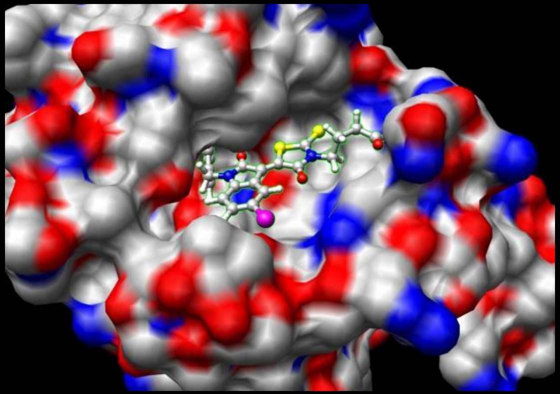
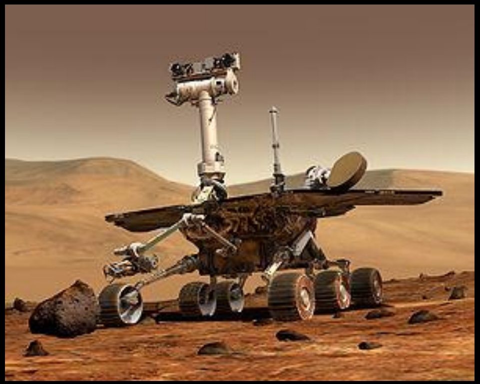
Motion Planning II - Configuration Space

Last time…

- We learned about how to plan paths for a point





- Real-world robots are complex, often articulated bodies
- What if we invented a space where the robots could be treated as points?

Outline

- Topology basics
- Configuration Space
- Obstacles
- Metrics

Basic Sets

- **Open set** – A set with no boundary. Every point in the set has an open neighborhood which is also in the set.
  - In $\boldsymbol{R}^n$, this open neighborhood is called an open ball:

    $$B(x,\rho) = \{x' \in \boldsymbol{R}^n \mid \|x' - x\| < \rho\}$$

  - The set $X \subseteq \boldsymbol{R}^n$ is open if

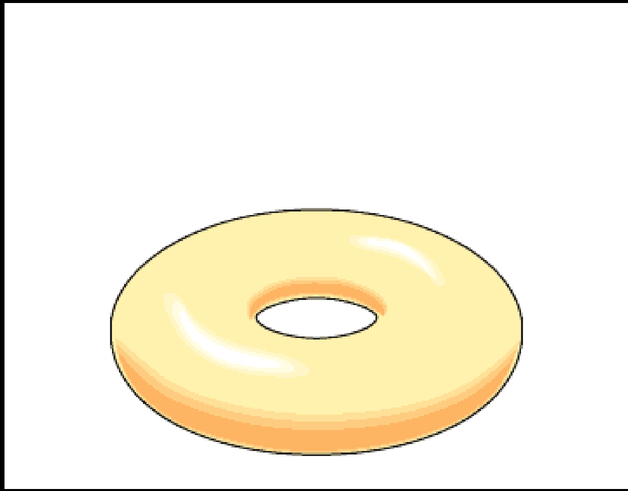
    $$\exists B(x,\rho) \subseteq X, \rho > 0 \qquad \forall x \in X$$

  - Example: $X = \{x \in \boldsymbol{R} \mid 1 < x < 5\}$
- **Closed Set** – A set with a boundary. A closed set is the complement of some open set and vice versa.
  - Example: $X = \{x \in \boldsymbol{R} \mid 1 \leq x \leq 5\}$
    - What is the complement of this set?

Topological Spaces

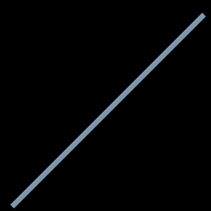
- A set X is called a topological space if there is a collection of open subsets of X for which the following hold:
  1. The union of any number of open sets is an open set.
  2. The intersection of a finite number of open sets is an open set.
  3. Both X and $\emptyset$ are open sets.

- Two topological spaces X and Y are homeomorphic if there is a bijective (one-to-one and onto) function $f: X \rightarrow Y$ and both f and f^{-1} are continuous.
  - Intuitively, you can think of f as a continuous function that warps X into Y
  - f is called a homeomorphism



Homeomorphisms

- Homeomorphisms can not add or remove holes!
- Which are homeomorphic?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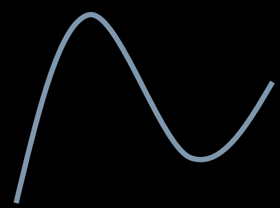


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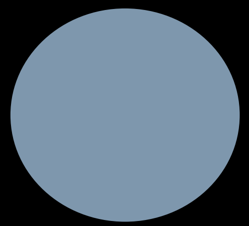

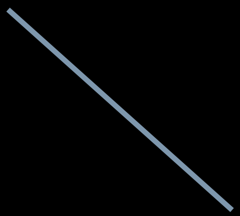
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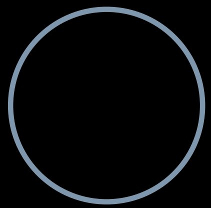

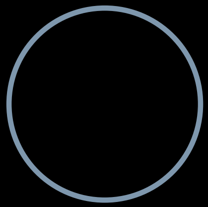
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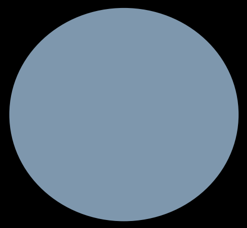


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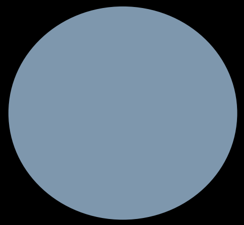

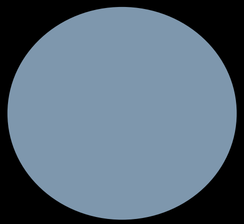
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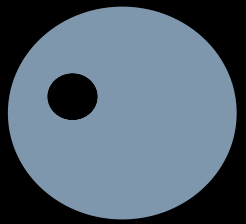


