

1 Self-Energy

The Dyson equation is

$$\hat{G}^{-1} = \hat{G}^{(0)-1} - \hat{\Sigma} \quad (1)$$

we have

$$\hat{G}^{(0)-1} = i\varepsilon_n - \xi\hat{\tau}_3 - \Delta\hat{\tau}_1 \quad (2)$$

$$\hat{\Sigma} \equiv \Sigma_\varepsilon + \Sigma_\Delta\hat{\tau}_1 \quad (3)$$

$$i\tilde{\varepsilon}_n \equiv i\varepsilon_n - \Sigma_\varepsilon \rightarrow \tilde{\varepsilon}_n = \varepsilon_n + i\Sigma_\varepsilon \quad (4)$$

$$\tilde{\Delta} \equiv \Delta + \Sigma_\Delta \quad (5)$$

and

$$\hat{G} = (i\tilde{\varepsilon}_n - \xi\hat{\tau}_3 - \tilde{\Delta}\hat{\tau}_1)^{-1} = -\frac{i\tilde{\varepsilon}_n + \xi\hat{\tau}_3 + \tilde{\Delta}\hat{\tau}_1}{\tilde{\varepsilon}_n^2 + \xi^2 + \tilde{\Delta}^2} \quad (6)$$

so from $\hat{G}^{(0)}$ to \hat{G} , the change is $\Delta \rightarrow \tilde{\Delta}$ and $\varepsilon_n \rightarrow \tilde{\varepsilon}_n$.

We multiply \hat{G} from left and right to the Dyson equation, we have

$$1 = \hat{G}\hat{G}^{(0)-1} - \hat{G}\hat{\Sigma} \rightarrow 1 = \hat{\tau}_3\hat{G}\hat{G}^{(0)-1}\hat{\tau}_3 - \hat{\tau}_3\hat{G}\hat{\Sigma}\hat{\tau}_3 \quad (7)$$

$$1 = \hat{G}^{(0)-1}\hat{G} - \hat{\Sigma}\hat{G} \rightarrow 1 = \hat{G}^{(0)-1}\hat{\tau}_3\hat{\tau}_3\hat{G} - \hat{\Sigma}\hat{\tau}_3\hat{\tau}_3\hat{G} \quad (8)$$

Subtract the two equations, we have

$$\left[\hat{G}^{(0)-1}\hat{\tau}_3 - \hat{\Sigma}\hat{\tau}_3, \hat{\tau}_3\hat{G} \right] = 0 \quad (9)$$

$$\rightarrow \left[i\varepsilon_n\hat{\tau}_3 + i\Delta\hat{\tau}_2 - \hat{\Sigma}\hat{\tau}_3, \hat{\tau}_3\hat{G} \right] = 0 \quad (10)$$

In quasiclassical limit, we assume $\hat{\Sigma}$ does not depend on ξ , and we can apply $\int d\xi$ to get the homogeneous Eilenberger equation

$$\left[i\varepsilon_n\hat{\tau}_3 + i\Delta\hat{\tau}_2 - \hat{\Sigma}\hat{\tau}_3, \hat{\mathcal{G}} \right] = 0 \quad (11)$$

where the quasiclassical Green's function is

$$\hat{\mathcal{G}} = \int d\xi \hat{\tau}_3 \hat{G} = -\pi \frac{i\tilde{\varepsilon}_n\hat{\tau}_3 + i\tilde{\Delta}\hat{\tau}_2}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \quad (12)$$

The second-order self-energy is

$$\hat{\Sigma}^{(2)} = \hat{\Sigma}_{vv}^{(2)} + \hat{\Sigma}_{vm}^{(2)} + \hat{\Sigma}_{mm}^{(2)} + \hat{\Sigma}_{nn}^{(2)} \equiv \hat{\Sigma}_{\text{elastic}}^{(2)} + \hat{\Sigma}_{\text{inelastic}}^{(2)} \quad (13)$$

We define the scattering rates as

$$\frac{\hbar}{2\pi\tau_{el}(E, T)} \equiv \hbar \left(\frac{1}{\tau_{vv}} + \frac{2N_{ge}}{\tau_{vm}} + \frac{1}{\tau_{mm}} \right) \quad (14)$$

and

$$\frac{\hbar}{2\pi\tau_{in}} \equiv \frac{\hbar}{\tau_{nn}} \quad (15)$$

The elastic part is proportional to the quasiclassical Green's function,

$$\hat{\Sigma}_{\text{elastic}}^{(2)}(\varepsilon_n) = \frac{\hbar}{2\tau_{el}} \frac{-i\varepsilon_n + \Delta\hat{\tau}_1}{\sqrt{\Delta^2 + \varepsilon_n^2}} \rightarrow \hat{\Sigma}_{\text{elastic}}^{(2)}\hat{\tau}_3 = \frac{\hbar}{2\pi\tau_{el}} \hat{\mathcal{G}}^{(0)} \quad (16)$$

After renormalization, $\hat{\mathcal{G}}^{(0)} \rightarrow \hat{\mathcal{G}}$, the elastic self-energy drops out of the Eilenberger equation.

