THE ALGEBRAIC CLOSURE OF THE POWER SERIES FIELD IN POSITIVE CHARACTERISTIC

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ABSTRACT. For K an algebraically closed field, let $K((t))$ denote the quotient field of the power series ring over K. The "Newton-Puiseux theorem" states that if K has characteristic 0, the algebraic closure of $K((t))$ is the union of the fields $K((t^{1/n}))$ over $n \in \mathbb{N}$. We answer a question of Abhyankar by constructing an algebraic closure of $K((t))$ for any field K of positive characteristic explicitly in terms of certain generalized power series.

1. INTRODUCTION

For K a field, which unless otherwise specified will be algebraically closed, let $K((t))$ denote the field of formal power series over K (that is, expressions of the form $\sum_{i=m}^{\infty} x_i t^i$ for some $m \in \mathbb{Z}$ and $x_i \in K$, with the usual arithmetic operations). A classical theorem [10, Proposition II.8] attributed to Puiseux, but essentially known to Newton, states that if K has characteristic zero, then the algebraic closure of $K((t))$ is isomorphic to

$$\bigcup_{i=1}^{\infty} K((t^{1/i})).$$

Hereafter, we will take K to have characteristic $p > 0$. In this case, Chevalley [3] noted that the Artin-Schreier polynomial $x^p - x - t^{-1}$ has no root in the Newton-Puiseux field. In fact, the Newton-Puiseux field is precisely the perfect closure of the maximal tamely ramified extension of $K((t))$.

Abhyankar [1] pointed out that under a suitable generalization of the notion of power series, Chevalley's polynomial should acquire the root

$$x = t^{-1/p} + t^{-1/p^2} + t^{-1/p^3} + \cdots.$$

The generalization we use here was introduced by Hahn [4], and we will only give a brief introduction here; for detailed treatments, see [7] or [9].

A *generalized power series* (or simply "series") is an expression of the form $\sum_{i \in \mathbb{Q}} x_i t^i$ with $x_i \in K$, where the set of i such that $x_i \neq 0$ (the *support* of the

series) is a *well-ordered* subset of $\mathbb{Q}$, that is, one every nonempty subset of which has a least element. We add and multiply generalized power series in the natural way:

$$\sum_i x_i t^i + \sum_j y_j t^j = \sum_k (x_k + y_k) t^k,$$

$$\sum_i x_i t^i \cdot \sum_{j \in T} y_j t^j = \sum_k \left(\sum_{i+j=k} x_i y_j \right) t^k.$$

Note that multiplication makes sense because for any k, there are only finitely many pairs i, j with $i + j = k$ and $x_i y_j \neq 0$. Also, both the sum and product have well-ordered supports, so the generalized power series form a ring under these operations.

The ring of generalized power series is quite large, so one might reasonably expect it to contain an algebraic closure of $K((t))$. A stronger assertion was proved independently by Huang [5], Rayner [8] and Ştefănescu [11]. (Huang's Ph.D. thesis, written under Abhyankar, does not appear to have been published.)

Theorem 1 (Huang, Rayner, Ştefănescu). *Let L be the set of generalized power series of the form $f = \sum_{i \in S} x_i t^i$ ($x_i \in K$), where the set S (which depends on f) has the following properties:*

1. *Every nonempty subset of S has a least element (i.e. S is well-ordered).*
2. *There exists a natural number m such that every element of mS has denominator a power of p.*

Then L is an algebraically closed field.

The purpose of this paper is to refine this result by determining precisely which series of the form described by Theorem 1 are algebraic over $K((t))$. Such series will satisfy two additional restrictions: a further condition on the support of the series, and (unlike in characteristic zero) a condition on the coefficients themselves. A prototype of the latter condition is the following result, also due independently to Huang and to Ştefănescu [12].

