PYP Answer - MA1102R

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1. (a) We note that

$$f'(x) = \begin{cases} e^{x-3}(1-x) & \text{if } x \leq 3 \\ 20 - 16x + 3x^2 & \text{if } 3 < x < 5 \end{cases}$$

By definition of critical point, we solve $f'(x) = 0$, and we have $x = 1$ or $x = \frac{10}{3}$. Also, f' does not exists at $x = 3$. So the x coordinate of each critical point is $1, 3, \frac{10}{3}$.

(b) We note that $f(x) > 0$ for all $x \leq 3$. Also, Therefore, there is no absolute

Interval	$(-\infty, 1)$	1	$(1, 3)$	3	$(3, \frac{10}{3})$	$\frac{10}{3}$	$(\frac{10}{3}, 5)$	5
f	$\nearrow$	e^{-2}	$\searrow$	-1	$\searrow$	$-\frac{32}{27}$	$\nearrow$	9

maximum value of f, and the minimum value of f is $-\frac{32}{27}$ at $x = \frac{10}{3}$.

(c) We calculate f''.

$$f''(x) = \begin{cases} -e^{x-3}x & \text{if } x \leq 3 \\ -16 + 6x & \text{if } 3 < x < 5 \end{cases}$$

So $f''(x) > 0$ gives $x < 0$ and $3 < x < 5$.

(d)

$$\begin{aligned} \int_{-\infty}^{3} |f(x)| \mathrm{d}x &= \int_{-\infty}^{2} f(x) \mathrm{d}x - \int_{2}^{3} f(x) \mathrm{d}x \\ &= [e^{x-3}(3-x)]_{-\infty}^{2} + [-e^{x-3}(3-x)]_{2}^{3} \\ &= e^{-1} - 0 + e^{-1} \\ &= 2e^{-1} \end{aligned}$$

2. (a) i. Rearranging, we have $10 \int \frac{1}{x^2} \mathrm{d}x = \int (\frac{1}{t^2} - 1) \mathrm{d}t$. Therefore, we have $-10x^{-1} = -t^{-1} - t + c$. Substituting $x = 4, t = 2$ into the solution, we have $c = 0$. So $x = \frac{10t}{1+t^2}$.

ii. $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-10(t^2-1)}{(t^2+1)^2}$. Therefore, $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ gives $t = 1$. We can easily check that $x' > 0$ for $t \in (0, 1)$ and $x' < 0$ for $t \in (1, \infty)$. So the maximum distance is $x(1) = 5$.

(b) Since $z = y^{-2}$, $\frac{\mathrm{d}z}{\mathrm{d}y} = -2y^-3$. Multiply $\frac{\mathrm{d}z}{\mathrm{d}y}$ on both side of the equation, we have $x^2\frac{\mathrm{d}z}{\mathrm{d}x} + 2xz = -12\ln(x)$. Dividing both size by x^2 arrives at the result.
Using formula, we have $P(x) = \int \frac{2}{x}\mathrm{d}x = 2\ln x$. Then $v(x) = e^{P(x)} = x^2$. And $y^{-2} = z = \frac{1}{x^2}\int -12\ln(x)\mathrm{d}x = -\frac{12}{x^2}(x\ln x - x + c)$. Subsituting $x = 1, y = 1$, we have $c = \frac{11}{12}$. So $y = \sqrt{\frac{1}{-\frac{12}{x^2}(x\ln x - x + \frac{11}{12})}}$.

3. (a) Integrating by part, we have $I_n = [(2-\ln x)^n x]_1^{e^2} - \int_1^{e^2} nx(2-\ln x)^{n-1}(-\frac{1}{x})\mathrm{d}x = nI_{n-1} - 2^n$.

(b) $I_0 = \int_1^{e^2} 1\mathrm{d}x = e^2 - 1$. $I_1 = e^2 - 1 - 2 = e^2 - 3$. $I_2 = 2(e^2-3) - 4 = 2e^2 - 10$.

(c) $R = 1\times 4 + \int_1^{e^2} y\mathrm{d}x$. Let $u = \ln x$, then $R = 4 + \int_0^2 (2-u)^2\mathrm{d}u = 4 + [-\frac{1}{3}(2-u)^3]_0^2 = \frac{20}{3}$.

(d) Employ the cylindrical shell method, $V = \int_0^{e^2} 2\pi xy\mathrm{d}x = \int_0^1 2\pi x(4)\mathrm{d}x + \int_1^{e^2} 2\pi(2-\ln x)^2\mathrm{d}x = 4\pi + 2\pi I_2 = 4\pi e^2 - 16\pi$.

4. (a) Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{|\frac{1}{\sqrt{a}-\frac{1}{8}\sqrt{a}}|}$. Then $|x-a| < \delta \Rightarrow$

$$\begin{aligned}
|\sin\sqrt{x} - \sin\sqrt{a}| &= |2\sin\frac{\sqrt{x}-\sqrt{a}}{2}\cos\frac{\sqrt{x}+\sqrt{a}}{2}| \\
&\le |\sqrt{x}-\sqrt{a})(1 - \frac{1}{8}(\sqrt{x}+\sqrt{a})^2| \\
&= \delta|\frac{1}{\sqrt{x}+\sqrt{a}} - \frac{1}{8}(\sqrt{x}+\sqrt{a})| \\
&\le \delta|\frac{1}{\sqrt{a}-\frac{1}{8}\sqrt{a}}| \quad \text{due to monotone increasing} \\
\le \epsilon
\end{aligned}$$

(b) By Mean Value Theorem, we have, there exists $c \in [0, 1102]$, such that $g'(c) = 0$. Therefore, $\frac{1}{2}(f(c))^{-\frac{1}{2}}f'(c)f(1102-c) - f(c)^{\frac{1}{2}}f'(1102-c) = 0$. The result follows from rearraging of the previous equation.

(c) i. We know that $f(a) = a < \lambda a_(1-\lambda)b < b = f(b)$. Therefore, by intermediate value theorem, there exists $c \in (a,b)$ such that $f(c) = \lambda a + (1-\lambda)b$.

ii. We have $\alpha \in (a,c)$ such that $f'(\alpha) = \frac{f(c)-f(a)}{c-a} = \frac{(1-\lambda)(b-a)}{c-a}$. Simiarly, we have $\beta \in (c,b)$ such that $f'(\beta) = \frac{f(b)-f(c)}{b-c} = \frac{\lambda(b-a)}{b-c}$. Substituting these value into the equation, we have our result.

5. (a) LHS$=\frac{x^2-6x+9+216+36x}{(x-3)^2} = \left(\frac{x+15}{x-3}\right)^2$=RHS.
Therefore, arc length $L = \int_4^5 \frac{x+15}{x-3}\mathrm{d}x = 1 + 18\ln 2$.

(b)

$$\begin{aligned}
\int_2^{2017} \frac{1}{[x]^2-[x]}\mathrm{d}x &= \sum_{i=2}^{2016}\int_i^{i+1}\frac{1}{[x]^2-[x]}\mathrm{d}x \\
&= \sum_{i=2}^{2016}\int_i^{i+1}\frac{1}{i^2-i}\mathrm{d}x \\
&= \sum_{i=2}^{2016}\frac{1}{i^2-i} \\
&= \sum_{i=2}^{2016}\left(\frac{1}{i-1}-\frac{1}{i}\right) \\
&= 1-\frac{1}{2016} \\
&= \frac{2015}{2016}
\end{aligned}$$

(c) Note,

$$\begin{aligned}
\ln(\lim_{x\to 0}(1+\int_{2x}^{4x}\sin(t^2)\mathrm{d}t)^{\csc(4x^3)}) &= \lim_{x\to 0}\ln(1+\int_{2x}^{4x}\sin(t^2)\mathrm{d}t)^{\csc(4x^3)} \\
&= \lim_{x\to 0}\csc(4x^3)\ln(1+\int_{2x}^{4x}\sin(t^2)\mathrm{d}t) \\
&= \lim_{x\to 0}\frac{\ln(1+\int_{2x}^{4x}\sin(t^2)\mathrm{d}t)}{\sin(4x^3)} \\
&= \frac{1}{4}\lim_{x\to 0}\frac{4x^3}{\sin(4x^3)}\frac{\ln\left(1+\int_{2x}^{4x}\sin(t^2)dt\right)}{\int_{2x}^{4x}\sin(t^2)dt}\frac{\int_{2x}^{4x}\sin(t^2)dt}{x^3} \\
&= \frac{1}{4}1\times\frac{56}{3}\times 1 \\
&= \frac{14}{3}
\end{aligned}$$

Therefore, the required limit is $e^{\frac{14}{3}}$.

(d) Let $c = \frac{1}{t}\int_0^t f(x)\mathrm{d}x$. Then,

$$\begin{aligned}
\int_0^t (f(x) - c)^2 \,\mathrm{d}x &= \int_0^t f(x)^2\mathrm{d}x + c^2 dx - 2cf(x)\mathrm{d}x \\
&= \int_0^t f(x)^2\mathrm{d}x + tc^2 - 2c\int_0^t f(x)\mathrm{d}x \\
&= \int_0^t f(x)^2\mathrm{d}x + tc^2 - 2tc^2 \\
&= \int_0^t f(x)^2\mathrm{d}x - tc^2 \\
&= int_0^t f(x)^2\mathrm{d}x - \frac{1}{t}(\int_0^t f(x)\mathrm{d}x)^2 \geq 0
\end{aligned}$$

The result follows the last inequality.
We then take $f(x) = \frac{1}{1+x}$. Then subsituting it into the inequality shown, we have $\frac{t}{1+t} \geq t(\ln(1+t))^2$. We then take the square root to get the result.