

MEAN-FIELD THEORY

The mean-field Nambu-Gor'kov Hamiltonian is

$$H = \frac{1}{2} \iint d^3\mathbf{r} d^3\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r}') \begin{pmatrix} \xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\hat{1} & \hat{\Delta}(\mathbf{r},\mathbf{r}') \\ \hat{\Delta}^\dagger(\mathbf{r},\mathbf{r}') & -\xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\hat{1} \end{pmatrix} \hat{\Psi}(\mathbf{r}) \quad (1)$$

$$\equiv \frac{1}{2} \iint d^3\mathbf{r} d^3\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r}') \hat{\tau}_3 \left(\xi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') + \hat{\Delta}(\mathbf{r},\mathbf{r}') \right) \hat{\Psi}(\mathbf{r}). \quad (2)$$

where

$$\hat{\Delta}(\mathbf{r},\mathbf{r}') = \begin{pmatrix} 0 & \hat{\Delta}(\mathbf{r},\mathbf{r}') \\ -\hat{\Delta}^\dagger(\mathbf{r},\mathbf{r}') & 0 \end{pmatrix} \quad (3)$$

When the system is homogeneous, the two-point gap function only depends on the relative coordinate, i.e., $\hat{\Delta}(\mathbf{r},\mathbf{r}') = \hat{\Delta}(\mathbf{r}-\mathbf{r}')$. In this case, we can transform to momentum space, and the Hamiltonian becomes

$$H = \frac{1}{2} \sum_{\mathbf{p}} \hat{\Psi}_{\mathbf{p}}^\dagger \hat{\tau}_3 (\xi_{\mathbf{p}} + \hat{\Delta}(\mathbf{p})) \hat{\Psi}_{\mathbf{p}}. \quad (4)$$

where $\hat{\Psi}_{\mathbf{p}}^\dagger = (c_{\mathbf{p}\uparrow}^\dagger, c_{\mathbf{p}\downarrow}^\dagger, c_{-\mathbf{p}\uparrow}, c_{-\mathbf{p}\downarrow})$.

FINITE TEMPERATURE

The imaginary time Green's function is

$$\hat{G}(\mathbf{r}\tau, \mathbf{r}'\tau') = \hat{G}(\mathbf{r}, \mathbf{r}'; \tau - \tau') = - \left\langle T_\tau \hat{\Psi}(\mathbf{r}, \tau) \hat{\Psi}^\dagger(\mathbf{r}', \tau') \right\rangle \equiv \begin{pmatrix} \hat{G} & \hat{F} \\ \hat{F} & \hat{G} \end{pmatrix} \quad (5)$$

The unperturbed Matsubara Green's function is

$$\hat{G}_0(\mathbf{p}, i\epsilon_n) = (i\epsilon_n - \hat{\tau}_3 \xi_{\mathbf{p}})^{-1} = - \frac{i\epsilon_n + \hat{\tau}_3 \xi_{\mathbf{p}}}{\epsilon_n^2 + \xi_{\mathbf{p}}^2}. \quad (6)$$

In bulk, the Gorkov Green's function satisfies

$$\hat{G}^{-1}(\mathbf{p}, i\epsilon_n) = \hat{G}_0^{-1}(\mathbf{p}, i\epsilon_n) - \hat{\tau}_3 \hat{\Delta}(\mathbf{p}) = i\epsilon_n - \hat{\tau}_3 \xi_{\mathbf{p}} - \hat{\tau}_3 \hat{\Delta}(\mathbf{p}). \quad (7)$$

which gives

$$\hat{G}(\mathbf{p}, i\epsilon_n) = - \frac{i\epsilon_n + \hat{\tau}_3 \xi_{\mathbf{p}} + \hat{\tau}_3 \hat{\Delta}(\mathbf{p})}{\epsilon_n^2 + \xi_{\mathbf{p}}^2 + |\Delta|^2} \quad (8)$$

