NATIONAL UNIVERSITY OF SINGAPORE

MA3269 – MATHEMATICAL FINANCE I

(Semester 1 : AY2016/2017)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Write your matriculation/student number only. **Do not write your name**.
2. This examination paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
3. This examination carries a total of **65** marks.
4. Answer **ALL** questions.
5. This is a **CLOSED BOOK** examination.
6. You are allowed to use four A4-size, double-sided help sheets.
7. You may use **non-programmable** calculators. However, you should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

In this question, r_x, μ_x, σ_x and $\mathbf{w}_x$ denote respectively the rate of return, mean, standard deviation and weight vector of a portfolio of risky assets labelled x.

A financial market consisting of n ($n > 2$) risky assets and a risk-free asset satisfies the Capital Asset Pricing Model. Portfolios g and m are respectively the global minimum-variance portfolio of risky assets and the market (tangency) portfolio. Portfolios x and y are distinct portfolios on the minimum-variance frontier for risky assets.

In the $\sigma - \mu$ plane, the Capital Market Line passes through the origin.

It is given further that $\mu_g = \frac{1}{6}$, $\mu_x = 0$, $\sigma_g^2 = \frac{1}{12}$ and $\sigma_x^2 = \sigma_y^2 = \frac{1}{4}$.

Find

(i) the risk-free rate ,

(ii) the equation of the efficient frontier for risky assets, giving your answer in the form $\mu = f(\sigma)$,

(iii) the equation of the asymptote to the efficient frontier curve ,

(iv) μ_y ,

(v) μ_m and σ_m^2 ,

(vi) the beta of the portfolio whose weight vector is $\frac{1}{4}\left(\mathbf{w}_g + \mathbf{w}_m + \mathbf{w}_x + \mathbf{w}_y\right)$,

(vii) $\operatorname{corr}\left(r_y, r_m\right)$, the correlation between r_y and r_m .

Question 2 [15 marks]

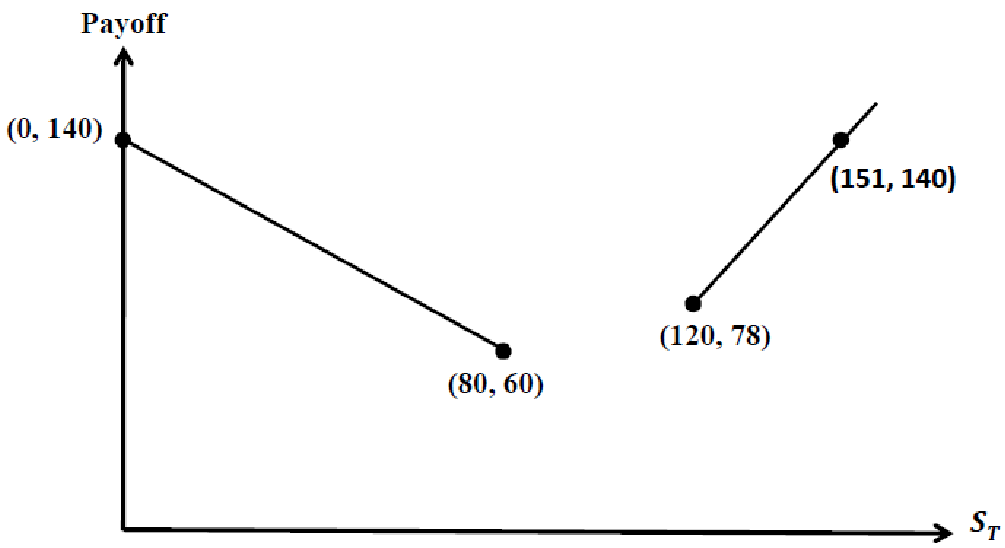
In this question, all options have the same underlying non-dividend-paying stock and time to maturity T. Let S_T denote the terminal price of the underlying stock.

(a) Let $K_2 > K_1 > 0$. A capped call option has payoff function $\max\left(\ \min(S_T, K_2) - K_1, 0\ \right)$.

(i) Draw the payoff table and sketch the payoff diagram for this option.

(ii) Let r be the continuously compounded risk-free rate. Describe a portfolio consisting of the risk-free asset and two put options that has the same payoff as the above option.

(b) The diagram below shows part of the payoff diagram for a portfolio created by writing n_1 put options with strike price 80, holding n_2 call options with strike price 90, holding n_3 put options with strike price 100 and writing 1 call option with strike price K, where $100 < K < 120$.



(i) Determine and **write down** the values of n_1, n_2, n_3 and K. **No working is required.**

(ii) Draw the payoff diagram for $80 \le S_T \le 120$.

