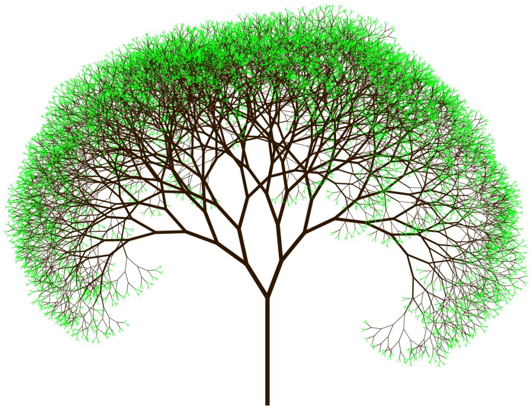
Hausdorff Measure and Hausdorff Dimension



Final Project in course
Math 710 Measure and Integration
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1 Introduction

The notion of dimension seems to be intuitive and simple. However, without a detailed study in analysis, some of the notions in geometry, such as dimension, turn out to be difficult to precisely define. For instance, one way to define the dimension of a square on a plane is to count how may copies of a square we need to produce the square scaled by an integer n. It is not hard to see that we need n^2 squares to do so. For instance, if we have scaled the square by a factor of 2, then we need 4 squares to produce that 2-scaled square. What about a line and a cube? A line needs exactly n lines to produce the n-scaled line, and after a bit of thought and drawing, it is easy to see that the cube needs n^3 cubes to produce the n-scaled cube. From these, we can read off the exponents to define the dimension of that set. For instance, the dimension of the cube is $\log_n n^3 = 3$, in alignment with our intuition. Now, recall one of our most interesting set on a real line, the Cantor set. Because of its construction being iterating same copies of itself, we only need 2 copies of it to produce the 3-scaled Cantor set. This implies that according to our guiding principle, the dimension of a Cantor set is $\log_3 2 = \log 2/\log 3$. This is surprising, firstly because the Cantor set is still a nontrivial (uncountable) subset of a real line yet having a dimension other than 1, and secondly, because the dimension is non-integer. This example invites us to the following question. How can we rigorously define the dimension of sets in $\mathbb{R}^d$? It is possible to give a satisfying answer on this question by introducing the notion of Hausdorff dimension, introduced in 1918 by mathematician Felix Hausdorff.

2 Exterior Hausdorff measure

We first introduce some basic notation that we will use.

Definition 2.1. *For any subset U of $\mathbb{R}^d$, let diamU denote its diameter, that is diamU $= \sup\{|x-y| : x, y \in U\}$ and diam$\varnothing = 0$.*

In order to define the Hausdorff measure, we first start with the definition of the exterior Hausdorff measure.

Definition 2.2. *For each $\alpha > 0$, $\delta > 0$ and $E \subset \mathbb{R}^d$, define the Hausdorff premeasure*

$$\mathcal{H}_\alpha^\delta(E) = \inf\left\{\sum_{k=1}^{\infty}(\mathrm{diam}\, F_k)^\alpha : E \subset \bigcup_{k=1}^{\infty} F_k,\ \textit{diam}\ F_k \leq \delta\ \textit{all}\ k\right\},$$

and define the exterior α-dimensional Hausdorff measure

$$m_\alpha^*(E) = \lim_{\delta\to 0}\mathcal{H}_\alpha^\delta(E).$$

Next, we examine some basic properties of the exterior Hausdorff measure. The proofs are analogous to those of the exterior Lebesgue measure.

Proposition 1 (Null empty set)**.** $m_\alpha^*(\varnothing) = 0$

Proof. This directly follows from the fact that diam$\varnothing = 0$. ☐

Proposition 2 (Monotonicity)**.** *If* $E_1 \subset E_2$ *, then* $m_\alpha^*(E_1) \leq m_\alpha^*(E_2)$ *.*

Proof. Note that since E_2 is a cover of E_1, the set of covers of E_1 contains the set of covers of E_2. Thus, taking the infimum, we have $\mathcal{H}_\alpha^\delta(E_1) \leq \mathcal{H}_\alpha^\delta(E_2)$ for each δ. Taking $\delta \to 0$, we have $m_\alpha^*(E_1) \leq m_\alpha^*(E_2)$. □

