The Dyson equation is

$$\widehat{G}^{-1} = \widehat{G}^{(0)-1} - \widehat{\Sigma} \tag{1}$$

we have

$$\widehat{G}^{(0)-1} = i\varepsilon_n - \xi\widehat{\tau}_3 - \Delta\widehat{\tau}_1 \tag{2}$$

$$\widehat{\Sigma} \equiv \Sigma_\varepsilon + \Sigma_\Delta\widehat{\tau}_1 \tag{3}$$

$$i\tilde{\varepsilon}_n \equiv i\varepsilon_n - \Sigma_\varepsilon \to \tilde{\varepsilon}_n = \varepsilon_n + i\Sigma_\varepsilon \tag{4}$$

$$\tilde{\Delta} \equiv \Delta + \Sigma_\Delta \tag{5}$$

and

$$\widehat{G} = (i\tilde{\varepsilon}_n - \xi\widehat{\tau}_3 - \tilde{\Delta}\widehat{\tau}_1)^{-1} = -\frac{i\tilde{\varepsilon}_n + \xi\widehat{\tau}_3 + \tilde{\Delta}\widehat{\tau}_1}{\tilde{\varepsilon}_n^2 + \xi^2 + \tilde{\Delta}^2} \tag{6}$$

so from $\widehat{G}^{(0)}$ to $\widehat{G}$, the change is $\Delta \to \tilde{\Delta}$ and $\varepsilon_n \to \tilde{\varepsilon}_n$.

We multiply $\widehat{G}$ from left and right to the Dyson equation, we have

$$1 = \widehat{G}\widehat{G}^{(0)-1} - \widehat{G}\widehat{\Sigma} \to 1 = \widehat{\tau}_3\widehat{G}\widehat{G}^{(0)-1}\widehat{\tau}_3 - \widehat{\tau}_3\widehat{G}\widehat{\Sigma}\widehat{\tau}_3 \tag{7}$$

$$1 = \widehat{G}^{(0)-1}\widehat{G} - \widehat{\Sigma}\widehat{G} \to 1 = \widehat{G}^{(0)-1}\widehat{\tau}_3\widehat{\tau}_3\widehat{G} - \widehat{\Sigma}\widehat{\tau}_3\widehat{\tau}_3\widehat{G} \tag{8}$$

Subtract the two equations, we have

$$\left[\widehat{G}^{(0)-1}\widehat{\tau}_3 - \widehat{\Sigma}\widehat{\tau}_3, \widehat{\tau}_3\widehat{G}\right] = 0 \tag{9}$$

$$\to \left[i\varepsilon_n\widehat{\tau}_3 + i\Delta\widehat{\tau}_2 - \widehat{\Sigma}\widehat{\tau}_3, \widehat{\tau}_3\widehat{G}\right] = 0 \tag{10}$$

In quasiclassical limit, we assume $\widehat{\Sigma}$ does not depend on ξ, and we can apply $\int \mathrm{d}\xi$ to get the homogeneous Eilenberger equation

$$\left[i\varepsilon_n\widehat{\tau}_3 + i\Delta\widehat{\tau}_2 - \widehat{\Sigma}\widehat{\tau}_3, \widehat{\mathcal{G}}\right] = 0 \tag{11}$$

where the quasiclassical Green's function is

$$\widehat{\mathcal{G}} = \int \mathrm{d}\xi\widehat{\tau}_3\widehat{G} = -\pi\frac{i\tilde{\varepsilon}_n\widehat{\tau}_3 + i\tilde{\Delta}\widehat{\tau}_2}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \tag{12}$$

The second-order self-energy is

$$\widehat{\Sigma}^{(2)} = \widehat{\Sigma}^{(2)}_{vv} + \widehat{\Sigma}^{(2)}_{vm} + \widehat{\Sigma}^{(2)}_{mm} + \widehat{\Sigma}^{(2)}_{nn} \equiv \widehat{\Sigma}^{(2)}_{\text{elastic}} + \widehat{\Sigma}^{(2)}_{\text{inelastic}} \tag{13}$$

We define the scattering rates as

$$\frac{\hbar}{2\pi\tau_{el}(E,T)} \equiv \hbar\left(\frac{1}{\tau_{vv}} + \frac{2N_{ge}}{\tau_{vm}} + \frac{1}{\tau_{mm}}\right) \tag{14}$$

and

$$\frac{\hbar}{2\pi\tau_{in}} \equiv \frac{\hbar}{\tau_{nn}} \tag{15}$$

The elastic part is proportional to the quasiclassical Green's function,

$$\widehat{\Sigma}^{(2)}_{\text{elastic}}(\varepsilon_n) = \frac{\hbar}{2\tau_{el}}\frac{-i\varepsilon_n + \Delta\widehat{\tau}_1}{\sqrt{\Delta^2 + \varepsilon_n^2}} \to \widehat{\Sigma}^{(2)}_{\text{elastic}}\widehat{\tau}_3 = \frac{\hbar}{2\pi\tau_{el}}\widehat{\mathcal{G}}^{(0)} \tag{16}$$

After renormalization, $\widehat{\mathcal{G}}^{(0)} \to \widehat{\mathcal{G}}$, the elastic self-energy drops out of the Eilenberger equation.

