1 Self-Energy

The Dyson equation is

$$\widehat{G}^{-1} = \widehat{G}^{(0)-1} - \widehat{\Sigma} \tag{1}$$

we have

$$\widehat{G}^{(0)-1} = i\varepsilon_n - \xi\widehat{\tau}_3 - \Delta\widehat{\tau}_1 \tag{2}$$

$$\widehat{\Sigma} \equiv \Sigma_\varepsilon + \Sigma_\Delta\widehat{\tau}_1 \tag{3}$$

$$i\tilde{\varepsilon}_n \equiv i\varepsilon_n - \Sigma_\varepsilon \rightarrow \tilde{\varepsilon}_n = \varepsilon_n + i\Sigma_\varepsilon \tag{4}$$

$$\tilde{\Delta} \equiv \Delta + \Sigma_\Delta \tag{5}$$

and

$$\widehat{G} = (i\tilde{\varepsilon}_n - \xi\widehat{\tau}_3 - \tilde{\Delta}\widehat{\tau}_1)^{-1} = -\frac{i\tilde{\varepsilon}_n + \xi\widehat{\tau}_3 + \tilde{\Delta}\widehat{\tau}_1}{\tilde{\varepsilon}_n^2 + \xi^2 + \tilde{\Delta}^2} \tag{6}$$

so from $\widehat{G}^{(0)}$ to $\widehat{G}$, the change is $\Delta \rightarrow \tilde{\Delta}$ and $\varepsilon_n \rightarrow \tilde{\varepsilon}_n$.

We multiply $\widehat{G}$ from left and right to the Dyson equation, we have

$$1 = \widehat{G}\widehat{G}^{(0)-1} - \widehat{G}\widehat{\Sigma} \rightarrow 1 = \widehat{\tau}_3\widehat{G}\widehat{G}^{(0)-1}\widehat{\tau}_3 - \widehat{\tau}_3\widehat{G}\widehat{\Sigma}\widehat{\tau}_3 \tag{7}$$

$$1 = \widehat{G}^{(0)-1}\widehat{G} - \widehat{\Sigma}\widehat{G} \rightarrow 1 = \widehat{G}^{(0)-1}\widehat{\tau}_3\widehat{\tau}_3\widehat{G} - \widehat{\Sigma}\widehat{\tau}_3\widehat{\tau}_3\widehat{G} \tag{8}$$

Subtract the two equations, we have

$$\left[\widehat{G}^{(0)-1}\widehat{\tau}_3 - \widehat{\Sigma}\widehat{\tau}_3, \widehat{\tau}_3\widehat{G}\right] = 0 \tag{9}$$

$$\rightarrow \left[i\varepsilon_n\widehat{\tau}_3 + i\Delta\widehat{\tau}_2 - \widehat{\Sigma}\widehat{\tau}_3, \widehat{\tau}_3\widehat{G}\right] = 0 \tag{10}$$

In quasiclassical limit, we assume $\widehat{\Sigma}$ does not depend on ξ, and we can apply $\int \mathrm{d}\xi$ to get the homogeneous Eilenberger equation

$$\left[i\varepsilon_n\widehat{\tau}_3 + i\Delta\widehat{\tau}_2 - \widehat{\Sigma}\widehat{\tau}_3, \widehat{\mathcal{G}}\right] = 0 \tag{11}$$

where the quasiclassical Green's function is

$$\widehat{\mathcal{G}} = \int \mathrm{d}\xi\widehat{\tau}_3\widehat{G} = -\pi\frac{i\tilde{\varepsilon}_n\widehat{\tau}_3 + i\tilde{\Delta}\widehat{\tau}_2}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \tag{12}$$

The second-order self-energy is

$$\widehat{\Sigma}^{(2)} = \widehat{\Sigma}^{(2)}_{vv} + \widehat{\Sigma}^{(2)}_{vm} + \widehat{\Sigma}^{(2)}_{mm} + \widehat{\Sigma}^{(2)}_{nn} \equiv \widehat{\Sigma}^{(2)}_{\text{elastic}} + \widehat{\Sigma}^{(2)}_{\text{inelastic}} \tag{13}$$

