

# 1 Self-Energy

The Dyson equation is

$$\hat{G}^{-1} = \hat{G}^{(0)-1} - \hat{\Sigma} \quad (1)$$

we have

$$\hat{G}^{(0)-1} = i\varepsilon_n - \xi\hat{\tau}_3 - \Delta\hat{\tau}_1 \quad (2)$$

$$\hat{\Sigma} \equiv \Sigma_\varepsilon + \Sigma_\Delta\hat{\tau}_1 \quad (3)$$

$$i\tilde{\varepsilon}_n \equiv i\varepsilon_n - \Sigma_\varepsilon \rightarrow \tilde{\varepsilon}_n = \varepsilon_n + i\Sigma_\varepsilon \quad (4)$$

$$\tilde{\Delta} \equiv \Delta + \Sigma_\Delta \quad (5)$$

and

$$\hat{G} = (i\tilde{\varepsilon}_n - \xi\hat{\tau}_3 - \tilde{\Delta}\hat{\tau}_1)^{-1} = -\frac{i\tilde{\varepsilon}_n + \xi\hat{\tau}_3 + \tilde{\Delta}\hat{\tau}_1}{\tilde{\varepsilon}_n^2 + \xi^2 + \tilde{\Delta}^2} \quad (6)$$

so from  $\hat{G}^{(0)}$  to  $\hat{G}$ , the change is  $\Delta \rightarrow \tilde{\Delta}$  and  $\varepsilon_n \rightarrow \tilde{\varepsilon}_n$ .

We multiply  $\hat{G}$  from left and right to the Dyson equation, we have

$$1 = \hat{G}\hat{G}^{(0)-1} - \hat{G}\hat{\Sigma} \rightarrow 1 = \hat{\tau}_3\hat{G}\hat{G}^{(0)-1}\hat{\tau}_3 - \hat{\tau}_3\hat{G}\hat{\Sigma}\hat{\tau}_3 \quad (7)$$

$$1 = \hat{G}^{(0)-1}\hat{G} - \hat{\Sigma}\hat{G} \rightarrow 1 = \hat{G}^{(0)-1}\hat{\tau}_3\hat{\tau}_3\hat{G} - \hat{\Sigma}\hat{\tau}_3\hat{\tau}_3\hat{G} \quad (8)$$

Subtract the two equations, we have

$$\left[ \hat{G}^{(0)-1}\hat{\tau}_3 - \hat{\Sigma}\hat{\tau}_3, \hat{\tau}_3\hat{G} \right] = 0 \quad (9)$$

$$\rightarrow \left[ i\varepsilon_n\hat{\tau}_3 + i\Delta\hat{\tau}_2 - \hat{\Sigma}\hat{\tau}_3, \hat{\tau}_3\hat{G} \right] = 0 \quad (10)$$

In quasiclassical limit, we assume  $\hat{\Sigma}$  does not depend on  $\xi$ , and we can apply  $\int d\xi$  to get the homogeneous Eilenberger equation

$$\left[ i\varepsilon_n\hat{\tau}_3 + i\Delta\hat{\tau}_2 - \hat{\Sigma}\hat{\tau}_3, \hat{\mathcal{G}} \right] = 0 \quad (11)$$

where the quasiclassical Green's function is

$$\hat{\mathcal{G}} = \int d\xi \hat{\tau}_3 \hat{G} = -\pi \frac{i\tilde{\varepsilon}_n\hat{\tau}_3 + i\tilde{\Delta}\hat{\tau}_2}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \quad (12)$$

The second-order self-energy is

$$\hat{\Sigma}^{(2)} = \hat{\Sigma}_{vv}^{(2)} + \hat{\Sigma}_{vm}^{(2)} + \hat{\Sigma}_{mm}^{(2)} + \hat{\Sigma}_{nn}^{(2)} \equiv \hat{\Sigma}_{\text{elastic}}^{(2)} + \hat{\Sigma}_{\text{inelastic}}^{(2)} \quad (13)$$

