Introduction :: Probability.

1. Probability.

→ Rocket 자세 추정 → random signal.

1) what ?

→ the amount (quantity) of possibility.

→ from 0 to 1.

← Low → High

0 : never occurrence → no information.

1 : always occurrence

2) why ?

→ uncertainty of occurrence (phenomenon)

→ unpredictable.

→ but, with some possibility.

3) when or where ?

→ unpredictable, random phenomenon. (↔ deterministic).

→ ex) electrical noise, communication signals, event occurrence, weather condition, stock market,

4) How ?

→ define mathematical variables → Random Variables.

→ model probability distribution → function of RVs.

→ prediction, estimation, recognition, ...

→ mean, variance.

NO.

2. Statistics

1) what ?

→ modeling real data to mathematical probability dist.

→ analysis of real random data.

2) why ?

→ analysis of real random data.

→ modeling, fitting, estimation.

→ simplifying large data set (mean, variance...)

3) when or where ?

→ . confidence of random samples.

4) How ?

→ mathematical function of real data → distribution.

→ estimation.

→ confidence test. of samples.

⋮

MOOKEUK

Chapter 1. Basic Probability Concepts.

1.2 Sample Space and Events.

- Experiment (trial) → outcome.
- Sample space (SS) : set of all outcomes of experiment. → Union set.
- Event : subset of sample space
  defines some specific cases.

- eg: 3 coins. tossing.
  → SS : {TTT, TTH, ... HHT, HHH} → 8 outcomes.
  → 1 head event → {HTT, THT, TTH}

- set operations for events.
  - $A \cup B$: events that belong to A or B → union
  - $A \cap B$: " " A and B. → intersection
  - $A - B$: " " A but not B. → difference.

1.3 Definitions of Probability

$P(A)$: prob. of event A.

1.3.1 Axiomatic definition

1) Axiom 1 : $0 \leqq P(A) \leqq 1$.
2) Axiom 2 : $P(S) = 1$.
3) Axiom 3 : For any n mutually exclusive events, $A_1, A_2, \cdots A_n$,
$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$

1.3.2 Relative - Frequency Definition

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n}$$ (n times trial, and n_A times outcomes of event A.)

1.3.3 classical Definition.

$$P(A) = \frac{N_A}{N}$$ using a priori information of events and experiment.

eg; 3 coins tossing, A : 1 head. → $\frac{3}{8}$.

1.5. Elementary Set Theory.

- set operation ⇄ probability operation. (event, sample space → set)

$\left(\begin{array}{l} \cdot \text{ union : } A \cup B \\ \cdot \text{ intersection : } A \cap B \\ \text{Complement : } \bar{A} \end{array}\right\}$ Venn Diagram approach.

- Commutitive & Associative Laws.
- De Morgan's Law

$$\begin{cases} \overline{(A \cap B)} = \bar{A} \cup \bar{B} \\ \overline{(A \cup B)} = \bar{A} \cap \bar{B} \end{cases}$$

- Distributitive Law

$$\begin{cases} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{cases}$$

1.6 Properties of Probability.

from the axiomatic definition,

1.) $P(\overline{A}) = 1 - P(A)$

$\Rightarrow$ A and $\overline{A}$ are mutually exclusive and $A \cup \overline{A} = S$.

$P(A \cup \overline{A}) = P(A) + P(\overline{A}) = P(S) = 1.$

2.) $P(\emptyset) = 0.$ (null event).

3.) If $A \subset B$, $P(A) \leqq P(B)$

4.) $A = A_1 \cup A_2 \cup \cdots \cup A_n \Rightarrow P(A) = P(A_1) + \cdots + P(A_n)$

5.) $P(A \cup B) = P(A) + P(B) - P(A \cap B).$

1.7 Conditional Probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

given condition (or variable.)

1.7.1 Total Probability and Bayes' Theorem.

- Partition $\{A_1, A_2, \cdots, A_n\} \Rightarrow \begin{cases} A_i \cap A_j = \emptyset & i \neq j \\ \bigcup_{i=1}^{n} A_i = S \end{cases}$

Proposition 1.1

- $\{A_1, A_2, \cdots, A_n\} \Rightarrow$ partition of sample space S,
- $P(A_i) > 0$.

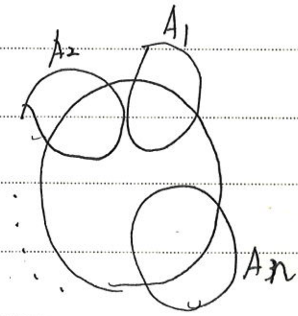
$$\Rightarrow P(A) = \sum_{i=1}^{n} P(A_i) P(A \mid A_i)$$

$$= P(A_1)P(A \mid A_1) + \cdots + P(A_n)P(A \mid A_n).$$

pf) P(mutually exclusive events) $\Rightarrow \sum_{all} P(\text{event})$

$$P(A) = P(A \cap A_1) + P(A \cap A_2) + \cdots P(A \cap A_n) \quad -①$$

↳ $(A \cap A_i)$ and $(A \cap A_j)$ are mutually exclusive.



since $P(A \cap A_i) = P(A \mid A_i) P(A_i)$, $-②$

② → ①

$$\Rightarrow P(A) = \sum_{i=1}^{n} P(A_i) P(A \mid A_i)$$

<Example 1.5>

- supplier A → 1000, defective → 0.05
- supplier B → 2000, 〃 → 0.10
- supplier C → 3000, 〃 → 0.10

$\Rightarrow P(D) = ?$

$$\Rightarrow \begin{cases} P(\text{Def} \mid A) = 0.05, \\ P(D \mid B) = 0.10 \\ P(D \mid C) = 0.10 \end{cases} \qquad \begin{cases} P(A) = \dfrac{1000}{3000 + 2000 + 1000} = \dfrac{1}{6} \\ P(B) = \dfrac{2}{6} = \dfrac{1}{3} \\ P(C) = \dfrac{3}{6} = \dfrac{1}{2}. \end{cases}$$

$$P(D) = P(D \text{ from } A) + P(D \text{ from } B) + P(D \text{ from } C)$$
$$= P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)$$
$$= 0.05 \cdot \frac{1}{6} + 0.10 \cdot \frac{1}{3} + 0.10 \cdot \frac{1}{2} \ //$$

∘ Bayes' Theorem. (Bayesian rule)

· given event A, which mutually exclusive event occurred?

↳ $P(A_k | A) = ?$ we know the conditional pdf, $P(A|A_k)$ $k=1,2\cdots n$,

$$P(A_k|A) = \cancel{\frac{P(A|A_k)P(A_k)}{P(A)}} = \frac{P(A \cap A_k)}{P(A)}$$
$$= \frac{P(A|A_k)P(A_k)}{P(A)}$$
$$= \frac{P(A|A_k)P(A_k)}{\sum_{i=1}^{n} P(A_i)P(A|A_i)}$$

<Example 1.6>.

When given a randomly selected IC is defective, what is prob. that it come from A?

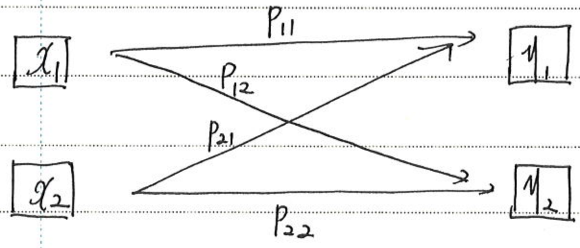
$$P(A|D) = ?$$
$$= \frac{P(D|A)P(A)}{P(D)} = \frac{P(D|A)P(A)}{P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)}$$
$$= \frac{0.05 \times \frac{1}{6}}{0.05\times\frac{1}{6} + 0.1\times\frac{1}{3} + 0.1\times\frac{1}{2}} \ //$$

< Example 1.7 > Binary Symmetric channel.

(input symbols $\{x_1, x_2, \cdots\}$, output symbols $\{y_1, y_2, \cdots\}$
(transition probability P_{ij} : input $x_i \underset{\uparrow}{\rightarrow}$ output y_i.

channel.

$$\Rightarrow P_{ij} = P(y_i \mid x_i)$$



0) · $P_{error} = P(x_1)\,P(y_2 \mid x_1) + P(x_2)\,P(y_1 \mid x_2)$ //

a) given y_2 received, what prob. of x_1 transmitted ?

$$P(x_1 \mid y_2) = ?$$
$$= \frac{P(y_2 \mid x_1)\,P(x_1)}{P(y_2)} = \frac{P(y_2 \mid x_1)\,P(x_1)}{P(x_1)\,P(y_2 \mid x_1) + P(x_2)\,P(y_2 \mid x_2)}$$

c) given y_1 received, what prob. of x_1 transmitted ?

$$P(x_1 \mid y_1) = ?$$
$$= \frac{P(y_1 \mid x_1)\,P(x_1)}{P(y_1)} = \frac{P(y_1 \mid x_1)\,P(x_1)}{P(x_1)\,P(y_1 \mid x_1) + P(x_2)\,P(y_1 \mid x_2)} \ //$$

e) unconditioned error ? $P_{erro} = P(x_1)\,P(y_2 \mid x_1) + P(x_2)\,P(y_1 \mid x_2)$.

1.8 Independent Events

If event B and A are independent,

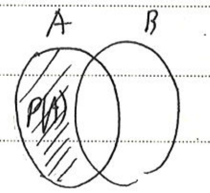
↳ $P(B|A) \triangleq P(B)$ and $P(A|B) \triangleq P(A)$.

$\Rightarrow P(B|A) = P(B \cap A)/P(A) = P(B)$

↳ $P(A \cap B) = P(A)\,P(B)$.

- independendace is different from mutually exclusive!
  ↳ independence is related with multiple trials.
  mutually exclusiveness " " between single events.

- If A and B are independent,
  $\bar{A}$ and, B, $\bar{B}$ and $\bar{A}$ are also independent!
  $\bar{B}$ and A,



$$
\begin{aligned}
P(A \cap \bar{B}) &= P(A) - P(A \cap B) = P(A) - P(A)P(B) \\
&= P(A)\cdot(1 - P(B)) \\
&= P(A)\,P(\bar{B}).
\end{aligned}
$$

1.9 Combined Experiments.

- For two experiments with sample spaces, S_1 and S_2.

↳ the combined sample space : $S_1 \times S_2$: Cartesian Product.
$= \{ (x_i, y_i) \}$.

⟨Ex 1.17⟩ : one coin + one dice tossing $\Rightarrow \{(H,1), (H,2) \cdots, (T,1) \cdots (T,6)\}$.

1.10 Basic Combinatorial Analysis.

1.10.1 Permutation (순열)

◦ $n!$ → $\overbrace{\circ\ \circ\ \circ\ \cdots\ \circ}^{n}$ arrangement.

$n \cdot (n-1) \cdots 1$

◦ ${}_nP_r = \dfrac{n!}{(n-r)!}$ $\qquad \overbrace{\circ\ \circ\ \cdots\ \circ}^{r}$

$n \cdot (n-1) \cdots (n-r+1)$

$0! = 1.$

◦ Theorem :

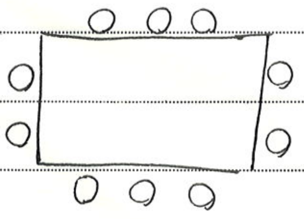
$n_1 + n_2 + \cdots + n_k = n.$ $\quad n_k$: subgroup of n.

$$N_k = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Ex: $R=5,\ B=2,\ G=3.$ $\qquad \dfrac{10!}{5!\,2!\,3!}$

1.10.2 ◦ Circular Arrangement.

$$\frac{n!}{n} = (n-1)!$$



$$\frac{10!}{10} \times 5.$$

1.10.4 Combinations.

$$ {}_nC_r = \binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!\,r!} = {}_nC_{n-r}. $$

$$ \binom{n}{r} \overset{d}{=} \frac{{}_nP_r}{r!} $$

$$ \binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} $$

m = boys
n = girls
$\Rightarrow$?

1.10.5 Binomial Theorem.

$$ (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}. $$

$$ f(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \Rightarrow x=1 \; ; \quad 2^n = \sum_{k=0}^{n} \binom{n}{k} $$

$$ \frac{d}{dx} f(x) = n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k \cdot x^{k-1} \Rightarrow x=1 \; ; \quad n(2)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}. $$

1.10.6 Stirling's Formula

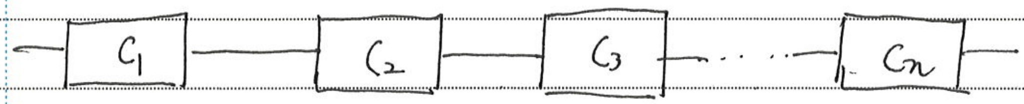
$$ n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n $$

1.11 Reliability Applications.

reliability : duration of the useful life of systems.

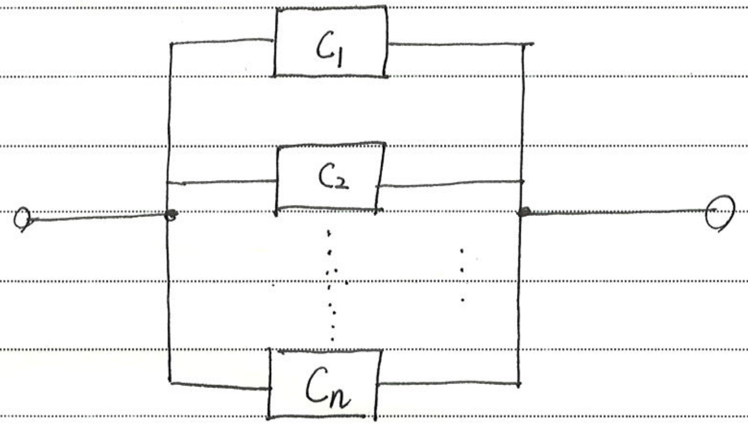
$R(t)$: prob. that a system will be functioning at time t.

- series connection



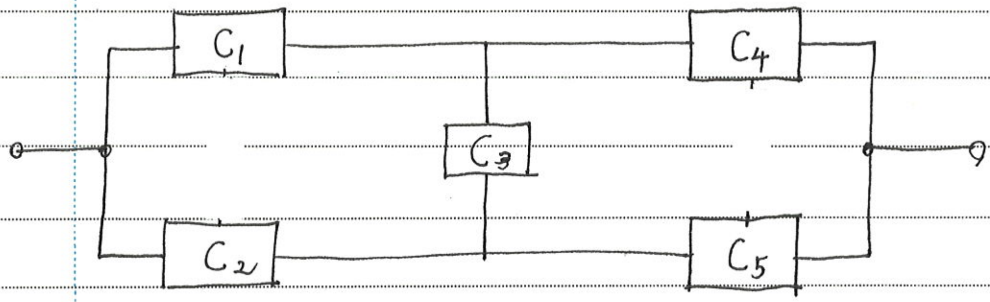
$$R(t) = \prod_{i=1}^{n} R_i(t).$$

- Parallel connection



$$R(t) = 1 - \prod_{i=1}^{n} (1 - R_i(t))$$

Ex. 1.36



(i) C_3 : fail $\Rightarrow R_Y$.

$$R_Y = 1 - (1 - R_1 R_4)(1 - R_2 R_5)$$

(ii) C_3 : operational $\Rightarrow R_X$

$$R_X = [1-(1-R_1)(1-R_2)][1-(1-R_4)(1-R_5)]$$

$$R = R_X R_3 + R_Y (1-R_3)$$
$$= R_1R_4 + R_2R_5 + R_1R_3R_5 + R_2R_3R_4 - \cdots\cdots + 2R_1R_2R_3R_4R_5$$

HW. chapter 1.

1.2 1.7

1.3 1.10, 12, 15

1.5

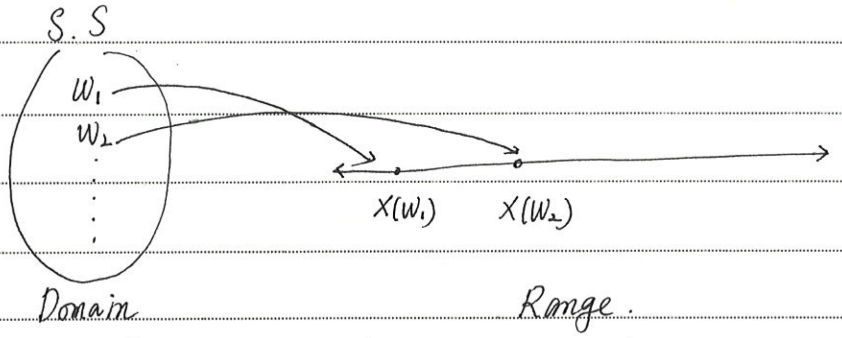
1.6 ✓1.26, 1.27, ✓1.29

1.7 ✓1.34, 1.36, ✓1.37

1.11 1.49

2.2 Definition of R.V

R.V



mapping each outcome of random experiment to a real value.

$X(w_i) \rightarrow X_i$

Ex: $\begin{cases} X(\text{head}) = 1 \\ X(\text{tail}) = 0 \end{cases}$

By the conventional,

$\begin{cases} \text{R.V} : X, \\ \text{a fixed real value} : x \end{cases}$

2.3 Events Defined by R.V.

Let A_x an event of S.S.

$A_x = \{ w \mid X(w) = x \}$.

$p = P(A_x)$.

ex) $[X \leq x] = \{ w \mid X(w) \leq x \}$

$[a < X < b] = \{ w \mid a < X(w) < b \}$

- Probability assignment.

$$P(X \leq x) = P(\{w \mid X(w) \leq x\})$$

$$P(X > x) = 1 - P(X \leq x).$$

Ex 2.2 2 coins tossing, X = # of heads.

$\Rightarrow S = \{TT, TH, HT, HH\}$ $X = \{0, 1, 2\}$

$$P(X=1) = P(w = TH \text{ or } w = HT) = \frac{1}{2}.$$
$$P(X \leq 1) = P(0) + P(1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

2.4 Distribution Functions.

For an R.V. X and a real number x,

- Cumulative Distribution Function (CDF)

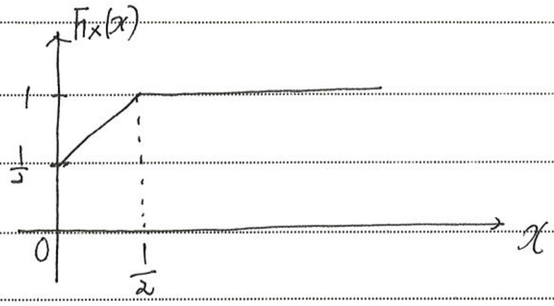
$$F_X(x) = P[X \leq x] = P(\{w \mid X(w) \leq x\}) \quad -\infty < x < \infty$$

1. if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$: non-increasing function
2. $0 \leq F_X(x) \leq 1.$
3. $F_X(\infty) = 1.$
4. $F_X(-\infty) = 0$
5. $P[a < X \leq b] = F_X(b) - F_X(a)$
6. $P[X > a] = 1 - F_X(a).$

Ex 2.3

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x + \frac{1}{2}, & 0 \leq x \leq \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases}$$

(a)



(b) $P[X > \frac{1}{4}] = 1 - P(X \leq \frac{1}{4})$

$= 1 - F_X(\frac{1}{4})$

$= 1 - \frac{3}{4} = \frac{1}{4}.$

2.5 Discrete Random Variables

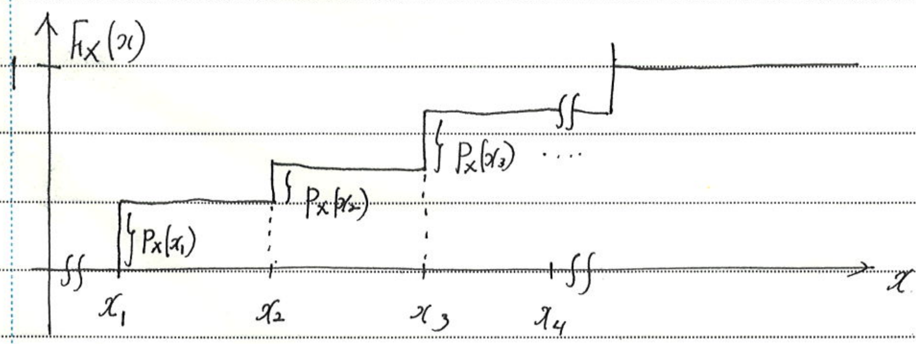
- probability mass function (PMF) → Finite or Countably infinite set.

$$P_X(x) = P[X = x].$$ δ(x) function 사용

For CDF of X,

$$F_X(x) = P[X \leq x] = \sum_{k \leq x} P_X(k)$$

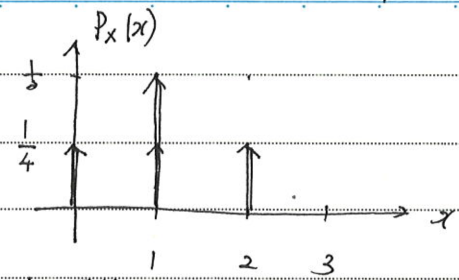
The CDF of discrete R.V. is a step function.



