INTRODUCTION

In cylindrical coordinates, we have

$$\begin{pmatrix} \eta_r \\ \eta_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \tag{1}$$

and

$$\begin{pmatrix} \partial_r \\ \frac{1}{r}\partial_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \tag{2}$$

In bulk, for the two time-reversed ground states,

$$(\eta_x^{\text{bulk}}, \eta_y^{\text{bulk}}) = \eta_0(1, \pm i) \tag{3}$$

$$(\eta_r^{\text{bulk}}, \eta_\phi^{\text{bulk}}) = e^{\pm i\phi}\eta_0(1, \pm i) \tag{4}$$

In GL theory, the order parameter is determined by minimizing the free energy functional

$$F[\boldsymbol{\eta}] = \int \mathrm{d}^3 r \, (f_{\text{bulk}} + f_{\text{grad}}) \tag{5}$$

where

$$f_{\text{bulk}}[\boldsymbol{\eta}] = \alpha \boldsymbol{\eta} \cdot \boldsymbol{\eta}^* + \beta_1 (\boldsymbol{\eta} \cdot \boldsymbol{\eta}^*)^2 + \beta_2 |\boldsymbol{\eta} \cdot \boldsymbol{\eta}|^2, \tag{6}$$

and

$$f_{\text{grad}}^{\text{Car}}[\boldsymbol{\eta}] = \kappa_1 (\partial_i \eta_j)(\partial_i \eta_j)^* + \kappa_2 (\partial_i \eta_i)(\partial_j \eta_j)^* + \kappa_3 (\partial_i \eta_j)(\partial_j \eta_i)^* \tag{7}$$

In cylindrical basis, we have $\partial_i^{\text{Car}} = R_{ik}\partial_k$ and $\eta_j^{\text{Car}} = R_{jl}\eta_l$, where R_{ij} is the 2d rotational matrix. The form of the bulk free energy term doesn't change. The gradient term becomes

$$\begin{aligned} f_{\text{grad}}^{\text{Cyl}}[\boldsymbol{\eta}] =& \kappa_1 (\partial_i \eta_j)(\partial_i \eta_j)^* + \kappa_2 (\partial_i \eta_i)(\partial_j \eta_j)^* + \kappa_3 (\partial_i \eta_j)(\partial_j \eta_i)^* \\ &+ \frac{2}{r} \operatorname{Re}\left\{ \kappa_1 \left(\eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \frac{\partial_\phi}{r} \eta_r \right) + \kappa_2 \eta_r^* \partial_j \eta_j + \kappa_3 \left(\eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \partial_r \eta_\phi \right) \right\} \\ &+ \frac{1}{r^2} \left[\kappa_1 \eta_j^* \eta_j + (\kappa_2 + \kappa_3) \eta_r^* \eta_r \right]. \end{aligned} \tag{8}$$

Take the κ_1 term for example, note that $R_{ij}R_{ik} = R^T_{ji}R_{ik} = \delta_{jk}$

$$
\begin{aligned}
&(\partial_i^{\text{Car}}\eta_j^{\text{Car}})(\partial_i^{\text{Car}}\eta_j^{\text{Car}})^* && (9)\\
=&\left[R_{ik}\partial_k(R_{jl}\eta_l)\right]\left[R_{im}\partial_m(R_{jn}\eta_n^*)\right] && (10)\\
=&\left[\partial_k(R_{jl}\eta_l)\right]\left[\partial_k(R_{jn}\eta_n^*)\right] && (11)\\
=&\left[(\partial_k R_{jl})\eta_l + R_{jl}(\partial_k\eta_l)\right]\left[(\partial_k R_{jn})\eta_n^* + R_{jn}(\partial_k\eta_n^*)\right] && (12)\\
=&(\partial_k R_{jl})\eta_l(\partial_k R_{jn})\eta_n^* + (\partial_k R_{jl})\eta_l R_{jn}(\partial_k\eta_n^*) + R_{jl}(\partial_k\eta_l)(\partial_k R_{jn})\eta_n^* + R_{jl}R_{jn}(\partial_k\eta_l)(\partial_k\eta_n^*)
\end{aligned}
$$

Note that $\partial_r R(\phi) = 0$, $\partial_\phi R(\phi) = R(\phi)R(\pi/2) = R(\pi/2)R(\phi)$. Finally we have

$$
\begin{aligned}
&(\partial_i^{\text{Car}}\eta_j^{\text{Car}})(\partial_i^{\text{Car}}\eta_j^{\text{Car}})^* && (13)\\
=&\frac{1}{r^2}\eta_l\eta_l^* + \frac{1}{r}R_{nl}^{\pi/2}\eta_l\left(\frac{\partial_\phi}{r}\eta_n^*\right) + \frac{1}{r}R_{ln}^{\pi/2}\eta_n^*\left(\frac{\partial_\phi}{r}\eta_l\right) + (\partial_k\eta_l)(\partial_k\eta_l)^* && (14)\\
=&\frac{1}{r^2}\eta_l\eta_l^* + \frac{2}{r}\,\mathrm{Re}\left\{R_{ln}^{\pi/2}\eta_n^*\left(\frac{\partial_\phi}{r}\eta_l\right)\right\} + (\partial_k\eta_l)(\partial_k\eta_l)^* && (15)\\
=&\frac{1}{r^2}\eta_l\eta_l^* + \frac{2}{r}\,\mathrm{Re}\left\{\eta_r^*\frac{\partial_\phi}{r}\eta_\phi - \eta_\phi^*\frac{\partial_\phi}{r}\eta_r\right\} + (\partial_k\eta_l)(\partial_k\eta_l)^* && (16)
\end{aligned}
$$

Similarly, for the κ_2 terms, we have

$$
\begin{aligned}
&(\partial_i^{\text{Car}}\eta_i^{\text{Car}})(\partial_j^{\text{Car}}\eta_j^{\text{Car}})^* && (17)\\
=&\left[R_{ik}\partial_k(R_{il}\eta_l)\right]\left[R_{jm}\partial_m(R_{jn}\eta_n^*)\right] && (18)\\
=&\left[R_{ik}(\partial_k R_{il})\eta_l + R_{ik}R_{il}(\partial_k\eta_l)\right]\left[R_{jm}(\partial_m R_{jn})\eta_n^* + R_{jm}R_{jn}(\partial_m\eta_n^*)\right] && (19)\\
=&\left[\frac{1}{r}\eta_r + \partial_k\eta_k\right]\left[\frac{1}{r}\eta_r^* + \partial_m\eta_m^*\right] && (20)\\
=&\frac{1}{r^2}\eta_r\eta_r^* + \frac{2}{r}\,\mathrm{Re}\{\eta_r^*\partial_k\eta_k\} + (\partial_k\eta_k)(\partial_m\eta_m)^* && (21)
\end{aligned}
$$