Theorem 2 (Huang, Ştefănescu). *The series $\sum_{i=1}^{\infty} x_i t^{-1/p^i}$ ($x_i \in \overline{\mathbb{F}_p}$) is algebraic over $\overline{\mathbb{F}_p}((t))$ if and only if the sequence $\{x_i\}$ is eventually periodic.*

Our main result (Theorem 8) implies Theorem 2 as well as other results of Benhissi [2], Huang, and Vaidya [13]. (It does not directly imply Theorem 1, but our approach can be easily adapted to give a short proof of that theorem.) From it we can derive various other consequences about the structure of algebraic generalized power series; in particular, one can describe them simply when $K = \overline{\mathbb{F}_p}$ (Theorem 15).

It should be noted that an analogous description of the algebraic closure of a mixed-characteristic complete discrete valuation ring can be given; see [6] for details.

2. Lemmas

We begin with two preparatory lemmas. The first lemma is a routine exercise in Galois theory.

Lemma 3. *Every finite normal extension of $K((t))$ is contained in a tower of Artin-Schreier extensions over $K((t^{1/n}))$ for some $n \in \mathbb{N}$.*

Proof. Let L be a finite normal extension of $K((t))$ of inseparable degree q; then L is Galois over $K((t^{1/q}))$. We now appeal to results from [10, Chap. IV] on extensions of complete fields:

1. A finite Galois extension of complete fields inducing the trivial extension on residue fields is totally ramified.
2. The wild inertia group of such an extension is a p-group.
3. The quotient of the inertia group by the wild inertia group is cyclic of degree prime to p.

Let M be the maximal subextension of L tamely ramified over $K((t^{1/q}))$ and let m be the degree of M over $K((t^{1/q}))$. By Kummer theory (since K contains an m-th root of unity), $M = K((t^{1/q}))(x^{1/m})$ for some $x \in K((t^{1/q}))$, but this directly implies $M = K((t^{1/qm}))$.

Now L is a p-power extension of M, so to complete the proof of the lemma, we need only show that L can be expressed as a tower of Artin-Schreier extensions over M. Since every nontrivial p-group has a nontrivial center, we can find a normal series

$$\mathrm{Gal}(L/M) = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{1\}$$

in which $[G_{i-1} : G_i] = p$ for $i = 1, \ldots, n$. The corresponding subfields form a tower of degree p Galois extensions from M to L. Every Galois extension of degree p in characteristic p is an Artin-Schreier extension (by the additive version of Hilbert's Satz 90), completing the proof. ∎

The second lemma characterizes sequences satisfying the "linearized recurrence relation" (hereafter abbreviated LRR)

$$(1) \qquad d_0 c_n + d_1 c_{n+1}^p + \cdots + d_k c_{n+k}^{p^k} = 0$$

for $n \geq 0$. Of course we may assume $d_k \neq 0$; if we are willing to neglect the first few terms of the series, we may also assume $d_0 \neq 0$.

Lemma 4. *Let k be a positive integer and let $d_0, \ldots, d_k$ be elements of K with $d_0, d_k \neq 0$.*

1. *The roots of the polynomial $P(x) = d_0 x + d_1 x^p + \cdots + d_k x^{p^k}$ form a vector space of dimension k over $\mathbb{F}_p$.*
2. *Let $z_1, \ldots, z_k$ be a basis of the aforementioned vector space. Then a sequence $\{c_n\}$ satisfies (1) if and only if it has the form*

$$(2) \qquad c_n = z_1 \lambda_1^{1/p^n} + \cdots + z_k \lambda_k^{1/p^n}$$

for some $\lambda_1, \ldots, \lambda_k \in K$.

Proof. 1. The roots of $P(x)$ form an $\mathbb{F}_p$-vector space because $P(x+y) = P(x)+P(y)$ (that is, P is linearized), and the dimension is k because $P'(x) = d_0$ has no zeroes, so $P(x)$ has distinct roots.

