PYP Answer - MA2101 AY1718Sem1

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January 30, 2018

1. Basically let $Z = P^{-1}Y$, i.e., $Y = PZ$. We can then write $Y' = PY$ as $Z' = P^{-1}APZ = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} Z$, i.e.,

$$\begin{cases} z_1' &= z_1 + 2z_2 \\ z_2' &= -z_2 \end{cases}$$

Therefore, $z_2 = Ae^{-x}$. And $z_1' - z_1 = 2Ae^{-x}$. Applying formula given, we have $z_1 = Be^x - Ae^{-x}$.

2. (a) Solving $|\lambda I - A| = \det \begin{pmatrix} \lambda - 2 & 2 \\ 2 & \lambda - 5 \end{pmatrix} = \lambda^2 - 7\lambda + 6 = 0$ where λ is the eigenvalue, we have $\lambda_1 = 1$ and $\lambda_2 = 6$.

   (b) We need to solve $\begin{pmatrix} \lambda - 2 & 2 \\ 2 & \lambda - 5 \end{pmatrix} \mathbf{x} = 0$ for $\mathbf{x}$. For $\lambda_1 = 1$, we have $\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \mathbf{x} = 0 \Rightarrow B_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Similarly, $B_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

   (c) Let $P = \begin{pmatrix} \mathbf{B_1}^t & \mathbf{B_2}^t \end{pmatrix}$. Then one admissible D will be $\begin{pmatrix} 30 & 0 \\ 0 & 5 \end{pmatrix} = P^tAP$.

3. (a) Suppose the contrary that C is linearly dependent, then there exists an index set where $\sum_{i \in I, |I| < \infty} \mathbf{c}_i = 0$ has non-trivial solution, where $\mathbf{c}_i \in C$. Then by applying T to both sides, we arrive to the conclusion that $\sum_{i \in I, |I| < \infty} \mathbf{d}_i = 0$ has non-trivial solution, which contradicts with the fact that D is a linearly independent subset. Therefore, C is a linearly independent subset of V.

   (b) Since $T : V \to W$ is subjective linear transformation, we only need to verify that T is injective. Suppose $0 \neq \mathbf{v} \in N(T)$, i.e., $T(\mathbf{v}) = 0$. Since C is a basis of V, we can write $\mathbf{v} = \sum_{i \in I} a_i \mathbf{c}_i$ for some index set I. Therefore, $0 = T(\sum_{i \in I} a_i \mathbf{c}_i) = \sum_{i \in I} a_i T(\mathbf{c}_i) = \sum_{i \in I} a_i \mathbf{d}_i$. Since D is linearly independent, then we have $a_i = 0$ for all $i \in I$. In other words, $\mathbf{v} = \mathbf{0}$. So the kernel of T is $\{\mathbf{0}\}$, showing injectivity of T.

4. (a) A map f from vector space V to vector space U is an isomorphism if (1) f is a linear transformation, (2) f is injective and (3) f is surjective.

(b) H is a vector space over F if for all vector $\mathbf{v}_1, \mathbf{v}_2 \in H$, $\mathbf{v}_1 + \mathbf{v}_2 \in H$, and for all $\alpha \in F$, $\alpha \mathbf{v}_1 \in H$.

(c) ϕ is defined as below

$$\begin{aligned} \phi : H &\to M_n(F) \\ f &\mapsto [f]_{B_{St}} \end{aligned}$$

We verify the isomorphism of ϕ first.
(1) Linear Transformation: we pick any $f_1, f_2 \in H$ and $\alpha_1, \alpha_2 \in F$, $\phi(\alpha_1 f_1 + \alpha_2 f_2) = [\alpha_1 f_1 + \alpha_2 f_2]_{B_{St}} = \alpha_1 [f_1]_{B_{St}} + \alpha_2 [f_2]_{B_{St}} = \alpha_1 \phi(f_1) + \alpha_2 \phi(f_2)$. Shown.
(2) Injectivity: suppose $f \in N(\phi)$, i.e., $\phi(f) = [f]_{B_{St}} = 0$. This means that f is the zero map, which shows $N(\phi) = \{\mathbf{0}\}$, implying injectivity.
(3) Surjectivity: For any matrix $A \in M_n(F)$, we can construct linear transformation $T(\mathbf{v}) = A\mathbf{v}$. We can easily check this linear transformation T has the representation matrix A under standard basis.
These three parts combine to show ϕ is an isomorphism.
We then construct ψ to be

$$\begin{aligned} \psi : M_n(F) &\to H \\ A &\mapsto T : \mathbf{v} \to A\mathbf{v} \end{aligned}$$

