MA2104 – Multivariable Calculus

Semester 1: AY2018/19

Time Allowed: 2 Hours

Instructions to students:

- Please write only your student number. Do not write your name.
- This examination paper contains FIVE questions and comprises of TEN printed pages.
- Answer ALL questions. This examination carries a total of 100 marks.
- All solutions should be written on this exam.
- Unless otherwise stated, candidates should show all their work.
- This is a closed book examination. Candidates are allowed to bring ONE A4-sized double-sided help sheet.
- Candidates may use calculators.
- Good luck!

1. (12 marks) Indicate if the following statements are true or false. No explanation is required.

   a. (2 marks) Let p be a point in $\mathbb{R}^3$ whose x-value is negative. If the spherical coordinates for p are (ρ, θ, ϕ), then $\phi < 0$.

   True / False

   b. (2 marks) The graph of any differentiable function is an orientable surface.

   True / False

   c. (2 marks) Let D be a domain in $\mathbb{R}^3$ and let f be a continuous function on D. Then

$$\iiint_D |f|\, dV \geq \iiint_D f\, dV.$$

   True / False

   d. (2 marks) Let f be a continuous function of two variables. If a, b, c, d are constants so that $b > a$ and $d > c$, then

$$\int_a^b \int_c^d f(x, y) dx dy = \int_c^d \int_a^b f(x, y) dy dx.$$

   True / False

   e. (2 marks) Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$, and let C be the unit circle centered at the origin. Suppose that for every point $(\cos\theta, \sin\theta)$ along C, we have

$$\mathbf{F}(\cos\theta, \sin\theta) = \langle -\sin\theta, \cos\theta \rangle.$$

   Then $\mathbf{F}$ is not conservative.

   True / False

   f. (2 marks) If the integral of a vector field $\mathbf{F}$ along C is 6, then the integral of $\mathbf{F}$ along $-C$ is also 6.

   True / False

2. (12 marks) Write down the answer to the following questions. No explanation is required.

a. (3 marks) Let $g(y) = \begin{cases} \sqrt{2-y} & \text{if } 1 \leq y \leq 2 \\ 1 & \text{if } 0 \leq y \leq 1 \end{cases}$. Rewrite the integral

$$\int_0^2 \int_0^{g(y)} f(x,y)\,dx\,dy$$

in the order $dy\,dx$.

Ans: ____________________

b. (3 marks) Let D be the square in $\mathbb{R}^2$ whose four vertices are $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, and let C be its boundary curve, oriented in the positive direction. Integrate the vector field $\mathbf{F}(x,y) = \langle e^{\sin(x)}, 3x + y^{\cos(y)} \rangle$ along C.

Ans: ____________________

c. (3 marks) Let T be a planar transformation from the (u,v)-plane to the (x,y)-plane that is a bijection, let S be a domain in the (u,v)-plane, and let $R = T(S)$. If $\frac{\partial(x,y)}{\partial(u,v)} = 2$ at every point, and the area of R is 8, what is the area of S?

Ans: ____________________

d. (3 marks) Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two vector fields on $\mathbb{R}^3$ that satisfy

$$\mathbf{F}_1 = \mathbf{F}_2 + \langle \sin x, e^{-y}, z\cos z \rangle.$$

Let C be a closed simple oriented curve in $\mathbb{R}^3$. If the integral of $\mathbf{F}_1$ along C is 7, what is the integral of $\mathbf{F}_2$ along C?

Ans: ____________________

3. (30 marks) You bought a chocolate cake whose base is a disk of radius 2, and whose top is a hemisphere of radius 2. Based on the way the cake was made, you know that the density of any point in the cake is proportional to its height above the base of the cake. Model the cake as the solid in $\mathbb{R}^3$ above the (x, y)-plane, and inside the sphere of radius 2 centered at the origin.

   a. (10 marks) Suppose that the mass of the cake is 1. Calculate the density function f of the cake.

b. (10 marks) You cut away the parts of the cake that lies outside the cylinder of radius 1 centered about the z-axis. Suppose that the density function of the cake is $f(x, y, z) = 2z$. Calculate the mass of the remaining part of the cake.

c. (10 marks) You cut out the slice of the cake bounded by the (x, z)-plane and the plane $y = x + 2$. Calculate the volume of this slice.
Hint: Model the cake so that its top is the upper hemisphere centered at $(2, 0, 0)$.

4. (26 marks) Let $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ be a vector field on $\mathbb{R}^3$.

   a. (6 marks) Is $\mathbf{F}$ a conservative vector field on $\mathbb{R}^3$? Justify your answer.

   b. (10 marks) Let S be the graph of the function $g(x, y) = 1 - y^2 - x^2$ over the unit disc D, and let $\mathbf{S}$ be S equipped with the upward orientation. Calculate the flux of $\mathbf{F}$ across $\mathbf{S}$.

c. (10 marks) Let E be the cube whose faces are the planes $x = -1$, $x = 1$, $y = -1$, $y = 1$, $z = -1$, and $z = 1$. Let S_1 be the face of E in the $z = 1$ plane, and let S_2 be the surface formed by the remaining five faces of E. Finally, let $\mathbf{S}_1$ and $\mathbf{S}_2$ be S_1 and S_2 equipped with the orientations that point away from E. Calculate the flux of $\mathbf{F}$ across $\mathbf{S}_2$.

5. a. (10 marks) Let E be the solid bounded by the planes $y+x=2$, $y+x=0$, $2x+z=1$, $2x+z=3$, $x=2y-1$ and $x=2y+1$. Calculate the volume of E.

b. (10 marks) Write the integral

$$\int_{-1}^{1} \int_{0}^{1-y^2} \int_{0}^{1-z} f(x, y, z) dx\, dz\, dy$$

as an interated integral in the order $dy\, dz\, dx$.