Kalman Filter and Extended Kalman Filter

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May 31, 2015





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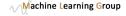






Kalman Filter

- Kalman filter uses a series of measurements observed over time, containing noise and other inaccuracies.
- It produces estimates of unknown variables that tend to be more precise than those based on a single measurement alone.
- ▶ It operates recursively on streams of noisy input data.

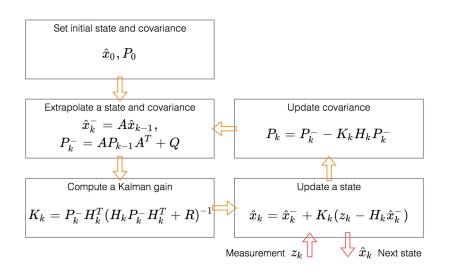


Preliminaries

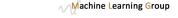
- ► Input (Measurement) z_k
- ▶ Output (State) \hat{x}_k
- ▶ System Model A, H_k, Q, R
- $ightharpoonup z_k = H_k x_k + v_k$ where $v_k \sim \mathcal{N}(0, R)$
- $x_{k+1} = Ax_k + w_k$ where $w_k \sim \mathcal{N}(0, Q)$



Block Diagram of Kalman Filter







Proof of Kalman Filter: State and Covariance Extrapolation

► State extrapolation is

$$\hat{x}_{k+1}^{-} = \mathbb{E}[x_{k+1}|I_k]$$

$$= \mathbb{E}[Ax_k + w_k|I_k]$$

$$= A\hat{x}_k.$$

Covariance extrapolation is

$$\begin{aligned} P_{k+1}^{-} &= \mathbb{E}[(x_{k+1} - \hat{x}_{k}^{-})(x_{k+1} - \hat{x}_{k}^{-})^{T} | I_{k}] \\ &= \mathbb{E}[(x_{k+1} - \hat{x}_{k}^{-})^{2} | I_{k}] \\ &= \mathbb{E}[(Ax_{k} + w_{k} - A\hat{x}_{k})^{2} | I_{k}] \\ &= \mathbb{E}[(A(x_{k} - \hat{x}_{k}) + w_{k})^{2} | I_{k}] \\ &= AP_{k}A^{T} + Q. \end{aligned}$$



► The next state given the current state and the measured state can be expressed by the equation with the given terms,

$$\hat{x}_{k+1} = K'_{k+1} \hat{x}_k + K_{k+1} z_{k+1}.$$

▶ The error term $\tilde{x}_k = \hat{x}_k - x_k$ has zero mean and covariance P_k .



The error term is arranged to

$$\hat{x}_{k+1} - x_{k+1} &= K'_{k+1} \hat{x}_k + K_{k+1} z_{k+1} - x_{k+1} \\
&= K'_{k+1} \hat{x}_k + K_{k+1} (H_{k+1} x_{k+1} + v_{k+1}) \\
&- x_{k+1} - K'_{k+1} x_k + K'_{k+1} x_k.$$

Rearranging term is

$$= K'_{k+1}(\hat{x}_k - x_k) + K_{k+1}(H_{k+1}(Ax_k + w_k) + v_{k+1}) - (Ax_k + w_k) + K'_{k+1}x_k$$

$$= K'_{k+1}(\hat{x}_k - x_k) + (K_{k+1}H_{k+1}A - A + K'_{k+1})x_k + (K_{k+1}H_{k+1} - I)w_k + K_{k+1}v_{k+1}.$$





▶ The expectation of the error term is

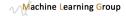
$$\mathbb{E}[\hat{x}_{k+1} - x_{k+1}] = [K_{k+1}H_{k+1}A - A + K'_{k+1}]\mathbb{E}[x_k] = 0.$$

Therefore,

$$K_{k+1}H_{k+1}A - A + K'_{k+1} = 0$$

 $K'_{k+1} = (I - K_{k+1}H_{k+1})A$

is satisfied.



The next state is

$$\begin{split} \hat{x}_{k+1} &= (I - K_{k+1} H_{k+1}) A \hat{x}_k + K_{k+1} z_{k+1} \\ &= \underbrace{A \hat{x}_k}_{\text{extrapolated state}} + \underbrace{K_{k+1} (z_{k+1} - H_{k+1} A \hat{x}_k)}_{\text{residual of measurement and prediction}} \\ &= \hat{x}_{k+1}^- + K_{k+1} (z_{k+1} - H_{k+1} \hat{x}_{k+1}^-) \quad \text{by } \hat{x}_{k+1}^- = A \hat{x}_k. \end{split}$$





