SCIENTIA SINICA
Vol. XIV, No. 10, 1965

MATHEMATICS

ON THE SHORTEST ARBORESCENCE OF A DIRECTED GRAPH*

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ABSTRACT

Kruskal[1] and Sollin[3] have given the algorithm for finding the shortest spanning subtree of a graph. In many practical problems, we should consider not only the "line-segment" but also the "directional line-segment". For example, we may prepare a scheme of routes to a canal under certain conditions.

In this paper an algorithm for finding the shortest arborescence of a directed graph is established by induction.

I. THE PROBLEM

Given a directed graph $G = \{X; U\}$, where $X = \{x_1, x_2, \cdots, x_n\}$ is a set of vertices and $U \subset X \times X$ is a set of arcs. An arborescence of a directed graph is defined as follows:

Definition[2]. An arborescence of a directed graph $G = \{X; U\}$ is a subgraph $H = \{X; V\}$ of G which contains no cycle such that

(a) there is a particular vertex called the root, which is not the terminal vertex of any arc in V;

(b) for any other vertex x_i, there is one and only one arc in V, whose terminal vertex is x_i.

When there is an arborescence $H = \{X; V\}$, for any vertex x_i, we associate a non-negative integral number $h(x_i)$, called the generational number of x_i relating to H, where $h(x_i)$ is defined as follows:

If x_0 is the root of H, let $h(x_0) = 0$.

If there is an arc $(x_i, x_j) \in V$ and the generational number of x_i has been defined, then define $h(x_j) = h(x_i) + 1$.

Since an arborescence is a tree, all the properties of the tree are true to the arborescence. In particular, when any arc $(x_i, x_j) \notin V$ is added to V, there will appear a cycle and only a cycle designated by μ. Set

$$h(x_{i_0}) = \min_{x_i \in \mu} h(x_i).$$

If $x_{i_0} = x_j$, then μ is a loop; otherwise, in H there exists a path $P(x_{i_0}, x_j)$ from x_{i_0} to x_j. This means that in H there exists an arc $(x_{i'}, x_j) \in P(x_{i_0}, x_j) \subset \mu$. This is an important property for the proof of our algorithm.

* Received Nov. 7, 1964.

For any arc $(x_i, x_j) \in U$, we associate a nonnegative real number l_{ij} (not necessarily $l_{ij} = l_{ji}$), called the length of the arc. For an arborescence $H = \{X; V\}$ (or a cycle C), we define

$$l(H) = \sum_{(x_i, x_j) \in V} l_{ij} \quad \left(\text{or } l(C) = \sum_{(x_i, x_j) \in C} l_{ij}\right),$$

where $l(H)$ (or $l(C)$) is called the length of H (or C). The shortest arborescence H^* of G means that for any arborescence H of G, we have $l(H^*) \leqslant l(H)$. The purpose of this paper is to establish an algorithm for finding the shortest arborescence of a directed graph.

II. ALGORITHM OF THE PROBLEM

We show the algorithm through the following example:

Given a directed graph $G = \{X; U\}$, as indicated in Fig. 1, where $|X| = n = 9$ and the numbers in brackets are the lengths of the arcs.

Set

$$U^-(x_i) = \{(x_j, x_i) \mid (x_j, x_i) \in U, \\ j = 1, 2, \cdots, n\}.$$

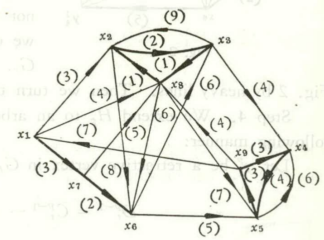

Fig. 1

Step 1. For any vertex x_i whose $U^-(x_i) \neq \phi$, take the arc u_i such that

$$l(u_i) = \min_{u \in U^-(x_i)} l(u)$$

(if such arcs are more than one, take one of them arbitrarily). The set of these arcs is designated by W_0. In our concrete case $W_0 = \{(x_9, x_1), (x_8, x_2), (x_2, x_3), (x_9, x_4), (x_4, x_5), (x_7, x_6), (x_1, x_7), (x_3, x_8), (x_5, x_9)\}$.

