Kalman Filter and Extended Kalman Filter

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Kalman Filter

- Kalman filter uses a series of measurements observed over time, containing noise and other inaccuracies.
- It produces estimates of unknown variables that tend to be more precise than those based on a single measurement alone.
- It operates recursively on streams of noisy input data.

Preliminaries

- Input (Measurement) - z_k
- Output (State) - $\hat{x}_k$
- System Model - A, H_k, Q, R
- $z_k = H_k x_k + v_k$ where $v_k \sim \mathcal{N}(0, R)$
- $x_{k+1} = Ax_k + w_k$ where $w_k \sim \mathcal{N}(0, Q)$

Block Diagram of Kalman Filter

Set initial state and covariance

$$\hat{x}_0, P_0$$

Extrapolate a state and covariance

$$\hat{x}_k^- = A\hat{x}_{k-1},$$
$$P_k^- = AP_{k-1}A^T + Q$$

Compute a Kalman gain

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R)^{-1}$$

Update a state

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H_k \hat{x}_k^-)$$

Measurement z_k

$\hat{x}_k$ Next state

Update covariance

$$P_k = P_k^- - K_k H_k P_k^-$$

Proof of Kalman Filter: State and Covariance Extrapolation

- State extrapolation is

$$\begin{aligned}\hat{x}_{k+1}^{-} &= \mathbb{E}[x_{k+1}|I_k] \\ &= \mathbb{E}[Ax_k + w_k|I_k] \\ &= A\hat{x}_k.\end{aligned}$$

- Covariance extrapolation is

$$\begin{aligned}P_{k+1}^{-} &= \mathbb{E}[(x_{k+1} - \hat{x}_k^{-})(x_{k+1} - \hat{x}_k^{-})^T|I_k] \\ &= \mathbb{E}[(x_{k+1} - \hat{x}_k^{-})^2|I_k] \\ &= \mathbb{E}[(Ax_k + w_k - A\hat{x}_k)^2|I_k] \\ &= \mathbb{E}[(A(x_k - \hat{x}_k) + w_k)^2|I_k] \\ &= AP_kA^T + Q.\end{aligned}$$

Proof of Kalman Filter: State Update

- The next state given the current state and the measured state can be expressed by the equation with the given terms,

$$\hat{x}_{k+1} = K'_{k+1}\hat{x}_k + K_{k+1}z_{k+1}.$$

- The error term $\tilde{x}_k = \hat{x}_k - x_k$ has zero mean and covariance P_k.

Proof of Kalman Filter: State Update

- The error term is arranged to

$$\begin{aligned}\hat{x}_{k+1} - x_{k+1} &= K'_{k+1}\hat{x}_k + K_{k+1}z_{k+1} - x_{k+1} \\ &= K'_{k+1}\hat{x}_k + K_{k+1}(H_{k+1}x_{k+1} + v_{k+1}) \\ &\quad - x_{k+1} - K'_{k+1}x_k + K'_{k+1}x_k.\end{aligned}$$

- Rearranging term is

$$\begin{aligned}&= K'_{k+1}(\hat{x}_k - x_k) + K_{k+1}(H_{k+1}(Ax_k + w_k) + v_{k+1}) \\ &\quad - (Ax_k + w_k) + K'_{k+1}x_k\end{aligned}$$

$$\begin{aligned}&= K'_{k+1}(\hat{x}_k - x_k) + (K_{k+1}H_{k+1}A - A + K'_{k+1})x_k \\ &\quad + (K_{k+1}H_{k+1} - I)w_k + K_{k+1}v_{k+1}.\end{aligned}$$

Proof of Kalman Filter: State Update

- The expectation of the error term is

$$\mathbb{E}[\hat{x}_{k+1} - x_{k+1}] = [K_{k+1} H_{k+1} A - A + K'_{k+1}] \mathbb{E}[x_k] = 0.$$

- Therefore,

$$\begin{aligned} K_{k+1} H_{k+1} A - A + K'_{k+1} &= 0 \\ K'_{k+1} &= (I - K_{k+1} H_{k+1}) A \end{aligned}$$

is satisfied.

