GAP PROFILE

We define $t \equiv T/T_{c0}$, $\delta \equiv \Delta/T_{c0}$, $\epsilon_n = (2n+1)\pi t$. The dimensionless gap equation is

$$\ln t = \pi t \sum_n \left(\frac{1}{\sqrt{\delta^2 + \epsilon_n^2}} - \frac{1}{|\epsilon_n|} \right). \tag{1}$$

The perturbed dimensionless gap equation is

$$\ln t = \pi t \sum_n \left(\frac{1 + \sigma_\Delta/\delta}{\sqrt{(\delta + \sigma_\Delta)^2 - (i\epsilon_n - \sigma_\varepsilon)^2}} - \frac{1}{|\epsilon_n|} \right). \tag{2}$$

$$\ln t = \pi t \sum_n \left(\frac{1 + \sigma_\Delta/\delta}{\sqrt{\delta^2 (1 + \sigma_\Delta/\delta)^2 + \epsilon_n^2 (1 - \sigma_\varepsilon / i\epsilon_n)^2}} - \frac{1}{|\epsilon_n|} \right). \tag{3}$$

We define the dimensionless scattering time $1/t_{xy} \equiv \hbar/(\tau_{xy} T_{c0})$, dimensionless energy splitting $e \equiv E/T_{c0}$, the dimensionless kinetic energy $x \equiv \xi/T_{c0}$, and the dimensionless transformed kinetic energy $y(x) \equiv \mathrm{sgn}(x)\sqrt{x^2 + \delta^2}$. We have $N_{ge} \equiv N_g - N_e = \tanh(e/2t)$, and $n_F(-y) - n_F(y) = \tanh(y/2t)$. We define the average over energy splittings as $\langle \ldots \rangle_e \equiv \int \mathrm{d}e\, p(e)(\ldots)$. The dimensionless second-order self-energy is

$$-\frac{\sigma_\varepsilon(n)}{i\epsilon_n} \equiv -\frac{\Sigma_\varepsilon^{(2)}(\varepsilon_n)/T_{c0}}{i\varepsilon_n/T_{c0}} = -\frac{\Sigma_\varepsilon^{(2)}(\varepsilon_n)}{i\varepsilon_n} = \left(\frac{1}{t_{vv}} + \frac{2\langle N_{ge} \rangle_e}{t_{vm}} + \frac{1}{t_{mm}} \right) \frac{\pi}{\sqrt{\delta^2 + \epsilon_n^2}} + \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh(y/2t)}{(y-e)^2 + \epsilon_n^2} \right\rangle_e \tag{4}$$

$$\frac{\sigma_\Delta(n)}{\delta} \equiv \frac{\Sigma_\Delta^{(2)}(\varepsilon_n)/T_{c0}}{\Delta/T_{c0}} = \frac{\Sigma_\Delta^{(2)}(\varepsilon_n)}{\Delta} = \left(\frac{1}{t_{vv}} + \frac{2\langle N_{ge} \rangle_e}{t_{vm}} + \frac{1}{t_{mm}} \right) \frac{\pi}{\sqrt{\delta^2 + \epsilon_n^2}} + \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh(y/2t)}{(y-e)^2 + \epsilon_n^2} \frac{y-e}{y} \right\rangle_e . \tag{5}$$

Given the scattering times t_{vv}, t_{vm}, t_{mm}, t_{nn} and the energy splitting distribution $p(e)$, our goal is to solve for the $\delta - t$ relation.

