GAP PROFILE

We define $t \equiv T/T_{c0}$, $\delta \equiv \Delta/T_{c0}$, $\epsilon_n = (2n+1)\pi t$. The dimensionless gap equation is

$$\ln t = \pi t \sum_n \left(\frac{1}{\sqrt{\epsilon_n^2 + \delta^2}} - \frac{1}{|\epsilon_n|} \right). \tag{1}$$

We define the dimensionless scattering time $1/t_{xy} \equiv \hbar/(\tau_{xy} T_{c0})$, dimensionless energy splitting $e \equiv E/T_{c0}$, the dimensionless kinetic energy $x \equiv \xi/T_{c0}$, and the dimensionless transformed kinetic energy $y(x) \equiv \mathrm{sgn}(x)\sqrt{x^2+\delta^2}$. We have $N_{ge} \equiv N_g - N_e = \tanh(e/2t)$, and $n_F(-y) - n_F(y) = \tanh(y/2t)$. We define the average over energy splittings as $\langle \ldots \rangle_e \equiv \int \mathrm{d}e p(e)(\ldots)$. The dimensionless second-order self-energy is

$$\begin{aligned}
\sigma_\varepsilon(n) \equiv \frac{i\Sigma_\varepsilon^{(2)}(\varepsilon_n)}{\varepsilon_n} &= \left(\frac{1}{t_{vv}} + \frac{2\langle N_{ge} \rangle_e}{t_{vm}} + \frac{1}{t_{mm}} \right) \frac{\pi}{\sqrt{\delta^2 + \epsilon_n^2}} \\
&\quad + \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh(y/2t)}{(y-e)^2 + \epsilon_n^2} \right\rangle_e \qquad (2) \\
\sigma_\Delta(n) \equiv \frac{\Sigma_\Delta^{(2)}(\varepsilon_n)}{\Delta} &= \left(\frac{1}{t_{vv}} + \frac{2\langle N_{ge} \rangle_e}{t_{vm}} + \frac{1}{t_{mm}} \right) \frac{\pi}{\sqrt{\delta^2 + \epsilon_n^2}} \\
&\quad + \frac{1}{t_{nn}} \int \mathrm{d}x \left\langle \frac{1 - \tanh(e/2t)\tanh(y/2t)}{(y-e)^2 + \epsilon_n^2} \frac{y-e}{y} \right\rangle_e . \qquad (3)
\end{aligned}$$

The perturbed dimensionless gap equation is

$$\ln t = \pi t \sum_n \left(\frac{1 + \sigma_\Delta}{\sqrt{\epsilon_n^2 (1+\sigma_\varepsilon)^2 + \delta^2 (1+\sigma_\Delta)^2}} - \frac{1}{|\epsilon_n|} \right). \tag{4}$$

Given the scattering times t_{vv}, t_{vm}, t_{mm}, t_{nn} and the energy splitting distribution $p(e)$, our goal is to solve for the $\delta - t$ relation.

DENSITY OF STATES

The dimensionless analytically continuated self-energy is

$$
\begin{aligned}
-\sigma_\varepsilon(\epsilon) \equiv -\frac{\Sigma_\varepsilon^{(2)}(\varepsilon)}{\varepsilon} &= \left(\frac{1}{t_{vv}} + \frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}} + \frac{1}{t_{mm}}\right)\frac{\pi}{\sqrt{\delta^2-(\epsilon+i0^+)^2}} \\
&\quad + \frac{1}{t_{nn}}\int \mathrm{d}x \left\langle \frac{1-\tanh(e/2t)\tanh\left(y/2t\right)}{(y-e)^2-(\epsilon+i0^+)^2}\right\rangle_e \\
\sigma_\Delta(\epsilon) \equiv \frac{\Sigma_\Delta^{(2)}(\varepsilon)}{\Delta} &= \left(\frac{1}{t_{vv}} + \frac{2\left\langle N_{ge}\right\rangle_e}{t_{vm}} + \frac{1}{t_{mm}}\right)\frac{\pi}{\sqrt{\delta^2-(\epsilon+i0^+)^2}} \\
&\quad + \frac{1}{t_{nn}}\int \mathrm{d}x \left\langle \frac{1-\tanh(e/2t)\tanh\left(y/2t\right)}{(y-e)^2-(\epsilon+i0^+)^2}\frac{y-e}{y}\right\rangle_e
\end{aligned}
\tag{5}
$$

where $\epsilon = \varepsilon/T_{c_0}$ is the dimensionless energy.

