NATIONAL UNIVERSITY OF SINGAPORE

MA2104 - MULTIVARIABLE CALCULUS

(Semester 2: AY 2018-2019)

Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/student number only. Do not write your name.
2. This examination paper contains a total of **SIX (6)** questions and comprises **THREE (3)** printed pages.
3. Answer **ALL** questions.
4. Please start each question on a new page.
5. This is a CLOSED BOOK (with helpsheet) examination.
6. Students are allowed to use one A4 size helpsheet (both sides).
7. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1. [20 marks]

Let P be the point $(3, 3, 3)$ in $\mathbb{R}^3$.

(a) Find the distance from P to the line $\ell : \; x = 2y = z$.

(b) Let S be the surface $z = x^2 - y^2 + 3$.

(i) Find an equation of the tangent plane π to S at P.

(ii) Show that the line $\ell_1 : \; x = y, z = 3$ lies in the intersection of S and π.

(iii) Find symmetric equations of another line ℓ_2 different from ℓ_1 that passes through P and lies in the intersection of S and π.

Question 2. [15 marks]

(a) Let $f(x, y) = x^3 + y^3 + 3xy$. Find all critical points of f. At each of these critical points, determine whether f has a local maximum, a local minimum, or a saddle point.

(b) Find the maximum and minimum values of $f(x, y) = 2x + y$ subject to the constraint $x^2 + 2xy + 2y^2 = 5$.

Question 3. [15 marks]

(a) By using the transformation $T(x, y) = (x+y, y-2x)$, evaluate the double integral

$$\iint_R \sqrt{x+y}\,(y-2x)^2\,dxdy,$$

where R is the triangle in the xy-plane with vertices $A(0, 0)$, $B(3, 0)$ and $C(0, 3)$.

(b) Let $\mathbf{F}(x, y, z) = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ and let C be the curve from $(0, 0, 0)$ to $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ with parametric equations:

$$x = t + t\cos(t), \;\; y = t\sin(t), \;\; z = t, \qquad 0 \le t \le \frac{\pi}{2}.$$

Find a potential function of $\mathbf{F}$. Hence, or otherwise, evaluate the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

Question 4. [15 marks]

(a) Using Green's Theorem, evaluate the line integral

$$\oint_C (7y - e^{\sin x})\, dx + [9x - \cos(y^3 + 7y)]\, dy,$$

where C is the circle of radius 2 centred at the point $(1,1)$ and is given the counterclockwise orientation.

(b) Find the volume of the solid bounded below by the cone $\sqrt{3}z = \sqrt{x^2+y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 2z$.

Question 5. [15 marks]

(a) Rewrite the following iterated integral in the order $dydxdz$:

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)dzdydx.$$

(b) Let $\mathbf{F}(x,y,z) = \langle y^3, x, z^3\rangle$. Let C be the curve of intersection of the surface $z = xy$ and the cylinder $x^2 + y^2 = 1$. C is oriented in the counterclockwise sense when viewed from above. Evaluate

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

Question 6. [20 marks]

(a) Let $f(x,y,z)$ and $g(x,y,z)$ be functions having continuous 2nd order partial derivatives on $\mathbb{R}^3$. Let Σ be a smooth oriented surface in $\mathbb{R}^3$ with boundary C which is a simple closed curve oriented with the positive orientation. Using Stokes' theorem, or otherwise, prove that

$$\int_C f\nabla g\cdot d\mathbf{r} = \int_{-C} g\nabla f\cdot d\mathbf{r}.$$

(b) Let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F}(x,y,z) = \left\langle \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} + z^2 \right\rangle,$$

where $(x,y,z) \neq (0,0,0)$. Let S be the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ oriented with the outward pointing normal. Using the divergence theorem, or otherwise, evaluate

$$\iint_S \mathbf{F}\cdot d\mathbf{S}.$$

END OF PAPER