

INTRODUCTION

In cylindrical coordinates, we have

$$\begin{pmatrix} \eta_r \\ \eta_\phi \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \quad (1)$$

and

$$\begin{pmatrix} \partial_r \\ \frac{1}{r}\partial_\phi \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \quad (2)$$

In bulk, for the two time-reversed ground states,

$$(\eta_x^{\text{bulk}}, \eta_y^{\text{bulk}}) = \eta_0(1, \pm i) \quad (3)$$

$$(\eta_r^{\text{bulk}}, \eta_\phi^{\text{bulk}}) = e^{\pm i\phi} \eta_0(1, \pm i) \quad (4)$$

In GL theory, the order parameter is determined by minimizing the free energy functional

$$F[\boldsymbol{\eta}] = \int d^3r (f_{\text{bulk}} + f_{\text{grad}}) \quad (5)$$

where

$$f_{\text{bulk}}[\boldsymbol{\eta}] = \alpha \boldsymbol{\eta} \cdot \boldsymbol{\eta}^* + \beta_1 (\boldsymbol{\eta} \cdot \boldsymbol{\eta}^*)^2 + \beta_2 |\boldsymbol{\eta} \cdot \boldsymbol{\eta}|^2, \quad (6)$$

and

$$f_{\text{grad}}^{\text{Car}}[\boldsymbol{\eta}] = \kappa_1 (\partial_i \eta_j) (\partial_i \eta_j)^* + \kappa_2 (\partial_i \eta_i) (\partial_j \eta_j)^* + \kappa_3 (\partial_i \eta_j) (\partial_j \eta_i)^* \quad (7)$$

In cylindrical basis, we have $\partial_i^{\text{Car}} = R_{ik} \partial_k$ and $\eta_j^{\text{Car}} = R_{jl} \eta_l$, where R_{ij} is the 2d rotational matrix. The form of the bulk free energy term doesn't change. The gradient term becomes

$$\begin{aligned} f_{\text{grad}}^{\text{Cyl}}[\boldsymbol{\eta}] = & \kappa_1 (\partial_i \eta_j) (\partial_i \eta_j)^* + \kappa_2 (\partial_i \eta_i) (\partial_j \eta_j)^* + \kappa_3 (\partial_i \eta_j) (\partial_j \eta_i)^* \\ & + \frac{2}{r} \text{Re} \left\{ \kappa_1 \left(\eta_r^* \frac{\partial \phi}{r} \eta_\phi - \eta_\phi^* \frac{\partial \phi}{r} \eta_r \right) + \kappa_2 \eta_r^* \partial_j \eta_j + \kappa_3 \left(\eta_r^* \frac{\partial \phi}{r} \eta_\phi - \eta_\phi^* \partial_r \eta_\phi \right) \right\} \\ & + \frac{1}{r^2} [\kappa_1 \eta_j^* \eta_j + (\kappa_2 + \kappa_3) \eta_r^* \eta_r]. \end{aligned} \quad (8)$$

Take the κ_1 term for example, note that $R_{ij}R_{ik} = R_{ji}^T R_{ik} = \delta_{jk}$

$$(\partial_i^{\text{Car}} \eta_j^{\text{Car}})(\partial_i^{\text{Car}} \eta_j^{\text{Car}})^* \quad (9)$$

$$= \left[R_{ik} \partial_k (R_{jl} \eta_l) \right] \left[R_{im} \partial_m (R_{jn} \eta_n^*) \right] \quad (10)$$

$$= \left[\partial_k (R_{jl} \eta_l) \right] \left[\partial_k (R_{jn} \eta_n^*) \right] \quad (11)$$

$$= \left[(\partial_k R_{jl}) \eta_l + R_{jl} (\partial_k \eta_l) \right] \left[(\partial_k R_{jn}) \eta_n^* + R_{jn} (\partial_k \eta_n^*) \right] \quad (12)$$

$$= (\partial_k R_{jl}) \eta_l (\partial_k R_{jn}) \eta_n^* + (\partial_k R_{jl}) \eta_l R_{jn} (\partial_k \eta_n^*) + R_{jl} (\partial_k \eta_l) (\partial_k R_{jn}) \eta_n^* + R_{jl} R_{jn} (\partial_k \eta_l) (\partial_k \eta_n^*)$$

Note that $\partial_r R(\phi) = 0$, $\partial_\phi R(\phi) = R(\phi) R(\pi/2) = R(\pi/2) R(\phi)$. Finally we have

$$(\partial_i^{\text{Car}} \eta_j^{\text{Car}})(\partial_i^{\text{Car}} \eta_j^{\text{Car}})^* \quad (13)$$

$$= \frac{1}{r^2} \eta_l \eta_l^* + \frac{1}{r} R_{nl}^{\pi/2} \eta_l \left(\frac{\partial_\phi}{r} \eta_n^* \right) + \frac{1}{r} R_{ln}^{\pi/2} \eta_n^* \left(\frac{\partial_\phi}{r} \eta_l \right) + (\partial_k \eta_l) (\partial_k \eta_l)^* \quad (14)$$

$$= \frac{1}{r^2} \eta_l \eta_l^* + \frac{2}{r} \text{Re} \left\{ R_{ln}^{\pi/2} \eta_n^* \left(\frac{\partial_\phi}{r} \eta_l \right) \right\} + (\partial_k \eta_l) (\partial_k \eta_l)^* \quad (15)$$

$$= \frac{1}{r^2} \eta_l \eta_l^* + \frac{2}{r} \text{Re} \left\{ \eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \frac{\partial_\phi}{r} \eta_r \right\} + (\partial_k \eta_l) (\partial_k \eta_l)^* \quad (16)$$

Similarly, for the κ_2 terms, we have

$$(\partial_i^{\text{Car}} \eta_i^{\text{Car}})(\partial_j^{\text{Car}} \eta_j^{\text{Car}})^* \quad (17)$$

$$= \left[R_{ik} \partial_k (R_{il} \eta_l) \right] \left[R_{jm} \partial_m (R_{jn} \eta_n^*) \right] \quad (18)$$

$$= \left[R_{ik} (\partial_k R_{il}) \eta_l + R_{ik} R_{il} (\partial_k \eta_l) \right] \left[R_{jm} (\partial_m R_{jn}) \eta_n^* + R_{jm} R_{jn} (\partial_m \eta_n^*) \right] \quad (19)$$

$$= \left[\frac{1}{r} \eta_r + \partial_k \eta_k \right] \left[\frac{1}{r} \eta_r^* + \partial_m \eta_m^* \right] \quad (20)$$

$$= \frac{1}{r^2} \eta_r \eta_r^* + \frac{2}{r} \text{Re} \{ \eta_r^* \partial_k \eta_k \} + (\partial_k \eta_k) (\partial_m \eta_m)^* \quad (21)$$