Step 2. If $|W_0| < n - 1$, then the process stops, and there is no arborescence of G; if $|W_0| \geqslant n - 1$, then we choose $n - 1$ arcs in W_0 and the set of these $n - 1$ arcs is designated by V_0 such that

$$\max_{v \in V_0} l(v) \leqslant \min_{u \in W_0 - V_0} l(u).$$

The choice may not be unique. In our concrete case $V_0 = W_0 - \{(x_9, x_1)\}$, (x_9, x_1) is the longest arc in W_0, $(l_{91} = 7)$.

Step 3. If there is no loop in V_0, then the process stops and $H_0 = \{X; V_0\}$ is the shortest arborescence of graph (the proof is trivial, so it is omitted); otherwise, there exist some loops in V, for example, $C_1^0, C_2^0, \cdots, C_{h_0}^0$, where

$$C_t^0 = \{u_{t_1}, u_{t_2}, \cdots, u_{t_{i_t}}\}, \quad t = 1, 2, \cdots, h_0.$$

We retract C_t^0 into a single vertex y_t^1, and the new graph obtained from the graph G by such retraction is designated by $G_1 = \{X_1; U_1\}$. In our concrete case, $h_0 = 2$, G_1 is shown by Fig. 2:

$$C_1^0 = \{(x_8, x_2), (x_2, x_3), (x_3, x_8)\},$$
$$C_2^0 = \{(x_9, x_4), (x_4, x_5), (x_5, x_9)\}.$$

The length of the arc u_i of G_1 is redefined as follows:

$$l^1(u_i) = \begin{cases} l(u_i) & \text{if the terminal vertex of } u_i \text{ is not a retractive vertex,} \\ l(u_i) - l(u_{t_i}) + d_t & \text{if the terminal vertex of } u_i \text{ is a retractive vertex } y_t^1, \text{ and } u_{t_i} \text{ is the arc of } C_t^0 \text{ which has the same terminal vertex as } u_i \text{ (in graph } G), \end{cases}$$

where $d_t = \max\limits_{u \in C_t^0} l(u)$.

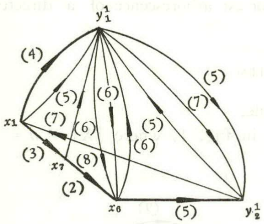

Fig. 2

In Fig. 2, the numbers in brackets are obtained by this formula.

We repeat Steps 1, 2, 3, until the process stops (it must be stopped at finite steps). A graph $G_p = \{X_p:U_p\}$ is obtained. If the process stops at Step 2, the arborescence of G_p (and hence G) will not exist; otherwise (i.e., the process stops at Step 3), we obtain the shortest arborescence $H_p = \{X_p:V_p\}$ of G_p. In our concrete case, $p = 1$, H_p is shown in Fig. 2 by heavy lines. Then we turn to the following step.

Step 4. We extend H_p to an arborescence H_{p-1} of G_{p-1} (note that $G_0 = G$) in the following manner:

Let y_t^p be a retractive vertex in G_p. The corresponding loop in G_{p-1} is C_t^{p-1}. Define

$$D_t^{p-1} = C_t^{p-1} - \{u_{t_i}\}, \quad t = 1, 2, \cdots, h_{p-1}.$$

When y_t^p is the root of H_p, u_{t_i} is one of the longest arcs in C_t^{p-1}; otherwise, in G_p there is an arc $u_i \in V_p$ whose terminal vertex is y_t^p, and u_i is an arc in G_{p-1} ($u_i \notin C_t^{p-1}$) also. Suppose its terminal vertex is $x (x \in C_t^{p-1})$ in G_{p-1}. Then u_{t_i} is the arc in C_t^{p-1} whose terminal vertex is x.