We define the scattering rates as

$$\frac{\hbar}{2\pi\tau_{el}(E, T)} \equiv \hbar \left( \frac{1}{\tau_{vv}} + \frac{2N_{ge}}{\tau_{vm}} + \frac{1}{\tau_{mm}} \right) \quad (14)$$

and

$$\frac{\hbar}{2\pi\tau_{in}} \equiv \frac{\hbar}{\tau_{nn}} \quad (15)$$

The elastic part is proportional to the quasiclassical Green's function,

$$\hat{\Sigma}_{\text{elastic}}^{(2)}(\varepsilon_n) = \frac{\hbar}{2\tau_{el}} \frac{-i\varepsilon_n + \Delta\hat{\tau}_1}{\sqrt{\Delta^2 + \varepsilon_n^2}} \rightarrow \hat{\Sigma}_{\text{elastic}}^{(2)}\hat{\tau}_3 = \frac{\hbar}{2\pi\tau_{el}} \hat{\mathcal{G}}^{(0)} \quad (16)$$

After renormalization,  $\hat{\mathcal{G}}^{(0)} \rightarrow \hat{\mathcal{G}}$ , the elastic self-energy drops out of the Eilenberger equation.

For the inelastic part, if we evaluate the two Matsubara sums first, we have

$$\hat{\Sigma}_{\text{inelastic}}^{(2)}(\varepsilon_n) = \frac{\hbar}{2\pi\tau_{in}} \int d\xi \left( 1 - N_{ge}[n_F(-\xi_\Delta) - n_F(\xi_\Delta)] \right) \frac{-i\varepsilon_n + \Delta\hat{\tau}_1 \frac{\xi_\Delta - E}{\xi_\Delta}}{(\xi_\Delta - E)^2 + \varepsilon_n^2} \quad (17)$$

where we denote the transformed kinetic energy as  $\xi_\Delta \equiv \text{sgn}(\xi)\sqrt{\xi^2 + \Delta^2}$ .

If we evaluate the  $\xi$  integral first, we have

$$\hat{\Sigma}_{\text{inelastic}}^{(2)}(\varepsilon_n) = -\frac{\hbar}{2\pi\tau_{in}} \sum_{a \neq b} T^2 \sum_{n_1, n_2} D_a(\varepsilon_{n_1}) D_b(\varepsilon_{n_2}) \hat{\mathcal{G}}^{(0)}(\varepsilon_n - \varepsilon_{n_1} + \varepsilon_{n_2}) \hat{\tau}_3 \quad (18)$$

We define the Bosonic Matsubara frequency as  $\omega_m \equiv \varepsilon_{n_1} - \varepsilon_{n_2}$ .

$$\hat{\Sigma}_{\text{inelastic}}^{(2)}(\varepsilon_n) = -\frac{\hbar}{2\pi\tau_{in}} \sum_{a \neq b} T^2 \sum_{m, n_2} D_a(\omega_m + \varepsilon_{n_2}) D_b(\varepsilon_{n_2}) \hat{\mathcal{G}}^{(0)}(\varepsilon_n - \omega_m) \hat{\tau}_3 \quad (19)$$

Note that

$$T \sum_{n_2} D_a(\omega_m + \varepsilon_{n_2}) D_b(\varepsilon_{n_2}) \quad (20)$$

$$= T \sum_{n_2} \frac{1}{i\varepsilon_{n_2} + i\omega_m + \mu_f - \varepsilon_a} \times \frac{1}{i\varepsilon_{n_2} + \mu_f - \varepsilon_b} \quad (21)$$

$$= \frac{n_F(\varepsilon_a - i\omega_m - \mu_f) - n_F(\varepsilon_b - \mu_f)}{(\varepsilon_a - \varepsilon_b) - i\omega_m} \quad (22)$$

$$= \frac{n_a - n_b}{\varepsilon_{ab} - i\omega_m} \quad (23)$$

where we denote  $n_a \equiv n_F(\varepsilon_a - \mu_f)$ . Now sum over the two levels