Proposition 3 (Subadditivity)**.** *For each* $\alpha > 0$ *and any countable family of subsets* $\{E_j\}_{j=1}^\infty$ *of* $\mathbb{R}^d$

$$m_\alpha^*\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty m_\alpha^*(E_j).$$

Proof. Let $\delta > 0$, $\epsilon > 0$, and $j \in \mathbb{N}$. By supremum in the definition of the Hausdorff measure, there exists a countable cover $E_j \subset \cup_{k=1}^\infty F_{j,k}$ such that $\text{diam} F_{j,k} \leq \delta$ and $\sum_{k=1}^\infty (\text{diam} F_{j,k})^\alpha \leq \mathcal{H}_\alpha^\delta(E_j) + \epsilon/2^j$. Then,

$$\mathcal{H}_\alpha^\delta(\cup_{j=1}^\infty E_j) \leq \sum_{j,k=1}^\infty (\text{diam} F_{j,k})^\alpha \leq \sum_{j=1}^\infty \mathcal{H}_\alpha^\delta(E_j) + \epsilon$$

where we used the fact that $\{F_{j,k}\}_{j,k=1}^\infty$ covers $\cup_{j=1}^\infty E_j$. Since ϵ is arbitrary, we have $\mathcal{H}_\alpha^\delta(\cup_{j=1}^\infty E_j) \leq \sum_{j=1}^\infty \mathcal{H}_\alpha^\delta(E_j)$. Taking the supremum, we have $m_\alpha^*(\cup_{j=1}^\infty E_j) \leq \sum_{j=1}^\infty m_\alpha^*(E_j)$. □

Remark 1. *Note that by null empty set property, monotonicity, and subadditivity,* m_α^* *satisfies the definition of exterior measure.*

Proposition 4 (Distant additivity)**.** *If* $d(E_1, E_2) > 0$ *, then* $m_\alpha^*(E_1 \cup E_2) = m_\alpha^*(E_1) + m_\alpha^*(E_2)$ *.*

Proof. Firstly, note that by subadditivity, we have $m_\alpha^*(E_1 \cup E_2) \leq m_\alpha^*(E_1) + m_\alpha^*(E_2)$. Secondly, we show that $m_\alpha^*(E_1 \cup E_2) \geq m_\alpha^*(E_1) + m_\alpha^*(E_2)$. Let $0 < \epsilon < d(E_1, E_2)$ and $\{F_j\}_{j=1}^\infty$ be a cover of $E_1 \cup E_2$ with diameter less than $\delta < \epsilon$. Let $F_j^1 = E_1 \cap F_j$ and $F_j^2 = E_2 \cap F_j$. Then, $E_1 \subset \cup_{j=1}^\infty F_j^1$ and $E_2 \subset \cup_{j=1}^\infty F_j^2$ are disjoint covers, implying that $\sum_{j=1}^\infty (\text{diam} F_j^1)^\alpha + \sum_{j=1}^\infty (\text{diam} F_j^2)^\alpha \leq \sum_{j=1}^\infty (\text{diam} F_j)^\alpha$. Thus, taking infimum over all the coverings of $E_1 \cup E_2$ with diameter less than δ, we have $\mathcal{H}_\alpha^\delta(E_1 \cup E_2) \leq \mathcal{H}_\alpha^\delta(E_1) + \mathcal{H}_\alpha^\delta(E_2)$. By taking $\delta \to 0$, we have $m_\alpha^*(E_1 \cup E_2) \leq m_\alpha^*(E_1) + m_\alpha^*(E_2)$. □

Remark 2. *By Proposition 3 and 4,* m_α^* *satisfies the definition of metric exterior measure.*

3 Hausdorff measure

We are now ready to define the Hausdorff measure via Caratheodory's criterion.