We check ψ is an isomorphism by verifying ψ is the inverse of ϕ. In fact, for all arbitrary A, we always have $\phi(\psi(A)) = A$ and also $\psi\phi(A) = A$. So there are indeed inverse. This implies that ψ is an isomorphism.

(d) $\dim_F H = \dim M_n(F) = n^2$.

5. (a) $T^* : V \to V$ is the adjoint of $T : V \to V$ if $\langle T(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{u}, T^*(\mathbf{v}) \rangle$.

(b) B is an orthonormal basis of the inner product space V if $[T^*]_B = ([T]_B)^*$.

(c) This is not true.

(d) We evaluate

$$\begin{aligned} &\langle \mathbf{u}, T^*(\mathbf{v}) \rangle \\ =&\langle T(\mathbf{u}, \mathbf{v}) \rangle \\ =&\langle A[\mathbf{u}]_B, \mathbf{v} \rangle \\ =&(A[\mathbf{u}]_B)^t \overline{[\mathbf{v}]_B} \\ =&[\mathbf{u}]_B^t A^t \overline{[\mathbf{v}]_B} \end{aligned}$$

6. (a) The conjugate $\overline{\lambda}$ of complex number $\lambda = a + bi$ equals $a - bi$.

(b) A complex matrix A in $M_n(\mathbb{C})$ is unitary if

$$AA^* = I_n$$

(c) False. $UY = \lambda Y \Rightarrow U^*UY = \lambda U^*Y \Rightarrow U^*Y = \frac{1}{\lambda} Y \neq \overline{\lambda} Y$ in general.

(d) True. Consider $\langle Y, U^*\overline{Y}\rangle = \langle UY, \overline{Y}\rangle = \lambda\langle Y, \overline{Y}\rangle = \langle Y, \overline{\lambda Y}\rangle$, for all Y. Then, we arrive at $U^*\overline{Y} = \overline{\lambda Y}$.

7. (a) Yes. Since $p(A) = 0$ for $q(x) = x^3 - x = x(x+1)(x-1)$, the minimum polynomial which divides q contains only simple zeroes. Thus, A is diagonalisable.

   (b) So the characteristic polynomial is of the form $q(x) = x^{k_1}(x-1)^{k_2}(x+1)^{k_3}$, where $0 \leq k_1, k_2, k_3 \leq 3$ and $k_1 + k_2 + k_3 = 3$. All possible Jordan canonical forms up to isomorphism are

$$D[1,1,1], D[1,1,0], D[1,1,-1], D[1,0,0], D[1,-1,0], D[1,-1,-1]$$

$$D[0,0,0], D[0,0,-1], D[0,-1,-1], D[-1,-1,-1]$$

8. (a) T is a normal operator if $TT^* = T^*T$.

   (b) Let T be normal. By Schur's theorem, we may write $T = U^*DU$ where U is unitary and D is upper triangular. We claim that D is i fact diagonal. To see this, note that since $T^*T = TT^*$, $D^*D = DD^*$. Hence, we need to show that an upper triangular normal matrix is diagonal. The key is to compare the diagonal entries of DD^* and D^*D. Let d_{ii} be ith diagonal entry of D, and let $\mathbf{a}_i$ denote its ith row. Now the diagonal entries of DD^* are $|\mathbf{a}_1|^2, \ldots, |\mathbf{a}_n|^2$. On the other hand, the diagonal entries of D^*D are $|t_{11}|^2, \ldots, |t_{nn}|^2$. It follows that $|\mathbf{a}_i|^2 = |t_{ii}|^2$ for each i, and consequently T has to be diagonal. Therefore T is unitarily diagonalisable. QED.