MEAN-FIELD THEORY

The mean-field Nambu-Gor'kov Hamiltonian is

$$H = \frac{1}{2}\iint \mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{r}'\widehat{\Psi}^\dagger(\mathbf{r}') \begin{pmatrix} \xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\hat{1} & \hat{\Delta}(\mathbf{r},\mathbf{r}') \\ \hat{\Delta}^\dagger(\mathbf{r},\mathbf{r}') & -\xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\hat{1} \end{pmatrix} \widehat{\Psi}(\mathbf{r}) \tag{1}$$

$$\equiv \frac{1}{2}\iint \mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{r}'\widehat{\Psi}^\dagger(\mathbf{r}')\widehat{\tau}_3 \left(\xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') + \widehat{\Delta}(\mathbf{r},\mathbf{r}')\right) \widehat{\Psi}(\mathbf{r}). \tag{2}$$

where

$$\widehat{\Delta}(\mathbf{r},\mathbf{r}') = \begin{pmatrix} 0 & \hat{\Delta}(\mathbf{r},\mathbf{r}') \\ -\hat{\Delta}^\dagger(\mathbf{r},\mathbf{r}') & 0 \end{pmatrix} \tag{3}$$

When the system is homogeneous, the two-point gap function only depends on the relative coordinate, i.e., $\hat{\Delta}(\mathbf{r},\mathbf{r}') = \hat{\Delta}(\mathbf{r}-\mathbf{r}')$. In this case, we can transform to momentum space, and the Hamiltonian becomes

$$H = \frac{1}{2}\sum_{\mathbf{p}} \widehat{\Psi}^\dagger_{\mathbf{p}}\widehat{\tau}_3(\xi_{\mathbf{p}} + \widehat{\Delta}(\mathbf{p}))\widehat{\Psi}_{\mathbf{p}}. \tag{4}$$

where $\widehat{\Psi}^\dagger_{\mathbf{p}} = (c^\dagger_{\mathbf{p}\uparrow}, c^\dagger_{\mathbf{p}\downarrow}, c_{-\mathbf{p}\uparrow}, c_{-\mathbf{p}\downarrow})$.

FINITE TEMPERATURE

The imaginary time Green's function is

$$\widehat{G}(\mathbf{r}\tau,\mathbf{r}'\tau') = \widehat{G}(\mathbf{r},\mathbf{r}';\tau-\tau') = -\left\langle T_\tau \widehat{\Psi}(\mathbf{r},\tau)\widehat{\Psi}^\dagger(\mathbf{r}',\tau')\right\rangle \equiv \begin{pmatrix} \hat{G} & \hat{F} \\ \underline{\hat{F}} & \underline{\hat{G}} \end{pmatrix} \tag{5}$$

The unperturbed Matsubara Green's function is

$$\widehat{G}_0(\mathbf{p}, i\epsilon_n) = (i\epsilon_n - \widehat{\tau}_3\xi_{\mathbf{p}})^{-1} = -\frac{i\epsilon_n + \widehat{\tau}_3\xi_{\mathbf{p}}}{\epsilon_n^2 + \xi_{\mathbf{p}}^2}. \tag{6}$$

In bulk, the Gorkov Green's function satisfies

$$\widehat{G}^{-1}(\mathbf{p}, i\epsilon_n) = \widehat{G}_0^{-1}(\mathbf{p}, i\epsilon_n) - \widehat{\tau}_3\widehat{\Delta}(\mathbf{p}) = i\epsilon_n - \widehat{\tau}_3\xi_{\mathbf{p}} - \widehat{\tau}_3\widehat{\Delta}(\mathbf{p}). \tag{7}$$

which gives

$$\widehat{G}(\mathbf{p}, i\epsilon_n) = -\frac{i\epsilon_n + \widehat{\tau}_3\xi_{\mathbf{p}} + \widehat{\tau}_3\widehat{\Delta}(\mathbf{p})}{\epsilon_n^2 + \xi_{\mathbf{p}}^2 + |\Delta|^2} \tag{8}$$

The bulk quasiclassical Green's function is

$$\widehat{g}(\mathbf{p}, i\epsilon_n) = \int \mathrm{d}\xi_p \widehat{G}(\mathbf{p}, i\epsilon_n)\widehat{\tau}_3 = -\pi\frac{i\epsilon_n\widehat{\tau}_3 - \widehat{\Delta}(\mathbf{p})}{\sqrt{|\Delta|^2+\epsilon_n^2}} = \frac{\pi}{\sqrt{|\Delta|^2+\epsilon_n^2}}\begin{pmatrix} -i\epsilon_n & \hat{\Delta} \\ -\hat{\Delta}^\dagger & i\epsilon_n \end{pmatrix} \equiv \begin{pmatrix} \hat{g} & \hat{f} \\ \underline{\hat{f}} & \underline{\hat{g}} \end{pmatrix} \tag{9}$$

and it satisfies the normalization condition

$$\widehat{g}(\mathbf{p}, i\epsilon_n)^2 = -\pi^2\widehat{1}. \tag{10}$$

The bulk quasiclassical Green's function satisfies

$$\left[i\epsilon_n\widehat{\tau}_3 - \widehat{\Delta}, \widehat{g}\right] = 0 \tag{11}$$

But when near the boundary or impurities, the quasiclassical Green's function $\widehat{g}(\mathbf{R}, \mathbf{p}; \epsilon)$ and the gap function $\widehat{\Delta}(\mathbf{R}, \mathbf{p})$ are inhomogeneous, and are described by the Eilenberger equation and gap equation.

$$\left[i\epsilon_n\widehat{\tau}_3 - \widehat{\Delta}(\mathbf{R}, \mathbf{p}), \widehat{g}(\mathbf{R}, \mathbf{p}; i\epsilon_n)\right] + i\hbar\mathbf{v}_p\cdot\nabla\widehat{g}(\mathbf{R}, \mathbf{p}; i\epsilon_n) = 0 \tag{12}$$

and

$$\hat{\Delta}(\mathbf{R}, \mathbf{p}) = T\sum_{\epsilon_n=-\omega_c}^{\omega_c}\left\langle v(\mathbf{p}, \mathbf{p}')\hat{f}(\mathbf{R}, \mathbf{p}', \epsilon_n)\right\rangle_{p'} \tag{13}$$

Here $\mathbf{R} = \frac{1}{2}(\mathbf{r} + \mathbf{r}')$ is the center of the pair.

CURRENT DENSITY

The one-particle density matrix is

$$\rho_{\uparrow\uparrow}(\mathbf{r}, \mathbf{r}') \equiv \left\langle \psi_\uparrow^\dagger(\mathbf{r}')\psi_\uparrow(\mathbf{r})\right\rangle = G_{\uparrow\uparrow}(\mathbf{r}, \mathbf{r}'; 0^-) = T\sum_{n=-\infty}^{\infty} G_{\uparrow\uparrow}(\mathbf{r}, \mathbf{r}'; i\epsilon_n)e^{-i\epsilon_n 0^-} \tag{14}$$

The local current density is

$$\mathbf{j}(\mathbf{R}) = 2N_f T\sum_n \langle \mathbf{v}_p g(\mathbf{R}, \mathbf{p}, \epsilon_n)\rangle_p \tag{15}$$

Multiple it by the quasiparticle mass $p_f = m^* v_f$, and we get the mass current density:

$$\mathbf{j}_m(\mathbf{R}) \equiv m^*\mathbf{j}(\mathbf{R}) = \frac{2N_f v_f p_f}{4}\hbar \times \frac{2T}{\xi(T)|\Delta(T)|}\sum_n \langle \hat{\mathbf{p}} g(\mathbf{R}, \mathbf{p}, \epsilon_n)\rangle_p \tag{16}$$

where $\xi(T) = \hbar v_f/2|\Delta(T)|$ is the coherence length, and $n = 2N_f p_f v_f$ is the particle density. The ground state angular momentum in a cylinder is

$$\mathbf{L} = \int_0^R \mathrm{d}^3 r \mathbf{r} \times \mathbf{j}_m(\mathbf{r}) \tag{17}$$

$$= 2\pi h \hat{\mathbf{z}} \int_0^R \mathrm{d}r\, r^2 j_m(r) \tag{18}$$

$$\approx 2\pi R^2 h \hat{\mathbf{z}} \int_{R-\xi}^R \mathrm{d}r\, j_m(r) \tag{19}$$

$$\approx 2V \hat{\mathbf{z}} \int_{R-\xi}^R \mathrm{d}r\, j_m(r) \tag{20}$$

$$= \frac{N}{2} \hbar \hat{\mathbf{z}} \times \frac{2T}{\xi(T)|\Delta(T)|} \int_{R-\xi}^R \mathrm{d}r \sum_n \langle p_y g(\mathbf{r}, \mathbf{p}, \epsilon_n) \rangle_p \tag{21}$$

GAP EQUATION

Now we neglect the spin structure, i.e. discard the little hats. In bulk, we have

$$\Delta(\mathbf{p}) = T \sum_{\epsilon_n = -\omega_c}^{\omega_c} \left\langle v(\mathbf{p}, \mathbf{p}') \frac{\pi \Delta(\mathbf{p}')}{\sqrt{|\Delta|^2 + \epsilon_n^2}} \right\rangle_{p'} \tag{22}$$

Move everything to the right hand side and subtract Eq. (13) from it, we get

$$0 = T \sum_{\epsilon_n = -\infty}^{\infty} \left[\left\langle v(\mathbf{p}, \mathbf{p}') \frac{f(\mathbf{R}, \mathbf{p}', \epsilon_n)}{\Delta(\mathbf{R}, \mathbf{p})} \right\rangle_{p'} - \left\langle v(\mathbf{p}'', \mathbf{p}') \frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}} \frac{\Delta(\mathbf{p}')}{\Delta(\mathbf{p}'')} \right\rangle_{p'} \right] \tag{23}$$

which gives

$$\Delta(\mathbf{R}, \mathbf{p}) = \frac{T \sum_{\epsilon_n}^{\infty} \left\langle v(\mathbf{p}, \mathbf{p}') \hat{f}(\mathbf{R}, \mathbf{p}', \epsilon_n) \right\rangle_{p'}}{T \sum_{\epsilon_n}^{\infty} \left\langle v(\mathbf{p}'', \mathbf{p}') \frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}} \frac{\Delta(\mathbf{p}')}{\Delta(\mathbf{p}'')} \right\rangle_{p'}} \tag{24}$$

We have the Digamma function

$$K(T) = T \sum_{\epsilon_n = -\omega_c}^{\omega_c} \frac{\pi}{|\epsilon_n|} \approx \ln\left(1.13 \frac{\omega_c}{T}\right) \tag{25}$$

In bulk at T_c, we have

$$\Delta(\mathbf{p}) = K(T_c) \langle v(\mathbf{p}, \mathbf{p}') \Delta(\mathbf{p}') \rangle_{p'} \tag{26}$$

and we have

$$K(T_c) - K(T) = \ln(T/T_c) = \left.\frac{\Delta(\mathbf{p}'')}{\langle v(\mathbf{p}'',\mathbf{p}')\Delta(\mathbf{p}')\rangle_{p'}}\right|_{T_c} - T\sum_{\epsilon_n=-\omega_c}^{\omega_c}\frac{\pi}{|\epsilon_n|} \tag{27}$$

Substitute in Eq. (13), we get another equation to determine the gap inhomogeneity:

$$\ln\frac{T}{T_c} = T\sum_{\epsilon_n=-\infty}^{\infty}\left[\frac{\langle v(\mathbf{p},\mathbf{p}')f(\mathbf{R},\mathbf{p}',\epsilon_n)\rangle_{p'}/\Delta(\mathbf{R},\mathbf{p})}{\langle v(\mathbf{p}'',\mathbf{p}')\Delta(\mathbf{p}')\rangle_{p'}/\Delta(\mathbf{p}'')\big|_{T_c}} - \frac{\pi}{|\epsilon_n|}\right] \tag{28}$$

If you want temperature dependence of bulk gap, you can substitute in Eq. (22) to get

$$\ln\frac{T}{T_c} = T\sum_{\epsilon_n=-\infty}^{\infty}\left[\frac{\pi\left\langle v(\mathbf{p},\mathbf{p}')\Delta(\mathbf{p}')/\Delta(\mathbf{p})\sqrt{|\Delta|^2+\epsilon_n^2}\right\rangle_{p'}}{\langle v(\mathbf{p}'',\mathbf{p}')\Delta(\mathbf{p}')\rangle_{p'}/\Delta(\mathbf{p}'')\big|_{T_c}} - \frac{\pi}{|\epsilon_n|}\right] \tag{29}$$

RICCATI PARAMETERIZATIONS

The Riccati parameterization of the Green's function is

$$\widehat{g} = -i\pi\widehat{N}\begin{pmatrix} 1+\hat{a}\underline{\hat{a}} & 2\hat{a} \\ -2\underline{\hat{a}} & -1-\underline{\hat{a}}\hat{a}\end{pmatrix} \tag{30}$$

where $\widehat{N}$ is

$$\widehat{N} = \begin{pmatrix}(1-\hat{a}\underline{\hat{a}})^{-1} & 0 \\ 0 & (1-\underline{\hat{a}}\hat{a})^{-1}\end{pmatrix} \tag{31}$$

The inverse is

$$\hat{a} = (\hat{g} - i\pi)^{-1}\hat{f} \tag{32}$$

$$\underline{\hat{a}} = (\underline{\hat{g}} + i\pi)^{-1}\underline{\hat{f}} \tag{33}$$

Substitute in the bulk Green's function to get the bulk Ricatti amplitude:

$$\hat{a} = \frac{-\hat{\Delta}}{i\epsilon_n + i\sqrt{|\Delta|^2+\epsilon_n^2}} \tag{34}$$

$$\underline{\hat{a}} = \frac{-\hat{\Delta}^\dagger}{i\epsilon_n + i\sqrt{|\Delta|^2+\epsilon_n^2}} \tag{35}$$

The matrix Ricatti equation is

$$i\hbar\mathbf{v}_p\cdot\boldsymbol{\nabla}\hat{a} + 2i\epsilon_n\hat{a} + \hat{a}\hat{\Delta}^\dagger\hat{a} + \hat{\Delta} = 0 \tag{36}$$

$$i\hbar\mathbf{v}_p\cdot\boldsymbol{\nabla}\underline{\hat{a}} - 2i\epsilon_n\underline{\hat{a}} - \underline{\hat{a}}\hat{\Delta}\underline{\hat{a}} - \hat{\Delta}^\dagger = 0 \tag{37}$$

^{3}HE-A

For ^{3}He-A, the spin structure is $\hat{\mathbf{d}}\cdot(i\vec{\sigma}\hat{\sigma}_y) = \hat{\sigma}_x$, and we have two components of the gap function,

$$\hat{\Delta}(\mathbf{R},\mathbf{p}) = \hat{\sigma}_x(\Delta_1(\mathbf{R})p_x + \Delta_2(\mathbf{R})p_y) \tag{38}$$

In bulk we have $\hat{\Delta}(\mathbf{R},\mathbf{p}) = \hat{\sigma}_x\Delta(p_x + ip_y) = \hat{\sigma}_x\Delta e^{i\phi_p}$. The interaction is $v(\mathbf{p},\mathbf{p}') = 3v_0\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}'$. Note that

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos\phi e^{i\phi} = \frac{1}{2} \tag{39}$$

In 2d case ($|\Delta|^2$ does not depend on $\mathbf{p}$), Eq. (24) becomes

$$\Delta(\mathbf{R},\mathbf{p}) = \frac{2\sum_{\epsilon_n}^{\infty}\langle\mathbf{p}\cdot\mathbf{p}'f(\mathbf{R},\mathbf{p}',\epsilon_n)\rangle_{p'}}{\sum_{\epsilon_n}^{\infty}\frac{\pi}{\sqrt{|\Delta|^2+\epsilon_n^2}}} \tag{40}$$

The 2d bulk gap equation is

$$\ln\frac{T}{T_c} = T\sum_{\epsilon_n=-\infty}^{\infty}\left[\frac{\pi}{\sqrt{|\Delta|^2+\epsilon_n^2}} - \frac{\pi}{|\epsilon_n|}\right] \tag{41}$$

S-WAVE PAIRING

For s-wave pairing, the spin structure is $i\hat{\sigma}_y$, and we have

$$\hat{\Delta}(\mathbf{R},\mathbf{p}) = i\hat{\sigma}_y\Delta(\mathbf{R}) \tag{42}$$

In bulk we have $\hat{\Delta}(\mathbf{R},\mathbf{p}) = i\hat{\sigma}_y\Delta$. The interaction is $v(\mathbf{p},\mathbf{p}') = v_s$, and Eq. (24) reduces to

$$\Delta(\mathbf{R}) = \frac{\sum_n\langle f(\mathbf{R},\mathbf{p}',\epsilon_n)\rangle_{p'}}{\sum_n\frac{\pi}{\sqrt{|\Delta|^2+\epsilon_n^2}}} \tag{43}$$

^{3}HE-A EDGE GAP PROFILE

We assume translational invariance along the edge, i.e. the y direction. We also assume the gap to be real(imaginary) along the $p_x(p_y)$ direction.

$$\hat{\Delta}(x,\mathbf{p}) = \hat{\sigma}_x(\Delta_1(x)p_x + i\Delta_2(x)p_y) \tag{44}$$

Here $\Delta_1(x)$ and $\Delta_2(x)$ are real functions, and we have

$$\widehat{\Delta}(x,\mathbf{p}) = \begin{pmatrix} 0 & \hat{\sigma}_x(\Delta_1(x)p_x + i\Delta_2(x)p_y) \\ -\hat{\sigma}_x(\Delta_1(x)p_x - i\Delta_2(x)p_y) & 0 \end{pmatrix} \tag{45}$$

$$= i\hat{\sigma}_x(\Delta_2 p_y \widehat{\tau}_1 + \Delta_1 p_x \widehat{\tau}_2) \tag{46}$$

We can always write the anomalous Green's function as

$$\hat{f}(\mathbf{R},\mathbf{p},\epsilon_n) = \hat{\sigma}_x\Big(f_1(\mathbf{R},\mathbf{p},\epsilon_n) + if_2(\mathbf{R},\mathbf{p},\epsilon_n)\Big) \tag{47}$$

where $f_{1,2}$ are real functions, and we have

$$\widehat{g} = \hat{g}\widehat{\tau}_3 + i\hat{\sigma}_x(f_2\widehat{\tau}_1 + f_1\widehat{\tau}_2) \tag{48}$$

In this case, we can even write the gap equation as

$$\Delta_{1,2}(x) = T\sum_{\epsilon_n=-\omega_c}^{\omega_c} \langle v(\mathbf{p}_{x,y},\mathbf{p}')f_{1,2}(x,\mathbf{p}',\epsilon_n)\rangle_{p'} \tag{49}$$

and the gap iteration becomes

$$\Delta_{1,2}(x) = \frac{2\sum_n \langle \mathbf{p}_{x,y}\cdot\mathbf{p}' f_{1,2}(x,\mathbf{p}',n)\rangle_{p'}}{\sum_n \frac{\pi}{\sqrt{|\Delta|^2+\epsilon_n^2}}} \tag{50}$$

Numerical solution

The bulk gap temperature dependence is

$$\ln t = \pi t\sum_{n=-\infty}^{\infty}\left[\frac{1}{\sqrt{\delta(t)^2+e_n^2}} - \frac{1}{|e_n|}\right] \tag{51}$$

The dimensionless iteration of the gap function is

$$\delta_{1,2}(x) = \frac{2\sum_n \langle \mathbf{p}_{x,y}\cdot\mathbf{p}' f_{1,2}(x,\mathbf{p}',n)\rangle_{p'}}{\sum_n \frac{\pi}{\sqrt{1+e_n^2/\delta(t)^2}}} \tag{52}$$

where $t = T/T_c$, $\delta_{1,2}(x) = \Delta_{1,2}(x)/|\Delta(T)|$, $e_n = \epsilon_n/T_c$ and $\delta(t) = |\Delta(T)|/T_c$. The Ricatti equation is

$$ip_x\partial_x a(\mathbf{p},x) + \frac{ie_n}{\delta(t)}a(\mathbf{p},x) + \frac{\delta^*(\mathbf{p},x)}{2}a^2(\mathbf{p},x) + \frac{\delta(\mathbf{p},x)}{2} = 0 \tag{53}$$

$$ip_x\partial_x \underline{a}(\mathbf{p},x) - \frac{ie_n}{\delta(t)}\underline{a}(\mathbf{p},x) - \frac{\delta(\mathbf{p},x)}{2}\underline{a}^2(\mathbf{p},x) - \frac{\delta^*(\mathbf{p},x)}{2} = 0 \tag{54}$$

where $\xi = \hbar v_f / 2|\Delta(T)|$ and $\vec{\partial} = \xi \boldsymbol{\nabla}$. We can get g and $f_{1,2}$ from

$$g = -i\pi \frac{1 + a\underline{a}}{1 - a\underline{a}} \tag{55}$$

$$f = f_1 + i f_2 = -i\pi \frac{2a}{1 - a\underline{a}} \tag{56}$$

We first assume an initial guess for the gap function $\delta(\mathbf{r}, \hat{\mathbf{p}})$. For each trajectory $\hat{\mathbf{p}}$ along which the quasiparticle propagates, we can solve the Eilenberger equation to get the Green's function $g(\mathbf{r}, \hat{\mathbf{p}}, n)$ and the anomalous Green's function $f(\mathbf{r}, \hat{\mathbf{p}}, n)$. Then we iterate to update the gap function $\delta(\mathbf{r}, \hat{\mathbf{p}})$ by solving the gap equation. Repeat until convergence.

ANALYTIC CONTINUATION

For the integration contour C being an infinite circle, assume function $g(z)$ has no poles inside the contour and vanishes at infinity, the residue theorem gives

$$\oint_C \mathrm{d}z n_F(z) g(z) = 2\pi i \sum_{\text{poles}} \mathrm{Res}\{n_F(z) g(z)\} = 2\pi i \,(-T) \sum_{n=-\infty}^{\infty} g(\epsilon_n) \tag{57}$$

The residue of the Fermi distribution function $\mathrm{Res}\{n_F(z)\} = -T$. We can seperate the contour into upper and lower half circles, which gives

$$\begin{aligned}
\oint_C \mathrm{d}z n_F(z) g(z) &= \left(\oint_{\text{upper}} + \oint_{\text{lower}} \right) \mathrm{d}z n_F(z) g(z) && (58)\\
&= \int_{-\infty}^{\infty} \mathrm{d}\epsilon n_F(\epsilon) g^R(\epsilon + i0^+) + \int_{\infty}^{-\infty} \mathrm{d}\epsilon n_F(\epsilon) g^A(\epsilon - i0^+) && (59)\\
&= \int_{-\infty}^{\infty} \mathrm{d}\epsilon n_F(\epsilon) \left[g^R(\epsilon + i0^+) - g^A(\epsilon - i0^+) \right] && (60)\\
&= \int_{-\infty}^{\infty} \mathrm{d}\epsilon n_F(\epsilon) 2i \,\mathrm{Im}\{g^R(\epsilon + i0^+)\} && (61)
\end{aligned}$$

and we get

$$T \sum_{n=-\infty}^{\infty} g(\epsilon_n) = \int_{-\infty}^{\infty} \mathrm{d}\epsilon n_F(\epsilon) \left(-\frac{1}{\pi} \mathrm{Im}\{g^R(\epsilon + i0^+)\} \right) \tag{62}$$

Analytically continue Eq. (15) to real frequency, we get

$$\mathbf{j}(\mathbf{R}) = 2N_f \int \mathrm{d}\epsilon n_F(\epsilon) \left\langle \mathbf{v}_p \left(-\frac{1}{\pi} \mathrm{Im}\{g(\mathbf{R}, \mathbf{p}, \epsilon)\} \right) \right\rangle_p \tag{63}$$