

MEAN-FIELD THEORY

The mean-field Nambu-Gor'kov Hamiltonian is

$$H = \frac{1}{2} \iint d^3\mathbf{r} d^3\mathbf{r}' \widehat{\Psi}^\dagger(\mathbf{r}') \begin{pmatrix} \xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\hat{1} & \widehat{\Delta}(\mathbf{r}, \mathbf{r}') \\ \widehat{\Delta}^\dagger(\mathbf{r}, \mathbf{r}') & -\xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\hat{1} \end{pmatrix} \widehat{\Psi}(\mathbf{r}) \quad (1)$$

$$\equiv \frac{1}{2} \iint d^3\mathbf{r} d^3\mathbf{r}' \widehat{\Psi}^\dagger(\mathbf{r}') \widehat{\tau}_3 \left(\xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') + \widehat{\Delta}(\mathbf{r}, \mathbf{r}') \right) \widehat{\Psi}(\mathbf{r}). \quad (2)$$

where

$$\widehat{\Delta}(\mathbf{r}, \mathbf{r}') = \begin{pmatrix} 0 & \widehat{\Delta}(\mathbf{r}, \mathbf{r}') \\ -\widehat{\Delta}^\dagger(\mathbf{r}, \mathbf{r}') & 0 \end{pmatrix} \quad (3)$$

When the system is homogeneous, the two-point gap function only depends on the relative coordinate, i.e., $\widehat{\Delta}(\mathbf{r}, \mathbf{r}') = \widehat{\Delta}(\mathbf{r} - \mathbf{r}')$. In this case, we can transform to momentum space, and the Hamiltonian becomes

$$H = \frac{1}{2} \sum_{\mathbf{p}} \widehat{\Psi}_{\mathbf{p}}^\dagger \widehat{\tau}_3 (\xi_{\mathbf{p}} + \widehat{\Delta}(\mathbf{p})) \widehat{\Psi}_{\mathbf{p}}. \quad (4)$$

where $\widehat{\Psi}_{\mathbf{p}}^\dagger = (c_{\mathbf{p}\uparrow}^\dagger, c_{\mathbf{p}\downarrow}^\dagger, c_{-\mathbf{p}\uparrow}, c_{-\mathbf{p}\downarrow})$.

FINITE TEMPERATURE

The imaginary time Green's function is

$$\widehat{G}(\mathbf{r}\tau, \mathbf{r}'\tau') = \widehat{G}(\mathbf{r}, \mathbf{r}'; \tau - \tau') = - \left\langle T_\tau \widehat{\Psi}(\mathbf{r}, \tau) \widehat{\Psi}^\dagger(\mathbf{r}', \tau') \right\rangle \equiv \begin{pmatrix} \widehat{G} & \widehat{F} \\ \underline{\widehat{F}} & \underline{\widehat{G}} \end{pmatrix} \quad (5)$$

The unperturbed Matsubara Green's function is

$$\widehat{G}_0(\mathbf{p}, i\epsilon_n) = (i\epsilon_n - \widehat{\tau}_3 \xi_{\mathbf{p}})^{-1} = -\frac{i\epsilon_n + \widehat{\tau}_3 \xi_{\mathbf{p}}}{\epsilon_n^2 + \xi_{\mathbf{p}}^2}. \quad (6)$$

In bulk, the Gorkov Green's function satisfies

$$\widehat{G}^{-1}(\mathbf{p}, i\epsilon_n) = \widehat{G}_0^{-1}(\mathbf{p}, i\epsilon_n) - \widehat{\tau}_3 \widehat{\Delta}(\mathbf{p}) = i\epsilon_n - \widehat{\tau}_3 \xi_{\mathbf{p}} - \widehat{\tau}_3 \widehat{\Delta}(\mathbf{p}). \quad (7)$$

which gives

$$\widehat{G}(\mathbf{p}, i\epsilon_n) = -\frac{i\epsilon_n + \widehat{\tau}_3 \xi_{\mathbf{p}} + \widehat{\tau}_3 \widehat{\Delta}(\mathbf{p})}{\epsilon_n^2 + \xi_{\mathbf{p}}^2 + |\Delta|^2} \quad (8)$$