We define the scattering rates as

$$\frac{\hbar}{2\pi\tau_{el}(E,T)} \equiv \hbar\left(\frac{1}{\tau_{vv}} + \frac{2N_{ge}}{\tau_{vm}} + \frac{1}{\tau_{mm}}\right) \tag{14}$$

and

$$\frac{\hbar}{2\pi\tau_{in}} \equiv \frac{\hbar}{\tau_{nn}} \tag{15}$$

The elastic part is proportional to the quasiclassical Green's function,

$$\widehat{\Sigma}^{(2)}_{\text{elastic}}(\varepsilon_n) = \frac{\hbar}{2\tau_{el}}\frac{-i\varepsilon_n + \Delta\widehat{\tau}_1}{\sqrt{\Delta^2 + \varepsilon_n^2}} \rightarrow \widehat{\Sigma}^{(2)}_{\text{elastic}}\widehat{\tau}_3 = \frac{\hbar}{2\pi\tau_{el}}\widehat{\mathcal{G}}^{(0)} \tag{16}$$

After renormalization, $\widehat{\mathcal{G}}^{(0)} \rightarrow \widehat{\mathcal{G}}$, the elastic self-energy drops out of the Eilenberger equation.

For the inelastic part, if we evaluate the two Matsubara sums first, we have

$$\widehat{\Sigma}^{(2)}_{\text{inelastic}}(\varepsilon_n) = \frac{\hbar}{2\pi\tau_{in}}\int \mathrm{d}\xi\left(1 - N_{ge}[n_F(-\xi_\Delta) - n_F(\xi_\Delta)]\right)\frac{-i\varepsilon_n + \Delta\widehat{\tau}_1\frac{\xi_\Delta - E}{\xi_\Delta}}{(\xi_\Delta - E)^2 + \varepsilon_n^2} \tag{17}$$

where we denote the transformed kinetic energy as $\xi_\Delta \equiv \mathrm{sgn}(\xi)\sqrt{\xi^2 + \Delta^2}$.

If we evaluate the ξ integral first, we have

$$\widehat{\Sigma}^{(2)}_{\text{inelastic}}(\varepsilon_n) = -\frac{\hbar}{2\pi\tau_{in}}\sum_{a\neq b} T^2 \sum_{n_1,n_2} D_a(\varepsilon_{n_1}) D_b(\varepsilon_{n_2}) \widehat{\mathcal{G}}^{(0)}(\varepsilon_n - \varepsilon_{n_1} + \varepsilon_{n_2})\widehat{\tau}_3 \tag{18}$$

We define the Bosonic Matsubara frequency as $\omega_m \equiv \varepsilon_{n_1} - \varepsilon_{n_2}$.

$$\widehat{\Sigma}^{(2)}_{\text{inelastic}}(\varepsilon_n) = -\frac{\hbar}{2\pi\tau_{in}}\sum_{a\neq b} T^2 \sum_{m,n_2} D_a(\omega_m + \varepsilon_{n_2}) D_b(\varepsilon_{n_2}) \widehat{\mathcal{G}}^{(0)}(\varepsilon_n - \omega_m)\widehat{\tau}_3 \tag{19}$$

Note that

$$\begin{aligned}
&T\sum_{n_2} D_a(\omega_m + \varepsilon_{n_2}) D_b(\varepsilon_{n_2}) && (20)\\
=&T\sum_{n_2} \frac{1}{i\varepsilon_{n_2} + i\omega_m + \mu_f - \varepsilon_a} \times \frac{1}{i\varepsilon_{n_2} + \mu_f - \varepsilon_b} && (21)\\
=&\frac{n_F(\varepsilon_a - i\omega_m - \mu_f) - n_F(\varepsilon_b - \mu_f)}{(\varepsilon_a - \varepsilon_b) - i\omega_m} && (22)\\
=&\frac{n_a - n_b}{\varepsilon_{ab} - i\omega_m} && (23)
\end{aligned}$$

where we denote $n_a \equiv n_F(\varepsilon_a - \mu_f)$. Now sum over the two levels

$$\begin{aligned}
&\sum_{a\neq b} \frac{n_a - n_b}{\varepsilon_{ab} - i\omega_m} && (24)\\
=&\frac{N_{eg}}{E - i\omega_m} + \frac{N_{eg}}{E + i\omega_m} && (25)\\
=&-2N_{ge}\frac{E}{E^2 + \omega_m^2} && (26)
\end{aligned}$$

and we have

$$\begin{aligned}
\widehat{\Sigma}^{(2)}_{\text{inelastic}}(\varepsilon_n) =& \frac{\hbar}{2\pi\tau_{in}} \times 2N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \widehat{\mathcal{G}}^{(0)}(\varepsilon_n - \omega_m)\widehat{\tau}_3 && (27)\\
\Sigma^{(2)}_{\varepsilon,in}(\varepsilon_n) + \Sigma^{(2)}_{\Delta,in}(\varepsilon_n)\widehat{\tau}_1 =& \frac{\hbar}{\tau_{in}} N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \frac{-i(\varepsilon_n - \omega_m) + \Delta\widehat{\tau}_1}{\sqrt{\Delta^2 + (\varepsilon_n - \omega_m)^2}} && (28)
\end{aligned}$$