$$\sum_{a \neq b} \frac{n_a - n_b}{\varepsilon_{ab} - i\omega_m} \quad (24)$$

$$= \frac{N_{eg}}{E - i\omega_m} + \frac{N_{eg}}{E + i\omega_m} \quad (25)$$

$$= -2N_{ge} \frac{E}{E^2 + \omega_m^2} \quad (26)$$

and we have

$$\hat{\Sigma}_{\text{inelastic}}^{(2)}(\varepsilon_n) = \frac{\hbar}{2\pi\tau_{in}} \times 2N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \hat{\mathcal{G}}^{(0)}(\varepsilon_n - \omega_m) \hat{\tau}_3 \quad (27)$$

$$\Sigma_{\varepsilon, in}^{(2)}(\varepsilon_n) + \Sigma_{\Delta, in}^{(2)}(\varepsilon_n) \hat{\tau}_1 = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \frac{-i(\varepsilon_n - \omega_m) + \Delta \hat{\tau}_1}{\sqrt{\Delta^2 + (\varepsilon_n - \omega_m)^2}} \quad (28)$$

After renormalization,  $\hat{\mathcal{G}}^{(0)} \rightarrow \hat{\mathcal{G}}$ , and  $\hat{\Sigma}_{\text{inelastic}}^{(2)} \rightarrow \hat{\Sigma}_{\text{inelastic}}^{(2, re)}$ , which includes a set of diagrams with particular ‘Saturn’ pattern.

$$\hat{\Sigma}_{\text{inelastic}}^{(2, re)}(\varepsilon_n) = \frac{\hbar}{\pi\tau_{in}} N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \hat{\mathcal{G}}(\varepsilon_n - \omega_m) \hat{\tau}_3 \quad (29)$$

$$\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n) + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_n) \hat{\tau}_1 = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_m \frac{E}{E^2 + \omega_m^2} \frac{-i(\varepsilon_n - \omega_m) + \Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n - \omega_m) + \left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_n - \omega_m)\right) \hat{\tau}_1}{\sqrt{\left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_n - \omega_m)\right)^2 + \left(\varepsilon_n - \omega_m + i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n - \omega_m)\right)^2}} \quad (30)$$

In order to do analytic continuation, we introduce  $\varepsilon_l \equiv \varepsilon_n - \omega_m$ ,

$$\hat{\Sigma}_{\text{inelastic}}^{(2, re)}(\varepsilon_n) = \frac{\hbar}{\pi\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \hat{\mathcal{G}}(\varepsilon_l) \hat{\tau}_3 \quad (31)$$

$$\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n) + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_n) \hat{\tau}_1 = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l) + \left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l)\right) \hat{\tau}_1}{\sqrt{\left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l)\right)^2 + \left(\varepsilon_l + i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l)\right)^2}} \quad (32)$$

and we substitute  $i\varepsilon_n \rightarrow \varepsilon + i0^+$ , we have

$$\hat{\Sigma}_{\text{inelastic}}^{(2, re)}(\varepsilon) = \frac{\hbar}{\pi\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 - (\varepsilon - i\varepsilon_l)^2} \hat{\mathcal{G}}(\varepsilon_l) \hat{\tau}_3 \quad (33)$$

$$\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon) + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon) \hat{\tau}_1 = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 - (\varepsilon - i\varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l) + \left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l)\right) \hat{\tau}_1}{\sqrt{\left(\Delta + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l)\right)^2 + \left(\varepsilon_l + i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l)\right)^2}} \quad (34)$$

If we turn off the mean-field pairing, i.e.  $\Delta = 0$  in Eq. (54), then the whole system becomes a normal metal, and the question becomes whether the interaction between electrons and TLSs can induce a superconducting state below a critical temperature, i.e.  $\Sigma_{\Delta, in}^{(2, re)}(\varepsilon) > 0$  when  $T < T_c^{tls}$ ,

$$\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n) + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_n)\hat{\tau}_1 = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{-i\varepsilon_l + \Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l) + \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l)\hat{\tau}_1}{\sqrt{\Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l)^2 + \left(\varepsilon_l + i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l)\right)^2}} \quad (35)$$