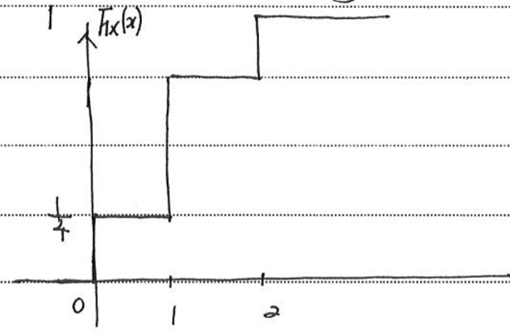
$\sum_{x} P(X = x) = 1$

Ex. 2.4

$$P_X(x) = \begin{cases} \frac{1}{4}, & x=0 \\ \frac{1}{2}, & x=1 \\ \frac{1}{4}, & x=2 \end{cases}$$

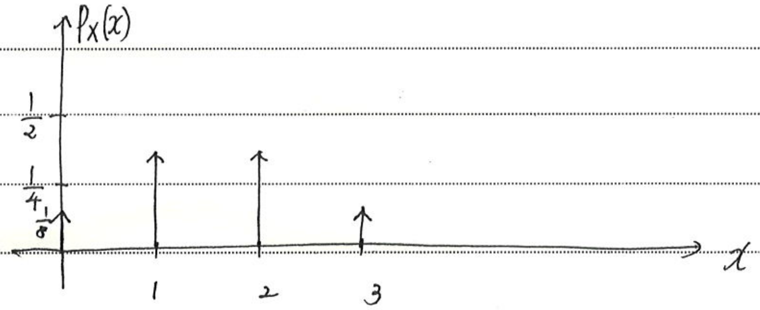


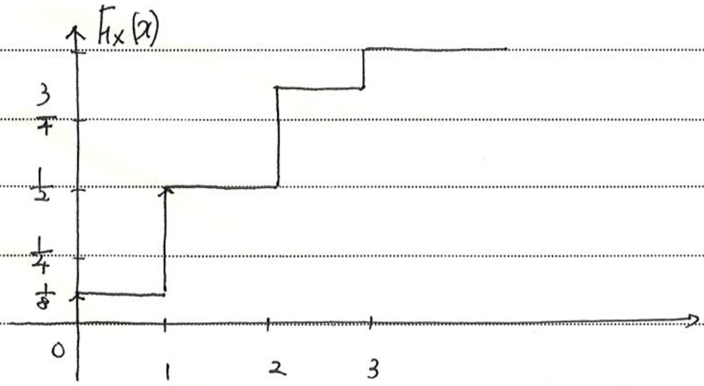
$$F_X(x) = \begin{cases} 0 & x<0 \\ \frac{1}{4} & 1 \leq x < 1 \\ \frac{3}{4} & 1 \leq x < 2 \\ 1 & 2 \leq x \end{cases}$$



Ex 2.5 X: # of heads in 3 tossing of a fair coin.

(a) PMF $P_X(x) = \begin{cases} \frac{1}{8} = \binom{3}{0}\left(\frac{1}{2}\right)^3, & x=0 \\ \frac{3}{8} = \binom{3}{1}\left(\frac{1}{2}\right)^3, & x=1 \\ \frac{3}{8} = \binom{3}{2}\left(\frac{1}{2}\right)^3, & x=2 \\ \frac{1}{8} = \binom{3}{3}\left(\frac{1}{2}\right)^3, & x=3. \end{cases}$





$$F_X(x) = \begin{cases} 0, & x<0 \\ \frac{1}{8}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{7}{8}, & 2 \leq x < 3 \\ 1, & 3 \leq x \end{cases}$$

CDF ↔ PMF 관계 설명.

2.6 Continuous R.V.

continuous R.V → uncountable set.

$$P(X \in A) \triangleq \int_A f_X(x)\,dx \quad , \qquad f_X(x) \geq 0.$$

$f_X(x)$: probability density function. (PDF)

$$f_X(x) \triangleq \frac{dF_X(x)}{dx}.$$

$P(X=x) = \lim_{\Delta x \to 0} F_X(x+\Delta x) - F_X(x) \to 0$ cont. or $P(X=x)$ dis.

$\frac{P(X=x)}{\Delta x} \triangleq f_X(x) \triangleq \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x}$ //

1. $f_X(x) \geq 0$
2. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$

$f_X(x)\cdot\Delta x \to P(X=x)$ //

3. $P[a \leq X \leq b] = \int_a^b f_X(x)\,dx$,
   $P[X=a] = \int_a^a f_X(x)\,dx = 0.$ for continuous R.V.

4. $P[X<a] = P[X \leq a] = F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx.$

Ex 2.11
$$f_X(x) = \begin{cases} A(2x - x^2) & , \quad 0 < x < 2 \\ 0 & , \quad \text{otherwise.} \end{cases}$$

(a) A ?

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

$$\int_0^2 A(2x - x^2)\,dx = A\left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{4}{3}\cdot A = 1.$$

$$A = \frac{3}{4} //$$

(b) $P[X>1] = \int_1^\infty f_X(x)\,dx = \int_1^2 \frac{3}{4}(2x - x^2)\,dx$

$= \frac{3}{4}\left(x^2 - \frac{1}{3}x^3\right)\Big]_1^2 = \frac{1}{2}$.

Ex 2.15 $f_T(t) = \begin{cases} \frac{1}{6}, & 2 \leq t \leq 8 \\ 0, & \text{otherwise} \end{cases}$ → uniform dist.

(a) CDF of T ?

$F_T(t) = \int_{-\infty}^{t} f_T(\tilde{t})\,d\tilde{t} = \int_{-\infty}^{t} \frac{1}{6}\,d\tilde{t}$

$= \int_2^{t} \frac{1}{6}\,d\tilde{t} = \frac{1}{6}(t-2)$

$F_T(t) = \begin{cases} 0 & t < 2 \\ \frac{1}{6}(t-2) & 2 \leq t \leq 8 \\ 1 & 8 \leq t \end{cases}$

(b) $P[t<5] = F_T(5) = \int_2^5 f_T(t)\,dt = \frac{1}{2}$ //

HW #2.

2.2, 2.5, 2.8, 2.11
2.14, 2.15, 2.21
2.24, 2.26, 2.31
2.32, 2.35

Ex. 2.16 $F_X(x) = \begin{cases} 0 & x < 2 \\ A(x-2) & 2 \leq x < 6 \\ 1 & 6 \leq x \end{cases}$

(a) $A = ?$

$A(6-2) = 1 \Rightarrow A = \frac{1}{4}$.

(b) $P[3 \leq X \leq 5] = F_X(5) - F_X(3) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.

$= \int_3^5 \frac{1}{4}(x-2)\,dx = \frac{1}{2}$.

3. Moments of Random Variables

3.1 Introduction.

- Arithmetic average.

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_N}{N}$$

- for different frequencies, $\{w_i\}$

$$\bar{X} = \frac{w_1 X_1 + w_2 X_2 + \cdots + w_N X_N}{w_1 + w_2 + \cdots + w_N}$$

- mean → a measure of central tendency.
- for random variables : mean → expectation. (central tendancy)
  variance → spread out of R.V.

3.2 Expectation.

$$E(X) = \begin{cases} \sum_i x_i P(x_i) & \text{for disirete R.V.'s} \\ \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx & \text{for continuous R.V.'s} \end{cases}$$

Ex. 3.3 $P_K(k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad K = 0, 1, 2, \cdots \quad (\lambda > 0)$

$$E[K] = \sum_{K=0}^{\infty} K \frac{\lambda^K}{K!} e^{-\lambda}$$

→ Taylor Series.

$$= \sum_{K=1}^{\infty} \frac{\lambda^K}{(K-1)!} e^{-\lambda} = \sum_{K=0} \lambda \cdot \frac{\lambda^K}{K!} e^{-\lambda} = \lambda.$$

Ex 3.4 $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geqq 0 \\ 0, & x < 0 \end{cases}$ → exponential dist.
↳ life time.

$$E[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx$$
$$= -\frac{\lambda}{\lambda} x e^{-\lambda x}\Big]_0^\infty + \int_0^\infty e^{-\lambda x}\, dx$$
$$= -\frac{1}{\lambda} e^{-\lambda x}\Big]_0^\infty = \frac{1}{\lambda}$$

3.4 Moments of R.V.

∘ nth moment of R.V. X

$$E[X^n] = \begin{cases} \sum_i x_i^n P_X(x_i) \\ \int_{-\infty}^\infty x^n f_X(x)\, dx \end{cases}$$

∘ Central moment

$$E[(X-\bar{X})^n] = \begin{cases} \sum_i (x_i - \bar{X})^n P_X(x_i) \\ \int_{-\infty}^\infty (x-\bar{X})^n f_X(x)\, dx \end{cases}$$

for $n=2$, variance σ_X^2
↳ how much the R.V. is spread out ~.

Proposition 3.2

Let a, b constants, X, random variable.

$$Y = aX + b$$

$$E[Y] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\, dx$$

$$= a\int_{-\infty}^{\infty} x \cdot f_X(x)\, dx + b = aE[X] + b = E[aX+b]$$

→ 먼저 $E[g(x)] = \int g(x) \cdot f_X(x)\, dx$

Proposition 3.3

◦ for R.V. X, mean $E[X]$

$$g_3(X) = g_1(X) + g_2(X)$$

$$E[g_3(X)] = \int_{-\infty}^{\infty} g_3(x) \cdot f_X(x)\, dx$$

$$= \int g_1(x) f_X(x)\, dx + \int g_2(x) f_X(x)\, dx$$

$$= E[g_1(X)] + E[g_2(X)]$$

$$\sigma_X^2 = E[(X-\bar{X})^2] = E[X^2 - 2X\bar{X} + \bar{X}^2]$$

$$= E[X^2] - 2\bar{X}E[X] + \bar{X}^2$$

$$= E[X^2] - \bar{X}^2$$

Ex. 3.6.

$$f_X(x) = \begin{cases} \frac{1}{4}, & 2 \le x \le 6 \\ 0, & \text{else} \end{cases}$$

$$E[X] = \int_2^6 x \cdot \frac{1}{4}\, dx = \frac{1}{8}x^2\Big]_2^6 = 4.$$

$$\sigma_X^2 = \int_2^6 (x-4)^2 \frac{1}{4}\, dx = \frac{1}{12}(x-4)^3\Big]_2^6$$

$$= \frac{8-(-8)}{12} = \frac{16}{12} = \frac{4}{3}.$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \int_2^6 x^2 \cdot \frac{1}{4}\, dx - (4)^2$$

$$= \frac{1}{12}x^3\Big]_2^6 - 16 = \frac{216-8}{12} - 16 = \frac{4}{3}.$$

Ex. 3.7. $P_X(k) = \dfrac{\lambda^k}{k!}e^{-\lambda}, \quad k = 0, 1, 2, \cdots$

$$E[X] = \lambda.$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \sum_{k=0}^{\infty} k \cdot \lambda \frac{\cdot \lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda \cdot e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!}$$

$$\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \sum_{k=1}^{\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!} = \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda e^{\lambda}$$

2가지 풀이

$$\frac{d}{d\lambda}\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}\right] = \underline{\sum_{k=1}^{\infty} \frac{k \cdot \lambda^{k-1}}{(k-1)!}} = \frac{d}{d\lambda}[\lambda e^{\lambda}] = \underline{e^{\lambda}(1+\lambda)},$$

$(k-1+1) \cdots$

$$E[X^2] = \lambda e^{-\lambda} \cdot e^{\lambda}(1+\lambda) = \lambda^2 + \lambda.$$

$$\sigma_X^2 = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Ex. 3.13. prob. success p. until success, # of trials K.

$$P_K(k) = (1-p)^{k-1} \cdot p. \quad K = 1, 2, 3, \cdots$$

a. PMF. ?

$$\sum_{K=1}^{\infty} P_K(k) = 1.$$

$$\sum_{K=1}^{\infty} (1-p)^{k-1} \cdot p = p \cdot \frac{1}{1-(1-p)} = 1.$$

b. $P[K>5] = ?$

$$P[K>5] = \sum_{K=6}^{\infty} (1-p)^{K-1} \cdot p = \frac{p \cdot (1-p)^5}{1-(1-p)} = (1-p)^5.$$

c. $E[X] = ?$

$$E[X] = \sum_{K=1}^{\infty} K \cdot (1-p)^{K-1} \cdot p$$

where $\sum_{K=1}^{\infty} (1-p)^K = \frac{(1-p)}{1-(1-p)} = \frac{1-p}{p}$

$$\frac{d}{dp}\left[\sum_{K=1}^{\infty} (1-p)^K\right] = -\sum_{K=1}^{\infty} K \cdot (1-p)^{K-1} = \frac{-p-(1-p)}{p^2} = -\frac{1}{p^2}.$$

$$E[X] = \sum_{K=1}^{\infty} K \cdot (1-p)^{K-1} \cdot p = \frac{1}{p^2} \cdot p = \frac{1}{p}.$$

$$\frac{d}{dp}\left[\sum_{k=1}^{\infty} k(1-p)^{k-1}\right] = -\sum_{k=1}^{\infty} k(k-1)(1-p)^{k-2} = -\frac{2}{p^3}$$

$\times(1-p)$ $$\sum_{k=1}^{\infty} (k^2-k)(1-p)^{k-1} = \frac{2}{p^3}(1-p)$$

$$\sum_{k=1}^{\infty} k^2 (1-p)^{k-1} = \sum_{k=1}^{\infty} k(1-p)^{k-1} + \frac{2-2p}{p^3}$$

$$= \frac{1}{p^2} + \frac{2}{p^3} - \frac{2}{p^2} = \frac{2}{p^3} - \frac{1}{p^2}.$$

$$E[X^2] = \sum_{k=1}^{\infty} k^2 p (1-p)^{k-1} = \frac{2}{p^2} - \frac{1}{p}$$

$$\sigma_X^2 = E[X^2] - E[X]^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} \,_{//}$$

3.5 Conditional Expectations

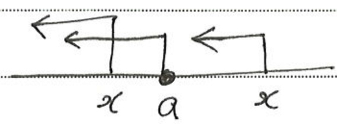
3.5, 3.8, 3.11
3.16, 3.19, 3.20
3.22, 3.25

$$E[X|A] = \begin{cases} \int_{-\infty}^{\infty} x\, f_X(x|A)\, dx \\ \sum x_i\, P_X(x_i|A) \end{cases}$$

Ex. 3.14 $\quad A = \{ X \leq a \}$

$$f_X(x|A) = \frac{d}{dx} F_X(x|A) = \frac{d}{dx} P(X \leq x \mid A) = \frac{d}{dx} \frac{P(X \leq x \cap A)}{P(A)}$$

$$= \frac{d}{dx} \frac{P(X \leq a \cap X \leq x)}{P(X \leq a)}$$



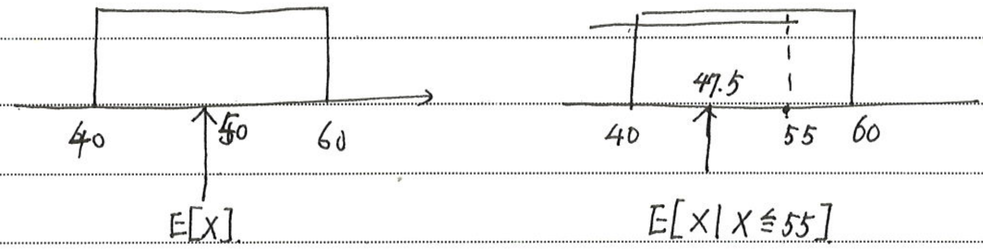
$$= \begin{cases} \dfrac{d}{dx} \dfrac{P(X \leq x)}{P(X \leq a)} = \dfrac{f_X(x)}{F_X(a)}, & x \leq a \\ \dfrac{d}{dx} \dfrac{P(X \leq a)}{P(X \leq a)} = 0 & x > a \end{cases}$$

Ex. 3.15.

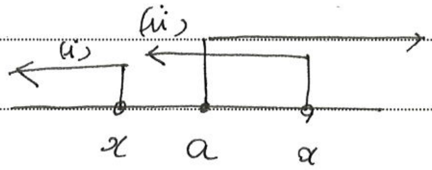
$$f_X(x) = \begin{cases} \frac{1}{20}, & 40 \le x \le 60 \\ 0, & \end{cases}$$

$$E[X \mid X \leqq 55] = \int_{40}^{55} x \cdot f_X(x \mid x \leqq 55)\, dx$$

$$= \frac{\int_{40}^{55} x \cdot \frac{1}{20}\, dx}{F_X(55)} = 47.5.$$



Ex.) $E[X \mid X \geqq a]$



(i) $x < a$.

$$F_X(x \mid X \geqq a) = P(X \leqq x \mid X \geqq a)$$

$$= \frac{P(\emptyset)}{P(X \geqq a)} = 0.$$

(ii) $a \leqq x$

$$F_X(x \mid X \geqq a) = \frac{P(X \leqq x \cap X \geqq a)}{P(X \geqq a)} = \frac{P(a \leqq X \leqq x)}{P(X \geqq a)}$$

$$= \frac{F_X(x) - F_X(a)}{1 - F_X(a)}$$

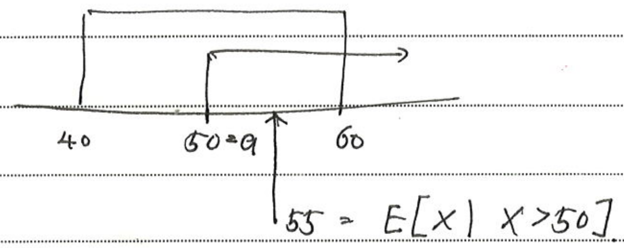
$$f_X(x \mid X > a) = \frac{f_X(x)}{1 - F_X(a)} \quad (a \leqq x).$$

$$E[X \mid X>a] = \int_a^{\infty} x \cdot f_X(x \mid x>a)\, dx$$

$$= \int_a^{\infty} x \cdot \frac{f_X(x)}{1 - F_X(a)}\, dx$$

eg. $a = 50.$ $f_X(x) = \frac{1}{20}$ in Ex 3.15,

$$\int_{50}^{60} \frac{1}{10} x \, dx = \frac{1}{20}\left[60^2 - 50^2\right] = 55.$$



3.6 Chebyshev Inequality

$$P\left[\, |X - E[X]| \geq a \,\right] \leq \frac{\sigma_X^2}{a^2}, \quad a>0.$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\, dx.$$

$$= \int_{|x-E[X]| \geq a} (x - E[X])^2 \cdot f_X(x)\, dx + \int_{|x-E[X]| < a} (x - E[X])^2 \cdot f_X(x)\, dx$$

$$\geq \int_{|x-E[X]| \geq a} a^2 \cdot f_X(x)\, dx = a^2 \int_{|x-E[X]| \geq a} f_X(x)\, dx$$

$$= a^2 \cdot P\left(|X - E[X]| \geq a \right)$$

Chap. 4. Special Prob. Distributions

4.2 Bernoulli Distribution.

$P[\text{success}] = p$
$P[\text{failure}] = 1-p$ $\Rightarrow P_X(x) = \begin{cases} 1-p, & x=0 \\ p, & x=1 \end{cases}$

$E[X] = p.$
$\sigma_X^2 = p - p^2 = p(1-p).$

4.3 Binomial Distribution.

- n trials of independent Bernoulli trial, with p.
  X: # of trials with p out of n trials.
  $B(n, p)$, $P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$. $x = 0, 1, 2, \cdots, n$.