After renormalization, $\widehat{\mathcal{G}}^{(0)} \to \widehat{\mathcal{G}}$, and $\widehat{\Sigma}^{(2)}_{\text{inelastic}} \to \widehat{\Sigma}^{(2,re)}_{\text{inelastic}}$, which includes a set of diagrams with particular 'Saturn' pattern.

$$\begin{aligned}
\widehat{\Sigma}^{(2,re)}_{\text{inelastic}}(\varepsilon_n) =& \frac{\hbar}{\pi\tau_{in}} N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \widehat{\mathcal{G}}(\varepsilon_n - \omega_m)\widehat{\tau}_3 && (29)\\
\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n) + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n)\widehat{\tau}_1 =& \frac{\hbar}{\tau_{in}} N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \frac{-i(\varepsilon_n - \omega_m) + \Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n - \omega_m) + \left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n - \omega_m)\right)\widehat{\tau}_1}{\sqrt{\left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n - \omega_m)\right)^2 + \left(\varepsilon_n - \omega_m + i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n - \omega_m)\right)^2}} && (30)
\end{aligned}$$

In order to do analytic continuation, we introduce $\varepsilon_l \equiv \varepsilon_n - \omega_m$,

$$\begin{aligned}
\widehat{\Sigma}^{(2,re)}_{\text{inelastic}}(\varepsilon_n) =& \frac{\hbar}{\pi\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \widehat{\mathcal{G}}(\varepsilon_l)\widehat{\tau}_3 && (31)\\
\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n) + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n)\widehat{\tau}_1 =& \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l) + \left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)\right)\widehat{\tau}_1}{\sqrt{\left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)\right)^2 + \left(\varepsilon_l + i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l)\right)^2}} && (32)
\end{aligned}$$

and we substitute $i\varepsilon_n \to \varepsilon + i0^+$, we have

$$\begin{aligned}
\widehat{\Sigma}^{(2,re)}_{\text{inelastic}}(\varepsilon) =& \frac{\hbar}{\pi\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 - (\varepsilon - i\varepsilon_l)^2} \widehat{\mathcal{G}}(\varepsilon_l)\widehat{\tau}_3 && (33)\\
\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon) + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon)\widehat{\tau}_1 =& \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 - (\varepsilon - i\varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l) + \left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)\right)\widehat{\tau}_1}{\sqrt{\left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)\right)^2 + \left(\varepsilon_l + i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l)\right)^2}} && (34)
\end{aligned}$$

2 TLS induced T_c^{tls} without mean-field pairing

If we turn off the mean-field pairing, i.e. $\Delta = 0$ in Eq. (56), then the whole system becomes a normal metal, and the question becomes whether the interaction between electrons and TLSs can induce a superconducting state below a critical temperature, i.e. $\Sigma^{(2,re)}_{\Delta,in}(\varepsilon) > 0$ when $T < T_c^{tls}$,

$$\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n) + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n)\widehat{\tau}_1 = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l) + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)\widehat{\tau}_1}{\sqrt{\Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)^2 + \left(\varepsilon_l + i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l)\right)^2}} \tag{35}$$

When $T \geq T_c^{tls}$, we have $\Sigma^{(2,re)}_{\Delta,in} = 0$. For the diagonal part $\Sigma^{(2,re)}_{\varepsilon,in}$, we have

$$i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \mathrm{sgn}\left(\varepsilon_l + i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l)\right) \tag{36}$$

We propose an ansatz that $\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n)$ is an odd function of ε_n, and has the same sign as ε_n, i.e. $\mathrm{sgn}(\varepsilon_n) = \mathrm{sgn}\left(i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n)\right)$, and $\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n) = -\Sigma^{(2,re)}_{\varepsilon,in}(-\varepsilon_n)$. Then we have

$$i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \mathrm{sgn}(\varepsilon_l) \tag{37}$$

We seperate the positive and negative parts of ε_l in the sum, and we have

$$i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\varepsilon_l > 0} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} - \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\varepsilon_l < 0} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \tag{38}$$

We define $\omega_m \equiv \varepsilon_n - \varepsilon_l = 2m\pi T$, and

$$i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m < \varepsilon_n} \frac{E}{E^2 + \omega_m^2} - \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m > \varepsilon_n} \frac{E}{E^2 + \omega_m^2} \tag{39}$$

$$= \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m > -\varepsilon_n} \frac{E}{E^2 + \omega_m^2} - \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m > \varepsilon_n} \frac{E}{E^2 + \omega_m^2} \tag{40}$$

$$= \frac{\hbar}{\tau_{in}} N_{ge} \mathrm{sgn}(\varepsilon_n) T \sum_{|\omega_m| < |\varepsilon_n|} \frac{E}{E^2 + \omega_m^2} \tag{41}$$

This solution is consistent with the ansatz.