When  $T \geq T_c^{tls}$ , we have  $\Sigma_{\Delta, in}^{(2, re)} = 0$ . For the diagonal part  $\Sigma_{\varepsilon, in}^{(2, re)}$ , we have

$$i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \text{sgn}(\varepsilon_l + i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l)) \quad (36)$$

We propose an ansatz that  $\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n)$  is an odd function of  $\varepsilon_n$ , and has the same sign as  $\varepsilon_n$ , i.e.  $\text{sgn}(\varepsilon_n) = \text{sgn}(i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n))$ , and  $\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n) = -\Sigma_{\varepsilon, in}^{(2, re)}(-\varepsilon_n)$ . Then we have

$$i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \text{sgn}(\varepsilon_l) \quad (37)$$

We separate the positive and negative parts of  $\varepsilon_l$  in the sum, and we have

$$i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\varepsilon_l > 0} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} - \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\varepsilon_l < 0} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \quad (38)$$

We define  $\omega_m \equiv \varepsilon_n - \varepsilon_l = 2m\pi T$ , and

$$i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m < \varepsilon_n} \frac{E}{E^2 + \omega_m^2} - \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m > \varepsilon_n} \frac{E}{E^2 + \omega_m^2} \quad (39)$$

$$= \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m > -\varepsilon_n} \frac{E}{E^2 + \omega_m^2} - \frac{\hbar}{\tau_{in}} N_{ge} T \sum_{\omega_m > \varepsilon_n} \frac{E}{E^2 + \omega_m^2} \quad (40)$$

$$= \frac{\hbar}{\tau_{in}} N_{ge} \text{sgn}(\varepsilon_n) T \sum_{|\omega_m| < |\varepsilon_n|} \frac{E}{E^2 + \omega_m^2} \quad (41)$$

This solution is consistent with the ansatz.

For the off-diagonal part, when  $T \rightarrow T_c^{tls}$ , we have  $i\Sigma_{\varepsilon, in}^{(2, re)} \gg \Sigma_{\Delta, in}^{(2, re)} \rightarrow 0$ , and

$$\Sigma_{\Delta, in}^{(2, re)}(\varepsilon_n) = \frac{\hbar}{\tau_{in}} N_{ge} T_c^{tls} \sum_l \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{\Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l)}{\left| \varepsilon_l + i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l) \right|} \Bigg|_{T_c^{tls}} \quad (42)$$

This is just an eigenvalue problem. For  $\Sigma_{\Delta, in}^{(2, re)}(\varepsilon_n)$  to have a non-trivial solution, we should have

$$\sum_l \left( M_{nl} - \delta_{nl} \right) \Sigma_{\Delta, in}^{(2, re)}(\varepsilon_l) = 0 \leftrightarrow \det(M_{nl} - \delta_{nl}) = 0 \quad (43)$$

where

$$M_{nl} = \frac{\hbar}{\tau_{in}} N_{ge} T_c^{tls} \frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \frac{1}{\left| \varepsilon_l + i\Sigma_{\varepsilon, in}^{(2, re)}(\varepsilon_l) \right|} \Bigg|_{T_c^{tls}} \quad (44)$$

With mean-field pairing, in the weak-coupling limit, we assume the interaction kernel becomes

$$\frac{E}{E^2 + (\varepsilon_n - \varepsilon_l)^2} \rightarrow \frac{1}{E} \Theta(E - |\varepsilon_l|) \quad (45)$$

then in Eq. (32), we have  $\Sigma_{\varepsilon, in}^{(2, re)} = 0$  because the right-hand side is an odd function of  $\varepsilon_l$ , and the off-diagonal self-energy does not have a dependence on energy  $\varepsilon_n$ ,

$$\Sigma_{\Delta, in}^{(2, re)} = \frac{\hbar}{\tau_{in} E} N_{ge} T \sum_{|\varepsilon_l| < |E|} \frac{\Delta + \Sigma_{\Delta, in}^{(2, re)}}{\sqrt{\left( \Delta + \Sigma_{\Delta, in}^{(2, re)} \right)^2 + \varepsilon_l^2}} \quad (46)$$