For the inelastic part, if we evaluate the two Matsubara sums first, we have

$$\hat{\Sigma}_{\text{inelastic}}^{(2)}(\varepsilon_n) = \frac{\hbar}{2\pi\tau_{in}} \int d\xi \left(1 - N_{ge}[n_F(-\xi_\Delta) - n_F(\xi_\Delta)] \right) \frac{-i\varepsilon_n + \Delta\hat{\tau}_1 \frac{\xi_\Delta - E}{\xi_\Delta}}{(\xi_\Delta - E)^2 + \varepsilon_n^2} \quad (17)$$

where we denote the transformed kinetic energy as $\xi_\Delta \equiv \text{sgn}(\xi)\sqrt{\xi^2 + \Delta^2}$.

If we evaluate the ξ integral first, we have

$$\hat{\Sigma}_{\text{inelastic}}^{(2)}(\varepsilon_n) = -\frac{\hbar}{2\pi\tau_{in}} \sum_{a \neq b} T^2 \sum_{n_1, n_2} D_a(\varepsilon_{n_1}) D_b(\varepsilon_{n_2}) \hat{\mathcal{G}}^{(0)}(\varepsilon_n - \varepsilon_{n_1} + \varepsilon_{n_2}) \hat{\tau}_3 \quad (18)$$

We define the Bosonic Matsubara frequency as $\omega_m \equiv \varepsilon_{n_1} - \varepsilon_{n_2}$.

$$\hat{\Sigma}_{\text{inelastic}}^{(2)}(\varepsilon_n) = -\frac{\hbar}{2\pi\tau_{in}} \sum_{a \neq b} T^2 \sum_{m, n_2} D_a(\omega_m + \varepsilon_{n_2}) D_b(\varepsilon_{n_2}) \hat{\mathcal{G}}^{(0)}(\varepsilon_n - \omega_m) \hat{\tau}_3 \quad (19)$$

Note that

$$T \sum_{n_2} D_a(\omega_m + \varepsilon_{n_2}) D_b(\varepsilon_{n_2}) \quad (20)$$

$$= T \sum_{n_2} \frac{1}{i\varepsilon_{n_2} + i\omega_m + \mu_f - \varepsilon_a} \times \frac{1}{i\varepsilon_{n_2} + \mu_f - \varepsilon_b} \quad (21)$$

$$= \frac{n_F(\varepsilon_a - i\omega_m - \mu_f) - n_F(\varepsilon_b - \mu_f)}{(\varepsilon_a - \varepsilon_b) - i\omega_m} \quad (22)$$

$$= \frac{n_a - n_b}{\varepsilon_{ab} - i\omega_m} \quad (23)$$

where we denote $n_a \equiv n_F(\varepsilon_a - \mu_f)$. Now sum over the two levels

$$\sum_{a \neq b} \frac{n_a - n_b}{\varepsilon_{ab} - i\omega_m} \quad (24)$$

$$= \frac{N_{eg}}{E - i\omega_m} + \frac{N_{eg}}{E + i\omega_m} \quad (25)$$

$$= -2N_{ge} \frac{E}{E^2 + \omega_m^2} \quad (26)$$

and we have

$$\hat{\Sigma}_{\text{inelastic}}^{(2)}(\varepsilon_n) = \frac{\hbar}{2\pi\tau_{in}} \times 2N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \hat{\mathcal{G}}^{(0)}(\varepsilon_n - \omega_m) \hat{\tau}_3 \quad (27)$$

$$\Sigma_{\varepsilon, in}^{(2)}(\varepsilon_n) + \Sigma_{\Delta, in}^{(2)}(\varepsilon_n) \hat{\tau}_1 = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \frac{-i(\varepsilon_n - \omega_m) + \Delta \hat{\tau}_1}{\sqrt{\Delta^2 + (\varepsilon_n - \omega_m)^2}} \quad (28)$$

After renormalization, $\hat{\mathcal{G}}^{(0)} \rightarrow \hat{\mathcal{G}}$, and $\hat{\Sigma}_{\text{inelastic}}^{(2)} \rightarrow \hat{\Sigma}_{\text{inelastic}}^{(2, re)}$, which includes a set of diagrams with particular ‘Saturn’ pattern.