(c) A non-dividend-paying stock has a current price of \$10. One month later, the stock price will either move up by 10% or move down by 5%. The continuously-compounded risk-free rate is 0.06. A one-month European option has payoff function $\max\left(\dfrac{(S_1-1)^2}{10} - 7,\ 0\right)$, where S_1 is the stock price at the end of one month. Use a single-period binomial model to price this option. Give your answer to **three significant figures**.

Question 3 [15 marks]

(a) A trader with an initial wealth of $\$1$ and utility function

$$U(w) = \ln(w), \quad w > 0,$$

is offered a lottery that pays $\$(ax)$ or $\$(-bx)$, with equal probabilities, where $a > 0,\ b > 0$ and $0 < bx < 1$. Let the certainty equivalent of the lottery be denoted by $C(x)$.

(i) Find $C(x)$ in terms of a, b and x. Simplify your answer.

(ii) For the case when $a \le b$, show that $C(x)$ is a strictly decreasing function of x.

(iii) For the case when $a > b$, find in terms of a and b the range of values of x for which the trader will avoid the lottery.

(iv) For the case when $a > b$, find the maximum value of $C(x)$ in terms of a and b. Deduce that the trader will play the lottery when $C(x)$ attains its maximum value.

(b) Let U_1 and U_2 be twice differentiable concave utility functions defined on $(0, \infty)$ and let U be defined on $\{U_2(x) : x \in \mathbf{R}\}$ by

$$U(x) = U_1(U_2^{-1}(x)),$$

where U_2^{-1} denotes the inverse of U_2.

Let R_1 and R_2 denote the Arrow-Pratt absolute risk aversion functions associated with U_1 and U_2 respectively. It is given that $R_1(x) > R_2(x)$ for all $x > 0$.

Prove that

(i) $$\frac{d^2U}{dx^2} = \frac{U_1'(t)}{\left(U_2'(t) \right)^2}\left(R_2(t) - R_1(t)\right), \text{ where } t = U_2^{-1}(x),$$

(ii) U is a concave utility function.

Question 4 [15 marks]

In this question, r_x, μ_x, σ_x and $\mathbf{w}_x$ denote respectively the rate of return, mean, standard deviation and weight vector of a portfolio of risky assets labelled x. The covariance and correlation between r_x and r_y are denoted respectively by σ_{xy} and ρ_{xy}. Two portfolios x and y are said to be orthogonal if $\sigma_{xy} = 0$.

A financial market consisting of n ($n > 2$) risky assets and a risk-free asset satisfies the Capital Asset Pricing Model. Let $\boldsymbol{\mu}$ and $\mathbf{C}$ denote respectively the mean vector and covariance matrix of the risky assets. Let g and m be the global minimum-variance portfolio of risky assets and the market (tangency) portfolio respectively. Let x be a portfolio of risky assets **not** lying on the minimum-variance frontier for risky assets and y be a portfolio on the minimum-variance frontier for risky assets such that $\mu_y = \mu_x \; (\neq \mu_g)$

Let $a = \mathbf{1}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{1}$, $b = \mathbf{1}^{\mathrm{T}}\mathbf{C}^{-1}\boldsymbol{\mu} > 0$ and $c = \boldsymbol{\mu}^{\mathrm{T}}\mathbf{C}^{-1}\boldsymbol{\mu}$, where $\mathbf{1}$ is the column vector of ones.

(i) By expressing $\mathbf{w}_y$ in the form $\alpha \dfrac{\mathbf{C}^{-1}\boldsymbol{\mu}}{b} + (1-\alpha)\dfrac{\mathbf{C}^{-1}\mathbf{1}}{a}$, for some $\alpha \in \mathbf{R}$, show that $\sigma_{xy} = \sigma_y^2$.

(ii) Let $r_p = \gamma\, r_y + (1-\gamma) r_g$, where $\gamma \in \mathbf{R}$, be the rate of return of a portfolio p on the minimum-variance frontier for risky assets. Show that

$$\rho_{px} = \frac{\gamma\,\sigma_y^2 + (1-\gamma)\,\sigma_g^2}{\sigma_x\sqrt{\gamma^2\sigma_y^2 + (1-\gamma^2)\sigma_g^2}}\,.$$

Hence, show that the minimum-variance portfolio of risky assets having the highest correlation with portfolio x is portfolio y.

(iii) Show that $\mathbf{w}_x$ can be expressed in the form $\mathbf{w}_g + \mathbf{w}_a + \mathbf{w}_b$, where the mean of portfolio a is zero and portfolios g, a and b are pair-wise orthogonal.

(iv) Suppose μ_x is greater than the risk-free rate. Let z be the portfolio on the Capital Market Line such that $\mu_z = \mu_x$. Show that

$$\rho_{xz} = \frac{\sigma_z}{\sigma_x}.$$

END OF PAPER