For the κ_3 terms, we have

$$
\begin{aligned}
&(\partial_i^{\mathrm{Car}}\eta_j^{\mathrm{Car}})(\partial_j^{\mathrm{Car}}\eta_i^{\mathrm{Car}})^* && (22)\\
=&\Big[R_{ik}\partial_k(R_{jl}\eta_l)\Big]\Big[R_{jm}\partial_m(R_{in}\eta_n^*)\Big] && (23)\\
=&\Big[R_{ik}(\partial_k R_{jl})\eta_l + R_{ik}R_{jl}(\partial_k\eta_l)\Big]\Big[R_{jm}(\partial_m R_{in})\eta_n^* + R_{jm}R_{in}(\partial_m\eta_n^*)\Big] && (24)\\
=&R_{ik}(\partial_k R_{jl})\eta_l R_{jm}(\partial_m R_{in})\eta_n^* + R_{ik}(\partial_k R_{jl})\eta_l R_{jm}R_{in}(\partial_m\eta_n^*)\\
&+ R_{ik}R_{jl}(\partial_k\eta_l)R_{jm}(\partial_m R_{in})\eta_n^* + R_{ik}R_{jl}(\partial_k\eta_l)R_{jm}R_{in}(\partial_m\eta_n^*) && (25)\\
=&\frac{1}{r^2}R^{\pi/2}_{\phi l}\eta_l R^{\pi/2}_{\phi n}\eta_n^* + \frac{1}{r}R^{\pi/2}_{ml}\eta_l(\partial_m\eta_\phi^*) + \frac{1}{r}R^{\pi/2}_{kn}\eta_n^*(\partial_k\eta_\phi) + (\partial_k\eta_l)(\partial_l\eta_k)^* && (26)\\
=&\frac{1}{r^2}\eta_r\eta_r^* + \frac{2}{r}\,\mathrm{Re}\Big\{R^{\pi/2}_{kn}\eta_n^*(\partial_k\eta_\phi)\Big\} + (\partial_k\eta_l)(\partial_l\eta_k)^* && (27)\\
=&\frac{1}{r^2}\eta_r\eta_r^* + \frac{2}{r}\,\mathrm{Re}\Big\{\eta_r^*\frac{\partial_\phi}{r}\eta_\phi - \eta_\phi^*\partial_r\eta_\phi\Big\} + (\partial_k\eta_l)(\partial_l\eta_k)^* && (28)
\end{aligned}
$$

So the free energy density is

$$
f_{\mathrm{bulk}}[\boldsymbol{\eta}] = \alpha\boldsymbol{\eta}\cdot\boldsymbol{\eta}^* + \beta_1(\boldsymbol{\eta}\cdot\boldsymbol{\eta}^*)^2 + \beta_2|\boldsymbol{\eta}\cdot\boldsymbol{\eta}|^2, \tag{29}
$$

and

$$
\begin{aligned}
f^{\mathrm{Cyl}}_{\mathrm{grad}}[\boldsymbol{\eta}] =&\kappa_1(\partial_i\eta_j)(\partial_i\eta_j)^* + \kappa_2(\partial_i\eta_i)(\partial_j\eta_j)^* + \kappa_3(\partial_i\eta_j)(\partial_j\eta_i)^*\\
&+\frac{2}{r}\,\mathrm{Re}\Big\{\kappa_1\left(\eta_r^*\frac{\partial_\phi}{r}\eta_\phi - \eta_\phi^*\frac{\partial_\phi}{r}\eta_r\right) + \kappa_2\eta_r^*\partial_j\eta_j + \kappa_3\left(\eta_r^*\frac{\partial_\phi}{r}\eta_\phi - \eta_\phi^*\partial_r\eta_\phi\right)\Big\}\\
&+\frac{1}{r^2}\left[\kappa_1\eta_j^*\eta_j + (\kappa_2+\kappa_3)\eta_r^*\eta_r\right].
\end{aligned} \tag{30}
$$

EULER-LAGRANGE EQUATIONS

The GL differential equations $\delta F/\delta\eta_k^* = 0$.

If we do the functional variation of the free energy in cylindrical basis, we will get

$$
\begin{aligned}
&\alpha\eta_k + 2(\beta_1\eta_i\eta_i^*\eta_k + \beta_2\eta_i^2\eta_k^*) - \left(\kappa_1\partial_i^2\eta_k + \kappa_2\partial_k\partial_i\eta_i + \kappa_3\partial_j\partial_k\eta_j\right)\\
&-\frac{1}{r}\left[(2\kappa_1+\kappa_3)\frac{\partial_\phi}{r}\left(\eta_r\delta_{k\phi} - \eta_\phi\delta_{kr}\right) + \kappa_1\partial_r\eta_k + (\kappa_2+\kappa_3)\partial_k\eta_r\right]\\
&\qquad\qquad+\frac{1}{r^2}[\kappa_1\eta_k + (\kappa_2+\kappa_3)\eta_r\delta_{kr}] = 0.
\end{aligned} \tag{31}
$$

For the κ_1 term in first line of Eq. (30)

$$F \sim \iint (\partial_i \eta_j)(\partial_i \eta_j)^* r \mathrm{d}\phi \mathrm{d}r = \text{surface term} - \iint \eta_j^* \partial_i (r \partial_i \eta_j) \mathrm{d}\phi \mathrm{d}r \tag{32}$$

$$= \text{surface term} - \iint \eta_j^* \frac{1}{r} \partial_i (r \partial_i \eta_j) r \mathrm{d}\phi \mathrm{d}r \tag{33}$$

the corresponding functional variation is

$$\delta F/\delta \eta_k^* \sim -\frac{1}{r}\partial_i(r\partial_i \eta_k) = -\frac{1}{r}(\partial_i r)(\partial_i \eta_k) - \partial_i^2 \eta_k = -\frac{1}{r}\partial_r \eta_k - \partial_i^2 \eta_k \tag{34}$$

For the κ_2 term in first line of Eq. (30)

$$\delta F/\delta \eta_k^* \sim -\frac{1}{r}\partial_k(r\partial_i \eta_i) = -\frac{1}{r}(\partial_k r)(\partial_i \eta_i) - \partial_k \partial_i \eta_i = -\delta_{kr}\frac{1}{r}\partial_i \eta_i - \partial_k \partial_i \eta_i \tag{35}$$