$$\hat{\Sigma}_{\text{inelastic}}^{(2, re)}(\varepsilon_n) = \frac{\hbar}{\pi\tau_{in}} N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \hat{\mathcal{G}}(\varepsilon_n - \omega_m) \hat{\tau}_3 \quad (29)$$

$$\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n) + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_n) \hat{\tau}_1 = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \frac{-i(\varepsilon_n - \omega_m) + \Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n - \omega_m) + \left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_n - \omega_m)\right) \hat{\tau}_1}{\sqrt{\left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_n - \omega_m)\right)^2 + \left(\varepsilon_n - \omega_m + i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n - \omega_m)\right)^2}} \quad (30)$$

In order to do analytic continuation, we introduce $\varepsilon_l \equiv \varepsilon_n - \omega_m$,

$$\hat{\Sigma}_{\text{inelastic}}^{(2, re)}(\varepsilon_n) = \frac{\hbar}{\pi\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \hat{\mathcal{G}}(\varepsilon_l) \hat{\tau}_3 \quad (31)$$

$$\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n) + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_n) \hat{\tau}_1 = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l) + \left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l)\right) \hat{\tau}_1}{\sqrt{\left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l)\right)^2 + \left(\varepsilon_l + i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l)\right)^2}} \quad (32)$$

and we substitute $i\varepsilon_n \rightarrow \varepsilon + i0^+$, we have

$$\hat{\Sigma}_{\text{inelastic}}^{(2, re)}(\varepsilon) = \frac{\hbar}{\pi\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 - (\varepsilon - i\varepsilon_l)^2} \hat{\mathcal{G}}(\varepsilon_l) \hat{\tau}_3 \quad (33)$$

$$\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon) + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon) \hat{\tau}_1 = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 - (\varepsilon - i\varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l) + \left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l)\right) \hat{\tau}_1}{\sqrt{\left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l)\right)^2 + \left(\varepsilon_l + i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l)\right)^2}} \quad (34)$$

2 TLS induced T_c^{tls} without mean-field pairing

If we turn off the mean-field pairing, i.e. $\Delta = 0$ in Eq. (56), then the whole system becomes a normal metal, and the question becomes whether the interaction between electrons and TLSs can induce a superconducting state below a critical temperature, i.e. $\Sigma_{\Delta,in}^{(2,re)}(\varepsilon) > 0$ when $T < T_c^{tls}$,

$$\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) + \Sigma_{\Delta,in}^{(2,re)}(\varepsilon_n)\hat{\tau}_1 = \frac{\hbar}{\tau_{in}}N_{ge}T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_l) + \Sigma_{\Delta,in}^{(2,re)}(\varepsilon_l)\hat{\tau}_1}{\sqrt{\Sigma_{\Delta,in}^{(2,re)}(\varepsilon_l)^2 + \left(\varepsilon_l + i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_l)\right)^2}} \quad (35)$$

When $T \geq T_c^{tls}$, we have $\Sigma_{\Delta,in}^{(2,re)} = 0$. For the diagonal part $\Sigma_{\varepsilon,in}^{(2,re)}$, we have

$$i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}}N_{ge}T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \text{sgn}(\varepsilon_l + i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_l)) \quad (36)$$

We propose an ansatz that $\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n)$ is an odd function of ε_n , and has the same sign as ε_n , i.e. $\text{sgn}(\varepsilon_n) = \text{sgn}(i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n))$, and $\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) = -\Sigma_{\varepsilon,in}^{(2,re)}(-\varepsilon_n)$. Then we have

$$i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}}N_{ge}T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \text{sgn}(\varepsilon_l) \quad (37)$$

We separate the positive and negative parts of ε_l in the sum, and we have

$$i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}}N_{ge}T \sum_{\varepsilon_l > 0} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} - \frac{\hbar}{\tau_{in}}N_{ge}T \sum_{\varepsilon_l < 0} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \quad (38)$$

We define $\omega_m \equiv \varepsilon_n - \varepsilon_l = 2m\pi T$, and

$$i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}}N_{ge}T \sum_{\omega_m < \varepsilon_n} \frac{E}{E^2 + \omega_m^2} - \frac{\hbar}{\tau_{in}}N_{ge}T \sum_{\omega_m > \varepsilon_n} \frac{E}{E^2 + \omega_m^2} \quad (39)$$

$$= \frac{\hbar}{\tau_{in}}N_{ge}T \sum_{\omega_m > -\varepsilon_n} \frac{E}{E^2 + \omega_m^2} - \frac{\hbar}{\tau_{in}}N_{ge}T \sum_{\omega_m > \varepsilon_n} \frac{E}{E^2 + \omega_m^2} \quad (40)$$

$$= \frac{\hbar}{\tau_{in}}N_{ge}\text{sgn}(\varepsilon_n)T \sum_{|\omega_m| < |\varepsilon_n|} \frac{E}{E^2 + \omega_m^2} \quad (41)$$

This solution is consistent with the ansatz.