Common Topological Spaces

- The real numbers: $\boldsymbol{R}^1$

Number of dimensions

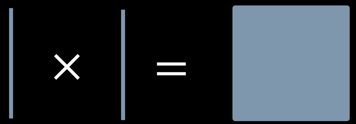
Symbolic name for the space

- The unit circle: $\boldsymbol{S}^1 = \{(x, y) \in \boldsymbol{R}^2 \mid x^2 + y^2 = 1\}$
  - This is NOT the same as a *disc:* $\{(x, y) \in \boldsymbol{R}^2 \mid x^2 + y^2 \leq 1\}$

Cartesian Product

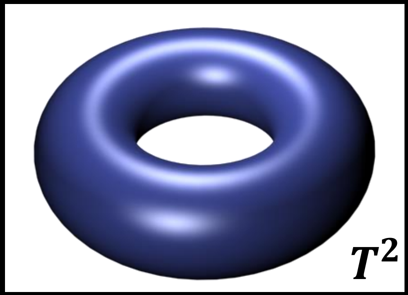
- Can make more complex spaces using the Cartesian product: Every $x \in X, y \in Y$ makes an $(x, y) \in X \times Y$. For example:

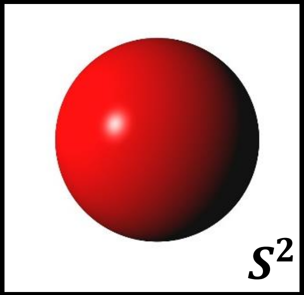
$$\boldsymbol{R}^1 \times \boldsymbol{R}^1 = \boldsymbol{R}^2$$



- What is $\boldsymbol{R}^1 \times \boldsymbol{S}^{\boldsymbol{1}}$?
  - A hollow cylinder

- BE CAREFUL: Results of Cartesian product are not always obvious. This is NOT the same thing as adding up exponents in multiplication.
  - Example: What is $\boldsymbol{S}^{\boldsymbol{1}} \times \boldsymbol{S}^{\boldsymbol{1}}$?
    - Hint: try to visualize the shape
  - This creates a 2D Taurus, $\boldsymbol{S}^{\boldsymbol{1}} \times \boldsymbol{S}^{\boldsymbol{1}} = \boldsymbol{T}^{\boldsymbol{2}}$
  - $\boldsymbol{S}^{\boldsymbol{2}}$ is a sphere

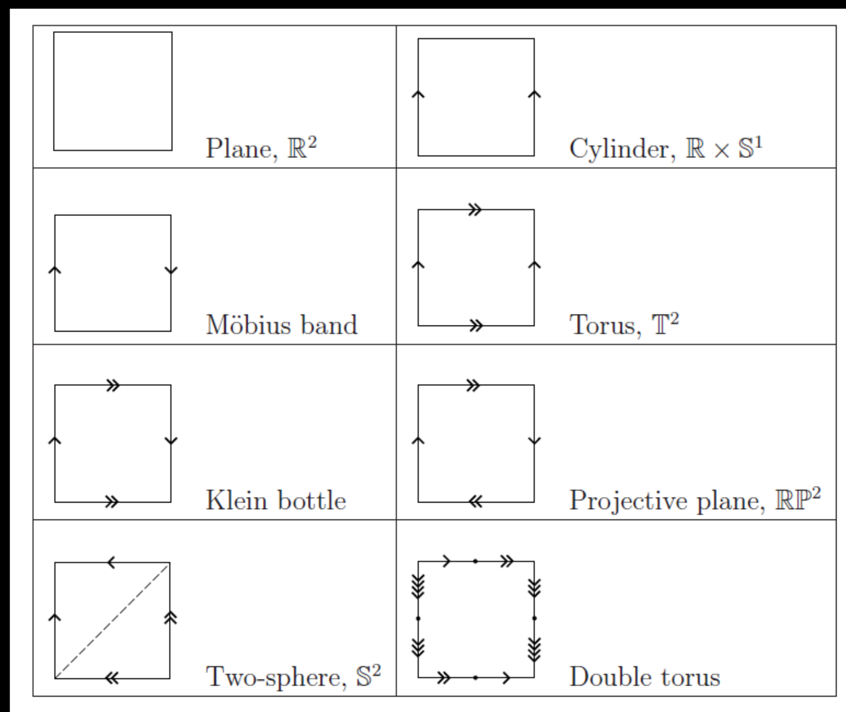




- Also, $\boldsymbol{S}^1 = \boldsymbol{T}^1$ but $\boldsymbol{S}^{n>1} \neq \boldsymbol{T}^{n>1}$

More complex spaces

- Can create more complex topological spaces by Cartesian products and “gluing” boundaries:



Configuration Space

Definitions

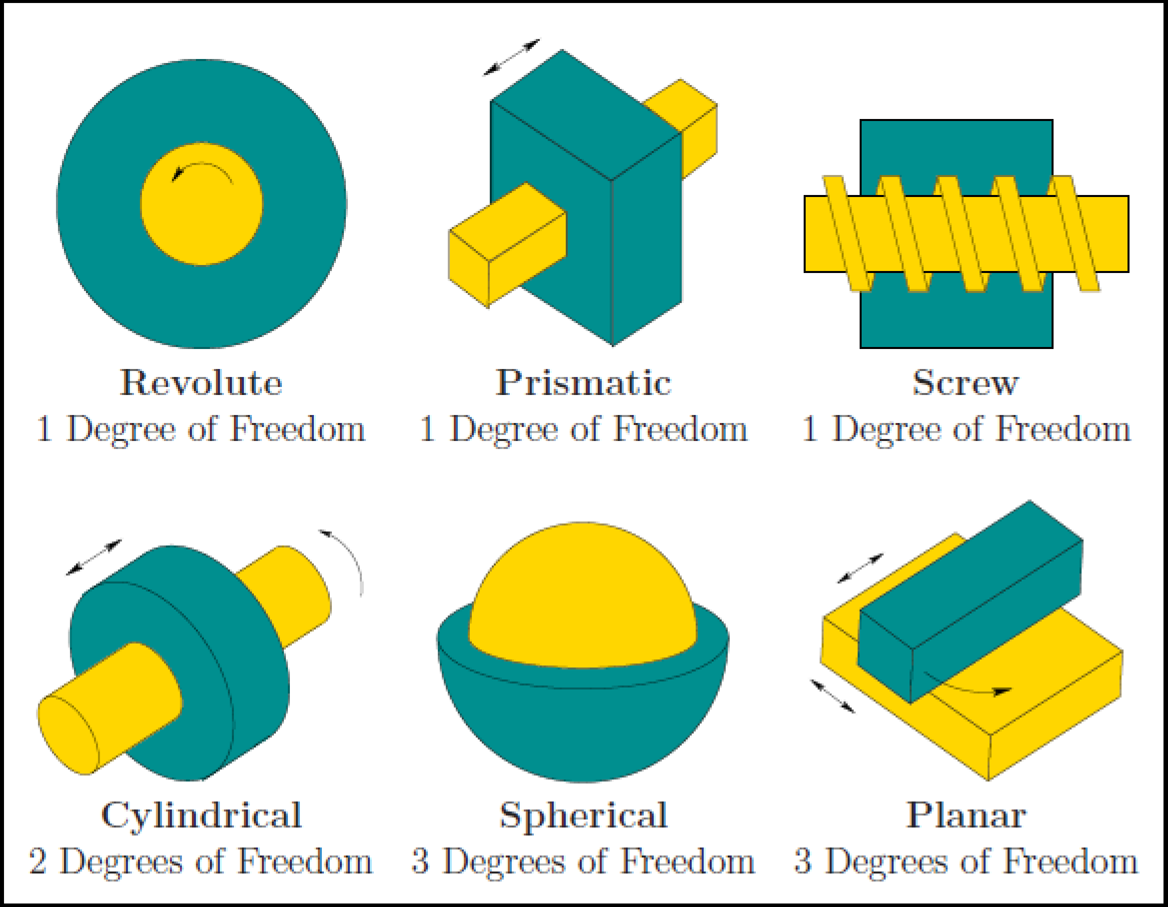
- The **configuration** of a moving object is a specification of the position of **every** point on the object.
  - A configuration q is usually expressed as a vector of the Degrees of Freedom (DOF) of the robot