$$\sum_{x=0}^{n} \binom{n}{x} p^x \cdot (1-p)^{n-x} = \{ p + (1-p) \}^n = 1.$$

$$E[X] = \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x \cdot (1-p)^{n-x}$$

$$(i) \sum_{x=0}^{n} x \frac{n!}{(n-x)!\, x!} p^x \cdot (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(n-x)!\,(x-1)!} p^x \cdot (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} n \cdot p \frac{(n-1)!}{(n-1-(x-1))!\,(x-1)!} p^{x-1} (1-p)^{(n-1)-(x-1)}$$

$$= np \cdot \sum_{x=0}^{n-1} \frac{(n-1)!}{(n-1-x)!\ x!}\ p^{x} (1-p)^{n-1-x} \qquad x-1 \to x.$$

$$= np \cdot \{ p + (1-p) \}^{n-1} = np.$$

$$E[X^2] = \sum_{x=0}^{n} x^2 \binom{n}{x} p^{x} \cdot (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x^2 \cdot \frac{n!}{(n-x)!\ x!}\ p^{x} \cdot (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{(n-x)!\ (x-1)!}\ p^{x} \cdot (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} (x-1+1) \frac{n!}{(n-x)!\ (x-1)!}\ p^{x} \cdot (1-p)^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n \cdot (n-1)\ (n-2)!}{\underset{(n-2)-(x-2)}{(n-x)!}\ (x-2)!}\ p^2 \cdot p^{x-2} \cdot (1-p)^{n-x} + \sum_{x=1}^{n} \frac{n\ (n-1)!}{\underset{(n-1)-(x-1)}{(n-x)!}\ (x-1)!}\ p \cdot p^{x-1} (1-p)^{n-x}$$

$$= \sum_{x=0}^{n-2} \frac{n \cdot (n-1) p^2 \cdot (n-2)!}{\cancel{(n-x)!}\ (n-2-x)!\ x!}\ p^{x} \cdot (1-p)^{n-x-2} + \sum_{x=0}^{n-1} \frac{n \cdot (n-1)!}{\cancel{(n-x)!}\ (n-1-x)!\ x!}\ p \cdot p^{x-1} (1-p)^{n-1-x}$$

$$= n(n-1)p^2 \cdot \{ p + (1-p) \}^{n-2} + np \{ p + (1-p) \}^{n-1}$$

$$= n^2p^2 - np^2 + np.$$

$$\sigma_x^2 = E[X^2] - E[X]^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p).$$

(iii) Binomial theorem.

$$\{pt + (1-p)\}^n = \sum_{x=0}^{n} \binom{n}{x} p^x \cdot t^x \cdot (1-p)^{n-x}.$$

$$\frac{d}{dt} \Rightarrow n \cdot p \{pt + (1-p)\}^{n-1} = \sum_{x=0}^{n} \binom{n}{x} \cdot x \cdot p^x t^{x-1} (1-p)^{n-x}.$$

$$t=1, \quad np = \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x \cdot (1-p)^{n-x} = E[X].$$

$$\frac{d^2}{dt^2} = n(n-1)p^2 \{pt + (1-p)\}^{n-2} = \sum_{x=0}^{n} \binom{n}{x} x \cdot (x-1) p^x \cdot (1-p)^{n-x} \cdot t^{x-2}$$

$$t=1. \quad n^2p^2 - np^2 = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x \cdot (1-p)^{n-x} - \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x}$$

$$= E[X^2] - E[X].$$

$$E[X^2] = n^2p^2 - np^2 + np.$$

$$\sigma_X^2 = E[X^2] - E[X]^2 = n^2p^2 - np^2 + np - (np)^2$$
$$= np(1-p)_{//}$$

4.4. Geometric Distribution.

X : # of Bernoulli trials until first success.

$$p_X(x) = p \cdot (1-p)^{x-1}, \quad x = 1, 2, \cdots$$

By ~~binomial~~ series of geometric sequences and differentiation

$$E[X] = \frac{1}{p}, \quad \sigma_X^2 = \frac{1-p}{p^2} _{//}$$

4.4.2 Forgetfulness (Memoryless) Property

⇒ Consider the # of K of additional trials until the first success

n까지 failure, additional K trials

$$P(X=K+n \mid X>n) = \frac{P(X=n+K \cap X>n)}{P(X>n)}$$

$$= \frac{P(X=n+K)}{1-P(X\leq n)}$$

$$= \frac{p\cdot(1-p)^{n+K-1}}{(1-p)^n} =$$

$$= p\cdot(1-p)^{K-1} = P_X(X=K)$$

4.7. Poisson Distribution.

$$P_X(X=k) = \frac{\lambda^k}{K!}e^{-\lambda} \qquad K=0,1,2,\dots$$

$E[X] = \lambda$

$\sigma_X^2 = \lambda.$

4.7.1. Poisson Approximation to the Binomial Distribution.

$$P_X(\cancel{X}=x) = \binom{n}{x}p^x\cdot(1-p)^{n-x}, \Rightarrow E[X] = np. = \lambda.$$

$$p = \frac{\lambda}{n}.$$

$$P_X(X=x) = \binom{n}{x} p^x \cdot (1-p)^{n-x} = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n\cdot(n-1)(n-2)\cdots(n-x+1)}{x!\cdot n^x} \lambda^x \cdot \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{x-1}{n})}{x!} \lambda^x \cdot \left(1-\frac{\lambda}{n}\right)^n \cdot \left(1-\frac{\lambda}{n}\right)^{-x}$$

for large $n \to \infty$.

$$\lim_{n\to\infty} P_X(X=x) = \lim_{n\to\infty} \frac{1}{x!} \lambda^x \left(1-\frac{\lambda}{n}\right)^n$$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$$= \lim_{n\to\infty} \frac{\lambda^x}{x!} \left(1+\frac{-\lambda}{n}\right)^{n(-\lambda)(-\frac{1}{\lambda})} = \lim_{n\to\infty} \frac{\lambda^x}{x!} \left(1+\frac{1}{\frac{n}{-\lambda}}\right)^{-\frac{n}{\lambda}\cdot(-\lambda)}$$

$$= \frac{\lambda^x}{x!} e^{-\lambda}$$

$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{\frac{n}{a}\cdot a}$

$= \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{\frac{n}{a}\cdot a} = e^a$

4.8. Exponential Distribution.

→ reliability distribution.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \Rightarrow \lambda e^{-\lambda x} u(x)$$

$$F_X(x) = P(X \leq x) = \int_0^x \lambda e^{-\lambda u}\, du = \left[1 - e^{-\lambda x}\right] u(x), \ (x \geq 0)$$

$$E[X] = \int_0^\infty x\cdot\lambda e^{-\lambda x}\, dx = \cancel{-x e^{-\lambda x}\Big]_0^\infty} + \int_0^\infty e^{-\lambda x}\, dx$$

$$= \frac{1}{\lambda}$$

$$E[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x}\, dx = -x^n e^{-\lambda x}\Big]_0^\infty + \int_0^\infty n \cdot x^{n-1} e^{-\lambda x}\, dx$$

$$= -\frac{n}{\lambda} x^{n-1} e^{-\lambda x}\Big]_0^\infty + \int_0^\infty \frac{n(n-1)}{\lambda} x^{n-2} e^{-\lambda x}\, dx$$

$$= \cdots\cdots = \frac{n!}{\lambda^n}$$

$$I_n = \frac{n}{\lambda} I_{n-1}$$

$$\begin{cases} I_n = \int \lambda x^n e^{-\lambda x}\, dx \\ I_1 = \frac{1}{\lambda} \end{cases}$$

$$E[X^2] = \frac{1}{\lambda^2}$$

$$\sigma_x^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

4.8.1 Forgetfulness Property of Exponential Distribution.

For a life time of system, with exponential dist. (mean $= \frac{1}{\lambda}$)

, consider the system has not failed by time t, then for a $s>0$

$$P(X \leq t+s \mid X>t) = \frac{P(X \leq t+s \cap X>t)}{P(X>t)}$$

$$= \frac{P(t < X \leq t+s)}{1 - F_X(t)}$$

$$= \frac{e^{-\lambda t} - e^{-(t+s)\lambda}}{e^{-\lambda t}} = 1 - e^{-\lambda s}$$

$$= F_X(s)$$

→ memoryless (remember only present)

$$P(X \leq x \mid X>t) = \frac{e^{-\lambda t} - e^{-\lambda x}}{e^{-\lambda t}} = 1 - e^{-\lambda(x-t)} = F_X(x \mid X>t)$$

$$f_X(x \mid X>t) = \frac{d}{dx} F_X(x \mid x>t) = \lambda e^{-\lambda(x-t)}$$

$\hookrightarrow \Delta t.$

4.9 Erlang Distribution

E.D is a generalization of the exponential dist.

- exponential dist ⇒ the time (interval) between adjacent events.
- Erlang dist. ⇒ 〃 between any event and the K^{th} following event.

K^{th}-order Erlang (Erlang-K) R.V. with a parameter λ.

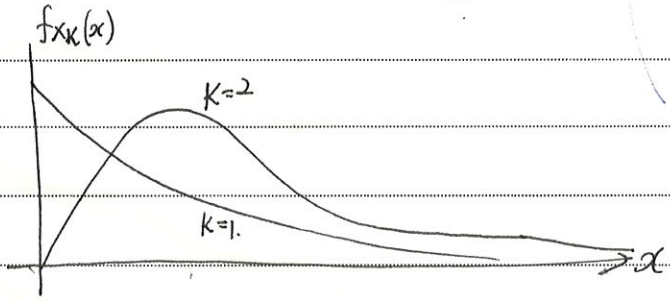
$$f_{X_K}(x) = \begin{cases} \dfrac{\lambda^k \cdot x^{k-1} e^{-\lambda x}}{(k-1)!}, & K = 1, 2, 3, \cdots, \quad x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$F_{X_K}(x) = P(X_K \leq x) = \int_0^x \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} dt$$

$$= 1 - \sum_{j=0}^{K-1} \frac{(\lambda x)^j e^{-\lambda x}}{j!}$$ → HW.

$K \to \infty$ then $F_{X_K}(x) \to 0$.

$\lim_{k \to \infty} \sum_{j=0}^{K-1} \frac{(\lambda x)^j}{j!} \to e^{\lambda x}$ "



Set $u = \lambda x$.

$$E[X_K] = \int_0^\infty x \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} dx = \frac{1}{(k-1)!} \int_0^\infty (\lambda x)^k e^{-\lambda x} dx$$

$$\frac{du}{dx} = \lambda.$$

$$\Rightarrow \frac{1}{(k-1)!} \int_0^\infty u^k e^{-u} \frac{1}{\lambda} du.$$

Gamma function is defined as

$$\Gamma(k+1) = \int_0^\infty u^k e^{-u} du = -u^k e^{-u} \Big]_0^\infty + k \cdot \int_0^\infty u^{k-1} e^{-u} du$$

where the boundary term $-u^k e^{-u} \Big]_0^\infty$ is marked $\to 0$.

$$= k\Gamma(k). = k\cdot(k-1)\Gamma(k-1) = \cdots$$

$$\Gamma(k+1) = k! \qquad k = 0, 1, 2, \cdots$$

$$E[X] = \frac{1}{\lambda} \frac{1}{(k-1)!} \Gamma(k+1) = \frac{1}{\lambda} \frac{1}{(k-1)!} k! = \frac{k}{\lambda}.$$

$$E[X_k^2] = \int_0^\infty x^2 \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} dx = \frac{1}{(k-1)!} \int_0^\infty \frac{\lambda^{k+1} x^{k+1}}{\lambda} e^{-\lambda x} dx$$

$$= \frac{1}{(k-1)!} \frac{1}{\lambda} \int_0^\infty (\lambda x)^{k+1} e^{-\lambda x} dx$$

$$= \frac{1}{(k-1)!} \frac{1}{\lambda} \int_0^\infty \frac{1}{\lambda} t^{k+1} e^{-t} dt = \frac{1}{(k-1)!} \frac{1}{\lambda^2} \Gamma(k+2)$$

$$= \frac{1}{\lambda^2} k\cdot(k+1).$$

$$\sigma_X^2 = \frac{k(k+1)}{\lambda^2} - \frac{k^2}{\lambda^2} = \frac{k}{\lambda^2}$$

4.10 Uniform Distribution → Quantization error.
↳ uniform.

for $[a, b]$.

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{else.} \end{cases}$$

$$F_X(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b \\ 0, & \cancel{\text{else}}\ x < a \\ 1, & x \ge b \end{cases}$$

$E[X] = \dfrac{a+b}{2}$ (Arithmetic mean) → 구간값 (a, b)에 대한 함수

$$E[X^2] = \int_a^b \frac{1}{b-a} \cdot x^2\, dx = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

$$\sigma_X^2 = E[X^2] - E[X]^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12} \qquad b-a = \Delta$$

4.10.1 Discrete Uniform Distribution.

$$P_K(k) = \begin{cases} \dfrac{1}{N}, & k = a, a+1, \cdots, a+N-1. \\ 0, & \text{others.} \end{cases}$$

$$E[K] = \frac{1}{2}\left((a+N-1) + a\right).$$

$$E[K^2] = \frac{1}{6}\left(2N^2 + 6aN + 6a^2 - 6a - 3N + 1\right)$$

$\sigma_K^2 = \dfrac{1}{12}(N^2 - 1)$ → function of # N.

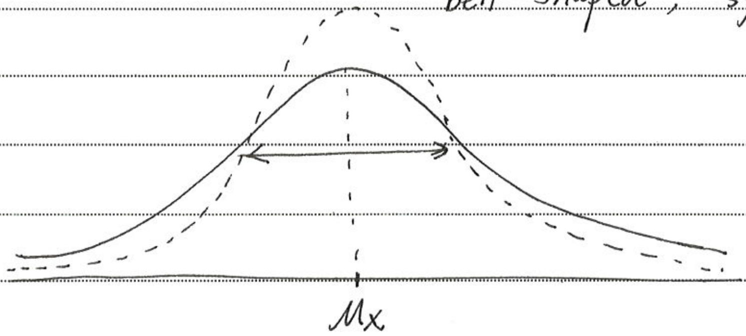
4.11 Normal Distribution.

- continuous R.V. X.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{1}{2\sigma_X^2}(x-\mu_X)^2}, \qquad -\infty < x < \infty.$$

parameter : σ_X^2, μ_X. $\rightarrow N(\mu_X ; \sigma_X^2)$

bell-shaped, symmetric at mean.



spreadness $\overset{\alpha}{\Leftrightarrow}$ σ_X^2.

CDF. $F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{1}{2\sigma_X^2}(t-\mu_X)^2} dt.$

let $u = \frac{t-\mu_X}{\sigma_X}$

$$= \int_{-\infty}^{\frac{x-\mu_X}{\sigma_X}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du. = \Phi\left(\frac{x-\mu_X}{\sigma_X}\right).$$

$\hookrightarrow N(0; 1^2) \rightarrow$ standard Normal Dist.

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$ for $N(0;1)$.

$$\Phi(-x) = 1 - \Phi(x).$$

Finally, we use Lookup table for $\Phi(x)$.

Ex. 4.23. $X = N(3 ; 9)$.

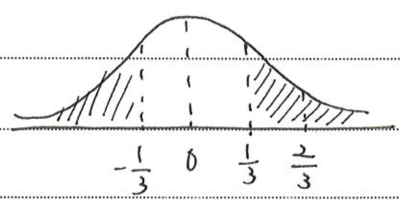
$$P(2 < X < 5) = F_X(5) - F_X(2)$$

$$= \Phi\left(\frac{5-3}{3}\right) - \Phi\left(\frac{2-3}{3}\right)$$

$$= \Phi\left(\tfrac{2}{3}\right) - \Phi\left(-\tfrac{1}{3}\right)$$

$$= \Phi\left(\tfrac{2}{3}\right) - \left\{1 - \Phi\left(\tfrac{1}{3}\right)\right\}$$

$$= \Phi\left(\tfrac{2}{3}\right) + \Phi\left(\tfrac{1}{3}\right) - 1.$$



4.11.1 Normal Approximation to Binomial Dist.

$$B(n; p) \xrightarrow{\text{for large } n} N(np ; np(1-p)) \qquad (n > 5).$$

$$P[a \le X \le b] \longrightarrow P\left[\frac{a - np}{\sqrt{np(1-p)}} \le Z \le \frac{b - np}{\sqrt{np(1-p)}}\right].$$

4.11.2. Error Function.

let $y = \dfrac{x - \mu_x}{\sqrt{2}\,\sigma_x}$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x-\mu_x}{\sqrt{2}\sigma_x}} e^{-y^2} dy.$$

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$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy. \qquad \text{error function.}$$

$\mathrm{erf}(\overset{2}{x}) = 1.$ for $x \to \infty$

↳ HW

Complementary error function.

$$\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}\,dy.$$

for a normal R.V., X

$$P(X > V) = \int_V^\infty \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}\,dx$$

$$Y = \frac{X-\mu_X}{\sqrt{2}\sigma_X}$$

$$= \int_{\frac{V-\mu_X}{\sqrt{2}\sigma_X}}^\infty \frac{1}{\sqrt{\pi}} e^{-y^2}\,dy.$$

$$= \frac{1}{2}\,\text{erfc}\left(\frac{V-\mu_X}{\sqrt{2}\sigma_X}\right)$$

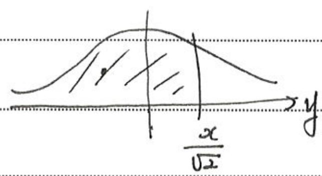
$$\begin{cases} \Phi(x) = \frac{1}{2}\left[1+\text{erf}\left(\frac{x}{\sqrt{2}}\right)\right]. \\ \text{erf}(x) = 2\Phi(x\sqrt{2}) - 1 \end{cases}$$

pf) $\Rightarrow$ $$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du.$$

$u = \sqrt{2}y. \quad du = \sqrt{2}\,dy.$

$$= \int_{-\infty}^{\frac{x}{\sqrt{2}}} \frac{1}{\sqrt{2\pi}} e^{-y^2}\sqrt{2}\,dy.$$

$$= \int_{-\infty}^{\frac{x}{\sqrt{2}}} \frac{1}{\sqrt{\pi}} e^{-y^2}\,dy.$$

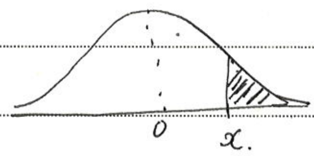


$$2\Phi(x) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{\sqrt{2}}} e^{-y^2}\,dy = 1 + \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{2}}} e^{-y^2}\,dy.$$

$$\Phi(x) = \frac{1}{2}\left(1+\text{erf}\left(\frac{x}{\sqrt{2}}\right)\right)_{//}$$

4. 11. 3. Q - Function.

N(0, 1)



$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{y^2}{2}} \, dy.$$

$$Q(-x) = 1 - Q(x)$$

$$Q(x) = 1 - \Phi(x)$$

Chap. 4. HW.

- derive ; Erlang dist. $F_{X_K}(x) = 1 - \sum_{j=0}^{K-1} \frac{(\lambda x)^j e^{-\lambda x}}{j!}$ in p. 137.
- derive ; $\Phi(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$

4. 14, 4. 17, 4. 24, 4. 37, 4. 41, 4. 46, 4. 47, 4. 53, 4. 59, 4. 62

derive $\text{erf}^2(x) = 1$

4.5. Pascal (Negative Binomial) Distribution

$X = n \Rightarrow$ # of trials until k-th success.

$\Rightarrow$ first, $k-1$ successes out of $n-1$ trials $\rightarrow$ Binomial $(n-1) \to k-1$.
finally, k-th success at n-th trial.

$$P(X=n) = \binom{n-1}{k-1} p^{k-1} \cdot (1-p)^{\overset{(n-1)-(k-1)}{n-k}} \cdot p$$

$$= \binom{n-1}{k-1} p^k (1-p)^{n-k} \quad , \quad k = 1, 2, \cdots, \quad n = k, k+1, \cdots$$

$k=1 \rightarrow$ geometric distribution. $(n \geq k)$.