For the κ_3 terms, we have

$$(\partial_i^{\text{Car}} \eta_j^{\text{Car}})(\partial_j^{\text{Car}} \eta_i^{\text{Car}})^* \quad (22)$$

$$= \left[R_{ik} \partial_k (R_{jl} \eta_l) \right] \left[R_{jm} \partial_m (R_{in} \eta_n^*) \right] \quad (23)$$

$$= \left[R_{ik} (\partial_k R_{jl}) \eta_l + R_{ik} R_{jl} (\partial_k \eta_l) \right] \left[R_{jm} (\partial_m R_{in}) \eta_n^* + R_{jm} R_{in} (\partial_m \eta_n^*) \right] \quad (24)$$

$$= R_{ik} (\partial_k R_{jl}) \eta_l R_{jm} (\partial_m R_{in}) \eta_n^* + R_{ik} (\partial_k R_{jl}) \eta_l R_{jm} R_{in} (\partial_m \eta_n^*) \\ + R_{ik} R_{jl} (\partial_k \eta_l) R_{jm} (\partial_m R_{in}) \eta_n^* + R_{ik} R_{jl} (\partial_k \eta_l) R_{jm} R_{in} (\partial_m \eta_n^*) \quad (25)$$

$$= \frac{1}{r^2} R_{\phi l}^{\pi/2} \eta_l R_{\phi n}^{\pi/2} \eta_n^* + \frac{1}{r} R_{ml}^{\pi/2} \eta_l (\partial_m \eta_\phi^*) + \frac{1}{r} R_{kn}^{\pi/2} \eta_n^* (\partial_k \eta_\phi) + (\partial_k \eta_l) (\partial_l \eta_k)^* \quad (26)$$

$$= \frac{1}{r^2} \eta_r \eta_r^* + \frac{2}{r} \text{Re} \left\{ R_{kn}^{\pi/2} \eta_n^* (\partial_k \eta_\phi) \right\} + (\partial_k \eta_l) (\partial_l \eta_k)^* \quad (27)$$

$$= \frac{1}{r^2} \eta_r \eta_r^* + \frac{2}{r} \text{Re} \left\{ \eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \partial_r \eta_\phi \right\} + (\partial_k \eta_l) (\partial_l \eta_k)^* \quad (28)$$

So the free energy density is

$$f_{\text{bulk}}[\boldsymbol{\eta}] = \alpha \boldsymbol{\eta} \cdot \boldsymbol{\eta}^* + \beta_1 (\boldsymbol{\eta} \cdot \boldsymbol{\eta}^*)^2 + \beta_2 |\boldsymbol{\eta} \cdot \boldsymbol{\eta}|^2, \quad (29)$$

and

$$f_{\text{grad}}^{\text{Cyl}}[\boldsymbol{\eta}] = \kappa_1 (\partial_i \eta_j) (\partial_i \eta_j)^* + \kappa_2 (\partial_i \eta_i) (\partial_j \eta_j)^* + \kappa_3 (\partial_i \eta_j) (\partial_j \eta_i)^* \\ + \frac{2}{r} \text{Re} \left\{ \kappa_1 \left(\eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \frac{\partial_\phi}{r} \eta_r \right) + \kappa_2 \eta_r^* \partial_j \eta_j + \kappa_3 \left(\eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \partial_r \eta_\phi \right) \right\} \\ + \frac{1}{r^2} [\kappa_1 \eta_j^* \eta_j + (\kappa_2 + \kappa_3) \eta_r^* \eta_r]. \quad (30)$$

EULER-LAGRANGE EQUATIONS

The GL differential equations $\delta F / \delta \eta_k^* = 0$.

If we do the functional variation of the free energy in cylindrical basis, we will get

$$\alpha \eta_k + 2(\beta_1 \eta_i \eta_i^* \eta_k + \beta_2 \eta_i^2 \eta_k^*) - (\kappa_1 \partial_i^2 \eta_k + \kappa_2 \partial_k \partial_i \eta_i + \kappa_3 \partial_j \partial_k \eta_j) \\ - \frac{1}{r} \left[(2\kappa_1 + \kappa_3) \frac{\partial_\phi}{r} (\eta_r \delta_{k\phi} - \eta_\phi \delta_{kr}) + \kappa_1 \partial_r \eta_k + (\kappa_2 + \kappa_3) \partial_k \eta_r \right] \\ + \frac{1}{r^2} [\kappa_1 \eta_k + (\kappa_2 + \kappa_3) \eta_r \delta_{kr}] = 0. \quad (31)$$

For the κ_1 term in first line of Eq. (30)

$$F \sim \iint (\partial_i \eta_j)(\partial_i \eta_j)^* r d\phi dr = \text{surface term} - \iint \eta_j^* \partial_i (r \partial_i \eta_j) d\phi dr \quad (32)$$

$$= \text{surface term} - \iint \eta_j^* \frac{1}{r} \partial_i (r \partial_i \eta_j) r d\phi dr \quad (33)$$

the corresponding functional variation is

$$\delta F / \delta \eta_k^* \sim -\frac{1}{r} \partial_i (r \partial_i \eta_k) = -\frac{1}{r} (\partial_i r) (\partial_i \eta_k) - \partial_i^2 \eta_k = -\frac{1}{r} \partial_r \eta_k - \partial_i^2 \eta_k \quad (34)$$

For the κ_2 term in first line of Eq. (30)

$$\delta F / \delta \eta_k^* \sim -\frac{1}{r} \partial_k (r \partial_i \eta_i) = -\frac{1}{r} (\partial_k r) (\partial_i \eta_i) - \partial_k \partial_i \eta_i = -\delta_{kr} \frac{1}{r} \partial_i \eta_i - \partial_k \partial_i \eta_i \quad (35)$$