2. The set of sequences satisfying (1) forms a K-vector space with scalar multiplication given by the formula

$$\lambda \cdot (c_0, c_1, c_2, \ldots) = (c_0 \lambda, c_1 \lambda^{1/p}, c_2 \lambda^{1/p^2}, \ldots)$$

(but not with the usual scalar multiplication, which will cause some difficulties later). The dimension of this space is clearly k, since $c_0, \ldots, c_{k-1}$ determine the entire sequence. On the other hand, the sequences satisfying (2) form a k-dimensional subspace, since the Moore determinant

$$\det \begin{pmatrix} z_1 & \cdots & z_k \\ z_1^p & \cdots & z_k^p \\ \vdots & & \vdots \\ z_1^{p^{k-1}} & \cdots & z_k^{p^{k-1}} \end{pmatrix}$$

is nonzero whenever $z_1, \ldots, z_k$ are linearly independent over $\mathbb{F}_p$. Thus all solutions of (1) are given by (2). $\qquad\square$

Corollary 5. *If the sequences $\{c_n\}$ and $\{c_n'\}$ satisfy LRRs with coefficients $d_0, \ldots,$ d_k and $d_0', \ldots, d_\ell'$, respectively, then the sequences $\{c_n + c_n'\}$ and $\{c_n c_n'\}$ satisfy LRRs with coefficients depending only on the d_i and d_i'.*

Proof. Let $z_1, \ldots, z_k$ and $y_1, \ldots, y_\ell$ be $\mathbb{F}_p$-bases for the roots of the polynomials $d_0 x + \cdots + d_k x^{p^k}$ and $d_0' x + \cdots + d_\ell' x^{p^\ell}$, respectively. Then for suitable λ_i and μ_j,

$$c_n = \sum_i z_i \lambda_i^{1/p^n},$$

$$c_n' = \sum_j y_j \mu_j^{1/p^n},$$

$$c_n + c_n' = \sum_i z_i \lambda_i^{1/p^n} + \sum_j y_j \mu_j^{1/p^n},$$

$$c_n c_n' = \sum_{i,j} (z_i y_j)(\lambda_i \mu_j)^{1/p^n}.$$

In other words, $\{c_n + c_n'\}$ and $\{c_n c_n'\}$ satisfy LRRs whose coefficients are those of the polynomials whose roots comprise the $\mathbb{F}_p$-vector space spanned by $z_i + y_j$ and $z_i y_j$, respectively. In particular, these coefficients depend only on the d_i and d_j', and not on the particular sequences $\{c_n\}$ and $\{c_n'\}$. $\qquad\square$

3. The main result: Algebraic series over $K((t))$

We now construct the sets on which algebraic series are supported. For $a \in \mathbb{N}$ and $b, c \geq 0$, define the set

$$S_{a,b,c} = \left\{ \frac{1}{a}(n - b_1 p^{-1} - b_2 p^{-2} - \cdots) : n \geq -b, b_i \in \{0, \ldots, p-1\}, \sum b_i \leq c \right\}.$$

Since $S_{a,b,c}$ visibly satisfies the conditions of Theorem 1, any series supported on $S_{a,b,c}$ belongs to the field L of that theorem.

Theorem 6. *The ring of series supported on $S_{a,b,c}$ for some a, b, c contains an algebraic closure of $K((t))$.*

We do not give an independent proof of this result, as it follows immediately from Theorem 8, which we prove directly. Beware that the ring in Theorem 6 is not a field! For example, the inverse of $x = \sum_{i=1}^{\infty} x_i t^{-1/p^i}$ is not supported on any $S_{a,b,c}$ unless x is algebraic over $K((t))$.

As noted earlier, to isolate the algebraic closure of $K((t))$ inside the ring of generalized power series, it does not suffice to constrain the support of the series;

we must also impose a "periodicity" condition on the coefficients. Such a condition should resemble the criterion of Theorem 2, but with two key differences: it should apply to an arbitrary field K, and it must constrain the coefficients of a series supported on $S_{a,b,c}$, which cannot be naturally organized into a single sequence.

The first difference is addressed by Vaidya's generalization of Theorem 2 [13, Lemma 4.1.1]: for K arbitrary, the series $\sum_{i=1}^{\infty} x_i t^{-1/p^i}$ is algebraic over $K((t))$ if and only if the sequence $\{x_i\}$ satisfies an LRR. To address the second difference, we must impose Vaidya's criterion on many different sequences of coefficients in a uniform way, so that the criterion actually forces the series to be algebraic. The following definition fulfills this demand.