- To be PMF (or PDF), for a fixed k.

$$\sum_{n=k}^{\infty} P(X=n) = 1.$$

$$\sum_{n=k}^{\infty} \binom{n-1}{k-1} p^k (1-p)^{n-k} = 1.$$

Let, $1-p = q$, $p = 1-q$.

$$\sum_{n=k}^{\infty} \binom{n-1}{k-1} (1-q)^k q^{n-k} = 1.$$

$$(1-q)^{-k} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} q^{n-k} = \sum_{m=0}^{\infty} \binom{k+m-1}{k-1} q^m$$

$\hookrightarrow$ Maclaurin's series.

$$E[X_k] = \sum_{n=k}^{\infty} P(X_k = n) \cdot n$$

$$= \sum_{n=k}^{\infty} \binom{n-1}{k-1} n \cdot p^k \cdot (1-p)^{n-k}.$$

$1-p = q.$

$$= \sum_{n=k}^{\infty} \binom{n-1}{k-1} n \cdot (1-q)^k q^{n-k}$$

$$E[X_k] = (1-q)^k \underline{\sum_{n=k}^{\infty} \binom{n-1}{k-1} n \cdot q^{n-k}}.$$

$$\frac{d}{dq} \Rightarrow \frac{k}{(1-q)^{k+1}} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} (n-k) q^{n-k-1}$$

$$= \frac{1}{q} \sum_{n=k}^{\infty} \binom{n-1}{k-1} n \cdot q^{n-k} - \frac{k}{q} \sum_{n=k}^{\infty} \binom{n-1}{k-1} q^{n-k}$$

$$\times (1-q)^k. \quad \frac{k}{(1-q)} = \frac{1}{q} (1-q)^k \cdot \sum_{n=k}^{\infty} \binom{n-1}{k-1} n \cdot q^{n-k} - \frac{k}{q} (1-q)^k \sum_{n=k}^{\infty} \binom{n-1}{k-1} q^{n-k}$$

$$\frac{k}{1-q} = \frac{E[X_k]}{q} - \frac{k}{q} \Rightarrow E[X_k] = \frac{k}{1-q} = \frac{k}{p} \, //$$

(k - times).

for $k=1$, geometric dist. $E[X_1] = \frac{1}{p}$. k

For variance, first find $E[X_k^2]$

$$\frac{d^2}{dq^2} \Rightarrow \frac{k(k+1)}{(1-q)^{k+2}} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} (n-k)(n-k-1) q^{n-k-2}$$

$$= \frac{1}{q^2} \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(n^2 - n(2k+1) + k(k+1)\right) q^{n-k}$$

$$= \frac{1}{q^2} \sum_{n=k}^{\infty} \binom{n-1}{k-1} n^2 \cdot q^{n-k} - \frac{1}{q^2} \sum_{n=k}^{\infty} \binom{n-1}{k-1} (2k+1) \cdot n \, q^{n-k} + \frac{k(k+1)}{q^2} \sum_{n=k}^{\infty} \binom{n-1}{k-1} q^{n-k}$$

$$x \cdot (1-q)^k \cdot \frac{k(k+1)}{(1-q)^2} = \frac{1}{q^2}(1-q)^k \sum_{n=k}^{\infty} \binom{n-1}{k-1} n^2 q^{n-k}$$

$$- \frac{2k+1}{q^2}(1-q)^k \sum_{n=k}^{\infty} \binom{n-1}{k-1} n \cdot q^{n-k}$$

$$+ \frac{k(k+1)}{q^2}(1-q)^k \sum_{n=k}^{\infty} \binom{n-1}{k-1} q^{n-k}$$

$$= \frac{1}{q^2} E[X_k^2] - \frac{2k+1}{q^2} E[X_k] + \frac{k+1}{q^2} \,//$$

$$E[X_k^2] = \frac{q^2 k(k+1)}{(1-q)^2} + (2k+1)\frac{k}{1-q} - k(k+1)$$

$$= \frac{1}{(1-q)^2} \cdot [k^2 + kq] = \frac{k^2 + k(1-p)}{p^2}$$

$$\sigma_{X_k}^2 = E[X_k^2] - E[X_k]^2 = \frac{k^2 + k(1-p)}{p^2} - \frac{k^2}{p^2} = \frac{k(1-p)}{p^2}$$

↓

for geometric dist. $(k=1)$

k time. variance $\left(\frac{1-p}{p^2}\right)$.

Ex 4.9.

→ m successes before r fails.

⇒ # of trials : from m ~~fr~~ to $m+r-1$.

final trial should be successful.

Let n be # of trials. $m \leq n \leq m+r-1$.

$$\sum_{n=m}^{m+r-1} \binom{n-1}{m-1} p^{m-1} \cdot (1-p)^{n-m} \cdot \underset{\substack{\uparrow \\ \text{last trial.}}}{p} = \sum_{n=m}^{m+r-1} \binom{n-1}{m-1} p^m \cdot (1-p)^{n-m} \,//$$

〈별해〉

$$E[X] = \sum_{n=k}^{\infty} n\binom{n-1}{k-1} p^k (1-p)^{n-k} \qquad \Leftarrow \sum_{n=k}^{\infty} \binom{n-1}{k-1} p^k (1-p)^{n-k} = 1$$

↳ n trials → k successes ⇒ $n \geq k$.

$$= \sum_{n=k}^{\infty} \frac{n \cdot (n-1)!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$= \sum_{n=k}^{\infty} \frac{n! \cdot k}{(n-k)!\,k!} p^k (1-p)^{n-k}$$

$$= \sum_{n=k}^{\infty} \frac{k}{p}\binom{n}{k} p^{k+1} (1-p)^{n-k}$$

$$= \frac{k}{p} \underbrace{\sum_{n=k}^{\infty} \binom{n}{k} p^{k+1} (1-p)^{n-k}}_{1}$$

→ P($n+1$ trials → $k+1$ successes) ⇒ $n \geq k$.

$$= \binom{n}{k} p^{k+1} (1-p)^{n-k}, \quad n \geq k.$$

$$= \frac{k}{p}$$

$$E[X^2] = \sum_{n=k}^{\infty} n^2 \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= \sum_{n=k}^{\infty} \frac{n^2 (n-1)!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

유지

$$= \sum_{n=k}^{\infty} \frac{n \cdot n!}{(n-k)!\,k!} \frac{k}{p} p^{k+1} (1-p)^{n-k} = \sum_{n=k}^{\infty} \frac{(n+1-1)\,n!}{(n-k)!\,k!} \frac{k}{p} p^{k+1} (1-p)^{n-k}$$

$$= \sum_{n=k}^{\infty} \frac{(n+1)\,n!}{(n-k)!\,k!} \frac{k}{p} p^{k+1} (1-p)^{n-k} - \sum_{n=k}^{\infty} \frac{n!}{(n-k)!\,k!} \frac{k}{p} p^{k+1} (1-p)^{n-k}$$

$$= \sum_{n=k}^{\infty} \frac{(n+1)!}{(n-k)!\,(k+1)!} \frac{k(k+1)}{p^2} p^{k+2} (1-p)^{n-k} - \sum_{n=k}^{\infty} \frac{k}{p} \underbrace{\binom{n}{k} p^{k+1} (1-p)^{n-k}}_{P(n+1 \text{ trials} \to k+1 \text{ successes}),\ n \geq k}$$

$=$

$$= \frac{k(k+1)}{p^2} \underbrace{\sum_{n=k}^{\infty} \binom{n+1}{k+1} p^{k+2} (1-p)^{\overset{(n+1)-(k+1)}{n-k}}}_{} - \frac{k}{p}$$

$P(n+2 \text{ trials} \underset{n \geq k}{\rightarrow} k+2 \text{ successes})$

$\binom{n+1}{k+1} p^{k+1} (1-p)^{(n-k)} \cdot p$

$$= \frac{k^2+k}{p^2} - \frac{k}{p}$$

$$\sigma_X^2 = E[X^2] - E[X]^2 = \frac{k^2+k}{p^2} - \frac{k}{p} - \frac{k^2}{p^2}$$

$$= \frac{k}{p^2} - \frac{k}{p} = \frac{k(1-p)}{p^2}$$

Chap. 5 Multiple R.V.

5.2 Joint CDF of Bivariate R.V.

For two S.S and corresponding two R.V X and Y.

joint CDF : $F_{XY}(x, y) = P[X \leq x, \ Y \leq y]$

marginal CDF. : $F_X(x)$, $F_Y(y)$

5.2.1. Properties of joint CDF.

1. $0 \leq F_{XY}(x, y) \leq 1$.

2. if $x_1 \leq x_2$ and $y_1 \leq y_2$, then

$$\begin{cases} F_{XY}(x_1, y_1) \leq F_{XY}(x_2, y_1) \leq F_{XY}(x_2, y_2) \\ F_{XY}(x_1, y_1) \leq F_{XY}(x_1, y_2) \leq F_{XY}(x_2, y_2) \end{cases}$$

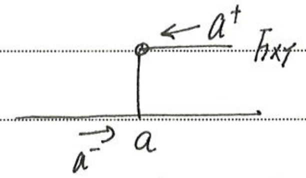
3. $\lim_{\substack{x \to \infty \\ y \to \infty}} F_{XY}(x, y) = F_{XY}(\infty, \infty) = 1$.

4. $\lim_{x \to -\infty} F_{XY}(x, y) = F_{XY}(-\infty, y) = 0$

$\lim_{y \to -\infty} F_{XY}(x, y) = F_{XY}(x, -\infty) = 0$

5. $\lim_{x \to a^+} F_{XY}(x, y) = F_{XY}(a, y)$. right-side continuous.

$\lim_{y \to b^+} F_{XY}(x, y) = F_{XY}(x, b)$



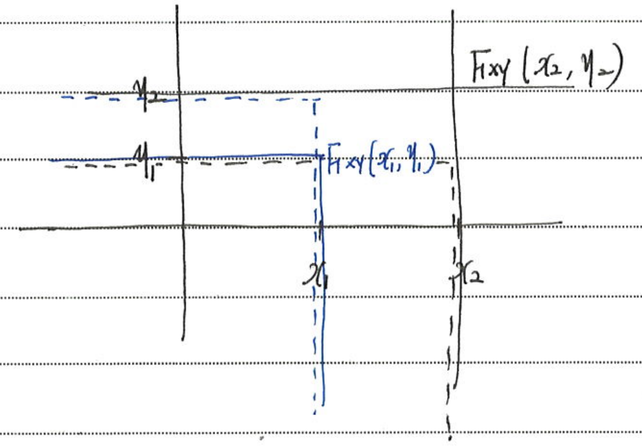
6. $P(x_1 < X \leq x_2, \ Y \leq y) = F_{XY}(x_2, y) - F_{XY}(x_1, y)$

6. $P[X \le x, \quad y_1 < Y \le y_2] = F_{XY}(x, y_2) - F_{XY}(x, y_1)$

7. $x_1 \le x_2$ and $y_1 \le y_2$.

$P[x_1 < X \le x_2, \quad y_1 < Y \le y_2]$

$= F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1)$

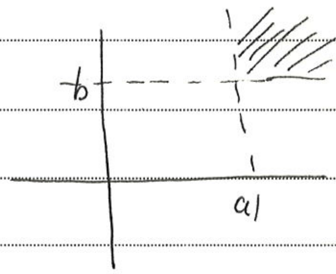


∘ Marginal CDFs of X and Y.

$$\begin{cases} F_X(x) = F_{XY}(x, \infty) \\ F_Y(y) = F_{XY}(\infty, y). \end{cases}$$

Ex. 5.1. $P[X > a, \quad Y > b]$

$= 1 - F_X(a) - F_Y(b) + F_{XY}(a, b).$



5.3 Discrete R. V.

$P_{XY}(x, y) = P[X = x, \quad Y = y].$

1. $0 \le P_{XY}(x, y) \le 1.$

2. $\sum_x \sum_y P_{XY}(x, y) = 1$

3. $\sum_{x \le a} \sum_{y \le b} P_{XY}(x, y) = F_{XY}(a, b)$.

4. $\begin{cases} P_X(x) = \sum_y P_{XY}(x, y) \\ P_Y(y) = \sum_x P_{XY}(x, y). \end{cases}$

5. Independence. $P(A \cap B) = P(A) \cdot P(B)$.

$P_{XY}(x, y) = P_X(x) \cdot P_Y(y)$.

Ex 5.2 $P_{XY}(x, y) = \begin{cases} k(2x + y), & x = 1, 2, \quad y = 1, 2. \\ 0, & \text{otherwise}. \end{cases}$

(a) $k = ?$

$\sum_x \sum_y P_{XY}(x, y) = 1. = k(3 + 4 + 5 + 6), \qquad k = \frac{1}{18}$

(b) $\begin{cases} P_X(x) = \sum_y P_{XY}(x, y) = \frac{1}{18}(2x+1) + \frac{1}{18}(2x+2) = \frac{1}{18}(4x+3). \quad x=1,2 \\ P_Y(y) = \sum_x P_{XY}(x, y) = \frac{1}{18}(2+y) + \frac{1}{18}(4+y) = \frac{1}{18}(2y+6) \end{cases}$

(c) $P_{XY}(x, y) \neq P_X(x) \cdot P_Y(y)$. $\rightarrow$ not independent.

Ex 5.3. $\begin{cases} X \to \text{Poisson dist.} & \text{mean} = \lambda. \\ Y \to \text{robbery} . \ p \end{cases}$

$$P_{XY}(x, y) = P[X=x, Y=y] = \frac{\lambda^{x} e^{-\lambda}}{x!} \binom{x}{y} p^{y} (1-p)^{x-y}.$$

$$P_Y(y) = \sum_x P_{XY}(x, y) = \sum_x \frac{\lambda^{x} e^{-\lambda}}{x!} \frac{x!}{(x-y)!\, y!} p^y (1-p)^{x-y}.$$

$$= \frac{\lambda^y e^{-\lambda}}{y!} p^y \cdot \sum_x \frac{\lambda^{x-y}}{(x-y)!} (1-p)^{x-y}.$$

$$= \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_x \frac{[\lambda \cdot (1-p)]^{x-y}}{(x-y)!} \qquad x-y = k.$$

$$= \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_k \frac{[\lambda(1-p)]^k}{k!} = \frac{(\lambda p)^y e^{-\lambda}}{y!} e^{\lambda(1-p)} = \frac{(\lambda p)^y \cdot e^{-\lambda p}}{y!}$$

↳ poisson dist. with mean $= \lambda p$.

Ex 5.4) 3 coins tossing : X : 1st toss $\begin{cases} \text{head} \Rightarrow 1, \\ \text{else} \Rightarrow 0 \end{cases}$ $\quad Y$: # of heads. $P_{XY}(x,y)$?

5.4 Continuous R.V.

· joint PDF : $f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$

$$F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(u, v)\, du\, dv$$

1. $f_{XY}(x, y) \geqq 0 \quad$ for $\forall x, y$.

2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\, dy = 1.$

3. $P[x_1 < X \leqq x_2,\ y_1 < Y \leqq y_2] = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{XY}(x, y)\, dx\, dy.$

- marginal PDF.

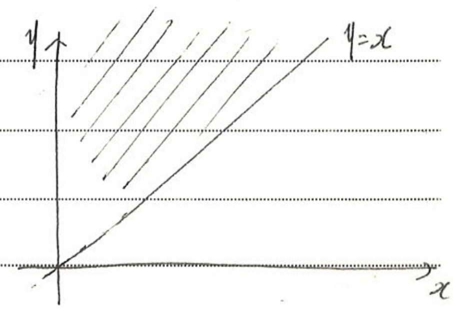
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx.$$

- independent.

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

Ex 5.6 $f_{XY}(x, y) = \begin{cases} 2e^{-(x+y)}, & 0 \leq x \leq y, \quad 0 \leq y < \infty \\ 0, & \text{others} \end{cases}$



$$f_X(x) = \int f_{XY}(x, y)\, dy.$$

$$= \int_{x}^{\infty} 2e^{-x} \cdot e^{-y} \cdot dy$$

$$= 2e^{-x} \left[-e^{-y} \right]_{x}^{\infty} = 2e^{-2x}, \quad (x \geq 0).$$

$$f_Y(y) = \int 2e^{-x} e^{-y}\, dx = \int_{0}^{y} 2e^{-x} e^{-y}\, dx = 2e^{-y} \left[-e^{-x} \right]_{0}^{y}$$

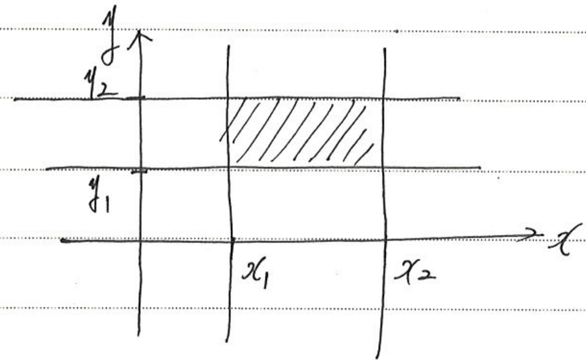
$$= 2e^{-y}(1 - e^{-y}), \quad (y \geq 0).$$

$f_{XY}(x, y) \neq f_X(x) \cdot f_Y(y)$. → not independent.

5.5 Determining Prob. from a joint CDF.

given a joint CDF, $F_{XY}(x, y)$

$P(x_1 < X \leq x_2,\ y_1 < Y \leq y_2)$



$$= F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$$

$$= \begin{cases} \displaystyle\int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x, y)\, dx\, dy \\ \displaystyle\sum_{y \geq y_1}^{y_2} \sum_{x \geq x_1}^{x_2} P_{XY}(x, y) \end{cases}$$

Ex. 5.7 >

$$F_{XY}(x, y) = \begin{cases} \frac{1}{8} & x=1,\ y=1 \\ \frac{5}{8} & x=1,\ y=2 \\ \frac{1}{4} & x=2,\ y=1 \\ 1 & x=2,\ y=2 \end{cases}$$

1) Joint PMF ?

$P_{XY}(1, 1) = \frac{1}{8} = F_{XY}(1, 1)$

$F_{XY}(1, 2) = P(X \leq 1,\ Y \leq 2) = P(X=1, Y=1) + P(X=1, Y=2)$

$= \frac{1}{8} + P_{XY}(x=1, y=2) = \frac{5}{8}$.

$\Rightarrow P_{XY}(1, 2) = \frac{1}{2}$

$F_{XY}(2, 1) = P_{XY}(1, 1) + P_{XY}(2, 1) = \frac{1}{4}$

$\Rightarrow \frac{1}{8} + P_{XY}(2, 1) = \frac{1}{4}$

$\Rightarrow P_{XY}(2, 1) = \frac{1}{8}$

$F_{XY}(2,2) = 1 = P_{XY}(1,1) + P_{XY}(1,2) + P_{XY}(2,1) + P_{XY}(2,2)$

$= \frac{1}{8} + \frac{1}{2} + \frac{1}{8} + P_{XY}(2,2) = 1$

$\Rightarrow P_{XY}(2,2) = \frac{1}{4}$

2) Marginal PMF.

$P_X(x) = \sum_{y} P_{XY}(x,y) = F_{XY}(x, \infty)$

$P_X(1) = P_{XY}(1,1) + P_{XY}(1,2) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$

$P_X(2) = P_{XY}(2,1) + P_{XY}(2,2) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

$\Rightarrow 1$

$P_Y(1) = P_{XY}(1,1) + P_{XY}(2,1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

$P_Y(2) = P_{XY}(1,2) + P_{XY}(2,2) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$\Rightarrow 1.$

3) Independent ?

$P_{XY}(1,1) = P_X(1) \cdot P_Y(1) \Leftrightarrow \frac{1}{8} \neq \frac{5}{8} \cdot \frac{1}{4}$ $\rightarrow$ not indep.

5.6 Conditional Distributions

5.6.1 Discrete R.V.

$$P_{Y|X}(y|x) = \frac{P[X=x,\ Y=y]}{P(X=x)} = \frac{P_{XY}(x,y)}{P_X(x)}$$

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)}$$

$$F_{X|Y}(x|y) = P[X \le x \mid Y=y] = \sum_{u \le x} P_{X|Y}(u|y)$$

$$F_{Y|X}(y|x) = P[Y \le y \mid X=x] = \sum_{u \le y} P_{Y|X}(u|x)$$

5.6.2. Conditional PDF of Continuous R.V.

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}, \qquad f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$\Rightarrow \quad f(y|x) = \frac{\partial}{\partial y} F_Y(y \mid X=x) = \frac{\partial}{\partial y} P(Y \le y \mid X=x)$$

$$= \lim_{\Delta x \to 0} \frac{\partial}{\partial y} P(Y \le y \mid x < X \le x+\Delta x)$$

$$= \lim_{\Delta x \to 0} \frac{\partial}{\partial y} \frac{P(Y \le y,\ x < X \le x+\Delta x)}{P(x < X \le x+\Delta x)} \quad \to \quad F_X(x+\Delta x) - F_X(x)$$

$$= \lim_{\Delta x \to 0} \frac{\partial}{\partial y} \frac{F_{XY}(x+\Delta x, y) - F_{XY}(x,y)}{f_X(x)\,\Delta x} \quad \Leftarrow$$

$$= \cancel{\lim_{\Delta x \to 0}} \frac{\partial^2}{\partial y\,\partial x} \frac{F_{XY}(x,y)}{f_X(x)} = \frac{f_{XY}(x,y)}{f_X(x)}$$

∘ independence.

$$f_{X|Y}(x|y) = f_X(x), \quad f_{Y|X}(y|x) = f_Y(y)$$
$$\Rightarrow \quad f_{XY}(x,y) = f_X(x)\cdot f_Y(y).$$

Ex 5.9.

$$f_{XY}(x,y) = \begin{cases} x\cdot e^{-x(y+1)}, & 0 \le x < \infty,\ 0 \le y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

first, $f_X(x) = \int_0^\infty x\cdot e^{-x(y+1)}\,dy = x\cdot e^{-x}\int_0^\infty e^{-xy}\,dy$

$$= xe^{-x}\cdot\left[-\frac{1}{x}e^{-xy}\right]_0^\infty = e^{-x}$$

$$f_Y(y) = \int_0^\infty x e^{-x(y+1)}\,dx = \left[\frac{-1}{y+1}x\, e^{-x(y+1)}\right]_0^\infty + \int_0^\infty \frac{1}{y+1}e^{-x(y+1)}\,dx$$

$$= -\frac{1}{(y+1)^2}e^{-x(y+1)}\Big|_0^\infty = \frac{1}{(y+1)^2}$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = x(y+1)^2 e^{-x(y+1)}$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = x\cdot e^{-xy}$$

5.6.3 Conditional Means & Variances.

∘ for discrete case,

$$\mu_{Y|X} = E[Y|X=x] = \sum y\cdot P_{Y|X}(y|x)$$
$$\sigma_{Y|X}^2 = E[Y^2|X=x] - \mu_{Y|X}^2$$

- for continuous case,

$$\mu_{Y|X} = E[Y|X=x] = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\, dy$$

Ex 5.10

$$f_{XY}(x,y) = \begin{cases} \dfrac{e^{-(\frac{x}{y})} e^{-y}}{y}, & 0 \leq x < \infty, \quad 0 \leq y < \infty \\ 0, & \text{otherwise} \end{cases}$$

find $E[X|Y=y]$.