The bulk quasiclassical Green's function is

$$\widehat{g}(\mathbf{p}, i\epsilon_n) = \int d\xi_p \widehat{G}(\mathbf{p}, i\epsilon_n) \widehat{\tau}_3 = -\pi \frac{i\epsilon_n \widehat{\tau}_3 - \widehat{\Delta}(\mathbf{p})}{\sqrt{|\Delta|^2 + \epsilon_n^2}} = \frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}} \begin{pmatrix} -i\epsilon_n & \widehat{\Delta} \\ -\widehat{\Delta}^\dagger & i\epsilon_n \end{pmatrix} \equiv \begin{pmatrix} \widehat{g} & \widehat{f} \\ \widehat{f} & \widehat{g} \end{pmatrix} \quad (9)$$

and it satisfies the normalization condition

$$\widehat{g}(\mathbf{p}, i\epsilon_n)^2 = -\pi^2 \widehat{1}. \quad (10)$$

The bulk quasiclassical Green's function satisfies

$$[i\epsilon_n \widehat{\tau}_3 - \widehat{\Delta}, \widehat{g}] = 0 \quad (11)$$

But when near the boundary or impurities, the quasiclassical Green's function $\widehat{g}(\mathbf{R}, \mathbf{p}; \epsilon)$ and the gap function $\widehat{\Delta}(\mathbf{R}, \mathbf{p})$ are inhomogeneous, and are described by the Eilenberger equation and gap equation.

$$[\epsilon \widehat{\tau}_3 - \widehat{\Delta}(\mathbf{R}, \mathbf{p}), \widehat{g}(\mathbf{R}, \mathbf{p}; \epsilon)] + i\hbar \mathbf{v}_p \cdot \nabla \widehat{g}(\mathbf{R}, \mathbf{p}; \epsilon) = 0 \quad (12)$$

and

$$\widehat{\Delta}(\mathbf{R}, \mathbf{p}) = T \sum_{\epsilon_n=-\omega_c}^{\omega_c} \left\langle v(\mathbf{p}, \mathbf{p}') \widehat{f}(\mathbf{R}, \mathbf{p}', \epsilon_n) \right\rangle_{p'} \quad (13)$$

Here $\mathbf{R} = \frac{1}{2}(\mathbf{r} + \mathbf{r}')$ is the center of the pair. In bulk, we have

$$\Delta(\mathbf{p}) = T \sum_{\epsilon_n=-\omega_c}^{\omega_c} \left\langle v(\mathbf{p}, \mathbf{p}') \frac{\pi \Delta(\mathbf{p}')}{\sqrt{|\Delta|^2 + \epsilon_n^2}} \right\rangle_{p'} \quad (14)$$

We have the Digamma function

$$K(T) = T \sum_{\epsilon_n=-\omega_c}^{\omega_c} \frac{\pi}{|\epsilon_n|} \approx \ln \left(1.13 \frac{\omega_c}{T} \right) \quad (15)$$

In bulk at T_c , we have

$$\Delta(\mathbf{p}) = K(T_c) \langle v(\mathbf{p}, \mathbf{p}') \Delta(\mathbf{p}') \rangle_{p'} \quad (16)$$

and we have

$$K(T_c) - K(T) = \ln(T/T_c) = \frac{\Delta(\mathbf{p}'')}{\langle v(\mathbf{p}'', \mathbf{p}') \Delta(\mathbf{p}') \rangle_{p'}} \Big|_{T_c} - T \sum_{\epsilon_n=-\omega_c}^{\omega_c} \frac{\pi}{|\epsilon_n|} \quad (17)$$

Substitute in Eq. (13), we get

$$\ln \frac{T}{T_c} = T \sum_{\epsilon_n=-\infty}^{\infty} \left[\frac{\langle v(\mathbf{p}, \mathbf{p}') f(\mathbf{R}, \mathbf{p}', \epsilon_n) \rangle_{p'} / \Delta(\mathbf{R}, \mathbf{p})}{\langle v(\mathbf{p}'', \mathbf{p}') \Delta(\mathbf{p}') \rangle_{p'} / \Delta(\mathbf{p}'')|_{T_c}} - \frac{\pi}{|\epsilon_n|} \right] \quad (18)$$

If you want temperature dependence of bulk gap, you can substitute in Eq. (14) to get

$$\ln \frac{T}{T_c} = T \sum_{\epsilon_n=-\infty}^{\infty} \left[\frac{\pi \left\langle v(\mathbf{p}, \mathbf{p}') \Delta(\mathbf{p}') / \Delta(\mathbf{p}) \sqrt{|\Delta|^2 + \epsilon_n^2} \right\rangle_{p'}}{\langle v(\mathbf{p}'', \mathbf{p}') \Delta(\mathbf{p}') \rangle_{p'} / \Delta(\mathbf{p}'')|_{T_c}} - \frac{\pi}{|\epsilon_n|} \right] \quad (19)$$