For the inelastic part, if we evaluate the two Matsubara sums first, we have

$$\widehat{\Sigma}^{(2)}_{\text{inelastic}}(\varepsilon_n) = \frac{\hbar}{2\pi\tau_{in}}\int \mathrm{d}\xi\left(1 - N_{ge}[n_F(-\xi_\Delta) - n_F(\xi_\Delta)]\right)\frac{-i\varepsilon_n + \Delta\widehat{\tau}_1\frac{\xi_\Delta - E}{\xi_\Delta}}{(\xi_\Delta - E)^2 + \varepsilon_n^2} \tag{17}$$

where we denote the transformed kinetic energy as $\xi_\Delta \equiv \mathrm{sgn}(\xi)\sqrt{\xi^2 + \Delta^2}$.

If we evaluate the ξ integral first, we have

$$\widehat{\Sigma}^{(2)}_{\text{inelastic}}(\varepsilon_n) = -\frac{\hbar}{2\pi\tau_{in}} \sum_{a\neq b} T^2 \sum_{n_1,n_2} D_a(\varepsilon_{n_1}) D_b(\varepsilon_{n_2}) \widehat{\mathcal{G}}^{(0)}(\varepsilon_n - \varepsilon_{n_1} + \varepsilon_{n_2})\widehat{\tau}_3 \tag{18}$$

We define the Bosonic Matsubara frequency as $\omega_m \equiv \varepsilon_{n_1} - \varepsilon_{n_2}$.

$$\widehat{\Sigma}^{(2)}_{\text{inelastic}}(\varepsilon_n) = -\frac{\hbar}{2\pi\tau_{in}} \sum_{a\neq b} T^2 \sum_{m,n_2} D_a(\omega_m + \varepsilon_{n_2}) D_b(\varepsilon_{n_2}) \widehat{\mathcal{G}}^{(0)}(\varepsilon_n - \omega_m)\widehat{\tau}_3 \tag{19}$$

Note that

$$\begin{aligned}
&T\sum_{n_2} D_a(\omega_m + \varepsilon_{n_2}) D_b(\varepsilon_{n_2}) && (20)\\
=&T\sum_{n_2} \frac{1}{i\varepsilon_{n_2} + i\omega_m + \mu_f - \varepsilon_a} \times \frac{1}{i\varepsilon_{n_2} + \mu_f - \varepsilon_b} && (21)\\
=&\frac{n_F(\varepsilon_a - i\omega_m - \mu_f) - n_F(\varepsilon_b - \mu_f)}{(\varepsilon_a - \varepsilon_b) - i\omega_m} && (22)\\
=&\frac{n_a - n_b}{\varepsilon_{ab} - i\omega_m} && (23)
\end{aligned}$$

where we denote $n_a \equiv n_F(\varepsilon_a - \mu_f)$. Now sum over the two levels

$$\begin{aligned}
&\sum_{a\neq b} \frac{n_a - n_b}{\varepsilon_{ab} - i\omega_m} && (24)\\
=&\frac{N_{eg}}{E - i\omega_m} + \frac{N_{eg}}{E + i\omega_m} && (25)\\
=&-2N_{ge}\frac{E}{E^2 + \omega_m^2} && (26)
\end{aligned}$$

and we have

$$\begin{aligned}
\widehat{\Sigma}^{(2)}_{\text{inelastic}}(\varepsilon_n) =& \frac{\hbar}{2\pi\tau_{in}} \times 2N_{ge}T\sum_m \frac{E}{E^2 + \omega_m^2} \widehat{\mathcal{G}}^{(0)}(\varepsilon_n - \omega_m)\widehat{\tau}_3 && (27)\\
\Sigma^{(2)}_{\varepsilon,in}(\varepsilon_n) + \Sigma^{(2)}_{\Delta,in}(\varepsilon_n)\widehat{\tau}_1 =& \frac{\hbar}{\tau_{in}} N_{ge}T\sum_m \frac{E}{E^2 + \omega_m^2} \frac{-i(\varepsilon_n - \omega_m) + \Delta\widehat{\tau}_1}{\sqrt{\Delta^2 + (\varepsilon_n - \omega_m)^2}} && (28)
\end{aligned}$$

