Q1

a. False

b. True

c. True

d. True

e. True

f. False

Q2

a.

$$\int_0^1 \int_0^{2-x^2} f(x,y)\ dy\ dx$$

b. 3

c. 4

d. 7

Q3

a. Let the density, in circular coordinates, be $f(r,\theta,z) = kz$

$$\begin{aligned}\int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}} kzr\ dz\ dr\ d\theta &= 2\pi\int_0^2 \frac{1}{2}kr(4-r^2)\ dr\\ &= \pi k\int_0^2 4r - r^3\ dr\\ &= \pi k\left[2r^2 - \frac{1}{4}r^4\right]_0^2\\ &= 4\pi k\\ &= 1\end{aligned}$$

$$\therefore k = \frac{1}{4\pi}$$
$$\therefore f(r,\theta,z) = \frac{z}{4\pi}$$

b.

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} 2zr\ dz\ dr\ d\theta = \frac{7}{2}\pi$$

c. Take volume of half a sphere minus a spherical cap, and divide that by 2

$$\begin{aligned}\frac{1}{2}\left(\frac{1}{2}\frac{4}{3}\pi(2)^3 - \frac{\pi(2-\sqrt{2})^2}{3}(6-(2-\sqrt{2}))\right) &= \frac{8\pi}{3} - \frac{\pi(2-\sqrt{2})^2(4+\sqrt{2})}{3}\\ &= \frac{8\pi}{3} - \frac{2\pi(8-5\sqrt{2})}{3}\end{aligned}$$

Q4

a. Yes. The potential function

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$$

works

b. Let E be the solid bounded by S and D. Note that the flux across D is 0, as $z = 0$ in D. Hence, the flux across S is the same as the flux across the surface of E, and we can apply Divergence Theorem.

$$\nabla \cdot F = 3$$

$$\iint_S F \cdot dA = \iiint_E 3 \; dV = \pi$$

c. 20

Q5

a. Change coordinates to u, v, z, where $u = x + y, v = x - 2y$, then the volume has $-1 \leq v \leq 1, 0 \leq u \leq 2, \frac{2}{3}(2u + v) - 3 \leq z \leq \frac{2}{3}(2u + v) - 1$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left| \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \right| = -3$$

$$\int_{-1}^{1} \int_{0}^{2} \int_{\frac{2}{3}(2u+v)-3}^{\frac{2}{3}(2u+v)-1} \frac{1}{3} \; dz \; du \; dv = \frac{8}{3}$$

b.

$$-1 \leq y \leq 1$$

$$0 \leq z \leq 1 - y^2$$

$$0 \leq x \leq 1 - z$$

Hence

$$0 \leq x \leq 1 - z \leq 1$$

$$0 \leq z \leq 1 - x$$

$$y^2 \leq 1 - z \implies -\sqrt{1-z} \leq y \leq \sqrt{1-z}$$

Hence

$$\int_{-1}^{1} \int_{0}^{1-y^2} \int_{0}^{1-z} f(x,y,z) \; dx \; dz \; dy = \int_{0}^{1} \int_{0}^{1-x} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} f(x,y,z) \; dy \; dz \; dx$$