RICCATI PARAMETERIZATIONS

The Riccati parameterization of the Green's function is

$$\hat{g} = -i\pi \hat{N} \begin{pmatrix} 1 + \hat{a}\hat{a} & 2\hat{a} \\ -2\hat{a} & -1 - \hat{a}\hat{a} \end{pmatrix} \quad (20)$$

where \hat{N} is

$$\hat{N} = \begin{pmatrix} (1 - \hat{a}\hat{a})^{-1} & 0 \\ 0 & (1 - \hat{a}\hat{a})^{-1} \end{pmatrix} \quad (21)$$

The matrix Riccati equation is

$$i\hbar \mathbf{v}_p \cdot \nabla \hat{a} + 2\epsilon \hat{a} + \hat{a} \hat{\Delta}^\dagger \hat{a} + \hat{\Delta} = 0 \quad (22)$$

$$i\hbar \mathbf{v}_p \cdot \nabla \hat{a} - 2\epsilon \hat{a} - \hat{a} \hat{\Delta} \hat{a} - \hat{\Delta}^\dagger = 0 \quad (23)$$

³HE-A

For ³He-A, the spin structure is $\hat{\mathbf{d}} \cdot (i\vec{\sigma}\hat{\sigma}_y) = \hat{\sigma}_x$, and we have two components of the gap function,

$$\hat{\Delta}(\mathbf{r}, \mathbf{p}) = \hat{\sigma}_x (\Delta_1(\mathbf{r}) p_x + \Delta_2(\mathbf{r}) p_y) \quad (24)$$

In bulk we have $\hat{\Delta}(\mathbf{r}, \mathbf{p}) = \hat{\sigma}_x \Delta(p_x + ip_y) = \hat{\sigma}_x \Delta e^{i\phi_p}$. The interaction is $v(\mathbf{p}, \mathbf{p}') = 3v_0 \hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'$, and the gap-equation reduces to

$$\ln \frac{T}{T_c} = T \sum_{\epsilon_n=-\infty}^{\infty} \left[\frac{2 \langle \mathbf{p}_{x,y} \cdot \mathbf{p}' f(\mathbf{r}, \mathbf{p}', \epsilon_n) \rangle_{p'}}{\Delta_{1,2}(\mathbf{r})} - \frac{\pi}{|\epsilon_n|} \right] \quad (25)$$

The 2d bulk gap equation ($|\Delta|^2$ does not depend on \mathbf{p}) is

$$\ln \frac{T}{T_c} = T \sum_{\epsilon_n=-\infty}^{\infty} \left[\frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}} - \frac{\pi}{|\epsilon_n|} \right] \quad (26)$$

S-WAVE PAIRING

For s-wave pairing, the spin structure is $i\hat{\sigma}_y$, and we have

$$\hat{\Delta}(\mathbf{r}, \mathbf{p}) = i\hat{\sigma}_y \Delta(\mathbf{r}) \quad (27)$$

In bulk we have $\hat{\Delta}(\mathbf{r}, \mathbf{p}) = i\hat{\sigma}_y \Delta$. The interaction is $v(\mathbf{p}, \mathbf{p}') = v_s$, and the gap-equation reduces to

$$\ln \frac{T}{T_c} = T \sum_{\epsilon_n=-\infty}^{\infty} \left[\frac{f(\mathbf{r}, \epsilon_n)}{\Delta(\mathbf{r})} - \frac{\pi}{|\epsilon_n|} \right] \quad (28)$$

³HE-A EDGE GAP PROFILE

We assume translational invariance along the edge, i.e. the y direction. We also assume the gap to be real(imaginary) along the $p_x(p_y)$ direction.

$$\hat{\Delta}(x, \mathbf{p}) = \hat{\sigma}_x (\Delta_1(x)p_x + i\Delta_2(x)p_y) \quad (29)$$