After renormalization, $\widehat{\mathcal{G}}^{(0)} \to \widehat{\mathcal{G}}$, and $\widehat{\Sigma}^{(2)}_{\text{inelastic}} \to \widehat{\Sigma}^{(2,re)}_{\text{inelastic}}$, which includes a set of diagrams with particular 'Saturn' pattern.

$$\begin{aligned}
\widehat{\Sigma}^{(2,re)}_{\text{inelastic}}(\varepsilon_n) =& \frac{\hbar}{\pi\tau_{in}} N_{ge}T\sum_m \frac{E}{E^2 + \omega_m^2} \widehat{\mathcal{G}}(\varepsilon_n - \omega_m)\widehat{\tau}_3 && (29)\\
\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n) + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n)\widehat{\tau}_1 =& \frac{\hbar}{\tau_{in}} N_{ge}T\sum_m \frac{E}{E^2 + \omega_m^2} \frac{-i(\varepsilon_n - \omega_m) + \Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n - \omega_m) + \left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n - \omega_m)\right)\widehat{\tau}_1}{\sqrt{\left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n - \omega_m)\right)^2 + \left(\varepsilon_n - \omega_m + i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n - \omega_m)\right)^2}} && (30)
\end{aligned}$$

In order to do analytic continuation, we introduce $\varepsilon_l \equiv \varepsilon_n - \omega_m$,

$$\begin{aligned}
\widehat{\Sigma}^{(2,re)}_{\text{inelastic}}(\varepsilon_n) =& \frac{\hbar}{\pi\tau_{in}} N_{ge}T\sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \widehat{\mathcal{G}}(\varepsilon_l)\widehat{\tau}_3 && (31)\\
\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_n) + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_n)\widehat{\tau}_1 =& \frac{\hbar}{\tau_{in}} N_{ge}T\sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l) + \left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)\right)\widehat{\tau}_1}{\sqrt{\left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)\right)^2 + \left(\varepsilon_l + i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l)\right)^2}} && (32)
\end{aligned}$$

and we substitute $i\varepsilon_n \to \varepsilon + i0^+$, we have

$$\begin{aligned}
\widehat{\Sigma}^{(2,re)}_{\text{inelastic}}(\varepsilon) =& \frac{\hbar}{\pi\tau_{in}} N_{ge}T\sum_l \frac{E}{E^2 - (\varepsilon - i\varepsilon_l)^2} \widehat{\mathcal{G}}(\varepsilon_l)\widehat{\tau}_3 && (33)\\
\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon) + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon)\widehat{\tau}_1 =& \frac{\hbar}{\tau_{in}} N_{ge}T\sum_l \frac{E}{E^2 - (\varepsilon - i\varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l) + \left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)\right)\widehat{\tau}_1}{\sqrt{\left(\Delta + \Sigma^{(2,re)}_{\Delta,in}(\varepsilon_l)\right)^2 + \left(\varepsilon_l + i\Sigma^{(2,re)}_{\varepsilon,in}(\varepsilon_l)\right)^2}} && (34)
\end{aligned}$$

If we turn off the mean-field pairing, i.e. $\Delta = 0$ in Eq. (47), then the whole system becomes a normal metal, and the question becomes whether the interaction between electrons and TLSs can induce a superconducting state below a critical temperature, i.e. $\Sigma_{\Delta,in}^{(2,re)}(\varepsilon) > 0$ when $T < T_c^{tls}$,

$$\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) + \Sigma_{\Delta,in}^{(2,re)}(\varepsilon_n)\widehat{\tau}_1 = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_l) + \Sigma_{\Delta,in}^{(2,re)}(\varepsilon_l)\widehat{\tau}_1}{\sqrt{\Sigma_{\Delta,in}^{(2,re)}(\varepsilon_l)^2 + \left(\varepsilon_l + i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_l)\right)^2}} \tag{35}$$

When $T \geq T_c^{tls}$, we have $\Sigma_{\Delta,in}^{(2,re)} = 0$. For the diagonal part $\Sigma_{\varepsilon,in}^{(2,re)}$, we have

$$i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \mathrm{sgn}\left(\varepsilon_l + i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_l)\right) \tag{36}$$