For the off-diagonal part, when $T \rightarrow T_c^{tls}$, we have $i\Sigma_{\varepsilon,in}^{(2,re)} \gg \Sigma_{\Delta,in}^{(2,re)} \rightarrow 0$, and

$$\Sigma_{\Delta,in}^{(2,re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}}N_{ge}T_c^{tls} \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{\Sigma_{\Delta,in}^{(2,re)}(\varepsilon_l)}{\left|\varepsilon_l + i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_l)\right|} \Bigg|_{T_c^{tls}} \quad (42)$$

This is just an eigenvalue problem. For $\Sigma_{\Delta,in}^{(2,re)}(\varepsilon_n)$ to have a non-trivial solution, we should have

$$\sum_l \left(M_{nl} - \delta_{nl}\right) \Sigma_{\Delta,in}^{(2,re)}(\varepsilon_l) = 0 \leftrightarrow \det(M_{nl} - \delta_{nl}) = 0 \quad (43)$$

where

$$M_{nl} = \frac{\hbar}{\tau_{in}}N_{ge}T_c^{tls} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{1}{\left|\varepsilon_l + i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_l)\right|} \Bigg|_{T_c^{tls}} \quad (44)$$

3 TLS induced T_c shift with mean-field pairing

With mean-field pairing, interaction with TLSs can induce a shift of T_{c0} to T_c . When $T \rightarrow T_c$, we have $\Delta \ll T_c$ and $\Sigma_{\Delta,in}^{(2,re)} \ll T_c$, the solution of the diagonal self-energy $\Sigma_{\varepsilon,in}^{(2,re)}$ is still Eq. (41). For the off-diagonal part, the expression is similar to Eq. (42), which is

$$\frac{\Sigma_{\Delta,in}^{(2,re)}(\varepsilon_n)}{\Delta} = \frac{\hbar}{\tau_{in}}N_{ge}T_c \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{1 + \Sigma_{\Delta,in}^{(2,re)}(\varepsilon_l)/\Delta}{\left|\varepsilon_l + i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_l)\right|} \Bigg|_{T_c} \quad (45)$$

The renormalized gap equation Eq. (59) becomes

$$\ln \frac{T_c}{T_{c0}} = \pi T_c \sum_n \left(\frac{1 + \Sigma_\Delta/\Delta}{|\varepsilon_n + i\Sigma_\varepsilon|} - \frac{1}{|\varepsilon_n|} \right) \quad (46)$$

4 Weak-Coupling Limit

In weak-coupling limit, we assume the interaction kernel becomes

$$\frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \rightarrow \frac{1}{E} \Theta(E - |\varepsilon_l|) \quad (47)$$

then the self-energy will not depend on energy ε_n . In the normal state, i.e. when $T \geq T_c$, the equation for the diagonal self-energy is similar to Eq. (36), which is

$$i\Sigma_{\varepsilon, in}^{(2, re)} = \frac{\hbar}{\tau_{in} E} N_{ge} T \sum_{|\varepsilon_l| < E} \text{sgn}(\varepsilon_l + i\Sigma_{\varepsilon, in}^{(2, re)}) \quad (48)$$

A trivial solution is $\Sigma_{\varepsilon, in}^{(2, re)} = 0$. In this case, when $T \rightarrow T_c$, the off-diagonal self-energy is determined by

$$\Sigma_{\Delta, in}^{(2, re)} = \frac{\hbar}{\tau_{in} E} N_{ge} T_c \sum_{|\varepsilon_l| < |E|} \frac{\Delta + \Sigma_{\Delta, in}^{(2, re)}}{|\varepsilon_l|} \Big|_{T_c} = \frac{\hbar}{\tau_{in} E} N_{ge} \left(\Delta + \Sigma_{\Delta, in}^{(2, re)} \right) T_c \sum_{|\varepsilon_l| < |E|} \frac{1}{|\varepsilon_l|} \Big|_{T_c} \quad (49)$$