For the off-diagonal part, when $T \to T_c^{tls}$, we have $i\Sigma^{(2,re)}_{\varepsilon,in} \gg \Sigma^{(2,re)}_{\Delta,in} \to 0$, and

$$\Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T_c^{tls} \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{\Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)}{\left|\varepsilon_l + i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l)\right|}\Bigg|_{T_c^{tls}} \tag{42}$$

This is just an eigenvalue problem. For $\Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n)$ to have a non-trivial solution, we should have

$$\sum_l \Big(M_{nl} - \delta_{nl}\Big)\Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l) = 0 \leftrightarrow \det(M_{nl} - \delta_{nl}) = 0 \tag{43}$$

where

$$M_{nl} = \frac{\hbar}{\tau_{in}} N_{ge} T_c^{tls} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{1}{\left|\varepsilon_l + i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l)\right|}\Bigg|_{T_c^{tls}} \tag{44}$$

3 TLS induced T_c shift with mean-field pairing

With mean-field pairing, interaction with TLSs can induce a shift of T_{c_0} to T_c. When $T \to T_c$, we have $\Delta \ll T_c$ and $\Sigma^{(2,re)}_{\Delta,in} \ll T_c$, the solution of the diagonal self-energy $\Sigma^{(2,re)}_{\varepsilon,in}$ is still Eq. (41). For the off-diagonal part, the expression is similar to Eq. (42), which is

$$\frac{\Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n)}{\Delta} = \frac{\hbar}{\tau_{in}} N_{ge} T_c \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{1 + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)/\Delta}{\left|\varepsilon_l + i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l)\right|}\Bigg|_{T_c} \tag{45}$$

The renormalized gap equation Eq. (59) becomes

$$\ln \frac{T_c}{T_{c_0}} = \pi T_c \sum_n \left(\frac{1 + \Sigma_\Delta/\Delta}{|\varepsilon_n + i\Sigma_\varepsilon|} - \frac{1}{|\varepsilon_n|} \right) \tag{46}$$

4 Weak-Coupling Limit

In weak-coupling limit, we assume the interaction kernel becomes

$$\frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \to \frac{1}{E}\Theta(E - |\varepsilon_l|) \tag{47}$$

then the self-energy will not depend on energy ε_n. In the normal state, i.e. when $T \geq T_c$, the equation for the diagonal self-energy is similar to Eq. (36), which is

$$i\Sigma^{(2,re)}_{\varepsilon,in} = \frac{\hbar}{\tau_{in}E} N_{ge} T \sum_{|\varepsilon_l|<E} \operatorname{sgn}\left(\varepsilon_l + i\Sigma^{(2,re)}_{\varepsilon,in}\right) \tag{48}$$

A trivial solution is $\Sigma^{(2,re)}_{\varepsilon,in} = 0$. In this case, when $T \to T_c$, the off-diagonal self-energy is determined by

$$\Sigma^{(2,re)}_{\Delta,in} = \frac{\hbar}{\tau_{in}E} N_{ge} T_c \sum_{|\varepsilon_l|<|E|} \left.\frac{\Delta + \Sigma^{(2,re)}_{\Delta,in}}{|\varepsilon_l|}\right|_{T_c} = \frac{\hbar}{\tau_{in}E} N_{ge} \left(\Delta + \Sigma^{(2,re)}_{\Delta,in}\right) T_c \sum_{|\varepsilon_l|<|E|} \left.\frac{1}{|\varepsilon_l|}\right|_{T_c} \tag{49}$$

4.1 TLS induced T_c^{tls}

If we once again turn off the mean-field pairing $\Delta = 0$, the TLS induced T_c^{tls} is determined by

$$1 = \frac{\hbar}{\tau_{in}E} N_{ge} T_c^{tls} \sum_{|\varepsilon_l|<E} \left.\frac{1}{|\varepsilon_l|}\right|_{T_c^{tls}} \tag{50}$$

This is similar to the BCS gap equation, except that the cutoff is E instead of the Debye frequency ε_D, and the interaction strength is proportional to $1/E$ and N_{ge}. Use the Digamma function in Eq. (57), we have

$$T_c^{tls} = \frac{2e^\gamma}{\pi} E \exp\left(-\frac{\pi E}{N_{ge}\hbar/\tau_{in}}\right) \approx 1.13E \exp\left.\left(-\frac{\pi E}{N_{ge}\hbar/\tau_{in}}\right)\right|_{T_c^{tls}} \tag{51}$$

