PYP Answer - MA1104 AY1617Sem2

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1. (a) Let $F(x, y, z) := xz + 2x^2y + y^2z^3$. One normal vector of the tangent plane is the gradient vector $\nabla F(2, 1, 1)$.
   Since $\nabla F = \langle z + 4xy, 2x^2 + 2z^3y, x + 3y^2z^2\rangle$, $\mathbf{n} := \nabla F(2, 1, 1) = \langle 9, 10, 5\rangle$.
   Then the equation of the plane is
   $$9x + 10y + 5z = 9 \times 2 + 10 \times 1 + 5 \times 1 = 33$$
   (b) The tangent line is not only perpendicular to $\mathbf{n}$, but also perpendicular to $\langle 1, 1, 1\rangle$. Therefore, one directional vector for the tangent line is $\mathbf{d} := \langle 9, 10, 5\rangle \times \langle 1, 1, 1\rangle = \langle 5, -4, -1\rangle$. It has length $\sqrt{5^2 + (-4)^2 + (-1)^2} = \sqrt{42}$.
   Then the required unit vector can be $\frac{\langle 5,-4,-1\rangle}{\sqrt{42}}$ or $-\frac{\langle 5,-4,-1\rangle}{\sqrt{42}}$.
   (c) The parametric equation of the tangent line is $\mathbf{r} = \langle 5, -4, -1\rangle\lambda + \langle 2, 1, 1\rangle$. By letting $z = 0$, we have $0 = -\lambda + 1 \Rightarrow \lambda = 1$.
   Therefore, the intersection is $(7, -3, 0)$.

2. (a) By chain rule,
   $$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\\ &= (2x + y^3)v^2 + 3xy^2\\ &= 58\end{aligned}$$
   Similarly, we have $\frac{\partial z}{\partial v} = 151$ and $\frac{\partial z}{\partial w} = 27$.
   (b) i. We note that $z = \sqrt{4 - x^2 - y^2} := g(x, y)$, with $\frac{\partial g}{\partial x} = -(1 - x^2 - y^2)^{-\frac{1}{2}}x$ and$\frac{\partial g}{\partial y} = -(1 - x^2 - y^2)^{-\frac{1}{2}}y$

By surface area formula,

$$\begin{aligned}
\text{Area}(S) &= \int_S 1\mathrm{d}S \\
&= \iint_D 1\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\mathrm{d}A \\
&= \iint_D \sqrt{\frac{1}{4-x^2-y^2}}\mathrm{d}A \text{ where } D = \{(x,y) : x^2+y^2 < 3\} \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} (4-r^2)^{-\frac{1}{2}} \cdot r\mathrm{d}r\mathrm{d}\theta \\
&= 2\pi \times \left[-(4-r^2)^{\frac{1}{2}}\right]_0^{\sqrt{3}} \\
&= 2\pi
\end{aligned}$$

ii.

3. (a) Note the extreme value must occur at the boundary of D: $g(x,y,z) := x^2+y^2+2z^2 = 6$. Employ Lagrange multiplier, $\nabla f = \langle 3y, 3x, 6\rangle$ and $\nabla g = \lambda\langle 2x, 2y, 4z\rangle$, we arrive at

$$\begin{cases} 3y &= 2\lambda x \\ 3x &= 2\lambda y \\ 6 &= 4\lambda z \end{cases}$$

The first two equation gives $9xy = 4\lambda^2 xy$. Suppose $xy = 0$, then since $\lambda \neq 0$, both x and y are 0, which does not produce points on the boundary of D. As a result, we have $9 = 4\lambda^2 \Rightarrow \lambda = \pm\frac{3}{2}$.
There are four solutions.
$P_1 = (\sqrt{2}, \sqrt{2}, 1)$; $f(P_1) = 12$. $P_2 = (-\sqrt{2}, -\sqrt{2}, 1)$ with $f(P_2) = 12$. These two points produce the maximum value of 12.
$P_3 = (\sqrt{2}, -\sqrt{2}, -1)$ with $f(P_3) = -12$ and $P_4 = (-\sqrt{2}, \sqrt{2}, -1)$ with $f(P_4) = -12$. These two points produce the minimum value of -12.

(b) $\|\nabla f\| = 3\sqrt{x^2+y^2+4}$. It achieves its maximum value when

$$\begin{cases} \frac{\partial\|\nabla f\|}{\partial x} &= 0 \\ \frac{\partial\|\nabla f\|}{\partial y} &= 0 \\ \frac{\partial\|\nabla f\|}{\partial z} &= 0 \end{cases} \Rightarrow \begin{cases} 3(x^2+y^2+4)^{-\frac{1}{2}}x &= 0 \\ 3(x^2+y^2+4)^{-\frac{1}{2}}y &= 0 \\ 0 &= 0 \end{cases}$$

So the set of points are $\{(x,y,z) : x = 0 \text{ and } y = 0 \text{ and } -\sqrt{3} \leq z \leq \sqrt{3}\}$.