Definition 1. Let $T_c = S_{1,0,c} \cap (-1, 0)$. A function $f: T_c \to K$ is *twist-recurrent of order k*, for some positive integer k, if there exist $d_0, \ldots, d_k \in K$ such that the LRR (1) holds for any sequence $\{c_n\}$ of the form

$$(3) \qquad c_n = f(-b_1 p^{-1} - \cdots - b_{j-1} p^{-j+1} - p^{-n}(b_j p^{-j} + \cdots)) \quad (n \geq 0)$$

for $j \in \mathbb{N}$ and $b_1, b_2, \ldots \in \{0, \ldots, p-1\}$ with $\sum b_i \leq c$.

An example may clarify the definition: one such sequence is

$$f(-1.2021), f(-1.20201), f(-1.202001), f(-1.2020001), \ldots,$$

where the arguments of f are written in base p. If the value of k is not relevant, we simply say f is twist-recurrent.

Definition 2. A series $x = \sum x_i t^i$ is *twist-recurrent* if the following conditions hold:

1. There exist $a, b, c \in \mathbb{N}$ such that x is supported on $S_{a,b,c}$.
2. For some (any) a, b, c for which x is supported on $S_{a,b,c}$, and for each integer $m \geq -b$, the function $f_m: T_c \to K$ given by $f_m(z) = x_{(m+z)/a}$ is twist-recurrent of order k for some k.
3. The functions f_m span a finite-dimensional vector space over K.

Note that condition 3 implies that the choice of k in the second condition can be made independently of m. It does not imply, however, that the coefficients $d_0, \ldots, d_k$ can be chosen independently of m (except in the case $K = \overline{\mathbb{F}_p}$, as we shall see in the next section).

Lemma 7. *Every twist-recurrent series supported on $S_{1,b,c}$ can be written as a finite $K((t))$-linear combination of twist-recurrent series supported on T_c, and every such linear combination is twist-recurrent.*

Proof. If $x = \sum_i x_i t^i$ is a twist-recurrent series supported on $S_{1,b,c}$, define the functions $f_m: T_c \to K$ for $m \geq -b$ by the formula $f_m(z) = x_{m+z}$, as in Definition 2. By condition 3 of the definition, the f_m span a finite-dimensional vector space of functions from T_c to K; let $g_1, \ldots, g_r$ be a basis for that space, and write $f_m =$

$k_{m1}g_1 + \cdots + k_{mr}g_r$. Now writing

$$x = \sum_{m \geq -b} \sum_{i \in T_c} f_m(i)t^{m+i}$$

$$= \sum_{m \geq -b} \sum_{i \in T_c} t^{m+i} \left(\sum_{j=1}^{r} k_{mj}g_j(i) \right)$$

$$= \sum_{j=1}^{r} \left(\sum_{m \geq -b} k_{mj}t^m \right) \left(\sum_{i \in T_c} g_j(i)t^i \right)$$

expresses x as a finite $K((t))$-linear combination of twist-recurrent series supported on T_c. Conversely, to show that a linear combination of twist-recurrent series on T_c is twist-recurrent, it suffices to observe that by Corollary 5, the sum of two twist-recurrent functions is twist-recurrent (thus verifying condition 2 for the sum, the other two being evident). □

Theorem 8. *The twist-recurrent series form an algebraic closure of $K((t))$.*

Proof. We verify the following three assertions:

1. Every twist-recurrent series is algebraic over $K((t))$.
2. The twist-recurrent series are closed under addition and scalar multiplication.
3. If y is twist-recurrent and $x^p - x = y$, then x is twist-recurrent.

From these, it will follow that the twist-recurrent series form a ring algebraic over $K((t))$ (which is automatically then a field) closed under Artin-Schreier extensions; by Lemma 3, this field is algebraically closed.

Before proceeding, we note that for each assertion, it suffices to work with series supported on $S_{a,b,c}$ with $a = 1$.