The bulk quasiclassical Green's function is

$$\hat{g}(\mathbf{p}, i\epsilon_n) = \int d\xi_p \hat{G}(\mathbf{p}, i\epsilon_n) \hat{\tau}_3 = -\pi \frac{i\epsilon_n \hat{\tau}_3 - \hat{\Delta}(\mathbf{p})}{\sqrt{|\Delta|^2 + \epsilon_n^2}} = \frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}} \begin{pmatrix} -i\epsilon_n & \hat{\Delta} \\ -\hat{\Delta}^\dagger & i\epsilon_n \end{pmatrix} \equiv \begin{pmatrix} \hat{g} & \hat{f} \\ \hat{f} & \hat{g} \end{pmatrix} \quad (9)$$

and it satisfies the normalization condition

$$\hat{g}(\mathbf{p}, i\epsilon_n)^2 = -\pi^2 \hat{1}. \quad (10)$$

The bulk quasiclassical Green's function satisfies

$$\left[i\epsilon_n \hat{\tau}_3 - \hat{\Delta}, \hat{g} \right] = 0 \quad (11)$$

But when near the boundary or impurities, the quasiclassical Green's function $\hat{g}(\mathbf{R}, \mathbf{p}; \epsilon)$ and the gap function $\hat{\Delta}(\mathbf{R}, \mathbf{p})$ are inhomogeneous, and are described by the Eilenberger equation and gap equation.

$$\left[i\epsilon_n \hat{\tau}_3 - \hat{\Delta}(\mathbf{R}, \mathbf{p}), \hat{g}(\mathbf{R}, \mathbf{p}; i\epsilon_n) \right] + i\hbar \mathbf{v}_p \cdot \nabla \hat{g}(\mathbf{R}, \mathbf{p}; i\epsilon_n) = 0 \quad (12)$$

and

$$\hat{\Delta}(\mathbf{R}, \mathbf{p}) = T \sum_{\epsilon_n = -\omega_c}^{\omega_c} \left\langle v(\mathbf{p}, \mathbf{p}') \hat{f}(\mathbf{R}, \mathbf{p}', \epsilon_n) \right\rangle_{p'} \quad (13)$$

Here $\mathbf{R} = \frac{1}{2}(\mathbf{r} + \mathbf{r}')$ is the center of the pair.

CURRENT DENSITY

The one-particle density matrix is

$$\rho_{\uparrow\uparrow}(\mathbf{r}, \mathbf{r}') \equiv \left\langle \psi_{\uparrow}^\dagger(\mathbf{r}') \psi_{\uparrow}(\mathbf{r}) \right\rangle = G_{\uparrow\uparrow}(\mathbf{r}, \mathbf{r}'; 0^-) = T \sum_{n=-\infty}^{\infty} G_{\uparrow\uparrow}(\mathbf{r}, \mathbf{r}'; i\epsilon_n) e^{-i\epsilon_n 0^-} \quad (14)$$

The local current density is

$$\mathbf{j}(\mathbf{R}) = 2N_f T \sum_n \langle \mathbf{v}_p g(\mathbf{R}, \mathbf{p}, \epsilon_n) \rangle_p \quad (15)$$

Multiple it by the quasiparticle mass $p_f = m^* v_f$, and we get the mass current density:

$$\mathbf{j}_m(\mathbf{R}) \equiv m^* \mathbf{j}(\mathbf{R}) = \frac{2N_f v_f p_f}{4} \hbar \times \frac{2T}{\xi(T) |\Delta(T)|} \sum_n \langle \hat{\mathbf{p}} g(\mathbf{R}, \mathbf{p}, \epsilon_n) \rangle_p \quad (16)$$

where $\xi(T) = \hbar v_f / 2|\Delta(T)|$ is the coherence length, and $n = 2N_f p_f v_f$ is the particle density. The ground state angular momentum in a cylinder is

$$\mathbf{L} = \int_0^R d^3r \mathbf{r} \times \mathbf{j}_m(\mathbf{r}) \quad (17)$$

$$= 2\pi \hbar \hat{\mathbf{z}} \int_0^R dr r^2 j_m(r) \quad (18)$$

$$\approx 2\pi R^2 \hbar \hat{\mathbf{z}} \int_{R-\xi}^R dr j_m(r) \quad (19)$$

$$\approx 2V \hat{\mathbf{z}} \int_{R-\xi}^R dr j_m(r) \quad (20)$$

$$= \frac{N}{2} \hbar \hat{\mathbf{z}} \times \frac{2T}{\xi(T)|\Delta(T)|} \int_{R-\xi}^R dr \sum_n \langle p_y g(\mathbf{r}, \mathbf{p}, \epsilon_n) \rangle_p \quad (21)$$