We propose an ansatz that $\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n)$ is an odd function of ε_n, and has the same sign as ε_n, i.e. $\mathrm{sgn}(\varepsilon_n) = \mathrm{sgn}\left(i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n)\right)$, and $\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) = -\Sigma_{\varepsilon,in}^{(2,re)}(-\varepsilon_n)$. Then we have

$$i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \mathrm{sgn}(\varepsilon_l) \tag{37}$$

We seperate the positive and negative parts of ε_l in the sum, and we have

$$i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\varepsilon_l > 0} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} - \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\varepsilon_l < 0} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \tag{38}$$

We define $\omega_m \equiv \varepsilon_n - \varepsilon_l = 2m\pi T$, and

$$i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m < \varepsilon_n} \frac{E}{E^2 + \omega_m^2} - \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m > \varepsilon_n} \frac{E}{E^2 + \omega_m^2} \tag{39}$$

$$= \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m > -\varepsilon_n} \frac{E}{E^2 + \omega_m^2} - \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m > \varepsilon_n} \frac{E}{E^2 + \omega_m^2} \tag{40}$$

$$= \frac{\hbar}{\tau_{in}} N_{ge} \mathrm{sgn}(\varepsilon_n) T \sum_{|\omega_m| < |\varepsilon_n|} \frac{E}{E^2 + \omega_m^2} \tag{41}$$

This solution is consistent with the ansatz.

For the off-diagonal part, when $T \to T_c^{tls}$, we have $i\Sigma_{\varepsilon,in}^{(2,re)} \gg \Sigma_{\Delta,in}^{(2,re)} \to 0$, and

$$\Sigma_{\Delta,in}^{(2,re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T_c^{tls} \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{\Sigma_{\Delta,in}^{(2,re)}(\varepsilon_l)}{\left|\varepsilon_l + i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_l)\right|}\Bigg|_{T_c^{tls}} \tag{42}$$

This is just an eigenvalue problem. For $\Sigma_{\Delta,in}^{(2,re)}(\varepsilon_n)$ to have a non-trivial solution, we should have

$$\sum_l \left(M_{nl} - \delta_{nl}\right)\Sigma_{\Delta,in}^{(2,re)}(\varepsilon_l) = 0 \leftrightarrow \det(M_{nl} - \delta_{nl}) = 0 \tag{43}$$

where

$$M_{nl} = \frac{\hbar}{\tau_{in}} N_{ge} T_c^{tls} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{1}{\left|\varepsilon_l + i\Sigma_{\varepsilon,in}^{(2,re)}(\varepsilon_l)\right|}\Bigg|_{T_c^{tls}} \tag{44}$$

The quasiclassical Green's function is

$$\widehat{\mathcal{G}} \equiv \int \mathrm{d}\xi \widehat{\tau}_3 \widehat{G} = -\pi \frac{i\tilde{\varepsilon}_n\widehat{\tau}_3 + i\tilde{\Delta}\widehat{\tau}_2}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} = \begin{pmatrix} g & f \\ -f^* & -g \end{pmatrix} \tag{45}$$

The density of states is

$$N(\varepsilon) = -\frac{1}{\pi}\mathrm{Im}\{g(\varepsilon^R)\} = \mathrm{Im}\left\{\frac{\tilde{\varepsilon}}{\sqrt{\tilde{\Delta}^2 - (\tilde{\varepsilon} + i0^+)^2}}\right\} \tag{46}$$

The gap equation is

$$\Delta = -v_0 T \sum_{\varepsilon_n}^{\varepsilon_c} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2} \operatorname{Tr}\left\{ \widehat{G} \widehat{\tau}_1 \right\} \equiv g\pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{\tilde{\Delta}}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \tag{47}$$

The digamma function is

$$K(T) \equiv \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{|\varepsilon_n|} \approx \ln\left(1.13 \frac{\varepsilon_c}{T}\right) \tag{48}$$

Then we have

$$\frac{1}{g} = \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{\tilde{\Delta}/\Delta}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \overset{\text{clean limit}}{=} \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{\sqrt{\Delta^2 + \varepsilon_n^2}} \overset{T \to T_{c_0}}{=} \pi T_{c_0} \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{|(2n+1)\pi T_{c_0}|} \approx \ln\left(1.13 \frac{\varepsilon_c}{T_{c_0}}\right) \tag{49}$$

Subtract the previous two equations, we have

$$\ln \frac{T}{T_{c_0}} = \pi T \sum_n \left(\frac{1 + \Sigma_\Delta/\Delta}{\sqrt{(\Delta + \Sigma_\Delta)^2 + (\varepsilon_n + i\Sigma_\varepsilon)^2}} - \frac{1}{|\varepsilon_n|} \right) \tag{50}$$