Definition 3.1. *The* α*-dimensional Hausdorff measure* $m_\alpha(E)$ *of* $\mathbb{R}^d$ *is the restriction of a metric exterior measure* $m_\alpha^*(E)$ *to Borel sets in* $\mathbb{R}^d$*.*

Remark 3. *Since the exterior Hausdorff measure* m_α^* *is a metric exterior measure, its restriction to Borel sets in* $\mathbb{R}^d$ *is a measure.*

4 Basic properties of Hausdorff measure

Now we have defined the Hausdorff measure, we state some of the standard measure-theoretic properties. Some of these follow from the Caratheodory's criterion.

Proposition 5 (Additivity). *If $\{E_j\}_{j=1}^{\infty}$ is a countable family of disjoint Borel sets and $E = \cup_{j=1}^{\infty} E_j$, then $m_\alpha(E) = \sum_{j=1}^{\infty} m_\alpha(E_j)$.**

Proposition 6 (Invariance). *(Invariance) Hausdorff measure is invariant under translations and rotations. Moreover, it scales as follows: $m_\alpha(\lambda E) = \lambda^\alpha m_\alpha(E)$ for all $\lambda > 0$.*

Proof. By invariance of $\text{diam}(S)$ uner translations and rotations, and satisfies $\text{diam}(\lambda S) = \lambda\text{diam}(S)$ for all $\lambda > 0$. □

Proposition 7. *The quantity $m_0(E)$ counts the number of points in E, while $m_1(E) = m(E)$ for all Borel sets $E \subset \mathbb{R}$. (Here, m denotes the Lebesgue measure on $\mathbb{R}$).**

Proof. This directly follows from the definition of the Hausdorff measure. □

We now quantitatively compare the Hausdorff measure with the Lebesgue measure in $\mathbb{R}^d$. We first introduce some relevant contants.

Definition 4.1. *In $\mathbb{R}^d$, we define a dimensional constant $c_d = \frac{m(B)}{(diam B)^d} = \frac{v_d}{2^d}$ where B is the unit ball and v_d denotes the volume of the unit ball. Here, $v_d = \frac{\Gamma\left(\frac{1}{2}\right)^d}{\Gamma\left(\frac{d}{2}+1\right)} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$.*

Proposition 8. *If E is a Borel subset of $\mathbb{R}^d$ and $m(E)$ is its Lebesgue measure, then $m_d(E) \approx m(E)$ in the sense that $c_d m_d(E) \leq m(E) \leq 2^d c_d m_d(E)$.*

Proof. Since if $m_d(E) = \infty$, then the inequality is trivially satisfied, it suffices to consider the case $m_d(E) < \infty$. We first show that $m_d(E) \leq c_d^{-1} m(E)$. Note that for each Borel subset E, $\epsilon > 0$, and $\delta > 0$, there exists a covering $E \subset \cup_{j=1}^{\infty}$ where B_j are balls of diameter less than δ and $\sum_{j=1}^{\infty} m(B_j) \leq m(E)+\epsilon$. (This is justified, for example, in Exercise 3.26 in Stein and Schakarchi, Measure Theory, Integration, and Hilbert Spaces.) Then,

$$\mathcal{H}_d^\delta(E) \leq \sum_{j=1}^{\infty} (\text{diam} B_j)^d = c_d^{-1} \sum_{j=1}^{\infty} m(B_j) \leq c_d^{-1}(m(E) + \epsilon)$$

and upon taking $\delta \to 0$ and $\epsilon \to 0$, we have $m_d(E) \leq c_d^{-1} m(E)$. Next, we show that $m(E) \leq 2^d c_d m_d(E)$. Begin by noting that there exists coverings $E \subset \cup_{j=1}^{\infty} F_j \subset \cup_{j=1}^{\infty} B_j$ of E that satisfies $\sum_{j=1}^{\infty} (\text{diam} F_j)^d \leq m_d(E) + \epsilon$, which follows from the definition of Hausdorff measure, and $F_j \subset B_j$ and $\text{diam} B_j = 2\text{diam} F_j$ by defining B_j to be closed balls centered at a point of F_j. Now,

$$m(E) \leq \sum_{j=1}^{\infty} m(B_j) = c_d \sum_{j=1}^{\infty} \leq 2^d c_d (m_d(E) + \epsilon),$$

where in the first inequality we used the fact that $E \subset \cup_{j=1}^{\infty} B_j$. Hence, $m(E) \leq 2^d c_d (m_d(E)+\epsilon)$, where taking $\epsilon \to 0$, we have $m(E) \leq 2^d c_d m_d(E)$. This completes the proof. □

