INTRODUCTION

In cylindrical coordinates, we have

$$\begin{pmatrix} \eta_r \\ \eta_\phi \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \tag{1}$$

and

$$\begin{pmatrix} \partial_r \\ \frac{1}{r} \partial_\phi \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$
 (2)

In bulk, for the two time-reversed ground states,

$$(\eta_x^{\text{bulk}}, \eta_y^{\text{bulk}}) = \eta_0(1, \pm i) \tag{3}$$

$$(\eta_r^{\text{bulk}}, \eta_\phi^{\text{bulk}}) = e^{\pm i\phi} \eta_0(1, \pm i) \tag{4}$$

In GL theory, the order parameter is determined by minimizing the free energy functional

$$F[\boldsymbol{\eta}] = \int d^3 r \left(f_{\text{bulk}} + f_{\text{grad}} \right) \tag{5}$$

where

$$f_{\text{bulk}}[\boldsymbol{\eta}] = \alpha \boldsymbol{\eta} \cdot \boldsymbol{\eta}^* + \beta_1 (\boldsymbol{\eta} \cdot \boldsymbol{\eta}^*)^2 + \beta_2 |\boldsymbol{\eta} \cdot \boldsymbol{\eta}|^2,$$
 (6)

and

$$f_{\text{grad}}^{\text{Car}}[\boldsymbol{\eta}] = \kappa_1(\partial_i \eta_j)(\partial_i \eta_j)^* + \kappa_2(\partial_i \eta_i)(\partial_j \eta_j)^* + \kappa_3(\partial_i \eta_j)(\partial_j \eta_i)^*$$
(7)

In cylindrical basis, we have $\partial_i^{\text{Car}} = R_{ik}\partial_k$ and $\eta_j^{\text{Car}} = R_{jl}\eta_l$, where R_{ij} is the 2d rotational matrix. The form of the bulk free energy term doesn't change. The gradient term becomes

$$f_{\text{grad}}^{\text{Cyl}}[\boldsymbol{\eta}] = \kappa_{1}(\partial_{i}\eta_{j})(\partial_{i}\eta_{j})^{*} + \kappa_{2}(\partial_{i}\eta_{i})(\partial_{j}\eta_{j})^{*} + \kappa_{3}(\partial_{i}\eta_{j})(\partial_{j}\eta_{i})^{*}$$

$$+ \frac{2}{r}\operatorname{Re}\left\{\kappa_{1}\left(\eta_{r}^{*}\frac{\partial_{\phi}}{r}\eta_{\phi} - \eta_{\phi}^{*}\frac{\partial_{\phi}}{r}\eta_{r}\right) + \kappa_{2}\eta_{r}^{*}\partial_{j}\eta_{j} + \kappa_{3}\left(\eta_{r}^{*}\frac{\partial_{\phi}}{r}\eta_{\phi} - \eta_{\phi}^{*}\partial_{r}\eta_{\phi}\right)\right\}$$

$$+ \frac{1}{r^{2}}\left[\kappa_{1}\eta_{j}^{*}\eta_{j} + (\kappa_{2} + \kappa_{3})\eta_{r}^{*}\eta_{r}\right].$$
(8)

Take the κ_1 term for example, note that $R_{ij}R_{ik} = R_{ji}^TR_{ik} = \delta_{jk}$

$$(\partial_i^{\text{Car}} \eta_i^{\text{Car}})(\partial_i^{\text{Car}} \eta_i^{\text{Car}})^* \tag{9}$$

$$= \left[R_{ik} \partial_k (R_{jl} \eta_l) \right] \left[R_{im} \partial_m (R_{jn} \eta_n^*) \right]$$
(10)

$$= \left[\partial_k(R_{jl}\eta_l) \right] \left[\partial_k(R_{jn}\eta_n^*) \right] \tag{11}$$

$$= \left[(\partial_k R_{jl}) \eta_l + R_{jl} (\partial_k \eta_l) \right] \left[(\partial_k R_{jn}) \eta_n^* + R_{jn} (\partial_k \eta_n^*) \right]$$
(12)

$$= (\partial_k R_{jl}) \eta_l (\partial_k R_{jn}) \eta_n^* + (\partial_k R_{jl}) \eta_l R_{jn} (\partial_k \eta_n^*) + R_{jl} (\partial_k \eta_l) (\partial_k R_{jn}) \eta_n^* + R_{jl} R_{jn} (\partial_k \eta_l) (\partial_k \eta_n^*)$$

Note that $\partial_r R(\phi) = 0$, $\partial_{\phi} R(\phi) = R(\phi) R(\pi/2) = R(\pi/2) R(\phi)$. Finally we have

$$(\partial_i^{\text{Car}} \eta_j^{\text{Car}}) (\partial_i^{\text{Car}} \eta_j^{\text{Car}})^* \tag{13}$$

$$= \frac{1}{r^2} \eta_l \eta_l^* + \frac{1}{r} R_{nl}^{\pi/2} \eta_l \left(\frac{\partial_\phi}{r} \eta_n^* \right) + \frac{1}{r} R_{ln}^{\pi/2} \eta_n^* \left(\frac{\partial_\phi}{r} \eta_l \right) + (\partial_k \eta_l) (\partial_k \eta_l)^*$$

$$(14)$$

$$= \frac{1}{r^2} \eta_l \eta_l^* + \frac{2}{r} \operatorname{Re} \left\{ R_{ln}^{\pi/2} \eta_n^* \left(\frac{\partial_{\phi}}{r} \eta_l \right) \right\} + (\partial_k \eta_l) (\partial_k \eta_l)^*$$
(15)

$$= \frac{1}{r^2} \eta_l \eta_l^* + \frac{2}{r} \operatorname{Re} \left\{ \eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \frac{\partial_\phi}{r} \eta_r \right\} + (\partial_k \eta_l) (\partial_k \eta_l)^*$$
(16)