GAP EQUATION

Now we neglect the spin structure, i.e. discard the little hats. In bulk, we have

$$\Delta(\mathbf{p}) = T \sum_{\epsilon_n = -\omega_c}^{\omega_c} \left\langle v(\mathbf{p}, \mathbf{p}') \frac{\pi \Delta(\mathbf{p}')}{\sqrt{|\Delta|^2 + \epsilon_n^2}} \right\rangle_{p'} \quad (22)$$

Move everything to the right hand side and subtract Eq. (13) from it, we get

$$0 = T \sum_{\epsilon_n = -\infty}^{\infty} \left[\left\langle v(\mathbf{p}, \mathbf{p}') \frac{f(\mathbf{R}, \mathbf{p}', \epsilon_n)}{\Delta(\mathbf{R}, \mathbf{p})} \right\rangle_{p'} - \left\langle v(\mathbf{p}'', \mathbf{p}') \frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}} \frac{\Delta(\mathbf{p}')}{\Delta(\mathbf{p}'')} \right\rangle_{p'} \right] \quad (23)$$

which gives

$$\Delta(\mathbf{R}, \mathbf{p}) = \frac{T \sum_{\epsilon_n}^{\infty} \left\langle v(\mathbf{p}, \mathbf{p}') \hat{f}(\mathbf{R}, \mathbf{p}', \epsilon_n) \right\rangle_{p'}}{T \sum_{\epsilon_n}^{\infty} \left\langle v(\mathbf{p}'', \mathbf{p}') \frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}} \frac{\Delta(\mathbf{p}')}{\Delta(\mathbf{p}'')} \right\rangle_{p'}} \quad (24)$$

We have the Digamma function

$$K(T) = T \sum_{\epsilon_n = -\omega_c}^{\omega_c} \frac{\pi}{|\epsilon_n|} \approx \ln \left(1.13 \frac{\omega_c}{T} \right) \quad (25)$$

In bulk at T_c , we have

$$\Delta(\mathbf{p}) = K(T_c) \langle v(\mathbf{p}, \mathbf{p}') \Delta(\mathbf{p}') \rangle_{p'} \quad (26)$$

and we have

$$K(T_c) - K(T) = \ln(T/T_c) = \frac{\Delta(\mathbf{p}'')}{\langle v(\mathbf{p}'', \mathbf{p}')\Delta(\mathbf{p}') \rangle_{p'} \big|_{T_c}} - T \sum_{\epsilon_n = -\omega_c}^{\omega_c} \frac{\pi}{|\epsilon_n|} \quad (27)$$

Substitute in Eq. (13), we get another equation to determine the gap inhomogeneity:

$$\ln \frac{T}{T_c} = T \sum_{\epsilon_n = -\infty}^{\infty} \left[\frac{\langle v(\mathbf{p}, \mathbf{p}')f(\mathbf{R}, \mathbf{p}', \epsilon_n) \rangle_{p'} / \Delta(\mathbf{R}, \mathbf{p})}{\langle v(\mathbf{p}'', \mathbf{p}')\Delta(\mathbf{p}') \rangle_{p'} / \Delta(\mathbf{p}'') \big|_{T_c}} - \frac{\pi}{|\epsilon_n|} \right] \quad (28)$$

If you want temperature dependence of bulk gap, you can substitute in Eq. (22) to get

$$\ln \frac{T}{T_c} = T \sum_{\epsilon_n = -\infty}^{\infty} \left[\frac{\pi \left\langle v(\mathbf{p}, \mathbf{p}')\Delta(\mathbf{p}') / \Delta(\mathbf{p}) \sqrt{|\Delta|^2 + \epsilon_n^2} \right\rangle_{p'}}{\langle v(\mathbf{p}'', \mathbf{p}')\Delta(\mathbf{p}') \rangle_{p'} / \Delta(\mathbf{p}'') \big|_{T_c}} - \frac{\pi}{|\epsilon_n|} \right] \quad (29)$$

RICCATI PARAMETERIZATIONS

The Riccati parameterization of the Green's function is

$$\hat{g} = -i\pi \hat{N} \begin{pmatrix} 1 + \hat{a}\hat{a} & 2\hat{a} \\ -2\hat{a} & -1 - \hat{a}\hat{a} \end{pmatrix} \quad (30)$$

where \hat{N} is

$$\hat{N} = \begin{pmatrix} (1 - \hat{a}\hat{a})^{-1} & 0 \\ 0 & (1 - \hat{a}\hat{a})^{-1} \end{pmatrix} \quad (31)$$