DENSITY OF STATES

The dimensionless analytically continuated self-energy is

$$
\begin{aligned}
-\sigma_\varepsilon(\epsilon) \equiv -\frac{\Sigma^{(2)}_\varepsilon(\varepsilon)}{T_{c_0}} &= \left(\frac{1}{t_{vv}} + \frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}} + \frac{1}{t_{mm}}\right) \frac{\pi\epsilon}{\sqrt{\delta^2 - (\epsilon + i0^+)^2}} \\
&\quad + \frac{1}{t_{nn}}\epsilon \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh\left(y/2t\right)}{(y-e)^2 - (\epsilon + i0^+)^2} \right\rangle_e \\
\frac{\sigma_\Delta(\epsilon)}{\delta} \equiv \frac{\Sigma^{(2)}_\Delta(\varepsilon)/T_{c_0}}{\Delta/T_{c_0}} = \frac{\Sigma^{(2)}_\Delta(\varepsilon)}{\Delta} &= \left(\frac{1}{t_{vv}} + \frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}} + \frac{1}{t_{mm}}\right) \frac{\pi}{\sqrt{\delta^2 - (\epsilon + i0^+)^2}} \\
&\quad + \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh\left(y/2t\right)}{(y-e)^2 - (\epsilon + i0^+)^2} \frac{y-e}{y} \right\rangle_e
\end{aligned} \tag{6}
$$

where $\epsilon = \varepsilon/T_{c_0}$ is the dimensionless energy.

The unperturbed retarded quasiclassical Green's function is

$$
g^{(0)}(\varepsilon^R) = -\pi \frac{\varepsilon}{\sqrt{\Delta^2 - (\varepsilon + i0^+)^2}}. \tag{7}
$$

The second-order retarded quasiclassical Green's function is

$$
g^{(2)}(\varepsilon^R) = -\pi \frac{\varepsilon - \Sigma^{(2)}_\varepsilon(\varepsilon)}{\sqrt{(\Delta + \Sigma^{(2)}_\Delta(\varepsilon))^2 - (\varepsilon - \Sigma^{(2)}_\varepsilon(\varepsilon) + i0^+)^2}}, \tag{8}
$$

and the density of state is

$$
\begin{aligned}
N(\varepsilon) &= -\frac{1}{\pi}\,\mathrm{Im}\big\{g^{(2)}(\varepsilon^R)\big\} && (9) \\
&= \mathrm{Im}\left\{ \frac{\varepsilon - \Sigma^{(2)}_\varepsilon(\varepsilon)}{\sqrt{(\Delta + \Sigma^{(2)}_\Delta(\varepsilon))^2 - (\varepsilon - \Sigma^{(2)}_\varepsilon(\varepsilon) + i0^+)^2}} \right\} && (10) \\
&= \mathrm{Im}\left\{ \frac{\epsilon - \sigma_\varepsilon}{\sqrt{\delta(1 + \sigma_\Delta/\delta)^2 - [\epsilon - \sigma_\varepsilon + i0^+]^2}} \right\}. && (11)
\end{aligned}
$$

When we introduce the effective temperature $t^* \equiv T^*/T_{c_0}$ for the nonequilibrium distribution of the TLS, we can directly replace $N_{ge} = \tanh(e/2t)$ with $N^*_{ge} = \tanh(e/2t^*)$ in the above equations.

Note that we have

$$\operatorname{Im}\left\{\frac{\epsilon}{(y-e)^2-(\epsilon+i0^+)^2}\right\} \tag{12}$$

$$=\operatorname{Im}\left\{\frac{\epsilon}{(y-e)^2-\epsilon^2-2i\epsilon 0^+}\right\} \tag{13}$$

$$=\pi|\epsilon|\delta\left((y-e)^2-\epsilon^2\right) \tag{14}$$

$$=\frac{\pi}{2}\Big[\delta(y-e-\epsilon)+\delta(y-e+\epsilon)\Big] \tag{15}$$

and

$$\operatorname{Im}\left\{\frac{\epsilon}{\sqrt{\delta^2-(\epsilon+i0^+)^2}}\right\} \tag{16}$$

$$=\operatorname{Im}\left\{\frac{\epsilon}{\sqrt{\delta^2-\epsilon^2-2i\epsilon 0^+}}\right\} \tag{17}$$