$\Rightarrow$ first find $f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$.

$$f_Y(y) = \int_{\cancel{-\infty}}^{\infty} f_{XY}(x,y)\, dx = \int_{\cancel{-\infty}}^{\infty} \frac{e^{-(\frac{x}{y})} e^{-y}}{y}\, dx$$

$$= \left[-e^{-\frac{x}{y}} e^{-y}\right]_0^{\infty} = e^{-y}.$$

$$E[X|Y=y] = \int_0^{\infty} x \cdot f_{X|Y}(x|y)\, dx$$

$$= \int_0^{\infty} x \cdot \frac{e^{-\frac{x}{y}} \cdot \cancel{e^{-y}}}{\cancel{e^{-y}} \cdot y}\, dx$$

$$= \left(\frac{1}{y}\right)\left[-x \cdot y\, e^{-\frac{x}{y}}\right]_0^{\infty} + \int_0^{\infty} e^{-\frac{x}{y}}\, dx$$

$$= y_{\,\prime\prime} \longrightarrow g(Y)$$

$E[g(X)] = \int g(x) f_X(x)\, dx, \qquad E[g(X,Y)] = \iint g(x,y) f_{XY}(x,y)\, dx\, dy$

Ex. 5.11. $E[E[X|Y]] = ?$

$$E[E[X|Y]] = E\left[\int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx.\right]$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx.\right] f_Y(y)\, dy.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, f_Y(y)\, dx\, dy.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot \frac{f_{XY}(x,y)}{\cancel{f_Y(y)}} \cancel{f_Y(y)}\, dy \cdot dx$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = E[X].$$

5.6.4. Simple Rule for Independence.

If PDF X and Y, $f_{XY}(x,y)$ = const. × x-factor × y-factor in the rectangular region (Sample space).

then, X and Y are independent.

5.7 Covariance & Correlation Coefficient.

$$\mathrm{Cov}(X,Y) = \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)]$$

$$= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y].$$

$$= E[XY] - \mu_X \mu_Y.$$

if X and Y are independent,

$$\left(\begin{array}{l} E[XY] = \mu_X \mu_Y. \Leftarrow \iint xy\, f_X(x) f_Y(y)\, dx\, dy \\ \sigma_{XY} = 0. \end{array} \right.$$

∘ X and Y are independent $\underset{X}{\overset{O}{\rightleftarrows}}$ $\sigma_{XY} = 0$

∘ uncorrelated, if $\sigma_{XY} = 0$.

∘ correlation coefficient, $\rho_{XY} = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}$, $|\rho_{XY}| \leq 1$.

proof). by Cauchy - Schwarz Inequality.

$$(ax+by+cz)^2 \leq (a^2+b^2+c^2)\cdot(x^2+y^2+z^2)$$

$$|\vec{X}\cdot\vec{Y}|^2 \leq |\vec{X}|^2\cdot|\vec{Y}|^2$$

$$\left[\int f(x)\cdot g(x)\,dx\right]^2 \leq \left(\int f(x)^2\,dx\right)\left(\int g(x)^2\,dx\right).$$

$$\int (af(x)+g(x))^2\,dx \geq 0$$

$$= a^2\int f(x)^2\,dx + 2a\int f(x)\cdot g(x)\,dx + \int g(x)^2\,dx \geq 0$$

$$D/4 = \left[\int f(x)\cdot g(x)\,dx\right]^2 - \int f(x)^2\,dx \cdot \int g(x)^2\,dx \leq 0.$$

$$\sigma_{XY}^2 = \left[\iint (X-\mu_X)(Y-\mu_X)\, f_{XY}(x,y)\,dx\,dy\right]^2$$

$$\leq \left(\iint (x-\mu_X)^2 f_{XY}(x,y)\,dx\,dy\right)\cdot\left(\iint (Y-\mu_Y)^2 f_{XY}(x,y)\,dx\,dy\right)$$

$$= \sigma_X^2\cdot\sigma_Y^2.$$

$$\rho_{XY}^2 = \frac{\sigma_{XY}^2}{\sigma_X^2\sigma_Y^2} \leq 1. \Rightarrow |\rho_{XY}| \leq 1.$$

↳ How well two R.V.s are matched.

if $\begin{cases} \rho_{XY} \Rightarrow 1 & Y = bX + a, \quad b>0 \\ \rho_{XY} \Rightarrow -1 & Y = bX + a, \quad b<0 \end{cases} \Rightarrow$ linearly correlated.

$\begin{cases} \rho_{XY} > 0 \ ; & X\uparrow \rightarrow Y\uparrow \\ \rho_{XY} < 0 \ ; & X\uparrow \rightarrow Y\downarrow \end{cases}$

$\rho_{XY} = 0$; No linear relation between X & Y

Ex. 5.13

$$f_{XY}(x,y) = \begin{cases} 25e^{-5y}, & 0<x<0.2, \quad y>0 \\ 0, & \text{otherwise.} \end{cases}$$

$$f_X(x) = \int_0^\infty 25\cdot e^{-5y}\,dy = 5, \quad 0<x<0.2$$

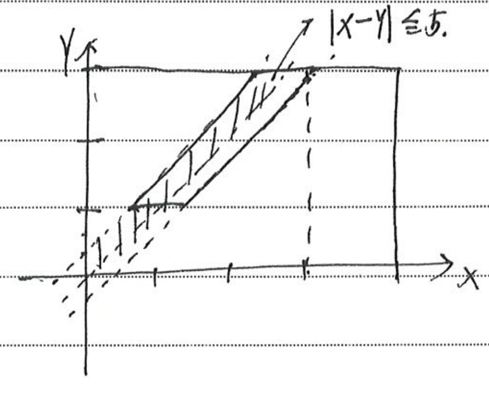
$$f_Y(y) = \int_0^{0.2} 25e^{-5y}\,dx = 5\cdot e^{-5y}, \quad 0<y$$

$\begin{cases} \sigma_{XY} = E[XY] - \mu_X\mu_Y = 0 & \Rightarrow \text{uncorrelated} \\ f_{XY}(x,y) = f_X(x)\cdot f_Y(y) & \rightarrow \text{independent.} \end{cases}$

Ex. 5.14. arrival time X, Y. for Hans, Ann.

$$f_X(x) = \begin{cases} \frac{1}{60}, & 0 \le X \le 60 \\ 0, & \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{30}, & 15 \le Y \le 45. \\ 0, & \end{cases}$$



meet within 5 min. difference,

$$P(|X-Y| \le 5) = \frac{1}{6}.$$

5.8 Many R.V.s

Joint CDF. $F_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n)$

Joint PMF. $P_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots x_n) = P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)$

joint PDF : $f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n) = \dfrac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} F_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n)$

- $f_{X_n | X_1, \cdots, X_{n-1}}(x_n | x_1, x_2, \cdots, x_{n-1}) = \dfrac{f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n)}{f_{X_1 \cdots X_{n-1}}(x_1, \cdots, x_{n-1})}$

- independence,

$$f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots x_n) = \prod_{i=1}^{n} f_{X_i}(x_i).$$

5.9. Multinomial Distributions.

Iterative n trials, $k_1, k_2 \cdots, k_m$, $\sum_{i=1}^{m} k_i = n$.

m - outcomes with $p_1, p_2, \cdots p_m$, $\sum_{i=1}^{m} p_i = 1$.

$$P(k_1, k_2, \cdots, k_m) = \binom{n}{k_1\ k_2 \cdots k_m} p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$$

$$= \frac{n!}{k_1!\, k_2! \cdots k_m!} p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$$

HW. 5.4, 5.9, 5.10, 5.11, 5.13, 5.19, 5.22, 5.23

◎ Joint Gaussian RVs. (Bivariate)

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right\}$$

ρ : correlation coeff. $= \dfrac{\sigma_{XY}}{\sigma_X\sigma_Y}$.

$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)]$.

- maximum is obtained at $(x,y) = (\mu_X, \mu_Y)$.

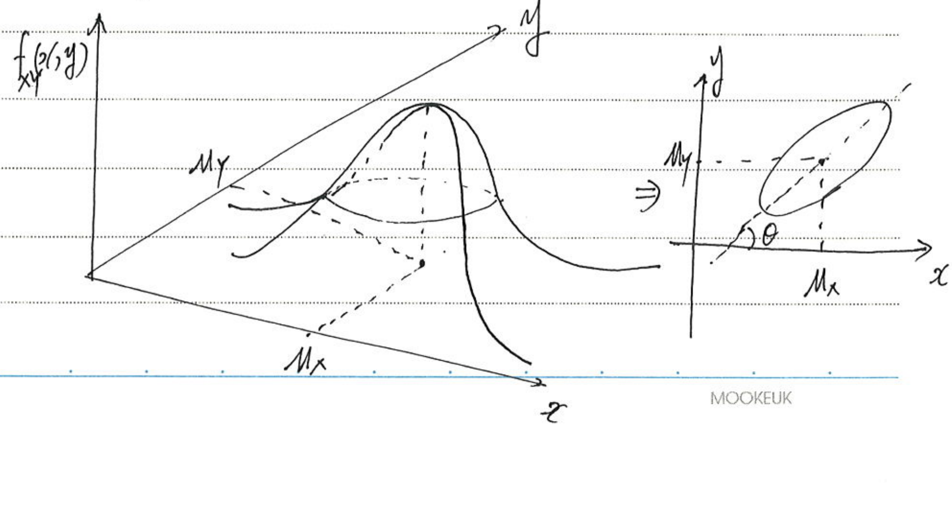
$$f_{XY}(x,y) = f_{XY}(\mu_X,\mu_Y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}.$$

- if $\rho = 0$, $\rightarrow$ uncorrelated,

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_Y} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

$$= f_X(x)\cdot f_Y(y) \rightarrow \text{independent.}$$

$\Rightarrow$ for joint Gaussian distribution,
uncorrelated $\underset{\circ}{\overset{\circ}{\rightleftarrows}}$ independent.

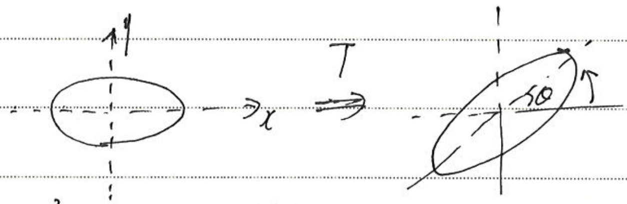


rotation angle

$$\theta = \frac{1}{2}\tan^{-1}\left[\frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2-\sigma_Y^2}\right].$$

pf). set $\mu_X = \mu_Y \Rightarrow 0$,,

by rotation transform $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{T} \begin{pmatrix} x \\ y \end{pmatrix}$



$$\frac{x^2}{2\sigma_X^2} + \frac{y^2}{2\sigma_Y^2} \xrightarrow{T} \frac{1}{2(1-\rho^2)}\left(\frac{x'^2}{\sigma_X^2} - \frac{2\rho\, x'y'}{\sigma_X\sigma_Y} + \frac{y'^2}{\sigma_Y^2}\right) \rightarrow ①$$

$$\begin{cases} x' = x\cos\theta - y\sin\theta, \\ y' = x\sin\theta + y\cos\theta \end{cases} \longrightarrow \text{insert into left-side in (1).}$$

and set the term of $x'y' = 0$.

$$\frac{1}{\sigma_X^2}\left(x^2C^2 - 2xyCS + y^2S^2\right) + \frac{1}{\sigma_Y^2}\left(x^2S^2 + 2xyCS + y^2C^2\right) - \frac{2\rho}{\sigma_X\sigma_Y}\left(x^2CS - y^2CS + xy(C^2-S^2)\right)$$

xy term : $-\frac{2CS}{\sigma_X^2} + \frac{2CS}{\sigma_Y^2} - \frac{2\rho}{\sigma_X\sigma_Y}(C^2-S^2) = 0.$

$\otimes\, \sigma_X^2\sigma_Y^2$

$$CS\cdot(\sigma_X^2 - \sigma_Y^2) = \rho\sigma_X\sigma_Y(C^2-S^2)$$

$$\frac{1}{2}\sin 2\theta\,(\sigma_X^2 - \sigma_Y^2) = \rho\sigma_X\sigma_Y\cos 2\theta.$$

$$\tan 2\theta = \frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2-\sigma_Y^2} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2-\sigma_Y^2}\right)$$

◎ N multiple Gaussian RV's.

$f_{X_i}(x_i) \rightarrow N(\mu_i, \sigma_i^2)$

↳ 정규분포.

$f_{X_1 X_2 \cdots X_N}(x_1, x_2, \cdots, x_N)$

$$= \frac{|C_X^{-1}|^{\frac{1}{2}}}{(2\pi)^{\frac{N}{2}}} \exp\left\{ - \frac{[X-\bar{X}]^T C_X^{-1} [X-\bar{X}]}{2} \right\}$$

$$[X-\bar{X}] = \begin{bmatrix} X_1 - \mu_{X_1} \\ X_2 - \mu_{X_2} \\ \vdots \\ X_N - \mu_{X_N} \end{bmatrix}$$

$$C_X = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & & & \\ \vdots & & & \\ C_{N1} & & & C_{NN} \end{bmatrix}$$

→ ~~Correlation~~ Covariance matrix.

$$C_{ij} = E[(X_i - \mu_{X_i})(X_j - \mu_{X_j})]$$

$C_{ii} = \sigma_{X_i}^2$

Example) $N=2$.

$$C_X = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix} \leftarrow \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}$$

$$C_X^{-1} = \frac{1}{(1-\rho^2)\sigma_X^2\sigma_Y^2} \begin{bmatrix} \sigma_Y^2 & -\rho\sigma_X\sigma_Y \\ -\rho\sigma_X\sigma_Y & \sigma_X^2 \end{bmatrix} = \frac{1}{(1-\rho^2)} \begin{bmatrix} 1/\sigma_X^2 & -\frac{\rho}{\sigma_X\sigma_Y} \\ -\frac{\rho}{\sigma_X\sigma_Y} & 1/\sigma_Y^2 \end{bmatrix}$$

$$|C_X^{-1}| = \frac{1}{(1-\rho^2)^2}\left(\frac{1}{\sigma_X^2\sigma_Y^2} - \frac{\rho^2}{\sigma_X^2\sigma_Y^2}\right)$$

$$= \frac{1}{(1-\rho^2)}\frac{1}{\sigma_X^2\sigma_Y^2}\,_{//}$$

$$f_{XY}(x,y) = \frac{1}{(2\pi)^1\cdot\sqrt{1-\rho^2}\,\sigma_X\sigma_Y}\exp\left\{-\frac{\begin{bmatrix}X-\mu_X\\Y-\mu_Y\end{bmatrix}^T\frac{1}{(1-\rho^2)}\begin{bmatrix}\frac{1}{\sigma_X^2} & -\frac{\rho}{\sigma_X\sigma_Y}\\ -\frac{\rho}{\sigma_X\sigma_Y} & \frac{1}{\sigma_Y^2}\end{bmatrix}\begin{bmatrix}X-\mu_X\\Y-\mu_Y\end{bmatrix}}{2}\right\}$$

$$= \frac{1}{(2\pi)\sqrt{1-\rho^2}\,\sigma_X\sigma_Y}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)\right\}_{//}$$

Chapter. 6 Functions of R.V.

6.2 Functions of R.V.

$$Y = g(X) \Rightarrow \begin{cases} F_Y(y)\ ? \\ f_Y(y)\ ? \end{cases} \text{ given } F_X(x),\ f_X(x)$$

6.2.1. Linear Functions.

$$Y = g(X) = aX + b.$$

(i) $a > 0$,

$$F_Y(y) = P(Y \le y) = P(aX + b \le y)$$

$$= P\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(\frac{y-b}{a}\right) = \frac{1}{a}f_X\left(\frac{y-b}{a}\right)$$

(ii) $a < 0$.

$$F_Y(y) = P(aX + b \le y) = P\left(X \ge \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = -\frac{1}{a}f_X\left(\frac{y-b}{a}\right)$$

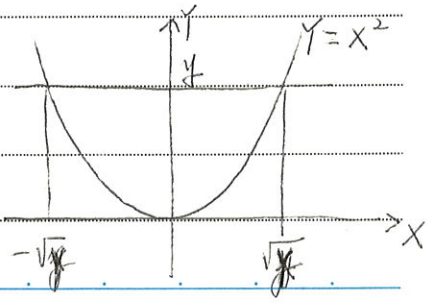
$$f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right)$$

1) monotonically increasing case

2) monotonically decreasing case.

6.2.2 Power Functions.

$$Y = X^2$$



$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$
$$= P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

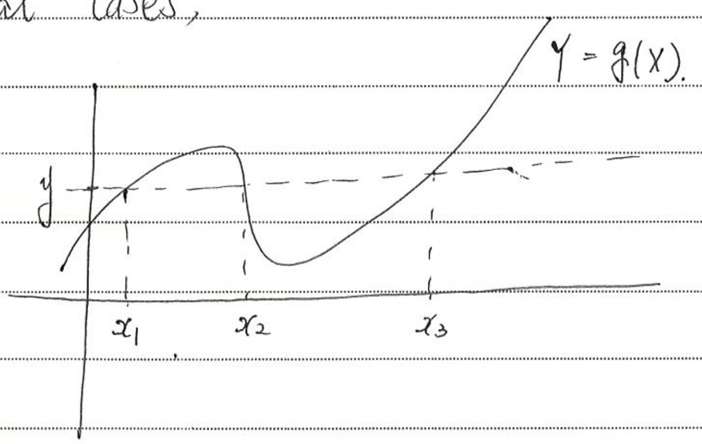
$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right)$$
$$= \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) \quad , y>0$$

Ex 6.2.

$$f_X(x) \sim N(0,1) \quad , \quad Y = X^2$$

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}\right)$$
$$= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \quad , \quad (y>0)$$

◦ For general cases,



$Y = g(X)$ → 3 intersections for y.

$$F_Y(y) = P(Y \le y) = P(X \le x_1) + P(x_2 < X \le x_3)$$
$$= F_X(x_1) + F_X(x_3) - F_X(x_2).$$

$$f_Y(y) = \frac{d}{dy}\left(F_X(x_1) + F_X(x_3) - F_X(x_2) \right)$$

$$= \frac{dx_1}{dy} f_X(x_1) + \frac{dx_3}{dy} f_X(x_3) - \frac{dx_2}{dy} f_X(x_2)$$

$$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_3)}{g'(x_3)} + \frac{f_X(x_2)}{|g'(x_2)|}$$

$g'(x_2) < 0.$

$$= \sum_{i=1}^{3} \frac{f_X(x_i)}{|g'(x_i)|}, \quad g(x_i) = y.$$

6.3 Expectation of a Function of one R.V.

$Y = g(X)$, given $f_X(x)$.

$$E[Y] = E[g(X)] = \begin{cases} \sum g(x)\, p(x) \\ \int g(x) f_X(x)\, dx \end{cases}$$

$\int y \cdot f_Y(y)\, dy$

for a linear function, $g(X) = aX + b, = Y$

$$E[Y] = E[aX+b] = aE[X] + b.$$

$$Var[Y] = E[(Y - E[Y])^2]$$

$$= \int [(ax+b) - (aE(X)+b)]^2\, dx$$

$$= \int a^2 (x - E[X])^2\, dx = a^2 \sigma_X^2$$

Convolution.

$$y(t) = \int x(\tau) \cdot h(t-\tau)\, d\tau$$ → 좌우 reflection & shift.

$$y[n] = \sum x[k] \cdot h[n-k].$$

examples.