If we once again turn off the mean-field pairing, we have

$$1 = \frac{\hbar}{\tau_{in} E} N_{ge} T \sum_{|\varepsilon_l| < |E|} \frac{1}{\sqrt{\left(\Sigma_{\Delta, in}^{(2, re)}\right)^2 + \varepsilon_l^2}} \quad (47)$$

When  $T \rightarrow T_c^{tls}$ , we have  $\Sigma_{\Delta, in}^{(2, re)} \ll T_c^{tls}$ , and

$$1 = \frac{\hbar}{\tau_{in} E} N_{ge} T_c^{tls} \sum_{|\varepsilon_l| < E} \frac{1}{|\varepsilon_l|} \quad (48)$$

This is similar to the BCS gap equation, except that the cutoff is  $E$  instead of the Debye frequency  $\varepsilon_D$ , and the interaction strength is proportional to  $1/E$  and  $N_{ge}$ . Use the Digamma function in Eq. (55), we have

$$T_c^{tls} = \frac{2e^\gamma}{\pi} E \exp\left(-\frac{\pi E}{N_{ge} \hbar / \tau_{in}}\right) \approx 1.13 E \exp\left(-\frac{\pi E}{N_{ge} \hbar / \tau_{in}}\right) \quad (49)$$

Compare with the BCS gap equation near  $T_c$ , we have

$$1 = g N(0) T_c \sum_{|\varepsilon_n| < \varepsilon_D} \frac{1}{|\varepsilon_n|} \quad (50)$$

and the solution is

$$T_c = \frac{2e^\gamma}{\pi} \varepsilon_D \exp\left(-\frac{\pi}{g N(0)}\right) \approx 1.13 \varepsilon_D \exp\left(-\frac{\pi}{g N(0)}\right) \quad (51)$$

## 2 Density of States

The quasiclassical Green's function is

$$\hat{G} \equiv \int d\xi \hat{\tau}_3 \hat{G} = -\pi \frac{i\tilde{\varepsilon}_n \hat{\tau}_3 + i\tilde{\Delta} \hat{\tau}_2}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} = \begin{pmatrix} g & f \\ -f^* & -g \end{pmatrix} \quad (52)$$

The density of states is

$$N(\varepsilon) = -\frac{1}{\pi} \text{Im}\{g(\varepsilon^R)\} = \text{Im}\left\{\frac{\tilde{\varepsilon}}{\sqrt{\tilde{\Delta}^2 - (\tilde{\varepsilon} + i0^+)^2}}\right\} \quad (53)$$

## 3 Gap Equation

The gap equation is

$$\Delta = -v_0 T \sum_{\varepsilon_n}^{\varepsilon_c} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2} \text{Tr}\{\hat{G} \hat{\tau}_1\} \equiv g \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{\tilde{\Delta}}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \quad (54)$$

The digamma function is

$$K(T) \equiv \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{|\varepsilon_n|} \approx \ln\left(1.13 \frac{\varepsilon_c}{T}\right) \quad (55)$$

Then we have

$$\frac{1}{g} = \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{\tilde{\Delta}/\Delta}{\sqrt{\tilde{\Delta}^2 + \tilde{\varepsilon}_n^2}} \stackrel{\text{clean limit}}{=} \pi T \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{\sqrt{\Delta^2 + \varepsilon_n^2}} \stackrel{T \rightarrow T_{c0}}{=} \pi T_{c0} \sum_{\varepsilon_n}^{\varepsilon_c} \frac{1}{|(2n+1)\pi T_{c0}|} \approx \ln\left(1.13 \frac{\varepsilon_c}{T_{c0}}\right) \quad (56)$$

Subtract the previous two equations, we have

$$\ln \frac{T}{T_{c0}} = \pi T \sum_n \left( \frac{1 + \Sigma_{\Delta}/\Delta}{\sqrt{(\Delta + \Sigma_{\Delta})^2 + (\varepsilon_n + i\Sigma_{\varepsilon})^2}} - \frac{1}{|\varepsilon_n|} \right) \quad (57)$$