MA1102R AY1718 Sem 1 Answers

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1. (i)

$$f(x) = (x^3 + 4x^2 + 11x + 14)e^{-x}$$

$$\begin{aligned} f'(x) =& -(x^3 + 4x^2 + 11x + 14)e^{-x} + (3x^2 + 8x + 11)e^{-x} > 0 \\ \iff& (3x^2 + 8x + 11) - (x^3 + 4x^2 + 11x + 14) > 0 \\ \iff& -x^3 - x^2 - 3x - 3 > 0 \\ \iff& (-x-1)(x^2+3) > 0 \\ \iff& -1 > x \end{aligned}$$

$\therefore f$ is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$

(ii) $f(-1) = 6e$ is a local maximum. There is no local minimum.

(iii)

$$f'(x) = (-x^3 - x^2 - 3x - 3)e^{-x}$$

$$\begin{aligned} f''(x) =& (-3x^2 - 2x - 3)e^{-x} - (-x^3 - x^2 - 3x - 3)e^{-x} \\ =& (x^3 - 2x^2 + x)e^{-x} > 0 \\ \iff& x^3 - 2x^2 + x > 0 \\ \iff& x(x-1)^2 > 0 \\ \iff& x > 0 \end{aligned}$$

f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$

(iv) $f(0) = 14$
$(0, 14)$

2. (a) For any $\epsilon > 0$, choose $\delta = \min(\epsilon, 1)$
Then for all x such that $0 < |x - 2| < \delta$

$$\begin{aligned} \left|\frac{x}{x^2+2} - \frac{1}{3}\right| = \left|\frac{3x - (x^2+2)}{3(x^2+2)}\right| &= \left|\frac{(x-1)(x-2)}{3(x^2+2)}\right| \\ &< |x-1|\left|\frac{1}{3(x^2+2)}\right|\epsilon && \text{since } |x-2| < \epsilon \\ &< 2 \times \frac{1}{3 \times 2} \times \epsilon && \text{since } |x-1| < 2 \text{ and } \frac{1}{x^2+2} < \frac{1}{2} \\ &< \epsilon \end{aligned}$$

(b)

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^3}{n^2(n^2+i^2)} &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{(\frac{i}{n})^3}{1+(\frac{i}{n})^2} \\
&= \int_0^1 \frac{x^3}{1+x^2}\ dx \\
&= \int_0^1 x-\frac{x}{1+x^2}\ dx \\
&= \left[\frac{1}{2}x-\frac{1}{2}\ln(x^2+1)\right]_0^1 \\
&= \frac{1}{2}-\frac{1}{2}\ln 2
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to0^+}\left(\frac{e^2-1}{x}\right)^{1/x} &= \lim_{x\to0^+}\exp\left(\frac{1}{x}\ln\left(\frac{e^x-1}{x}\right)\right) \\
&= \lim_{x\to0^+}\exp\left(\left(\frac{x}{e^x-1}\right)\left(\frac{e^x}{x}-\frac{e^x-1}{x^2}\right)\right) && \text{By L'Hôpital's rule} \\
&= \lim_{x\to0^+}\exp\left(\frac{xe^x-e^x+1}{x(e^x-1)}\right) \\
&= \lim_{x\to0^+}\exp\left(\frac{xe^x}{xe^x+e^x-1)}\right) && \text{By L'Hôpital's rule} \\
&= \lim_{x\to0^+}\exp\left(\frac{e^x}{e^x+\frac{e^x-1}{x}}\right) \\
&= \exp\left(\frac{1}{2}\right) \\
&= \sqrt{e}
\end{aligned}$$

3. Let the angle of the sector be θ

$$2r+r\theta=50 \implies \theta=\frac{50}{r}-2$$

$$\begin{aligned}
\text{Area} &= \frac{1}{2}r^2\theta \\
&= \frac{1}{2}r^2\left(\frac{50}{r}-2\right) \\
&= 25r-r^2 \\
&= r(25-r) \\
&\le \left(\frac{25}{2}\right)^2 && \text{By AMGM inequality, with equality at r=12.5}
\end{aligned}$$

$r=12.5$ m

4. (a)
$$\ln y = (\sec x)\ln(\tan x) + (\tan x)\ln(\sec x)$$
$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \sin x \ln(\tan x) + \operatorname{cosec} x \sec^2 x + \sec^2 x \ln(\sec x) + \sin^2 x \sec^2 x$$
If $x = \frac{\pi}{4}$, then $y = 1^{\sqrt{2}}\sqrt{2}^1 = \sqrt{2}$
Sub $x = \frac{\pi}{4}$ and $y = \sqrt{2}$ to the equation
$$\frac{1}{\sqrt{2}}\frac{dy}{dx} = 0 + 2\sqrt{2} + 2\ln\sqrt{2} + 1$$
$$\frac{dy}{dx} = 4 + 2\sqrt{2}\ln\sqrt{2} + \sqrt{2}$$