For the κ_3 term in first line of Eq. (30)

$$\delta F / \delta \eta_k^* \sim -\frac{1}{r} \partial_i (r \partial_k \eta_i) = -\frac{1}{r} (\partial_i r) (\partial_k \eta_i) - \partial_i \partial_k \eta_i = -\frac{1}{r} \partial_k \eta_r - \partial_i \partial_k \eta_i \quad (36)$$

For the κ_1 term in second line of Eq. (30)

$$F \sim \iint \frac{1}{r} \left(\eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \frac{\partial_\phi}{r} \eta_r + \eta_r \frac{\partial_\phi}{r} \eta_\phi^* - \eta_\phi \frac{\partial_\phi}{r} \eta_r^* \right) r d\phi dr \quad (37)$$

the corresponding functional variation is

$$\delta F / \delta \eta_k^* \sim \frac{1}{r} \left(2\delta_{kr} \frac{\partial_\phi}{r} \eta_\phi - 2\delta_{k\phi} \frac{\partial_\phi}{r} \eta_r \right) \quad (38)$$

The κ_2 term in second line of Eq. (30) is

$$F \sim \iint \frac{1}{r} (\eta_r^* \partial_j \eta_j + \eta_r \partial_j \eta_j^*) r d\phi dr \quad (39)$$

the corresponding functional variation is

$$\delta F / \delta \eta_k^* \sim \frac{1}{r} (\delta_{kr} \partial_j \eta_j - \partial_k \eta_r) \quad (40)$$

The κ_3 term in second line of Eq. (30) is

$$F \sim \iint \frac{1}{r} \left(\eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \partial_r \eta_\phi + \eta_r \frac{\partial_\phi}{r} \eta_\phi^* - \eta_\phi \partial_r \eta_\phi^* \right) r d\phi dr \quad (41)$$

the corresponding functional variation is

$$\delta F / \delta \eta_k^* \sim \frac{1}{r} \left(\delta_{kr} \frac{\partial_\phi}{r} \eta_\phi - \cancel{\delta_{k\phi} \partial_r \eta_\phi} - \delta_{k\phi} \frac{\partial_\phi}{r} \eta_r + \cancel{\delta_{k\phi} \partial_r \eta_\phi} \right) \quad (42)$$

Adding all these contributions together, we can get the GL equations in cylindrical basis.

In cartesian basis the GL equations are

$$\alpha\eta_k + 2(\beta_1\eta_i\eta_i^*\eta_k + \beta_2\eta_i^2\eta_k^*) - (\kappa_1\partial_i^2\eta_k + \kappa_2\partial_k\partial_i\eta_i + \kappa_3\partial_j\partial_k\eta_j) = 0 \quad (43)$$

If we transform it to cylindrical basis, we can also get

$$\begin{aligned} & \alpha\eta_k + 2(\beta_1\eta_i\eta_i^*\eta_k + \beta_2\eta_i^2\eta_k^*) - (\kappa_1\partial_i^2\eta_k + \kappa_2\partial_k\partial_i\eta_i + \kappa_3\partial_j\partial_k\eta_j) \\ & - \frac{1}{r} \left[(2\kappa_1 + \kappa_3) \frac{\partial\phi}{r} (\eta_r\delta_{k\phi} - \eta_\phi\delta_{kr}) + \kappa_1\partial_r\eta_k + (\kappa_2 + \kappa_3)\partial_k\eta_r \right] \\ & + \frac{1}{r^2} [\kappa_1\eta_k + (\kappa_2 + \kappa_3)\eta_r\delta_{kr}] = 0. \end{aligned} \quad (44)$$

The κ_1 term is just a Laplacian term

$$\partial_i^{\text{Car}} \partial_i^{\text{Car}} \eta_k^{\text{Car}} = R_{il} \partial_l (R_{im} \partial_m (R_{kn} \eta_n)) \quad (45)$$

$$= R_{il} (\partial_l R_{im}) \partial_m (R_{kn} \eta_n) + R_{il} R_{im} \partial_l \partial_m (R_{kn} \eta_n) \quad (46)$$

$$= \frac{1}{r} \partial_r (R_{kn} \eta_n) + \partial_l^2 (R_{kn} \eta_n) \quad (47)$$

$$= R_{kn} \frac{1}{r} \partial_r \eta_n + R_{kn} \partial_l^2 \eta_n + 2(\partial_l R_{kn})(\partial_l \eta_n) + \eta_n \partial_l^2 R_{kn} \quad (48)$$

$$= R_{kn} \left(\frac{1}{r} \partial_r \eta_n + \partial_l^2 \eta_n + \frac{2}{r} R_{nm}^{\pi/2} \frac{\partial\phi}{r} \eta_m - \frac{1}{r^2} \eta_n \right) \quad (49)$$

$$= R_{kn} \left(\frac{1}{r} \partial_r \eta_n + \partial_l^2 \eta_n + \frac{2}{r} \delta_{n\phi} \frac{\partial\phi}{r} \eta_r - \frac{2}{r} \delta_{nr} \frac{\partial\phi}{r} \eta_\phi - \frac{1}{r^2} \eta_n \right) \quad (50)$$

The κ_2 term is

$$\partial_k^{\text{Car}} \partial_i^{\text{Car}} \eta_i^{\text{Car}} = R_{kl} \partial_l (R_{im} \partial_m (R_{in} \eta_n)) \quad (51)$$

$$= R_{kl} \partial_l (R_{im} (\partial_m R_{in}) \eta_n + R_{im} R_{in} (\partial_m \eta_n)) \quad (52)$$

$$= R_{kl} \partial_l \left(\frac{1}{r} \eta_r + \partial_m \eta_m \right) \quad (53)$$

$$= R_{kl} \left(\frac{1}{r} \partial_l \eta_r - \delta_{lr} \frac{1}{r^2} \eta_r + \partial_l \partial_m \eta_m \right) \quad (54)$$