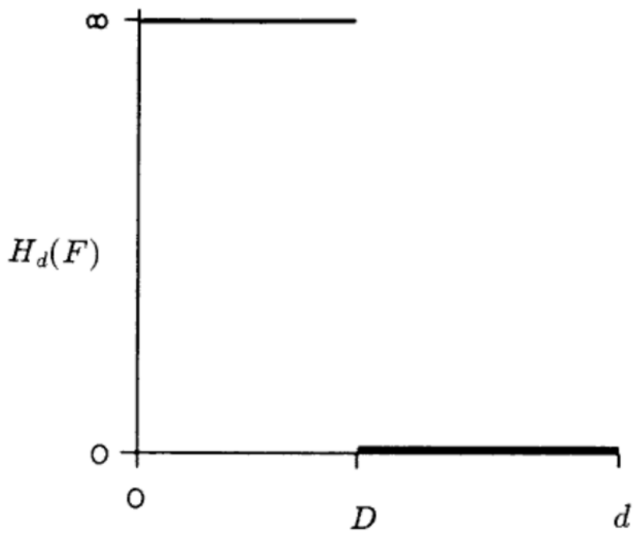

Figure 1: The discontinuity of Hausdorff measure.

Indeed, one can show that there is a stronger relationship between the Hausdorff measure and the Lebesgue measure. We will only state the theorem here. The proof includes a use of isodiametric inequality, stating that among all sets of a given diameter, the ball has largest volume. (The isodiametric inequality is justified in Problem 7.2 of Stein and Schakarchi, Measure, Integration, and Hilbert Spaces.)

Proposition 9. *If E is a Borel subset of $\mathbb{R}^d$, then $c_d m_d(E) = m(E)$ for some constant c_d that depends only on the dimension d. In other words, $m(E) = \beta_d m_d(E)$ where β_d is the volume of the unit diameter ball.*

We now move on to an important property of Hausdorff measure. One question we might ask ourselves is that what will happen as we vary α in the Hausdorff measure m_α^*? It turns out that answering this question naturally directs us to define the notion of dimension based on Hausdorff measure.

Proposition 10. *Let E be a Borel subset of $\mathbb{R}^d$.*
i) If $m_\alpha^*(E) < \infty$ and $\beta > \alpha$, then $m_\beta^*(E) = 0$.
ii) If $m_\alpha^*(E) > 0$ and $\beta < \alpha$, then $m_\beta^*(E) = \infty$.

Proof. We first show that if $m_\alpha^*(E) < \infty$ and $\alpha < \beta$, then $m_\beta^*(E) = 0$. Let $\delta > 0$ and $\{F_j\}_{j=1}^\infty$ be an arbitrary cover of E with $\text{diam}F_j \leq \delta$. Note that since $(\text{diam}F_j)^\beta = (\text{diam}F_j)^{\beta-\alpha}(\text{diam}F_j)^\alpha \leq \delta^{\beta-\alpha}(\text{diam}F_j)^\alpha$, by taking the infimum over all the covers $\{F_j\}_{j=1}^\infty$ of E with diameter less than δ, we have $\mathcal{H}_\beta^\delta(E) \leq \delta^{\beta-\alpha}\mathcal{H}_\alpha^\delta(E)$. By taking the supremum over δ, we have $\delta^{\beta-\alpha}\mathcal{H}_\alpha^\delta(E) \leq \delta^{\beta-\alpha}m_\alpha^*(E)$. Then, taking $\delta \to 0$, we have $m_\beta^*(E) = 0$, where we used the fact that $m_\alpha^*(E) < \infty$ and $\beta - \alpha > 0$. Now, note that by taking the contrapositive of the fact we have just proved, we have that if $m_\alpha^*(E) > 0$ and $\beta < \alpha$, then $m_\beta^*(E) = \infty$. Note that the case of $\alpha = \beta$ is trivial. $\square$

As a result, we have the following graph of Hausdorff measure $\mathcal{H}_\alpha(F) = H_d(F)$ for any Borel set F. Before we move on to the definition of the Hausdorff dimension, we consider some basic examples.