Compare with the BCS gap equation near T_c, we have

$$1 = gN(0)T_c \sum_{|\varepsilon_n|<\varepsilon_D} \frac{1}{|(2n+1)\pi T_c|} \tag{52}$$

and the solution is

$$T_c = \frac{2e^\gamma}{\pi} \varepsilon_D \exp\left(-\frac{\pi}{gN(0)}\right) \approx 1.13\varepsilon_D \exp\left(-\frac{\pi}{gN(0)}\right) \tag{53}$$

4.2 TLS induced T_c shift

In weak-coupling limit, with mean-field pairing, when $T \to T_c$, if the diagonal self-energy $\Sigma^{(2,re)}_{\varepsilon,in} = 0$, according to the gap equation Eq. (56), the off-diagonal self-energy $\Sigma^{(2,re)}_{\Delta,in}/\Delta = 0$ should be satisfied, otherwise the infinite sum in the gap equation will diverge, and T_c is not shifted by the interaction with TLSs. Meanwhile, we can actually solve the off-diagonal self-energy from Eq. (49), and the solution is not garanteed to be zero. Why will the weak-coupling limit lead to this inconsistency?

5 Density of States

The quasiclassical Green's function is

$$\widehat{\mathcal{G}} \equiv \int \mathrm{d}\xi \widehat{\tau}_3 \widehat{G} = -\pi \frac{i\tilde{\varepsilon}_n \widehat{\tau}_3 + i\tilde{\Delta}\widehat{\tau}_2}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} = \begin{pmatrix} g & f \\ -f^* & -g \end{pmatrix} \tag{54}$$

The density of states is

$$N(\varepsilon) = -\frac{1}{\pi} \operatorname{Im}\{g(\varepsilon^R)\} = \operatorname{Im}\left\{ \frac{\tilde{\varepsilon}}{\sqrt{\tilde{\Delta}^2 - (\tilde{\varepsilon} + i0^+)^2}} \right\} \tag{55}$$

6 Gap Equation

The gap equation is

$$\Delta = -v_0 T \sum_{\varepsilon_n}^{\varepsilon_c} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2} \operatorname{Tr}\Big\{ \widehat{G} \widehat{\tau}_1 \Big\} \equiv g\pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{\tilde{\Delta}}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \tag{56}$$

The digamma function is

$$K(T) \equiv \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{|\varepsilon_n|} \approx \ln\Big(1.13 \frac{\varepsilon_c}{T}\Big) \tag{57}$$

Then we have

$$\frac{1}{g} = \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{\tilde{\Delta}/\Delta}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \overset{\text{clean limit}}{=} \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{\sqrt{\Delta^2 + \varepsilon_n^2}} \overset{T \to T_{c_0}}{=} \pi T_{c_0} \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{|(2n+1)\pi T_{c_0}|} \approx \ln\left(1.13 \frac{\varepsilon_c}{T_{c_0}}\right) \tag{58}$$

Subtract the previous two equations, we have

$$\ln \frac{T}{T_{c_0}} = \pi T \sum_n \left(\frac{1 + \Sigma_\Delta/\Delta}{\sqrt{(\Delta + \Sigma_\Delta)^2 + (\varepsilon_n + i\Sigma_\varepsilon)^2}} - \frac{1}{|\varepsilon_n|} \right) \tag{59}$$

7 The TLS model

The Hamiltonian is

$$H = H_{\mathrm{BCS}} + H_{\mathrm{imp}} + H_{\text{e-imp}}\,, \tag{60}$$

where

$$H_{\mathrm{imp}}(\{\mathbf{X}_j\}) = \sum_{j=1}^{N} H_{\mathrm{imp},j}(\mathbf{X}_j) = \sum_{j=1}^{N} \left\{ \frac{|\mathbf{P}_j|^2}{2M} + U_j(\mathbf{X}_j) \right\}, \tag{61}$$

U_j is the potential for each impurity, which have different centers $\mathbf{R}_j$ and orientations $\hat{\mathbf{a}}_j$. $\mathbf{X}_j$ is the position of the j-th impurity. The interaction between electrons and impurities is

$$H_{\text{e-imp}}(\{\mathbf{X}_j\}) = \sum_\alpha \int d^3 r\, \psi_\alpha^\dagger(\mathbf{r}) \sum_j V(\mathbf{r} - \mathbf{X}_j) \psi_\alpha(\mathbf{r})\,, \tag{62}$$