The κ_3 term is

$$\partial_j^{\text{Car}} \partial_k^{\text{Car}} \eta_j^{\text{Car}} = R_{jl} \partial_l (R_{km} \partial_m (R_{jn} \eta_n)) \quad (55)$$

$$= R_{jl} (\partial_l R_{km}) \partial_m (R_{jn} \eta_n) + R_{jl} R_{km} \partial_l \partial_m (R_{jn} \eta_n) \quad (56)$$

$$= R_{j\phi} \frac{1}{r} R_{ki} R_{im}^{\pi/2} \partial_m (R_{jn} \eta_n) + R_{jl} R_{ki} \partial_l \partial_i (R_{jn} \eta_n) \quad (57)$$

$$= R_{ki} \left[R_{j\phi} \frac{1}{r} R_{im}^{\pi/2} \partial_m (R_{jn} \eta_n) + R_{jl} \partial_l \partial_i (R_{jn} \eta_n) \right] \quad (58)$$

$$= R_{ki} \left[R_{j\phi} \frac{1}{r} R_{im}^{\pi/2} (\partial_m R_{jn}) \eta_n + R_{j\phi} \frac{1}{r} R_{im}^{\pi/2} R_{jn} (\partial_m \eta_n) \right] \quad (59)$$

$$\begin{aligned} & + R_{jl} (\partial_l \partial_i R_{jn}) \eta_n + R_{jl} (\partial_l R_{jn}) (\partial_i \eta_n) + R_{jl} (\partial_i R_{jn}) (\partial_l \eta_n) + R_{jl} R_{jn} (\partial_l \partial_i \eta_n) \Big] \\ & = R_{ki} \left[\frac{1}{r^2} R_{i\phi}^{\pi/2} \eta_r + \frac{1}{r} R_{im}^{\pi/2} \partial_m \eta_\phi \right. \\ & \quad \left. + \delta_{i\phi} R_{jl} \eta_n \partial_l \left(\frac{1}{r} R_{jn} \right) + \frac{1}{r} \partial_i \eta_r + \delta_{i\phi} \frac{1}{r} R_{ln}^{\pi/2} \partial_l \eta_n + \partial_l \partial_i \eta_l \right] \\ & = R_{ki} \left[-\frac{1}{r^2} \delta_{ir} \eta_r + \cancel{\frac{1}{r} \delta_{i\phi} \partial_r \eta_\phi} - \frac{1}{r^2} \delta_{ir} \partial_\phi \eta_\phi \right. \\ & \quad \left. + \delta_{i\phi} R_{jl} \eta_n \left(\partial_l \frac{1}{r} \right) R_{jn} + \delta_{i\phi} R_{jl} \eta_n \frac{1}{r} (\partial_l R_{jn}) + \frac{1}{r} \partial_i \eta_r + \delta_{i\phi} \frac{1}{r^2} \partial_\phi \eta_r - \cancel{\delta_{i\phi} \frac{1}{r} \partial_r \eta_\phi} + \partial_l \partial_i \eta_l \right] \\ & = R_{ki} \left[-\frac{1}{r^2} \delta_{ir} \eta_r - \frac{1}{r^2} \delta_{ir} \partial_\phi \eta_\phi \right. \\ & \quad \left. - \cancel{\delta_{i\phi} \eta_r \frac{1}{r^2}} + \cancel{\delta_{i\phi} \eta_r \frac{1}{r^2}} + \frac{1}{r} \partial_i \eta_r + \delta_{i\phi} \frac{1}{r^2} \partial_\phi \eta_r + \partial_l \partial_i \eta_l \right] \\ & = R_{ki} \left[-\frac{1}{r^2} \delta_{ir} \eta_r - \frac{1}{r^2} \delta_{ir} \partial_\phi \eta_\phi + \frac{1}{r} \partial_i \eta_r + \delta_{i\phi} \frac{1}{r^2} \partial_\phi \eta_r + \partial_l \partial_i \eta_l \right] \end{aligned} \quad (60)$$

PHYSICAL OBSERVABLES

In Cartesian basis we denote $\mathbf{r} = (x, y)$. A Galilean boost with velocity \mathbf{u} introduce a local gauge transformation $\eta_i(\mathbf{r}) \xrightarrow{\mathbf{u}} \eta_i(\mathbf{r}) e^{-iM\mathbf{u}\cdot\mathbf{r}/\hbar}$, where M is the mass of a pair of Helium atoms. The phase gradient correspond to a velocity field $\mathbf{v} = \frac{\hbar}{M} \nabla \theta$ which transform as $\mathbf{v} \xrightarrow{\mathbf{u}} \mathbf{v} - \mathbf{u}$ under the Galilean boost. We also have $\partial_i \xrightarrow{\mathbf{u}} \partial_i - iM u_i / \hbar$. The GL free energy density also transforms as $f \xrightarrow{\mathbf{u}} f - \mathbf{j} \cdot \mathbf{u} + \mathcal{O}(u^2)$, where $\mathbf{j}_k = \frac{2M}{\hbar} \text{Im} \{ \kappa_1 \eta_j^* \partial_k \eta_j + \kappa_2 \eta_k^* \partial_j \eta_j + \kappa_3 \eta_j^* \partial_j \eta_k \}$ is the superfluid mass current density or the momentum density. We transform it to cylindrical

basis and get

$$\mathbf{j}_k = \frac{2M}{\hbar} \left[\text{Im} \{ \kappa_1 \eta_j^* \partial_k \eta_j + \kappa_2 \eta_k^* \partial_j \eta_j + \kappa_3 \eta_j^* \partial_j \eta_k \} \right. \\ \left. + \text{Im} \{ (2\kappa_1 + \kappa_2 + \kappa_3) \delta_{k\phi} \eta_\phi^* \eta_r / r \} \right], \quad (61)$$

In weak-coupling limit, where $\beta_1 = 2\beta_2$ and $\kappa_1 = \kappa_2 = \kappa_3$, the angular momentum density $l_z = r j_\phi$ becomes

$$l_z = \frac{2M\kappa_1}{\hbar} \left[\text{Im} \{ 3\eta_\phi^* \partial_\phi \eta_\phi + \eta_r^* \partial_\phi \eta_r + 4\eta_\phi^* \eta_r \} \right. \\ \left. + \text{Im} \{ r\eta_\phi^* \partial_r \eta_r + r\eta_r^* \partial_r \eta_\phi \} \right]. \quad (62)$$

ROTATING FRAME

We can stabilize these low-flow states by rotating the annulus at certain angular velocities Ω_m . Transform into the rotating frame with angular velocity $\mathbf{\Omega} = \hat{\mathbf{z}}\Omega$, and the free energy becomes

$$F' = F - \mathbf{L} \cdot \mathbf{\Omega}. \quad (63)$$

The critical angular velocity Ω_m which increase the winding number from $m-1$ to m should satisfy $F'(m, \Omega_m) = F'(m-1, \Omega_m)$, which gives

$$\Omega_m^\pm = \frac{F_\pm(m) - F_\pm(m-1)}{L_\pm(m) - L_\pm(m-1)}. \quad (64)$$

UNIFORM-FLOW APPROXIMATION

We can have superflow in the annulus $\mathbf{v}(\mathbf{r}) = \frac{\hbar}{M} \nabla \theta$ where M is the mass of a pair of Helium atoms.