Example 1. 1. *If I is a finite line segment in $\mathbb{R}^d$, then $0 < m_1(I) < \infty$. This directly follows from the proposition on the comparison between the Hausdorff measure and the Lebesgue measure.*

2. *If Q is a k-cube in $\mathbb{R}^d$, then $0 < m_k(Q) < \infty$. Here, a k-cube in $\mathbb{R}^d$ with $k \leq d$ is the product of k non-trivial intervals and $d - k$ points in $\mathbb{R}^d$. This also directly follows from the proposition on the comparison between the Hausdorff measure and the Lebesgue measure.*

3. *If $\mathcal{O}$ is a non-empty open set in $\mathbb{R}^d$, then $m_\alpha(\mathcal{O}) = \infty$ whenever $\alpha < d$. This follows from the fact that $m_d(\mathcal{O}) > 0$ and the second part of the previous proposition.*

4. *If $\alpha > d$, then $m_\alpha(E) = 0$ for every Borel subset. This follows from the fact that $m_\alpha(B) = 0$ for each ball B, the monotonicity of Hausdorff measure, and that $m_\alpha(\mathbb{R}^d) = 0$.*

5 Definition of Hausdorff dimension

Definition 5.1. *By the last proposition, for each Borel subset E of $\mathbb{R}^d$, there exists a unique α such that*

$$m_\beta(E) = \begin{cases} \infty & \text{if } \beta < \alpha \\ 0 & \text{if } \alpha < \beta. \end{cases}$$

In other words,

$$\alpha = \sup\{\beta : m_\beta(E) = \infty\} = \inf\{\beta : m_\beta(E) = 0\}.*$$

We say that E has Hausdorff dimension α, or more succinctly, E has dimension α. We denote such unique α as $\alpha = \dim E$. If E is bounded and the inequalities are strict, that is, $0 < m_\alpha(E) < \infty$, then we say that E has strict Hausdorff dimension α.

Remark 4. *In general, $0 \leq \dim E \leq \infty$. Also, if a set has a fractional dimension, that is, non-integral dimension, then it is commonly called a fractal. It turns out that calculating the Hausdorff measure of a set is a difficult problem. However, in some cases, it is possible to bound this measure from above and below to determine the dimension of the set in question.*

We now prepare some background to consider an important example for Hausdorff dimension, which is the Cantor set.

Definition 5.2 (Lipschitz condition)**.** *A function f from a metric space (X, d_X) to a metric space (Y, d_Y) satisfies a Lipschitz condition if there exists $M > 0$ such that*

$$d_Y(f(x), f(y)) \leq M d_X(x, y) \quad \text{for all } x, y \in X.$$

More generally, a function f satisfies a Lipschitz condition with exponent γ (or is Hölder γ) if

$$d_Y(f(x), f(y)) \leq M d_X(x, y)^\gamma \quad \text{for all} x, y \in X.$$

Remark 5. *The only interesting case is when $0 < \gamma \leq 1$, in which case the function is called a contraction (this is further explained in Exercise 7.3. in Stein and Shakarchi, Measure, Integration, and Hilbert Spaces). This is* *because if $f : [0,1] \to \mathbb{R}$ satisfies a Lipschitz condition of exponent $\gamma > 1$, then f is a constant. To see this, let $x < y$ be distinct points in $[0,1]$ and partition $[x,y]$ into n equally spaced subintervals $\{x_0 = x < x_1 < \cdots < x_n = y\}$. Then, $d_Y(f(x), f(y)) = \sum_{i=1}^{n} d_Y(f(x_{i-1}), f(x_i)) \leq \sum_{i=1}^{n} M d_X(x_{i-1}, x_i)^\gamma = M \sum_{i=1}^{n} (\frac{y-x}{n})^\gamma = M(y-x)n^{1-\gamma}$. Taking $n \to \infty$ and using the fact that $\gamma > 1$, we have our desired statement.*

We introduce two lemmas that we will use in order to calculate the Hausdorff dimension of the Cantor set.