1) $x(t) = e^{-t}u(t)$, $h(t) = e^{-2t}u(t)$

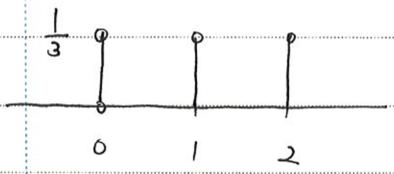
2) $x(t) = e^{-t+2}u(t-2)$, $h(t) = e^{-2t}u(t)$.

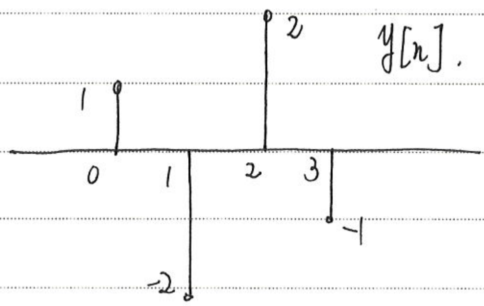
3) $x(t) = u(t) - u(t-2)$, $h(t) = u(t) - u(t-1)$

4) $x(t) = u(t+1) - u(t-1)$, $h(t) = e^{-t}u(t)$.

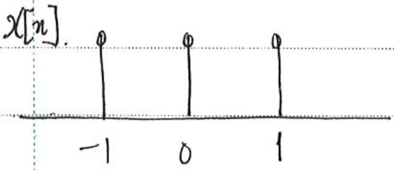
o for discrete case.

5) $x[n]$





6) $x[n]$.



⇕

$y[n]$.

7) $x[n] = \left(\frac{1}{3}\right)^n u[n]$,
$y[n] = (2)^n u[n]$

◦ Convolution 개념부터 ... 소개. (예제 2~3개).

6.4 Sums of Independent R.V.

For two independent R.V.s X and Y

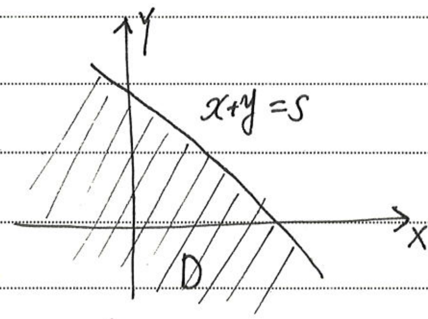
$$S = g(X, Y) = X + Y \quad , \quad \text{given } f_{XY}(x, y)$$

$$F_S(s) = P(S \leq s) = P(X + Y \leq s) = \iint_D f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{s-y} f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{s-y} f_X(x) \cdot dx \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} F_X(s-y) \cdot f_Y(y)\, dy$$



$$f_S(s) = \frac{d}{ds} F_S(s) = \int_{-\infty}^{\infty} f_X(s-y) f_Y(y)\, dy =$$

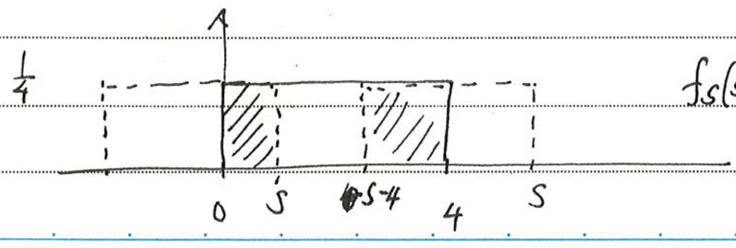
$$= \int_{-\infty}^{\infty} f_X(x) f_Y(s-x)\, dx = \cancel{f_X(u) * f_Y(u)}$$

$$= f_X(s) * f_Y(s)$$

⇒ convolution.

Ex. 6.3

$$f_X(x) = f_Y(y) = \begin{cases} \frac{1}{4}, & 0 < x, y < 4 \\ 0, & 0 \end{cases}$$



$$f_S(s) = \begin{cases} \frac{1}{16} s & , 0 \leq s \leq 4 \\ \frac{1}{16}(8-s) & , 4 \leq s \leq 8 \\ 0 & \text{others} \end{cases}$$

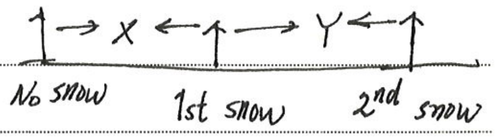
Ex. 6.5.

· time X between consecutive snowstorms (events)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0. \\ 0, & \end{cases}$$

time U between ~~two~~ second snowstorms.

$U = X + Y$. (assume X and Y are independent).



$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y \geq 0 \\ 0, & \end{cases}$$ ← due to memoryless property.

$$f_U(u) = f_X(u) * f_Y(u)$$

$$= \int_0^{\infty} f_X(x)\, f_Y(u-x)\, dx$$

$$= \int_0^{u} \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(u-x)}\, dx$$

$$= \lambda^2 \int_0^{u} e^{-\lambda u}\, dx = \lambda^2 \cdot u e^{-\lambda u}, \quad (u \geq 0)$$

$$\int_0^{u} \lambda^2 x e^{-\lambda x} \cdot \lambda e^{-\lambda(u-x)}\, dx$$

$$= \lambda^3 \int_0^{u} x \cdot e^{-\lambda u}\, dx$$

$$= \lambda^3 \cdot \frac{u^2}{2} e^{-\lambda u} \quad u(\lambda u).$$

⇒ Erlang-2 distribution.

$$f_{X_k}(x) = \frac{\lambda^k \cdot x^{k-1} e^{-\lambda x}}{(k-1)!} \quad \text{(Erlang-K.)} \leftarrow E\text{-}3$$

6.4.1 Moments of the Sum of RVs.

$S = X + Y.$

$$E(S) = E[X+Y]$$
$$= \iint (x+y) f_{XY}(x,y)\,dx\,dy$$
$$= \iint x \cdot f_{XY}(x,y)\,dx\,dy + \iint y\, f_{XY}(x,y)\,dx\,dy$$
$$= \int x \cdot f_X(x)\,dx + \int y\, f_Y(y)\,dy = E[X] + E[Y].$$

$$\sigma_S^2 = E[(X+Y-E[X]-E[Y])^2]$$
$$= E[(X-\mu_X)^2 + 2(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2].$$
$$= \sigma_X^2 + \sigma_Y^2 + 2\,COV(X,Y)$$

if X and Y are $\begin{cases}\text{independent}\\ \text{uncorrelated}\end{cases}$, $\sigma_S^2 = \sigma_X^2 + \sigma_Y^2.$

6.4.2 Sum of Discrete RVs.

$Z = X + Y.$

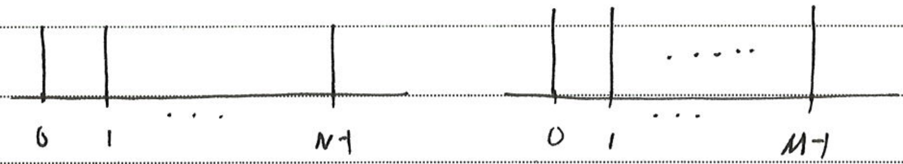
$$P_Z(z) = P[Z=z] = P[X+Y=z]$$
$$= \sum_k P[X=k,\ Y=z-k].$$
$$= \sum_{k \le z} P_{XY}(k, z-k)$$

if X and Y are independent.

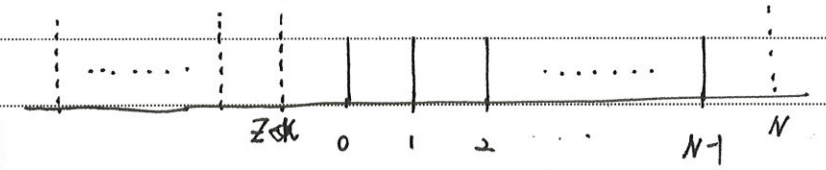
$$P_Z(z) = \sum_{k \le z} P_X(k)\,P_Y(z-k).$$

Ex 6.8 $\begin{cases} P_X(x) = \frac{1}{M}, & x = 0, 1, 2, \cdots M-1 \\ P_Y(y) = \frac{1}{N}, & y = 0, 1, 2, \cdots, N-1 \end{cases}$

$N > M, \quad Z = X + Y$



$$P_Z = \sum_{k \le Z} P_X(k) P_Y(Z-k)$$



(i) $0 \leq Z \leq M-1$

$$P_Z(z) = \frac{z+1}{N \cdot M}$$

(ii) $M \leq z \leq N-1$

$$P_Z(z) = \frac{1}{N}$$

(iii) $N \leq z \leq N+M-2$

$$P_Z(z) = \frac{M - (z-(N-1))}{N \cdot M} = \frac{M+N-z-1}{N \cdot M}$$

iv else

$$P_Z(z) = 0.$$

6.4.3. Sum of Independent R.Vs

Binomial.

$$P_Z(z) = \sum_{k \le z} P_X(k) \cdot P_Y(z-k) \qquad \begin{cases} X \sim (n, p) \\ Y \sim (m, p) \end{cases}$$

$$= \sum_{k \le z} \binom{n}{k} p^k (1-p)^{n-k} \binom{m}{z-k} p^{z-k} (1-p)^{m-z+k}$$

$$= \sum_{k \le z} \binom{n}{k} \binom{m}{z-k} p^{z} (1-p)^{n+m-z}.$$

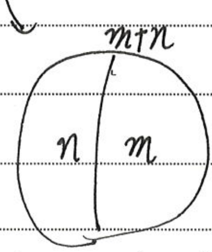
$$= \sum_{k \le z} \frac{n!}{k!\,(n-k)!} \frac{m!}{(z-k)!\,(m-z+k)!} \cdot p^{z} \cdot (1-p)^{n+m-z}$$

$$\sum_{k \le z} \binom{n}{k}\binom{m}{z-k} = \binom{n}{0}\binom{m}{z} + \binom{n}{1}\binom{m}{z-1} + \cdots + \binom{n}{z}\binom{m}{0} = \binom{n+m}{z}.$$

$$\Rightarrow \binom{n+m}{z} p^{z} \cdot (1-p)^{n+m-z}.$$

↳ binomial. dist.

$B(n+m, p)$.



z combinations out of $(n+m)$

ex: boys: n, girls: m. → z select!

0	z
1	$z-1$
⋮	⋮
+ z	0

$$\binom{n+m}{z} p^{z} \cdot (1-p)^{n+m-z}.$$

6.4.4. Sum of Indep. Poisson R.V.s

λ_X, λ_Y, $Z = X + Y$.

$$P_Z(Z = X+Y) = \sum_{k \le z} P_X(k) \cdot P_Y(z-k)$$

$$= \sum_{k \le z} \frac{\lambda_X^{k} e^{-\lambda_X}}{k!} \cdot \frac{\lambda_Y^{z-k} e^{-\lambda_Y}}{(z-k)!}$$

$$= \sum_{k \le z} \frac{(\lambda_X^k \cdot \lambda_Y^{z-k})}{k!\,(z-k)!} e^{-\lambda_X - \lambda_Y}$$

$$= \underline{\sum_{k \le z} \frac{z!}{k!\,(z-k)!} \left(\frac{\lambda_X}{\cancel{\lambda_Y}}\right)^k \cdot \lambda_Y^{z-k}} \frac{1}{z!} e^{-(\lambda_X + \lambda_Y)}$$

↳ 이항정리

$$\doteq$$

$$= \underline{\sum_{k \le z} \binom{z}{k} \lambda_X^k \lambda_Y^{z-k}} \frac{1}{z!} e^{-(\lambda_X + \lambda_Y)}$$

$$= (\lambda_X + \lambda_Y)^z \frac{1}{z!} e^{-(\lambda_X + \lambda_Y)}$$

$$P_Z(z) = \frac{(\lambda_X + \lambda_Y)^z}{z!} e^{-(\lambda_X + \lambda_Y)} = \frac{\lambda_Z^z}{z!} e^{-\lambda_Z}. \qquad \lambda_Z = \lambda_X + \lambda_Y.$$

↳ poisson.

6.8 Two Functions of Two Random Variables.

- $X, Y \sim f_{XY}(x, y)$

$$\begin{cases} V = g(X, Y) \\ W = h(X, Y) \end{cases}$$

$$f_{VW}(v, w) = \sum_{x_i, y_i} \frac{f_{XY}(x_i, y_i)}{|J(x_i, y_i)|}$$

$\{(x_i, y_i)\}$ are solutions of $V = g(x, y)$ and $W = h(x, y)$

$$J(x, y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix}_{x_i, y_i},$$

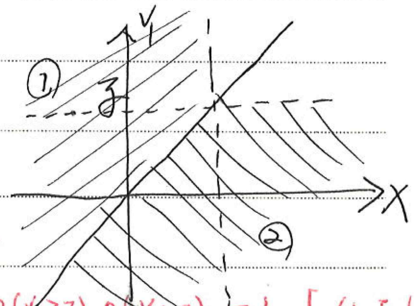
6.5. Minimum of two independent RV's.

$Z = \min(X, Y)$. given X and Y.

$F_Z(z) = P(Z \leq z) = P(\min(X,Y) \leq z)$

$= P([X \leq z \cap X \leq Y] \cup [Y \leq z \cap X > Y])$

① ②



$= F_X(z) + F_Y(z) - F_{XY}(z, z)$

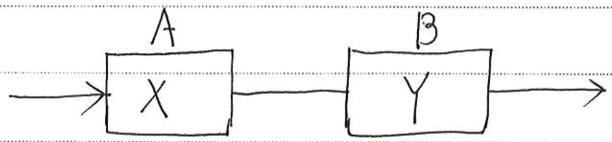
$= F_X(z) + F_Y(z) - F_X(z) \cdot F_Y(z)$

$\Leftarrow 1 - P(X > z, Y > z) = 1 - P(X > z)\,P(Y > z) = 1 - [(1 - F_X(z))(1 - F_Y(z))]$

$f_Z(z) = f_X(z) + f_Y(z) - f_X(z) F_Y(z) - f_Y(z) F_X(z)$

$= f_X(z)(1 - F_Y(z)) + f_Y(z)(1 - F_X(z))$.

o Example. > Series connection of failure time.



X: RV of time until failure A.
Y: RV of time until failure B.

Z: time until failure.

$\Rightarrow Z = \min(X, Y)$.

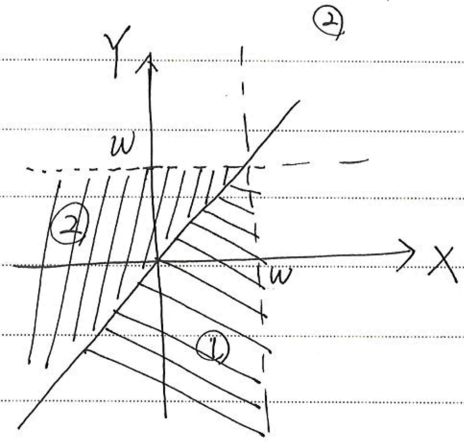
6.6 Maximum of two RV's.

$W = \max(X, Y)$, given X and Y.

$\Rightarrow F_W(w) = P(W \leq w) = P(\max(X,Y) \leq w)$

$= P([X \leq w \cap X \geq Y] \cup [Y \leq w \cap X < Y])$

① ②



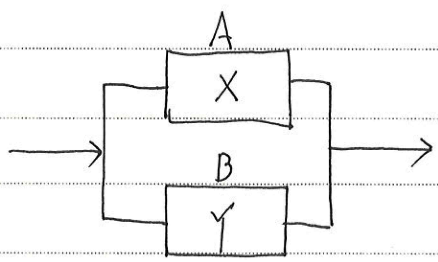
$= P(X \leq w, Y \leq w)$

$= F_{XY}(w, w)$

for independent X, Y, $= F_X(w) F_Y(w)$

$f_W(w) = f_X(w) \cdot F_Y(w) + f_Y(w) \cdot F_X(w)$

Example > parallel connection of time failure



$W = \max[X, Y]$

6.8 Two functions of two RV's.

· $X, Y \sim f_{XY}(x, y)$.

$\begin{cases} U = g(X,Y) \\ W = h(X,Y) \end{cases} \Rightarrow$ to find X and Y using U and W, we need two functions for two variables.

Generally,

$$f_{UW}(u,w) = \sum_{(x_i, y_i)} \frac{f_{XY}(x_i, y_i)}{|J(x_i, y_i)|}$$

where $\{(x_i, y_i)\}$ are solutions of $U = g(x,y)$ and $W = h(x,y)$.

$$J(x_i, y_i) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix}_{(x_i, y_i)} = \frac{\partial g}{\partial x} \cdot \frac{\partial h}{\partial y} - \frac{\partial g}{\partial y} \cdot \frac{\partial h}{\partial x}.$$

Ex. 6.16 > $\begin{cases} U = X + Y \\ W = X - Y. \end{cases}$

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2, \quad \frac{u+w}{2} = x, \quad \frac{u-w}{2} = y.$$

$$f_{UW}(u,w) = \frac{1}{|-2|} f_{XY}\left(\frac{u+w}{2}, \frac{u-w}{2}\right),$$

6.8.1 Application of the Transformation Method.

$U = g(X, Y)$ for $f_U(u)$.

Let's define an auxiliary function $W = X$ or $W = Y$.
Then find $f_{UW}(u, w)$ and

$$f_U(u) = \int_{-\infty}^{\infty} f_{UW}(u, w)\, dw.$$

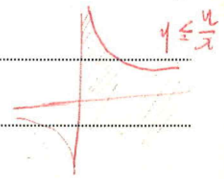
Ex. 6.20

$$U = XY \quad \rightarrow \quad F_U(u) = P(U \leqq u) = P(XY \leqq u)$$

$$\iint_{xy \leq u} f_{XY}(x, y)\, dx\, dy$$

$\Rightarrow$ define $W = X$.

$$J = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x = -w.$$



$$f_{UW}(u, w) = \sum \frac{f_{XY}(x_i, y_i)}{|J_i|} = \frac{f_{XY}\left(w, \frac{u}{w}\right)}{|w|}$$

$$f_U(u) = \int_{-\infty}^{\infty} \frac{f_{XY}\left(w, \frac{u}{w}\right)}{|w|}\, dw$$

6.9 Laws of Large Numbers.

Let $X_1, X_2, X_3, \cdots, X_n$: a seq. of mutually indep. & identical (iid) distributed R.Vs.

$S_n = X_1 + X_2 + \cdots + X_n$

$\overline{S_n} = S_n/n$: sample mean.

1. Weak Law. → discrete 하게 값이 커질 수 있더라도 확률적으로는 → 0.
(weak)

$$\begin{cases} \lim_{n\to\infty} P\left[|\bar{S}_n - \mu_X| > \varepsilon \right] = 0 \\ \lim_{n\to\infty} P\left[|\bar{S}_n - \mu_X| \leq \varepsilon \right] = 1, \quad \text{for } \forall \varepsilon > 0. \end{cases}$$

↳ 값 자체가 $\leq \varepsilon$. (strong)

2. Strong Law.

$$\begin{cases} P\left[\lim_{n\to\infty} |\bar{S}_n - \mu_X| > \varepsilon \right] = 0 \\ P\left[\lim_{n\to\infty} |\bar{S}_n - \mu_X| \leq \varepsilon \right] = 1, \quad \text{for } \forall \varepsilon > 0 \end{cases}$$

6.10 Central Limit Theorem.

→ Gaussian pdf Linear transform도 Gaussian.

For iid seq. $X_1, X_2, \cdots, X_n$.

$S_n = X_1 + X_2 + \cdots + X_n$

$\bar{S}_n = E[S_n] = n\mu_X, \quad \sigma_{S_n}^2 = n\sigma_X^2$

$$Z_n = \frac{S_n - \bar{S}_n}{\sigma_{S_n}} = \frac{S_n - n\mu_X}{\sqrt{n}\,\sigma_X}$$

$$\lim_{n\to\infty} F_{Z_n}(z) \sim N(0, 1)$$

↳ converges to normal distribution

HW #6.

6.4, 6.6, 6.8, 6.9, 6.12, ~~6.14~~, 6.15, 6.21

◦ Generating Gaussian Distribution.

◦ Set x, y are (uniform in $(0,1)$,
independent.

$\Rightarrow$ $f_X(x) = f_Y(y) = 1$, $\quad f_{XY}(x,y) = 1$. $\quad 0<x<1,\ 0<y<1$.

