PYP Answer - MA2101 AY1516Sem1

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1. The characteristic polynomial of A is

$$p(x) = |xI - A| = \det\begin{pmatrix} x & -2 \\ -2 & x-3 \end{pmatrix} = (x-4)(x+1)$$

So the eigenvalues of A are 4 or -1. Solving $(xI - A)v = 0$ for eigenvalues $x = 4, -1$ respectively, we have $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ associated with $x = 4$ and $v = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ associated with $x = -1$. So

$$D = \begin{pmatrix} 4 & \\ & -1 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

2. Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Y = PZ = \begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. So $Y' = PZ'$ and we note that

$$\begin{cases} y_1 = z_1 + z_2 \\ y_2 = z_1 \end{cases}$$

and $PZ' = APZ$, which implies $Z' = P^{-1}APZ = \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} Z$.
Therefore,

$$\begin{cases} z_1' = 2z_1 + z_2 & (1) \\ z_2' = 2z_2 & (2) \end{cases}$$

Solving (2),

$$z_2 = Ae^{2x}$$

and solving (1) using hint,

$$z_1 = Axe^{2x} + Be^{-2x}$$

Substituting back, we have

$$\begin{cases} y_1 = Axe^{2x} + Ae^{2x} + Be^{-2x} \\ y_2 = Axe^{2x} + Be^{-2x} \end{cases}$$

3. (a) We show that $T^{-1}(W)$ respects additiona and scalar multiplication.
For $u_1, u_2 \in T^{-1}(W)$ and $c \in F$, we have $T(u_1), T(u_2) \in W$, and $T(u_1+u_2) \in W$, so $u_1 + u_2 \in T^{-1}(W)$. Also, $cT(u_1) \in W$ and $T(cu_1) \in W$, so $cu_1 \in T^{-1}(W)$, by definition of $T^{-1}(W)$.

   (b) $\dim U = \text{rank}T + \text{nullity}T = \dim V + \text{nullity}T$, by the rank nullity theorem and surjectivity of T.
Now consider $T \mid_{T^{-1}W} : T^{-1}(W) \to W$. This is again surjective, and together with the rank nullity theorem we have

$$\dim T^{-1}(W) = \text{rank}T \mid_{T^{-1}(W)} + \text{nullity}T \mid_{T^{-1}(W)} = \dim W + \text{nullity}T \mid_{T^{-1}(W)}$$

Combining these we get,

$$\dim U + \dim W = \dim V + \dim T^{-1}(W) + \text{nullity}T - \text{nullity}T \mid_{T^{-1}(W)}$$

However, we know that $\text{nullity}T = \text{nullity}T \mid_{T^{-1}(W)}$ because $0 \in W$. Hence, we prove the result.

4. (a) Suppose $Qv = \lambda v$, then

$$\langle Qv, Qv \rangle = \langle \lambda v, \lambda v = \lambda\overline{\lambda}\langle v, v \rangle$$

Since Q is orthogonal, we have

$$\langle Qv, Qv \rangle = \langle QQ^t v, v \rangle = \langle v, v \rangle$$

Hence, $\lambda\overline{\lambda} = 1$ for all eigenvalue λ. And since the determinant of a real matrix must have real coefficients, by Fundamental Theorem of Algebra, it consists of at least 1 real root, and for that real root, we have $\lambda^2 = 1$.

   (b) False, consider $\begin{pmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix}$, which has 1 real and 2 complex eigenvalue, so $\lambda^2 \neq 1$ for those two imaginery eigenvalues.

5. (a) Suppose $v_1, v_2 \in W^\perp$ and $c \in \mathbb{R}$. We shall show that $W^\perp$ respects addition and scalar multiplication.

Addition Since $v_1, v_2 \in W^\perp$, $\langle v_i, w \rangle = 0$ for $i = 1, 2$. So $\langle v_1 + v_2, w \rangle = 0$, and therefore, $v_1 + v_2 \in W^\perp$.

Scalar Multiplication We note that $c\langle v_1, w \rangle = 0$ gives $\langle cv_1, w \rangle = 0$, so $cv_1 \in W^\perp$.

   (b) Yes. For any $v \in W^\perp$,i.e., $\langle v, w \rangle = 0$, we want to show that $T(v) \in W^\perp$, i.e., $\langle T(v), w \rangle = 0$.
Note $\langle T(v), w \rangle = \langle v, T^*(w) \rangle = \langle v, w' \rangle$ for all $w \in W$ since W is T^* invariant. And since $v \in W^\perp$, $\langle T(v), w \rangle = 0$ for all $w \in W$, and this proves the claim.

   (c) Let V be 2×1 real matrix, and inner product be the dot product. Let $T(v) = \begin{pmatrix} 1 & 3 \\ & 2 \end{pmatrix}$ and thus $T^* = \begin{pmatrix} 1 & \\ 3 & 2 \end{pmatrix}$. $W = \text{Span}\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\}$ and $W^\perp = \text{Span}\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\}$.
However, $T^* \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \end{pmatrix} \notin W^\perp$. Hence, the claim is false.

6. Basically $f(x)$ kills A. Also, as A is self adjoint, A is diagonalisable, so the minimum polynomial has at most degree 1 for each factor. Therefore, possible $m_A(x)$ are

$$
\begin{aligned}
m_A(x) &= x-1 \\
m_A(x) &= x-2 \\
m_A(x) &= x-3 \\
m_A(x) &= (x-1)(x-2) \\
m_A(x) &= (x-1)(x-3) \\
m_A(x) &= (x-2)(x-3) \\
m_A(x) &= (x-1)(x-2)(x-3)
\end{aligned}
$$

7. (a) For any $k_m \in K_m, T^m(k_m) = 0$, therefore, $T^{m+1}(k_m) = T(0) = 0$. So $k_m \in K_{m+1}$ and therefore $K_m \subseteq K_{m+1}$.

   (b) $\dim K_r$ admits a non-decreasing sequence as r increases. And we require, $\dim K_r \leq \dim K_{r+1} \leq \dim K_{r+2} \leq \cdots \leq \dim V$.Therefore, there is at most $\dim V$ strict inequality. Therefore, after some $r \geq 1$, $K_r = K_{r+s}$ for all $s \geq 1$.

   (c) No. Let V be the vector space of infinite sequence of real numbers $(x_0, x_1, x_2 \ldots)$, under componentwise addition and scalar multiplication. The linear transformation is the right shift operator $T : (x_0, x_1, x_2, \ldots) \mapsto (0, x_0, x_1, \ldots)$.
   Consider $v = T^s(1, 0, 0, 0, \ldots) \in K_s$ but not in K_{s+1}, as the $s+1$th coordinate in v is 1 but that in K_{s+1} can only be 0. Therefore, $K_s \neq K_{s+1}$ for all $s \geq 1$.

8. (a) No. A is not necessarily self-adjoint. Let $A = \begin{pmatrix} 1 & 0.5 \\ 0.3 & 1 \end{pmatrix}$.

   bi By Principal Axis Theorem, we can take orthonormal basis B such that $[A]_B$ is diagonal: $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Therefore, we can choose $[G]_B = \mathrm{diag}(\lambda_i^{\frac{1}{4}}, \ldots, \lambda_n^{\frac{1}{4}})$.

   bii By Principal Axis Theorem, $A = PDP^*Q^*DQ$ where D is diagonal. Write $D = M^2$, where $M_{ii} = D_{ii}^{\frac{1}{2}}$ for $i = 1, \ldots, n$ with 0 on other entries. Then $A = Q^*MMQ = Q^*M * MQ = (MQ)^*MQ = H^*H$.