ST2131/MA2216 AY1718 Sem 2 Answers

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1. (a)

$$P(X = i) = pq^{i-1} = (1-q)q^{i-1} = q^{i-1} - q^i$$

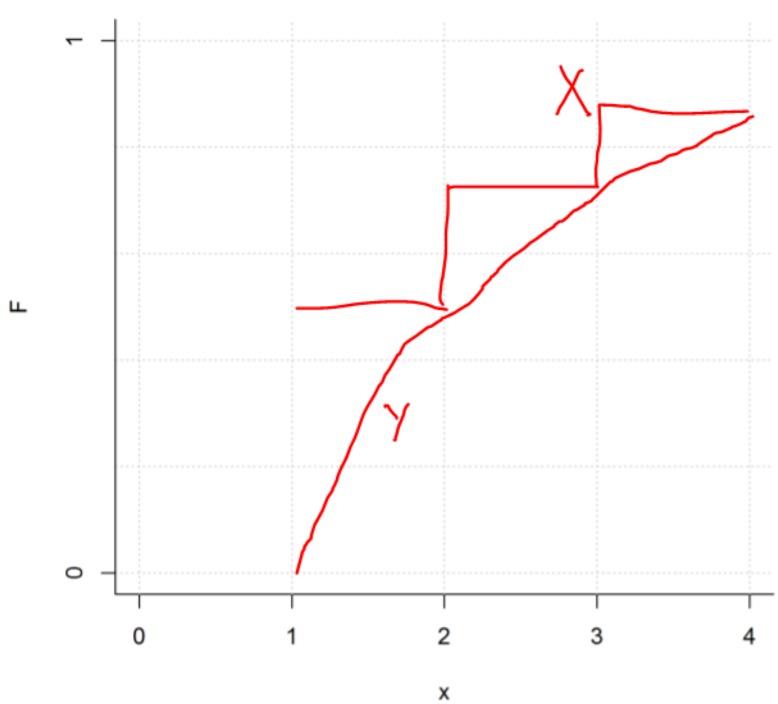
(b)

$$\begin{aligned} P(X \leq i) &= (q^0 - q^1) + (q^1 - q^2) + \cdots + (q^{i-1} - q^i) \\ &= 1 - q^i \end{aligned}$$

(c)

$$\begin{aligned} P(W = w) &= P(w \leq F_Y^{-1}(U) < w + 1) \\ &= P(F_Y(w) \leq U < F_Y(w+1)) \\ &= P(1 - q^{w-1} \leq U < 1 - q^w) \\ &= q^{w-1} - q^w \\ &= Geo(p) \end{aligned}$$

(d)



2. Let C be the total sum of car weights

$$C \sim Z(115 \times 3, 115 \times 0.3^2)$$

$$W - C \sim Z(400 - 115 \times 3, 40^2 + 115 \times 0.3^2)$$

$$\begin{aligned} P(W < C) &= P(W - C < 0) \\ &= P(Z < \frac{0 - 400 - 115 \times 3}{\sqrt{40^2 + 115 \times 0.3^2})} \\ &= P(Z < -1.3706) \\ &= 0.0852 \end{aligned}$$

3. (a)

$$
\begin{aligned}
E(Y) &= \int_0^1 E(N(x, x^2))\ dx \\
&= \int_0^1 x\ dx \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
E(Y^2) &= \int_0^1 E(N(x, x^2)^2)\ dx \\
&= \int_0^1 2x^2\ dx \\
&= \frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
Var(Y) &= \frac{2}{3} - (\frac{1}{2})^2 \\
&= \frac{5}{12}
\end{aligned}
$$

$$
E(X) = \frac{1}{2}
$$

$$
\begin{aligned}
E(XY) &= \int_0^1 E(XY|Y = x)\ dx \\
&= \int_0^1 x^2\ dx \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
cov(X, Y) &= \frac{1}{3} - (\frac{1}{2})(\frac{1}{2}) \\
&= \frac{1}{12}
\end{aligned}
$$

(b)

$$
U = g(X, Y) = Y/X, V = h(X, Y) = X
$$

$$
x = v, y = uv
$$

$$
J(x, y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 1 & 0 \end{vmatrix} = -\frac{1}{x}
$$

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}(x,y)|-x| \\
&= \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2x^2}(y-x)^2\} \\
&= \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2v^2}(uv-v)^2\} \\
&= \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2v^2}(u^2v^2-2uv^2+v^2)\} \\
&= \frac{1}{\sqrt{2\pi}} \exp\{u-u^2/2-1/2\}
\end{aligned}
$$

Which is not even dependent on v. Hence, they are independent.

4. Note that $|Y| \sim Unif(0,1)$

$$
\begin{aligned}
P(Z<k) &= \int_0^1 P(X/y<k)\ dy \\
&= \int_0^1 P(X<ky)\ dy
\end{aligned}
$$

If $0<k<2$, then $ky<2$, hence, $P(X<ky)=ky/2$

$$
P(Z<k) = \int_0^1 P(X<ky)\ dy = \int_0^1 ky/2\ dy = k/4
$$

$$
\therefore f_Z(k) = 1/4
$$

Else, if $k \geq 2$, then $2/k \leq 1$

$$
\begin{aligned}
P(Z<k) &= \int_0^1 P(X<ky)\ dy \\
&= \int_0^{2/k} P(X<ky)\ dy + \int_{2/k}^1 P(X<ky)\ dy \\
&= \int_0^{2/k} ky/2\ dy + \int_{2/k}^1 1\ dy \\
&= (k/4)(2/k)^2 + (1-2/k) \\
&= 1-1/k
\end{aligned}
$$

$$
\therefore f_Z(k) = 1/k^2
$$

5. Define $R_0 := 0$

Suppose $E(R_i)=i$ for all $i=0,1,\cdots,n-1$. We want to show that $E(R_n)=n$ by induction. The base case of $i=0$ is trivial.

$$
\begin{aligned}
E(R_n) &= E(1 + \sum_{i=0}^{n} P(X_n = i) R_{n-i}) \\
&= 1 + \sum_{i=0}^{n} P(X_n = i) E(R_{n-i}) \\
&= 1 + \sum_{i=0}^{n} P(X_n = i)(n - i) + P(X_n = 0)(E(R_n) - n) \\
&= 1 + n \sum_{i=0}^{n} P(X_n = i) - \sum_{i=0}^{n} i P(X_n = i) + P(X_n = 0)(E(R_n) - n) \\
&= 1 + n(1) - E(X_n) + P(X_n = 0)(E(R_n) - n) \\
&= 1 + n(1) - 1 + P(X_n = 0)(E(R_n) - n) \\
&= n + P(X_n = 0)(E(R_n) - n)
\end{aligned}
$$

$$
\therefore E(R_n) = n + P(X_n = 0)(E(R_n) - n)
$$

$$
\therefore E(R_n) = \frac{n - nP(X_n = 0)}{1 - P(X_n = 0)} = n
$$