Similarly, for the κ_2 terms, we have

$$(\partial_i^{\text{Car}} \eta_i^{\text{Car}})(\partial_j^{\text{Car}} \eta_i^{\text{Car}})^* \tag{17}$$

$$= \left[R_{ik} \partial_k (R_{il} \eta_l) \right] \left[R_{jm} \partial_m (R_{jn} \eta_n^*) \right]$$
(18)

$$= \left[R_{ik}(\partial_k R_{il}) \eta_l + R_{ik} R_{il}(\partial_k \eta_l) \right] \left[R_{jm}(\partial_m R_{jn}) \eta_n^* + R_{jm} R_{jn}(\partial_m \eta_n^*) \right]$$
(19)

$$= \left[\frac{1}{r} \eta_r + \partial_k \eta_k \right] \left[\frac{1}{r} \eta_r^* + \partial_m \eta_m^* \right]$$
 (20)

$$= \frac{1}{r^2} \eta_r \eta_r^* + \frac{2}{r} \operatorname{Re} \{ \eta_r^* \partial_k \eta_k \} + (\partial_k \eta_k) (\partial_m \eta_m)^*$$
(21)

For the κ_3 terms, we have

$$(\partial_i^{\text{Car}} \eta_i^{\text{Car}}) (\partial_i^{\text{Car}} \eta_i^{\text{Car}})^* \tag{22}$$

$$= \left[R_{ik} \partial_k (R_{jl} \eta_l) \right] \left[R_{jm} \partial_m (R_{in} \eta_n^*) \right]$$
(23)

$$= \left[R_{ik}(\partial_k R_{jl}) \eta_l + R_{ik} R_{jl}(\partial_k \eta_l) \right] \left[R_{jm}(\partial_m R_{in}) \eta_n^* + R_{jm} R_{in}(\partial_m \eta_n^*) \right]$$
(24)

$$=R_{ik}(\partial_k R_{jl})\eta_l R_{jm}(\partial_m R_{in})\eta_n^* + R_{ik}(\partial_k R_{jl})\eta_l R_{jm} R_{in}(\partial_m \eta_n^*)$$

$$+ R_{ik}R_{il}(\partial_k\eta_l)R_{im}(\partial_mR_{in})\eta_n^* + R_{ik}R_{il}(\partial_k\eta_l)R_{im}R_{in}(\partial_m\eta_n^*)$$
(25)

$$= \frac{1}{r^2} R_{\phi l}^{\pi/2} \eta_l R_{\phi n}^{\pi/2} \eta_n^* + \frac{1}{r} R_{ml}^{\pi/2} \eta_l (\partial_m \eta_\phi^*) + \frac{1}{r} R_{kn}^{\pi/2} \eta_n^* (\partial_k \eta_\phi) + (\partial_k \eta_l) (\partial_l \eta_k)^*$$
 (26)

$$= \frac{1}{r^2} \eta_r \eta_r^* + \frac{2}{r} \operatorname{Re} \left\{ R_{kn}^{\pi/2} \eta_n^* (\partial_k \eta_\phi) \right\} + (\partial_k \eta_l) (\partial_l \eta_k)^*$$
(27)

$$= \frac{1}{r^2} \eta_r \eta_r^* + \frac{2}{r} \operatorname{Re} \left\{ \eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \partial_r \eta_\phi \right\} + (\partial_k \eta_l) (\partial_l \eta_k)^*$$
(28)

So the free energy density is

$$f_{\text{bulk}}[\boldsymbol{\eta}] = \alpha \boldsymbol{\eta} \cdot \boldsymbol{\eta}^* + \beta_1 (\boldsymbol{\eta} \cdot \boldsymbol{\eta}^*)^2 + \beta_2 |\boldsymbol{\eta} \cdot \boldsymbol{\eta}|^2, \tag{29}$$

and

$$f_{\text{grad}}^{\text{Cyl}}[\boldsymbol{\eta}] = \kappa_{1}(\partial_{i}\eta_{j})(\partial_{i}\eta_{j})^{*} + \kappa_{2}(\partial_{i}\eta_{i})(\partial_{j}\eta_{j})^{*} + \kappa_{3}(\partial_{i}\eta_{j})(\partial_{j}\eta_{i})^{*}$$

$$+ \frac{2}{r}\operatorname{Re}\left\{\kappa_{1}\left(\eta_{r}^{*}\frac{\partial_{\phi}}{r}\eta_{\phi} - \eta_{\phi}^{*}\frac{\partial_{\phi}}{r}\eta_{r}\right) + \kappa_{2}\eta_{r}^{*}\partial_{j}\eta_{j} + \kappa_{3}\left(\eta_{r}^{*}\frac{\partial_{\phi}}{r}\eta_{\phi} - \eta_{\phi}^{*}\partial_{r}\eta_{\phi}\right)\right\}$$

$$+ \frac{1}{r^{2}}\left[\kappa_{1}\eta_{j}^{*}\eta_{j} + (\kappa_{2} + \kappa_{3})\eta_{r}^{*}\eta_{r}\right].$$
(30)

EULER-LAGRANGE EQUATIONS

The GL differential equations $\delta F/\delta \eta_k^* = 0$.