4.1 TLS induced T_c^{tls}

If we once again turn off the mean-field pairing $\Delta = 0$, the TLS induced T_c^{tls} is determined by

$$1 = \frac{\hbar}{\tau_{in} E} N_{ge} T_c^{tls} \sum_{|\varepsilon_l| < E} \frac{1}{|\varepsilon_l|} \Big|_{T_c^{tls}} \quad (50)$$

This is similar to the BCS gap equation, except that the cutoff is E instead of the Debye frequency ε_D , and the interaction strength is proportional to $1/E$ and N_{ge} . Use the Digamma function in Eq. (57), we have

$$T_c^{tls} = \frac{2e^\gamma}{\pi} E \exp\left(-\frac{\pi E}{N_{ge} \hbar / \tau_{in}}\right) \approx 1.13 E \exp\left(-\frac{\pi E}{N_{ge} \hbar / \tau_{in}}\right) \Big|_{T_c^{tls}} \quad (51)$$

Compare with the BCS gap equation near T_c , we have

$$1 = gN(0)T_c \sum_{|\varepsilon_n| < \varepsilon_D} \frac{1}{|(2n+1)\pi T_c|} \quad (52)$$

and the solution is

$$T_c = \frac{2e^\gamma}{\pi} \varepsilon_D \exp\left(-\frac{\pi}{gN(0)}\right) \approx 1.13 \varepsilon_D \exp\left(-\frac{\pi}{gN(0)}\right) \quad (53)$$

4.2 TLS induced T_c shift

In weak-coupling limit, with mean-field pairing, when $T \rightarrow T_c$, if the diagonal self-energy $\Sigma_{\varepsilon, in}^{(2, re)} = 0$, according to the gap equation Eq. (56), the off-diagonal self-energy $\Sigma_{\Delta, in}^{(2, re)}/\Delta = 0$ should be satisfied, otherwise the infinite sum in the gap equation will diverge, and T_c is not shifted by the interaction with TLSs. Meanwhile, we can actually solve the off-diagonal self-energy from Eq. (49), and the solution is not guaranteed to be zero. Why will the weak-coupling limit lead to this inconsistency?

5 Density of States

The quasiclassical Green's function is

$$\hat{G} \equiv \int d\xi \hat{\tau}_3 \hat{G} = -\pi \frac{i\tilde{\varepsilon}_n \hat{\tau}_3 + i\tilde{\Delta} \hat{\tau}_2}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} = \begin{pmatrix} g & f \\ -f^* & -g \end{pmatrix} \quad (54)$$

The density of states is

$$N(\varepsilon) = -\frac{1}{\pi} \text{Im}\{g(\varepsilon^R)\} = \text{Im}\left\{ \frac{\tilde{\varepsilon}}{\sqrt{\tilde{\Delta}^2 - (\tilde{\varepsilon} + i0^+)^2}} \right\} \quad (55)$$

6 Gap Equation

The gap equation is

$$\Delta = -v_0 T \sum_{\varepsilon_n}^{\varepsilon_c} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2} \text{Tr} \{ \hat{G} \hat{\tau}_1 \} \equiv g \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{\tilde{\Delta}}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \quad (56)$$

The digamma function is

$$K(T) \equiv \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{|\varepsilon_n|} \approx \ln \left(1.13 \frac{\varepsilon_c}{T} \right) \quad (57)$$

Then we have

$$\frac{1}{g} = \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{\tilde{\Delta}/\Delta}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \stackrel{\text{clean limit}}{=} \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{\sqrt{\Delta^2 + \varepsilon_n^2}} \stackrel{T \rightarrow T_{c0}}{=} \pi T_{c0} \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{|(2n+1)\pi T_{c0}|} \approx \ln \left(1.13 \frac{\varepsilon_c}{T_{c0}} \right) \quad (58)$$

Subtract the previous two equations, we have

$$\ln \frac{T}{T_{c0}} = \pi T \sum_n \left(\frac{1 + \Sigma_{\Delta}/\Delta}{\sqrt{(\Delta + \Sigma_{\Delta})^2 + (\varepsilon_n + i\Sigma_{\varepsilon})^2}} - \frac{1}{|\varepsilon_n|} \right) \quad (59)$$

7 The TLS model

The Hamiltonian is

$$H = H_{\text{BCS}} + H_{\text{imp}} + H_{\text{e-imp}}, \quad (60)$$

where

$$H_{\text{imp}}(\{\mathbf{X}_j\}) = \sum_{j=1}^N H_{\text{imp},j}(\mathbf{X}_j) = \sum_{j=1}^N \left\{ \frac{|\mathbf{P}_j|^2}{2M} + U_j(\mathbf{X}_j) \right\}, \quad (61)$$

U_j is the potential for each impurity, which have different centers \mathbf{R}_j and orientations $\hat{\mathbf{a}}_j$. \mathbf{X}_j is the position of the j -th impurity. The interaction between electrons and impurities is

$$H_{\text{e-imp}}(\{\mathbf{X}_j\}) = \sum_{\alpha} \int d^3 r \psi_{\alpha}^{\dagger}(\mathbf{r}) \sum_j V(\mathbf{r} - \mathbf{X}_j) \psi_{\alpha}(\mathbf{r}), \quad (62)$$

We approximate each impurity as a TLS with local strain,

$$H_{\text{imp}} \rightarrow H_{\text{TLS}} = \sum_j \frac{E_j}{2} \sigma_z(j), \quad (63)$$

Here $E_j = \sqrt{J_j^2 + \varepsilon_j^2}$. $J_j = \langle e | H_{\text{imp}} | e \rangle_j - \langle g | H_{\text{imp}} | g \rangle_j$ is the tunneling matrix element. ε_j is the strain. The interaction between electrons and TLSs is

$$H_{\text{e-TLS}} = \sum_{\alpha} \int d^3 r \psi_{\alpha}^{\dagger}(\mathbf{r}) \sum_j \left[v_j(\mathbf{r}) + m_j(\mathbf{r}) \sigma_z(j) + n_j(\mathbf{r}) \sigma_x(j) \right] \psi_{\alpha}(\mathbf{r}), \quad (64)$$

Here the interaction potentials $f_j(\mathbf{r}) = f(\mathbf{r} - \mathbf{R}_j, \mathbf{a}_j)$ where $f(\mathbf{r})$ refers to $v(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$. In momentum representation, we have $\psi_{\alpha}^{\dagger}(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}'} e^{-i\mathbf{k}' \cdot \mathbf{r}} c_{\mathbf{k}'\alpha}^{\dagger}$,

$$H_{\text{e-TLS}} = \frac{1}{V} \sum_{\mathbf{k}', \mathbf{k}; \alpha} c_{\mathbf{k}'\alpha}^{\dagger} \sum_j e^{-i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{R}_j} \left[v_{\mathbf{k}' - \mathbf{k}}(j) + m_{\mathbf{k}' - \mathbf{k}}(j) \sigma_z(j) + n_{\mathbf{k}' - \mathbf{k}}(j) \sigma_x(j) \right] c_{\mathbf{k}\alpha}, \quad (65)$$

where $f_{\mathbf{k}' - \mathbf{k}} \equiv \int d^3 r e^{-i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}} f(\mathbf{r})$ for the three electron-TLS interactions, which have dimension of [Energy \times Volume]. We use Abrikosov pseudo-fermion to factorize spin operators, $\vec{\sigma}(j) = \sum_{a,b} f_{j,a}^{\dagger} \vec{\sigma}_{ab} f_{j,b}$

$$H_{\text{e-TLS}} = \frac{1}{V} \sum_{\mathbf{k}', \mathbf{k}; \alpha} c_{\mathbf{k}'\alpha}^{\dagger} \sum_j e^{-i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{R}_j} \left[v_{\mathbf{k}' - \mathbf{k}}(j) + \sum_{a,b} f_{j,a}^{\dagger} A_{\mathbf{k}' - \mathbf{k}}^{ab}(j) f_{j,b} \right] c_{\mathbf{k}\alpha} \quad (66)$$

where

$$A_{\mathbf{k}' - \mathbf{k}}^{ab}(j) = m_{\mathbf{k}' - \mathbf{k}}(j) (\sigma_z)_{ab} + n_{\mathbf{k}' - \mathbf{k}}(j) (\sigma_x)_{ab} = \begin{pmatrix} m_{\mathbf{k}' - \mathbf{k}}(j) & n_{\mathbf{k}' - \mathbf{k}}(j) \\ n_{\mathbf{k}' - \mathbf{k}}(j) & -m_{\mathbf{k}' - \mathbf{k}}(j) \end{pmatrix}_{ab} \quad (67)$$

And we use Popov-Fedotov method to do the perturbation expansion. The first-order terms include a v and a m term. They are elastic and are just corrections to the chemical potential. Note that the **unperturbed** pseudo-fermion propagator $D_{ij,ab} \equiv \delta_{ij}\delta_{ab}D_{ja}$,

$$\hat{G}_{\mathbf{k}',\mathbf{k}}^{(1)} = \hat{G}_{\mathbf{k}'}^{(0)} \frac{1}{V} \sum_j e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}_j} \left[v_{\mathbf{k}'-\mathbf{k}}(j) - T \sum_{n;a,b} D_{jj,ab}(\varepsilon_n) A_{\mathbf{k}'-\mathbf{k}}^{ab}(j) \right] \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)} \quad (68)$$

Apply $\prod_{i=1}^N \int \frac{d^3 R_i}{V}$ to do position average, we have

$$\prod_{i=1}^N \int \frac{d^3 R_i}{V} \hat{G}_{\mathbf{k}',\mathbf{k}}^{(1)} = \hat{G}_{\mathbf{k}}^{(1)} \delta_{\mathbf{k}'\mathbf{k}} = \delta_{\mathbf{k}'\mathbf{k}} \hat{G}_{\mathbf{k}}^{(0)} \frac{1}{V} \sum_j \left[v_0(j) - T \sum_{n,a} D_{j,a}(\varepsilon_n) m_0^{aa}(j) \right] \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)} \quad (69)$$

In Maekawa's model, $m_0^{aa}(j) = 0$ and $v_0(j) = v_0$. We denote $\frac{1}{V} \sum_j \rightarrow n_s$,

$$\hat{G}_{\mathbf{k}}^{(1)} = \hat{G}_{\mathbf{k}}^{(0)} n_s v_0 \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)} \quad (70)$$

which is exactly the static impurity result. The second-order terms include the v-v, v-m, m-m, Saturn and the figure-eight diagrams.

$$\begin{aligned} \hat{G}_{\mathbf{k}'',\mathbf{k}}^{(\text{Saturn})}(\varepsilon_n) = & -\frac{1}{V^2} \sum_{ij} \sum_{abcdk'} e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\mathbf{R}_i} e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}_j} A_{\mathbf{k}''-\mathbf{k}'}^{cd}(i) A_{\mathbf{k}'-\mathbf{k}}^{ab}(j) \\ & \times T^2 \sum_{n_1, n_2} D_{ij,da}(\varepsilon_{n_1}) D_{ji,bc}(\varepsilon_{n_2}) \hat{G}_{\mathbf{k}''}^{(0)} \hat{\tau}_3 \hat{G}_{\mathbf{k}'}^{(0)}(\varepsilon_n + \varepsilon_{n_2} - \varepsilon_{n_1}) \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)}. \end{aligned} \quad (71)$$

In the Saturn diagram, the **unperturbed** pseudo-fermion Green's function already demands repeated scattering of the same impurity.

$$\hat{G}_{\mathbf{k}'',\mathbf{k}}^{(\text{Saturn})}(\varepsilon_n) = -\frac{1}{V^2} \sum_j e^{-i(\mathbf{k}''-\mathbf{k})\cdot\mathbf{R}_j} \sum_{abk'} A_{\mathbf{k}''-\mathbf{k}'}^{ba}(j) A_{\mathbf{k}'-\mathbf{k}}^{ab}(j) T^2 \sum_{n_1, n_2} D_{ja}(\varepsilon_{n_1}) D_{jb}(\varepsilon_{n_2}) \hat{G}_{\mathbf{k}''}^{(0)} \hat{\tau}_3 \hat{G}_{\mathbf{k}'}^{(0)}(\varepsilon_n + \varepsilon_{n_2} - \varepsilon_{n_1}) \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)}. \quad (72)$$

Apply $\prod_{i=1}^N \int \frac{d^3 R_i}{V}$ to do position average, we have

$$\hat{G}_{\mathbf{k}}^{(\text{Saturn})}(\varepsilon_n) = -\frac{1}{V^2} \sum_{jabk'} |A_{\mathbf{k}'-\mathbf{k}}^{ab}(j)|^2 T^2 \sum_{n_1, n_2} D_{ja}(\varepsilon_{n_1}) D_{jb}(\varepsilon_{n_2}) \hat{G}_{\mathbf{k}}^{(0)} \hat{\tau}_3 \hat{G}_{\mathbf{k}'}^{(0)}(\varepsilon_n + \varepsilon_{n_2} - \varepsilon_{n_1}) \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)}. \quad (73)$$

Apply orientation average $\prod_{i=1}^N \int \frac{d\Omega_{a_j}}{4\pi}$, which is equivalently averaging over the external momentum direction $\prod_{i=1}^N \int \frac{d\Omega_k}{4\pi}$. Note that $\frac{1}{V} \sum_{k'} \rightarrow \frac{1}{(2\pi)^3} \int d^3 k' \approx \int \frac{d\Omega_{k'}}{4\pi} N(0) \int d\xi_{k'}$,

$$\hat{G}_{\mathbf{k}}^{(\text{Saturn})}(\varepsilon_n) = -\frac{1}{V} \sum_{jab} N(0) \int d\xi_{k'} \langle |A_{\mathbf{k}'-\mathbf{k}}^{ab}(j)|^2 \rangle_{\Omega} T^2 \sum_{n_1, n_2} D_{ja}(\varepsilon_{n_1}) D_{jb}(\varepsilon_{n_2}) \hat{G}_{\mathbf{k}}^{(0)} \hat{\tau}_3 \hat{G}_{\mathbf{k}'}^{(0)}(\varepsilon_n + \varepsilon_{n_2} - \varepsilon_{n_1}) \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)}. \quad (74)$$