We approximate each impurity as a TLS with local strain,

$$H_{\mathrm{imp}} \to H_{\mathrm{TLS}} = \sum_j \frac{E_j}{2}\, \sigma_z(j)\,, \tag{63}$$

Here $E_j = \sqrt{J_j^2 + \varepsilon_j^2}$. $J_j = \langle e|H_{\mathrm{imp}}|e\rangle_j - \langle g|H_{\mathrm{imp}}|g\rangle_j$ is the tunneling matrix element. ε_j is the strain. The interaction between electrons and TLSs is

$$H_{\text{e-TLS}} = \sum_\alpha \int \mathrm{d}^3 r\, \psi_\alpha^\dagger(\mathbf{r}) \sum_j \Big[v_j(\mathbf{r}) + m_j(\mathbf{r})\, \sigma_z(j) + n_j(\mathbf{r})\, \sigma_x(j) \Big] \psi_\alpha(\mathbf{r})\,, \tag{64}$$

Here the interaction potentials $f_j(\mathbf{r}) = f(\mathbf{r} - \mathbf{R}_j, \mathbf{a}_j)$ where $f(\mathbf{r})$ refers to $v(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$. In momentum representation, we have $\psi_\alpha^\dagger(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}'} e^{-i\mathbf{k}'\cdot\mathbf{r}} c_{\mathbf{k}'\alpha}^\dagger$,

$$H_{\text{e-TLS}} = \frac{1}{V} \sum_{\mathbf{k}',\mathbf{k};\alpha} c_{\mathbf{k}'\alpha}^\dagger \sum_j e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}_j} \Big[v_{\mathbf{k}'-\mathbf{k}}(j) + m_{\mathbf{k}'-\mathbf{k}}(j)\, \sigma_z(j) + n_{\mathbf{k}'-\mathbf{k}}(j)\, \sigma_x(j) \Big] c_{\mathbf{k}\alpha}\,, \tag{65}$$

where $f_{\mathbf{k}'-\mathbf{k}} \equiv \int d^3 r\, e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\, f(\mathbf{r})$ for the three electron-TLS interactions, which have dimension of [Energy $\times$ Volume]. We use Abrikosov pseudo-fermion to factorize spin operators, $\vec{\sigma}(j) = \sum_{a,b} f_{j,a}^\dagger\, \vec{\sigma}_{ab}\, f_{j,b}$

$$H_{\text{e-TLS}} = \frac{1}{V} \sum_{\mathbf{k}',\mathbf{k};\alpha} c_{\mathbf{k}'\alpha}^\dagger \sum_j e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}_j} \left[v_{\mathbf{k}'-\mathbf{k}}(j) + \sum_{a,b} f_{j,a}^\dagger\, A_{\mathbf{k}'-\mathbf{k}}^{ab}(j)\, f_{j,b} \right] c_{\mathbf{k}\alpha} \tag{66}$$

where

$$A_{\mathbf{k}'-\mathbf{k}}^{ab}(j) = m_{\mathbf{k}'-\mathbf{k}}(j)\ (\sigma_z)_{ab} + n_{\mathbf{k}'-\mathbf{k}}(j)\ (\sigma_x)_{ab} = \begin{pmatrix} m_{\mathbf{k}'-\mathbf{k}}(j) & n_{\mathbf{k}'-\mathbf{k}}(j) \\ n_{\mathbf{k}'-\mathbf{k}}(j) & -m_{\mathbf{k}'-\mathbf{k}}(j) \end{pmatrix}_{ab} \tag{67}$$

And we use Popov-Fedotov method to do the perturbation expansion. The first-order terms include a v and a m term. They are elastic and are just corrections to the chemical potential. Note that the **unperturbed** pseudo-fermion propagator $D_{ij,ab} \equiv \delta_{ij}\delta_{ab}D_{ja}$,

$$\widehat{G}^{(1)}_{\mathbf{k}',\mathbf{k}} = \widehat{G}^{(0)}_{k'}\frac{1}{V}\sum_j e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}_j}\left[v_{\mathbf{k}'-\mathbf{k}}(j) - T\sum_{n;a,b} D_{jj,ab}(\varepsilon_n)A^{ab}_{\mathbf{k}'-\mathbf{k}}(j)\right]\widehat{\tau}_3\widehat{G}^{(0)}_k \tag{68}$$