For the κ_3 term in first line of Eq. (30)

$$\delta F/\delta \eta_k^* \sim -\frac{1}{r}\partial_i(r\partial_k \eta_i) = -\frac{1}{r}(\partial_i r)(\partial_k \eta_i) - \partial_i \partial_k \eta_i = -\frac{1}{r}\partial_k \eta_r - \partial_i \partial_k \eta_i \tag{36}$$

For the κ_1 term in second line of Eq. (30)

$$F \sim \iint \frac{1}{r}\left(\eta_r^* \frac{\partial_\phi}{r}\eta_\phi - \eta_\phi^* \frac{\partial_\phi}{r}\eta_r + \eta_r \frac{\partial_\phi}{r}\eta_\phi^* - \eta_\phi \frac{\partial_\phi}{r}\eta_r^*\right) r \mathrm{d}\phi \mathrm{d}r \tag{37}$$

the corresponding functional variation is

$$\delta F/\delta \eta_k^* \sim \frac{1}{r}\left(2\delta_{kr}\frac{\partial_\phi}{r}\eta_\phi - 2\delta_{k\phi}\frac{\partial_\phi}{r}\eta_r\right) \tag{38}$$

The κ_2 term in second line of Eq. (30) is

$$F \sim \iint \frac{1}{r}\left(\eta_r^* \partial_j \eta_j + \eta_r \partial_j \eta_j^*\right) r \mathrm{d}\phi \mathrm{d}r \tag{39}$$

the corresponding functional variation is

$$\delta F/\delta \eta_k^* \sim \frac{1}{r}\left(\delta_{kr}\partial_j \eta_j - \partial_k \eta_r\right) \tag{40}$$

The κ_3 term in second line of Eq. (30) is

$$F \sim \iint \frac{1}{r}\left(\eta_r^* \frac{\partial_\phi}{r}\eta_\phi - \eta_\phi^* \partial_r \eta_\phi + \eta_r \frac{\partial_\phi}{r}\eta_\phi^* - \eta_\phi \partial_r \eta_\phi^*\right) r \mathrm{d}\phi \mathrm{d}r \tag{41}$$

the corresponding functional variation is

$$\delta F/\delta \eta_k^* \sim \frac{1}{r}\left(\delta_{kr}\frac{\partial_\phi}{r}\eta_\phi \cancel{- \delta_{k\phi}\partial_r \eta_\phi} - \delta_{k\phi}\frac{\partial_\phi}{r}\eta_r \cancel{+ \delta_{k\phi}\partial_r \eta_\phi}\right) \tag{42}$$

Adding all these contributions together, we can get the GL equations in cylindrical basis.

In cartesian basis the GL equations are

$$\alpha\eta_k + 2(\beta_1\eta_i\eta_i^*\eta_k + \beta_2\eta_i^2\eta_k^*) - \left(\kappa_1\partial_i^2\eta_k + \kappa_2\partial_k\partial_i\eta_i + \kappa_3\partial_j\partial_k\eta_j\right) = 0 \tag{43}$$

If we transform it to cylindrical basis, we can also get

$$\begin{aligned}\alpha\eta_k + 2(\beta_1\eta_i\eta_i^*\eta_k + \beta_2\eta_i^2\eta_k^*) - \left(\kappa_1\partial_i^2\eta_k + \kappa_2\partial_k\partial_i\eta_i + \kappa_3\partial_j\partial_k\eta_j\right)\\ -\frac{1}{r}\left[(2\kappa_1+\kappa_3)\frac{\partial_\phi}{r}\left(\eta_r\delta_{k\phi} - \eta_\phi\delta_{kr}\right) + \kappa_1\partial_r\eta_k + (\kappa_2+\kappa_3)\partial_k\eta_r\right]\\ +\frac{1}{r^2}[\kappa_1\eta_k + (\kappa_2+\kappa_3)\eta_r\delta_{kr}] = 0.\end{aligned} \tag{44}$$

The κ_1 term is just a Laplacian term

$$\begin{aligned}\partial_i^{\mathrm{Car}}\partial_i^{\mathrm{Car}}\eta_k^{\mathrm{Car}} =& R_{il}\partial_l(R_{im}\partial_m(R_{kn}\eta_n)) &(45)\\ =& R_{il}(\partial_l R_{im})\partial_m(R_{kn}\eta_n) + R_{il}R_{im}\partial_l\partial_m(R_{kn}\eta_n) &(46)\\ =& \frac{1}{r}\partial_r(R_{kn}\eta_n) + \partial_l^2(R_{kn}\eta_n) &(47)\\ =& R_{kn}\frac{1}{r}\partial_r\eta_n + R_{kn}\partial_l^2\eta_n + 2(\partial_l R_{kn})(\partial_l\eta_n) + \eta_n\partial_l^2 R_{kn} &(48)\\ =& R_{kn}\left(\frac{1}{r}\partial_r\eta_n + \partial_l^2\eta_n + \frac{2}{r}R_{nm}^{\pi/2}\frac{\partial_\phi}{r}\eta_m - \frac{1}{r^2}\eta_n\right) &(49)\\ =& R_{kn}\left(\frac{1}{r}\partial_r\eta_n + \partial_l^2\eta_n + \frac{2}{r}\delta_{n\phi}\frac{\partial_\phi}{r}\eta_r - \frac{2}{r}\delta_{nr}\frac{\partial_\phi}{r}\eta_\phi - \frac{1}{r^2}\eta_n\right) &(50)\end{aligned}$$

The κ_2 term is

$$\begin{aligned}\partial_k^{\mathrm{Car}}\partial_i^{\mathrm{Car}}\eta_i^{\mathrm{Car}} =& R_{kl}\partial_l(R_{im}\partial_m(R_{in}\eta_n)) &(51)\\ =& R_{kl}\partial_l(R_{im}(\partial_m R_{in})\eta_n + R_{im}R_{in}(\partial_m\eta_n)) &(52)\\ =& R_{kl}\partial_l\left(\frac{1}{r}\eta_r + \partial_m\eta_m\right) &(53)\\ =& R_{kl}\left(\frac{1}{r}\partial_l\eta_r - \delta_{lr}\frac{1}{r^2}\eta_r + \partial_l\partial_m\eta_m\right) &(54)\end{aligned}$$

