NATIONAL UNIVERSITY OF SINGAPORE

ST2131/MA2216 PROBABILITY

(Semester 2 : AY 2017/2018)

Time Allowed : 2 Hours

INSTRUCTIONS TO STUDENTS

1. This assessment paper contains **FIVE (5)** questions and comprises **TWELVE (12)** printed pages.
2. Students are required to answer **ALL** questions. The total mark for this paper is 40. The marks allocation for each question is shown in the table below, and at the beginning of each question inside this booklet.
3. This is a CLOSED BOOK assessment. You are allowed to bring in one A4-sized formula sheet.
4. Non-programmable calculators may be used.
5. If you need extra space, you may use the blank pages on pages 10 and 11.
6. Please write your student number only in the space below. **Do not write your name.**

Student No: ______________________________

Question	Marks scored	Max. marks
1		10
2		6
3		10
4		9
5		5

Total
(40)

1. (10 points) Let X be a $Geo(p)$ random variable and let $q = 1 - p$.

   (a) Show that the pmf of X, in terms of q, is

$$P(X = i) = q^{i-1} - q^i, \quad i = 1, 2, 3, 4, \ldots$$

   (b) Show that the cdf of X, in terms of q, is

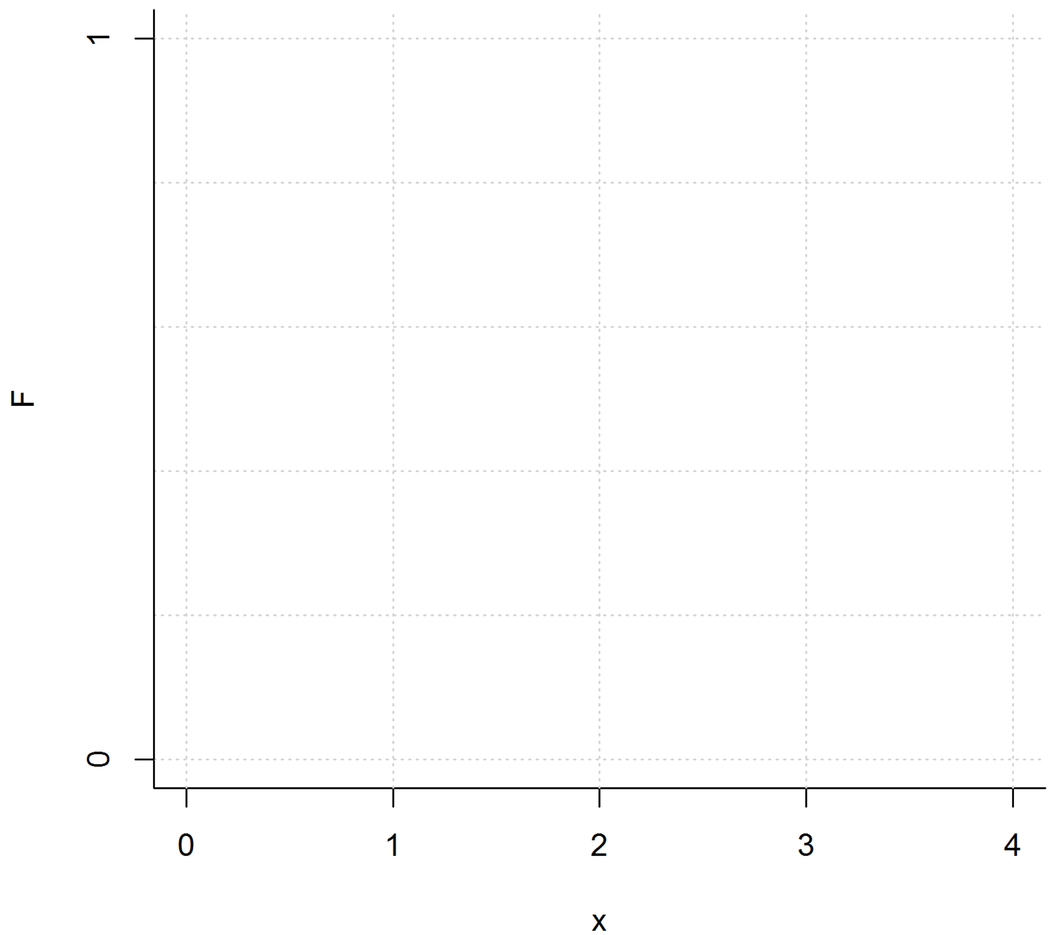
$$P(X \leq i) = 1 - q^i, \quad i = 1, 2, 3, 4, \ldots$$

(c) Now for a specified $q \in (0, 1)$, let Y be a **continuous** random variable, with cdf given by

$$F_Y(y) = P(Y \leq y) = \begin{cases} 0, & y \leq 1 \\ 1 - q^{y-1}, & y > 1 \end{cases}$$

For $U \sim Unif(0, 1)$, let $W = \lfloor F_Y^{-1}(U) \rfloor$. What is the distribution function of W? Recall that $\lfloor x \rfloor$ is the largest integer that is less than or equal to x.

(d) Sketch the cdfs of X and Y for $p = 0.5$ in the graph below.



2. (6 points) Civil engineers believe that W, the amount of weight (in units of 1000 kilograms) that a certain span of a bridge can withstand without structural damage resulting, is normally distributed with mean 400 and standard deviation 40. Suppose that the weight (again, in units of 1000 kilograms) of a car is a random variable with mean 3 and standard deviation 0.3. Use the Central Limit Theorem to estimate the probability of structural damage when there are 115 cars on the bridge.

   Assume that the 116 random variables involved: weights of cars and W, are independent.

3. (10 points) Suppose that the pdf of Y, conditional on $X = x$, is $N(x,\ x^2)$. Suppose further that the marginal distribution of X is $Unif(0, 1)$.

   (a) Use conditioning to find $E(Y)$, $\text{var}(Y)$ and $\text{cov}(X, Y)$.

(b) The joint pdf of X and Y can be derived to be

$$f_{X,Y}(x,y) = \frac{1}{x\sqrt{2\pi}} \exp\left\{-\frac{1}{2x^2}(y-x)^2\right\}, \quad y \in \mathbb{R},\ 0 < x < 1$$

Let $U = Y/X$ and $V = X$. Derive the joint pdf of U and V and use it to decide if U and V are independent.

4. (9 points) Let $X \sim Unif(0, 2)$ and $Y \sim Unif(-1, 1)$ be independent random variables. Find the distribution function of $Z = X/|Y|$.

   Consider the cases $z < 2$ and $z \geq 2$.

5. (5 points) Suppose that a group of n people throw their hats into the center of a room. The hats are mixed up, and each person randomly selects one. In our notes, we proved that the expected number of matches is 1.

   Now suppose that those who were lucky enough to choose their own hat depart, while the others (those without a match) put their selected hats in the center of the room, mix them up, and then reselect. Also, suppose that this process continues until each individual has his own hat.

   Let R_n be the number of rounds necessary when there are n individuals (and hats) initially present. Use induction to show that $E[R_n] = n$, for $n \geq 1$.

   Hint 1: Condition on the number of matches X_n in the initial round, when there are n individuals, but note that you don't have to explicitly work out the pmf of X_n.

   Hint 2: In the induction step, when proving for the case $n + 1$, it will be easier to assume that the result holds for all $k \leq n$ instead of just $k = n$. In other words, use strong mathematical induction.

– END OF QUESTIONS –

Extra space:

Extra space:

Normal Table: