

MA1102R AY1718 Sem 2 Answers

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1. i

$$\begin{aligned} & \ln \frac{e^x + 1}{e^x - 1} \text{ exists} \\ \iff & \frac{e^x + 1}{e^x - 1} > 0 \\ \iff & e^x - 1 > 0 \\ \iff & x > 0 \end{aligned}$$

$$\begin{aligned} & \sin^{-1} \frac{1}{\ln \frac{e^x + 1}{e^x - 1}} \text{ exists} \\ \iff & -\frac{\pi}{2} < \frac{1}{\ln \frac{e^x + 1}{e^x - 1}} < \frac{\pi}{2} \\ \iff & 0 < \frac{1}{\ln \frac{e^x + 1}{e^x - 1}} < \frac{\pi}{2}, \text{ since } \ln \frac{e^x + 1}{e^x - 1} > 0 \\ \iff & \ln \frac{e^x + 1}{e^x - 1} > \frac{2}{\pi} \\ \iff & \frac{e^x + 1}{e^x - 1} > e^{2/\pi} \\ \iff & e^x < \frac{e^{2/\pi} + 1}{e^{2/\pi} - 1} \\ \iff & x < \ln \frac{e^{2/\pi} + 1}{e^{2/\pi} - 1} \end{aligned}$$

$$\therefore 0 < x < \ln \frac{e^{2/\pi} + 1}{e^{2/\pi} - 1}$$

ii $f(x)$ is a one-to-one function on its maximal domain.

$$f^{-1}\left(\sin^{-1} \frac{1}{\ln \frac{e^x + 1}{e^x - 1}}\right) = x$$

Substitute $x = \ln(\frac{2}{y-1} + 1)$

$$\begin{aligned} f^{-1}(\sin^{-1} \frac{1}{\ln \frac{e^x+1}{e^x-1}}) &= f^{-1}(\sin^{-1} \frac{1}{\ln(1 + \frac{2}{e^x-1})}) \\ &= f^{-1}(\sin^{-1} \frac{1}{\ln(1 + \frac{2}{\frac{2}{y-1}+1-1})}) \\ &= f^{-1}(\sin^{-1} \frac{1}{\ln y}) = \ln(\frac{2}{y-1} + 1) \end{aligned}$$

Substitute $y = e^{1/\sin z}$

$$\begin{aligned} f^{-1}(\sin^{-1} \frac{1}{\ln y}) &= f^{-1}(z) \\ &= \ln(\frac{2}{y-1} + 1) \\ &= \ln(\frac{2}{e^{1/\sin z} - 1} + 1) \\ f^{-1}(x) &= \ln(\frac{2}{e^{1/\sin x} - 1} + 1) \end{aligned}$$

2.

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} \frac{a}{x-1} [3 \sin(x-1) - 2 \tan(\ln x)] \\ &= \lim_{x \rightarrow 1^-} \frac{a}{1} [3 \cos(x-1) - 2 \sec^2(\ln x) \frac{1}{x}] \\ &= a \end{aligned}$$

$\therefore a = b$

$$\begin{aligned} \lim_{x \rightarrow 1^-} f'(x) &= \lim_{x \rightarrow 1^-} \left[\frac{d}{dx} \left(\frac{a}{x-1} (3 \sin(x-1) - 2 \tan(\ln x)) \right) \right] \\ &= a \lim_{x \rightarrow 1^-} \left[\frac{3 \cos(x-1) - 2 \sec^2(\ln x)/x}{x-1} - \frac{3 \sin(x-1) - 2 \tan(\ln x)}{(x-1)^2} \right] \\ &= a \lim_{x \rightarrow 1^-} \left[\frac{2 \cos(x-1) - 2 \sec^2(\ln x)/x}{x-1} - \frac{3 \sin(x-1) - 2 \tan(\ln x) + (x-1) \cos(x-1)}{(x-1)^2} \right] \\ &= \dots \quad \text{This is left as an exercise for wolfram the reader} \\ &= a \end{aligned}$$

Define

$$d = \int_0^1 e^{[\ln(t+1)]^c} dx$$

$$\begin{aligned}
\lim_{x \rightarrow 1^-} f'(x) &= \frac{d}{dx} \left(\int_{4(x-1)}^{x^2} e^{x + [\ln(t+1)]^c} dx \right)_{x=1} \\
&= \left[e^x \frac{d}{dx} \left(\int_{4(x-1)}^{x^2} e^{[\ln(t+1)]^c} dx \right) \right]_{x=1} + \left[e^x \left(\int_{4(x-1)}^{x^2} e^{[\ln(t+1)]^c} dx \right) \right]_{x=1} \\
&= e^1 (2 \cdot e^{[\ln(2)]^c} - 4e^{[\ln 1]^c}) + e^1 \cdot d \\
&= e^1 (2 \cdot e^{[\ln(2)]^c} - 4) + e^1 \cdot d
\end{aligned}$$

On the other hand,

$$\lim_{x \rightarrow 1^+} f(x) = e^1 \cdot d$$

For f to be continuous,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) \Rightarrow a = e \cdot d$$

Since f is differentiable,

$$\lim_{x \rightarrow 1^-} f'(x) = a = \lim_{x \rightarrow 1^+} f'(x) = e \cdot d + e \cdot (2 \cdot e^{[\ln(2)]^c} - 4)$$

Therefore

$$2 \cdot e^{[\ln(2)]^c} - 4 = 0 \Rightarrow c = 1$$

$$\begin{aligned}
d &= \int_0^1 e^{[\ln(t+1)]^c} dx = \int_0^1 e^{\ln(t+1)} dx \\
&= \int_0^1 t + 1 dx \\
&= \frac{3}{2}
\end{aligned}$$

$$\therefore a = b = \frac{3}{2}e, c = 1$$

3.

$$f(1) = f(0) + f' \left(\frac{1}{2} \right) + Af''(c)$$

By Intermediate value theorem, $\exists a \in (0, 1)$ such that $f'(a) = f(1) - f(0)$

$$\therefore f' \left(\frac{1}{2} \right) + Af''(c) = f(1) - f(0) = f(a)$$

If $a = \frac{1}{2}$, then we can choose $A = 0$ and $c = 0.123$, and we are done

Otherwise, $a \neq \frac{1}{2}$

By Intermediate value theorem, $\exists b \in (a, \frac{1}{2})$ or $(\frac{1}{2}, a)$ such that $(\frac{1}{2} - a)f''(b) = f'(\frac{1}{2}) - f'(a)$

Choose $c = b$ and $A = \frac{1}{2} - a$

$$\therefore Af''(c) = \left(\frac{1}{2} - a\right) f''(b) = f(a) - f'\left(\frac{1}{2}\right)$$

$$\frac{1}{2} - 1 \leq A = \frac{1}{2} - a \leq \frac{1}{2} - 0$$

$$\therefore -\frac{1}{2} \leq A \leq \frac{1}{2}$$

4. i

$$\int (Ax^2 + B)^{-3/2} dx$$

Let $x = \sqrt{\frac{B}{A}} \tan \theta$, then $dx = \sqrt{\frac{B}{A}} \sec^2 \theta d\theta$

$$\begin{aligned} \int (Ax^2 + B)^{-3/2} dx &= \int \left(A \left(\sqrt{\frac{B}{A}} \tan \theta \right)^2 + B \right)^{-3/2} \sqrt{\frac{B}{A}} \sec^2 \theta d\theta \\ &= \int (B \tan^2 \theta + B)^{-3/2} \sqrt{\frac{B}{A}} \sec^2 \theta d\theta \\ &= \int (B \sec^2 \theta)^{-3/2} \sqrt{\frac{B}{A}} \sec^2 \theta d\theta \\ &= \int \frac{1}{B\sqrt{A} \sec \theta} d\theta \\ &= \frac{1}{B\sqrt{A}} (\sin \theta) + C \\ &= \frac{1}{B\sqrt{A}} \left(\frac{\tan \theta}{\sqrt{\tan^2 \theta + 1}} \right) + C \\ &= \frac{1}{B} \left(\frac{x}{\sqrt{Ax^2 + B}} \right) + C \end{aligned}$$

ii

$$\begin{aligned} y &= [(2x^2(2 + \sin t)^4 + 2 - \sin t)]^{-3/2} \\ [(2x^2(2 + \sin t)^4 + 2 - \sin t)]^{-3/2} &= y = \frac{1}{8} \\ \therefore 2x^2(2 + \sin t)^4 + 2 - \sin t &= 4 \\ \therefore x^2 &= \frac{2 + \sin t}{2(2 + \sin t)^4} = \frac{1}{2(2 + \sin t)^3} \\ \therefore x &= \pm \sqrt{\frac{1}{2(2 + \sin t)^3}} \end{aligned}$$

The intersection of the curves has x values $-\sqrt{\frac{1}{2(2+\sin t)^3}}$ and $\sqrt{\frac{1}{2(2+\sin t)^3}}$. Let $p = \sqrt{\frac{1}{2(2+\sin t)^3}}$

$$\begin{aligned}
\text{Area} &= \int_{-p}^p y \, dx - \frac{1}{8}(2p) \\
&= \left[\frac{x}{B\sqrt{Ax^2 + B}} \right]_{-p}^p - \frac{p}{4} && \text{where } A = 2(2\sin t)^4 \text{ and } B = 2 - \sin t \\
&= 2 \left[\frac{p}{B\sqrt{Ap^2 + B}} \right] - \frac{p}{4} \\
&= 2 \left[\frac{1}{B\sqrt{A + B/p^2}} \right] - \frac{p}{4} \\
&= \frac{2}{(2 - \sin t)\sqrt{2(2 + \sin t)^4 + (2 - \sin t)(2)(2 + \sin t)^3}} - \frac{1}{4\sqrt{2(2 + \sin t)^3}} \\
&= \frac{1}{(2 - \sin t)(2 + \sin t)\sqrt{2(2 + \sin t)}} - \frac{1}{4(2 + \sin t)\sqrt{2(2 + \sin t)}} \\
&= \frac{1}{(2 + \sin t)\sqrt{2(2 + \sin t)}} \left(\frac{1}{2 - \sin t} - \frac{1}{4} \right) \\
&= \frac{1}{(2 + \sin t)\sqrt{2(2 + \sin t)}} \left(\frac{2 + \sin t}{4(2 - \sin t)} \right) \\
&= \frac{1}{4(2 - \sin t)\sqrt{2(2 + \sin t)}}
\end{aligned}$$

iii Find the absolute min and max of $S(t) = \frac{1}{4(2 - \sin t)\sqrt{2(2 + \sin t)}}$

This is equivalent to finding the max and min of $\frac{1}{4(2-x)\sqrt{2(2+x)}}$ for $x \in [-1, 1]$

Let $f(x) = \frac{1}{4(2-x)\sqrt{2(2+x)}}$

$$f'(x) = \frac{1}{4\sqrt{2}} \left(\frac{1}{(2-x)^2\sqrt{2+x}} - \frac{1}{2(2-x)(2+x)^{3/2}} \right) = 0$$

$$\therefore (2+x) - \frac{1}{2}(2-x) = 0$$

$$\therefore x = -\frac{2}{3}$$

$$f(-1) = \frac{1}{12\sqrt{2}}, f\left(-\frac{2}{3}\right) = \frac{3\sqrt{6}}{128}, f(1) = \frac{1}{4\sqrt{6}}$$

Absolute minimal is $\frac{3\sqrt{6}}{128}$, maximal is $\frac{1}{4\sqrt{6}}$

5. i

$$\begin{aligned}
 I_{n+2} &= \int_0^{\pi/2} \sin^{n+2} x \, dx \\
 &= \int_0^{\pi/2} \sin x \sin^{n+1} x \, dx \\
 &= \left[(-\cos x) \sin^{n+1} x \right]_0^{\pi/2} + (n+1) \int_0^{\pi/2} \cos^2 x \sin^n x \, dx \\
 &= (n+1) \int_0^{\pi/2} (1 - \sin^2 x) \sin^n x \, dx \\
 &= (n+1)(I_n - I_{n+2})
 \end{aligned}$$

$$\therefore (n+2)I_{n+2} = (n+1)I_n$$

$$\therefore I_{n+2} = \frac{n+1}{n+2} I_n$$

ii

$$I_0 = \frac{\pi}{2}$$

$$I_1 = 1$$

$$\begin{aligned}
 I_9 &= \frac{8}{9} I_7 \\
 &= \frac{8}{9} \times \frac{6}{7} I_5 \\
 &= \frac{8 \times 6 \times 4 \times 2}{9 \times 7 \times 5 \times 3} \\
 &= \frac{128}{315}
 \end{aligned}$$

$$\begin{aligned}
 I_{10} &= \frac{9}{10} I_8 \\
 &= \frac{9}{10} \times \frac{7}{8} I_6 \\
 &= \frac{9 \times 7 \times 5 \times 3 \times 1}{10 \times 8 \times 6 \times 4 \times 2} I_0 \\
 &= \frac{63\pi}{512}
 \end{aligned}$$

6.

$$\begin{aligned}
 \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt &= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2} dt \\
 &= \int_0^{2\pi} \sqrt{2 - 2 \cos t} dt \\
 &= \int_0^{2\pi} \sqrt{2 - 2(1 - 2 \sin^2(t/2))} dt \\
 &= \int_0^{2\pi} 2 \sin(t/2) dt \\
 &= [-4 \cos(t/2)]_0^{2\pi} \\
 &= 8
 \end{aligned}$$

7. Let $y = 2 \tanh x - x$

We want to find all stationary points on y

$$\frac{dy}{dx} = 2 \operatorname{sech}^2 x - 1 = 0$$

$$\frac{4}{e^2 - e^{-x}} = 1$$

$$e^x + e^{-x} = 4$$

$$e^x = 2 \pm \sqrt{3}$$

$\therefore y$ has exactly 2 stationary points at $x = \ln(2 + \sqrt{3})$ and $x = \ln(2 - \sqrt{3})$

At $x = 100$, $y \approx -98$

At $x = \ln(2 + \sqrt{3})$, $y \approx 0.451$

At $x = \ln(2 - \sqrt{3})$, $y \approx -0.451$

At $x = -100$, $y \approx 98$

And since y is monotonous between those 3 intervals, there is exactly 1 solution for each interval.

\therefore the equation has exactly 3 solutions.

8.

$$(x^2 y - y) \frac{dy}{dx} + (x y^2 + x) = 0$$

$$\frac{dy}{dx} = -\frac{x(y^2 + 1)}{y(x^2 - 1)}$$

$$\int \frac{y}{y^2 + 1} dy = - \int \frac{x}{x^2 - 1} dx$$

$$\frac{1}{2} \ln(y^2 + 1) = -\frac{1}{2} \ln(x^2 - 1) + \frac{1}{2} C$$

$$\ln(y^2 + 1) = \ln\left(\frac{C}{x^2 - 1}\right)$$

$$y^2 + 1 = \frac{C}{x^2 - 1}$$

$$y = \sqrt{\frac{C}{x^2 - 1} - 1} \text{ or } y = -\sqrt{\frac{C}{x^2 - 1} - 1} \text{ (rejected since } y > 0 \text{ at } 0)$$

$$\therefore y = \sqrt{\frac{C}{x^2 - 1} - 1}$$

Sub $y = 1$ and $x = 0$

$$1 = \sqrt{-C - 1}$$

$$\therefore C = -2$$

$$\therefore y = \sqrt{-\frac{2}{x^2 - 1} - 1}$$