$$q = (q_1, q_2, \ldots, q_n)$$

- The **configuration space** C is the set of all possible configurations. Usually this is a topological space.
  - A configuration q is a point in C

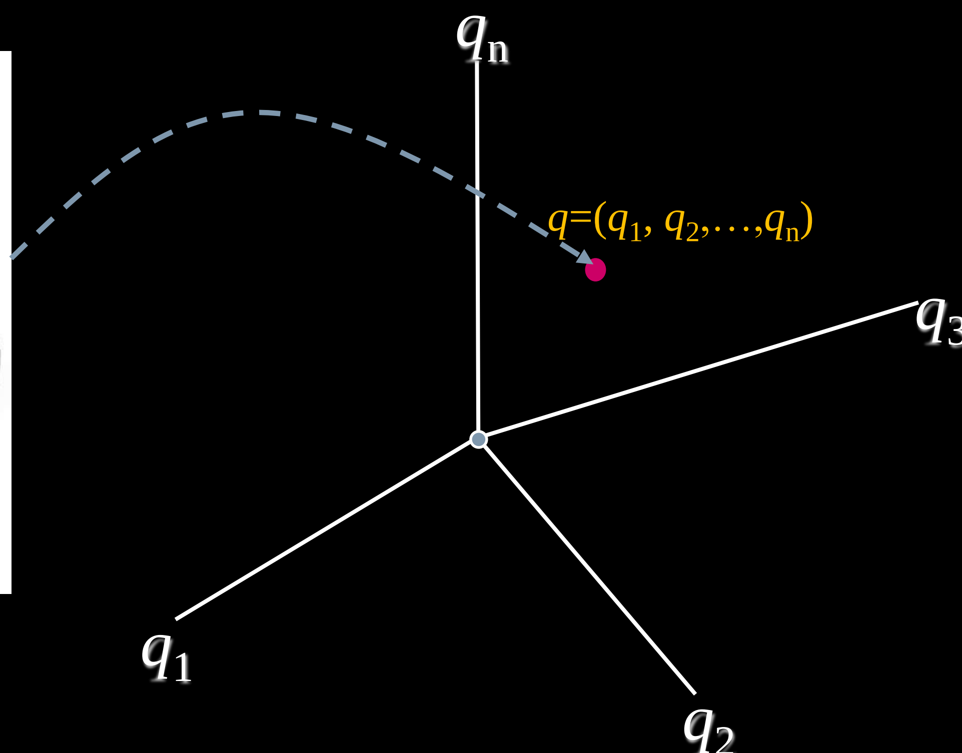
Degrees of Freedom

What is the topology of each of these (assume no joint limits)?

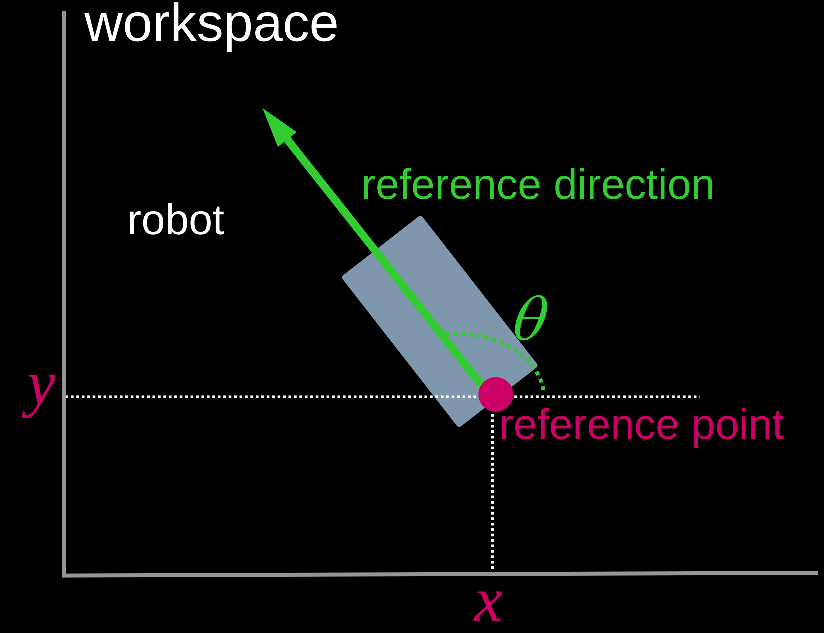


Configuration Space

- The **dimension of a configuration space** is the **minimum** number of DOF needed to specify the configuration of the object completely.



