MA1102R AY1718 Sem 2 Answers

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1.   i

$$\ln \frac{e^x + 1}{e^x - 1} \text{ exists}$$

$$\Longleftrightarrow \frac{e^x + 1}{e^x - 1} > 0$$

$$\Longleftrightarrow e^x - 1 > 0$$

$$\Longleftrightarrow x > 0$$

$$\sin^{-1} \frac{1}{\ln \frac{e^x+1}{e^x-1}} \text{ exists}$$

$$\Longleftrightarrow -\frac{\pi}{2} < \frac{1}{\ln \frac{e^x+1}{e^x-1}} < \frac{\pi}{2}$$

$$\Longleftrightarrow 0 < \frac{1}{\ln \frac{e^x+1}{e^x-1}} < \frac{\pi}{2}, \text{ since } \ln \frac{e^x + 1}{e^x - 1} > 0$$

$$\Longleftrightarrow \ln \frac{e^x + 1}{e^x - 1} > \frac{2}{\pi}$$

$$\Longleftrightarrow \frac{e^x + 1}{e^x - 1} > e^{2/\pi}$$

$$\Longleftrightarrow e^x < \frac{e^{2/\pi} + 1}{e^{2/\pi} - 1}$$

$$\Longleftrightarrow x < \ln \frac{e^{2/\pi} + 1}{e^{2/\pi} - 1}$$

$\therefore 0 < x < \ln \frac{e^{2/\pi}+1}{e^{2/\pi}-1}$

ii $f(x)$ is a one-to-one function on its maximal domain.

$$f^{-1}(\sin^{-1} \frac{1}{\ln \frac{e^x+1}{e^x-1}}) = x$$

Substitute $x = \ln(\frac{2}{y-1} + 1)$

$$f^{-1}(\sin^{-1} \frac{1}{\ln \frac{e^x+1}{e^x-1}}) = f^{-1}(\sin^{-1} \frac{1}{\ln(1 + \frac{2}{e^x-1})})$$

$$= f^{-1}(\sin^{-1} \frac{1}{\ln(1 + \frac{2}{\frac{2}{y-1}+1-1})})$$

$$= f^{-1}(\sin^{-1} \frac{1}{\ln y}) = \ln(\frac{2}{y-1} + 1)$$

Substitute $y = e^{1/\sin z}$

$$f^{-1}(\sin^{-1} \frac{1}{\ln y}) = f^{-1}(z)$$

$$= \ln(\frac{2}{y-1} + 1)$$

$$= \ln(\frac{2}{e^{1/\sin z} - 1} + 1)$$

$$f^{-1}(x) = \ln(\frac{2}{e^{1/\sin x} - 1} + 1)$$

2.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{a}{x-1}[3\sin(x-1) - 2\tan(\ln x)]$$

$$= \lim_{x \to 1^-} \frac{a}{1}[3\cos(x-1) - 2\sec^2(\ln x)\frac{1}{x}]$$

$$= a$$

$\therefore a = b$

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} \left[\frac{d}{dx}\left(\frac{a}{x-1}(3\sin(x-1) - 2\tan(\ln x)) \right) \right]$$

$$= a \lim_{x \to 1^-} \left[\frac{3\cos(x-1) - 2\sec^2(\ln x)/x}{x-1} - \frac{3\sin(x-1) - 2\tan(\ln x)}{(x-1)^2} \right]$$

$$= a \lim_{x \to 1^-} \left[\frac{2\cos(x-1) - 2\sec^2(\ln x)/x}{x-1} - \frac{3\sin(x-1) - 2\tan(\ln x) + (x-1)\cos(x-1)}{(x-1)^2} \right]$$

$$= \dots \qquad \text{This is left as an exercise for } \sout{\text{wolfram}} \text{ the reader}$$

$$= a$$

Define

$$d = \int_0^1 e^{[\ln(t+1)]^c} \, dx$$

$$\lim_{x \to 1^-} f'(x) = \frac{d}{dx}\left(\int_{4(x-1)}^{x^2} e^{x + [\ln(t+1)]^c}\ dx\right)_{x=1}$$

$$= \left[e^x \frac{d}{dx}\left(\int_{4(x-1)}^{x^2} e^{[\ln(t+1)]^c}\ dx\right)\right]_{x=1} + \left[e^x \left(\int_{4(x-1)}^{x^2} e^{[\ln(t+1)]^c}\ dx\right)\right]_{x=1}$$

$$= e^1 (2 \cdot e^{[\ln(2)]^c]} - 4e^{[\ln 1]^c}) + e^1 \cdot d$$

$$= e^1 (2 \cdot e^{[\ln(2)]^c]} - 4) + e^1 \cdot d$$

On the other hand,

$$\lim_{x \to 1^+} f(x) = e^1 \cdot d$$

For f to be continuous,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \Rightarrow a = e \cdot d$$

Since f is differentiable,

$$\lim_{x \to 1^-} f'(x) = a = \lim_{x \to 1^+} f'(x) = e \cdot d + e \cdot (2 \cdot e^{[\ln(2)]^c]} - 4)$$

