therefore \int_0^\infty \frac{x \ln x}{(1+x^2)^2} \ dx = 0$$

#### 7. (a)

$$y = \frac{1}{x} + \frac{1}{z} \implies z = 1 \text{ at } x = 1$$

$$\frac{dy}{dx} = -\frac{1}{x^2} - \frac{1}{z^2} \frac{dz}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{x^2} - \frac{1}{z^2} \frac{dz}{dx} = \left(\frac{1}{x} + \frac{1}{z}\right)^2 - \frac{1}{x} \left(\frac{1}{x} + \frac{1}{z}\right) - \frac{1}{x^2}$$

$$-\frac{1}{z^2} \frac{dy}{dx} = \frac{1}{z^2} + \frac{1}{xz}$$

$$\frac{dy}{dx} + \frac{z}{x} + 1 = 0$$

Let  $w = \frac{z}{x}$ . Then w = 1 at x = 1.

$$wx = z \implies x\frac{dw}{dx} + w = \frac{dz}{dx}$$
$$\frac{dz}{dx} = x\frac{dw}{dx} + w = -1 - w$$
$$\int \frac{1}{-1 - 2x} = \int \frac{1}{x}$$
$$-\frac{1}{2}\ln|1 + 2w| = \ln(x) + C$$

Substitute 
$$x = 1, w = 1$$

$$-\frac{1}{2}\ln 3 = C$$

Therefore,

$$-\frac{1}{2}\ln|1 + 2w| = \ln\frac{x}{\sqrt{3}}$$

$$\frac{1}{\sqrt{1 + 2w}} = \frac{x}{\sqrt{3}}$$

$$w = \frac{1}{2}\left(\frac{3}{x^2} - 1\right)$$

$$z = wx = \frac{3}{2x} - \frac{x}{2} = \frac{3 - x^2}{2x}$$

$$y = \frac{1}{x} + \frac{2x}{3 - x^2}$$

(b)

$$\int \frac{4h - h^2}{\sqrt{h}} dh = \int -1 dh$$
$$\frac{8}{3}h^{3/2} - \frac{2}{5}h^{5/2} + C = -t$$

Substitute t = 0, h = 4

$$\frac{64}{3} - \frac{64}{5} + C = 0 \implies C = -\frac{128}{15}$$

Therefore,

$$\frac{8}{3}h^{3/2} - \frac{2}{5}h^{5/2} - \frac{128}{15} = -t$$

Substitute h = 0

$$t = \frac{128}{15}$$

128/15 minutes

8. We first want to show that  $\forall x \leq 0.5, |f(x)| \leq Mx$ .

Suppose  $\exists a \in (0,1)$  such that |f(a)| > Mx, then by mean value theorem,  $\exists b \in (0,a)$  such that |f'(b)| = |(f(a) - 0)/(a - 0)| > M, a contradiction.

Therefore,  $\forall x \leq 0.5, |f(x)| \leq Mx$ .

Similarly, we can also show that  $\forall x \geq 0.5, |f(x)| \leq M(1-x)$ .

Suppose  $\exists a \in (0,1)$  such that |f(a)| > M(1-x), then by mean value theorem,  $\exists b \in (a,1)$  such that |f'(b)| = |(f(a)-0)/(a-1)| > M, a contradiction.

Hence,

$$\int_0^1 |f(x)| dx = \int_0^{0.5} |f(x)| dx + \int_{0.5}^1 |f(x)| dx$$

$$< \int_0^{0.5} Mx dx + \int_{0.5}^1 M(1 - x) dx$$

$$= \frac{1}{4}M$$