The inverse is

$$\hat{a} = (\hat{g} - i\pi)^{-1} \hat{f} \quad (32)$$

$$\hat{a} = (\hat{g} + i\pi)^{-1} \hat{f} \quad (33)$$

Substitute in the bulk Green's function to get the bulk Ricatti amplitude:

$$\hat{a} = \frac{-\hat{\Delta}}{i\epsilon_n + i\sqrt{|\Delta|^2 + \epsilon_n^2}} \quad (34)$$

$$\hat{a} = \frac{-\hat{\Delta}^\dagger}{i\epsilon_n + i\sqrt{|\Delta|^2 + \epsilon_n^2}} \quad (35)$$

The matrix Ricatti equation is

$$i\hbar \mathbf{v}_p \cdot \nabla \hat{a} + 2i\epsilon_n \hat{a} + \hat{a} \hat{\Delta}^\dagger \hat{a} + \hat{\Delta} = 0 \quad (36)$$

$$i\hbar \mathbf{v}_p \cdot \nabla \hat{a} - 2i\epsilon_n \hat{a} - \hat{a} \hat{\Delta} \hat{a} - \hat{\Delta}^\dagger = 0 \quad (37)$$

³HE-A

For ³He-A, the spin structure is $\hat{\mathbf{d}} \cdot (i\vec{\sigma}\hat{\sigma}_y) = \hat{\sigma}_x$, and we have two components of the gap function,

$$\hat{\Delta}(\mathbf{R}, \mathbf{p}) = \hat{\sigma}_x(\Delta_1(\mathbf{R})p_x + \Delta_2(\mathbf{R})p_y) \quad (38)$$

In bulk we have $\hat{\Delta}(\mathbf{R}, \mathbf{p}) = \hat{\sigma}_x\Delta(p_x + ip_y) = \hat{\sigma}_x\Delta e^{i\phi_p}$. The interaction is $v(\mathbf{p}, \mathbf{p}') = 3v_0\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'$. Note that

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \cos \phi e^{i\phi} = \frac{1}{2} \quad (39)$$

In 2d case ($|\Delta|^2$ does not depend on \mathbf{p}), Eq. (24) becomes

$$\Delta(\mathbf{R}, \mathbf{p}) = \frac{2 \sum_{\epsilon_n}^{\infty} \langle \mathbf{p} \cdot \mathbf{p}' f(\mathbf{R}, \mathbf{p}', \epsilon_n) \rangle_{p'}}{\sum_{\epsilon_n}^{\infty} \frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}}} \quad (40)$$

The 2d bulk gap equation is

$$\ln \frac{T}{T_c} = T \sum_{\epsilon_n=-\infty}^{\infty} \left[\frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}} - \frac{\pi}{|\epsilon_n|} \right] \quad (41)$$

S-WAVE PAIRING

For s-wave pairing, the spin structure is $i\hat{\sigma}_y$, and we have

$$\hat{\Delta}(\mathbf{R}, \mathbf{p}) = i\hat{\sigma}_y\Delta(\mathbf{R}) \quad (42)$$

In bulk we have $\hat{\Delta}(\mathbf{R}, \mathbf{p}) = i\hat{\sigma}_y\Delta$. The interaction is $v(\mathbf{p}, \mathbf{p}') = v_s$, and Eq. (24) reduces to

$$\Delta(\mathbf{R}) = \frac{\sum_n \langle f(\mathbf{R}, \mathbf{p}', \epsilon_n) \rangle_{p'}}{\sum_n \frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}}} \quad (43)$$

³HE-A EDGE GAP PROFILE

We assume translational invariance along the edge, i.e. the y direction. We also assume the gap to be real(imaginary) along the $p_x(p_y)$ direction.

$$\hat{\Delta}(x, \mathbf{p}) = \hat{\sigma}_x(\Delta_1(x)p_x + i\Delta_2(x)p_y) \quad (44)$$

Here $\Delta_1(x)$ and $\Delta_2(x)$ are real functions, and we have

$$\hat{\Delta}(x, \mathbf{p}) = \begin{pmatrix} 0 & \hat{\sigma}_x(\Delta_1(x)p_x + i\Delta_2(x)p_y) \\ -\hat{\sigma}_x(\Delta_1(x)p_x - i\Delta_2(x)p_y) & 0 \end{pmatrix} \quad (45)$$

$$= i\hat{\sigma}_x(\Delta_2 p_y \hat{\tau}_1 + \Delta_1 p_x \hat{\tau}_2) \quad (46)$$