Example: A Rigid 2D Mobile Robot



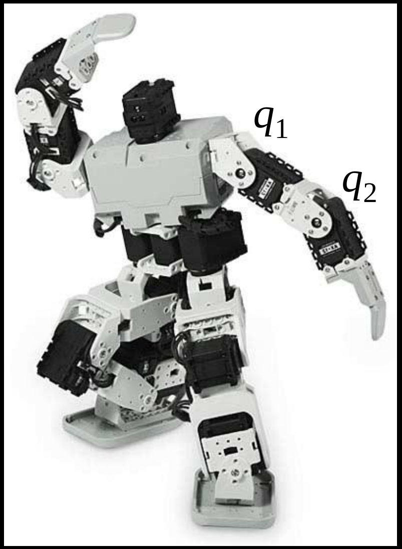
- 3-parameter specification: $q = (x, y, \theta)$ with $\theta \in [0, 2\pi)$.
  - 3D configuration space
  - Topology: $SE(2) = R^2 \times S^1$

Example: Rigid Robot in 3D workspace

- q = (position, rotation) = $(x, y, z, \textbf{???})$
- 3 representations for rotation
  - Euler Angles
  - Transform Matrices
  - Quaternions
- No matter the representation, rotation in 3D is 3 DOF
- C-space dimension: 6
- Topology: $\mathrm{SE}(3) = \mathrm{R}^3 \times \mathrm{SO}(3)$

Configuration Space for Articulated Objects

- An articulated object is a set of rigid bodies connected by joints.
- For articulated robots (arms, humanoids, etc.) the DOF are usually the joints of the robot
- Example: For a single revolute joint with no joint limits, what is the topology?
- What is the topology if it does have joint limits?



$q = (q_1, q_2, \ldots, q_n)$

Number of DOF = n

Paths and Trajectories

- A **path** in C is a continuous curve connecting two configurations q_{start} and q_{goal} :

$$\tau : s \in [0,1] \rightarrow \tau(s) \in C$$

such that $\tau(0) = q_{start}$ and $\tau(1) = q_{goal}$.

- A **trajectory** is a path parameterized by time:

$$\tau : t \in [0,T] \rightarrow \tau(t) \in C$$

Obstacles in C-space

- A configuration q is collision-free, or **free**, if the robot placed at q does not intersect any obstacles in the workspace.

- The **free space** C_{free} is a subset of C that contains all free configurations.

- A configuration space obstacle C_{obs} is a subset of C that contains all configurations where the robot collides with workspace obstacles or with itself (self-collision).

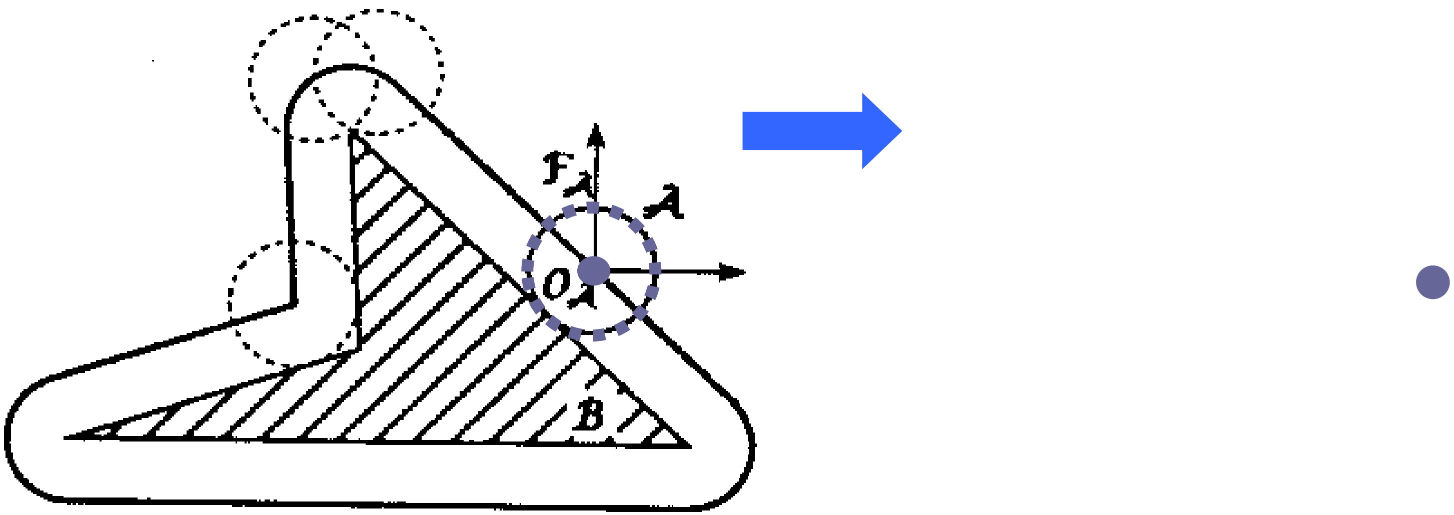
How do we compute C_{obs} ?

- Start with simple case: 2D translating robot
- Input:
  - Polygonal robot
  - Polygonal obstacle in environment
- Output:
  - Configuration space polygonal obstacle

Example: Disc in 2D workspace

Workspace (2D)