$\Rightarrow$ Set $\begin{cases} u = \sqrt{-2\ln x}\ \cos 2\pi y. \\ v = \sqrt{-2\ln x}\ \sin 2\pi y. \end{cases}$

<verification>.

$$\begin{array}{rl} & u^2 = -2\ln x \cdot \cos^2 2\pi y \\ +) & v^2 = -2\ln x \cdot \sin^2 2\pi y \\ \hline & u^2+v^2 = -2\ln x \end{array} \quad \Rightarrow \quad x = e^{-\frac{1}{2}(u^2+v^2)} \ //$$

$$\frac{v}{u} = \tan 2\pi y \quad \Rightarrow \quad y = \frac{1}{2\pi}\tan^{-1}\left(\frac{v}{u}\right).$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{-2}{2x\sqrt{-2\ln x}}\cos 2\pi y & -2\pi\sqrt{-2\ln x}\ \sin 2\pi y \\ \frac{-2}{2x\sqrt{-2\ln x}}\sin \pi y & 2\pi\sqrt{-2\ln x}\ \cos 2\pi y \end{vmatrix}$$

$$= \left| -\frac{2\pi\cos^2 2\pi y}{x} - \frac{2\pi\sin^2 2\pi y}{x} \right| = \frac{2\pi}{x}.$$

$$f_{UV}(u,v) = \frac{f_{XY}(x,y)}{J(u,v)} = \frac{1\cdot x}{2\pi} = \frac{1}{2\pi} e^{-\frac{u^2+v^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$$ → indep. bivariate Gaussian!

∘ Generating exponential distribution : $g(x)$ is not unique

X is uniform in $(0,1)$. → $F_X(x) = x$, $f_X(x) = 1$, $0<x<1$.

Y is exponential. $f_Y(y) = e^{-y}$, $F_Y(y) = 1-e^{-y}$, $y>0$.

(1) $Y = g(X)$ → monotonically decreasing function,

$$F_Y(y) = P(Y \le y) = 1 - e^{-y} = P(g(X) \le y) = P(X \ge g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$

x is uniform in $(0,1)$ $F_X(x) = x$.

$$\Rightarrow e^{-y} = F_X(g^{-1}(y)) = g^{-1}(y)$$

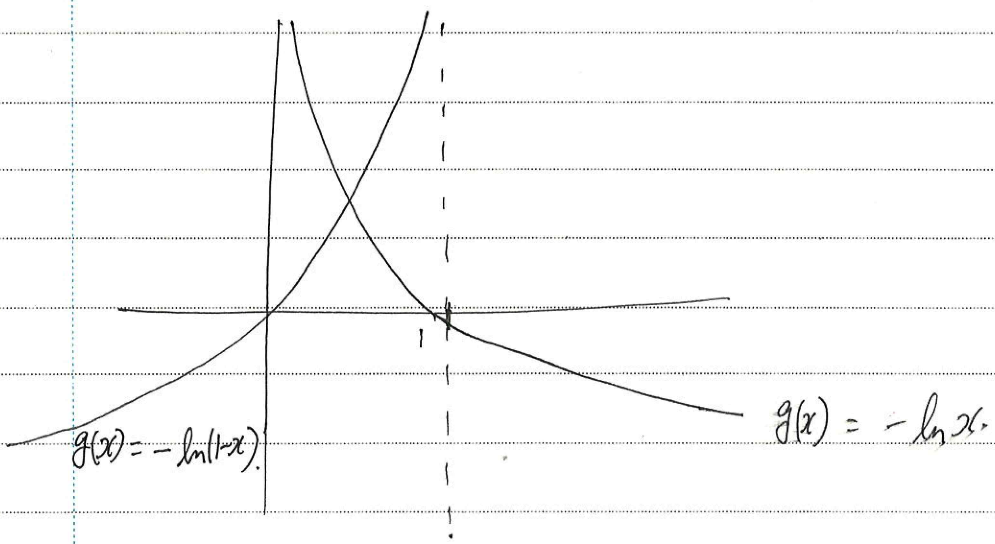
$$\Rightarrow y = g(x) = -\ln x$$

(2) $Y = g(X)$ → monotonically increasing function,

$$F_Y(y) = 1 - e^{-y} = P(g(X) \le y) = P(X \le g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$

$1 - e^{-y} = g^{-1}(y).$

$\Rightarrow \quad y = g(x) = -\ln(1-x)$ //



○ Histogram Equalization.

$F_X(x) \longrightarrow$ uniform $\quad F_Y(y) = \frac{y}{s}$ // $\quad f_Y(y) = \frac{1}{s}$ //

$Y = g(X)$

$F_Y(y) = P(Y \le y) = P(g(X) \le y)$ → monotonically increasing.

$= P(X \le g^{-1}(y))$

$= F_X(g^{-1}(y)) = \frac{y}{s}$ // → $F_X \circ g^{-1}(y) = y$

$= F_X(x) = \frac{1}{s} g(x)$ // ← $y = g(x)$ // $\quad g^{-1}(y) = F_X^{-1}(y).$

$g(y) = F_X(y).$

$\Rightarrow \frac{f_X(g^{-1}(y))}{g'(y)} = \frac{1}{s}.$

$\Rightarrow s \cdot f_X(g^{-1}(y)) = g'(y)$

$s F_X(\underbrace{g^{-1}(y)}_{X}) = g(y)$

$\underline{g(x) = s F_X(x)}$ //

Chapter 7. Transform Methods.

→ Fourier transform ⇒ time domain function → freq. domain function

⇒ analysis of energy for each freq. comp.

7.2 Characteristic Function.

$f_X(x)$: p.d.f of RV X. e^{-jwx} → Fourier transform

$$\Phi_X(w) = \int_{-\infty}^{\infty} f_X(x)\, e^{jwx}\, dx = E[e^{jwX}] \qquad j^2 = 1.$$

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(w)\, e^{-jwx}\, dw$$

$$\Phi_X(0) = 1.$$

∘ for discrete case $\quad \Phi_X(w) = \sum_{x=-\infty}^{\infty} P_X(x)\, e^{jwx}.$

Ex. 7.2 $\quad X \sim N(\mu_X, \sigma_X^2)$

$$\rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$

$$\Phi_X(w) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \cdot e^{jwx}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{x^2 - 2\mu_X x + \mu_X^2 - 2jwx\sigma_X^2}{2\sigma_X^2}}\, dx$$

~~where,~~ $\cancel{x^2 - 2(\mu_X + \tfrac{j}{2}w\sigma_X^2)x + \mu_X^2 = \left(x - (\mu_X + \tfrac{j}{2}w\sigma_X^2)\right)^2 + \mu_X^2 - (\mu_X + \tfrac{j}{2}w\sigma_X^2)^2}$

$\cancel{= \left(x - (\mu_X + \tfrac{j}{2}\sigma_X^2 w)\right)^2 - jw\mu_X + \frac{w^2\sigma_X^2}{4}}$

where $\quad x^2 - 2(\mu_X + jw\sigma_X^2)x + \mu_X^2 = [x - (\mu_X + jw\sigma_X^2)]^2 + \mu_X^2 - (\mu_X + jw\sigma_X^2)^2$

$$= (x - (\mu_X + jw\sigma_X^2))^2 - 2jw\mu_X\sigma_X^2 + w^2\sigma_X^4$$

$$\Phi_X(\omega) = \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-(\mu_X + j\omega\sigma_X^2))^2}{2\sigma_X^2}}}_{\Downarrow \; 1} e^{j\omega\mu_X - \frac{\omega^2\sigma_X^2}{2}} dx$$

$$= e^{j\omega\mu_X - \frac{\omega^2\sigma_X^2}{2}}$$

$$\hookrightarrow t = \frac{x-(\mu_X + j\omega\sigma_X^2)}{\sigma_X} \Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 1.$$

7.2.1 Moment Generating Property

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx.$$

$$\frac{d\Phi_X(\omega)}{d\omega} = \int_{-\infty}^{\infty} (jx) f_X(x) e^{j\omega x} dx$$

for $w=0.$ $\rightarrow$ $\left.\frac{d\Phi(\omega)}{d\omega}\right|_{w=0} = jE[X].$

$$\left.\frac{d^2\Phi_X(\omega)}{d\omega^2}\right|_{w=0} = -E[X^2].$$

$$\left.\frac{d^n\Phi_X(\omega)}{d\omega^n}\right|_{w=0} = (j)^n E[X^n].$$

Ex. 7.2 (b).

for Gaussian. $\Phi_X(\omega) = e^{j\omega\mu_X - \frac{\omega^2\sigma_X^2}{2}}$

$$\left.\frac{d}{d\omega}\Phi_X(\omega)\right|_{w=0} = \left.(j\mu_X - \omega\sigma_X^2) e^{j\omega\mu_X - \frac{\omega^2\sigma_X^2}{2}}\right|_{w=0}$$

$$= j\mu_X. = jE[X]$$

$$\left.\frac{d^2}{d\omega^2}\Phi_X(\omega)\right|_{w=0} = e^{j\omega\mu_X - \frac{\omega^2\sigma_X^2}{2}} \cdot \left.\left\{(j\mu_X - \omega\sigma_X^2)^2 - \sigma_X^2\right\}\right|_{w=0} = -\mu_X^2 - \sigma_X^2$$

$$= (j)^2(\mu_X^2 + \sigma_X^2)$$

Let. $f_Z(z) = f_X(z) * f_Y(z)$ $\qquad Z = X+Y.$

$$= \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx.$$

$$\Phi_Z(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx\; e^{j\omega z}\, dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, e^{j\omega x} \cdot e^{j\omega(z-x)}\, dx\, dz$$

$$= \int\int_{-\infty}^{\infty} \underbrace{f_Y(z-x)\, e^{j\omega(z-x)}\, dz}\cdot \underbrace{f_X(x)\, e^{j\omega x}}\, dx$$

$$= \Phi_Y(\omega)\cdot \Phi_X(\omega).$$

(→ first find $\Phi_Z(\omega)$ by $\Phi_X(\omega) \times \Phi_X(\omega)$,
then $f_Z(z) = \frac{1}{2\pi}\int \Phi_Z(\omega)\, e^{-j\omega z}\, d\omega.$)

7.3 S- Transform. (Laplace transform)

($f_X(x)$: for $x \geq 0.$
X : continuous R.V.

s-transform, $M_X(s) \stackrel{d}{=} E[e^{-sX}] = \int_0^{\infty} e^{-sx} f_X(x)\, dx.$

$$M_X(s)\Big|_{s=0} = 1.$$

∘ Moment Generating Property.

$$\frac{d^n}{ds^n} M_X(s)\Big|_{s=0} = (-1)^n E[X^n].$$

$s = j\omega \;\rightarrow\; \Phi_X(\omega).$

7.3.2. S-Transforms of Well-known PDF's.

◎ Exponential Distribution.

$$f_X(x) = \lambda \cdot e^{-\lambda x} u(x)$$

$$M_X(s) = E[e^{-sX}] = \int_0^\infty e^{-sx} \lambda \cdot e^{-\lambda x} dx$$

$$= \int_0^\infty \lambda \cdot e^{-(s+\lambda)x} dx = \frac{\lambda}{\lambda + s}$$

check!

$$-E[X] = \frac{d}{ds} M_X(s) \Big|_{s=0} = \frac{-\lambda}{(\lambda+s)^2}\Big|_{s=0} = -\frac{1}{\lambda}$$

$$E[X^2] \cancel{\sigma_X^2} = \frac{d^2}{ds^2} M_X(s)\Big|_{s=0} = \frac{2\lambda}{(\lambda+s)^3}\Big|_{s=0} = \frac{2}{\lambda^2}.$$

$$\hookrightarrow \sigma_X^2 = E[X^2] - E[X]^2 = \frac{1}{\lambda^2}$$

7.3.3 S-Transform of Sum of independent RV's

Let $X_1, X_2, \cdots, X_n$ be independent,

$$\rightarrow f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

$$Y = X_1 + X_2 + \cdots + X_n$$

$$M_Y(s) = E[e^{-sY}] = E[e^{-s(X_1 + X_2 + \cdots + X_n)}]$$

$$= \int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-s(x_1+x_2+\cdots+x_n)} f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n) dx_1 \cdots dx_n$$

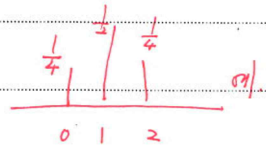
independent

$$= \int_0^\infty e^{-sx_1} f_{X_1}(x_1) dx_1 \cdot \int_0^\infty e^{-sx_2} f_{X_2}(x_2) dx_2 \cdots \int_0^\infty e^{-sx_n} f_{X_n}(x_n) dx_n$$

$$= M_{X_1}(s) \cdot M_{X_2}(s) \cdots M_{X_n}(s) = \prod_{i=1}^{n} M_{X_i}(s).$$

↳ convolution 연산을 대체.

7.4 Z - Transform



$P_X(x)$: PMF of discrete RV, X $(x \geq 0)$

$$G_X(z) = E[z^X] = \sum_{x=0}^{\infty} z^x P_X(x)$$ → convergence problem : $|z|<1$.

$$G_X(1) = 1.$$

$G_X(z)$ is a polynomial of z,

$$\Rightarrow P_X(0) + P_X(1)z + P_X(2)z^2 + P_X(3)z^3 + \cdots$$

↓ ↓ ↓ non-negative coefficients. $0 \leq P_X(x) \leq 1$

eg. $A(z) = 2z - 1$ → $A(1) = 1$, but coefficients $\begin{cases} 2 > 1, \\ -1 < 0 \end{cases}$

$$P_X(x) = \frac{1}{x!}\left[\frac{d^x}{dz^x} G_X(z)\right]_{z=0}, \quad x = 0, 1, 2, \cdots$$

↳ probability generating function.

Ex 7.6 > $G_X(z) = \dfrac{1-a}{1-az}$ $\quad 0 < a < 1.$ $\quad |az| < 1.$

$\cancel{P_X(x) = \frac{1}{x!}}$ $\Rightarrow (1-a) \cdot \sum_{x=0}^{\infty} (az)^x$ ~~az~~

$$= (1-a)(1 + az + a^2z^2 + \cdots).$$

$$P_X(x) = (1-a)a^x. \quad x = 0, 1, 2, \cdots$$

$$E[X] = \frac{d}{dz} G_X(z)\Big|_{z=1} = E[X \cdot z^{X-1}]\Big|_{z=1} \Leftarrow \sum_{x=0}^{\infty} x \cdot z^{x-1} P_X(x)\Big|_{z=1}$$

Ex. 7.8 $\quad G_K(z) = A\left[\frac{10+8z^2}{(2-z)}\right]$

1) $G_K(1) = 1. \Rightarrow A \cdot 18 = 1 \qquad A = \frac{1}{18}$

2) $E[K] = \frac{d}{dz} G_K(z)\Big|_{z=1} = \frac{1}{18} \cdot \frac{-8z^2+32z+10}{(2-z)^2}\Big|_{z=1} = \frac{17}{9}$

3) $P_K(1) = ?$

$$P_K(1) = \frac{d}{dz} G_K(z)\Big|_{z=0} = \frac{1}{18} \frac{-8z^2+32z+10}{(2-z)^2}\Big|_{z=0} = \frac{5}{36}$$

7.4.1 Moment - Generating Property of z-transform.

$$G_X(z) = \sum_{x=0}^{\infty} z^x P_X(x)$$

$$\frac{d}{dz} G_X(z) = \sum_{x=0}^{\infty} x \cdot z^{x-1} P_X(x) = \sum_{x=1}^{\infty} x \cdot z^{x-1} P_X(x)$$

$\hookrightarrow z=1. \qquad \frac{d}{dz} G_X(z)\Big|_{z=1} = E[X]$

$$\frac{d^2}{dz^2} G_X(z) = \sum_{x=1}^{\infty} x \cdot (x-1) z^{x-2} P_X(x) = \sum_{x=1}^{\infty} x^2 z^{x-2} P_X(x) - \sum_{x=1}^{\infty} x \cdot z^{x-2} P_X(x)$$

$$\frac{d^2}{dz^2} G_X(z)\Big|_{z=1} = E[X^2] - E[X]$$

$$E[X^2] = \frac{d^2}{dz^2} G_X(z)\Big|_{z=1} + \frac{d}{dz} G_X(z)\Big|_{z=1}$$

7.4.3 Z-Transform of Binomial Distribution

$$B(n, p) \rightarrow P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$G_X(z) = \sum_{x=0}^{\infty} z^x P_X(x) = \sum_{x=0}^{n} z^x \binom{n}{x} p^x \cdot (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (zp)^x (1-p)^{n-x}$$

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

$$= (1-p+zp)^n$$

7.4.4 Z-Transform of Geometric Distribution

$$P_X(x) = p \cdot (1-p)^{x-1} \qquad x = 1, 2, \cdots$$

$$G_X(z) = \sum_{x=1}^{\infty} z^x \cdot p (1-p)^{x-1}$$

$$= \sum_{x=1}^{\infty} zp \cdot [(1-p)z]^{x-1} = \frac{zp}{1-(1-p)z}$$

7.4.5 Z-Transform of Poisson Distribution

$$P_K(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad K = 0, 1, 2, \cdots$$

$$G_K(z) = \sum_{k=0}^{\infty} z^k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda z)^k}{k!} e^{-\lambda} = e^{\lambda z} \cdot e^{-\lambda} = e^{\lambda(z-1)}$$

7.4.6 Z-Transform of Sum of Independent RV's.

$$Y = X_1 + X_2 + \cdots + X_n \quad , \quad f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

$$G_Y(z) = \sum_{y=0}^{\infty} z^y f_Y(y)$$

$$= \sum_{y=0}^{\infty} z^{x_1 + x_2 + \cdots + x_n} f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

$$= \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\infty} \cdots \sum_{x_n=0}^{\infty} z^{x_1} z^{x_2} \cdots z^{x_n} f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

$$= G_{X_1}(z) \cdot G_{X_2}(z) \cdots G_{X_n}(z)$$

for IID pd RV'S,

HW 7.
7.3, 7.6, 7.8, 7.13, 7.16

$$G_Y(z) = [G_X(z)]^n$$

7.5 Random Sum of RV'S. (optional)

∘ for a fixed n, $Y = X_1 + X_2 + \cdots + X_n$, (for iid.)

$$\Rightarrow M_Y(s) = \prod_{i=1}^{n} M_{X_i}(s) = [M_X(s)]^n$$

$$\hookrightarrow M_{Y|n}(s|n) = [M_X(s)]^n$$

n: RV.

$$\Rightarrow M_Y(s) = \sum_{n=1}^{\infty} P_N(n) M_{Y|N=n}(s|n) = \sum_{n=1}^{\infty} P_N(n) [M_X(s)]^n$$

$$= G_N(M_X(s))$$

Chapter 8. Random Processes

8.1 Introduction

- Recall that random variables map the outcomes of random experiment to real numbers.

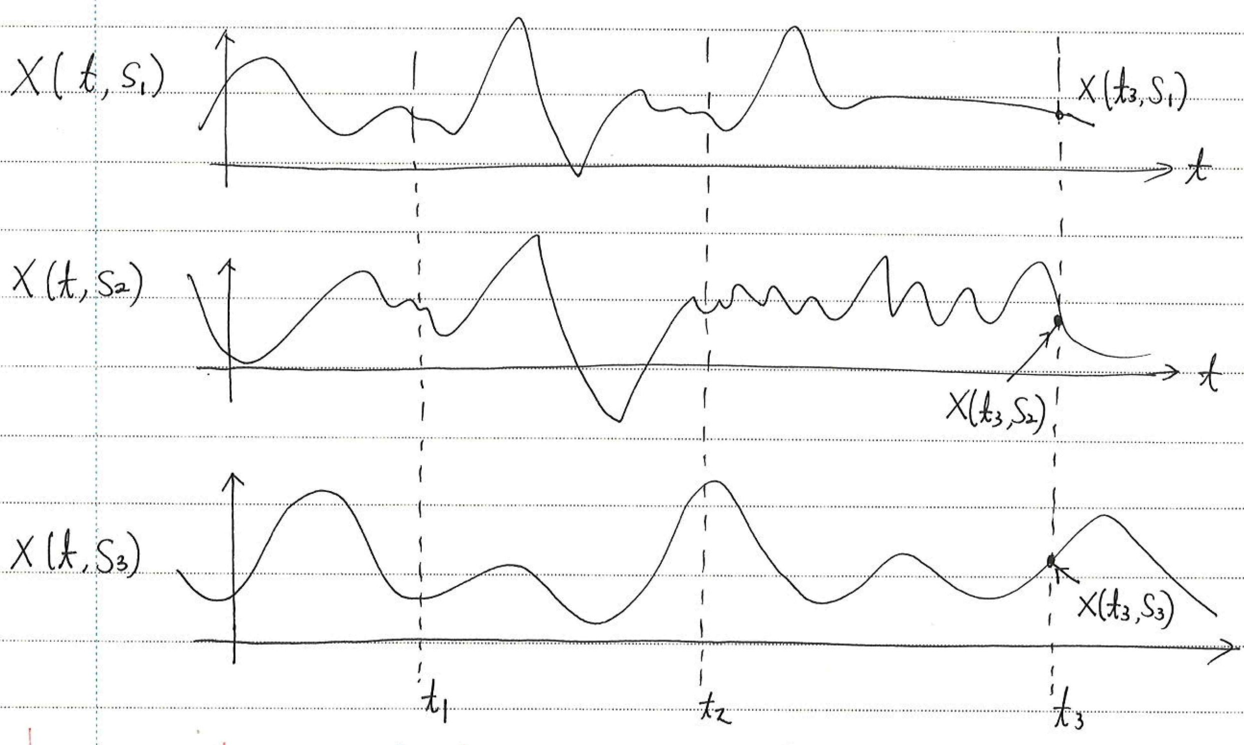
- Random process expand the concept of RV to include time.