The κ_3 term is

$$
\begin{aligned}
\partial_j^{\mathrm{Car}}\partial_k^{\mathrm{Car}}\eta_j^{\mathrm{Car}} =& R_{jl}\partial_l(R_{km}\partial_m(R_{jn}\eta_n)) && (55)\\
=& R_{jl}(\partial_l R_{km})\partial_m(R_{jn}\eta_n) + R_{jl}R_{km}\partial_l\partial_m(R_{jn}\eta_n) && (56)\\
=& R_{j\phi}\frac{1}{r}R_{ki}R^{\pi/2}_{im}\partial_m(R_{jn}\eta_n) + R_{jl}R_{ki}\partial_l\partial_i(R_{jn}\eta_n) && (57)\\
=& R_{ki}\left[R_{j\phi}\frac{1}{r}R^{\pi/2}_{im}\partial_m(R_{jn}\eta_n) + R_{jl}\partial_l\partial_i(R_{jn}\eta_n)\right] && (58)\\
=& R_{ki}\left[R_{j\phi}\frac{1}{r}R^{\pi/2}_{im}(\partial_m R_{jn})\eta_n + R_{j\phi}\frac{1}{r}R^{\pi/2}_{im}R_{jn}(\partial_m\eta_n)\right. && (59)\\
&\left. + R_{jl}(\partial_l\partial_i R_{jn})\eta_n + R_{jl}(\partial_l R_{jn})(\partial_i\eta_n) + R_{jl}(\partial_i R_{jn})(\partial_l\eta_n) + R_{jl}R_{jn}(\partial_l\partial_i\eta_n)\right]\\
=& R_{ki}\left[\frac{1}{r^2}R^{\pi/2}_{i\phi}\eta_r + \frac{1}{r}R^{\pi/2}_{im}\partial_m\eta_\phi\right.\\
&\left. + \delta_{i\phi}R_{jl}\eta_n\partial_l\left(\frac{1}{r}R_{jn}\right) + \frac{1}{r}\partial_i\eta_r + \delta_{i\phi}\frac{1}{r}R^{\pi/2}_{ln}\partial_l\eta_n + \partial_l\partial_i\eta_l\right]\\
=& R_{ki}\left[-\frac{1}{r^2}\delta_{ir}\eta_r \cancel{+\frac{1}{r}\delta_{i\phi}\partial_r\eta_\phi} - \frac{1}{r^2}\delta_{ir}\partial_\phi\eta_\phi\right.\\
&\left. + \delta_{i\phi}R_{jl}\eta_n\left(\partial_l\frac{1}{r}\right)R_{jn} + \delta_{i\phi}R_{jl}\eta_n\frac{1}{r}(\partial_l R_{jn}) + \frac{1}{r}\partial_i\eta_r + \delta_{i\phi}\frac{1}{r^2}\partial_\phi\eta_r \cancel{-\delta_{i\phi}\frac{1}{r}\partial_r\eta_\phi} + \partial_l\partial_i\eta_l\right]\\
=& R_{ki}\left[-\frac{1}{r^2}\delta_{ir}\eta_r - \frac{1}{r^2}\delta_{ir}\partial_\phi\eta_\phi\right.\\
&\left. \cancel{-\delta_{i\phi}\eta_r\frac{1}{r^2} + \delta_{i\phi}\eta_r\frac{1}{r^2}} + \frac{1}{r}\partial_i\eta_r + \delta_{i\phi}\frac{1}{r^2}\partial_\phi\eta_r + \partial_l\partial_i\eta_l\right]\\
=& R_{ki}\left[-\frac{1}{r^2}\delta_{ir}\eta_r - \frac{1}{r^2}\delta_{ir}\partial_\phi\eta_\phi + \frac{1}{r}\partial_i\eta_r + \delta_{i\phi}\frac{1}{r^2}\partial_\phi\eta_r + \partial_l\partial_i\eta_l\right] && (60)
\end{aligned}
$$

PHYSICAL OBSERVABLES

In Cartesian basis we denote $\mathbf{r} = (x, y)$. A Galilean boost with velocity $\mathbf{u}$ introduce a local gauge transformation $\eta_i(\mathbf{r}) \xrightarrow{\mathbf{u}} \eta_i(\mathbf{r})e^{-iM\mathbf{u}\cdot\mathbf{r}/\hbar}$, where M is the mass of a pair of Helium atoms. The phase gradient correspond to a velocity field $\mathbf{v} = \frac{\hbar}{M}\boldsymbol{\nabla}\theta$ which transform as $\mathbf{v} \xrightarrow{\mathbf{u}} \mathbf{v} - \mathbf{u}$ under the Galilean boost. We also have $\partial_i \xrightarrow{\mathbf{u}} \partial_i - iMu_i/\hbar$. The GL free energy density also transforms as $f \xrightarrow{\mathbf{u}} f - \mathbf{j}\cdot\mathbf{u} + \mathcal{O}(u^2)$, where $\mathbf{j}_k = \frac{2M}{\hbar}\,\mathrm{Im}\big\{\kappa_1\eta_j^*\partial_k\eta_j + \kappa_2\eta_k^*\partial_j\eta_j + \kappa_3\eta_j^*\partial_j\eta_k\big\}$ is the superfluid mass current density or the momentum density. We transform it to cylindrical basis and get

$$\mathbf{j}_k = \frac{2M}{\hbar}\Big[\mathrm{Im}\big\{\kappa_1\eta_j^*\partial_k\eta_j + \kappa_2\eta_k^*\partial_j\eta_j + \kappa_3\eta_j^*\partial_j\eta_k\big\}$$
$$+\mathrm{Im}\big\{(2\kappa_1+\kappa_2+\kappa_3)\delta_{k\phi}\eta_\phi^*\eta_r/r\big\}\Big], \tag{61}$$

In weak-coupling limit, where $\beta_1 = 2\beta_2$ and $\kappa_1 = \kappa_2 = \kappa_3$, the angular momentum density $l_z = rj_\phi$ becomes

$$l_z = \frac{2M\kappa_1}{\hbar}\Big[\mathrm{Im}\big\{3\eta_\phi^*\partial_\phi\eta_\phi + \eta_r^*\partial_\phi\eta_r + 4\eta_\phi^*\eta_r\big\}$$
$$+\mathrm{Im}\big\{r\eta_\phi^*\partial_r\eta_r + r\eta_r^*\partial_r\eta_\phi\big\}\Big]. \tag{62}$$

ROTATING FRAME

We can stabilize these low-flow states by rotating the annulus at certain angular velocities Ω_m. Transform into the rotating frame with angular velocity $\boldsymbol{\Omega} = \hat{\mathbf{z}}\Omega$, and the free energy becomes

$$F' = F - \mathbf{L}\cdot\boldsymbol{\Omega}. \tag{63}$$

The critical angular velocity Ω_m which increase the winding number from $m-1$ to m should satisfy $F'(m,\Omega_m) = F'(m-1,\Omega_m)$, which gives

$$\Omega_m^\pm = \frac{F_\pm(m) - F_\pm(m-1)}{L_\pm(m) - L_\pm(m-1)}. \tag{64}$$

UNIFORM-FLOW APPROXIMATION

We can have superflow in the annulus $\mathbf{v}(\mathbf{r}) = \frac{\hbar}{M}\boldsymbol{\nabla}\theta$ where M is the mass of a pair of Helium atoms.