Apply $\prod_{i=1}^N \int \frac{\mathrm{d}^3R_i}{V}$ to do position average, we have

$$\prod_{i=1}^N \int \frac{\mathrm{d}^3R_i}{V}\widehat{G}^{(1)}_{k'k} = \widehat{G}^{(1)}_k\delta_{k'k} = \delta_{k'k}\widehat{G}^{(0)}_k\frac{1}{V}\sum_j\left[v_0(j) - T\sum_{n,a} D_{j,a}(\varepsilon_n)m_0^{aa}(j)\right]\widehat{\tau}_3\widehat{G}^{(0)}_k \tag{69}$$

In Maekawa's model, $m_0^{aa}(j) = 0$ and $v_0(j) = v_0$. We denote $\frac{1}{V}\sum_j \to n_s$,

$$\widehat{G}^{(1)}_k = \widehat{G}^{(0)}_k n_s v_0\widehat{\tau}_3\widehat{G}^{(0)}_k \tag{70}$$

which is exactly the static impurity result. The second-order terms include the v-v, v-m, m-m, Saturn and the figure-eight diagrams.

$$\begin{aligned}\widehat{G}^{(\mathrm{Saturn})}_{\mathbf{k}'',\mathbf{k}}(\varepsilon_n) = &-\frac{1}{V^2}\sum_{ij}\sum_{abcdk'} e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\mathbf{R}_i}e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}_j}A^{cd}_{\mathbf{k}''-\mathbf{k}'}(i)A^{ab}_{\mathbf{k}'-\mathbf{k}}(j)\\ &\times T^2\sum_{n_1,n_2} D_{ij,da}(\varepsilon_{n_1})D_{ji,bc}(\varepsilon_{n_2})\widehat{G}^{(0)}_{k''}\widehat{\tau}_3\widehat{G}^{(0)}_{k'}(\varepsilon_n+\varepsilon_{n_2}-\varepsilon_{n_1})\widehat{\tau}_3\widehat{G}^{(0)}_k.\end{aligned} \tag{71}$$

In the Saturn diagram, the **unperturbed** pseudo-fermion Green's function already demands repeated scattering of the same impurity.

$$\widehat{G}^{(\mathrm{Saturn})}_{\mathbf{k}'',\mathbf{k}}(\varepsilon_n) = -\frac{1}{V^2}\sum_j e^{-i(\mathbf{k}''-\mathbf{k})\cdot\mathbf{R}_j}\sum_{abk'}A^{ba}_{\mathbf{k}''-\mathbf{k}'}(j)A^{ab}_{\mathbf{k}'-\mathbf{k}}(j)T^2\sum_{n_1,n_2} D_{ja}(\varepsilon_{n_1})D_{jb}(\varepsilon_{n_2})\widehat{G}^{(0)}_{k''}\widehat{\tau}_3\widehat{G}^{(0)}_{k'}(\varepsilon_n+\varepsilon_{n_2}-\varepsilon_{n_1})\widehat{\tau}_3\widehat{G}^{(0)}_k. \tag{72}$$

Apply $\prod_{i=1}^N \int \frac{\mathrm{d}^3R_i}{V}$ to do position average, we have

$$\widehat{G}^{(\mathrm{Saturn})}_{\mathbf{k}}(\varepsilon_n) = -\frac{1}{V^2}\sum_{jabk'}|A^{ab}_{\mathbf{k}'-\mathbf{k}}(j)|^2T^2\sum_{n_1,n_2} D_{ja}(\varepsilon_{n_1})D_{jb}(\varepsilon_{n_2})\widehat{G}^{(0)}_k\widehat{\tau}_3\widehat{G}^{(0)}_{k'}(\varepsilon_n+\varepsilon_{n_2}-\varepsilon_{n_1})\widehat{\tau}_3\widehat{G}^{(0)}_k. \tag{73}$$

Apply orientation average $\prod_{i=1}^N \int \frac{\mathrm{d}\Omega_{a_j}}{4\pi}$, which is equivalently averaging over the external momentum direction $\prod_{i=1}^N \int \frac{\mathrm{d}\Omega_k}{4\pi}$. Note that $\frac{1}{V}\sum_{k'} \to \frac{1}{(2\pi)^3}\int \mathrm{d}^3k' \approx \int \frac{\mathrm{d}\Omega_{k'}}{4\pi}N(0)\int \mathrm{d}\xi_{k'}$,

$$\widehat{G}^{(\mathrm{Saturn})}_k(\varepsilon_n) = -\frac{1}{V}\sum_{jab}N(0)\int \mathrm{d}\xi_{k'}\left\langle|A^{ab}_{\mathbf{k}'-\mathbf{k}}(j)|^2\right\rangle_\Omega T^2\sum_{n_1,n_2} D_{ja}(\varepsilon_{n_1})D_{jb}(\varepsilon_{n_2})\widehat{G}^{(0)}_k\widehat{\tau}_3\widehat{G}^{(0)}_{k'}(\varepsilon_n+\varepsilon_{n_2}-\varepsilon_{n_1})\widehat{\tau}_3\widehat{G}^{(0)}_k. \tag{74}$$