Lemma 5.1. *Suppose a function f defined on a compact set E satisfies a Lipschitz condition with exponent γ. Then,*

1. *$m_\beta(f(E)) \leq M^\beta m_\alpha(E)$ if $\beta = \alpha/\gamma$.*
2. *$\dim f(E) \leq \frac{1}{\gamma} \dim E$.*

Proof. We first prove $m_\beta(f(E)) \leq M^\beta m_\alpha(E)$ if $\beta = \alpha/\gamma$. Let $E \subset \cup_{i=1}^{\infty} F_i$ be a countable cover. Then, $f(E) \subset \cup_{i=1}^{\infty} f(E \cap F_i)$ is a countable cover. Note that $\text{diam} f(E \cap F_i) < M(\text{diam} F_k)^\gamma$, which implies that $\sum_{i=1}^{\infty} (\text{diam} f(E \cap F_i))^{\alpha/\gamma} \leq M^{\alpha/\gamma} \sum_{i=1}^{\infty} (\text{diam} F_i)^\alpha$. This proves our desired first result. Next, we show that $\dim f(E) \leq \frac{1}{\gamma} \dim E$. Let $\alpha = \dim E$ and $\beta = \dim f(E)$. Note that if $b < \alpha/\gamma$, then $b < \alpha/\gamma$ and $a < \alpha$ for some h. By the first part of the lemma, $m_b(f(E)) \leq M^b m_a(E)$ if $b < a/\gamma$. Also, $m_a(E) = 0$ if $a < \alpha$. Hence, $m_b(f(E)) = 0$ if $b < \alpha/\gamma$. Along with the fact that $\dim f(E) \leq \alpha/\gamma$, we have our desired second result. □

Lemma 5.2. *The Cantor function (or the Cantor-Lebesgue function or the devil's staircase) F on $\mathcal{C}$ satisfies a Lipschitz condition with exponent $\gamma = \log 2/\log 3$.*

Proof. Recall that the function F was constructed previously as the limit of a sequence $\{F_n\}_{n=1}^{\infty}$ of piecewise linear functions. Here, F_n is zero on the n -th gap of Cantor set and increases linearly on the n -th interval of Cantor set. We first show that $|F(x) - F(y)| \leq (\frac{3}{2})^n |x-y| + \frac{2}{2^n}$. To see this, first note that the slope of F_n is always bounded by $(3/2)^n$. This is because the function F_n increases by at most 2^{-n} on each interval of length 3^{-n}. Next, note that $|F_n(x) - F_n(y)| \leq (3/2)^n |x-y|$, $|F(x) - F_n(x)| \leq 1/2^n$, since the error at n-th stage is less than $1/2^n$. Thus, by triangle inequality, we have

$$|F(x) - F(y)| \leq F(x) - F_n(x)| + |F_n(x) - F_n(y)| + |F_n(y) - F(y)| \leq (3/2)^n |x-y| + \frac{2}{2^n}.$$

With this statement, we proceed to our main proof. Let us fix x and y and let n be a number such that $1 \leq 3^n |x-y| \leq 3$. Then, $|x-y| \leq 3^{-n}$ and $(3/2)^n |x-y| + \frac{2}{2^n} \leq 6 \cdot 2^{-n}$. Thus, $|F(x) - F(y)| \leq 5 \cdot 2^{-n} = 5(3^{-n})^\gamma \leq M|x-y|^\gamma$, where we used the fact that $\gamma = \log 2/\log 3$. Hence, the Cantor function F on $\mathcal{C}$ satisfies a Lipschitz condition with exponent $\gamma = \log 2/\log 3$. □

6 Cantor set

We are now ready to consider an important example, the Cantor set.