When we assume a uniform flow field along the azimuthal direction in the annulus, $\mathbf{v} = v\hat{\phi}$ and $v(r) = \frac{\hbar}{Mr}\partial_\phi\theta$, the order parameter in the annulus simply gains an extra phase factor along the azimuthal direction.

$$\boldsymbol{\eta}(r,\phi,v_s) = \boldsymbol{\eta}(r,v_s)e^{iMvr\phi/\hbar} = \boldsymbol{\eta}(r,v_s)e^{i(\partial_\phi\theta)\phi} \tag{65}$$

When the flow field is small, we denote

$$\boldsymbol{\eta}(r,\phi,v_s) = \boldsymbol{\eta}(r,v_s)e^{i(\partial_\phi\theta)\phi} = e^{\pm i\phi}\left(\eta_r^{(m)}(r), \pm i\eta_\phi^{(m)}(r)\right)e^{im\phi} \tag{66}$$

where m is an integer as the order parameter should be single-valued after 2π winding.

LOW-FLOW APPROXIMATION

For low flow states in an annulus, we further assume the radial profile won't be affected by the small flow field

$$\eta_i^{(m)}(r) = \eta_i^{(0)}(r) \tag{67}$$

For the denominator in Eq. (64), we have

$$L_\pm(m) - L_\pm(m-1) = \frac{2M\kappa_1}{\hbar}\int \mathrm{d}^3r \left(3\eta_\phi^{(0)}\eta_\phi^{(0)} + \eta_r^{(0)}\eta_r^{(0)}\right) \tag{68}$$

For the numerator in Eq. (64), we have to pick out terms associated with ∂_ϕ, which are

$$\begin{aligned} f_{\text{grad}} \sim & \frac{\kappa_1}{r^2}\left[3(\partial_\phi\eta_\phi)(\partial_\phi\eta_\phi)^* + (\partial_\phi\eta_r)(\partial_\phi\eta_r)^*\right] \\ & + \frac{\kappa_1}{r}\left[(\partial_\phi\eta_\phi)(\partial_r\eta_r)^* + (\partial_r\eta_r)(\partial_\phi\eta_\phi)^* + (\partial_\phi\eta_r)(\partial_r\eta_\phi)^* + (\partial_r\eta_\phi)(\partial_\phi\eta_r)^*\right] \\ & + \frac{2\kappa_1}{r^2}\,\mathrm{Re}\{3\eta_r^*\partial_\phi\eta_\phi - \eta_\phi^*\partial_\phi\eta_r\} \end{aligned} \tag{69}$$

Substitute in Eq. (66)

$$\begin{aligned} f_{\text{grad}}(m) \sim & \frac{\kappa_1}{r^2}(m\pm 1)^2\left[3\left(\eta_\phi^{(m)}\right)^2 + \left(\eta_r^{(m)}\right)^2\right] \\ & + \frac{2\kappa_1}{r}(1\pm m)\left[\eta_r^{(m)}\partial_r\eta_\phi^{(m)} - \eta_\phi^{(m)}\partial_r\eta_r^{(m)}\right] \\ & - \frac{8\kappa_1}{r^2}(1\pm m)\eta_r^{(m)}\eta_\phi^{(m)} \end{aligned} \tag{70}$$

For the second line, the corresponding total free energy is zero as long as $\eta_i^{(m)}(r)$ is an even function with respect to $r = R + D/2$,

$$2\pi\int_R^{R+D} r\mathrm{d}r \to \int_R^{R+D} \eta_i^{(m)}\partial_r\eta_j^{(m)}\mathrm{d}r = 0 \tag{71}$$

Then we will have

$$\begin{aligned} F_\pm(m) - F_\pm(m-1) = \int \mathrm{d}^3r\frac{\kappa_1}{r^2}\Bigg\{ & (2m-1)\left[3\left(\eta_\phi^{(0)}\right)^2 + \left(\eta_r^{(0)}\right)^2\right] \\ & \pm 2\left[3\left(\eta_\phi^{(0)}\right)^2 + \left(\eta_r^{(0)}\right)^2 - 4\eta_r^{(0)}\eta_\phi^{(0)}\right]\Bigg\} \end{aligned} \tag{72}$$

We denote the volume integral as $\langle ...\rangle_V = \int \mathrm{d}^3r(...)$. Then we have

$$\Omega_m^\pm = \frac{\hbar}{M}\frac{(m-\frac{1}{2})\left\langle \frac{1}{r^2}\left[3\left(\eta_\phi^{(0)}\right)^2 + \left(\eta_r^{(0)}\right)^2\right]\right\rangle_V \pm \left\langle \frac{1}{r^2}\left[3\left(\eta_\phi^{(0)}\right)^2 + \left(\eta_r^{(0)}\right)^2 - 4\eta_r^{(0)}\eta_\phi^{(0)}\right]\right\rangle_V}{\left\langle 3\left(\eta_\phi^{(0)}\right)^2 + \left(\eta_r^{(0)}\right)^2\right\rangle_V}$$

We denote $\langle ... \rangle_r \equiv \int_R^{R+D} \mathrm{d}r(...)$. When $R \gg D$, we can approximate $\langle \mathcal{O} \rangle_V \approx 2\pi R h \langle \mathcal{O} \rangle_r$ and $\left\langle \frac{1}{r^2}\mathcal{O} \right\rangle_V \approx \frac{2\pi h}{R} \langle \mathcal{O} \rangle_r$, where h is the z direction thickness, and we have

$$\Omega_m^{\pm} \approx \frac{\hbar}{MR^2}\left[\left(m-\frac{1}{2}\right) \pm \frac{\left\langle 3\left(\eta_\phi^{(0)}\right)^2 + \left(\eta_r^{(0)}\right)^2 - 4\eta_r^{(0)}\eta_\phi^{(0)} \right\rangle_r}{\left\langle 3\left(\eta_\phi^{(0)}\right)^2 + \left(\eta_r^{(0)}\right)^2 \right\rangle_r}\right] \tag{73}$$

