MEAN-FIELD THEORY

The mean-field Nambu-Gor'kov Hamiltonian is

$$H = \frac{1}{2}\iint \mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{r}'\widehat{\Psi}^\dagger(\mathbf{r}') \begin{pmatrix} \xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\hat{1} & \hat{\Delta}(\mathbf{r},\mathbf{r}') \\ \hat{\Delta}^\dagger(\mathbf{r},\mathbf{r}') & -\xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\hat{1} \end{pmatrix} \widehat{\Psi}(\mathbf{r}) \tag{1}$$

$$\equiv \frac{1}{2}\iint \mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{r}'\widehat{\Psi}^\dagger(\mathbf{r}')\widehat{\tau}_3 \left(\xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') + \widehat{\Delta}(\mathbf{r},\mathbf{r}')\right) \widehat{\Psi}(\mathbf{r}). \tag{2}$$

where

$$\widehat{\Delta}(\mathbf{r},\mathbf{r}') = \begin{pmatrix} 0 & \hat{\Delta}(\mathbf{r},\mathbf{r}') \\ -\hat{\Delta}^\dagger(\mathbf{r},\mathbf{r}') & 0 \end{pmatrix} \tag{3}$$

When the system is homogeneous, the two-point gap function only depends on the relative coordinate, i.e., $\hat{\Delta}(\mathbf{r},\mathbf{r}') = \hat{\Delta}(\mathbf{r}-\mathbf{r}')$. In this case, we can transform to momentum space, and the Hamiltonian becomes

$$H = \frac{1}{2}\sum_{\mathbf{p}} \widehat{\Psi}^\dagger_{\mathbf{p}}\widehat{\tau}_3(\xi_{\mathbf{p}} + \widehat{\Delta}(\mathbf{p}))\widehat{\Psi}_{\mathbf{p}}. \tag{4}$$

where $\widehat{\Psi}^\dagger_{\mathbf{p}} = (c^\dagger_{\mathbf{p}\uparrow}, c^\dagger_{\mathbf{p}\downarrow}, c_{-\mathbf{p}\uparrow}, c_{-\mathbf{p}\downarrow})$.

FINITE TEMPERATURE

The imaginary time Green's function is

$$\widehat{G}(x,x') = -\left\langle T_\tau \widehat{\Psi}(x)\widehat{\Psi}^\dagger(x')\right\rangle. \tag{5}$$

where $x = (\mathbf{r},\tau)$. The unperturbed Matsubara Green's function is

$$\widehat{G}_0(\mathbf{p}, i\epsilon_n) = (i\epsilon_n - \widehat{\tau}_3\xi_{\mathbf{p}})^{-1} = -\frac{i\epsilon_n + \widehat{\tau}_3\xi_{\mathbf{p}}}{\epsilon_n^2 + \xi_{\mathbf{p}}^2}. \tag{6}$$

The Gorkov Green's function satisfies

$$\widehat{G}^{-1}(\mathbf{p}, i\epsilon_n) = \widehat{G}_0^{-1}(\mathbf{p}, i\epsilon_n) - \widehat{\tau}_3\widehat{\Delta}(\mathbf{p}) = i\epsilon_n - \widehat{\tau}_3\xi_{\mathbf{p}} - \widehat{\tau}_3\widehat{\Delta}(\mathbf{p}). \tag{7}$$

which gives

$$\widehat{G}(\mathbf{p}, i\epsilon_n) = -\frac{i\epsilon_n + \widehat{\tau}_3\xi_{\mathbf{p}} + \widehat{\tau}_3\widehat{\Delta}(\mathbf{p})}{\epsilon_n^2 + \xi_{\mathbf{p}}^2 + |\Delta|^2} \tag{8}$$

The quasiclassical Green's function is

$$\widehat{g}(\mathbf{p}, i\epsilon_n) = \int \mathrm{d}\xi_p \widehat{G}(\mathbf{p}, i\epsilon_n)\widehat{\tau}_3 = -\pi \frac{i\epsilon_n \widehat{\tau}_3 - \widehat{\Delta}(\mathbf{p})}{\sqrt{|\Delta|^2 + \epsilon_n^2}} = \frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}} \begin{pmatrix} -i\epsilon_n & \hat{\Delta} \\ -\hat{\Delta}^\dagger & i\epsilon_n \end{pmatrix} \equiv \begin{pmatrix} \hat{g} & \hat{f} \\ -\hat{f}^\dagger & -\hat{g} \end{pmatrix} \tag{9}$$

and it satisfies the normalization condition

$$\widehat{g}(\mathbf{p}, i\epsilon_n)^2 = -\pi^2 \widehat{1}. \tag{10}$$

The bulk quasiclassical Green's function satisfies

$$\left[i\epsilon_n \widehat{\tau}_3 - \widehat{\Delta}, \widehat{g} \right] = 0 \tag{11}$$