$$=\frac{|\epsilon|}{\sqrt{\epsilon^2-\delta^2}}\Theta(\epsilon^2-\delta^2)\equiv\rho(\epsilon) \tag{18}$$

and $y=\operatorname{sgn}(x)\sqrt{x^2+\delta^2}$,

$$\int \mathrm{d}x=\int \mathrm{d}y\rho(y) \tag{19}$$

hence we have

$$\begin{aligned}-\operatorname{Im}\{\sigma_\varepsilon\}=&\left(\frac{1}{t_{vv}}+\frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}}+\frac{1}{t_{mm}}\right)\pi\rho(\epsilon)\\&+\frac{1}{t_{nn}}\frac{\pi}{2}\left\langle\rho(\epsilon+e)\left[1-N_{ge}\tanh\left(\frac{\epsilon+e}{2t}\right)\right]+\rho(-\epsilon+e)\left[1-N_{ge}\tanh\left(\frac{-\epsilon+e}{2t}\right)\right]\right\rangle_e\end{aligned} \tag{20}$$

and

$$\begin{aligned}\operatorname{Im}\{\sigma_\Delta\}/\delta=&\left(\frac{1}{t_{vv}}+\frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}}+\frac{1}{t_{mm}}\right)\pi\frac{\rho(\epsilon)}{\epsilon}\\&+\frac{1}{t_{nn}}\frac{\pi}{2}\left\langle\frac{\rho(\epsilon+e)}{\epsilon+e}\left[1-N_{ge}\tanh\left(\frac{\epsilon+e}{2t}\right)\right]-\frac{\rho(-\epsilon+e)}{-\epsilon+e}\left[1-N_{ge}\tanh\left(\frac{-\epsilon+e}{2t}\right)\right]\right\rangle_e\end{aligned} \tag{21}$$

T_c SHIFT

When $t\to t_c$ and $\delta\to 0$, we have $y\to x$, $\epsilon_n\to(2n+1)\pi t_c$, and the gap equation becomes

$$\ln t_c=\sum_n\frac{\pi t_c}{|\epsilon_n|}\left(\frac{1+\sigma_\Delta/\delta}{|1-\sigma_\varepsilon/i\epsilon_n|}-1\right). \tag{22}$$

and the dimensionless self-energy becomes

$$-\frac{\sigma_\varepsilon(n)}{i\epsilon_n} \equiv -\frac{\Sigma_\varepsilon^{(2)}(\varepsilon_n)/T_{c_0}}{i\varepsilon_n/T_{c_0}} = -\frac{\Sigma_\varepsilon^{(2)}(\varepsilon_n)}{i\varepsilon_n} = \left(\frac{1}{t_{vv}} + \frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}} + \frac{1}{t_{mm}}\right)\frac{\pi}{|\epsilon_n|} + \frac{1}{t_{nn}}\int \mathrm{d}x \left\langle \frac{1-\tanh(e/2t_c)\tanh\left(x/2t_c\right)}{(x-e)^2+\epsilon_n^2}\right\rangle_e \tag{23}$$

$$\frac{\sigma_\Delta(n)}{\delta} \equiv \frac{\Sigma_\Delta^{(2)}(\varepsilon_n)/T_{c_0}}{\Delta/T_{c_0}} = \frac{\Sigma_\Delta^{(2)}(\varepsilon_n)}{\Delta} = \left(\frac{1}{t_{vv}} + \frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}} + \frac{1}{t_{mm}}\right)\frac{\pi}{|\epsilon_n|} + \frac{1}{t_{nn}}\int \mathrm{d}x \left\langle \frac{1-\tanh(e/2t_c)\tanh\left(x/2t_c\right)}{(x-e)^2+\epsilon_n^2}\frac{x-e}{x}\right\rangle_e . \tag{24}$$