If we approximate $\left\langle 3\left(\eta_\phi^{(0)}\right)^2 + \left(\eta_r^{(0)}\right)^2 \right\rangle_r \approx 4\eta_0^2(D+b\xi)$, then

$$\Omega_m^{\pm} \approx \frac{\hbar}{MR^2}\left[\left(m-\frac{1}{2}\right) \pm \left(1 - \frac{\left\langle \eta_r^{(0)}\eta_\phi^{(0)} \right\rangle_r}{\eta_0^2(D+b\xi)}\right)\right] \tag{74}$$

If we assume $\eta_r^{(0)}$ and $\eta_\phi^{(0)}$ are even functions with respect to $r = R + D/2$, then we have

$$\left\langle \eta_r^{(0)}\eta_\phi^{(0)} \right\rangle_r = 2\int_R^{R+D/2} \eta_r^{(0)}\eta_\phi^{(0)} \mathrm{d}r \tag{75}$$

If we approximate $\eta_\phi^{(0)}\eta_r^{(0)} \approx \eta_0^2 \tanh(ax/\xi)$, where $x \equiv r - R$, then

$$\begin{aligned}
\left\langle \eta_r^{(0)}\eta_\phi^{(0)} \right\rangle_r &\approx 2\eta_0^2\xi \int_0^{D/2} \tanh\left(\frac{ax}{\xi}\right) \mathrm{d}\left(\frac{x}{\xi}\right) &(76)\\
&= \frac{2}{a}\eta_0^2\xi \ln\left(\cosh\left(ax/\xi\right)\right)\Big|_0^{D/2} &(77)\\
&= \frac{2}{a}\eta_0^2\xi \ln\left(\cosh\left(\frac{aD}{2\xi}\right)\right) &(78)
\end{aligned}$$

Note that $\cosh(x) = \frac{e^x+e^{-x}}{2} \approx e^x/2$ when $x \gg 1$. If we assume $D/\xi \gg 1$, then

$$\left\langle \eta_r^{(0)}\eta_\phi^{(0)} \right\rangle_r \approx \frac{2}{a}\eta_0^2\xi\left(\frac{aD}{2\xi} - \ln(2)\right) = \eta_0^2\left(D - \frac{2}{a}\ln(2)\xi\right) \tag{79}$$

which gives

$$\Omega_m^{\pm} \approx \frac{\hbar}{MR^2}\left[\left(m-\frac{1}{2}\right) \pm \left(\frac{2}{a}\ln(2) + b\right)\frac{\xi}{D}\right] \tag{80}$$

when $a = 1/3, b = 0$ and $D = 30\xi$, we have

$$\begin{aligned}
\Omega_1^{\pm} &\approx \frac{\hbar}{MR^2}\left(\frac{1}{2} \pm \frac{\ln(2)}{5}\right) &(81)\\
\Omega_1^{\pm} &\approx \frac{\hbar}{MR^2}\left(\frac{1}{2} \pm 0.139\right) &(82)
\end{aligned}$$

Exact numerical solution pf Eq. (73) gives

$$\Omega_1^{\pm} \approx \frac{\hbar}{MR^2}\left(\frac{1}{2} \pm 0.122\right) \tag{83}$$

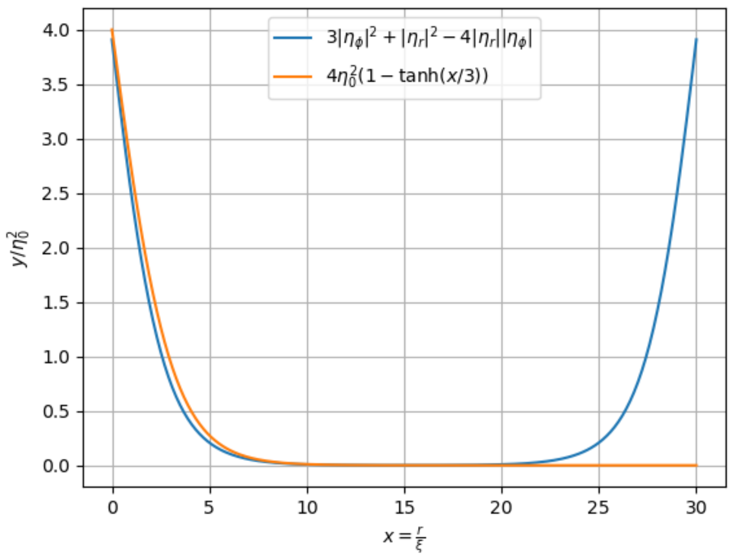

FIG. 1. Comparison of $y = 3\left(\eta_\phi^{(0)}\right)^2 + \left(\eta_r^{(0)}\right)^2 - 4\eta_r^{(0)}\eta_\phi^{(0)}$ and $y = \eta_0^2 \tanh(ax/\xi)$ for $a = 1/3$ and $D = 30\xi$.

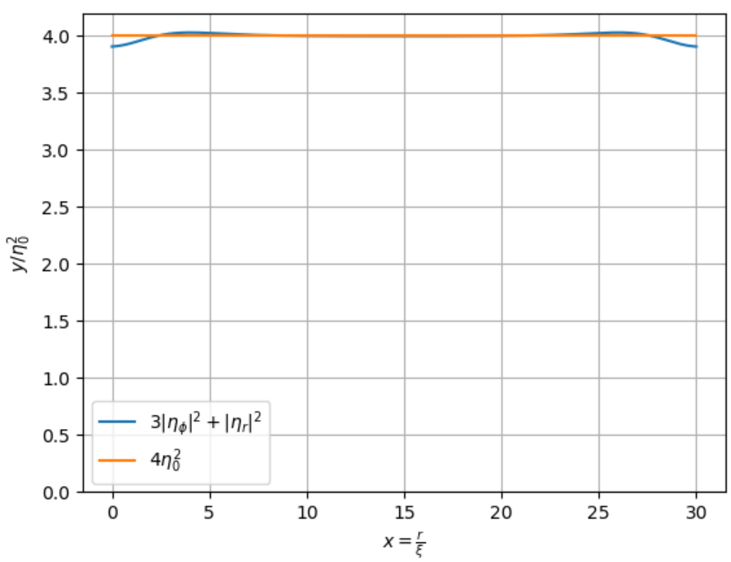

FIG. 2. Comparison of $y = 3\left(\eta_\phi^{(0)}\right)^2 + \left(\eta_r^{(0)}\right)^2$ and $y = 4\eta_0^2$ for $D = 30\xi$.

