PYP Answer - MA3269 AY1617Sem2

Ma Hongqiang

December 10, 2017

1. (a) $r_f = 0$ since CML has μ intercept of 0.

   (b) We have in general, $\sigma^2 = A(\mu - \mu_g)^2 + \sigma_g^2$, where A is some constant to be determined. By GMVP, we have

   $$\sigma^2 = A(\mu - \frac{1}{6})^2 + \frac{1}{12}$$

   And by portfolio x, we have

   $$\frac{1}{4} = A(0 - \frac{1}{6})^2 + \frac{1}{12}$$

   Solving, $A = 6$. Therefore, $\sigma^2 = A(\mu - \frac{1}{6})^2 + \frac{1}{12}$.
   Rearranging,

   $$\mu = \frac{1}{6} \pm \sqrt{\frac{1}{6}\sigma^2 - \frac{1}{72}}$$

   (c) From (ii), we can easily write down $\sigma = \sqrt{6}(\mu - \frac{1}{6})$.

   (d) From (ii), we can solve $\mu_y = \frac{1}{3}$.

   (e) From (ii), we have the following equations

   $$\begin{cases} \frac{a}{ac-b^2} = 6 \\ \frac{b}{a} = \frac{1}{6} \\ \frac{1}{a} = \frac{1}{12} \end{cases}$$

   Solving, we have

   $$\begin{cases} a = 12 \\ b = 2 \\ c = \frac{1}{2} \end{cases}$$

   Then, $\mu_m = \frac{c - r_f b}{b - r_f a} = \frac{c}{b} = \frac{1}{4}$ and $\sigma_m^2 = \frac{c}{b^2} = \frac{1}{8}$.

   (f) Using the definition of beta, $\beta_p = \frac{\mu_p - r_f}{\mu_m - r_f}$, we have

   $$\beta_g = \frac{2}{3} \quad \beta_m = 1 \quad \beta_x = 0 \quad \beta_y = \frac{4}{3}$$

Therefore, the required portfolio beta is

$$\beta = \frac{1}{4}(\beta_g + \beta_m + \beta_x + \beta_y) = \frac{1}{4}(\frac{2}{3} + 1 + 0 + \frac{4}{3}) = \frac{9}{8}$$

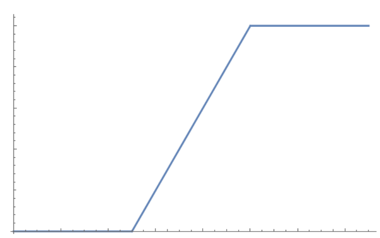
(g)

$$\begin{aligned} \text{corr}(r_y, r_m) &= \frac{\sigma_{ym}}{\sigma_y \sigma_m} \quad \text{by definition of correlation} \\ &= \frac{\frac{\mu_y}{\mu_m}\sigma_m^2}{\sigma_y \sigma_m} \quad \text{by CAPM} \\ &= \frac{2\sqrt{2}}{3} \end{aligned}$$

2. (a) i. The payoff table is The graph has turning point $(K_1, 0)$, $(K_2, K_2 - K_1)$ and

	$0 \le S_T < K_1$	$K_1 \le S_T < K_2$	$S_T \ge K_2$
Payoff	0	$S_T - K_1$	$K_2 - K_1$

is piecewise linear.



ii. One possible portfolio would be 1 long K_1-put, 1 short K_2-put and $e^{-rT}(K_2 - K_1)$ risk-free asset.

(b) i. $n_1 = 2 \;\; n_2 = 3 \;\; n_3 = 3 \;\; K = 108$

ii. It is a piecewise linear function with turning points (80,60), (90,30), (100,30), (108,54) and (120,78).

(c) We have $S_0 = 10, S_1^u = 11, S_1^d = 9.5$. Therefore, $u = 1.1, d = 0.95$ and $F_1^u = 3, F_1^d = 0.225$.
Also, we identify that $r = 0.06$ and $\delta t = \frac{1}{12}$.
By single-period binomial model, we have

$$q = \frac{e^{r\delta t} - d}{u - d} = 0.366750139$$

and hence

$$F_0 = e^{-r\delta t}(qF_1^u + (1-q)F_1^d) = 1.237$$

3. (a) i.

$$
\begin{aligned}
U(C(x)) &= \mathrm{E}[U(w_0 + X)] \\
\ln C(x) &= \frac{1}{2}\ln(1+ax) + \frac{1}{2}\ln(1-bx) \\
\ln(C(x)^2) &= \ln[(1+ax)(1-bx)] \\
C(x) &= [(1+ax)(1-bx)]^{\frac{1}{2}}
\end{aligned}
$$

The last square root opertion retains the positive root since $\min(w_0 + X) = \min(1 - bx) \geq 0$.

ii. Differentiate $C(x)$, we yield

$$
\frac{\mathrm{d}C(x)}{\mathrm{d}x} = \frac{1}{2}\frac{1}{C(x)}(-2abx - b + a)
$$

Obviously the first and second fraction are both positive. For the third term, note that, when $a \leq b$,

$$
\begin{aligned}
0 < bx < 1 \\
0 < 2abx < 2a \\
-2abx - b + a < 0 - b + a + 1 \leq -b + b = 0
\end{aligned}
$$

Therefore, the last term is negative so the derivative is negative, and hence $C(x)$ is a strictly decreasing function of x.

iii. The trader will avoid the lottery when $C(x) < w_0 = 1$. Therefore we need

$$
0 < (1+ax)(1-bx) < 1 \Rightarrow \frac{a-b}{ab} < x < \frac{1}{b}.
$$

iv. Let the derivative to equual 0. We need $2abx + b - a = 0$. Therefore, $x = \frac{a-b}{2ab} := \xi$.
Also, we check easily check taht for all $x > \xi$, the derivative is negative and for all $x < \xi$, derivative is positive. Therefore, $x = \xi$ is indeed the maximum. Then, since $\xi < \frac{a-b}{ab}$, the trader will indeed play the lottery.

