GAP PROFILE

We define $t \equiv T/T_{c0}$, $\delta \equiv \Delta/T_{c0}$, $\epsilon_n = (2n+1)\pi t$. The dimensionless gap equation is

$$\ln t = \pi t \sum_n \left(\frac{1}{\sqrt{\delta^2 + \epsilon_n^2}} - \frac{1}{|\epsilon_n|} \right). \tag{1}$$

The perturbed dimensionless gap equation is

$$\ln t = \pi t \sum_n \left(\frac{1 + \sigma_\Delta/\delta}{\sqrt{(\delta + \sigma_\Delta)^2 - (i\epsilon_n - \sigma_\varepsilon)^2}} - \frac{1}{|\epsilon_n|} \right). \tag{2}$$

$$\ln t = \pi t \sum_n \left(\frac{1 + \sigma_\Delta/\delta}{\sqrt{\delta^2(1 + \sigma_\Delta/\delta)^2 + \epsilon_n^2(1 - \sigma_\varepsilon/i\epsilon_n)^2}} - \frac{1}{|\epsilon_n|} \right). \tag{3}$$

We define the dimensionless scattering time $1/t_{xy} \equiv \hbar/(\tau_{xy} T_{c0})$, dimensionless energy splitting $e \equiv E/T_{c0}$, the dimensionless kinetic energy $x \equiv \xi/T_{c0}$, and the dimensionless transformed kinetic energy $y(x) \equiv \mathrm{sgn}(x)\sqrt{x^2 + \delta^2}$. We have $N_{ge} \equiv N_g - N_e = \tanh(e/2t)$, and $n_F(-y) - n_F(y) = \tanh(y/2t)$. We define the average over energy splittings as $\langle \ldots \rangle_e \equiv \int \mathrm{d}e p(e)(\ldots)$. The dimensionless second-order self-energy is

$$-\frac{\sigma_\varepsilon(n)}{i\epsilon_n} \equiv -\frac{\Sigma_\varepsilon^{(2)}(\varepsilon_n)/T_{c0}}{i\varepsilon_n/T_{c0}} = -\frac{\Sigma_\varepsilon^{(2)}(\varepsilon_n)}{i\varepsilon_n} = \left(\frac{1}{t_{vv}} + \frac{2\langle N_{ge} \rangle_e}{t_{vm}} + \frac{1}{t_{mm}} \right) \frac{\pi}{\sqrt{\delta^2 + \epsilon_n^2}}$$
$$+ \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh(y/2t)}{(y-e)^2 + \epsilon_n^2} \right\rangle_e \tag{4}$$

$$\frac{\sigma_\Delta(n)}{\delta} \equiv \frac{\Sigma_\Delta^{(2)}(\varepsilon_n)/T_{c0}}{\Delta/T_{c0}} = \frac{\Sigma_\Delta^{(2)}(\varepsilon_n)}{\Delta} = \left(\frac{1}{t_{vv}} + \frac{2\langle N_{ge} \rangle_e}{t_{vm}} + \frac{1}{t_{mm}} \right) \frac{\pi}{\sqrt{\delta^2 + \epsilon_n^2}}$$
$$+ \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh(y/2t)}{(y-e)^2 + \epsilon_n^2} \frac{y-e}{y} \right\rangle_e . \tag{5}$$

Given the scattering times t_{vv}, t_{vm}, t_{mm}, t_{nn} and the energy splitting distribution $p(e)$, our goal is to solve for the $\delta - t$ relation.

DENSITY OF STATES

The dimensionless analytically continuated self-energy is

$$
\begin{aligned}
-\sigma_\varepsilon(\epsilon) \equiv -\frac{\Sigma_\varepsilon^{(2)}(\varepsilon)}{T_{c_0}} &= \left(\frac{1}{t_{vv}} + \frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}} + \frac{1}{t_{mm}}\right) \frac{\pi\epsilon}{\sqrt{\delta^2-(\epsilon+i0^+)^2}} \\
&\quad + \frac{1}{t_{nn}}\epsilon \int \mathrm{d}x \left\langle \frac{1-\tanh(e/2t)\tanh\left(y/2t\right)}{(y-e)^2-(\epsilon+i0^+)^2} \right\rangle_e \\
\frac{\sigma_\Delta(\epsilon)}{\delta} \equiv \frac{\Sigma_\Delta^{(2)}(\varepsilon)/T_{c_0}}{\Delta/T_{c_0}} = \frac{\Sigma_\Delta^{(2)}(\varepsilon)}{\Delta} &= \left(\frac{1}{t_{vv}} + \frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}} + \frac{1}{t_{mm}}\right) \frac{\pi}{\sqrt{\delta^2-(\epsilon+i0^+)^2}} \\
&\quad + \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1-\tanh(e/2t)\tanh\left(y/2t\right)}{(y-e)^2-(\epsilon+i0^+)^2}\frac{y-e}{y} \right\rangle_e
\end{aligned}
\tag{6}
$$

where $\epsilon = \varepsilon/T_{c_0}$ is the dimensionless energy.