For convenience, set

$$X_t = X^t, \quad t = 0, 1, 2, \cdots, p \quad (\text{where } X_0 = X),$$
$$V_p = V^p, \quad H_p = H^p.$$

Then define

$$V^{p-1} = V^p + \sum_{t=1}^{h_{p-1}} D_t^{p-1},$$
$$H^{p-1} = \{X^{p-1}:V^{p-1}\}.$$

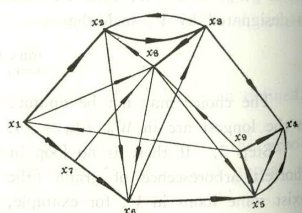

Fig. 3

Obviously H^{p-1} is an arborescence of G_{p-1} (note that V^{p-1} is different from V_{p-1}). We shall prove that H^{p-1} is the shortest arborescence of G_{p-1} in § III.

The process is repeated until the shortest arborescence of G (note $G_0 = G$) is obtained. In our concrete case, we obtain the shortest arborescence H by one step only, and H is shown in Fig. 3 by heavy lines.

III. Proof of the Algorithm

Let H^p be an arbitrary arborescence of G_p (not necessarily the shortest) and H^{p-1} be obtained from H^p by Step 4. We set

$$H^{p-1} = \phi(H^p),$$
$$\Phi^{p-1} = \{\phi(H^p) \mid H^p \text{ being any arborescence of } G_p\},$$

and denote the set of all shortest arborescences of G_{p-1} by S. Then we have the following

Lemma. *The intersection of Φ^{p-1} and S is not empty.*

Proof. We take a particular arborescence $H^{p-1} = \{X^{p-1}; V^{p-1}\}$ in S such that

$$|V^{p-1} \cap V_{p-1}| = \max_{\substack{H \subset S \\ H = (X^{p-1}:V)}} |V \cap V_{p-1}|,$$

where V_{p-1} defined by Step 2 is the fixed set of arcs of G_{p-1}. Then we shall prove that $H^{p-1} \in \Phi^{p-1}$.

If the assertion is not true, i.e., there exists a loop C_t^{p-1} such that $\{C_t^{p-1} - C_t^{p-1} \cap V^{p-1}\}$ contains at least two arcs, set

$$h(x_0) = \min_{x \in C_t^{p-1}} h(x),$$

where $h(x)$ defined in § I is the generational number of x (relating to H^{p-1}). We start from the original vertex x_0 and proceed along the loop C_t^{p-1}. Suppose that we first meet $U - V^{p-1}$ at the arc u as indicated in Fig. 4 (by hypothesis, the terminal vertex x_1 of u is not x_0). As H^{p-1} is an arborescence, the set $V^{p-1} + \{u\}$ contains one (and only one) cycle μ. The cycle cannot be a loop (otherwise, $h(x_0) > h(x_1)$ and since $h(x_0) = \min\limits_{x \in C_t^{p-1}} h(x)$, there follows a contradiction). By means of the property stated in § I, there is an arc $u' \in \mu$ whose terminal vertex is x_1 (obviously, $u' \in V^{p-1}$, $u' \notin V_{p-1}$). We can see that there is an arborescence $H' = \{X^{p-1}:V'\}$ where

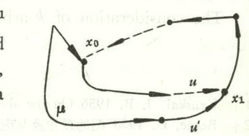

Fig. 4. The real lines are arcs of V^{p-1}.

$$V' = V^{p-1} + \{u\} - \{u'\}$$

such that

$$|V' \cap V_{p-1}| = |V^{p-1} \cap V_{p-1}| + 1.$$

Furthermore $l^{p-1}(H') \leqslant l^{p-1}(H^{p-1})$ (because V_{p-1} is constructed by Steps 1 and 2, hence $l^{p-1}(u') \geqslant l^{p-1}(u)$), and so follows a contradiction.