But when near the boundary or impurities, the quasiclassical Green's function $\widehat{g}(\mathbf{r}, \mathbf{p}; \epsilon)$ and the gap function $\widehat{\Delta}(\mathbf{r}, \mathbf{p})$ are inhomogeneous, and are described by the Eilenberger equation and gap equation.

$$\left[\epsilon \widehat{\tau}_3 - \widehat{\Delta}(\mathbf{r}, \mathbf{p}), \widehat{g}(\mathbf{r}, \mathbf{p}; \epsilon) \right] + i\hbar \mathbf{v}_p \cdot \nabla \widehat{g}(\mathbf{r}, \mathbf{p}; \epsilon) = 0 \tag{12}$$

and

$$\hat{\Delta}(\mathbf{r}, \mathbf{p}) = T \sum_{\epsilon_n = -\omega_c}^{\omega_c} \left\langle v(\mathbf{p}, \mathbf{p}') \hat{f}(\mathbf{r}, \mathbf{p}', \epsilon_n) \right\rangle_{p'} \tag{13}$$

In bulk, we have

$$\Delta(\mathbf{p}) = T \sum_{\epsilon_n = -\omega_c}^{\omega_c} \left\langle v(\mathbf{p}, \mathbf{p}') \frac{\pi \Delta(\mathbf{p}')}{\sqrt{|\Delta|^2 + \epsilon_n^2}} \right\rangle_{p'} \tag{14}$$

We have the Digamma function

$$K(T) = T \sum_{\epsilon_n = -\omega_c}^{\omega_c} \frac{\pi}{|\epsilon_n|} \approx \ln \left(1.13 \frac{\omega_c}{T} \right) \tag{15}$$

In bulk at T_c, we have

$$\Delta(\mathbf{p}) = K(T_c) \left\langle v(\mathbf{p}, \mathbf{p}') \Delta(\mathbf{p}') \right\rangle_{p'} \tag{16}$$

and we have

$$K(T_c) - K(T) = \ln (T/T_c) = \left. \frac{\Delta(\mathbf{p}'')}{\left\langle v(\mathbf{p}'', \mathbf{p}') \Delta(\mathbf{p}') \right\rangle_{p'}} \right|_{T_c} - T \sum_{\epsilon_n = -\omega_c}^{\omega_c} \frac{\pi}{|\epsilon_n|} \tag{17}$$

Substitute in Eq. (13), we get

$$\ln \frac{T}{T_c} = T \sum_{\epsilon_n = -\infty}^{\infty} \left[\frac{\left\langle v(\mathbf{p}, \mathbf{p}') f(\mathbf{r}, \mathbf{p}', \epsilon_n) \right\rangle_{p'} / \Delta(\mathbf{r}, \mathbf{p})}{\left\langle v(\mathbf{p}'', \mathbf{p}') \Delta(\mathbf{p}') \right\rangle_{p'} / \Delta(\mathbf{p}'') \big|_{T_c}} - \frac{\pi}{|\epsilon_n|} \right] \tag{18}$$

^{3}HE-A

For ^{3}He-A, the spin structure is $\hat{\mathbf{d}} \cdot (i\vec{\sigma}\hat{\sigma}_y) = \hat{\sigma}_x$, and we have two components of the gap function,

$$\hat{\Delta}(\mathbf{r}, \mathbf{p}) = \hat{\sigma}_x(\Delta_1(\mathbf{r})p_x + \Delta_2(\mathbf{r})p_y) \tag{19}$$

In bulk we have $\hat{\Delta}(\mathbf{r}, \mathbf{p}) = \hat{\sigma}_x\Delta(p_x + ip_y) = \hat{\sigma}_x\Delta e^{i\phi_p}$. The interaction is $v(\mathbf{p}, \mathbf{p}') = 3v_0\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'$, and the gap-equation reduces to

$$\ln\frac{T}{T_c} = T\sum_{\epsilon_n=-\infty}^{\infty}\left[\frac{2\left\langle \mathbf{p} \cdot \mathbf{p}' f(\mathbf{r}, \mathbf{p}', \epsilon_n)\right\rangle_{p'}}{\Delta(\mathbf{r}, \mathbf{p})} - \frac{\pi}{|\epsilon_n|}\right] \tag{20}$$

S-WAVE PAIRING

For s-wave pairing, the spin structure is $i\hat{\sigma}_y$, and we have

$$\hat{\Delta}(\mathbf{r}, \mathbf{p}) = i\hat{\sigma}_y\Delta(\mathbf{r}) \tag{21}$$

In bulk we have $\hat{\Delta}(\mathbf{r}, \mathbf{p}) = i\hat{\sigma}_y\Delta$. The interaction is $v(\mathbf{p}, \mathbf{p}') = v_s$, and the gap-equation reduces to

$$\ln\frac{T}{T_c} = T\sum_{\epsilon_n=-\infty}^{\infty}\left[\frac{f(\mathbf{r}, \epsilon_n)}{\Delta(\mathbf{r})} - \frac{\pi}{|\epsilon_n|}\right] \tag{22}$$