When we assume a uniform flow field along the azimuthal direction in the annulus, $\mathbf{v} = v \hat{\phi}$ and $v(r) = \frac{\hbar}{Mr} \partial_\phi \theta$, the order parameter in the annulus simply gains an extra phase factor along the azimuthal direction.

$$\boldsymbol{\eta}(r, \phi, v_s) = \boldsymbol{\eta}(r, v_s) e^{iMvr\phi/\hbar} = \boldsymbol{\eta}(r, v_s) e^{i(\partial_\phi \theta)\phi} \quad (65)$$

When the flow field is small, we denote

$$\boldsymbol{\eta}(r, \phi, v_s) = \boldsymbol{\eta}(r, v_s) e^{i(\partial_\phi \theta)\phi} = e^{\pm i\phi} \left(\eta_r^{(m)}(r), \pm i\eta_\phi^{(m)}(r) \right) e^{im\phi} \quad (66)$$

where m is an integer as the order parameter should be single-valued after 2π winding.

LOW-FLOW APPROXIMATION

For low flow states in an annulus, we further assume the radial profile won't be affected by the small flow field

$$\eta_i^{(m)}(r) = \eta_i^{(0)}(r) \quad (67)$$

For the denominator in Eq. (64), we have

$$L_{\pm}(m) - L_{\pm}(m-1) = \frac{2M\kappa_1}{\hbar} \int d^3r \left(3\eta_{\phi}^{(0)}\eta_{\phi}^{(0)} + \eta_r^{(0)}\eta_r^{(0)} \right) \quad (68)$$

For the numerator in Eq. (64), we have to pick out terms associated with ∂_{ϕ} , which are

$$\begin{aligned} f_{\text{grad}} \sim & \frac{\kappa_1}{r^2} \left[3(\partial_{\phi}\eta_{\phi})(\partial_{\phi}\eta_{\phi})^* + (\partial_{\phi}\eta_r)(\partial_{\phi}\eta_r)^* \right] \\ & + \frac{\kappa_1}{r} \left[(\partial_{\phi}\eta_{\phi})(\partial_r\eta_r)^* + (\partial_r\eta_r)(\partial_{\phi}\eta_{\phi})^* + (\partial_{\phi}\eta_r)(\partial_r\eta_{\phi})^* + (\partial_r\eta_{\phi})(\partial_{\phi}\eta_r)^* \right] \\ & + \frac{2\kappa_1}{r^2} \text{Re}\{3\eta_r^*\partial_{\phi}\eta_{\phi} - \eta_{\phi}^*\partial_{\phi}\eta_r\} \end{aligned} \quad (69)$$

Substitute in Eq. (66)

$$\begin{aligned} f_{\text{grad}}(m) \sim & \frac{\kappa_1}{r^2} (m \pm 1)^2 \left[3 \left(\eta_{\phi}^{(m)} \right)^2 + \left(\eta_r^{(m)} \right)^2 \right] \\ & + \frac{2\kappa_1}{r} (1 \pm m) \left[\eta_r^{(m)} \partial_r \eta_{\phi}^{(m)} - \eta_{\phi}^{(m)} \partial_r \eta_r^{(m)} \right] \\ & - \frac{8\kappa_1}{r^2} (1 \pm m) \eta_r^{(m)} \eta_{\phi}^{(m)} \end{aligned} \quad (70)$$

For the second line, the corresponding total free energy is zero as long as $\eta_i^{(m)}(r)$ is an even function with respect to $r = R + D/2$,

$$2\pi \int_R^{R+D} r dr \rightarrow \int_R^{R+D} \eta_i^{(m)} \partial_r \eta_j^{(m)} dr = 0 \quad (71)$$

Then we will have

$$\begin{aligned} F_{\pm}(m) - F_{\pm}(m-1) = & \int d^3r \frac{\kappa_1}{r^2} \left\{ (2m-1) \left[3 \left(\eta_{\phi}^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 \right] \right. \\ & \left. \pm 2 \left[3 \left(\eta_{\phi}^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 - 4\eta_r^{(0)}\eta_{\phi}^{(0)} \right] \right\} \end{aligned} \quad (72)$$

We denote the volume integral as $\langle \dots \rangle_V = \int d^3r (\dots)$. Then we have

$$\Omega_m^{\pm} = \frac{\hbar}{M} \frac{\left(m - \frac{1}{2} \right) \left\langle \frac{1}{r^2} \left[3 \left(\eta_{\phi}^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 \right] \right\rangle_V \pm \left\langle \frac{1}{r^2} \left[3 \left(\eta_{\phi}^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 - 4\eta_r^{(0)}\eta_{\phi}^{(0)} \right] \right\rangle_V}{\left\langle 3 \left(\eta_{\phi}^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 \right\rangle_V}$$

We denote $\langle \dots \rangle_r \equiv \int_R^{R+D} dr(\dots)$. When $R \gg D$, we can approximate $\langle \mathcal{O} \rangle_V \approx 2\pi R h \langle \mathcal{O} \rangle_r$ and $\langle \frac{1}{r^2} \mathcal{O} \rangle_V \approx \frac{2\pi h}{R} \langle \mathcal{O} \rangle_r$, where h is the z direction thickness, and we have

$$\Omega_m^\pm \approx \frac{\hbar}{MR^2} \left[\left(m - \frac{1}{2} \right) \pm \frac{\left\langle 3 \left(\eta_\phi^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 - 4\eta_r^{(0)} \eta_\phi^{(0)} \right\rangle_r}{\left\langle 3 \left(\eta_\phi^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 \right\rangle_r} \right] \quad (73)$$

If we approximate $\left\langle 3 \left(\eta_\phi^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 \right\rangle_r \approx 4\eta_0^2(D + b\xi)$, then

$$\Omega_m^\pm \approx \frac{\hbar}{MR^2} \left[\left(m - \frac{1}{2} \right) \pm \left(1 - \frac{\left\langle \eta_r^{(0)} \eta_\phi^{(0)} \right\rangle_r}{\eta_0^2(D + b\xi)} \right) \right] \quad (74)$$