Theorem 6.1. *The Cantor set $\mathcal{C}$ has Hausdorff dimension $\alpha = \log 2/\log 3$.*

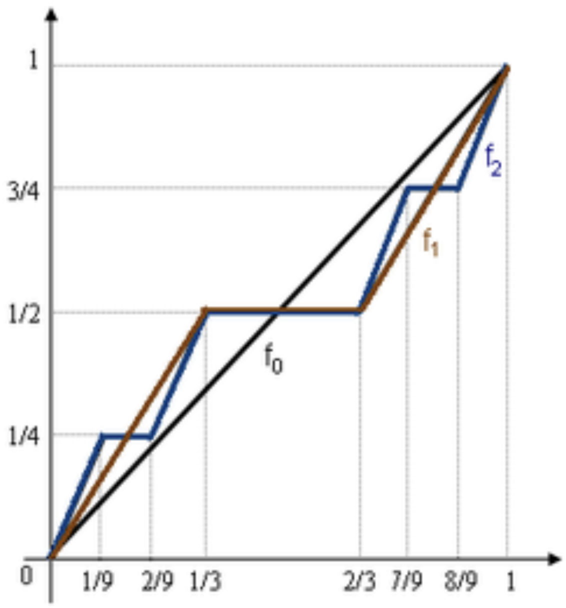

Figure 2: The Cantor function.

Proof. We first show that $m_\alpha(\mathcal{C} \le 1$. Since $\mathcal{C} = \cup_{k=1}^{\infty} C_k$, where C_k is a union of 2^k number of intervals of length 3^{-k}, given $\delta > 0$, if we choose K large enough such that $3^{-K} < \delta$, we have that C_K covers $\mathcal{C}$ and it consists of 2^K number of intervals of diameter $3^{-K} < \delta$. Hence, $\mathcal{H}_\alpha^\delta(\mathcal{C}) \le 2^K (3^{-K})^\alpha = 1$, where in the last equality, we used the fact that $\alpha = \log 2/\log 3$. By taking $\delta \to 0$, we have $m_\alpha(\mathcal{C}) \le 1$. Secondly, we show that $m_\alpha(\mathcal{C}) \ge 1$. By the first lemma among the two lemmas we proved for the Cantor set example, $m_1([0,1]) \le M^\beta m_\alpha(\mathcal{C})$. Here, we set $f = F$ to be the Cantor function on $E = \mathcal{C}$ with Lipschitz condition with exponent $\gamma = \alpha = \log 2/\log 3$. Also, since $m_1([0,1]) = 1$, $1 \le m_\alpha(\mathcal{C})$. Hence, $m_\alpha(\mathcal{C}) \ge 1$. Therefore, we have $0 < \dim \mathcal{C} = \log 2/\log 3 < \infty$. □

Remark 6. *The proof of the Cantor set example is typical in the sense that the inequality $m_\alpha(\mathcal{C}) < \infty$ is usually easier to obtain than $0 < m_\alpha(\mathcal{C})$. With extra effort, it is possible to show that the $\log 2/\log 3$ -dimensional Hausdorff measure of $\mathcal{C}$ is precisely 1. (This is outlined in Exercise 7.7. of Stein and Shakarchi, Measure, Integration, and Hilbert Spaces.)*

7 Sierpinski triangle

We consider another well-known example, which is the Sierpinski triangle.

Definition 7.1. *Let S_0 be a solid (filled-in) equilateral triangle with unit side length. Define a decreasing sequence $S_{k+1} \subset S_k$ of triangles S_k inductively as follows. For given S_k, we obtain S_{k+1} by removing the interiors of the three closed corner triangles among four equally sized equilateral triangles of S_k, maintaining the middle closed equilateral triangle. Note that this gives S_k, called k-th generation, to be a union of 3^k closed equilateral triangles of side length 2^{-k}. The vertex of a triangle is the lower left vertex of the triangle, so that k-th generation has 3^k vertices. The Sierpinski triangle is the compact set defined by $\mathcal{S} = \bigcap_{k=0}^{\infty} S_k$.*

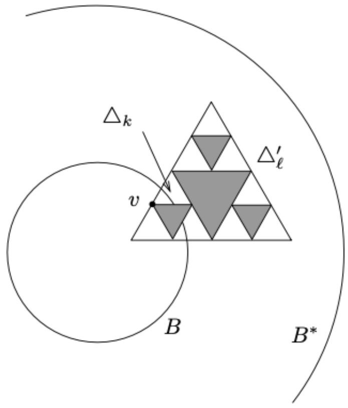

Figure 3: Figure in Lemma 7.1

Lemma 7.1. *Let $\mathcal{B}$ be a finite covering of $\mathcal{S}$ consisted of balls. Let $k \in \mathbb{N}$ such that $2^{-k} \leq \min_{B \in \mathcal{B}} diam B_j < 2^{-k+1}$. Then, if $B \in \mathcal{B}$ and $2^{-l} \leq diam B < 2^{-l+1}$ for some $l \leq k$, then B contains at most $c3^{k-l}$ vertices of the k-th generation for some constant $c > 0$.*

