# Parametric Affine Transformation

May 18 2018
Jungwon Kang

We present a 2D affine transformation, which is composed of transformations of scaling, shear, rotation, and translation. The affine transformation consists of the following four elementary transformations.

## □ Elementary transformation $M_{\mathrm{sc}}$, $M_{\mathrm{sh}}$, $M_{\mathrm{ro}}$, $M_{\mathrm{tr}}$

| Transformation | Parameter | Matrix |
|---|---|---|
| Scaling | $s_x, s_y$ | $M_{\mathrm{sc}} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| Shear | $h_x, h_y$ | $M_{\mathrm{sh}} = \begin{bmatrix} 1 & h_x & 0 \\ h_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| Rotation | $\theta$ | $M_{\mathrm{ro}} = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| Translation | $t_x, t_y$ | $M_{\mathrm{tr}} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$ |

## □ Composite transformation $M_a$

Based on the above elementary transformations, we build a transformation $M_a$:

$$M_a = M_{\mathrm{tr}} \cdot \widehat{M}_{\mathrm{ro}} \cdot \widehat{M}_{\mathrm{sh}} \cdot \widehat{M}_{\mathrm{sc}}$$

If a point $\mathbf{p} = [x, y, 1]^T$ exists, we can get a transformed point $\mathbf{p}' = M_a \cdot \mathbf{p}$.

Because we hope the rotation, shear and scaling can be done at an arbitrary point (e.g. the center point of given points), we use the following *translation-offset* applied elementary transformation:

$$\widehat{M}_{\mathrm{oo}} = M_{\mathrm{cen}} \cdot M_{\mathrm{oo}} \cdot M_{\mathrm{cen}}^{-1}$$

where $M_{\mathrm{oo}}$ is an elementary transformation, and $M_{\mathrm{cen}}$ is a matrix representing the translation-offset.

If we want to use the center point of all the given points, then $M_{\mathrm{cen}}$ is given as

$$M_{\mathrm{cen}} = \begin{bmatrix} 1 & 0 & t_{cen,x} \\ 0 & 1 & t_{cen,y} \\ 0 & 0 & 1 \end{bmatrix}$$

where $t_{cen,x} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{p}_x^{(i)}$, $t_{cen,y} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{p}_y^{(i)}$, and $\mathbf{p}_x^{(i)}$, $\mathbf{p}_y^{(i)}$ are x, y value of $i$th point, respectively.

□ Reference

http://www.cs.brandeis.edu/~cs155/Lecture_06.pdf
https://people.cs.clemson.edu/~dhouse/courses/401/notes/affines-matrices.pdf