If we assume $\eta_r^{(0)}$ and $\eta_\phi^{(0)}$ are even functions with respect to $r = R + D/2$, then we have

$$\left\langle \eta_r^{(0)} \eta_\phi^{(0)} \right\rangle_r = 2 \int_R^{R+D/2} \eta_r^{(0)} \eta_\phi^{(0)} dr \quad (75)$$

If we approximate $\eta_\phi^{(0)} \eta_r^{(0)} \approx \eta_0^2 \tanh(ax/\xi)$, where $x \equiv r - R$, then

$$\left\langle \eta_r^{(0)} \eta_\phi^{(0)} \right\rangle_r \approx 2\eta_0^2 \xi \int_0^{D/2} \tanh\left(\frac{ax}{\xi}\right) d\left(\frac{x}{\xi}\right) \quad (76)$$

$$= \frac{2}{a} \eta_0^2 \xi \ln(\cosh(ax/\xi)) \Big|_0^{D/2} \quad (77)$$

$$= \frac{2}{a} \eta_0^2 \xi \ln\left(\cosh\left(\frac{aD}{2\xi}\right)\right) \quad (78)$$

Note that $\cosh(x) = \frac{e^x + e^{-x}}{2} \approx e^x/2$ when $x \gg 1$. If we assume $D/\xi \gg 1$, then

$$\left\langle \eta_r^{(0)} \eta_\phi^{(0)} \right\rangle_r \approx \frac{2}{a} \eta_0^2 \xi \left(\frac{aD}{2\xi} - \ln(2) \right) = \eta_0^2 \left(D - \frac{2}{a} \ln(2) \xi \right) \quad (79)$$

which gives

$$\Omega_m^\pm \approx \frac{\hbar}{MR^2} \left[\left(m - \frac{1}{2} \right) \pm \left(\frac{2}{a} \ln(2) + b \right) \frac{\xi}{D} \right] \quad (80)$$

when $a = 1/3, b = 0$ and $D = 30\xi$, we have

$$\Omega_1^\pm \approx \frac{\hbar}{MR^2} \left(\frac{1}{2} \pm \frac{\ln(2)}{5} \right) \quad (81)$$

$$\Omega_1^\pm \approx \frac{\hbar}{MR^2} \left(\frac{1}{2} \pm 0.139 \right) \quad (82)$$

Exact numerical solution pf Eq. (73) gives

$$\Omega_1^\pm \approx \frac{\hbar}{MR^2} \left(\frac{1}{2} \pm 0.122 \right) \quad (83)$$

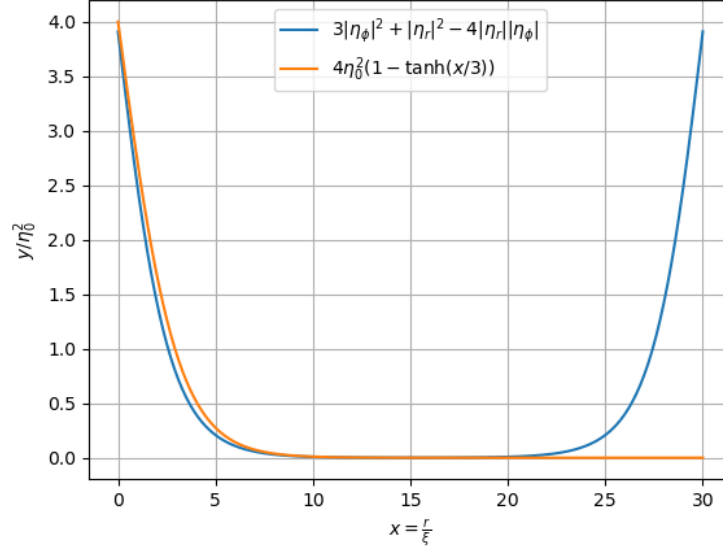


FIG. 1. Comparison of $y = 3 \left(\eta_{\phi}^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 - 4 \eta_r^{(0)} \eta_{\phi}^{(0)}$ and $y = \eta_0^2 \tanh(ax/\xi)$ for $a = 1/3$ and $D = 30\xi$.

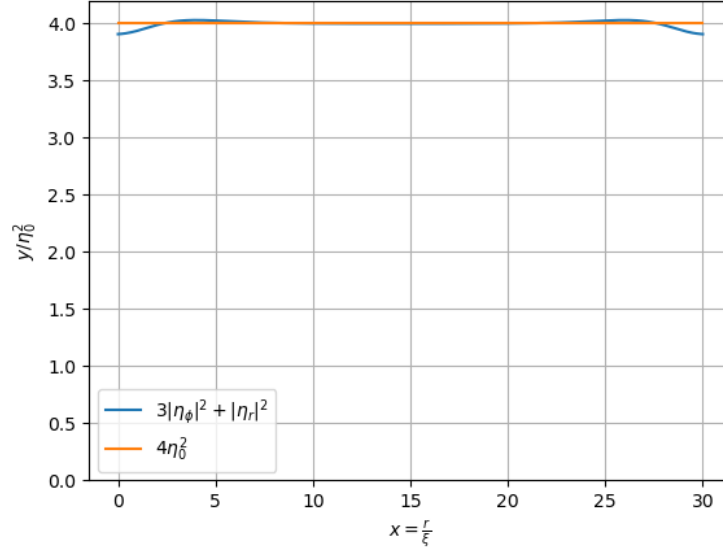


FIG. 2. Comparison of $y = 3 \left(\eta_{\phi}^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2$ and $y = 4\eta_0^2$ for $D = 30\xi$.