LONDON APPROXIMATION

We assume $\eta_i^{(m)}(r) = \eta_0$. The angular momentum increase is

$$L_\pm(m) - L_\pm(m-1) = \frac{8M}{\hbar}\kappa_1\eta_0^2\pi\left[(R+D)^2 - R^2\right]h \tag{84}$$

The free energy increase is

$$F_{\pm}(m) - F_{\pm}(m-1) = 8\pi\kappa_1\eta_0^2 \ln\left(\frac{R+D}{R}\right) h(2m-1) \tag{85}$$

where h is the thickness of the annulus. The critical angular velocity is

$$\Omega_m^{\pm} = \Omega_c \frac{\xi}{D} \frac{\ln(1+D/R)}{2+D/R}(2m-1), \tag{86}$$

where $\Omega_c = v_c/R$, $v_c = \hbar/M\xi$, $\xi^2 = \kappa_1/|\alpha|$. Actually, all of the GL parameters drop out

$$\Omega_m^{\pm} = \frac{\hbar}{MRD} \frac{\ln(1+D/R)}{2+D/R}(2m-1), \tag{87}$$

When $R \gg D$, we have

$$\Omega_m^{\pm} = \frac{\hbar}{MR^2}\left(m - \frac{1}{2}\right) \tag{88}$$

i.e.

$$\Omega_1 = \frac{\hbar}{MR^2}\frac{1}{2} \quad \text{for} \quad m=0 \to m=1 \tag{89}$$

$$\Omega_2 = \frac{\hbar}{MR^2}\frac{3}{2} \quad \text{for} \quad m=1 \to m=2 \tag{90}$$

$$\Omega_3 = \frac{\hbar}{MR^2}\frac{5}{2} \quad \text{for} \quad m=2 \to m=3 \tag{91}$$

$$\ldots \tag{92}$$

NUMERICAL SOLUTION

From the bulk free energy term we can get the bulk order parameter $\eta_0 = \frac{1}{2}\sqrt{\frac{|\alpha|}{\beta_1}}$. We also define the coherence length $\xi = \sqrt{\kappa_1/|\alpha|}$.

In weak-coupling limit, the GL equations in cylindrical basis are

$$\begin{aligned}\alpha\eta_k + 2\beta_1\eta_i\eta_i^*\eta_k + \beta_1\eta_i^2\eta_k^* - \kappa_1\left(\partial_i^2\eta_k + \partial_k\partial_i\eta_i + \partial_j\partial_k\eta_j\right)& \\ -\frac{\kappa_1}{r}\left[3\frac{\partial_\phi}{r}\left(\eta_r\delta_{k\phi} - \eta_\phi\delta_{kr}\right) + \partial_r\eta_k + 2\partial_k\eta_r\right]& \\ +\frac{\kappa_1}{r^2}\left[\eta_k + 2\eta_r\delta_{kr}\right] = 0.&\end{aligned} \tag{93}$$

The dimensionless GL equations in cylindrical basis are

$$\begin{aligned}-\eta_k + \frac{1}{2}\eta_i\eta_i^*\eta_k + \frac{1}{4}\eta_i^2\eta_k^* - \left(\partial_i^2\eta_k + \partial_k\partial_i\eta_i + \partial_j\partial_k\eta_j\right)& \\ -\frac{1}{r}\left[3\frac{\partial_\phi}{r}\left(\eta_r\delta_{k\phi} - \eta_\phi\delta_{kr}\right) + \partial_r\eta_k + 2\partial_k\eta_r\right]& \\ +\frac{1}{r^2}\left[\eta_k + 2\eta_r\delta_{kr}\right] = 0.&\end{aligned} \tag{94}$$

The r component of the GL equations is

$$-\eta_r + \frac{1}{2}\eta_i\eta_i^*\eta_r + \frac{1}{4}\eta_i^2\eta_r^* - (\partial_i^2\eta_r + \partial_r\partial_i\eta_i + \partial_j\partial_r\eta_j) - \frac{1}{r}\left[-3\frac{\partial_\phi}{r}\eta_\phi + 3\partial_r\eta_r\right] + \frac{3}{r^2}\eta_r = 0$$

$$-\eta_r + \frac{1}{2}\eta_i\eta_i^*\eta_r + \frac{1}{4}\eta_i^2\eta_r^* - (\partial_i^2\eta_r + \partial_r\partial_i\eta_i + \partial_j\partial_r\eta_j) + 3\left[\frac{1}{r^2}\partial_\phi\eta_\phi - \frac{1}{r}\partial_r\eta_r + \frac{\eta_r}{r^2}\right] = 0$$

$$-\eta_r + \frac{1}{2}\eta_i\eta_i^*\eta_r + \frac{1}{4}\eta_i^2\eta_r^* - 3\partial_r^2\eta_r - \frac{1}{r^2}\partial_\phi^2\eta_r - \frac{2}{r}\partial_r\partial_\phi\eta_\phi + \frac{4}{r^2}\partial_\phi\eta_\phi - \frac{3}{r}\partial_r\eta_r + \frac{3}{r^2}\eta_r = 0$$

The ϕ component of the GL equations is

$$-\eta_\phi + \frac{1}{2}\eta_i\eta_i^*\eta_\phi + \frac{1}{4}\eta_i^2\eta_\phi^* - \left(\partial_i^2\eta_\phi + \frac{\partial_\phi}{r}\partial_i\eta_i + \partial_j\frac{\partial_\phi}{r}\eta_j\right) - \frac{1}{r}\left[5\frac{\partial_\phi}{r}\eta_r + \partial_r\eta_\phi\right] + \frac{1}{r^2}\eta_\phi = 0$$

$$-\eta_\phi + \frac{1}{2}\eta_i\eta_i^*\eta_\phi + \frac{1}{4}\eta_i^2\eta_\phi^* - 3\frac{1}{r^2}\partial_\phi^2\eta_\phi - \partial_r^2\eta_\phi - \frac{2}{r}\partial_r\partial_\phi\eta_r - \frac{4}{r^2}\partial_\phi\eta_r - \frac{1}{r}\partial_r\eta_\phi + \frac{1}{r^2}\eta_\phi = 0$$

For the uniform flow approximation,

$$\boldsymbol{\eta}(r,\phi) = \left(\eta_r^{(n)}(r), \eta_\phi^{(n)}(r)\right)e^{in\phi} \tag{95}$$

which means that $\partial_\phi \to in$. We have

$$-\eta_r + \frac{1}{2}\eta_i\eta_i^*\eta_r + \frac{1}{4}\eta_i^2\eta_r^* - 3\partial_r^2\eta_r + \frac{n^2}{r^2}\eta_r - \frac{2in}{r}\partial_r\eta_\phi + \frac{4in}{r^2}\eta_\phi - \frac{3}{r}\partial_r\eta_r + \frac{3}{r^2}\eta_r = 0$$

$$-\eta_\phi + \frac{1}{2}\eta_i\eta_i^*\eta_\phi + \frac{1}{4}\eta_i^2\eta_\phi^* + \frac{3n^2}{r^2}\eta_\phi - \partial_r^2\eta_\phi - \frac{2in}{r}\partial_r\eta_r - \frac{4in}{r^2}\eta_r - \frac{1}{r}\partial_r\eta_\phi + \frac{1}{r^2}\eta_\phi = 0$$

We ignore the (n) superscript for the sake of simplicity. The solutions of $n = 1$ may corresponds to the $(p_x + ip_y, m = 0)$ state, or the $(p_x - ip_y, m = 2)$ state.