^{3}HE-A EDGE GAP PROFILE

We assume translational invariance along the edge, i.e. the y direction. We also assume the gap to be real(imaginary) along the $p_x(p_y)$ direction.

$$\hat{\Delta}(x, \mathbf{p}) = \hat{\sigma}_x(\Delta_1(x)p_x + i\Delta_2(x)p_y) \tag{23}$$

Here $\Delta_1(x)$ and $\Delta_2(x)$ are real functions, and we have

$$\widehat{\Delta}(\mathbf{r}, \mathbf{p}) = \begin{pmatrix} 0 & \hat{\sigma}_x(\Delta_1(x)p_x + i\Delta_2(x)p_y) \\ -\hat{\sigma}_x(\Delta_1(x)p_x - i\Delta_2(x)p_y) & 0 \end{pmatrix} \tag{24}$$

$$= i\hat{\sigma}_x(\Delta_2 p_y\widehat{\tau}_1 + \Delta_1 p_x\widehat{\tau}_2) \tag{25}$$

In this case, we can also write the anomalous Green's function as

$$\hat{f}(\mathbf{r},\mathbf{p},\epsilon_n) = \hat{\sigma}_x\Big(f_1(\mathbf{r},\mathbf{p},\epsilon_n) + if_2(\mathbf{r},\mathbf{p},\epsilon_n)\Big) \tag{26}$$

where $f_{1,2}$ are real functions, and we have

$$\widehat{g}(\mathbf{r},\mathbf{p},\epsilon_n) = \hat{g}\widehat{\tau}_3 + i\hat{\sigma}_x(f_2\widehat{\tau}_1 + f_1\widehat{\tau}_2) \tag{27}$$

In this case, we can even write the gap equation as

$$\Delta_{1,2}(x) = T\sum_{\epsilon_n=-\omega_c}^{\omega_c} \langle v(\mathbf{p}_{x,y},\mathbf{p}')f_{1,2}(x,\mathbf{p}',\epsilon_n)\rangle_{p'} \tag{28}$$

which becomes

$$\ln\frac{T}{T_c} = T\sum_{\epsilon_n=-\infty}^{\infty}\left[\frac{\langle \mathbf{p}_{x,y}\cdot\mathbf{p}' f_{1,2}(x,\mathbf{p}',\epsilon_n)\rangle_{p'}}{\Delta_{1,2}(x)} - \frac{\pi}{|\epsilon_n|}\right] \tag{29}$$

NUMERICAL SOLUTION

The $(1,1)$ and $(1,2)$ elements of the Eilenberger equation are

$$i\epsilon_n\hat{g} + \hat{\Delta}\hat{f}^\dagger + i\hbar\mathbf{v}_p\cdot\boldsymbol{\nabla}\hat{g} = 0 \tag{30}$$

$$i\epsilon_n\hat{f} + \hat{\Delta}\hat{g} + i\hbar\mathbf{v}_p\cdot\boldsymbol{\nabla}\hat{f} = 0 \tag{31}$$

Note that $\hat{\Delta}$ and $\hat{f}$ has the same spin structure, so we can substitute in $\hat{g}$ and $\hat{f}$, divide by T_c, and we get

$$i(2n+1)\pi t g(\mathbf{r},\hat{\mathbf{p}},n) + \delta(\mathbf{r},\hat{\mathbf{p}})f^*(\mathbf{r},\hat{\mathbf{p}},n) + i(\hat{\mathbf{p}}_x\partial_x + \hat{\mathbf{p}}_y\partial_y)g(\mathbf{r},\hat{\mathbf{p}},n) = 0 \tag{32}$$

$$i(2n+1)\pi t f(\mathbf{r},\hat{\mathbf{p}},n) + \delta(\mathbf{r},\hat{\mathbf{p}})g(\mathbf{r},\hat{\mathbf{p}},n) + i(\hat{\mathbf{p}}_x\partial_x + \hat{\mathbf{p}}_y\partial_y)f(\mathbf{r},\hat{\mathbf{p}},n) = 0 \tag{33}$$

where $t = T/T_c$, $\delta = \Delta/T_c$, $\xi = \hbar v_f/T_c$ and $\vec{\partial} = \xi\boldsymbol{\nabla}$.

We first assume an initial guess for the gap function $\delta(\mathbf{r},\hat{\mathbf{p}})$. For each trajectory $\hat{\mathbf{p}}$ along which the quasiparticle propagates, we can solve the Eilenberger equation to get the Green's function $g(\mathbf{r},\hat{\mathbf{p}},n)$ and the anomalous Green's function $f(\mathbf{r},\hat{\mathbf{p}},n)$. Then we iterate to update the gap function $\delta(\mathbf{r},\hat{\mathbf{p}})$ by solving the gap equation. Repeat until convergence.