

MA1102R AY1718 Sem 1 Answers

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1. (i)

$$f(x) = (x^3 + 4x^2 + 11x + 14)e^{-x}$$

$$f'(x) = -(x^3 + 4x^2 + 11x + 14)e^{-x} + (3x^2 + 8x + 11)e^{-x} > 0$$

$$\iff (3x^2 + 8x + 11) - (x^3 + 4x^2 + 11x + 14) > 0$$

$$\iff -x^3 - x^2 - 3x - 3 > 0$$

$$\iff (-x - 1)(x^2 + 3) > 0$$

$$\iff -1 > x$$

$\therefore f$ is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$

(ii) $f(-1) = 6e$ is a local maximum. There is no local minimum.

(iii)

$$f'(x) = (-x^3 - x^2 - 3x - 3)e^{-x}$$

$$f''(x) = (-3x^2 - 2x - 3)e^{-x} - (-x^3 - x^2 - 3x - 3)e^{-x}$$

$$= (x^3 - 2x^2 + x)e^{-x} > 0$$

$$\iff x^3 - 2x^2 + x > 0$$

$$\iff x(x - 1)^2 > 0$$

$$\iff x > 0$$

f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$

(iv) $f(0) = 14$
 $(0, 14)$

2. (a) For any $\epsilon > 0$, choose $\delta = \min(\epsilon, 1)$

Then for all x such that $0 < |x - 2| < \delta$

$$\left| \frac{x}{x^2 + 2} - \frac{1}{3} \right| = \left| \frac{3x - (x^2 + 2)}{3(x^2 + 2)} \right| = \left| \frac{(x - 1)(x - 2)}{3(x^2 + 2)} \right|$$

$$< |x - 1| \left| \frac{1}{3(x^2 + 2)} \right| \epsilon \quad \text{since } |x - 2| < \epsilon$$

$$< 2 \times \frac{1}{3 \times 2} \times \epsilon \quad \text{since } |x - 1| < 2 \text{ and } \frac{1}{x^2 + 2} < \frac{1}{2}$$

$$< \epsilon$$

(b)

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i^3}{n^2(n^2 + i^2)} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{\left(\frac{i}{n}\right)^3}{1 + \left(\frac{i}{n}\right)^2} \\&= \int_0^1 \frac{x^3}{1 + x^2} dx \\&= \int_0^1 x - \frac{x}{1 + x^2} dx \\&= \left[\frac{1}{2}x - \frac{1}{2} \ln(x^2 + 1) \right]_0^1 \\&= \frac{1}{2} - \frac{1}{2} \ln 2\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x \rightarrow 0^+} \left(\frac{e^2 - 1}{x} \right)^{1/x} &= \lim_{x \rightarrow 0^+} \exp \left(\frac{1}{x} \ln \left(\frac{e^x - 1}{x} \right) \right) \\&= \lim_{x \rightarrow 0^+} \exp \left(\left(\frac{x}{e^x - 1} \right) \left(\frac{e^x}{x} - \frac{e^x - 1}{x^2} \right) \right) \quad \text{By L'Hôpital's rule} \\&= \lim_{x \rightarrow 0^+} \exp \left(\frac{xe^x - e^x + 1}{x(e^x - 1)} \right) \\&= \lim_{x \rightarrow 0^+} \exp \left(\frac{xe^x}{xe^x + e^x - 1} \right) \quad \text{By L'Hôpital's rule} \\&= \lim_{x \rightarrow 0^+} \exp \left(\frac{e^x}{e^x + \frac{e^x - 1}{x}} \right) \\&= \exp \left(\frac{1}{2} \right) \\&= \sqrt{e}\end{aligned}$$

3. Let the angle of the sector be θ

$$2r + r\theta = 50 \implies \theta = \frac{50}{r} - 2$$

$$\begin{aligned}\text{Area} &= \frac{1}{2}r^2\theta \\&= \frac{1}{2}r^2 \left(\frac{50}{r} - 2 \right) \\&= 25r - r^2 \\&= r(25 - r) \\&\leq \left(\frac{25}{2} \right)^2 \quad \text{By AMGM inequality, with equality at } r=12.5\end{aligned}$$

$$r = 12.5 \text{ m}$$

4. (a)

$$\ln y = (\sec x) \ln(\tan x) + (\tan x) \ln(\sec x)$$

$$\frac{1}{y} \frac{dy}{dx} = \sec^2 x \sin x \ln(\tan x) + \operatorname{cosec} x \sec^2 x + \sec^2 x \ln(\sec x) + \sin^2 x \sec^2 x$$

