We define $t \equiv T/T_{c_0}$, $\delta \equiv \Delta/T_{c_0}$, $\epsilon_n = (2n+1)\pi t$. The dimensionless gap equation is

$$\ln t = \pi t \sum_n \left(\frac{1}{\sqrt{\epsilon_n^2 + \delta^2}} - \frac{1}{|\epsilon_n|} \right). \tag{1}$$

We define the dimensionless scattering time $1/t_{xy} \equiv \hbar/(\tau_{xy} T_{c_0})$, dimensionless energy splitting $e \equiv E/T_{c_0}$, the dimensionless kinetic energy $x \equiv \xi/T_{c_0}$, and the dimensionless transformed kinetic energy $y(x) \equiv \mathrm{sgn}(x)\sqrt{x^2+\delta^2}$. We have $N_{ge} \equiv N_g - N_e = \tanh(e/2t)$, and $n_F(-y) - n_F(y) = \tanh(y/2t)$. We define the average over energy splittings as $\langle \ldots \rangle_e \equiv \int \mathrm{d}e p(e)(\ldots)$. The dimensionless second-order self-energy is

$$\begin{aligned}
\sigma_\varepsilon(n) \equiv \frac{i\Sigma_\varepsilon^{(2)}(\varepsilon_n)}{\varepsilon_n} &= \left(\frac{1}{t_{vv}} + \frac{2\langle N_{ge} \rangle_e}{t_{vm}} + \frac{1}{t_{mm}} \right) \frac{\pi}{\sqrt{\delta^2 + \epsilon_n^2}} \\
&\quad + \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh(y/2t)}{(y-e)^2 + \epsilon_n^2} \right\rangle_e \qquad (2) \\
\sigma_\Delta(n) \equiv \frac{\Sigma_\Delta^{(2)}(\varepsilon_n)}{\Delta} &= \left(\frac{1}{t_{vv}} + \frac{2\langle N_{ge} \rangle_e}{t_{vm}} + \frac{1}{t_{mm}} \right) \frac{\pi}{\sqrt{\delta^2 + \epsilon_n^2}} \\
&\quad + \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh(y/2t)}{(y-e)^2 + \epsilon_n^2} \frac{y-e}{y} \right\rangle_e . \qquad (3)
\end{aligned}$$

The perturbed dimensionless gap equation is

$$\ln t = \pi t \sum_n \left(\frac{1+\sigma_\Delta}{\sqrt{\epsilon_n^2(1+\sigma_\varepsilon)^2 + \delta^2(1+\sigma_\Delta)^2}} - \frac{1}{|\epsilon_n|} \right). \tag{4}$$

Given the scattering times t_{vv}, t_{vm}, t_{mm}, t_{nn} and the energy splitting distribution $p(e)$, our goal is to solve for the $\delta - t$ relation.

The dimensionless analytically continuated self-energy is

$$\begin{aligned}
-\sigma_\varepsilon(\epsilon) \equiv -\frac{\Sigma_\varepsilon^{(2)}(\varepsilon)}{\varepsilon} &= \left(\frac{1}{t_{vv}} + \frac{2\langle N_{ge} \rangle_e}{t_{vm}} + \frac{1}{t_{mm}} \right) \frac{\pi}{\sqrt{\delta^2 - (\epsilon + i0^+)^2}} \\
&\quad + \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh(y/2t)}{(y-e)^2 - (\epsilon + i0^+)^2} \right\rangle_e \\
\sigma_\Delta(\epsilon) \equiv \frac{\Sigma_\Delta^{(2)}(\varepsilon)}{\Delta} &= \left(\frac{1}{t_{vv}} + \frac{2\langle N_{ge} \rangle_e}{t_{vm}} + \frac{1}{t_{mm}} \right) \frac{\pi}{\sqrt{\delta^2 - (\epsilon + i0^+)^2}} \\
&\quad + \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh(y/2t)}{(y-e)^2 - (\epsilon + i0^+)^2} \frac{y-e}{y} \right\rangle_e
\end{aligned} \tag{5}$$

where $\epsilon = \varepsilon/T_{c_0}$ is the dimensionless energy.

The unperturbed retarded quasiclassical Green's function is

$$g^{(0)}(\varepsilon^R) = -\pi \frac{\varepsilon}{\sqrt{\Delta^2 - (\varepsilon + i0^+)^2}}. \tag{6}$$

The second-order retarded quasiclassical Green's function is

$$g^{(2)}(\varepsilon^R) = -\pi \frac{\varepsilon - \Sigma_\varepsilon^{(2)}(\varepsilon)}{\sqrt{(\Delta + \Sigma_\Delta^{(2)}(\varepsilon))^2 - (\varepsilon - \Sigma_\varepsilon^{(2)}(\varepsilon) + i0^+)^2}}, \tag{7}$$

and the density of state is

$$N(\varepsilon) = -\frac{1}{\pi}\,\mathrm{Im}\big\{g^{(2)}(\varepsilon^R)\big\} \tag{8}$$

$$= \mathrm{Im}\left\{ \frac{\varepsilon - \Sigma_\varepsilon^{(2)}(\varepsilon)}{\sqrt{(\Delta + \Sigma_\Delta^{(2)}(\varepsilon))^2 - (\varepsilon - \Sigma_\varepsilon^{(2)}(\varepsilon) + i0^+)^2}} \right\} \tag{9}$$

$$= \mathrm{Im}\left\{ \frac{\epsilon(1 - \sigma_\varepsilon)}{\sqrt{\delta^2(1 + \sigma_\Delta)^2 - [\epsilon(1 - \sigma_\varepsilon) + i0^+]^2}} \right\}. \tag{10}$$

When we introduce the effective temperature $t^* \equiv T^*/T_{c_0}$ for the nonequilibrium distribution of the TLS, we can directly replace $N_{ge} = \tanh(e/2t)$ with $N_{ge}^* = \tanh(e/2t^*)$ in the above equations.