Here $\Delta_1(x)$ and $\Delta_2(x)$ are real functions, and we have

$$\hat{\Delta}(\mathbf{r}, \mathbf{p}) = \begin{pmatrix} 0 & \hat{\sigma}_x (\Delta_1(x)p_x + i\Delta_2(x)p_y) \\ -\hat{\sigma}_x (\Delta_1(x)p_x - i\Delta_2(x)p_y) & 0 \end{pmatrix} \quad (30)$$

$$= i\hat{\sigma}_x (\Delta_2 p_y \hat{\tau}_1 + \Delta_1 p_x \hat{\tau}_2) \quad (31)$$

In this case, we can also write the anomalous Green's function as

$$\hat{f}(\mathbf{r}, \mathbf{p}, \epsilon_n) = \hat{\sigma}_x \left(f_1(\mathbf{r}, \mathbf{p}, \epsilon_n) + i f_2(\mathbf{r}, \mathbf{p}, \epsilon_n) \right) \quad (32)$$

where $f_{1,2}$ are real functions, and we have

$$\hat{g}(\mathbf{r}, \mathbf{p}, \epsilon_n) = \hat{g} \hat{\tau}_3 + i\hat{\sigma}_x (f_2 \hat{\tau}_1 + f_1 \hat{\tau}_2) \quad (33)$$

In this case, we can even write the gap equation as

$$\Delta_{1,2}(x) = T \sum_{\epsilon_n=-\omega_c}^{\omega_c} \langle v(\mathbf{p}_{x,y}, \mathbf{p}') f_{1,2}(x, \mathbf{p}', \epsilon_n) \rangle_{p'} \quad (34)$$

which becomes

$$\ln \frac{T}{T_c} = T \sum_{\epsilon_n=-\infty}^{\infty} \left[\frac{2 \langle \mathbf{p}_{x,y} \cdot \mathbf{p}' f_{1,2}(x, \mathbf{p}', \epsilon_n) \rangle_{p'}}{\Delta_{1,2}(x)} - \frac{\pi}{|\epsilon_n|} \right] \quad (35)$$

NUMERICAL SOLUTION

The iteration of the gap function is

$$\Delta_{1,2}(x) = \frac{t \sum_n 2 \langle \mathbf{p}_{x,y} \cdot \mathbf{p}' f_{1,2}(x, \mathbf{p}', n) \rangle_{p'}}{\ln t + \sum_n 1/|2n+1|} \quad (36)$$

The (1, 1) and (1, 2) elements of the Eilenberger equation are

$$i\epsilon_n \hat{g} + \hat{\Delta} \hat{f}^\dagger + i\hbar \mathbf{v}_p \cdot \nabla \hat{g} = 0 \quad (37)$$

$$i\epsilon_n \hat{f} + \hat{\Delta} \hat{g} + i\hbar \mathbf{v}_p \cdot \nabla \hat{f} = 0 \quad (38)$$

Note that $\hat{\Delta}$ and \hat{f} has the same spin structure, so we can substitute in \hat{g} and \hat{f} , divide by T_c , and we get

$$i(2n+1)\pi t g(\mathbf{r}, \hat{\mathbf{p}}, n) + \delta(\mathbf{r}, \hat{\mathbf{p}}) f^*(\mathbf{r}, \hat{\mathbf{p}}, n) + i(\hat{\mathbf{p}}_x \partial_x + \hat{\mathbf{p}}_y \partial_y) g(\mathbf{r}, \hat{\mathbf{p}}, n) = 0 \quad (39)$$

$$i(2n+1)\pi t f(\mathbf{r}, \hat{\mathbf{p}}, n) + \delta(\mathbf{r}, \hat{\mathbf{p}}) g(\mathbf{r}, \hat{\mathbf{p}}, n) + i(\hat{\mathbf{p}}_x \partial_x + \hat{\mathbf{p}}_y \partial_y) f(\mathbf{r}, \hat{\mathbf{p}}, n) = 0 \quad (40)$$

where $t = T/T_c$, $\delta = \Delta/T_c$, $\xi = \hbar v_f/T_c$ and $\vec{\partial} = \xi \nabla$.

We first assume an initial guess for the gap function $\delta(\mathbf{r}, \hat{\mathbf{p}})$. For each trajectory $\hat{\mathbf{p}}$ along which the quasiparticle propagates, we can solve the Eilenberger equation to get the Green's function $g(\mathbf{r}, \hat{\mathbf{p}}, n)$ and the anomalous Green's function $f(\mathbf{r}, \hat{\mathbf{p}}, n)$. Then we iterate to update the gap function $\delta(\mathbf{r}, \hat{\mathbf{p}})$ by solving the gap equation. Repeat until convergence.