Proof. Let $3 * B$ be the concentric ball with respect to B with three times its diameter. Let Δ_k be a k-th generation which contains a vertex v that is contained in B, and let Δ_l be a l-th generation containing Δ_k. Then, note that $v \in \Delta_k \subset \Delta_l \subset 3 * B$, where the last containment directly follows from the triangle inequality, or simply geometry. (In the figure below, we use the notation B^* interchangably with $3 * B$.)

Let c be the maximum number of distinct l-th generations that are contained in $3 * B$, as both l-th generations and $3 * B$ have area proportional to 4^{-l}. Note that since Δ_l contains 3^{k-l} k-th generations, B contains at most $c3^{k-l}$ vertices of k-th generations. □

Theorem 7.2. *The Sierpinski triangle $\mathcal{S}$ has strict Hausdorff dimension $\alpha = \log 3 / \log 2$.*

Proof. Firstly, we show that $m_\alpha(\mathcal{S}) \leq 1$. For each $\delta > 0$, there exists $K \in \mathbb{N}$ such that $2^{-K} < \delta$. Because $\mathcal{S} \subset S_K$, where S_K is a union of 3^K triangles each with diameter $2^{-K} < \delta$, we have $\mathcal{H}^\delta_\alpha(\mathcal{S}) \leq 3^K(2^{-K})^\alpha = 1$. Note that in the last equality, we used the fact that $\alpha = \log 3 / \log 2$.

Secondly, we prove that $m_\alpha(\mathcal{S}) > 0$. Assume that $\{F_j\}_{j=1}^\infty$ is a countable cover of $\mathcal{S}$ with diam$F_j < \delta$. Then, $\sum_{j=1}^\infty (\text{diam} F_j)^\alpha = \sum_{j=1}^\infty (2\text{diam} B_j)^\alpha \geq \sum_{j=1}^N (\text{diam} B_j)^\alpha$, where $\{B_j\}_{j=1}^\infty$ is a collection of balls of twice the diameter with respect to $\{F_j\}_{j=1}^\infty$ such that $F_j \subset B_j$ and diamB_j = 2diamF_j, and in the last inequality, I used the fact that $\mathcal{S}$ is compact. Note that

$$\sum_{j=1}^N (\text{diam} B_j)^\alpha \geq \sum_l N_l 2^{-l\alpha}$$

where l in the sum runs for each j through the positive integers satisfying $2^{-l} \leq \text{diam} B_j \leq 2^{-l+1}$. Since $\mathcal{S}$ contains all vertices of the k-th generation and $c \sum_l N_l 3^{k-l}$ is the maximum number of vertices of k-th generation that are contained in $\mathcal{B}$, which cover $\mathcal{S}$, we have $c \sum_l N_l 3^{k-l} \geq 3^k$, implying that

$\sum_l N_l 3^{-l} \geq c$. Hence, $\sum_{j=1}^{N}(\text{diam}B_j)\alpha \geq c$, where we used the fact that $\alpha = \log 3/\log 2$. Therefore, $\sum_{j=1}^{\infty}(\text{diam}F_j)^\alpha \geq c > 0$, implying that $m_\alpha(\mathcal{S}) > 0$. □

Remark 7. *Although the Sierpinski triangle is simple in construction and is highly structured with symmetry, the exact* $\log 3/\log 2$*-dimensional Hausdorff measure of the Sierpinski triangle is unkown. There are only estimates regarding their upper and lower bounds.* [1]

8 References

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[1] For a recent result on this: Móra, Péter, Estimate of the Hausdorff measure of the Sierpinski triangle, Fractals 17, No. 2, 137-148 (2009). ZBL1178.28007.