(b) i. For this subquestion, x is a variable and is dropped for simplicity. Since $U = U_1 U_2^{-1}$,

$$
\frac{\mathrm{d}U}{\mathrm{d}x} = (U_1' \circ U_2^{-1})(U_2^{-1})'
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}^2U}{\mathrm{d}x^2} &= -(U_1'' \circ U_2^{-1})[(U_2^{-1})']^2 + (U_1' \circ U_2^{-1})(U_2^{-1})'' \\
&= \frac{-U_1''(t)U_1'(t)}{(U_2'(t))^2 U_1'(t)} + \frac{-U_2''(t)U_1'(t)}{(U_2'(t))^2 U_2'(t)} \\
&= RHS
\end{aligned}
$$

ii. Since $R_2(t) < R_1(t)$ for all $t > 0$, we indeed have a concave U. Next, since U_2 is positive and increasing U_2^{-1} is also positive and increasing and its composite with an positive and increasing U_1 is also positive and increasing.
Therefore, U is indeed a concave untility function.

(a) We first calculate $\sigma_y^2 = \mathbf{w}_y^{\mathrm{T}}\mathbf{C}\mathbf{w}_y$.

$$\begin{aligned}\sigma_y^2 &= (\alpha\frac{\mathbf{C}^{-1}\mu}{b} + (1-\alpha)\frac{\mathbf{C}^{-1}\mathbf{1}}{a})^{\mathrm{T}}\mathbf{C}(\alpha\frac{\mathbf{C}^{-1}\mu}{b} + (1-\alpha)\frac{\mathbf{C}^{-1}\mathbf{1}}{a})\\ &= \frac{\alpha^2}{b^2}c + \frac{1-\alpha^2}{a}\end{aligned}$$

Similarly, $\sigma_x^2 = \frac{\alpha}{b}\mu_y + \frac{1-\alpha}{a}$ (#).
Next, $\mu_y = \mu^{\mathrm{T}}\mathbf{w}_y = \alpha\frac{c}{b} + (1-\alpha)\frac{b}{a}$. Subsitute into (#) yields the result.

(b) Note that $\rho_{px} = \frac{\sigma_{px}}{\sigma_p\sigma_x}$. Therefore, we need to show

$$\frac{\sigma_{px}}{\sigma_p} = \frac{\gamma\sigma_y^2 + (1-\gamma)\sigma_g^2}{\sqrt{\gamma^2\sigma_y^2 + (1-\gamma^2)\sigma_g^2}}$$

We first calculate σ_p^2.

$$\sigma_p^2 = (\gamma\mathbf{w}_y + (1-\gamma)\mathbf{w}_g)^{\mathrm{T}}\mathbf{C}(\gamma\mathbf{w}_y + (1-\gamma)\mathbf{w}_g)$$
$$= \gamma^2\sigma_y^2 + (1-\gamma^2)\sigma_g^2$$

So we have verified the denominator. Next we show the nominator matches:

$$\begin{aligned}\sigma_{px} &= \mathbf{w}_p\mathbf{C}\mathbf{w}_x\\ &= \gamma\sigma_y^2 + (1-\gamma)\sigma_g^2\end{aligned}$$

We want to maximise ρ_{px}. By computing the derivative, we have

$$\frac{\mathrm{d}\rho_{px}}{\mathrm{d}\gamma} = \frac{(b(1-\gamma) + a\gamma)(2a\gamma - 2b\gamma)}{2c(a\gamma^2 + b(1-\gamma^2))^{\frac{3}{2}}} + \frac{a-b}{c\sqrt{a\gamma^2 + b(1-\gamma^2)}}$$

where $a = \sigma_y^2$, $b = \sigma_g^2$ and $c = \sigma_x$. Letting the derivative to equal 0, we have $\sigma = 1$. Therefore, the highest correlation is attained when $r_p = r_y$, i.e., by portfolio y.

(c) Indeed, we have $w_a = w_x - w_y$ and $w_b = w_y - w_g$.

(d) By one fund theorem, we have $\mathbf{w}_z = \alpha\mathbf{w}_m$. Therefore, $\sigma_{xz} = \alpha\sigma_{xm}$.
Since $\rho_{xz} = \frac{\sigma_{xz}}{\sigma_x\sigma_z}$. We will show the claim if we show that $\sigma_{xz} = \sigma_z^2$.
Note that $\sigma_z^2 = \alpha^2\sigma_m^2$. So we need to show $\sigma_{xm} = \alpha\sigma_m^2$.
This is indeed the case. Since $\beta_x = \beta_z = \alpha\beta_m = \alpha$. So we have our result.