We define the inelastic scattering time τ_{in} as $\frac{\hbar}{2\pi\tau} = n_sN(0)\left\langle|A|^2\right\rangle_\Omega$, assume the interaction potential only depends on the direction of the momentum transfer, define quasiclassical Green's function $\widehat{\mathcal{G}}$, and assume identical TLS energy splitting,

$$\widehat{G}^{(\mathrm{Saturn})}_k(\varepsilon_n) = -\sum_{ab}\left(\frac{\hbar}{2\pi\tau_m}\delta^{ab} + \frac{\hbar}{2\pi\tau_n}\sigma_x^{ab}\right)T^2\sum_{n_1,n_2} D_a(\varepsilon_{n_1})D_b(\varepsilon_{n_2})\widehat{G}^{(0)}_k\widehat{\mathcal{G}}^{(0)}(\varepsilon_n+\varepsilon_{n_2}-\varepsilon_{n_1})\widehat{\tau}_3\widehat{G}^{(0)}_k. \tag{75}$$

Later we can still do a configuration average over the TLS energy splitting. We will also see that the $\frac{\hbar}{2\pi\tau_m}\delta^{ab}$ term will not contribute to the self-energy. The figure-eight diagram is

$$\widehat{G}^{(\mathrm{figure\text{-}8})}_{\mathbf{k}'',\mathbf{k}}(\varepsilon_n) = \frac{1}{V^2}\sum_{ij}\sum_{abcdk'} e^{-i(\mathbf{k}''-\mathbf{k})\cdot\mathbf{R}_j}A^{cd}_0(i)A^{ab}_{\mathbf{k}''-\mathbf{k}}(j)T^2\sum_{n_1,n_2} D_{ij,da}(\varepsilon_{n_1})D_{ji,bc}(\varepsilon_{n_1})\widehat{G}^{(0)}_{k''}\widehat{\tau}_3\widehat{G}^{(0)}_{k'}(\varepsilon_{n_2})\widehat{\tau}_3\widehat{G}^{(0)}_k. \tag{76}$$

We can already see that sum over n_2 will give zero. Similarly we have

$$\widehat{G}^{(\mathrm{figure\text{-}8})}_{\mathbf{k}'',\mathbf{k}}(\varepsilon_n) = \frac{1}{V^2}\sum_{jabk'} e^{-i(\mathbf{k}''-\mathbf{k})\cdot\mathbf{R}_j}A^{ba}_0(j)A^{ab}_{\mathbf{k}''-\mathbf{k}}(j)T^2\sum_{n_1,n_2} D_{ja}(\varepsilon_{n_1})D_{jb}(\varepsilon_{n_1})\widehat{G}^{(0)}_{k''}\widehat{\tau}_3\widehat{G}^{(0)}_{k'}(\varepsilon_{n_2})\widehat{\tau}_3\widehat{G}^{(0)}_k. \tag{77}$$

apply position average

$$\widehat{G}^{(\mathrm{figure\text{-}8})}_{\mathbf{k}}(\varepsilon_n) = \frac{1}{V^2}\sum_{jabk'}|A^{ab}_0(j)|^2T^2\sum_{n_1,n_2} D_{ja}(\varepsilon_{n_1})D_{jb}(\varepsilon_{n_1})\widehat{G}^{(0)}_k\widehat{\tau}_3\widehat{G}^{(0)}_{k'}(\varepsilon_{n_2})\widehat{\tau}_3\widehat{G}^{(0)}_k. \tag{78}$$

apply orientation average. The quasiclassical Green's function is odd in the Matsubara energy. Sum over n_2 will give zero.

$$\widehat{G}^{(\mathrm{figure\text{-}8})}_k(\varepsilon_n) = \sum_{ab}\left(\frac{\hbar}{2\pi\tau_m}\delta^{ab} + \frac{\hbar}{2\pi\tau_n}\sigma_x^{ab}\right)T^2\sum_{n_1,n_2} D_a(\varepsilon_{n_1})D_b(\varepsilon_{n_1})\widehat{G}^{(0)}_k\widehat{\mathcal{G}}^{(0)}(\varepsilon_{n_2})\widehat{\tau}_3\widehat{G}^{(0)}_k = 0 \tag{79}$$