1. We proceed by induction on c (with vacuous base case $c = 0$); by Lemma 7, we need only consider a series $x = \sum x_i t^i$ supported on T_c. Choose $d_0, \ldots, d_k$ as in Definition 2, and let

$$y = d_0 x^{1/p^k} + d_1 x^{1/p^{k-1}} + \cdots + d_k x.$$

Clearly y is supported on T_c; in fact, we claim it is supported on $S_{p^k,0,c-1}$. To be precise, if $j = -\sum b_i p^{-i}$ belongs to T_c but not to $S_{p^k c, 0, c-1}$, then $\sum b_i = c$, and $b_i = 0$ for $i \leq k$. In particular $p^k j$ lies in T_c, and so

$$y_j = d_0 x_{p^k j}^{1/p^k} + d_1 x_{p^{k-1} j}^{1/p^{k-1}} + \cdots + d_k x_j = 0$$

because x is twist-recurrent.

We conclude that y is twist-recurrent (we have just verified condition 1, condition 2 follows from Corollary 5, and condition 3 is evident). By the induction hypothesis, y is algebraic over $K((t))$, as then are y^{p^k} and thus x.

2. Closure under addition follows immediately from Lemma 7; as for multiplication, it suffices to show that xy is twist-recurrent whenever $x = \sum x_i t^i$ and $y = \sum y_i t^i$ are twist-recurrent on T_c. We will prove this by showing that any sequence of the form

$$c_n = (xy)_{-b_0 - b_1 p^{-1} - \cdots - b_{j-1}p^{-(j-1)} - p^{-n}(b_j p^{-j} + \cdots)}$$

becomes, after some initial terms, the sum of a fixed number of pairwise products of similar sequences derived from x and y. Those sequences satisfy fixed LRRs, so $\{c_n\}$ will as well by Corollary 5.

To verify this claim, recall that $(xy)_k$ is the sum of $x_i y_j$ over all $i, j \in T_c$ with $i + j = k$. Writing the sum $(-i) + (-j)$ in base p, we notice that for n sufficiently large, there can be no carries across the "gap" between $p^{-(j-1)}$ and p^{-j-n}. (To be precise, the sum of the digits of $-k$ equals the sum of the digits of $(-i)$ and $(-j)$ minus $(p-1)$ times the number of carries.) Thus the number of ways to write $-k$ as $(-i) + (-j)$ is uniformly bounded, and moreover as k runs through a sequence of indices of the shape in (3), the possible i and j are constrained to a finite number of similar sequences. This proves the claim.

3. Since the map $x \mapsto x^p - x$ is additive, it suffices to consider the cases when y is supported on $(-\infty, 0)$ and $(0, \infty)$.

First, suppose y is supported on $(-\infty, 0) \cap S_{a,b,c}$ for some a, b, c; then

$$x = \sum_i \sum_{n=1}^{\infty} y_i^{1/p^n} t^{i/p^n} = \sum_i t^i \sum_n y_{ip^n}^{1/p^n}$$

is supported on $S_{a,b,b+c}$. We must show that if $-b \leq m \leq 0$, $b_i \in \{0, \ldots, p-1\}$ and $\sum b_i \leq c$, then for any j, the sequence

(4)
$$c_n = x_{m - b_1 p^{-1} - \cdots - b_{j-1} p^{-(j-1)} - p^{-n}(b_j p^{-j} + \cdots)}$$

satisfies a fixed LRR. If $m < 0$ or $j > 0$, then $\{c_n\}$ is the sum of a bounded number of sequences satisfying fixed LRRs, namely certain sequences of the y_i, so x is twist-recurrent by Corollary 5. If $m = j = 0$, then

$$c_{n+1}^p - c_n = y_{-b_1 p^{-1} - \cdots - b_{j-1} p^{-(j-1)} - p^{-n}(b_j p^{-j} + \cdots)};$$

if $\{c_{n+1}^p - c_n\}$ is twist-recurrent with coefficients $d_0, \ldots, d_k$, then $\{c_n\}$ is twist-recurrent with coefficients $-d_0, d_0 - d_1, \ldots, d_k - d_{k-1}$.