Therefore

$$2 \cdot e^{[\ln(2)]^c]} - 4 = 0 \Rightarrow c = 1$$

$$d = \int_0^1 e^{[\ln(t+1)]^c}\ dx = \int_0^1 e^{\ln(t+1)}\ dx$$

$$= \int_0^1 t + 1\ dx$$

$$= \frac{3}{2}$$

$$\therefore a = b = \frac{3}{2}e, c = 1$$

3.

$$f(1) = f(0) + f'\left(\frac{1}{2}\right) + Af''(c)$$

By Intermediate value theorem, $\exists a \in (0,1)$ such that $f'(a) = f(1) - f(0)$

$$\therefore f'\left(\frac{1}{2}\right) + Af''(c) = f(1) - f(0) = f(a)$$

If $a = \frac{1}{2}$, then we can choose $A = 0$ and $c = 0.123$, and we are done

Otherwise, $a \neq \frac{1}{2}$

By Intermediate value theorem, $\exists b \in (a, \frac{1}{2})$ or $(\frac{1}{2}, a)$ such that $(\frac{1}{2} - a)f''(b) = f'(\frac{1}{2}) - f'(a)$

Choose $c = b$ and $A = \frac{1}{2} - a$

$$\therefore Af''(c) = \left(\frac{1}{2} - a\right) f''(b) = f(a) - f'\left(\frac{1}{2}\right)$$

$$\frac{1}{2} - 1 \leq A = \frac{1}{2} - a \leq \frac{1}{2} - 0$$

$$\therefore -\frac{1}{2} \leq A \leq \frac{1}{2}$$

4.   i

$$\int (Ax^2 + B)^{-3/2} dx$$

Let $x = \sqrt{\frac{B}{A}} \tan\theta$, then $dx = \sqrt{\frac{B}{A}} \sec^2\theta \, d\theta$

$$\int (Ax^2 + B)^{-3/2} dx = \int \left(A \left(\sqrt{\frac{B}{A}} \tan\theta \right)^2 + B \right)^{-3/2} \sqrt{\frac{B}{A}} \sec^2\theta \, d\theta$$

$$= \int (B\tan^2\theta + B)^{-3/2} \sqrt{\frac{B}{A}} \sec^2\theta \, d\theta$$

$$= \int (B\sec^2\theta)^{-3/2} \sqrt{\frac{B}{A}} \sec^2\theta \, d\theta$$

$$= \int \frac{1}{B\sqrt{A}\sec\theta} d\theta$$

$$= \frac{1}{B\sqrt{A}} (\sin\theta) + C$$

$$= \frac{1}{B\sqrt{A}} \left(\frac{\tan\theta}{\sqrt{\tan^2\theta + 1}} \right) + C$$

$$= \frac{1}{B} \left(\frac{x}{\sqrt{Ax^2 + B}} \right) + C$$

ii

$$y = [(2x^2(2 + \sin t)^4 + 2 - \sin t)]^{-3/2}$$

$$[(2x^2(2 + \sin t)^4 + 2 - \sin t)]^{-3/2} = y = \frac{1}{8}$$

$$\therefore 2x^2(2 + \sin t)^4 + 2 - \sin t = 4$$

$$\therefore x^2 = \frac{2 + \sin t}{2(2 + \sin t)^4} = \frac{1}{2(2 + \sin t)^3}$$

$$\therefore x = \pm\sqrt{\frac{1}{2(2 + \sin t)^3}}$$

The intersection of the curves has x values $-\sqrt{\frac{1}{2(2+\sin t)^3}}$ and $\sqrt{\frac{1}{2(2+\sin t)^3}}$. Let
$p = \sqrt{\frac{1}{2(2+\sin t)^3}}$

$$\text{Area} = \int_{-p}^{p} y\, dx - \frac{1}{8}(2p)$$

$$= \left[\frac{x}{B\sqrt{Ax^2 + B}}\right]_{-p}^{p} - \frac{p}{4} \qquad \text{where } A = 2(2\sin t)^4 \text{ and } B = 2 - \sin t$$

$$= 2\left[\frac{p}{B\sqrt{Ap^2 + B}}\right] - \frac{p}{4}$$

$$= 2\left[\frac{1}{B\sqrt{A + B/p^2}}\right] - \frac{p}{4}$$

$$= \frac{2}{(2 - \sin t)\sqrt{2(2 + \sin t)^4} + (2 - \sin t)(2)(2 + \sin t)^3} - \frac{1}{4\sqrt{2(2 + \sin t)^3}}$$

$$= \frac{1}{(2 - \sin t)(2 + \sin t)\sqrt{2(2 + \sin t)}} - \frac{1}{4(2 + \sin t)\sqrt{2(2 + \sin t)}}$$

$$= \frac{1}{(2 + \sin t)\sqrt{2(2 + \sin t)}}\left(\frac{1}{2 - \sin t} - \frac{1}{4}\right)$$

$$= \frac{1}{(2 + \sin t)\sqrt{2(2 + \sin t)}}\left(\frac{2 + \sin t}{4(2 - \sin t)}\right)$$

$$= \frac{1}{4(2 - \sin t)\sqrt{2(2 + \sin t)}}$$

iii Find the absolute min and max of $S(t) = \frac{1}{4(2-\sin t)\sqrt{2(2+\sin t)}}$