configuration



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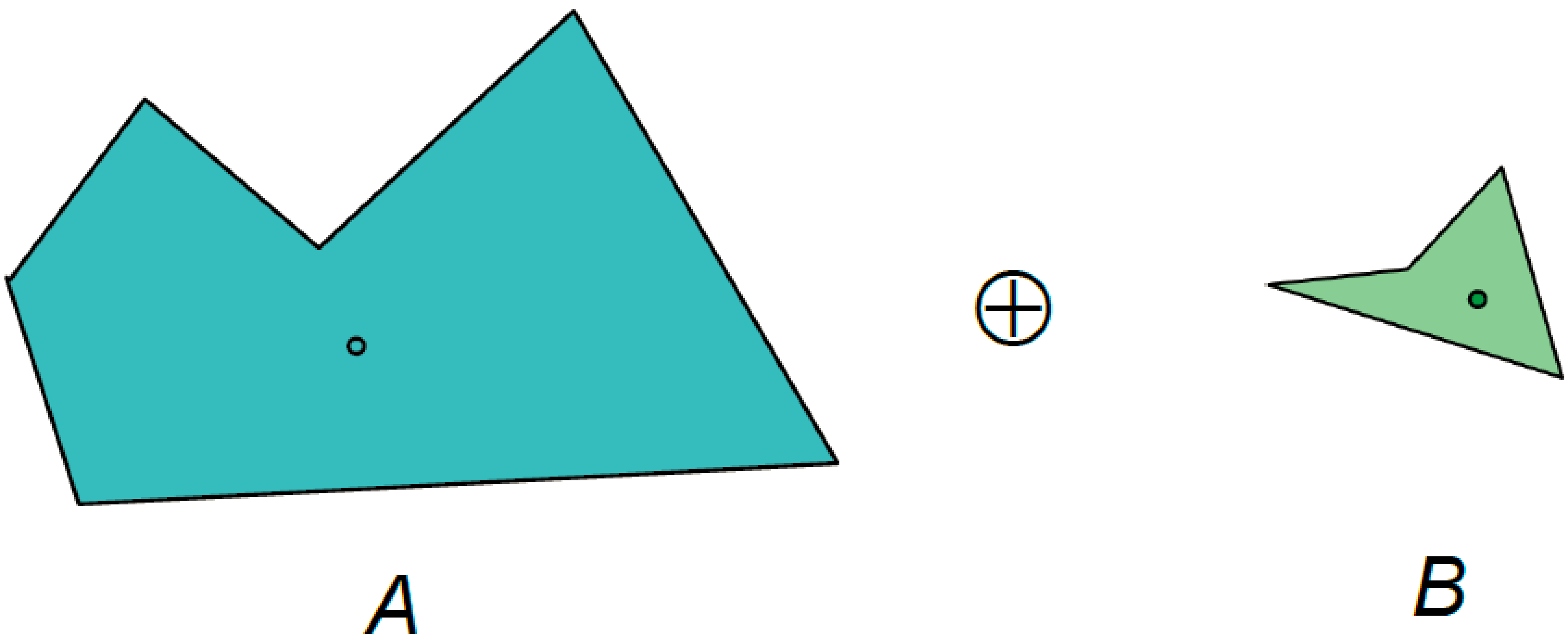
Dinesh Manocha

Minkowski Sum

$$A \oplus B = \{a + b \mid a \in A, b \in B\}$$



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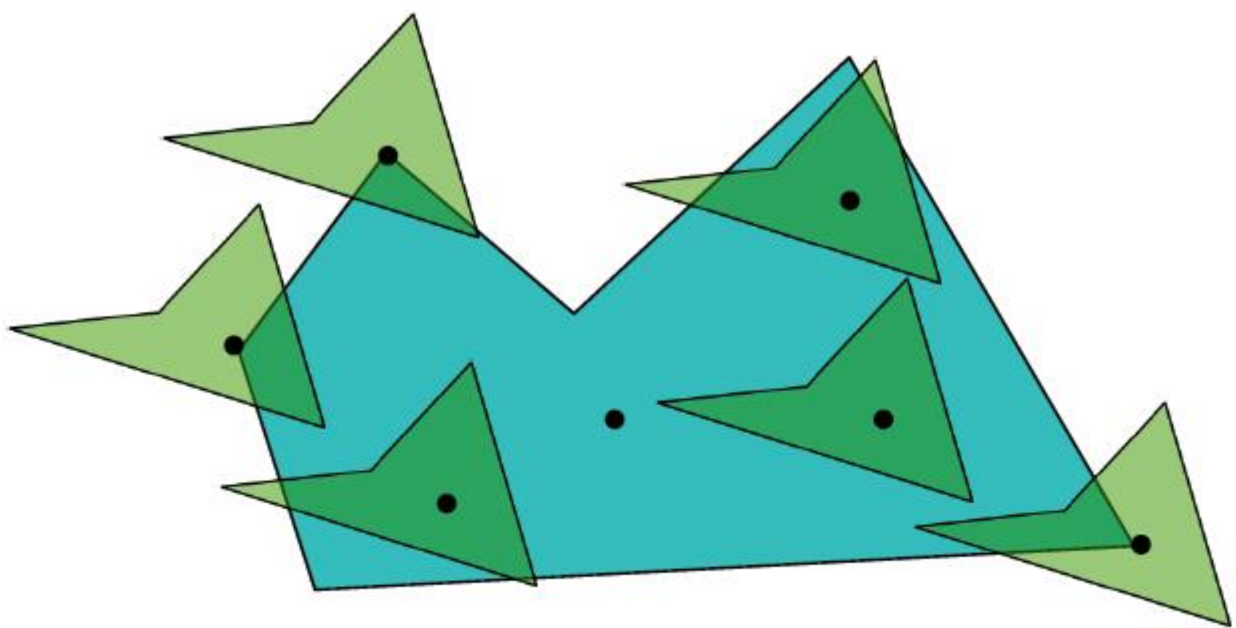