LONDON APPROXIMATION

We assume $\eta_i^{(m)}(r) = \eta_0$. The angular momentum increase is

$$L_{\pm}(m) - L_{\pm}(m-1) = \frac{8M}{\hbar} \kappa_1 \eta_0^2 \pi \left[(R+D)^2 - R^2 \right] h \quad (84)$$

The free energy increase is

$$F_{\pm}(m) - F_{\pm}(m-1) = 8\pi\kappa_1\eta_0^2 \ln\left(\frac{R+D}{R}\right)h(2m-1) \quad (85)$$

where h is the thickness of the annulus. The critical angular velocity is

$$\Omega_m^{\pm} = \Omega_c \frac{\xi}{D} \frac{\ln(1+D/R)}{2+D/R} (2m-1), \quad (86)$$

where $\Omega_c = v_c/R$, $v_c = \hbar/M\xi$, $\xi^2 = \kappa_1/|\alpha|$. Actually, all of the GL parameters drop out

$$\Omega_m^{\pm} = \frac{\hbar}{MRD} \frac{\ln(1+D/R)}{2+D/R} (2m-1), \quad (87)$$

When $R \gg D$, we have

$$\Omega_m^{\pm} = \frac{\hbar}{MR^2} \left(m - \frac{1}{2}\right) \quad (88)$$

i.e.

$$\Omega_1 = \frac{\hbar}{MR^2} \frac{1}{2} \quad \text{for } m=0 \rightarrow m=1 \quad (89)$$

$$\Omega_2 = \frac{\hbar}{MR^2} \frac{3}{2} \quad \text{for } m=1 \rightarrow m=2 \quad (90)$$

$$\Omega_3 = \frac{\hbar}{MR^2} \frac{5}{2} \quad \text{for } m=2 \rightarrow m=3 \quad (91)$$

$$\dots \quad (92)$$

NUMERICAL SOLUTION

From the bulk free energy term we can get the bulk order parameter $\eta_0 = \frac{1}{2}\sqrt{\frac{|\alpha|}{\beta_1}}$. We also define the coherence length $\xi = \sqrt{\kappa_1/|\alpha|}$.

In weak-coupling limit, the GL equations in cylindrical basis are

$$\begin{aligned} \alpha\eta_k + 2\beta_1\eta_i\eta_i^*\eta_k + \beta_1\eta_i^2\eta_k^* - \kappa_1(\partial_i^2\eta_k + \partial_k\partial_i\eta_i + \partial_j\partial_k\eta_j) \\ - \frac{\kappa_1}{r} \left[3\frac{\partial_\phi}{r} (\eta_r\delta_{k\phi} - \eta_\phi\delta_{kr}) + \partial_r\eta_k + 2\partial_k\eta_r \right] \\ + \frac{\kappa_1}{r^2} [\eta_k + 2\eta_r\delta_{kr}] = 0. \end{aligned} \quad (93)$$

The dimensionless GL equations in cylindrical basis are

$$\begin{aligned} -\eta_k + \frac{1}{2}\eta_i\eta_i^*\eta_k + \frac{1}{4}\eta_i^2\eta_k^* - (\partial_i^2\eta_k + \partial_k\partial_i\eta_i + \partial_j\partial_k\eta_j) \\ - \frac{1}{r} \left[3\frac{\partial_\phi}{r} (\eta_r\delta_{k\phi} - \eta_\phi\delta_{kr}) + \partial_r\eta_k + 2\partial_k\eta_r \right] \\ + \frac{1}{r^2} [\eta_k + 2\eta_r\delta_{kr}] = 0. \end{aligned} \quad (94)$$

The r component of the GL equations is

$$\begin{aligned}
-\eta_r + \frac{1}{2}\eta_i\eta_i^*\eta_r + \frac{1}{4}\eta_i^2\eta_r^* - (\partial_i^2\eta_r + \partial_r\partial_i\eta_i + \partial_j\partial_r\eta_j) - \frac{1}{r}\left[-3\frac{\partial_\phi}{r}\eta_\phi + 3\partial_r\eta_r\right] + \frac{3}{r^2}\eta_r &= 0 \\
-\eta_r + \frac{1}{2}\eta_i\eta_i^*\eta_r + \frac{1}{4}\eta_i^2\eta_r^* - (\partial_i^2\eta_r + \partial_r\partial_i\eta_i + \partial_j\partial_r\eta_j) + 3\left[\frac{1}{r^2}\partial_\phi\eta_\phi - \frac{1}{r}\partial_r\eta_r + \frac{\eta_r}{r^2}\right] &= 0 \\
-\eta_r + \frac{1}{2}\eta_i\eta_i^*\eta_r + \frac{1}{4}\eta_i^2\eta_r^* - 3\partial_r^2\eta_r - \frac{1}{r^2}\partial_\phi^2\eta_r - \frac{2}{r}\partial_r\partial_\phi\eta_\phi + \frac{4}{r^2}\partial_\phi\eta_\phi - \frac{3}{r}\partial_r\eta_r + \frac{3}{r^2}\eta_r &= 0
\end{aligned}$$

The ϕ component of the GL equations is

$$\begin{aligned}
-\eta_\phi + \frac{1}{2}\eta_i\eta_i^*\eta_\phi + \frac{1}{4}\eta_i^2\eta_\phi^* - \left(\partial_i^2\eta_\phi + \frac{\partial_\phi}{r}\partial_i\eta_i + \partial_j\frac{\partial_\phi}{r}\eta_j\right) - \frac{1}{r}\left[5\frac{\partial_\phi}{r}\eta_r + \partial_r\eta_\phi\right] + \frac{1}{r^2}\eta_\phi &= 0 \\
-\eta_\phi + \frac{1}{2}\eta_i\eta_i^*\eta_\phi + \frac{1}{4}\eta_i^2\eta_\phi^* - 3\frac{1}{r^2}\partial_\phi^2\eta_\phi - \partial_r^2\eta_\phi - \frac{2}{r}\partial_r\partial_\phi\eta_r - \frac{4}{r^2}\partial_\phi\eta_r - \frac{1}{r}\partial_r\eta_\phi + \frac{1}{r^2}\eta_\phi &= 0
\end{aligned}$$

For the uniform flow approximation,

$$\boldsymbol{\eta}(r, \phi) = \left(\eta_r^{(n)}(r), \eta_\phi^{(n)}(r)\right) e^{in\phi} \quad (95)$$

which means that $\partial_\phi \rightarrow in$. We have

$$\begin{aligned}
-\eta_r + \frac{1}{2}\eta_i\eta_i^*\eta_r + \frac{1}{4}\eta_i^2\eta_r^* - 3\partial_r^2\eta_r + \frac{n^2}{r^2}\eta_r - \frac{2in}{r}\partial_r\eta_\phi + \frac{4in}{r^2}\eta_\phi - \frac{3}{r}\partial_r\eta_r + \frac{3}{r^2}\eta_r &= 0 \\
-\eta_\phi + \frac{1}{2}\eta_i\eta_i^*\eta_\phi + \frac{1}{4}\eta_i^2\eta_\phi^* + \frac{3n^2}{r^2}\eta_\phi - \partial_r^2\eta_\phi - \frac{2in}{r}\partial_r\eta_r - \frac{4in}{r^2}\eta_r - \frac{1}{r}\partial_r\eta_\phi + \frac{1}{r^2}\eta_\phi &= 0
\end{aligned}$$

We ignore the (n) superscript for the sake of simplicity. The solutions of $n = 1$ may corresponds to the $(p_x + ip_y, m = 0)$ state, or the $(p_x - ip_y, m = 2)$ state.