This is equivalent to finding the max and min of $\frac{1}{4(2-x)\sqrt{2(2+x)}}$ for $x \in [-1, 1]$

Let $f(x) = \frac{1}{4(2-x)\sqrt{2(2+x)}}$

$$f'(x) = \frac{1}{4\sqrt{2}}\left(\frac{1}{(2 - x)^2\sqrt{2 + x}} - \frac{1}{2(2 - x)(2 + x)^{3/2}}\right) = 0$$

$$\therefore (2 + x) - \frac{1}{2}(2 - x) = 0$$

$$\therefore x = -\frac{2}{3}$$

$$f(-1) = \frac{1}{12\sqrt{2}}, f\left(-\frac{2}{3}\right) = \frac{3\sqrt{6}}{128}, f(1) = \frac{1}{4\sqrt{6}}$$

Absolute minimal is $\frac{3\sqrt{6}}{128}$, maximal is $\frac{1}{4\sqrt{6}}$

5.　　i

$$I_{n+2} = \int_0^{\pi/2} \sin^{n+2} x \, dx$$

$$= \int_0^{\pi/2} \sin x \sin^{n+1} x \, dx$$

$$= \left[(-\cos x) \sin^{n+1} x \right]_0^{\pi/2} + (n+1) \int_0^{\pi/2} \cos^2 x \sin^n x \, dx$$

$$= (n+1) \int_0^{\pi/2} (1 - \sin^2 x) \sin^n x \, dx$$

$$= (n+1)(I_n - I_{n+2})$$

$$\therefore (n+2)I_{n+2} = (n+1)I_n$$

$$\therefore I_{n+2} = \frac{n+1}{n+2} I_n$$

ii

$$I_0 = \frac{\pi}{2}$$

$$I_1 = 1$$

$$I_9 = \frac{8}{9} I_7$$

$$= \frac{8}{9} \times \frac{6}{7} I_5$$

$$= \frac{8 \times 6 \times 4 \times 2}{9 \times 7 \times 5 \times 3}$$

$$= \frac{128}{315}$$

$$I_{10} = \frac{9}{10} I_8$$

$$= \frac{9}{10} \times \frac{7}{8} I_6$$

$$= \frac{9 \times 7 \times 5 \times 3 \times 1}{10 \times 8 \times 6 \times 4 \times 2} I_0$$

$$= \frac{63\pi}{512}$$

6.

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2(1 - 2\sin^2(t/2))}\, dt$$

$$= \int_0^{2\pi} 2\sin(t/2)\, dt$$

$$= [-4\cos(t/2)]_0^{2\pi}$$

$$= 8$$

7. Let $y = 2\tanh x - x$

We want to find all stationary points on y

$$\frac{dy}{dx} = 2\operatorname{sech}^2 x - 1 = 0$$

$$\frac{4}{e^2 - e^{-x}} = 1$$

$$e^x + e^{-x} = 4$$

$$e^x = 2 \pm \sqrt{3}$$

$\therefore y$ has exactly 2 stationary points at $x = \ln(2 + \sqrt{3})$ and $x = \ln(2 - \sqrt{3})$

At $x = 100$, $y \approx -98$
At $x = \ln(2 + \sqrt{3})$, $y \approx 0.451$
At $x = \ln(2 - \sqrt{3})$, $y \approx -0.451$
At $x = -100$, $y \approx 98$

And since y is monotonous between those 3 intervals, there is exactly 1 solution for each interval.

$\therefore$ the equation has exactly 3 solutions.

8.

$$(x^2 y - y)\frac{dy}{dx} + (xy^2 + x) = 0$$

$$\frac{dy}{dx} = \frac{x(y^2 + 1)}{y(x^2 - 1)}$$

$$\int \frac{y}{y^2 + 1}\, dy = \int \frac{x}{x^2 - 1}\, dx$$

$$\frac{1}{2}\ln(y^2 + 1) = \frac{1}{2}\ln(x^2 - 1) + \frac{1}{2}C$$

$$\ln(y^2 + 1) = \ln(C(x^2 - 1))$$
$$y^2 + 1 = C(x^2 - 1)$$
$$y = \sqrt{C(x^2 - 1) - 1} \text{ or } y = -\sqrt{C(x^2 - 1) - 1} \text{ (rejected since } y > 0 \text{ at } 0)$$
$$\therefore y = \sqrt{C(x^2 - 1) - 1}$$

Sub $y = 1$ and $x = 0$

$$1 = \sqrt{-C - 1}$$
$$\therefore C = -2$$

$$\therefore y = \sqrt{-2(x^2 - 1) - 1}$$