Next, suppose y is supported on $(0, +\infty) \cap S_{a,b,c}$; then

$$x = -\sum_i \sum_{n=0}^{\infty} y_i^{p^n} t^{ip^n} = -\sum_i t^i \sum_n y_{i/p^n}^{p^n}$$

is also supported on $S_{a,b,c}$. For $i < p^k$, we have $y_{i/p^n} = 0$ for $n > k+c$, since the first c fractional digits of $-i/p^n$ in base p will be $p-1$. Thus each sequence defined by (4) is the sum of a bounded number of sequences satisfying fixed LRRs (the exact number and the coefficients of the LRRs depending on m), and so Corollary 5 again implies that x is twist-recurrent. □

4. Variations

Having completed the proof of the main theorem, we now formulate some variations of its statement, all of which follow as easy corollaries. Some of the modifications can be combined, but to avoid excessive repetition, we refrain from explicitly stating all possible combinations.

First, we describe the algebraic closure closure of $L((t))$ where L is an arbitrary perfect field of characteristic p, not necessarily algebraically closed.

Corollary 9. *Let L be a perfect (but not necessarily algebraically closed) field of characteristic p. Then the algebraic closure of $L((t))$ consists of all twist-recurrent series $x = \sum_i x_i t^i$ with x_i in a finite extension of L.*

Proof. The argument given for assertion 1 in the proof of Theorem 8 shows that any twist-recurrent series with coefficients in M is algebraic over $M((t))$. To show conversely that any series which is algebraic over $L((t))$ has coefficients in a finite extension of L, let E be a finite extension of $L((t))$, and M the integral closure of L in E. Then a slight modification of Lemma 3 implies that E can be expressed as a tower of Artin-Schreier extensions over $M((t^{1/n}))$ for some $n \in \mathbb{N}$. Now the argument given for assertion 3 in the proof of Theorem 8 shows that if y has coefficients in M and $x^p - x = y$, then y has coefficients in M except possibly for its constant coefficient, which may lie in an Artin-Schreier extension of M. We conclude that the coefficients of any element of E lie in a finite extension of L. $\square$

For L not perfect, the situation is more complicated, since if y has coefficients in M and $x^p - x = y$, x may have coefficients which generate inseparable extensions of M. We restrict ourselves to giving a necessary condition for algebraicity in this case.

Corollary 10. *Let L be a field of characteristic p, not necessarily perfect. If $x = \sum_i x_i t^i$ is a generalized power series which is algebraic over $L((t))$, then the following conditions must hold:*

1. *There exists a finite extension L' of L whose perfect closure contains all of the x_i.*
2. *For each i, let f_i be the smallest nonnegative integer such that $x_i^{p^{f_i}} \in L'$. Then $f_i - v_p(i)$ is bounded below.*

Next, we explicitly describe the t-adic completion of the algebraic closure of $K((t))$, which occurs more often in practice than its uncompleted counterpart. The proof is immediate from Theorem 8.

Corollary 11. *The completion of the algebraic closure of $K((t))$ consists of all series $x = \sum_{i \in I} x_i t^i$ such that for every $n \in \mathbb{N}$, the series $\sum_{i \in I \cap (-\infty, n)} x_i t^i$ is twist-recurrent (equivalently, satisfies conditions 1 and 2 of Definition 2).*

From the corollary it follows that truncating an algebraic series (that is, discarding all coefficients larger than some real number) gives an element in the completion of the algebraic closure. In fact a stronger statement is true; it appears possible but complicated to prove this without invoking Theorem 8.

Corollary 12. *Let $x = \sum_i x_i t^i$ be a generalized power series which is algebraic over $K((t))$. Then for any real number j, $\sum_{i<j} x_i t^i$ is also algebraic over $K((t))$.*

Proof. By the theorem, we may read "twist-recurrent" for "algebraic". Since $S_{a,b,c}$ is well ordered, there is a smallest element of $S_{a,b,c}$ which is greater than j, and replacing j by that element reduces us to the case that j is rational. (In fact it is even the case that all accumulation points of $S_{a,b,c}$ are rational.) Now simply note that the definition of twist-recurrence is stable under truncation. $\square$

Next, we note that if the field K is endowed with an absolute value $|\cdot|$, then for any real number r, we can consider the field $K((t))^c$ of power series with positive radius of convergence, which is to say, if $x = \sum_i x_i t^i$, then $r^i |x_i| \to 0$ as $i \to \infty$ for some $r > 0$. This definition extends without change to generalized power series.