BOUNDARY CONDITIONS

For boundary conditions, we have $\eta_r = 0$, and $\partial_r\eta_\phi = \frac{\eta_\phi}{r}$. We can derive this by requiring the sum of all the surface terms to vanish when deriving the Euler-Lagrange equations. Take Eq. (32) for example,

$$\text{surface term} = \iint \partial_i \left((\partial_i\eta_j)\eta_j^* r \right) d\phi dr \quad (96)$$

$$= \iint \left[\partial_r \left(r\eta_j^*(\partial_r\eta_j) \right) + \frac{\partial_\phi}{r} \left(r\eta_j^* \frac{\partial_\phi}{r} \eta_j \right) \right] d\phi dr \quad (97)$$

$$= \iint \left[\frac{1}{r} \partial_r \left(r\eta_j^*(\partial_r\eta_j) \right) + \frac{\partial_\phi}{r} \left(\eta_j^* \frac{\partial_\phi}{r} \eta_j \right) \right] r d\phi dr \quad (98)$$

$$= \iint \boldsymbol{\nabla} \cdot \mathbf{F} dA = \oint \mathbf{F} \cdot \hat{\mathbf{n}} ds \quad (99)$$

where $\mathbf{F} = (F_r, F_\phi) = \left(\eta_j^* \partial_r \eta_j, \eta_j^* \frac{\partial_\phi}{r} \eta_j \right)$. In an annulus or a cylinder, the boundary normal vector $\hat{\mathbf{n}} = \pm \hat{\mathbf{r}}$, and on boundaries we have $\eta_r = 0$, $F_r = \eta_\phi^* \partial_r \eta_\phi$, and $ds = r d\phi$, which gives

$$\text{surface term} = \oint_+ \eta_\phi^* (\partial_r \eta_\phi) r d\phi - \oint_- \eta_\phi^* (\partial_r \eta_\phi) r d\phi \quad (100)$$

The only remaining nonzero surface term is the κ_3 term in the second line of Eq. (30), which correspond to $F_r = -\frac{\eta_\phi^* \eta_\phi}{r}$. In weak-coupling limit, when $\partial_r \eta_\phi = \frac{\eta_\phi}{r}$, these two surface terms cancel each other.

η_\pm BASIS

In bulk, for the two time-reversed ground states, we have

$$(\eta_x, \eta_y) = (1, \pm i) \eta_0 \quad (101)$$

$$(\eta_r, \eta_\phi) = e^{\pm i\phi} (1, \pm i) \eta_0 \quad (102)$$

Now we want to define a new basis η_\pm such that for the $p + ip$ state, we have

$$(\eta_+, \eta_-) = (\eta_0, 0) \quad (103)$$

and for the $p - ip$ state, we have

$$(\eta_+, \eta_-) = (0, \eta_0) \quad (104)$$

The corresponding transformation is

$$\begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{-i\phi} & -ie^{-i\phi} \\ e^{i\phi} & ie^{i\phi} \end{pmatrix} \begin{pmatrix} \eta_r \\ \eta_\phi \end{pmatrix} \quad (105)$$

Under the uniform flow approximation, for winding number n , we have

$$\boldsymbol{\eta}(r, \phi) = \left(\eta_r^{(n)}(r), \eta_\phi^{(n)}(r) \right) e^{in\phi} \quad (106)$$

and

$$\eta_+(r, \phi) = \frac{1}{2} \left(\eta_r^{(n)}(r) - i \eta_\phi^{(n)}(r) \right) e^{i(n-1)\phi} \equiv \eta_+^{(m)}(r) e^{im\phi} \quad (107)$$

$$\eta_-(r, \phi) = \frac{1}{2} \left(\eta_r^{(n)}(r) + i \eta_\phi^{(n)}(r) \right) e^{i(n+1)\phi} \equiv \eta_-^{(p)}(r) e^{ip\phi} \quad (108)$$

where $m = n - 1$ and $p = n + 1$. The 4π phase winding difference, i.e. $p - m = 2$ can be understood from an angular momentum conservation perspective.

LARGE FLOW

When the flow field is large, we denote the superfluid flow field at the inner radius as $v_s = \frac{\hbar}{MR} \partial_\phi \theta$, and

$$\boldsymbol{\eta}(r, \phi, v_s) = \boldsymbol{\eta}^{(n)}(r) e^{in\phi} = \boldsymbol{\eta}(r, v_s) e^{i(\partial_\phi \theta)\phi} = \boldsymbol{\eta}(r, v_s) e^{i \frac{v_s R}{v_c \xi} \phi} \quad (109)$$

where the critical flow field $v_c = \hbar/M\xi$ and $\xi^2 = \kappa_1/|\alpha|$. We also have $n = \frac{v_s}{v_c} \frac{R}{\xi}$.

CARTESIAN BASIS

In Cartesian basis, weak-coupling limit dimensionless GL equations are

$$-\eta_k + \frac{1}{2} \eta_i \eta_i^* \eta_k + \frac{1}{4} \eta_i^2 \eta_k^* - (\partial_i^2 \eta_k + \partial_k \partial_i \eta_i + \partial_j \partial_k \eta_j) = 0 \quad (110)$$

The x component of the GL equations is

$$-\eta_x + \frac{3}{4} \eta_x^2 \eta_x^* + \frac{1}{2} \eta_y \eta_y^* \eta_x + \frac{1}{4} \eta_y^2 \eta_x^* - 3\partial_x^2 \eta_x - \partial_y^2 \eta_x - 2\partial_y \partial_x \eta_y = 0 \quad (111)$$

The y component of the GL equations is

$$-\eta_y + \frac{3}{4} \eta_y^2 \eta_y^* + \frac{1}{2} \eta_x \eta_x^* \eta_y + \frac{1}{4} \eta_x^2 \eta_y^* - 3\partial_y^2 \eta_y - \partial_x^2 \eta_y - 2\partial_x \partial_y \eta_x = 0 \quad (112)$$