We can also write the anomalous Green's function as corresponding components to $\Delta_{1,2}$

$$\hat{f}(\mathbf{R}, \mathbf{p}, \epsilon_n) = \hat{\sigma}_x \left(f_1(\mathbf{R}, \mathbf{p}, \epsilon_n) + i f_2(\mathbf{R}, \mathbf{p}, \epsilon_n) \right) \quad (47)$$

$$\underline{\hat{f}}(\mathbf{R}, \mathbf{p}, \epsilon_n) = -\hat{\sigma}_x \left(f_1(\mathbf{R}, \mathbf{p}, \epsilon_n) - i f_2(\mathbf{R}, \mathbf{p}, \epsilon_n) \right) \quad (48)$$

where $f_{1,2}$ are complex functions, and we have

$$\hat{g} = \hat{g}\hat{\tau}_3 + i\hat{\sigma}_x(f_2\hat{\tau}_1 + f_1\hat{\tau}_2) \quad (49)$$

where

$$f_1 = \frac{f - \underline{f}}{2} \quad \text{and} \quad f_2 = \frac{f + \underline{f}}{2i} \quad (50)$$

In this case, we can even write the gap equation as

$$\Delta_{1,2}(x) = T \sum_{\epsilon_n = -\omega_c}^{\omega_c} \langle v(\mathbf{p}_{x,y}, \mathbf{p}') f_{1,2}(x, \mathbf{p}', \epsilon_n) \rangle_{p'} \quad (51)$$

and the gap iteration becomes

$$\Delta_{1,2}(x) = \frac{2 \sum_n \langle \mathbf{p}_{x,y} \cdot \mathbf{p}' f_{1,2}(x, \mathbf{p}', n) \rangle_{p'}}{\sum_n \frac{\pi}{\sqrt{|\Delta|^2 + \epsilon_n^2}}} \quad (52)$$

We can only sum over the positive Matsubara frequencies and take real part because

$$f_{1,2}(\epsilon_n) = f_{1,2}^*(-\epsilon_n) \quad (53)$$

Numerical solution

The bulk gap temperature dependence is

$$\ln t = \pi t \sum_{n=-\infty}^{\infty} \left[\frac{1}{\sqrt{\delta(t)^2 + e_n^2}} - \frac{1}{|e_n|} \right] \quad (54)$$

The dimensionless iteration of the gap function is

$$\delta_{1,2}(x) = \frac{2 \sum_{n>0} \langle \mathbf{p}_{x,y} \cdot \mathbf{p}' \text{Re}\{f_{1,2}(x, \mathbf{p}', n)\} \rangle_{\mathbf{p}'}}{\sum_{n>0} \frac{\pi}{\sqrt{1+e_n^2/\delta(t)^2}}} \quad (55)$$

where $t = T/T_c$, $\delta_{1,2}(x) = \Delta_{1,2}(x)/|\Delta(T)|$, $e_n = \epsilon_n/T_c$ and $\delta(t) = |\Delta(T)|/T_c$. The Riccati equation is

$$ip_x \partial_x a(\mathbf{p}, x) + \frac{ie_n}{\delta(t)} a(\mathbf{p}, x) + \frac{\delta^*(\mathbf{p}, x)}{2} a^2(\mathbf{p}, x) + \frac{\delta(\mathbf{p}, x)}{2} = 0 \quad (56)$$

$$ip_x \partial_x \underline{a}(\mathbf{p}, x) - \frac{ie_n}{\delta(t)} \underline{a}(\mathbf{p}, x) - \frac{\delta(\mathbf{p}, x)}{2} \underline{a}^2(\mathbf{p}, x) - \frac{\delta^*(\mathbf{p}, x)}{2} = 0 \quad (57)$$

where $\xi = \hbar v_f/2|\Delta(T)|$, and $\partial_x = \xi \nabla_x$. The p_x can be interpreted as the projection of quasiparticle trajectory along the x direction. We can get g and $f_{1,2}$ from

$$g = -i\pi \frac{1 + a\underline{a}}{1 - a\underline{a}} \quad (58)$$

$$f = f_1 + if_2 = -i\pi \frac{2a}{1 - a\underline{a}} \quad (59)$$

We first assume an initial guess for the gap function $\delta(\mathbf{r}, \hat{\mathbf{p}})$. For each trajectory $\hat{\mathbf{p}}$ along which the quasiparticle propagates, we can solve the Eilenberger equation to get the Green's function $g(\mathbf{r}, \hat{\mathbf{p}}, n)$ and the anomalous Green's function $f(\mathbf{r}, \hat{\mathbf{p}}, n)$. Then we iterate to update the gap function $\delta(\mathbf{r}, \hat{\mathbf{p}})$ by solving the gap equation. Repeat until convergence.