We define the inelastic scattering time τ_{in} as $\frac{\hbar}{2\pi\tau} = n_s N(0) \langle |A|^2 \rangle_{\Omega}$, assume the interaction potential only depends on the direction of the momentum transfer, define quasiclassical Green's function $\hat{\mathcal{G}}$, and assume identical TLS energy splitting,

$$\hat{G}_{\mathbf{k}}^{(\text{Saturn})}(\varepsilon_n) = -\sum_{ab} \left(\frac{\hbar}{2\pi\tau_m} \delta^{ab} + \frac{\hbar}{2\pi\tau_n} \sigma_x^{ab} \right) T^2 \sum_{n_1, n_2} D_a(\varepsilon_{n_1}) D_b(\varepsilon_{n_2}) \hat{G}_{\mathbf{k}}^{(0)} \hat{\mathcal{G}}^{(0)}(\varepsilon_n + \varepsilon_{n_2} - \varepsilon_{n_1}) \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)}. \quad (75)$$

Later we can still do a configuration average over the TLS energy splitting. We will also see that the $\frac{\hbar}{2\pi\tau_m} \delta^{ab}$ term will not contribute to the self-energy. The figure-eight diagram is

$$\hat{G}_{\mathbf{k}'',\mathbf{k}}^{(\text{figure-8})}(\varepsilon_n) = \frac{1}{V^2} \sum_{ij} \sum_{abcdk'} e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\mathbf{R}_j} A_0^{cd}(i) A_{\mathbf{k}''-\mathbf{k}'}^{ab}(j) T^2 \sum_{n_1, n_2} D_{ij,da}(\varepsilon_{n_1}) D_{ji,bc}(\varepsilon_{n_1}) \hat{G}_{\mathbf{k}''}^{(0)} \hat{\tau}_3 \hat{G}_{\mathbf{k}'}^{(0)}(\varepsilon_{n_2}) \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)}. \quad (76)$$

Similarly we have

$$\hat{G}_{\mathbf{k}'',\mathbf{k}}^{(\text{figure-8})}(\varepsilon_n) = \frac{1}{V^2} \sum_{jabk'} e^{-i(\mathbf{k}''-\mathbf{k})\cdot\mathbf{R}_j} A_0^{ba}(j) A_{\mathbf{k}'-\mathbf{k}}^{ab}(j) T^2 \sum_{n_1, n_2} D_{ja}(\varepsilon_{n_1}) D_{jb}(\varepsilon_{n_1}) \hat{G}_{\mathbf{k}''}^{(0)} \hat{\tau}_3 \hat{G}_{\mathbf{k}'}^{(0)}(\varepsilon_{n_2}) \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)}. \quad (77)$$

Once again in Maekawa's model $A_0^{ab} = 0$. Apply position average

$$\hat{G}_{\mathbf{k}}^{(\text{figure-8})}(\varepsilon_n) = \frac{1}{V^2} \sum_{jabk'} |A_0^{ab}(j)|^2 T^2 \sum_{n_1, n_2} D_{ja}(\varepsilon_{n_1}) D_{jb}(\varepsilon_{n_1}) \hat{G}_{\mathbf{k}}^{(0)} \hat{\tau}_3 \hat{G}_{\mathbf{k}'}^{(0)}(\varepsilon_{n_2}) \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)}. \quad (78)$$

apply orientation average

$$\hat{G}_{\mathbf{k}}^{(\text{figure-8})}(\varepsilon_n) = \sum_{ab} \left(\frac{\hbar}{2\pi\tau_m} \delta^{ab} + \frac{\hbar}{2\pi\tau_n} \sigma_x^{ab} \right) T^2 \sum_{n_1, n_2} D_a(\varepsilon_{n_1}) D_b(\varepsilon_{n_1}) \hat{G}_{\mathbf{k}}^{(0)} \hat{\mathcal{G}}^{(0)}(\varepsilon_{n_2}) \hat{\tau}_3 \hat{G}_{\mathbf{k}}^{(0)} \quad (79)$$