4. (a) By a change of order of integration, the original integral equals

$$\begin{aligned}
&\int_0^1 \int_0^{\sqrt{y}} 2x \sin(y^2) \mathrm{d}x\mathrm{d}y \\
=&\int_0^1 [x^2 \sin(y^2)]_0^{\sqrt{y}} \mathrm{d}y \\
=&\int_0^1 y \sin(y^2) \mathrm{d}y \\
=&[-\frac{1}{2}\cos(y^2)]_0^1 \\
=&\frac{1}{2} - \frac{1}{2}\cos 1
\end{aligned}$$

(b)

(c) The iterated integral equals to

$$\int_0^3 \int_{3-y}^3 \int_0^{\sqrt{9-x^2}} f(x, y, z) \mathrm{d}z\mathrm{d}x\mathrm{d}y$$

5. (a)

$$\begin{aligned}
\mathrm{Area}(D) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{1+\cos\theta} 1\mathrm{d}r\mathrm{d}\theta - \frac{1}{2}\pi \times 1^2 \\
&= \frac{1}{2}\pi + 2
\end{aligned}$$

(b) By curl formula,

$$\mathrm{curl}(\mathbf{F}) = \langle -2z, -3x^2, -5 \rangle$$

(c) We note that C is a closed, clockwise-oriented curve on the surface $2xy - z = 0$. So we can take the downward pointing normal

$$\mathbf{n} = \frac{\langle 2y, 2x, -1 \rangle}{\sqrt{1 + 4x^2 + 4y^2}}$$

Employ Stokes' Theorem, the line integral equals to

$$\begin{aligned}
&\iint_{x^2+y^2\leq 1} \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}\mathrm{d}S \\
=&\iint_{x^2+y^2\leq 1} \underbrace{\langle -2z, -3x^2, -5 \rangle}_{\mathrm{curl}(\mathbf{F})} \cdot \underbrace{\frac{\langle 2y, 2x, -1 \rangle}{\sqrt{1 + 4x^2 + 4y^2}}}_{\mathbf{n}} \cdot \underbrace{\sqrt{1 + 4x^2 + 4y^2}\mathrm{d}x\mathrm{d}y}_{\mathrm{d}S} \\
=&\iint_{x^2+y^2\leq 1} (-8xy^2 - 6x^3 + 5)\mathrm{d}x\mathrm{d}y \\
=& 5\pi
\end{aligned}$$

6. (a) i. $\frac{\mathrm{d}}{\mathrm{d}x}(xy + e^x + x\cos y) = y + e^x + \cos y \neq \frac{\mathrm{d}}{\mathrm{d}y}(y + ye^x + \sin y) = 1 + e^x + \cos y$. Therefore, $\mathbf{G}$ is not conservative.

ii.

$$\begin{aligned}
\int_C \mathbf{G}\cdot \mathrm{d}\mathbf{r} &= \int_C \langle \frac{1}{2}y^2 + ye^x + \sin y, xy + e^x + x\cos y\rangle \cdot \mathrm{d}\mathbf{r} + \int_C \langle y - \frac{1}{2}y^2, 0\rangle \cdot \mathrm{d}\mathbf{r} \\
&= [\frac{1}{2}xy^2 + ye^x + x\sin y]_{(x,y)=(0,1)}^{(x,y)=(\frac{\pi}{2},0)} + \int_0^{\frac{\pi}{2}} \langle -\frac{1}{2}\cos^2 t, 0\rangle \cdot \langle 1, -\sin t\rangle \mathrm{d}t \\
&= -1 - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2(t)\mathrm{d}t \\
&= -1 - \frac{\pi}{8}
\end{aligned}$$

(b) i. S_1 has the following parametrisation $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, 7\rangle$, where $u \in [0,1]$ and $v \in [0, 2\pi]$. As a result, $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, u\rangle$. Therefore,

$$\begin{aligned}
\iint_{S_1} \mathbf{F}\cdot \mathrm{d}\mathbf{S} &= \int_0^1 \int_0^{2\pi} -2\times 7(u\cos v + u\sin v - 1)u\mathrm{d}v\mathrm{d}u \\
&= 14\pi
\end{aligned}$$

ii. We note that $\mathrm{div}\mathbf{F} = 2$. Therefore,

$$\begin{aligned}
\iiint_E \mathrm{div}\mathbf{F}\mathrm{d}V &= \iiint_E 2\mathrm{d}V \\
&= 2\iint_D (7 + x^2 + y^2)\mathrm{d}A \text{ where } D \text{ is the circle of radius 1 centered at origin} \\
&= 2\int_0^1 \int_0^{2\pi} 7 + r^2 \mathrm{d}\theta\mathrm{d}r \\
&= \frac{88}{3}\pi
\end{aligned}$$

iii. We parametrise the surface S_3 parallel to the z axis as following: $\mathbf{r}(u,v) = \langle \cos u, \sin u, v\rangle$ where $u \in [0, 2\pi]$ and $v \in [-1, 7]$. As a result, an outward pointing normal is $\mathbf{r}_u \times \mathbf{r}_v = \langle \cos u, \sin u, 0\rangle$. Then we evaluate

$$\iint_{S_3} \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_D \langle 0, 0, -2v(\cos u + \sin u - 1)\rangle \cdot \langle \cos u, \sin u, 0\rangle \mathrm{d}A = 0$$

Then, by divergence theorem, we have $\iint_{S_2} \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \frac{88}{3}\pi - 14\pi - 0 = \frac{46}{3}\pi$.