If $x = \frac{\pi}{4}$, then $y = 1^{\sqrt{2}} \sqrt{2}^1 = \sqrt{2}$

Sub $x = \frac{\pi}{4}$ and $y = \sqrt{2}$ to the equation

$$\frac{1}{\sqrt{2}} \frac{dy}{dx} = 0 + 2\sqrt{2} + 2 \ln \sqrt{2} + 1$$

$$\frac{dy}{dx} = 4 + 2\sqrt{2} \ln \sqrt{2} + \sqrt{2}$$

(b) For $x \neq 0$:

$$\begin{aligned} F'(x) &= \frac{d}{dx} \int_0^{x^2} f(t) dt \\ &= 2x f(x^2) \\ &= \frac{2 \sin(x^2)}{x} \end{aligned}$$

For $x = 0$:

$$F'(x) = 0$$

$\therefore F'(x) = 0$ for $x = \sqrt{k\pi}, k \in \mathbb{Z}$

To check if it is a local max or min, we check the concavity

For $x \neq 0$:

$$F''(x) = 4 \cos(x^2) - \frac{2 \sin(x^2)}{x^2}$$

For $x = 0$:

$$F''(x) = 2$$

$F''(\sqrt{k\pi}) = 4(-1)^k$ for $k \neq 0$ and $F''(0) = 2$

f attains local min at $x = \sqrt{k\pi}$ for even k and local max at $x = \sqrt{k\pi}$ for odd k .

(c)

$$f''(x) < 0 \implies f'(x) \text{ is decreasing} \implies f'(x) < 0 \implies f(x) \text{ is decreasing}$$

Either $\lim_{x \rightarrow \infty} f(x) = -\infty$ or $\lim_{x \rightarrow \infty} f(x) = k$ for some constant k

If $\lim_{x \rightarrow \infty} f(x) = k$, then $\lim_{x \rightarrow \infty} f'(x) = 0, \Rightarrow \Leftarrow$

$\therefore \lim_{x \rightarrow \infty} f(x) = -\infty$

$f(x)$ is decreasing and $\lim_{x \rightarrow \infty} f(x) = -\infty \implies$ exactly 1 root

5.

$$y^2 = 2x = 8 - x^2$$

$$\therefore x = 2, y = \pm 2$$

The curves intersect at $(2, 2)$ and $(2, -2)$

(i)

$$x^2 + y^2 = 8 \implies x = \sqrt{8 - y^2}$$

$$y^2 = 3x \implies x = \frac{1}{2}y^2$$

$$\begin{aligned} \text{Area} &= \int_{-2}^2 \sqrt{8 - y^2} - \frac{1}{2}y^2 \, dy \\ &= \int_{-2}^2 \sqrt{8 - y^2} \, dy - \left[-\frac{1}{6}y^3 \right]_{-2}^2 \quad \text{sub } y = \sqrt{8} \sin \theta \\ &= \int_{-\pi/4}^{\pi/4} 8 \cos^2 \theta \, d\theta - \frac{8}{3} \\ &= 4 \int_{-\pi/4}^{\pi/4} \cos(2\theta) + 1 \, d\theta - \frac{8}{3} \\ &= 2[\sin(2\theta) + 2\theta]_{-\pi/4}^{\pi/4} - \frac{8}{3} \\ &= \frac{4}{3} + 2\pi \end{aligned}$$

(ii)

$$x^2 + y^2 = 8 \implies y = \sqrt{8 - x^2}$$

$$y^2 = 3x \implies y = \sqrt{2x}$$

$$\begin{aligned} \text{Volume} &= 2 \left[\int_0^2 \sqrt{2x}(2\pi x) \, dx + \int_2^{\sqrt{8}} \sqrt{8 - x^2}(2\pi x) \, dx \right] \\ &= 4\pi \left[\left[\frac{2}{5} \sqrt{2} x^{5/2} \right]_0^2 + \left[-\frac{1}{3} (8 - x^2)^{3/2} \right]_2^{\sqrt{8}} \right] \\ &= 4\pi \left[\frac{16}{5} + \frac{8}{3} \right] \\ &= \frac{352}{15} \pi \end{aligned}$$

6. (i)

$$\begin{aligned} \int \frac{x \ln x}{(1 + x^2)^2} \, dx &= \left(-\frac{1}{2} \right) \frac{\ln x}{1 + x^2} - \int \left(-\frac{1}{2} \right) \frac{1}{x(1 + x^2)} \, dx \quad \text{sub } x = \tan \theta \\ &= -\frac{\ln x}{2(1 + x^2)} + \frac{1}{2} \int \frac{1}{\tan \theta \sec^2 \theta} \sec^2 \theta \, d\theta \\ &= -\frac{\ln x}{2(1 + x^2)} + \frac{1}{2} \int \frac{\cos \theta}{\sin \theta} \, d\theta \\ &= -\frac{\ln x}{2(1 + x^2)} + \frac{1}{2} \ln(\sin \theta) + C \\ &= -\frac{\ln x}{2(1 + x^2)} + \frac{1}{2} \ln \left(\frac{x}{\sqrt{1 + x^2}} \right) + C \end{aligned}$$