(b) For $x \neq 0$:
$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_0^{x^2} f(t)\ dt \\ &= 2xf(x^2) \\ &= \frac{2\sin(x^2)}{x} \end{aligned}$$
For $x = 0$:
$$F'(x) = 0$$
$\therefore F'(x) = 0$ for $x = \sqrt{k\pi}, k \in \mathbb{Z}$
To check if it is a local max or min, we check the concavity
For $x \neq 0$:
$$F''(x) = 4\cos(x^2) - \frac{2\sin(x^2)}{x^2}$$
For $x = 0$:
$$F''(x) = 2$$
$F''(\sqrt{k\pi} = 4(-1)^k$ for $k \neq 0$ and $F''(0) = 2$
f attains local min at $x = \sqrt{k\pi}$ for even k and local max at $x = \sqrt{k\pi}$ for odd k.

(c)
$$f''(x) < 0 \implies f'(x) \text{ is decreasing} \implies f'(x) < 0 \implies f(x) \text{ is decreasing}$$
Either $\lim_{x\to\infty} f(x) = -\infty$ or $\lim_{x\to\infty} f(x) = k$ for some constant k
If $\lim_{x\to\infty} f(x) = k$, then $\lim_{x\to\infty} f'(x) = 0$. However, $f'(1) = f'(0) + \int_0^1 f''(x)\,dx < f'(0) = 0$. Hence, $\lim_{x\to\infty} f'(x) > f'(1)$, a contradiction.
$\therefore \lim_{x\to\infty} f(x) = -\infty$
$f(x)$ is decreasing and $\lim_{x\to\infty} f(x) = -\infty \implies$ exactly 1 root

5.

$$y^2 = 2x = 8 - x^2$$

$$\therefore x = 2, y = \pm 2$$

The curves intersects at $(2,2)$ and $(2,-2)$

(i)

$$x^2 + y^2 = 8 \implies x = \sqrt{8 - y^2}$$

$$y^2 = 3x \implies x = \frac{1}{2}y^2$$

$$\begin{aligned}
\text{Area} &= \int_{-2}^{2} \sqrt{8 - y^2} - \frac{1}{2}y^2 \; dy \\
&= \int_{-2}^{2} \sqrt{8 - y^2} \; dy - \left[-\frac{1}{6}y^3 \right]_{-2}^{2} \qquad\qquad \text{sub } y = \sqrt{8}\sin\theta \\
&= \int_{-\pi/4}^{\pi/4} 8\cos^2\theta \; d\theta - \frac{8}{3} \\
&= 4\int_{-\pi/4}^{\pi/4} \cos(2\theta) + 1 \; d\theta - \frac{8}{3} \\
&= 2[\sin(2\theta) + 2\theta]_{-\pi/4}^{\pi/4} - \frac{8}{3} \\
&= \frac{4}{3} + 2\pi
\end{aligned}$$

(ii)

$$x^2 + y^2 = 8 \implies y = \sqrt{8 - x^2}$$

$$y^2 = 3x \implies y = \sqrt{2x}$$

$$\begin{aligned}
\text{Volume} &= 2\left[\int_0^2 \sqrt{2x}(2\pi x) \; dx + \int_2^{\sqrt{8}} \sqrt{8 - x^2}(2\pi x) \; dx \right] \\
&= 4\pi \left[\left[\frac{2}{5}\sqrt{2}x^{5/2} \right]_0^2 + \left[-\frac{1}{3}(8 - x^2)^{3/2} \right]_2^{\sqrt{8}} \right] \\
&= 4\pi \left[\frac{16}{5} + \frac{8}{3} \right] \\
&= \frac{352}{15}\pi
\end{aligned}$$

6. (i)

$$\begin{aligned}
\int \frac{x\ln x}{(1+x^2)^2}\,dx &= \left(-\frac{1}{2}\right)\frac{\ln x}{1+x^2} - \int\left(-\frac{1}{2}\right)\frac{1}{x(1+x^2)}\,dx \qquad \text{sub } x=\tan\theta \\
&= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\int \frac{1}{\tan\theta\sec^2\theta}\sec^2\,d\theta \\
&= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\int \frac{\cos\theta}{\sin\theta}\,d\theta \\
&= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\ln(\sin\theta) + C \\
&= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\ln\left(\frac{x}{\sqrt{1+x^2}}\right) + C
\end{aligned}$$

(ii)