For boundary conditions, we have $\eta_r = 0$, and $\partial_r\eta_\phi = \frac{\eta_\phi}{r}$. We can derive this by requiring the sum of all the surface terms to vanish when deriving the Euler-Lagrange equations. Take Eq. (32) for example,

$$\text{surface term} = \iint \partial_i\left((\partial_i\eta_j)\eta_j^* r\right)\mathrm{d}\phi\mathrm{d}r \tag{96}$$

$$= \iint \left[\partial_r\left(r\eta_j^*(\partial_r\eta_j)\right) + \frac{\partial_\phi}{r}\left(r\eta_j^*\frac{\partial_\phi}{r}\eta_j\right)\right]\mathrm{d}\phi\mathrm{d}r \tag{97}$$

$$= \iint \left[\frac{1}{r}\partial_r\left(r\eta_j^*(\partial_r\eta_j)\right) + \frac{\partial_\phi}{r}\left(\eta_j^*\frac{\partial_\phi}{r}\eta_j\right)\right]r\mathrm{d}\phi\mathrm{d}r \tag{98}$$

$$= \iint \boldsymbol{\nabla}\cdot\mathbf{F}\mathrm{d}A = \oint \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}s \tag{99}$$

where $\mathbf{F} = (F_r, F_\phi) = \left(\eta_j^*\partial_r\eta_j, \eta_j^*\frac{\partial_\phi}{r}\eta_j\right)$. In an annulus or a cylinder, the boundary normal vector $\hat{\mathbf{n}} = \pm\hat{\mathbf{r}}$, and on boundaries we have $\eta_r = 0$, $F_r = \eta_\phi^*\partial_r\eta_\phi$, and $\mathrm{d}s = r\mathrm{d}\phi$, which gives

$$\text{surface term} = \oint_+ \eta_\phi^*(\partial_r\eta_\phi)r\mathrm{d}\phi - \oint_- \eta_\phi^*(\partial_r\eta_\phi)r\mathrm{d}\phi \tag{100}$$

The only remaining nonzero surface term is the κ_3 in the second line of Eq. (30), which correspond to $F_r = -\frac{\eta_\phi^* \eta_\phi}{r}$. In weak-coupling limit, when $\partial_r \eta_\phi = \frac{\eta_\phi}{r}$, these two surface terms cancel each other.

$\eta_\pm$ BASIS

In bulk, for the two time-reversed ground states, we have

$$(\eta_x, \eta_y) = (1, \pm i)\eta_0 \tag{101}$$

$$(\eta_r, \eta_\phi) = e^{\pm i\phi}(1, \pm i)\eta_0 \tag{102}$$

Now we want to define a new basis $\eta_\pm$ such that for the $p + ip$ state, we have

$$(\eta_+, \eta_-) = (\eta_0, 0) \tag{103}$$

and for the $p - ip$ state, we have

$$(\eta_+, \eta_-) = (0, \eta_0) \tag{104}$$

The corresponding transformation is

$$\begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-i\phi} & -ie^{-i\phi} \\ e^{i\phi} & ie^{i\phi} \end{pmatrix}\begin{pmatrix} \eta_r \\ \eta_\phi \end{pmatrix} \tag{105}$$

Under the uniform flow approximation, for winding number n, we have

$$\boldsymbol{\eta}(r, \phi) = \left(\eta_r^{(n)}(r), \eta_\phi^{(n)}(r)\right)e^{in\phi} \tag{106}$$

and

$$\eta_+(r, \phi) = \frac{1}{2}\left(\eta_r^{(n)}(r) - i\eta_\phi^{(n)}(r)\right)e^{i(n-1)\phi} \equiv \eta_+^{(m)}(r)e^{im\phi} \tag{107}$$

$$\eta_-(r, \phi) = \frac{1}{2}\left(\eta_r^{(n)}(r) + i\eta_\phi^{(n)}(r)\right)e^{i(n+1)\phi} \equiv \eta_-^{(p)}(r)e^{ip\phi} \tag{108}$$

where $m = n - 1$ and $p = n + 1$. The 2π phase winding difference, i.e. $p - m = 2$ can be understood from an angular momentum conservation perspective.

LARGE FLOW

When the flow field is large, we denote the superfluid flow field at the inner radius as $v_s = \frac{\hbar}{MR}\partial_\phi \theta$, and

$$\boldsymbol{\eta}(r,\phi,v_s) = \boldsymbol{\eta}^{(n)}(r)e^{in\phi} = \boldsymbol{\eta}(r,v_s)e^{i(\partial_\phi\theta)\phi} = \boldsymbol{\eta}(r,v_s)e^{i\frac{v_s R}{v_c \xi}\phi} \tag{109}$$

where the critical flow field $v_c = \hbar/M\xi$ and $\xi^2 = \kappa_1/|\alpha|$. We also have $n = \frac{v_s}{v_c}\frac{R}{\xi}$.

CARTESIAN BASIS

In Cartesian basis, weak-coupling limit dimensionless GL equations are

$$-\eta_k + \frac{1}{2}\eta_i\eta_i^*\eta_k + \frac{1}{4}\eta_i^2\eta_k^* - \left(\partial_i^2\eta_k + \partial_k\partial_i\eta_i + \partial_j\partial_k\eta_j\right) = 0 \tag{110}$$

The x component of the GL equations is

$$-\eta_x + \frac{3}{4}\eta_x^2\eta_x^* + \frac{1}{2}\eta_y\eta_y^*\eta_x + \frac{1}{4}\eta_y^2\eta_x^* - 3\partial_x^2\eta_x - \partial_y^2\eta_x - 2\partial_y\partial_x\eta_y = 0 \tag{111}$$

The y component of the GL equations is

$$-\eta_y + \frac{3}{4}\eta_y^2\eta_y^* + \frac{1}{2}\eta_x\eta_x^*\eta_y + \frac{1}{4}\eta_x^2\eta_y^* - 3\partial_y^2\eta_y - \partial_x^2\eta_y - 2\partial_x\partial_y\eta_x = 0 \tag{112}$$