4-TH ORDER RUNGE-KUTTA METHOD

We can use the 4-th order Runge-Kutta method to solve the Riccati equation.

$$\partial_x a(\mathbf{p}, x) = \frac{i}{p_x} \left(\frac{ie_n}{\delta(t)} a(\mathbf{p}, x) + \frac{\delta_1(x)p_x - i\delta_2(x)p_y}{2} a^2(\mathbf{p}, x) + \frac{\delta_1(x)p_x + i\delta_2(x)p_y}{2} \right) \quad (60)$$

$$\partial_x \underline{a}(\mathbf{p}, x) = \frac{-i}{p_x} \left(\frac{ie_n}{\delta(t)} \underline{a}(\mathbf{p}, x) + \frac{\delta_1(x)p_x + i\delta_2(x)p_y}{2} \underline{a}^2(\mathbf{p}, x) + \frac{\delta_1(x)p_x - i\delta_2(x)p_y}{2} \right) \quad (61)$$

For an ODE

$$\frac{da}{dx} = f(x, a) \quad (62)$$

when step size is h , RK4 method gives

$$k_1 = f(x_n, a_n) \quad (63)$$

$$k_2 = f\left(x_n + \frac{h}{2}, a_n + \frac{h}{2}k_1\right) \quad (64)$$

$$k_3 = f\left(x_n + \frac{h}{2}, a_n + \frac{h}{2}k_2\right) \quad (65)$$

$$k_4 = f(x_n + h, a_n + hk_3) \quad (66)$$

$$a_{n+1} = a_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (67)$$

When iteratively solving the Ricatti equation and the gap equation, we can interpolate the gap function to get the half step value $\delta_{1,2}(x_n + h/2)$.

ANALYTIC CONTINUATION

For the integration contour C being an infinite circle, assume function $g(z)$ has no poles inside the contour and vanishes at infinity, the residue theorem gives

$$\oint_C dz n_F(z) g(z) = 2\pi i \sum_{\text{poles}} \text{Res}\{n_F(z) g(z)\} = 2\pi i (-T) \sum_{n=-\infty}^{\infty} g(\epsilon_n) \quad (68)$$

The residue of the Fermi distribution function $\text{Res}\{n_F(z)\} = -T$. We can separate the contour into upper and lower half circles, which gives

$$\oint_C dz n_F(z) g(z) = \left(\oint_{\text{upper}} + \oint_{\text{lower}} \right) dz n_F(z) g(z) \quad (69)$$

$$= \int_{-\infty}^{\infty} d\epsilon n_F(\epsilon) g^R(\epsilon + i0^+) + \int_{\infty}^{-\infty} d\epsilon n_F(\epsilon) g^A(\epsilon - i0^+) \quad (70)$$

$$= \int_{-\infty}^{\infty} d\epsilon n_F(\epsilon) [g^R(\epsilon + i0^+) - g^A(\epsilon - i0^+)] \quad (71)$$

$$= \int_{-\infty}^{\infty} d\epsilon n_F(\epsilon) 2i \text{Im}\{g^R(\epsilon + i0^+)\} \quad (72)$$

and we get

$$T \sum_{n=-\infty}^{\infty} g(\epsilon_n) = \int_{-\infty}^{\infty} d\epsilon n_F(\epsilon) \left(-\frac{1}{\pi} \text{Im}\{g^R(\epsilon + i0^+)\} \right) \quad (73)$$

Analytically continue Eq. (15) to real frequency, we get

$$\mathbf{j}(\mathbf{R}) = 2N_f \int d\epsilon n_F(\epsilon) \left\langle \mathbf{v}_p \left(-\frac{1}{\pi} \text{Im}\{g(\mathbf{R}, \mathbf{p}, \epsilon)\} \right) \right\rangle_p \quad (74)$$