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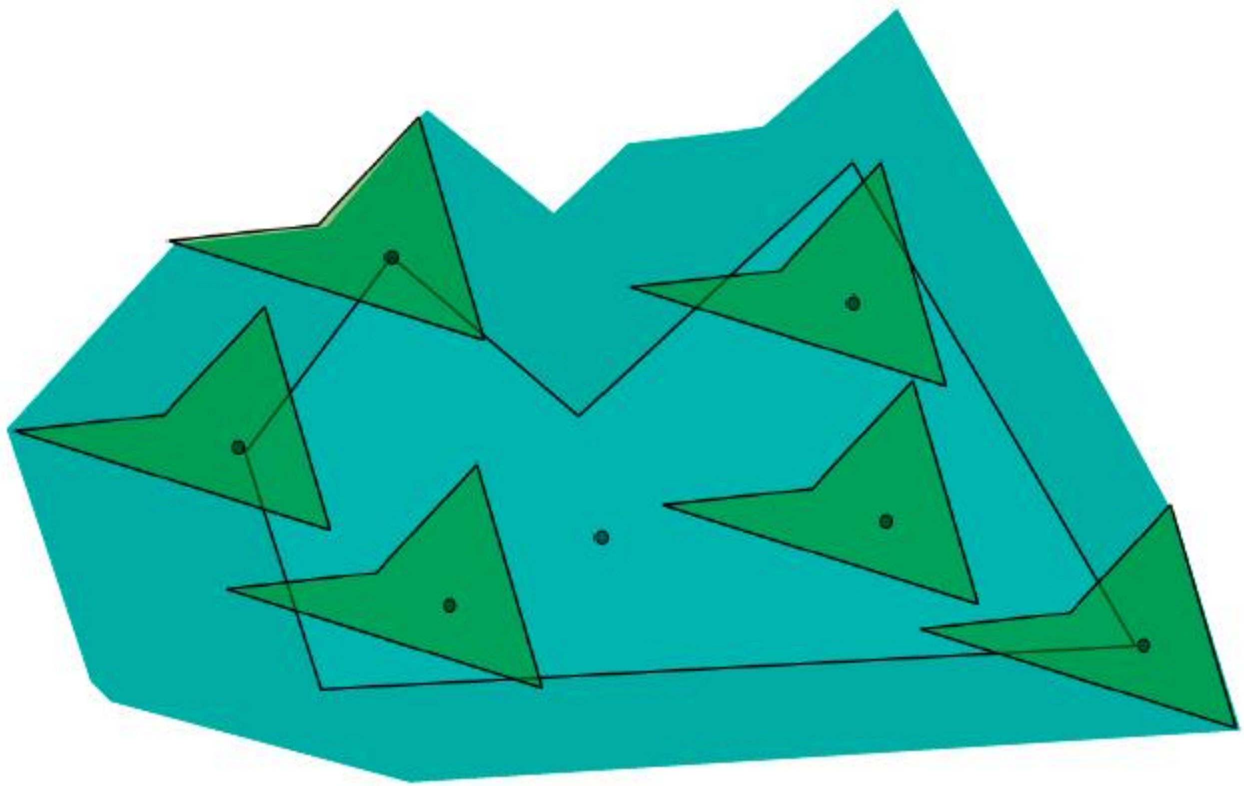
Minkowski Sum



Minkowski Sum

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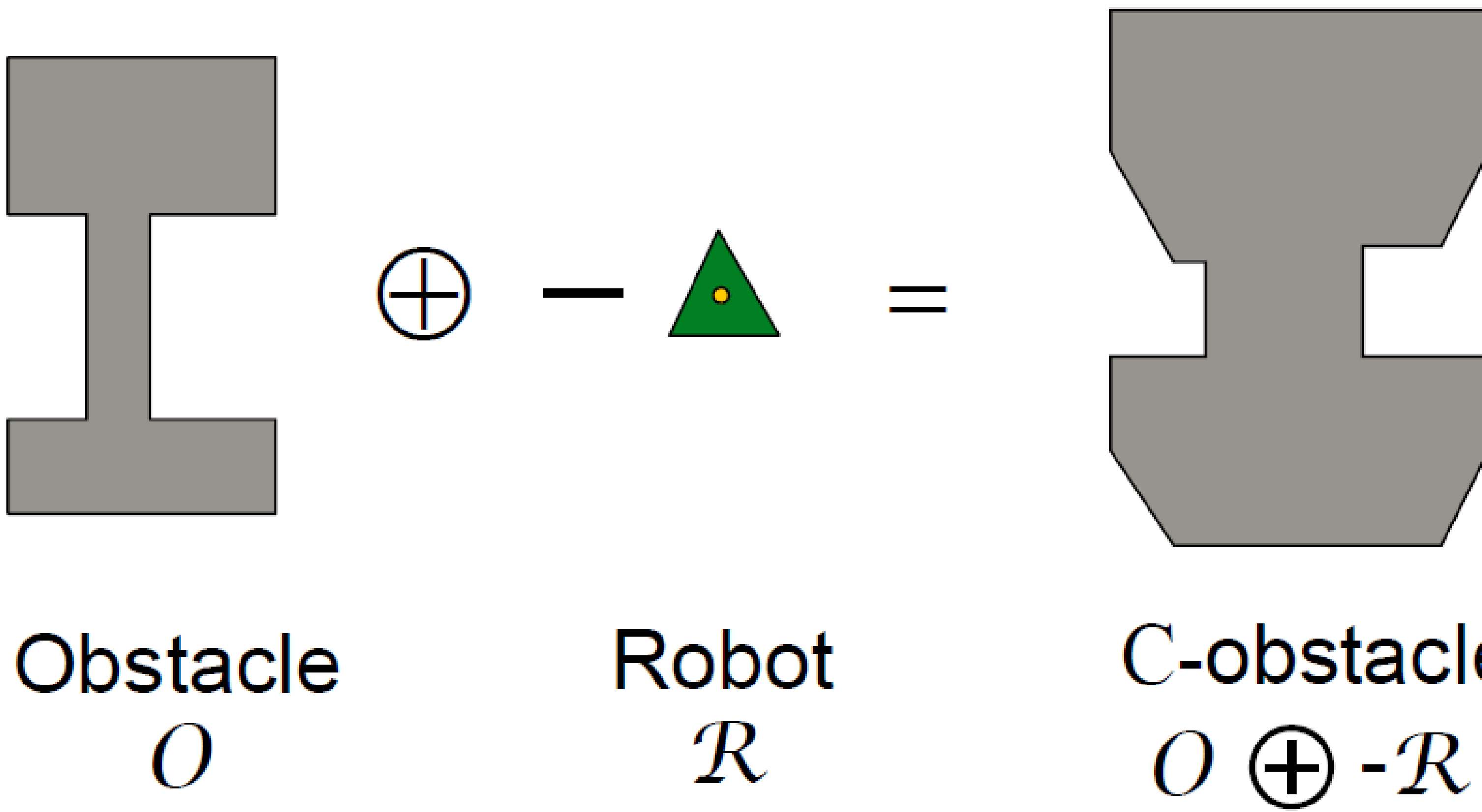
Configuration Space Obstacle