If we do the functional variation of the free energy in cylindrical basis, we will get

$$\alpha \eta_{k} + 2(\beta_{1} \eta_{i} \eta_{i}^{*} \eta_{k} + \beta_{2} \eta_{i}^{2} \eta_{k}^{*}) - (\kappa_{1} \partial_{i}^{2} \eta_{k} + \kappa_{2} \partial_{k} \partial_{i} \eta_{i} + \kappa_{3} \partial_{j} \partial_{k} \eta_{j})$$

$$-\frac{1}{r} \left[(2\kappa_{1} + \kappa_{3}) \frac{\partial_{\phi}}{r} (\eta_{r} \delta_{k\phi} - \eta_{\phi} \delta_{kr}) + \kappa_{1} \partial_{r} \eta_{k} + (\kappa_{2} + \kappa_{3}) \partial_{k} \eta_{r} \right]$$

$$+\frac{1}{r^{2}} [\kappa_{1} \eta_{k} + (\kappa_{2} + \kappa_{3}) \eta_{r} \delta_{kr}] = 0.$$
(31)

For the κ_1 term in first line of Eq. (30)

$$F \sim \iint (\partial_i \eta_j) (\partial_i \eta_j)^* r d\phi dr = \text{surface term} - \iint \eta_j^* \partial_i (r \partial_i \eta_j) d\phi dr$$
 (32)

= surface term
$$-\iint \eta_j^* \frac{1}{r} \partial_i (r \partial_i \eta_j) r d\phi dr$$
 (33)

the corresponding functional variation is

$$\delta F/\delta \eta_k^* \sim -\frac{1}{r}\partial_i(r\partial_i\eta_k) = -\frac{1}{r}(\partial_i r)(\partial_i\eta_k) - \partial_i^2\eta_k = -\frac{1}{r}\partial_r\eta_k - \partial_i^2\eta_k \tag{34}$$

For the κ_2 term in first line of Eq. (30)

$$\delta F/\delta \eta_k^* \sim -\frac{1}{r} \partial_k (r \partial_i \eta_i) = -\frac{1}{r} (\partial_k r)(\partial_i \eta_i) - \partial_k \partial_i \eta_i = -\delta_{kr} \frac{1}{r} \partial_i \eta_i - \partial_k \partial_i \eta_i$$
 (35)

For the κ_3 term in first line of Eq. (30)

$$\delta F/\delta \eta_k^* \sim -\frac{1}{r} \partial_i (r \partial_k \eta_i) = -\frac{1}{r} (\partial_i r)(\partial_k \eta_i) - \partial_i \partial_k \eta_i = -\frac{1}{r} \partial_k \eta_r - \partial_i \partial_k \eta_i$$
 (36)

For the κ_1 term in second line of Eq. (30)

$$F \sim \iint \frac{1}{r} \left(\eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \frac{\partial_\phi}{r} \eta_r + \eta_r \frac{\partial_\phi}{r} \eta_\phi^* - \eta_\phi \frac{\partial_\phi}{r} \eta_r^* \right) r d\phi dr$$
 (37)

the corresponding functional variation is

$$\delta F/\delta \eta_k^* \sim \frac{1}{r} \left(2\delta_{kr} \frac{\partial_\phi}{r} \eta_\phi - 2\delta_{k\phi} \frac{\partial_\phi}{r} \eta_r \right)$$
 (38)

The κ_2 term in second line of Eq. (30) is

$$F \sim \iint \frac{1}{r} \left(\eta_r^* \partial_j \eta_j + \eta_r \partial_j \eta_j^* \right) r d\phi dr$$
 (39)

the corresponding functional variation is

$$\delta F/\delta \eta_k^* \sim \frac{1}{r} \left(\delta_{kr} \partial_j \eta_j - \partial_k \eta_r \right)$$
 (40)

The κ_3 term in second line of Eq. (30) is

$$F \sim \iint \frac{1}{r} \left(\eta_r^* \frac{\partial_\phi}{r} \eta_\phi - \eta_\phi^* \partial_r \eta_\phi + \eta_r \frac{\partial_\phi}{r} \eta_\phi^* - \eta_\phi \partial_r \eta_\phi^* \right) r d\phi dr$$
 (41)

the corresponding functional variation is

$$\delta F/\delta \eta_k^* \sim \frac{1}{r} \left(\delta_{kr} \frac{\partial_{\phi}}{r} \eta_{\phi} - \delta_{k\phi} \partial_r \eta_{\phi} - \delta_{k\phi} \frac{\partial_{\phi}}{r} \eta_r + \delta_{k\phi} \partial_r \eta_{\phi} \right)$$
(42)

Adding all these contributions together, we can get the GL equations in cylindrical basis.

In cartesian basis the GL equations are

$$\alpha \eta_k + 2(\beta_1 \eta_i \eta_i^* \eta_k + \beta_2 \eta_i^2 \eta_k^*) - (\kappa_1 \partial_i^2 \eta_k + \kappa_2 \partial_k \partial_i \eta_i + \kappa_3 \partial_j \partial_k \eta_j) = 0$$
(43)

If we transform it to cylindrical basis, we can also get

$$\alpha \eta_{k} + 2(\beta_{1} \eta_{i} \eta_{i}^{*} \eta_{k} + \beta_{2} \eta_{i}^{2} \eta_{k}^{*}) - (\kappa_{1} \partial_{i}^{2} \eta_{k} + \kappa_{2} \partial_{k} \partial_{i} \eta_{i} + \kappa_{3} \partial_{j} \partial_{k} \eta_{j})$$

$$-\frac{1}{r} \left[(2\kappa_{1} + \kappa_{3}) \frac{\partial_{\phi}}{r} (\eta_{r} \delta_{k\phi} - \eta_{\phi} \delta_{kr}) + \kappa_{1} \partial_{r} \eta_{k} + (\kappa_{2} + \kappa_{3}) \partial_{k} \eta_{r} \right]$$

$$+\frac{1}{r^{2}} [\kappa_{1} \eta_{k} + (\kappa_{2} + \kappa_{3}) \eta_{r} \delta_{kr}] = 0.$$

$$(44)$$

The κ_1 term is just a Laplacian term

$$\partial_i^{\text{Car}} \partial_i^{\text{Car}} \eta_k^{\text{Car}} = R_{il} \partial_l (R_{im} \partial_m (R_{kn} \eta_n)) \tag{45}$$

$$=R_{il}(\partial_l R_{im})\partial_m(R_{kn}\eta_n) + R_{il}R_{im}\partial_l\partial_m(R_{kn}\eta_n)$$
(46)

$$= \frac{1}{r} \partial_r (R_{kn} \eta_n) + \partial_l^2 (R_{kn} \eta_n) \tag{47}$$

$$=R_{kn}\frac{1}{r}\partial_r\eta_n + R_{kn}\partial_l^2\eta_n + 2(\partial_lR_{kn})(\partial_l\eta_n) + \eta_n\partial_l^2R_{kn}$$
(48)

$$=R_{kn}\left(\frac{1}{r}\partial_r\eta_n + \partial_l^2\eta_n + \frac{2}{r}R_{nm}^{\pi/2}\frac{\partial_\phi}{r}\eta_m - \frac{1}{r^2}\eta_n\right)$$
(49)

$$=R_{kn}\left(\frac{1}{r}\partial_r\eta_n + \partial_l^2\eta_n + \frac{2}{r}\delta_{n\phi}\frac{\partial_\phi}{r}\eta_r - \frac{2}{r}\delta_{nr}\frac{\partial_\phi}{r}\eta_\phi - \frac{1}{r^2}\eta_n\right)$$
(50)

The κ_2 term is

$$\partial_k^{\text{Car}} \partial_i^{\text{Car}} \eta_i^{\text{Car}} = R_{kl} \partial_l (R_{im} \partial_m (R_{in} \eta_n))$$
(51)

$$=R_{kl}\partial_l(R_{im}(\partial_m R_{in})\eta_n + R_{im}R_{in}(\partial_m \eta_n))$$
(52)

$$=R_{kl}\partial_l\left(\frac{1}{r}\eta_r + \partial_m\eta_m\right) \tag{53}$$

$$=R_{kl}\left(\frac{1}{r}\partial_l\eta_r - \delta_{lr}\frac{1}{r^2}\eta_r + \partial_l\partial_m\eta_m\right)$$
(54)

The κ_3 term is

$$\begin{split} \partial_{j}^{\text{Car}} \partial_{k}^{\text{Car}} \eta_{j}^{\text{Car}} &= R_{jl} \partial_{l} (R_{km} \partial_{m} (R_{jn} \eta_{n})) \\ &= R_{jl} (\partial_{l} R_{km}) \partial_{m} (R_{jn} \eta_{n}) + R_{jl} R_{km} \partial_{l} \partial_{m} (R_{jn} \eta_{n}) \\ &= R_{j\phi} \frac{1}{r} R_{ki} R_{im}^{\pi/2} \partial_{m} (R_{jn} \eta_{n}) + R_{jl} R_{ki} \partial_{l} \partial_{i} (R_{jn} \eta_{n}) \\ &= R_{ki} \left[R_{j\phi} \frac{1}{r} R_{im}^{\pi/2} \partial_{m} (R_{jn} \eta_{n}) + R_{jl} \partial_{l} \partial_{i} (R_{jn} \eta_{n}) \right] \\ &= R_{ki} \left[R_{j\phi} \frac{1}{r} R_{im}^{\pi/2} (\partial_{m} R_{jn}) \eta_{n} + R_{j\phi} \frac{1}{r} R_{im}^{\pi/2} R_{jn} (\partial_{m} \eta_{n}) \right] \\ &= R_{ki} \left[R_{j\phi} \frac{1}{r} R_{im}^{\pi/2} (\partial_{m} R_{jn}) \eta_{n} + R_{j\phi} \frac{1}{r} R_{im}^{\pi/2} R_{jn} (\partial_{m} \eta_{n}) \right] \\ &= R_{ki} \left[R_{j\phi} \frac{1}{r} R_{im}^{\pi/2} (\partial_{m} R_{jn}) \eta_{n} + R_{j\phi} (\partial_{l} R_{jn}) (\partial_{l} \eta_{n}) + R_{jl} (\partial_{l} R_{jn}) (\partial_{l} \eta_{n}) + R_{jl} (\partial_{l} R_{jn}) (\partial_{l} \eta_{n}) \right] \\ &= R_{ki} \left[\frac{1}{r^{2}} R_{i\phi}^{\pi/2} \eta_{r} + \frac{1}{r} R_{im}^{\pi/2} \partial_{m} \eta_{\phi} \right. \\ &+ \delta_{i\phi} R_{jl} \eta_{n} \partial_{l} \left(\frac{1}{r} R_{jn} \right) + \frac{1}{r} \partial_{i\eta} \eta_{r} + \delta_{i\phi} \frac{1}{r} R_{ln}^{\pi/2} \partial_{l} \eta_{n} + \partial_{l} \partial_{i\eta} \right] \\ &= R_{ki} \left[-\frac{1}{r^{2}} \delta_{ir} \eta_{r} + \frac{1}{r} \delta_{i\phi} \partial_{r} \eta_{\phi} - \frac{1}{r^{2}} \delta_{ir} \partial_{\phi} \eta_{\phi} \right. \\ &+ \delta_{i\phi} R_{jl} \eta_{n} \left(\partial_{l} \frac{1}{r} \right) R_{jn} + \delta_{i\phi} R_{jl} \eta_{n} \frac{1}{r} \left(\partial_{l} R_{jn} \right) + \frac{1}{r} \partial_{i} \eta_{r} + \delta_{i\phi} \frac{1}{r^{2}} \partial_{\phi} \eta_{r} - \delta_{j\phi} \frac{1}{r} \partial_{r} \eta_{\phi} + \partial_{l} \partial_{i} \eta_{l} \right] \\ &= R_{ki} \left[-\frac{1}{r^{2}} \delta_{ir} \eta_{r} - \frac{1}{r^{2}} \delta_{ir} \partial_{\phi} \eta_{\phi} \right. \\ &- \delta_{i\phi} \eta_{p} \frac{1}{r^{2}} + \delta_{i\phi} \eta_{r} \frac{1}{r^{2}} + \frac{1}{r} \partial_{i} \eta_{r} + \delta_{i\phi} \frac{1}{r^{2}} \partial_{\phi} \eta_{r} + \partial_{l} \partial_{i} \eta_{l} \right] \\ &= R_{ki} \left[-\frac{1}{r^{2}} \delta_{ir} \eta_{r} - \frac{1}{r^{2}} \delta_{ir} \partial_{\phi} \eta_{\phi} + \frac{1}{r} \partial_{i} \eta_{r} + \delta_{i\phi} \frac{1}{r^{2}} \partial_{\phi} \eta_{r} + \partial_{l} \partial_{i} \eta_{l} \right] \end{aligned}$$

$$(60)$$

PHYSICAL OBSERVABLES

In Cartesian basis we denote $\mathbf{r} = (x, y)$. A Galilean boost with velocity \mathbf{u} introduce a local gauge transformation $\eta_i(\mathbf{r}) \stackrel{\mathbf{u}}{\to} \eta_i(\mathbf{r}) e^{-iM\mathbf{u}\cdot\mathbf{r}/\hbar}$, where M is the mass of a pair of Helium atoms. The phase gradient correspond to a velocity field $\mathbf{v} = \frac{\hbar}{M} \nabla \theta$ which transform as $\mathbf{v} \stackrel{\mathbf{u}}{\to} \mathbf{v} - \mathbf{u}$ under the Galilean boost. We also have $\partial_i \stackrel{\mathbf{u}}{\to} \partial_i - iMu_i/\hbar$. The GL free energy density also transforms as $f \stackrel{\mathbf{u}}{\to} f - \mathbf{j} \cdot \mathbf{u} + \mathcal{O}(u^2)$, where $\mathbf{j}_k = \frac{2M}{\hbar} \operatorname{Im} \left\{ \kappa_1 \eta_j^* \partial_k \eta_j + \kappa_2 \eta_k^* \partial_j \eta_j + \kappa_3 \eta_j^* \partial_j \eta_k \right\}$ is the superfluid mass current density or the momentum density. We transform it to cylindrical

basis and get

$$\mathbf{j}_{k} = \frac{2M}{\hbar} \left[\operatorname{Im} \left\{ \kappa_{1} \eta_{j}^{*} \partial_{k} \eta_{j} + \kappa_{2} \eta_{k}^{*} \partial_{j} \eta_{j} + \kappa_{3} \eta_{j}^{*} \partial_{j} \eta_{k} \right\} + \operatorname{Im} \left\{ (2\kappa_{1} + \kappa_{2} + \kappa_{3}) \delta_{k\phi} \eta_{\phi}^{*} \eta_{r} / r \right\} \right], \tag{61}$$

In weak-coupling limit, where $\beta_1 = 2\beta_2$ and $\kappa_1 = \kappa_2 = \kappa_3$, the angular momentum density $l_z = rj_\phi$ becomes

$$l_z = \frac{2M\kappa_1}{\hbar} \left[\operatorname{Im} \left\{ 3\eta_{\phi}^* \partial_{\phi} \eta_{\phi} + \eta_r^* \partial_{\phi} \eta_r + 4\eta_{\phi}^* \eta_r \right\} + \operatorname{Im} \left\{ r \eta_{\phi}^* \partial_r \eta_r + r \eta_r^* \partial_r \eta_{\phi} \right\} \right].$$
 (62)

ROTATING FRAME

We can stabilize these low-flow states by rotating the annulus at certain angular velocities Ω_m . Transform into the rotating frame with angular velocity $\Omega = \hat{\mathbf{z}}\Omega$, and the free energy becomes

$$F' = F - \mathbf{L} \cdot \mathbf{\Omega}. \tag{63}$$

The critical angular velocity Ω_m which increase the winding number from m-1 to m should satisfy $F'(m,\Omega_m)=F'(m-1,\Omega_m)$, which gives

$$\Omega_m^{\pm} = \frac{F_{\pm}(m) - F_{\pm}(m-1)}{L_{\pm}(m) - L_{\pm}(m-1)}.$$
(64)

UNIFORM-FLOW APPROXIMATION

We can have superflow in the annulus $\mathbf{v}(\mathbf{r}) = \frac{\hbar}{M} \nabla \theta$ where M is the mass of a pair of Helium atoms.

When we assume a uniform flow field along the azimuthal direction in the annulus, $\mathbf{v} = v\hat{\phi}$ and $v(r) = \frac{\hbar}{Mr}\partial_{\phi}\theta$, the order parameter in the annulus simply gains an extra phase factor along the azimuthal direction.

$$\boldsymbol{\eta}(r,\phi,v_s) = \boldsymbol{\eta}(r,v_s)e^{iMvr\phi/\hbar} = \boldsymbol{\eta}(r,v_s)e^{i(\partial_{\phi}\theta)\phi}$$
(65)

When the flow field is small, we denote

$$\boldsymbol{\eta}(r,\phi,v_s) = \boldsymbol{\eta}(r,v_s)e^{i(\partial_{\phi}\theta)\phi} = e^{\pm i\phi} \left(\eta_r^{(m)}(r), \pm i\eta_{\phi}^{(m)}(r)\right)e^{im\phi}$$
(66)

where m is an integer as the order parameter should be single-valued after 2π winding.

LOW-FLOW APPROXIMATION

For low flow states in an annulus, we further assume the radial profile won't be affected by the small flow field

$$\eta_i^{(m)}(r) = \eta_i^{(0)}(r) \tag{67}$$

For the denominator in Eq. (64), we have

$$L_{\pm}(m) - L_{\pm}(m-1) = \frac{2M\kappa_1}{\hbar} \int d^3r \left(3\eta_{\phi}^{(0)} \eta_{\phi}^{(0)} + \eta_r^{(0)} \eta_r^{(0)} \right)$$
 (68)

For the numerator in Eq. (64), we have to pick out terms associated with ∂_{ϕ} , which are

$$f_{\text{grad}} \sim \frac{\kappa_1}{r^2} \left[3(\partial_{\phi} \eta_{\phi})(\partial_{\phi} \eta_{\phi})^* + (\partial_{\phi} \eta_r)(\partial_{\phi} \eta_r)^* \right]$$

$$+ \frac{\kappa_1}{r} \left[(\partial_{\phi} \eta_{\phi})(\partial_r \eta_r)^* + (\partial_r \eta_r)(\partial_{\phi} \eta_{\phi})^* + (\partial_{\phi} \eta_r)(\partial_r \eta_{\phi})^* + (\partial_r \eta_{\phi})(\partial_{\phi} \eta_r)^* \right]$$

$$+ \frac{2\kappa_1}{r^2} \operatorname{Re} \left\{ 3\eta_r^* \partial_{\phi} \eta_{\phi} - \eta_{\phi}^* \partial_{\phi} \eta_r \right\}$$

$$(69)$$

Substitute in Eq. (66)

$$f_{\text{grad}}(m) \sim \frac{\kappa_1}{r^2} (m \pm 1)^2 \left[3 \left(\eta_{\phi}^{(m)} \right)^2 + \left(\eta_r^{(m)} \right)^2 \right]$$

$$+ \frac{2\kappa_1}{r} (1 \pm m) \left[\eta_r^{(m)} \partial_r \eta_{\phi}^{(m)} - \eta_{\phi}^{(m)} \partial_r \eta_r^{(m)} \right]$$

$$- \frac{8\kappa_1}{r^2} (1 \pm m) \eta_r^{(m)} \eta_{\phi}^{(m)}$$
(70)

For the second line, the corresponding total free energy is zero as long as $\eta_i^{(m)}(r)$ is an even function with respect to r = R + D/2,

$$2\pi \int_{R}^{R+D} r \mathrm{d}r \to \int_{R}^{R+D} \eta_i^{(m)} \partial_r \eta_j^{(m)} \mathrm{d}r = 0$$
 (71)

Then we will have

$$F_{\pm}(m) - F_{\pm}(m-1) = \int d^3r \frac{\kappa_1}{r^2} \left\{ (2m-1) \left[3 \left(\eta_{\phi}^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 \right] \right.$$

$$\pm 2 \left[3 \left(\eta_{\phi}^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 - 4 \eta_r^{(0)} \eta_{\phi}^{(0)} \right] \right\}$$
(72)

We denote the volume integral as $\langle ... \rangle_V = \int d^3 r(...)$. Then we have

$$\Omega_{m}^{\pm} = \frac{\hbar}{M} \frac{\left(m - \frac{1}{2}\right) \left\langle \frac{1}{r^{2}} \left[3 \left(\eta_{\phi}^{(0)} \right)^{2} + \left(\eta_{r}^{(0)} \right)^{2} \right] \right\rangle_{V} \pm \left\langle \frac{1}{r^{2}} \left[3 \left(\eta_{\phi}^{(0)} \right)^{2} + \left(\eta_{r}^{(0)} \right)^{2} - 4 \eta_{r}^{(0)} \eta_{\phi}^{(0)} \right] \right\rangle_{V}}{\left\langle 3 \left(\eta_{\phi}^{(0)} \right)^{2} + \left(\eta_{r}^{(0)} \right)^{2} \right\rangle_{V}}$$

We denote $\langle ... \rangle_r \equiv \int_R^{R+D} \mathrm{d}r(...)$. When $R \gg D$, we can approximate $\langle \mathcal{O} \rangle_V \approx 2\pi R h \langle \mathcal{O} \rangle_r$ and $\langle \frac{1}{r^2} \mathcal{O} \rangle_V \approx \frac{2\pi h}{R} \langle \mathcal{O} \rangle_r$, where h is the z direction thickness, and we have

$$\Omega_m^{\pm} \approx \frac{\hbar}{MR^2} \left[\left(m - \frac{1}{2} \right) \pm \frac{\left\langle 3 \left(\eta_{\phi}^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 - 4 \eta_r^{(0)} \eta_{\phi}^{(0)} \right\rangle_r}{\left\langle 3 \left(\eta_{\phi}^{(0)} \right)^2 + \left(\eta_r^{(0)} \right)^2 \right\rangle_r} \right]$$
(73)

If we use London approximation for $\left\langle 3\left(\eta_{\phi}^{(0)}\right)^2+\left(\eta_r^{(0)}\right)^2\right\rangle_r\approx 4\eta_0^2D$, then

$$\Omega_m^{\pm} \approx \frac{\hbar}{MR^2} \left[\left(m - \frac{1}{2} \right) \pm \left(1 - \frac{\left\langle \eta_r^{(0)} \eta_\phi^{(0)} \right\rangle_r}{\eta_0^2 D} \right) \right] \tag{74}$$

If we assume $\eta_r^{(0)}$ and $\eta_\phi^{(0)}$ are even functions with respect to r=R+D/2, then we have

$$\left\langle \eta_r^{(0)} \eta_\phi^{(0)} \right\rangle_r = 2 \int_R^{R+D/2} \eta_r^{(0)} \eta_\phi^{(0)} dr$$
 (75)

If we approximate $\eta_{\phi}^{(0)}\eta_{r}^{(0)} \approx \eta_{0}^{2} \tanh(ax/\xi)$, where $x \equiv r - R$, then

$$\left\langle \eta_r^{(0)} \eta_\phi^{(0)} \right\rangle_r \approx 2\eta_0^2 \xi \int_0^{D/2} \tanh\left(\frac{ax}{\xi}\right) d\left(\frac{x}{\xi}\right)$$
 (76)

$$= \frac{2}{a} \eta_0^2 \xi \ln\left(\cosh\left(ax/\xi\right)\right) \Big|_0^{D/2}$$
 (77)

$$= \frac{2}{a}\eta_0^2 \xi \ln \left(\cosh \left(\frac{aD}{2\xi} \right) \right) \tag{78}$$

Note that $\cosh(x) = \frac{e^x + e^{-x}}{2} \approx e^x/2$ when $x \gg 1$. If we assume $D/\xi \gg 1$, then

$$\left\langle \eta_r^{(0)} \eta_\phi^{(0)} \right\rangle_r \approx \frac{2}{a} \eta_0^2 \xi \left(\frac{aD}{2\xi} - \ln(2) \right) = \eta_0^2 D - \frac{2}{a} \ln(2) \eta_0^2 \xi$$
 (79)

which gives

$$\Omega_m^{\pm} \approx \frac{\hbar}{MR^2} \left[\left(m - \frac{1}{2} \right) \pm \frac{2}{a} \ln(2) \frac{\xi}{D} \right]$$
 (80)

when a = 1/3 and $D = 30\xi$, we have

$$\Omega_1^{\pm} \approx \frac{\hbar}{MR^2} \left(\frac{1}{2} \pm \frac{\ln(2)}{5} \right) \tag{81}$$

$$\Omega_1^{\pm} \approx \frac{\hbar}{MR^2} \left(\frac{1}{2} \pm 0.139 \right) \tag{82}$$

Exact numerical solution pf Eq. (73) gives

$$\Omega_1^{\pm} \approx \frac{\hbar}{MR^2} \left(\frac{1}{2} \pm 0.122 \right) \tag{83}$$

LONDON APPROXIMATION

We assume $\eta_i^{(m)}(r) = \eta_0$. The angular momentum increase is

$$L_{\pm}(m) - L_{\pm}(m-1) = \frac{8M}{\hbar} \kappa_1 \eta_0^2 \pi \left[(R+D)^2 - R^2 \right] h$$
 (84)

The free energy increase is

$$F_{\pm}(m) - F_{\pm}(m-1) = 8\pi\kappa_1 \eta_0^2 \ln\left(\frac{R+D}{R}\right) h(2m-1)$$
 (85)

where h is the thickness of the annulus. The critical angular velocity is

$$\Omega_m^{\pm} = \Omega_c \frac{\xi}{D} \frac{\ln(1 + D/R)}{2 + D/R} (2m - 1), \tag{86}$$

where $\Omega_c = v_c/R$, $v_c = \hbar/M\xi$, $\xi^2 = \kappa_1/|\alpha|$. Actually, all of the GL parameters drop out

$$\Omega_m^{\pm} = \frac{\hbar}{MRD} \frac{\ln(1 + D/R)}{2 + D/R} (2m - 1), \tag{87}$$

When $R \gg D$, we have

$$\Omega_m^{\pm} = \frac{\hbar}{MR^2} \left(m - \frac{1}{2} \right) \tag{88}$$

i.e.

$$\Omega_1 = \frac{\hbar}{MR^2} \frac{1}{2} \quad \text{for} \quad m = 0 \to m = 1$$
 (89)

$$\Omega_2 = \frac{\hbar}{MR^2} \frac{3}{2} \quad \text{for} \quad m = 1 \to m = 2 \tag{90}$$

$$\Omega_3 = \frac{\hbar}{MR^2} \frac{5}{2} \quad \text{for} \quad m = 2 \to m = 3 \tag{91}$$

$$\dots$$
 (92)

NUMERICAL SOLUTION

From the bulk free energy term we can get the bulk order parameter $\eta_0 = \frac{1}{2} \sqrt{\frac{|\alpha|}{\beta_1}}$. We also define the coherence length $\xi = \sqrt{\kappa_1/|\alpha|}$.

In weak-coupling limit, the GL equations in cylindrical basis are

$$\alpha \eta_{k} + 2\beta_{1} \eta_{i} \eta_{i}^{*} \eta_{k} + \beta_{1} \eta_{i}^{2} \eta_{k}^{*} - \kappa_{1} \left(\partial_{i}^{2} \eta_{k} + \partial_{k} \partial_{i} \eta_{i} + \partial_{j} \partial_{k} \eta_{j} \right)$$

$$- \frac{\kappa_{1}}{r} \left[3 \frac{\partial_{\phi}}{r} \left(\eta_{r} \delta_{k\phi} - \eta_{\phi} \delta_{kr} \right) + \partial_{r} \eta_{k} + 2 \partial_{k} \eta_{r} \right]$$

$$+ \frac{\kappa_{1}}{r^{2}} \left[\eta_{k} + 2 \eta_{r} \delta_{kr} \right] = 0.$$

$$(93)$$

The dimensionless GL equations in cylindrical basis are

$$-\eta_k + \frac{1}{2}\eta_i\eta_i^*\eta_k + \frac{1}{4}\eta_i^2\eta_k^* - (\partial_i^2\eta_k + \partial_k\partial_i\eta_i + \partial_j\partial_k\eta_j)$$

$$-\frac{1}{r}\left[3\frac{\partial_\phi}{r}\left(\eta_r\delta_{k\phi} - \eta_\phi\delta_{kr}\right) + \partial_r\eta_k + 2\partial_k\eta_r\right]$$

$$+\frac{1}{r^2}\left[\eta_k + 2\eta_r\delta_{kr}\right] = 0.$$
(94)

The r component of the GL equations is

$$-\eta_{r} + \frac{1}{2}\eta_{i}\eta_{i}^{*}\eta_{r} + \frac{1}{4}\eta_{i}^{2}\eta_{r}^{*} - (\partial_{i}^{2}\eta_{r} + \partial_{r}\partial_{i}\eta_{i} + \partial_{j}\partial_{r}\eta_{j}) - \frac{1}{r}\left[-3\frac{\partial_{\phi}}{r}\eta_{\phi} + 3\partial_{r}\eta_{r}\right] + \frac{3}{r^{2}}\eta_{r} = 0$$

$$-\eta_{r} + \frac{1}{2}\eta_{i}\eta_{i}^{*}\eta_{r} + \frac{1}{4}\eta_{i}^{2}\eta_{r}^{*} - (\partial_{i}^{2}\eta_{r} + \partial_{r}\partial_{i}\eta_{i} + \partial_{j}\partial_{r}\eta_{j}) + 3\left[\frac{1}{r^{2}}\partial_{\phi}\eta_{\phi} - \frac{1}{r}\partial_{r}\eta_{r} + \frac{\eta_{r}}{r^{2}}\right] = 0$$

$$-\eta_{r} + \frac{1}{2}\eta_{i}\eta_{i}^{*}\eta_{r} + \frac{1}{4}\eta_{i}^{2}\eta_{r}^{*} - 3\partial_{r}^{2}\eta_{r} - \frac{1}{r^{2}}\partial_{\phi}^{2}\eta_{r} - \frac{2}{r}\partial_{r}\partial_{\phi}\eta_{\phi} + \frac{4}{r^{2}}\partial_{\phi}\eta_{\phi} - \frac{3}{r}\partial_{r}\eta_{r} + \frac{3}{r^{2}}\eta_{r} = 0$$

The ϕ component of the GL equations is

$$-\eta_{\phi} + \frac{1}{2}\eta_{i}\eta_{i}^{*}\eta_{\phi} + \frac{1}{4}\eta_{i}^{2}\eta_{\phi}^{*} - \left(\partial_{i}^{2}\eta_{\phi} + \frac{\partial_{\phi}}{r}\partial_{i}\eta_{i} + \partial_{j}\frac{\partial_{\phi}}{r}\eta_{j}\right) - \frac{1}{r}\left[5\frac{\partial_{\phi}}{r}\eta_{r} + \partial_{r}\eta_{\phi}\right] + \frac{1}{r^{2}}\eta_{\phi} = 0$$
$$-\eta_{\phi} + \frac{1}{2}\eta_{i}\eta_{i}^{*}\eta_{\phi} + \frac{1}{4}\eta_{i}^{2}\eta_{\phi}^{*} - 3\frac{1}{r^{2}}\partial_{\phi}^{2}\eta_{\phi} - \partial_{r}^{2}\eta_{\phi} - \frac{2}{r}\partial_{r}\partial_{\phi}\eta_{r} - \frac{4}{r^{2}}\partial_{\phi}\eta_{r} - \frac{1}{r}\partial_{r}\eta_{\phi} + \frac{1}{r^{2}}\eta_{\phi} = 0$$

For the uniform flow approximation,

$$\boldsymbol{\eta}(r,\phi) = \left(\eta_r^{(n)}(r), \eta_\phi^{(n)}(r)\right) e^{in\phi} \tag{95}$$

which means that $\partial_{\phi} \to in$. We have

$$-\eta_r + \frac{1}{2}\eta_i\eta_i^*\eta_r + \frac{1}{4}\eta_i^2\eta_r^* - 3\partial_r^2\eta_r + \frac{n^2}{r^2}\eta_r - \frac{2in}{r}\partial_r\eta_\phi + \frac{4in}{r^2}\eta_\phi - \frac{3}{r}\partial_r\eta_r + \frac{3}{r^2}\eta_r = 0$$

$$-\eta_\phi + \frac{1}{2}\eta_i\eta_i^*\eta_\phi + \frac{1}{4}\eta_i^2\eta_\phi^* + \frac{3n^2}{r^2}\eta_\phi - \partial_r^2\eta_\phi - \frac{2in}{r}\partial_r\eta_r - \frac{4in}{r^2}\eta_r - \frac{1}{r}\partial_r\eta_\phi + \frac{1}{r^2}\eta_\phi = 0$$

We ignore the (n) superscript for the sake of simplicity. The solutions of n=1 may corresponds to the $(p_x+ip_y, m=0)$ state, or the $(p_x-ip_y, m=2)$ state.

LARGE FLOW

When the flow field is large, we denote the superfluid flow field at the inner radius as $v_s = \frac{\hbar}{MR} \partial_{\phi} \theta$, and

$$\boldsymbol{\eta}(r,\phi,v_s) = \boldsymbol{\eta}^{(n)}(r)e^{in\phi} = \boldsymbol{\eta}(r,v_s)e^{i(\partial_\phi\theta)\phi} = \boldsymbol{\eta}(r,v_s)e^{i\frac{v_sR}{v_c\xi}\phi}$$
(96)

where the critical flow field $v_c = \hbar/M\xi$ and $\xi^2 = \kappa_1/|\alpha|$. We also have $n = \frac{v_s}{v_c} \frac{R}{\xi}$.

CARTESIAN BASIS

In Cartesian basis, weak-coupling limit dimensionless GL equations are

$$-\eta_k + \frac{1}{2}\eta_i\eta_i^*\eta_k + \frac{1}{4}\eta_i^2\eta_k^* - \left(\partial_i^2\eta_k + \partial_k\partial_i\eta_i + \partial_j\partial_k\eta_j\right) = 0$$
(97)

The x component of the GL equations is

$$-\eta_x + \frac{3}{4}\eta_x^2 \eta_x^* + \frac{1}{2}\eta_y \eta_y^* \eta_x + \frac{1}{4}\eta_y^2 \eta_x^* - 3\partial_x^2 \eta_x - \partial_y^2 \eta_x - 2\partial_y \partial_x \eta_y = 0$$
 (98)

The y component of the GL equations is

$$-\eta_y + \frac{3}{4}\eta_y^2 \eta_y^* + \frac{1}{2}\eta_x \eta_x^* \eta_y + \frac{1}{4}\eta_x^2 \eta_y^* - 3\partial_y^2 \eta_y - \partial_x^2 \eta_y - 2\partial_x \partial_y \eta_x = 0$$
 (99)