The unperturbed retarded quasiclassical Green's function is

$$
g^{(0)}(\varepsilon^R) = -\pi\frac{\varepsilon}{\sqrt{\Delta^2-(\varepsilon+i0^+)^2}}. \tag{6}
$$

The second-order retarded quasiclassical Green's function is

$$
g^{(2)}(\varepsilon^R) = -\pi\frac{\varepsilon-\Sigma_\varepsilon^{(2)}(\varepsilon)}{\sqrt{(\Delta+\Sigma_\Delta^{(2)}(\varepsilon))^2-(\varepsilon-\Sigma_\varepsilon^{(2)}(\varepsilon)+i0^+)^2}}, \tag{7}
$$

and the density of state is

$$
\begin{aligned}
N(\varepsilon) &= -\frac{1}{\pi}\,\mathrm{Im}\big\{g^{(2)}(\varepsilon^R)\big\} && (8)\\
&= \mathrm{Im}\left\{\frac{\varepsilon-\Sigma_\varepsilon^{(2)}(\varepsilon)}{\sqrt{(\Delta+\Sigma_\Delta^{(2)}(\varepsilon))^2-(\varepsilon-\Sigma_\varepsilon^{(2)}(\varepsilon)+i0^+)^2}}\right\} && (9)\\
&= \mathrm{Im}\left\{\frac{\epsilon(1-\sigma_\varepsilon)}{\sqrt{\delta^2(1+\sigma_\Delta)^2-[\epsilon(1-\sigma_\varepsilon)+i0^+]^2}}\right\}. && (10)
\end{aligned}
$$

When we introduce the effective temperature $t^* \equiv T^*/T_{c_0}$ for the nonequilibrium distribution of the TLS, we can directly replace $N_{ge} = \tanh(e/2t)$ with $N^*_{ge} = \tanh(e/2t^*)$ in the above equations.

Note that we have

$$\begin{aligned}
&\mathrm{Im}\left\{\frac{\epsilon}{(y-e)^2-(\epsilon+i0^+)^2}\right\} && (11)\\
=&\mathrm{Im}\left\{\frac{\epsilon}{(y-e)^2-\epsilon^2-2i\epsilon 0^+}\right\} && (12)\\
=&\pi|\epsilon|\delta\left((y-e)^2-\epsilon^2\right) && (13)\\
=&\frac{\pi}{2}\Big[\delta(y-e-\epsilon)+\delta(y-e+\epsilon)\Big] && (14)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Im}\left\{\frac{\epsilon}{\sqrt{\delta^2-(\epsilon+i0^+)^2}}\right\} && (15)\\
=&\mathrm{Im}\left\{\frac{\epsilon}{\sqrt{\delta^2-\epsilon^2-2i\epsilon 0^+}}\right\} && (16)\\
=&\frac{|\epsilon|}{\sqrt{\epsilon^2-\delta^2}}\Theta(\epsilon^2-\delta^2)\equiv\rho(\epsilon) && (17)
\end{aligned}$$

and $y=\mathrm{sgn}(x)\sqrt{x^2+\delta^2}$,

$$\int \mathrm{d}x=\int \mathrm{d}y\rho(y) \tag{18}$$

hence we have

$$\begin{aligned}
-\epsilon\,\mathrm{Im}\{\sigma_\varepsilon\}=&\left(\frac{1}{t_{vv}}+\frac{2\langle N_{ge}\rangle_e}{t_{vm}}+\frac{1}{t_{mm}}\right)\pi\rho(\epsilon) && (19)\\
&+\frac{1}{t_{nn}}\frac{\pi}{2}\left\langle\rho(\epsilon+e)\left[1-N_{ge}\tanh\left(\frac{\epsilon+e}{2t}\right)\right]+\rho(-\epsilon+e)\left[1-N_{ge}\tanh\left(\frac{-\epsilon+e}{2t}\right)\right]\right\rangle_e
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Im}\{\sigma_\Delta\}=&\left(\frac{1}{t_{vv}}+\frac{2\langle N_{ge}\rangle_e}{t_{vm}}+\frac{1}{t_{mm}}\right)\pi\frac{\rho(\epsilon)}{\epsilon} && (20)\\
&+\frac{1}{t_{nn}}\frac{\pi}{2}\left\langle\frac{\rho(\epsilon+e)}{\epsilon+e}\left[1-N_{ge}\tanh\left(\frac{\epsilon+e}{2t}\right)\right]-\frac{\rho(-\epsilon+e)}{-\epsilon+e}\left[1-N_{ge}\tanh\left(\frac{-\epsilon+e}{2t}\right)\right]\right\rangle_e
\end{aligned}$$