C-obstacle is $\mathcal{O} \oplus -\mathcal{R}$

This means use $-(r \in R)$ instead of $r \in R$

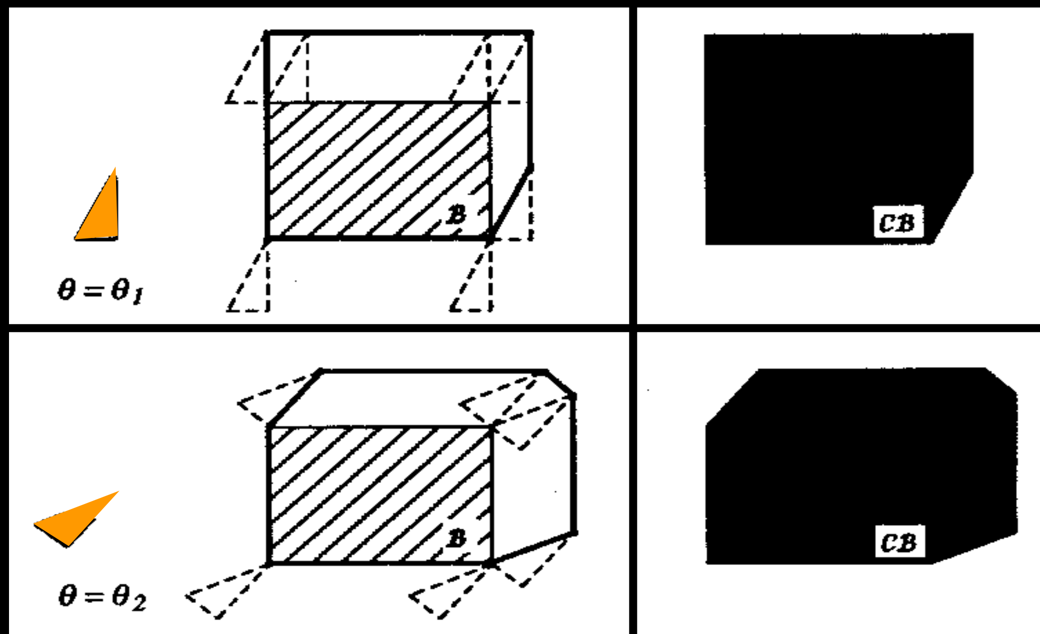


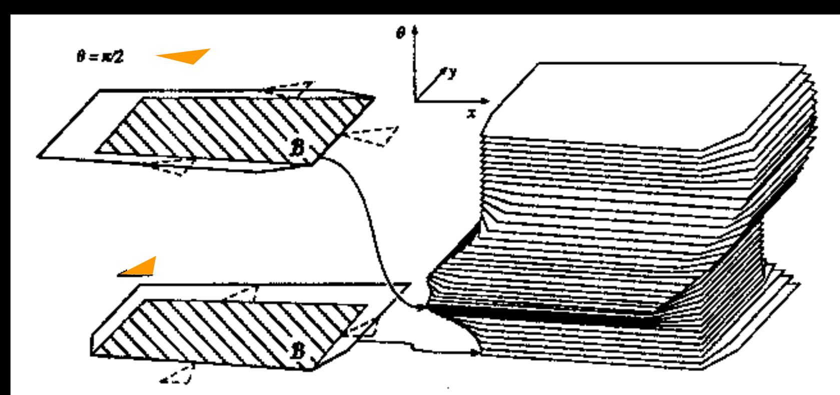
Like a convolution of the robot and obstacle

Minkowski Sums

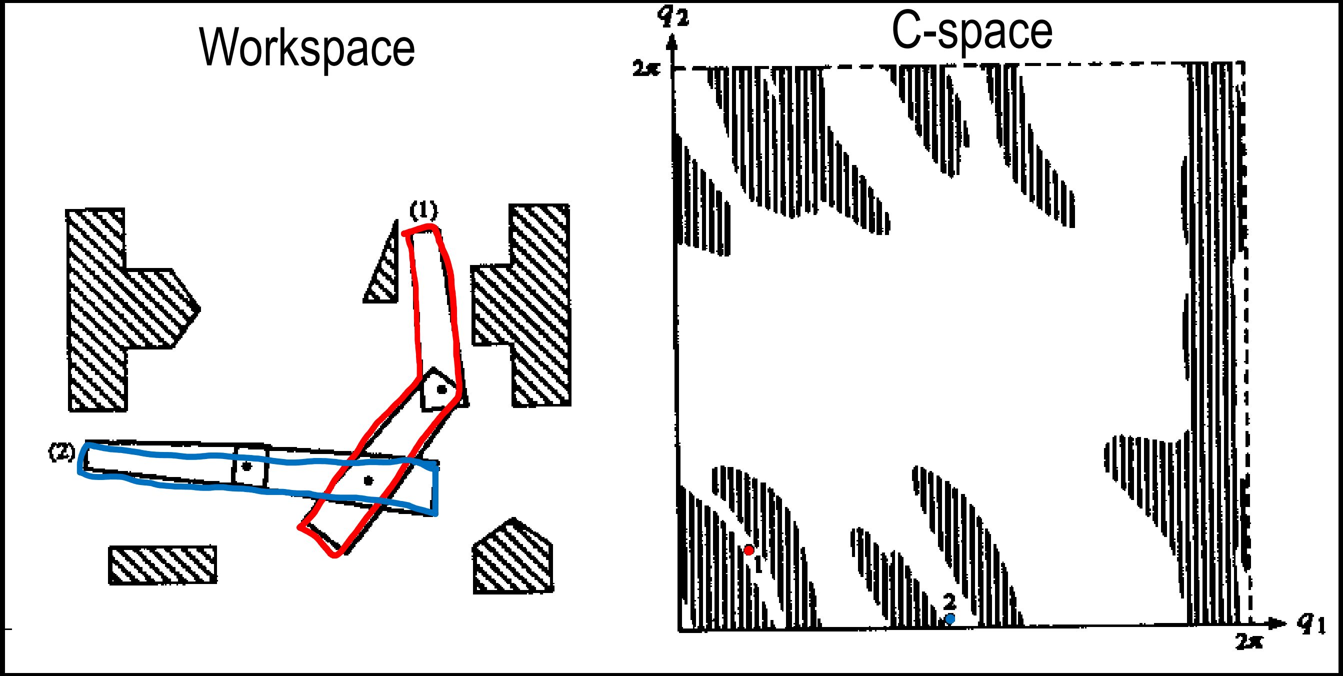
- Can Minkowski Sums be computed in higher dimensions efficiently?

Example: 2D Robot with Rotation





Configuration Space for Articulated Robots



How to compute C_{obs} for articulated bodies?