Note that we can simplify the expressions

$$\int \mathrm{d}x \left\langle \frac{1-\tanh(e/2t_c)\tanh\left(x/2t_c\right)}{(x-e)^2+\epsilon_n^2}\right\rangle_e \tag{25}$$

$$= \int \mathrm{d}y \left\langle \frac{1-\tanh(e/2t_c)\tanh\left((y+e)/2t_c\right)}{y^2+\epsilon_n^2}\right\rangle_e \tag{26}$$

$$= \frac{\pi}{|\epsilon_n|} - \left\langle \tanh(e/2t_c)\int \mathrm{d}y \frac{\tanh\left((y+e)/2t_c\right)}{y^2+\epsilon_n^2}\right\rangle_e \tag{27}$$

and

$$\int \mathrm{d}x \left\langle \frac{1-\tanh(e/2t_c)\tanh\left(x/2t_c\right)}{(x-e)^2+\epsilon_n^2}\frac{x-e}{x}\right\rangle_e \tag{28}$$

$$= \int \mathrm{d}y \left\langle \frac{1-\tanh(e/2t_c)\tanh\left((y+e)/2t_c\right)}{y^2+\epsilon_n^2}\frac{y}{y+e}\right\rangle_e \tag{29}$$

$$= \frac{1}{2}\int \mathrm{d}y \left\langle \frac{1-\tanh(e/2t_c)\tanh\left((y+e)/2t_c\right)}{y^2+\epsilon_n^2}\frac{y}{y+e} + \frac{1-\tanh(e/2t_c)\tanh\left((-y+e)/2t_c\right)}{y^2+\epsilon_n^2}\frac{y}{y-e}\right\rangle_e$$

$$= \frac{1}{2}\int \mathrm{d}y \left\langle \frac{y}{(y^2+\epsilon_n^2)(y^2-e^2)}\left[\Big(1-\tanh(e/2t_c)\tanh\left((y+e)/2t_c\right)\Big)(y-e) + \Big(1-\tanh(e/2t_c)\tanh\left((-y+e)/2t_c\right)\Big)(y+e)\right]\right\rangle_e \tag{30}$$

$$= \int \mathrm{d}y \left\langle \frac{y^2}{(y^2+\epsilon_n^2)(y^2-e^2)} - \frac{\tanh(e/2t_c)\tanh\left((y+e)/2t_c\right)y}{(y^2+\epsilon_n^2)(y+e)}\right\rangle_e \tag{31}$$

$$= \left\langle \frac{\pi|\epsilon_n|}{e^2+\epsilon_n^2}\right\rangle_e - \left\langle \tanh(e/2t_c)\int \mathrm{d}y \frac{\tanh((y+e)/2t_c)}{y^2+\epsilon_n^2}\frac{y}{y+e}\right\rangle_e \tag{32}$$

then we have

$$-\frac{\sigma_\varepsilon}{i\epsilon_n} = \left(\frac{1}{t_{vv}} + \frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}} + \frac{1}{t_{mm}}\right)\frac{\pi}{|\epsilon_n|} + \frac{1}{t_{nn}}\left(\frac{\pi}{|\epsilon_n|} - \left\langle \tanh(e/2t_c)\int \mathrm{d}y \frac{\tanh\left((y+e)/2t_c\right)}{y^2+\epsilon_n^2}\right\rangle_e\right)$$

and

$$\frac{\sigma_\Delta}{\delta} = \left(\frac{1}{t_{vv}} + \frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}} + \frac{1}{t_{mm}}\right)\frac{\pi}{|\epsilon_n|} + \frac{1}{t_{nn}}\left(\left\langle\frac{\pi|\epsilon_n|}{e^2+\epsilon_n^2}\right\rangle_e - \left\langle\tanh(e/2t_c)\int \mathrm{d}y\frac{\tanh((y+e)/2t_c)}{y^2+\epsilon_n^2}\frac{y}{y+e}\right\rangle_e\right)$$