Proof of Kalman Filter: Kalman Gain Computation

▶ The covariance of the error term is

$$P_{k+1} = \mathbb{E}[(x_{k+1} - \hat{x}_{k+1})(x_{k+1} - \hat{x}_{k+1})^{T}]$$

$$= \mathbb{E}[(x_{k+1} - \hat{x}_{k+1})^{2}]$$

$$= \mathbb{E}[(x_{k+1} - \hat{x}_{k+1}^{-} - K_{k+1}(z_{k+1} - H_{k+1}\hat{x}_{k+1}^{-}))^{2}]$$

$$= \mathbb{E}[(x_{k+1} - (I - K_{k+1}H_{k+1})\hat{x}_{k+1}^{-} - K_{k+1}z_{k+1})^{2}]$$

$$= \mathbb{E}[(x_{k+1} - (I - K_{k+1}H_{k+1})\hat{x}_{k+1}^{-} - K_{k+1}H_{k+1}x_{k+1} - K_{k+1}v_{k+1})^{2}]$$

$$= \mathbb{E}[((I - K_{k+1}H_{k+1})(x_{k+1} - \hat{x}_{k+1}^{-}) - K_{k+1}v_{k+1})^{2}]$$

$$= (I - K_{k+1}H_{k+1})\mathbb{E}[(x_{k+1} - \hat{x}_{k+1}^{-})^{2}](I - K_{k+1}H_{k+1})^{T} + K_{k+1}\mathbb{E}[v_{k+1}^{2}]K_{k+1}^{T}$$

$$= (I - K_{k+1}H_{k+1})P_{k+1}^{T}(I - K_{k+1}H_{k+1})^{T} + K_{k+1}RK_{k+1}^{T}$$



Proof of Kalman Filter: Kalman Gain Computation

▶ (con'd)

$$\begin{split} P_{k+1} &= P_{k+1}^{-} - K_{k+1} H_{k+1} P_{k+1}^{-} - P_{k+1}^{-} H_{k+1}^{T} K_{k+1}^{T} \\ &\quad + K_{k+1} (H_{k+1} P_{k+1}^{-} H_{k+1}^{T} + R) K_{k+1}^{T}. \end{split}$$

▶ It can be substituted from k + 1 to k,

$$P_{k} = P_{k}^{-} - K_{k} H_{k} P_{k}^{-} - P_{k}^{-} H_{k}^{T} K_{k}^{T} + K_{k} (H_{k} P_{k}^{-} H_{k}^{T} + R) K_{k}^{T}.$$

▶ The trace of P_k is

$$\nabla_{K} \operatorname{tr}(P_{k}) = -P_{k}^{-} H_{k}^{T} - P_{k}^{-} H_{k}^{T} + 2K_{k} H_{k} P_{k}^{-} H_{k}^{T} + 2K_{k} R = 0.$$

► Thus, Kalman gain is

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R)^{-1}.$$





Proof of Kalman Filter: Covariance Update

► The covariance of the error term can be expressed with substituting Kalman gain by

$$\begin{aligned} P_{k} &= P_{k}^{-} - K_{k} H_{k} P_{k}^{-} - P_{k}^{-} H_{k}^{T} K_{k}^{T} + K_{k} (H_{k} P_{k}^{-} H_{k}^{T} + R) K_{k}^{T} \\ &= P_{k}^{-} - K_{k} H_{k} P_{k}^{-} - P_{k}^{-} H_{k}^{T} K_{k}^{T} + P_{k}^{-} H_{k}^{T} K_{k}^{T} \\ &= (I - K_{k} H_{k}) P_{k}^{-}. \end{aligned}$$



Kalman Filter vs. Extended Kalman Filter

- ► Kalman Filter a linear system
- Extended Kalman Filter a nonlinear system





Extended Kalman Filter

▶ State and measurement are the functions of x_k ,

$$x_{k+1} = f_k(x_k) + w_k$$

$$y_k = h_k(x_k) + v_k,$$

where $w_k \sim \mathcal{N}(0, Q)$ and $v_k \sim \mathcal{N}(0, R)$.

▶ Let F(k) and H(k) be the Jacobian matrices of f(.) and h(.), denoted by

$$F(k) = \nabla f_k|_{\hat{x}(k|k)}$$

$$H(k+1) = \nabla h|_{\hat{x}(k=1|k)}.$$



Extended Kalman Filter

Predict Cycle

$$\hat{x}(k+1|k) = f_k(\hat{x}(k|k))$$

 $P(k+1|k) = F(k)P(k|k)F^{T}(k) + Q(k)$

Filtered Cycle

$$\begin{split} \hat{x}(k+1|k+1) &= \hat{x}(k+1|k) \\ &\quad + K(k+1)(y_{k+1} - h_{k+1}(\hat{x}(k+1|k))) \\ K(k+1) &= P(k+1|k)H^{T}(k+1) \\ &\quad * (H(k+1)P(k+1|k)H^{T}(k+1) + R(k+1))^{-1} \\ P(k+1|k+1) &= (I - K(k+1)H(k+1))P(k+1|k) \end{split}$$





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