Proof of Kalman Filter: State Update

- The next state is

$$
\begin{aligned}
\hat{x}_{k+1} &= (I - K_{k+1}H_{k+1})A\hat{x}_k + K_{k+1}z_{k+1} \\
&= \underbrace{A\hat{x}_k}_{\text{extrapolated state}} + \underbrace{K_{k+1}(z_{k+1} - H_{k+1}A\hat{x}_k)}_{\text{residual of measurement and prediction}} \\
&= \hat{x}_{k+1}^{-} + K_{k+1}(z_{k+1} - H_{k+1}\hat{x}_{k+1}^{-}) \quad \text{by } \hat{x}_{k+1}^{-} = A\hat{x}_k.
\end{aligned}
$$

Proof of Kalman Filter: Kalman Gain Computation

- The covariance of the error term is

$$
\begin{aligned}
P_{k+1} &= \mathbb{E}[(x_{k+1} - \hat{x}_{k+1})(x_{k+1} - \hat{x}_{k+1})^T] \\
&= \mathbb{E}[(x_{k+1} - \hat{x}_{k+1})^2] \\
&= \mathbb{E}[(x_{k+1} - \hat{x}^-_{k+1} - K_{k+1}(z_{k+1} - H_{k+1}\hat{x}^-_{k+1}))^2] \\
&= \mathbb{E}[(x_{k+1} - (I - K_{k+1}H_{k+1})\hat{x}^-_{k+1} - K_{k+1}z_{k+1})^2] \\
&= \mathbb{E}[(x_{k+1} - (I - K_{k+1}H_{k+1})\hat{x}^-_{k+1} - K_{k+1}H_{k+1}x_{k+1} \\
&\quad - K_{k+1}v_{k+1})^2] \\
&= \mathbb{E}[((I - K_{k+1}H_{k+1})(x_{k+1} - \hat{x}^-_{k+1}) - K_{k+1}v_{k+1})^2] \\
&= (I - K_{k+1}H_{k+1})\mathbb{E}[(x_{k+1} - \hat{x}^-_{k+1})^2](I - K_{k+1}H_{k+1})^T \\
&\quad + K_{k+1}\mathbb{E}[v^2_{k+1}]K^T_{k+1} \\
&= (I - K_{k+1}H_{k+1})P^-_{k+1}(I - K_{k+1}H_{k+1})^T + K_{k+1}RK^T_{k+1}
\end{aligned}
$$

Proof of Kalman Filter: Kalman Gain Computation

- (con'd)

$$P_{k+1} = P_{k+1}^{-} - K_{k+1}H_{k+1}P_{k+1}^{-} - P_{k+1}^{-}H_{k+1}^{T}K_{k+1}^{T} + K_{k+1}(H_{k+1}P_{k+1}^{-}H_{k+1}^{T} + R)K_{k+1}^{T}.$$

- It can be substituted from $k+1$ to k,

$$P_k = P_k^{-} - K_kH_kP_k^{-} - P_k^{-}H_k^TK_k^T + K_k(H_kP_k^{-}H_k^T + R)K_k^T.$$

- The trace of P_k is

$$\nabla_K \operatorname{tr}(P_k) = -P_k^{-}H_k^T - P_k^{-}H_k^T + 2K_kH_kP_k^{-}H_k^T + 2K_kR = 0.$$

- Thus, Kalman gain is

$$K_k = P_k^{-}H_k^T(H_kP_k^{-}H_k^T + R)^{-1}.$$

Proof of Kalman Filter: Covariance Update

- The covariance of the error term can be expressed with substituting Kalman gain by

$$
\begin{aligned}
P_k &= P_k^- - K_k H_k P_k^- - P_k^- H_k^T K_k^T + K_k (H_k P_k^- H_k^T + R) K_k^T \\
&= P_k^- - K_k H_k P_k^- - P_k^- H_k^T K_k^T + P_k^- H_k^T K_k^T \\
&= (I - K_k H_k) P_k^- .
\end{aligned}
$$

Kalman Filter vs. Extended Kalman Filter

- Kalman Filter - a linear system
- Extended Kalman Filter - a nonlinear system

Extended Kalman Filter

- State and measurement are the functions of x_k,

$$x_{k+1} = f_k(x_k) + w_k$$
$$y_k = h_k(x_k) + v_k,$$

where $w_k \sim \mathcal{N}(0, Q)$ and $v_k \sim \mathcal{N}(0, R)$.

- Let $F(k)$ and $H(k)$ be the Jacobian matrices of $f(.)$ and $h(.)$, denoted by

$$F(k) = \nabla f_k|_{\hat{x}(k|k)}$$
$$H(k+1) = \nabla h|_{\hat{x}(k=1|k)}.$$

Extended Kalman Filter

- Predict Cycle

$$\hat{x}(k+1|k) = f_k(\hat{x}(k|k))$$
$$P(k+1|k) = F(k)P(k|k)F^T(k) + Q(k)$$

- Filtered Cycle

$$\begin{aligned}
\hat{x}(k+1|k+1) = \hat{x}(k+1|k) \\
&+ K(k+1)(y_{k+1} - h_{k+1}(\hat{x}(k+1|k))) \\
K(k+1) = P(k+1|k)H^T(k+1) \\
&* (H(k+1)P(k+1|k)H^T(k+1) + R(k+1))^{-1} \\
P(k+1|k+1) = (I - K(k+1)H(k+1))P(k+1|k)
\end{aligned}$$

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