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{1}{2}\ln\frac{x}{\sqrt{1+x^2}} - \frac{\ln x}{2(1+x^2)}\right) &= \frac{1}{2}\ln(1) \\
&= 0 \\
\lim_{x\to 0^+}\left(\frac{1}{2}\ln\frac{x}{\sqrt{1+x^2}} - \frac{\ln x}{2(1+x^2)}\right) &= \lim_{x\to 0^+}\left(\frac{1}{2}\ln x - \frac{\ln x}{2(1+x^2)} - \frac{1}{2}\ln\sqrt{1+x^2}\right) \\
&= \lim_{x\to 0^+}\left((\ln x)\left(\frac{1}{2} - \frac{1}{2(1+x^2)}\right)\right) \\
&= \frac{1}{2}\lim_{x\to 0^+}\left(\frac{x^2\ln x}{1+x^2}\right) \\
&= \frac{1}{2}\lim_{x\to 0^+}(x^2\ln x) \\
&= \frac{1}{2}\lim_{x\to 0^+}\frac{1/x}{-2x^{-3}} \qquad \text{By L'Hôpital's rule} \\
&= 0
\end{aligned}$$

$$\therefore \int_0^\infty \frac{x\ln x}{(1+x^2)^2}\,dx = 0$$

7. (a)

$$y = \frac{1}{x} + \frac{1}{z} \implies z = 1 \text{ at } x = 1$$

$$\frac{dy}{dx} = -\frac{1}{x^2} - \frac{1}{z^2}\frac{dz}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{x^2} - \frac{1}{z^2}\frac{dz}{dx} = \left(\frac{1}{x}+\frac{1}{z}\right)^2 - \frac{1}{x}\left(\frac{1}{x}+\frac{1}{z}\right) - \frac{1}{x^2}$$

$$-\frac{1}{z^2}\frac{dy}{dx} = \frac{1}{z^2} + \frac{1}{xz}$$

$$\frac{dy}{dx} + \frac{z}{x} + 1 = 0$$

Let $w = \frac{z}{x}$. Then $w = 1$ at $x = 1$.

$$wx = z \implies x\frac{dw}{dx} + w = \frac{dz}{dx}$$

$$\frac{dz}{dx} = x\frac{dw}{dx} + w = -1 - w$$

$$\int \frac{1}{-1-2w}\ dw = \int \frac{1}{x}\ dx$$

$$-\frac{1}{2}\ln|1+2w| = \ln(x) + C$$

Substitute $x = 1, w = 1$

$$-\frac{1}{2}\ln 3 = C$$

Therefore,

$$-\frac{1}{2}\ln|1+2w| = \ln\frac{x}{\sqrt{3}}$$

$$\frac{1}{\sqrt{1+2w}} = \frac{x}{\sqrt{3}}$$

$$w = \frac{1}{2}\left(\frac{3}{x^2} - 1\right)$$

$$z = wx = \frac{3}{2x} - \frac{x}{2} = \frac{3-x^2}{2x}$$

$$y = \frac{1}{x} + \frac{2x}{3-x^2}$$

(b)

$$\int \frac{4h-h^2}{\sqrt{h}}\ dh = \int -1\ dt$$

$$\frac{8}{3}h^{3/2} - \frac{2}{5}h^{5/2} + C = -t$$

Substitute $t = 0, h = 4$

$$\frac{64}{3} - \frac{64}{5} + C = 0 \implies C = -\frac{128}{15}$$

Therefore,

$$\frac{8}{3}h^{3/2} - \frac{2}{5}h^{5/2} - \frac{128}{15} = -t$$

Substitute $h = 0$

$$t = \frac{128}{15}$$

128/15 minutes

8. We first want to show that $\forall x \leq 0.5, |f(x)| \leq Mx$.

   Suppose $\exists a \in (0,1)$ such that $|f(a)| > Mx$, then by mean value theorem, $\exists b \in (0,a)$ such that $|f'(b)| = |(f(a)-0)/(a-0)| > M$, a contradiction.

   Therefore, $\forall x \leq 0.5, |f(x)| \leq Mx$.

   Similarly, we can also show that $\forall x \geq 0.5, |f(x)| \leq M(1-x)$.

   Suppose $\exists a \in (0,1)$ such that $|f(a)| > M(1-x)$, then by mean value theorem, $\exists b \in (a,1)$ such that $|f'(b)| = |(f(a)-0)/(a-1)| > M$, a contradiction.

   Hence,

$$\begin{aligned}
\int_0^1 |f(x)|\ dx &= \int_0^{0.5} |f(x)|\ dx + \int_{0.5}^1 |f(x)|\ dx \\
&< \int_0^{0.5} Mx\ dx + \int_{0.5}^1 M(1-x)\ dx \\
&= \frac{1}{4}M
\end{aligned}$$