Theorem. *Suppose that $H^{p-1} = \phi(H^p)$; then H^{p-1} is the shortest arborescence of G_{p-1} if and only if H^p is the shortest arborescence of G_p.*

Proof. We denote the set of all arborescence of $G_{p-1}(G_p)$ by $A_{p-1}(A_p)$. The above lemma means

$$l^{p-1}(H^{p-1}) = \min_{H \in A_{p-1}} l^{p-1}(H) \Longleftrightarrow l^{p-1}(H^{p-1}) = \min_{H \in \Phi^{p-1}} l^{p-1}(H),$$

and from the formula of Step 3 we obtain

$$l^p(H^p) = l^{p-1}(H^{p-1}) + K_{p-1} \quad (H^{p-1} = \phi(H^p)),$$

where

$$K_{p-1} = \sum_{t=1}^{h_{p-1}} [l^{p-1}(C_t^{p-1}) - d_t^{p-1}]$$

is a constant. Hence

$$l^p(H^p) = \min_{H\in A_p} l^p(H) \Longleftrightarrow l^{p-1}(H^{p-1}) = \min_{H\in \phi^{p-1}} l^{p-1}(H) \Longleftrightarrow l^{p-1}(H^{p-1}) = \min_{H\in A_{p-1}} l^{p-1}(H).$$

The proof is completed.

(Note that $l^0(u) = l(u)$ throughout the paper.)

The algorithm can be proved inductively from this theorem.

When we wish to find the shortest arborescence of G which takes a given vertex x as root, we must change U to $U - U^-(x)$ only.

From the definition of arborescence in § I, by changing the phrase "a particular vertex" to "k particular vertices", we may obtain the definition of k-arborescence. The algorithm to find the shortest k-arborescence is analogous, while the difference is not essential, and so omitted.

For a class of directed graphs (including symmetrical graphs as its special case), the algorithm may be simpler, and Sollin's algorithm may be contained for the shortest tree as a special case.

The consideration of k-arborescence is first proposed by Ma Chung-fan.

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SCIENTIA SINICA
Vol. XIV, No. 10, 1965

MECHANICS

THERMO-ELASTIC STRESS AND DEFORMATION CAUSED BY A SPHERICAL INCLUSION IN AN INFINITE ELASTIC SLAB*

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Abstract

In this paper are studied the thermal stress and deformation caused by a spherical inclusion in an infinite elastic plate, the inclusion, and the plate having the same elastic behaviour but different coefficients of thermal expansion.

By means of the potential theory, the thermo-elastic displacement function is first found for both the interior and the exterior of the buried sphere; thereupon the pure elasticity boundary value problem concerned is solved in order to restore the free surface condition of the plate.

In the solution of the problem the integrals are left unintegrated, for it seems to the author that they can be handled conveniently only with numerical treatment.

I. Introduction

The thermal stress problem concerning a semi-infinite elastic solid with a spherical inclusion was solved by using the Love function[1] and the Galerkin stress function[2], wherein it was also assumed that the two materials concerned possess the same elastic property but different coefficients of thermal expansion.

The present work is to extend the same investigation to the infinite elastic plate through the Fourier transformation.

II. The Thermal Elastic Displacement Potential in an Infinite Elastic Solid with a Spherical Inclusion

As indicated by Fig. 1, within an infinite elastic plate $|z| \leqq b$, a sphere is buried without initial stress on the boundary surface between the sphere and the plate when the composite body is kept at zero temperature. The radius of the sphere is a, the z-axis of the coordinate system goes through the centre of the sphere, and c is the distance from the sphere centre to the coordinate origin. We have to set $b - c > a$ in order to let the sphere be totally concealed. The two materials of the composite body were assumed to have the same elastic property, but different coefficients of thermal expansion. For brevity, we introduce $\eta = \alpha_i - \alpha_a$, where α_i and α_a denote the coefficients of thermal expansion of the sphere and the plate respectively.

* Received Dec. 10, 1964.