The unperturbed retarded quasiclassical Green's function is

$$
g^{(0)}(\varepsilon^R) = -\pi \frac{\varepsilon}{\sqrt{\Delta^2-(\varepsilon+i0^+)^2}}. \tag{7}
$$

The second-order retarded quasiclassical Green's function is

$$
g^{(2)}(\varepsilon^R) = -\pi \frac{\varepsilon - \Sigma_\varepsilon^{(2)}(\varepsilon)}{\sqrt{(\Delta+\Sigma_\Delta^{(2)}(\varepsilon))^2-(\varepsilon-\Sigma_\varepsilon^{(2)}(\varepsilon)+i0^+)^2}}, \tag{8}
$$

and the density of state is

$$
\begin{aligned}
N(\varepsilon) &= -\frac{1}{\pi}\,\mathrm{Im}\big\{g^{(2)}(\varepsilon^R)\big\} &(9)\\
&= \mathrm{Im}\left\{\frac{\varepsilon-\Sigma_\varepsilon^{(2)}(\varepsilon)}{\sqrt{(\Delta+\Sigma_\Delta^{(2)}(\varepsilon))^2-(\varepsilon-\Sigma_\varepsilon^{(2)}(\varepsilon)+i0^+)^2}}\right\} &(10)\\
&= \mathrm{Im}\left\{\frac{\epsilon-\sigma_\varepsilon}{\sqrt{\delta(1+\sigma_\Delta/\delta)^2-[\epsilon-\sigma_\varepsilon+i0^+]^2}}\right\}. &(11)
\end{aligned}
$$

When we introduce the effective temperature $t^* \equiv T^*/T_{c_0}$ for the nonequilibrium distribution of the TLS, we can directly replace $N_{ge} = \tanh(e/2t)$ with $N_{ge}^* = \tanh(e/2t^*)$ in the above equations.

Note that we have

$$
\begin{aligned}
&\mathrm{Im}\left\{\frac{\epsilon}{(y-e)^2-(\epsilon+i0^+)^2}\right\} && (12)\\
=&\mathrm{Im}\left\{\frac{\epsilon}{(y-e)^2-\epsilon^2-2i\epsilon 0^+}\right\} && (13)\\
=&\pi|\epsilon|\delta\left((y-e)^2-\epsilon^2\right) && (14)\\
=&\frac{\pi}{2}\Big[\delta(y-e-\epsilon)+\delta(y-e+\epsilon)\Big] && (15)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Im}\left\{\frac{\epsilon}{\sqrt{\delta^2-(\epsilon+i0^+)^2}}\right\} && (16)\\
=&\mathrm{Im}\left\{\frac{\epsilon}{\sqrt{\delta^2-\epsilon^2-2i\epsilon 0^+}}\right\} && (17)\\
=&\frac{|\epsilon|}{\sqrt{\epsilon^2-\delta^2}}\Theta(\epsilon^2-\delta^2)\equiv\rho(\epsilon) && (18)
\end{aligned}
$$

and $y=\mathrm{sgn}(x)\sqrt{x^2+\delta^2}$,

$$\int \mathrm{d}x=\int \mathrm{d}y\rho(y) \tag{19}$$

hence we have

$$
\begin{aligned}
-\mathrm{Im}\{\sigma_\varepsilon\}=&\left(\frac{1}{t_{vv}}+\frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}}+\frac{1}{t_{mm}}\right)\pi\rho(\epsilon) && (20)\\
&+\frac{1}{t_{nn}}\frac{\pi}{2}\left\langle\rho(\epsilon+e)\left[1-N_{ge}\tanh\left(\frac{\epsilon+e}{2t}\right)\right]+\rho(-\epsilon+e)\left[1-N_{ge}\tanh\left(\frac{-\epsilon+e}{2t}\right)\right]\right\rangle_e
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Im}\{\sigma_\Delta\}/\delta=&\left(\frac{1}{t_{vv}}+\frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}}+\frac{1}{t_{mm}}\right)\pi\frac{\rho(\epsilon)}{\epsilon} && (21)\\
&+\frac{1}{t_{nn}}\frac{\pi}{2}\left\langle\frac{\rho(\epsilon+e)}{\epsilon+e}\left[1-N_{ge}\tanh\left(\frac{\epsilon+e}{2t}\right)\right]-\frac{\rho(-\epsilon+e)}{-\epsilon+e}\left[1-N_{ge}\tanh\left(\frac{-\epsilon+e}{2t}\right)\right]\right\rangle_e
\end{aligned}
$$