- For sample space S and $s \in S$,

  RV : $X(s)$

  Random Process : $X(t, s)$, $t \in T$ (time), $s \in S$ (outcome)

- time function $X(t, s)$ for $s \in S$.



↳ 하나의 random experiment outcome 으로 time function (sequence) 결정

- for fixed t → $X(t, s_1)$, $X(t, s_2)$, ⋯ ⇒ R.V.
- for fixed s → $X(t_1, s)$, $X(t_2, s)$, ⋯ ⇒ time function or sequence.

- Continuous - time random Process.
  discrete - time 〃 → random sequence.

$X(t,s)$ → states. $\{X(t,s)\}$ → state space.

usually, $\underset{\text{observation}}{Y(t,s)} = \underset{\text{desired signal}}{X(t,s)} + \underset{\text{noise}}{N(t,s)}$

8.3 Characterizing Random Process.

- suppress s (sample space parameter) → $X(t)$

- Joint CDF : characterize the random process.

→ $F_X(x_1, t_1) = F_X(x_1) = P[X(t_1) \leq x_1]$.

⋮

$F_X(x_n, t_n) = F_X(x_n) = P[X(t_n) \leq x_n]$.

wher, $0 < t_1 < t_2 \cdots < t_n$.

continuous ⇒
$F_X(x_1, x_2, \cdots, x_n ; t_1, t_2, \cdots, t_n)$
$f_X(x_1, x_2, \cdots, x_n ; t_1, t_2, \cdots, t_n)$

discrete ⇒ $P_X(x_1, x_2, \cdots x_n ; t_1, t_2, \cdots t_n) = P[X(t_1) = x_1, X(t_2) = x_2 \cdots, X(t_n) = x_n]$.

8.3.1 Mean & Autocorrelation Function

∘ mean

∘ ⇒ $\mu_X(t) = E[X(t)]$ → function of time (ensemble average)

∘ autocorrelation : measure of similarity between two observations of random process $X(t)$ at different points in time t and s.

$$R_{XX}(t,s) = E[X(t)\,X(s)] = E[X(s)\,X(t)] = R_{XX}(s,t)$$

$$R_{XX}(t,t) = E[X^2(t)]$$

∘ $s = t + \tau$ (τ: delay)

$$\Rightarrow R_{XX}(t, t+\tau) = E[X(t)\,X(t+\tau)]$$

∘ autocorrelation function of a deterministic periodic function of period T,

$$R_{XX}(t, t+\tau) = \frac{1}{2T}\int_{-T}^{T} f_X(t)\, f_X(t+\tau)\, dt$$

∘ for aperiodic function

$$R_{XX}(t, t+\tau) = \int_{-\infty}^{\infty} f_X(t)\, f_X(t+\tau)\, dt$$

→ autocorrelation function defines how much a signal is similar to a time-shifted version of itself.

8.3.2 Autocovariance Function

$$C_{XX}(t,s) = Cov(X(t), X(s)) = E[(X(t)-\mu_X(t))(X(s)-\mu_X(s))]$$

$$= R_{XX}(X(t), X(s)) - \mu_X(t)\,\mu_X(s)$$

If $X(t)$ and $X(s)$ are independent, $\rightarrow$ $R_{xx}(X(t), X(s)) = \mu_x(t)\,\mu_x(s)$

$\rightarrow$ $Cov(X(s), X(t)) = 0$

$\rightarrow$ $C_{xx}(t, s) = 0$

$\Rightarrow$ no coupling between $X(t)$ and $X(s)$

uncorrelated!

Ex 8.1

$X(t) = K\cos(wt)$, $(t \geq 0)$ $\rightarrow$ time function is determined by random output K.
variable

where, K : $f_K(k) = \frac{1}{2}$, $0 \leq k \leq 2$

(a) $E[X(t)] = E[K\cos(wt)]$

$= \underbrace{E[K]}_{1} \cdot \cos(wt) = \cos(wt)$

(b) $R_{xx}(t, s) = ?$

$R_{xx}(X(t), X(s)) = E[K\cos(wt) \cdot K\cos(ws)]$

$= E[K^2]\,\cos(wt) \cdot \cos(ws)$

$= \frac{4}{3} \cdot \cos(wt) \cdot \cos(ws)$

$\left(E[K^2] = \int_0^2 k^2 \frac{1}{2}\,dk = \frac{4}{3} \right)$

(c) $C_{xx}(t, s) = ?$

$C_{xx}(t, s) = R_{xx}(t, s) - \mu_x(t)\,\mu_x(s)$

$= \frac{4}{3}\cos(wt)\cos(ws) - \cos(wt) \cdot \cos(ws)$

$= \frac{1}{3}\cos(wt) \cdot \cos(ws)$

8.4 Cross correlation & Cross covariance Functions.

- two random process $X(t)$, $Y(t)$ → two random experiments that generate time function $X(t)$ and $Y(t)$, respectively.

- $R_{XY}(t,s)$: cross correlation

$$= E[X(t)Y(s)] = R_{YX}(s,t)$$

→ similarity measure of two different processes (signals) with time shift (t,s).

- If $R_{XY}(t,s) = 0$ → orthogonal process.

- If $X(t)$ and $Y(t)$ are independent,

$$R_{XY}(t,s) = E[X(t)Y(s)] = \mu_X(t)\cdot\mu_Y(s)$$

- Cross Covariance : $C_{XY}(t,s)$

$$C_{XY}(t,s) = E[(X(t)-\mu_X(t))(Y(s)-\mu_Y(s))]$$
$$= R_{XY}(t,s) - \mu_X(t)\mu_Y(s)$$

if uncorrelated or independent,

$$C_{XY}(t,s) = 0. \Rightarrow R_{XY}(t,s) = \mu_X(t)\mu_Y(s).$$

usually, $Y(t) = X(t) + \underbrace{N(t)}_{\text{noise}}$ (in communication systems)

Ex 8.2> $Y(t) = X(t) + N(t)$, $N(t)$: independent noise process.

∘ $R_{XY}(t,s) = E[X(t) \cdot Y(s)] = E[X(t)\{X(s) + N(s)\}]$

$= E[X(t)X(s)] + E[X(t)N(s)]$

$= R_{XX}(t,s) + \mu_X(t) \cdot \mu_N(s)$.

→ usually $\mu_N(t) = 0$ → thus, $R_{XY}(t,s) = R_{XX}(t,s)$

→ This means that the original signal $X(t)$ can be detected by cross correlation $C_{XY}(t,s)$ even in the noise environment.

$C_{XY}(t,s) = E[(X(t) - \mu_X(t))(Y(s) - \mu_Y(s))]$

$= E[X(t)Y(s)] - \mu_X(t)\mu_Y(s)$

$= R_{XY}(t,s) - \mu_X(t)(\mu_X(s) + \mu_N(s))$

$= R_{XX}(t,s) + \mu_X(t)\mu_N(s) - \mu_X(t)\mu_X(s) - \mu_X(t)\mu_N(s)$

$= R_{XX}(t,s) - \mu_X(t)\mu_X(s)$.

$= C_{XX}(t,s)$.

└→ covariance of $X(t)$ and $Y(t)$ ~~ar~~ is equivalent to auto covariance of X

∘ Review of Trigonometric functions.

$$\begin{cases} \sin(A \pm B) = \sin A \cos B \pm \sin B \cos A & -① \\ \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B & -② \end{cases}$$

By ①.

$$\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$$

By ②

$$\cos A \cdot \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$$

$$\sin A \cdot \sin B = -\frac{1}{2}\{\cos(A+B) - \cos(A-B)\}$$

Ex 8.3) $X(t) = A\cos(wt + \Theta)$ Θ : uniform $0 \sim 2\pi$.

$Y(t) = B\sin(wt + \Theta)$ A, B : constant.

$$R_{XY}(t,s) = E[X(t)Y(s)] = E[A\cos(wt+\Theta)\, B\sin(ws+\Theta)]$$

$$= AB \cdot E[\cos(wt+\Theta)\cdot\sin(ws+\Theta)]$$

$$= \frac{AB}{2} E\left[\sin\left(w(t+s)+2\Theta\right) + \sin\left(w(s-t)\right)\right]$$

$$= \underbrace{\frac{AB}{2} E[\sin(w(t+s)+2\Theta)]} - \frac{AB}{2}\sin(w(t-s))$$

$$E[\sin(w(t+s)+2\Theta)] = \int_0^{2\pi} \sin(w(t+s)+2\theta)\frac{1}{2\pi}\,d\theta = 0$$

$$\Rightarrow R_{XY}(t,s) = -\frac{1}{2}AB\sin(w(t-s))$$

$$= \frac{1}{2}AB\sin(w(s-t)) \qquad s = t+\tau.$$

$$= \frac{AB}{2}\sin(w\tau)$$

8.5 Stationary Random Processes.

→ stationary : statistical properties do not vary with time.

- strict-sense stationary (SSS)
- wide-sense stationary (WSS).

8.5.1. Strict-sense Stationary Processes.

Chapter 11. Statistics.

11.1 Introduction.

- probability models ⇒ completely defining a random phenomenon.
  ⇒ unique and precise!

- statistics is concerned with the relationship between (abstract) probabilistic model and actual physical random phenomenon.

⇒ applying the prob. models to real data (fitting)

1) descriptive statistics : presenting data or grouping.
   → mean, variance, median, mode ....

2) statistical inference
   → drawing conclusions or making generalization based on a set of observed data

⇒ Sampling theory : selecting samples from too large population.

⇒ estimation theory : predicting or estimating some parameters using available data

⇒ Hypothesis testing : evaluating or choosing the best models.

⇒ Curve fitting and regression : finding mathematical expressions that best represent data.

11. 2 Sampling Theory.

- population (모집단) : whole collection of data to be studied. too large generally.

- we just use the samples of population.

- The samples should represent the population sufficiently well.
  ⇒ each member of population has the same chance to be in the samples.
  ⇒ randomly selected, equally probable! → replacement,

- n samples ; $x_1, x_2, \cdots x_n$ → independent, identically distributed

  ⇒ $X_1, X_2, \cdots X_n$ are random variables, too!

- Parameters of interest from samples.
  ⇒ mean, variance..

11.2.1 Sample mean.

- Let, $X_1, X_2, X_3, \cdots, X_n$ (samples) be the RV's of size n.
- Sample mean (표본 평균) : $\overline{X}$

$$\overline{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

⇒ X_i and $\overline{X}$ are all random variables.

⇒ Assume that μ_X and σ_X^2 are mean and variance of whole population.

- mean of sample mean : $E[\overline{X}]$

$$E[\overline{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i]$$

i.i.d of X_i's

$$= \frac{1}{n}\, n\, E[X] = \mu_X$$

- When we estimate the mean ~~of~~ of population using the mean of sample mean, $E[\overline{X}] = \mu_X = E[X]$,
  → unbiased estimator.

- The variane of the sample mean.

$$\sigma_{\overline{X}}^2 = E[(\overline{X} - \mu_X)^2] = E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu_X\right)^2\right]$$

$$= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_X)\right)^2\right] = E\left[\frac{1}{n^2}\sum_{i=1}^{n}(X_i-\mu_X)^2 + 2\sum_{i}\sum_{\substack{j \\ i\neq j}}(X_i-\mu_X)(X_j-\mu_X)\right]$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} \underbrace{E\left[(X_i - \mu_X)^2\right]}_{\sigma_X^2} + 2\sum_{i}\sum_{j,\, i\neq j} \cancel{E\left[(X_i-\mu_X)(X_j-\mu_X)\right]}^{\,0} \right) \quad iid\ (indep.)$$

$$= \frac{1}{n}\sigma_X^2$$

∘ Usually $n \gg 30$ → By central limit theorem,

$\bar{X}$ is converged to a Gaussian distribution.

$$\Rightarrow N_{\bar{X}}\left(\mu_X, \frac{\sigma_X^2}{n}\right).$$

$$F_{\bar{X}}(x) = P(\bar{X} \leq x) = P\left(Z \leq \frac{x - \mu_X}{\frac{\sigma_X}{\sqrt{n}}}\right) = \Phi\left(\frac{x-\mu_X}{\sigma_X/\sqrt{n}}\right) //$$

Example 11.1 >

$$\begin{cases} X : f_X(x) = 2e^{-2x} \rightarrow \text{exponential dist.} \\ \bar{X} = \frac{1}{36}(X_1 + X_2 + \cdots + X_{36}), \Rightarrow n = 36 \end{cases}$$

(a) $E[\bar{X}] = E[X] = \frac{1}{\lambda} = \frac{1}{2} //$

$\sigma_X^2 = \frac{1}{\lambda^2} = \frac{1}{4}$.

$\sigma_{\bar{X}}^2 = \frac{\sigma_X^2}{n} = \frac{1}{144} //$ $\quad E[\bar{X}^2] = \frac{1}{144} + \frac{1}{4} = \frac{37}{144} //$

(b) $P\left(\frac{1}{4} \leq \bar{X} \leq \frac{3}{4}\right) = P\left(\frac{\frac{1}{4}-\frac{1}{2}}{\frac{1}{12}} \leq Z \leq \frac{\frac{3}{4}-\frac{1}{2}}{\frac{1}{12}}\right)$

standard normal dist. $= P(-3 \leq Z \leq 3)$

$= 2\cdot\left(\Phi(3) - \frac{1}{2}\right) //$

11.2.2 The Sample Variance.

- estimate the variance of population using the samples.

$$S^2 \triangleq \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 \rightarrow \text{random variable.}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[(X_i-\mu_X)-(\bar{X}-\mu_X)\right]^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_X)^2 - \frac{2}{n}\sum_{i=1}^{n}(X_i-\mu_X)(\bar{X}-\mu_X) + \frac{1}{n}\sum_{i=1}^{n}(\bar{X}-\mu_X)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_X)^2 - \frac{2}{n}(\bar{X}-\mu_X)\sum_{i=1}^{n}(X_i-\mu_X) + (\bar{X}-\mu_X)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_X)^2 - 2(\bar{X}-\mu_X)(\bar{X}-\mu_X) + (\bar{X}-\mu_X)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_X)^2 - (\bar{X}-\mu_X)^2$$

$$E[S^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_X)^2\right] - \underline{E[(\bar{X}-\mu_X)^2]} \rightarrow \text{var. of sample mean.}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\underbrace{E[(X_i-\mu_X)^2]}_{\text{var. of } X} - \sigma_{\bar{X}}^2$$

$$= \sigma_X^2 - \frac{\sigma_X^2}{n} = \frac{n-1}{n}\sigma_X^2 \;\; < \sigma_X^2 \rightarrow \text{biased estimate}$$

$$n\to\infty \rightarrow \sigma_X^2 = E[S^2]$$

- To obtain an unbiased estimate,

$$\hat{S}^2 = \frac{n}{n-1}S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$$

$$E[\hat{S}^2] = \sigma_X^2$$

11.2.3 Sampling Distributions.

- for $n \geqq 30$,

  $\bar{X} \rightarrow$ Normal distribution by Central limit theorem.

  $$N_{\bar{X}}\left(\mu_X, \frac{\sigma_X^2}{n}\right) \longrightarrow Z = \frac{\bar{X} - E[X]}{\frac{\sigma_X}{\sqrt{n}}}$$

- for $n < 30$,

  $\bar{X} \rightarrow$ Student's-t distribution by defining the normalized sample mean,

$$T = \frac{\bar{X} - E[X]}{\frac{\hat{S}}{\sqrt{n}}} = \frac{\bar{X} - E[X]}{\frac{S}{\sqrt{n-1}}} \qquad (\hat{S} \Rightarrow \sigma_x)$$

$$S = \sqrt{\frac{n-1}{n}}\,\sigma_X$$

where, $\begin{cases} S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2, & \text{sample variance.} \\ \hat{S}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \end{cases}$

$\rightarrow$ $n-1$ degrees of freedom.

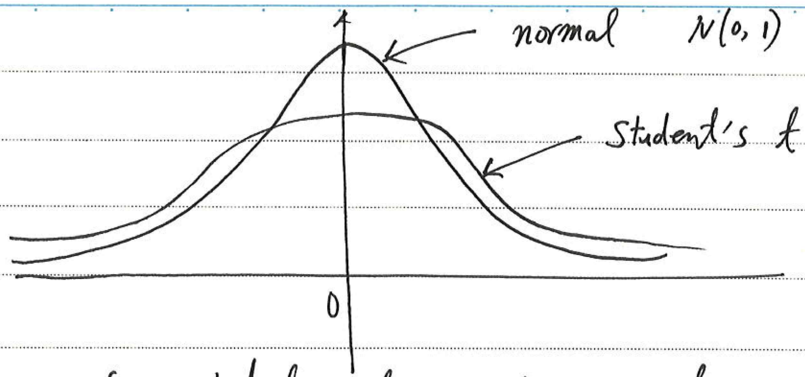
pdf of Student's t distribution, with n samples.

$$f_T(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{t^2}{\nu}\right)^{\frac{-(\nu+1)}{2}} \qquad \nu = n+1.$$

$$\Gamma(x) = (x-1)! = \int_0^\infty t^{x-1} e^{-x}\,dx$$

$$\Gamma(k+1) = \begin{cases} k\Gamma(k), & \text{any } k. \\ k!, & k \text{ integer} \end{cases}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi},$$



variance of Student's $t > 1$ $\lim_{n\to\infty} \to 1$

⊙ for sum of squared ~~variance~~ variables, Kai-squared dist.

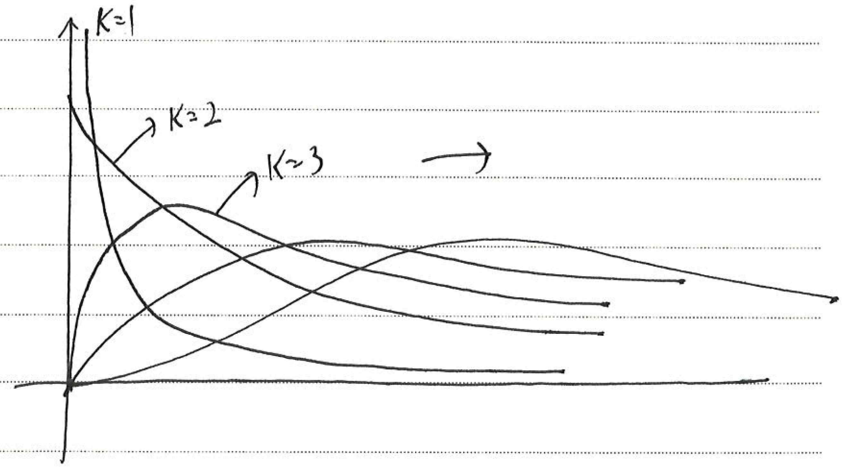
$Q = X_1^2 + X_2^2 + \cdots + X_K^2$ $X_i \to$ standard normal dist.

↳ χ^2- distribution.

$$f(x;k) = \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \qquad 0 < x$$

$E[Q] = k.$

$\sigma_Q^2 = 2k.$



→ 분산이 퍼져 있는 모양. → 평균을 기준으로 ...

11.3 Estimation Theory.

estimate mean & variance.

$$\begin{cases} \mu_X = E[\bar{X}] , & \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \\ \sigma_X^2 = E[\hat{S}^2] , & \hat{S}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \end{cases}$$

- Hypothesis test. → Q.

→ Normal dist. $(n \geq 30)$
→ Student's t dist. $(n < 30)$) $\bar{X}$ testing.

→ χ^2 - dist. $Q = X_1^2 + X_2^2 + \cdots + X_n^2$.
↳ variance testing. $(\hat{S}^2)$

Ex> 99%, $\sigma_X = 1$,

$$\Rightarrow P\left(\left(\bar{X} - 2.58\frac{\sigma_X}{\sqrt{n}}\right) \leq E[X] \leq \left(\bar{X} + 2.58\frac{\sigma_X}{\sqrt{n}}\right) \right) = 0.99$$

by ~~xx~~ standard normal dist.