(ii)

$$\lim_{x \rightarrow \infty} \left(\frac{1}{2} \ln \frac{x}{\sqrt{1+x^2}} - \frac{\ln x}{2(1+x^2)} \right) = \frac{1}{2} \ln(1) = 0$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \left(\frac{1}{2} \ln \frac{x}{\sqrt{1+x^2}} - \frac{\ln x}{2(1+x^2)} \right) &= \lim_{x \rightarrow 0^+} \left(\frac{1}{2} \ln x - \frac{\ln x}{2(1+x^2)} - \frac{1}{2} \ln \sqrt{1+x^2} \right) \\ &= \lim_{x \rightarrow 0^+} \left((\ln x) \left(\frac{1}{2} - \frac{1}{2(1+x^2)} \right) \right) \\ &= \frac{1}{2} \lim_{x \rightarrow 0^+} \left(\frac{x^2 \ln x}{1+x^2} \right) \\ &= \frac{1}{2} \lim_{x \rightarrow 0^+} (x^2 \ln x) \\ &= \frac{1}{2} \lim_{x \rightarrow 0^+} \frac{1/x}{-2x^{-3}} \quad \text{By L'Hôpital's rule} \\ &= 0 \end{aligned}$$

$$\therefore \int_0^\infty \frac{x \ln x}{(1+x^2)^2} dx = 0$$

7. (a)

$$y = \frac{1}{x} + \frac{1}{z} \implies z = 1 \text{ at } x = 1$$

$$\frac{dy}{dx} = -\frac{1}{x^2} - \frac{1}{z^2} \frac{dz}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{x^2} - \frac{1}{z^2} \frac{dz}{dx} = \left(\frac{1}{x} + \frac{1}{z} \right)^2 - \frac{1}{x} \left(\frac{1}{x} + \frac{1}{z} \right) - \frac{1}{x^2}$$

$$-\frac{1}{z^2} \frac{dy}{dx} = \frac{1}{z^2} + \frac{1}{xz}$$

$$\frac{dy}{dx} + \frac{z}{x} + 1 = 0$$

Let $w = \frac{z}{x}$. Then $w = 1$ at $x = 1$.

$$wx = z \implies x \frac{dw}{dx} + w = \frac{dz}{dx}$$

$$\frac{dz}{dx} = x \frac{dw}{dx} + w = -1 - w$$

$$\int \frac{1}{-1-2w} = \int \frac{1}{x}$$

$$-\frac{1}{2} \ln |1+2w| = \ln(x) + C$$

Substitute $x = 1, w = 1$

$$-\frac{1}{2} \ln 3 = C$$

Therefore,

$$-\frac{1}{2} \ln |1 + 2w| = \ln \frac{x}{\sqrt{3}}$$

$$\frac{1}{\sqrt{1 + 2w}} = \frac{x}{\sqrt{3}}$$

$$w = \frac{1}{2} \left(\frac{3}{x^2} - 1 \right)$$

$$z = wx = \frac{3}{2x} - \frac{x}{2} = \frac{3 - x^2}{2x}$$

$$y = \frac{1}{x} + \frac{2x}{3 - x^2}$$

(b)

$$\int \frac{4h - h^2}{\sqrt{h}} dh = \int -1 dh$$

$$\frac{8}{3} h^{3/2} - \frac{2}{5} h^{5/2} + C = -t$$

Substitute $t = 0, h = 4$

$$\frac{64}{3} - \frac{64}{5} + C = 0 \implies C = -\frac{128}{15}$$

Therefore,

$$\frac{8}{3} h^{3/2} - \frac{2}{5} h^{5/2} - \frac{128}{15} = -t$$

Substitute $h = 0$

$$t = \frac{128}{15}$$

128/15 minutes

8. We first want to show that $\forall x \leq 0.5, |f(x)| \leq Mx$.

Suppose $\exists a \in (0, 1)$ such that $|f(a)| > Mx$, then by mean value theorem, $\exists b \in (0, a)$ such that $|f'(b)| = |(f(a) - 0)/(a - 0)| > M$, a contradiction.

Therefore, $\forall x \leq 0.5, |f(x)| \leq Mx$.

Similarly, we can also show that $\forall x \geq 0.5, |f(x)| \leq M(1 - x)$.

Suppose $\exists a \in (0, 1)$ such that $|f(a)| > M(1 - x)$, then by mean value theorem, $\exists b \in (a, 1)$ such that $|f'(b)| = |(f(a) - 0)/(a - 1)| > M$, a contradiction.

Hence,

$$\begin{aligned}\int_0^1 |f(x)| \, dx &= \int_0^{0.5} |f(x)| \, dx + \int_{0.5}^1 |f(x)| \, dx \\ &< \int_0^{0.5} Mx \, dx + \int_{0.5}^1 M(1-x) \, dx \\ &= \frac{1}{4}M\end{aligned}$$