Corollary 13. *The algebraic closure of $K((t))^c$ consists of the twist-recurrent series with positive radius of convergence.*

5. Periodic series and the case $K = \overline{\mathbb{F}_p}$

Since the case $K = \overline{\mathbb{F}_p}$ is probably the most frequent to occur in practice, it is worth stating explicitly how Theorem 8 simplifies in this case. Specifically, instead of working with the definition of twist-recurrent functions, we can use a slightly less complicated definition.

A function $f: T_c \to \overline{\mathbb{F}_p}$ is *periodic of period M after N terms* if for every sequence $\{c_n\}$ defined as in (3),

$$c_{n+M} = c_n \qquad \forall\, n \geq N.$$

Lemma 14. *A function $f: T_c \to \overline{\mathbb{F}_p}$ is twist-recurrent if and only if it is periodic.*

Proof. The values of a function periodic after N terms, or of a function twist-recurrent with coefficients $d_0, \ldots, d_N$, are determined by its values on those numbers $i \in (-1, 0)$ such that no two nonzero digits of $-i$ in base p are separated by N or more zeroes. In particular, all values of the function lie in some finite field $\mathbb{F}_q$.

Now if f is periodic with period M after N terms, then f is clearly twist-recurrent of order $k = N + M \log_p q$. Conversely, suppose f is twist-recurrent for some $d_0, \ldots, d_k$; then f is periodic with period at most q^k after q^k terms, since a sequence satisfying (1) repeats as soon as a subsequence of k consecutive terms repeats. $\qquad\square$

Since the set of periodic series for given values of M and N is a vector space over $\overline{\mathbb{F}_p}$, we can restate condition 3 of Definition 2 as a simple uniformity condition. The resulting statement, given as the following theorem, can be deduced immediately from Theorem 8. In turn, from the theorem can be deduced some of the previously known results mentioned in the introduction, such as Theorem 2.

Theorem 15. *A series $x = \sum_{i \in S_{a,b,c}} x_i t^i$ is algebraic over $\overline{\mathbb{F}_p}((t))$ if and only if there exist $M, N \in \mathbb{N}$ such that for every integer $m \geq -b$, the function $f(z) = x_{(m+z)/a}$ is periodic of period M after N terms.*

6. Desideratum: An algebraic proof

Although the definition of a twist-recurrent function involves infinitely many conditions, such a function can be specified by a finite number of coefficients (the exact number depending on a, b, c). Thus it makes sense to ask for an algorithm to compute, for any real number r, enough of the coefficients of a root $\sum_{i \in I} x_i t^i$ of a given polynomial to determine all of the x_i for $i < r$.

In characteristic 0, such an algorithm exists and is well known: a standard application of Newton polygons allows one to compute the lowest-order term of each root, and one can then translate the roots of the polynomial to eliminate this term in the root of interest, compute the new lowest-order term, and repeat. This method works because the supports of the roots are well-behaved: there are only finitely many nonzero x_i with $i < r$. Since this is not generally true in characteristic p, a different strategy must be adopted. One approach would be to determine at the outset a finite set of indices I such that for any r, the values of x_i for $i \in I \cap (-\infty, r)$ determine the values of x_j for all $j < r$; then for each $i \in I$ in increasing order, use the Newton polygon to determine x_i, compute x_j for all j less than the next element of I, and translate the polynomial to eliminate all known low-order terms.

Unfortunately, it is not clear how to bound the size of the index set I in terms of the given polynomial. A bound of this sort would give both a usable algorithm

for computing roots and a direct proof (without Galois theory) of Theorem 8, and so would be of great interest.

Acknowledgments

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