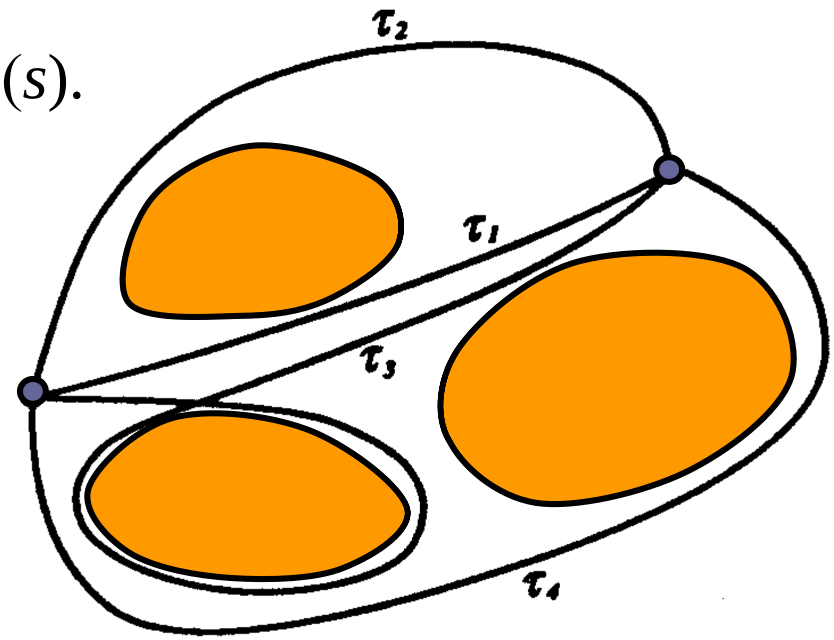
Break

Homotopic paths

- Two paths τ and τ' with the same endpoints are **homotopic** if one can be continuously deformed into the other through the free space F:

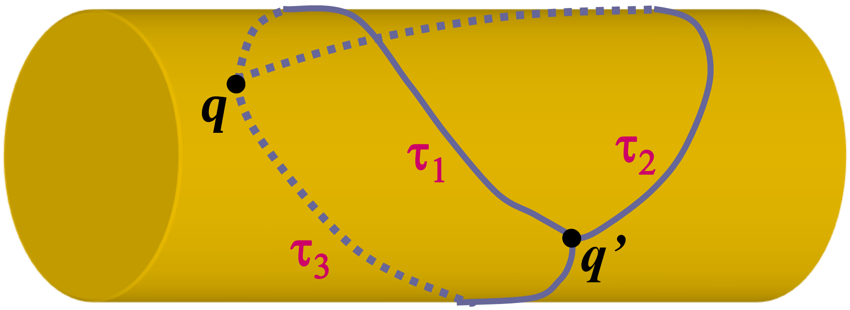
$$h:[0,1]\times[0,1]\rightarrow F$$

  with $h(s,0) = \tau(s)$ and $h(s,1) = \tau'(s)$.



- A homotopic class of paths contains all paths that are homotopic to one another.

Example



- τ_1 and τ_2 are homotopic
- τ_1 and τ_3 are not homotopic

- There are infinity homotopy classes here. Why?

Connectedness of C-Space

- C is **connected** if every two configurations can be connected by a path.

- C is **simply-connected** if any two paths connecting the same endpoints are homotopic. Examples: R^2 or R^3

- Otherwise C is multiply-connected.
  - Can you think of an example?

Metrics in configuration space

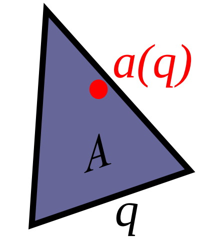
- A **metric** or **distance** function d in a configuration space C is a function

$$d:(q,q')\in C^2 \rightarrow d(q,q')\geq 0$$

such that
  - $d(q, q') = 0$ if and only if $q = q'$,
  - $d(q, q') = d(q', q)$,
  - $d(q,q')\leq d(q,q'')+d(q'',q')$.

Example

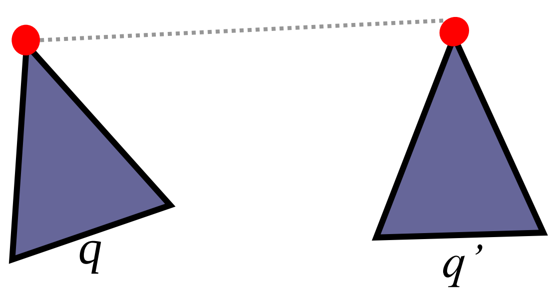
- Consider robot A and a point a on A
- $a(q)$: position of a in the workspace when A is at configuration q



- Example distance metric: d in C is defined by

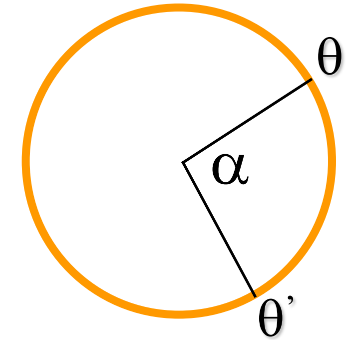
$$d(q, q') = \max_{a \in A} \| a(q) - a(q') \|$$

where $||a - b||$ denotes the Euclidean distance between points a and b in the workspace.



Examples in R² x S¹

- Consider $R^2 \times S^1$ (the C-space of a mobile robot)
  - $q = (x, y, \theta)$, $q' = (x', y', \theta')$ with $\theta, \theta' \in [0,2\pi)$
  - $\alpha = \min\{\, |\theta - \theta'|\,,\ 2\pi - |\theta - \theta'|\,\}$

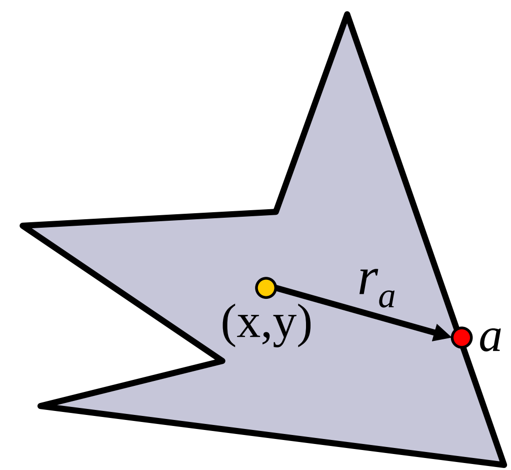


- $d(q,q') = \max_{a \in A} \| a(q) - a(q') \|$

$$= \max_{a \in A} \sqrt{(x-x')^2 + (y-y')^2 + \alpha r_a}$$

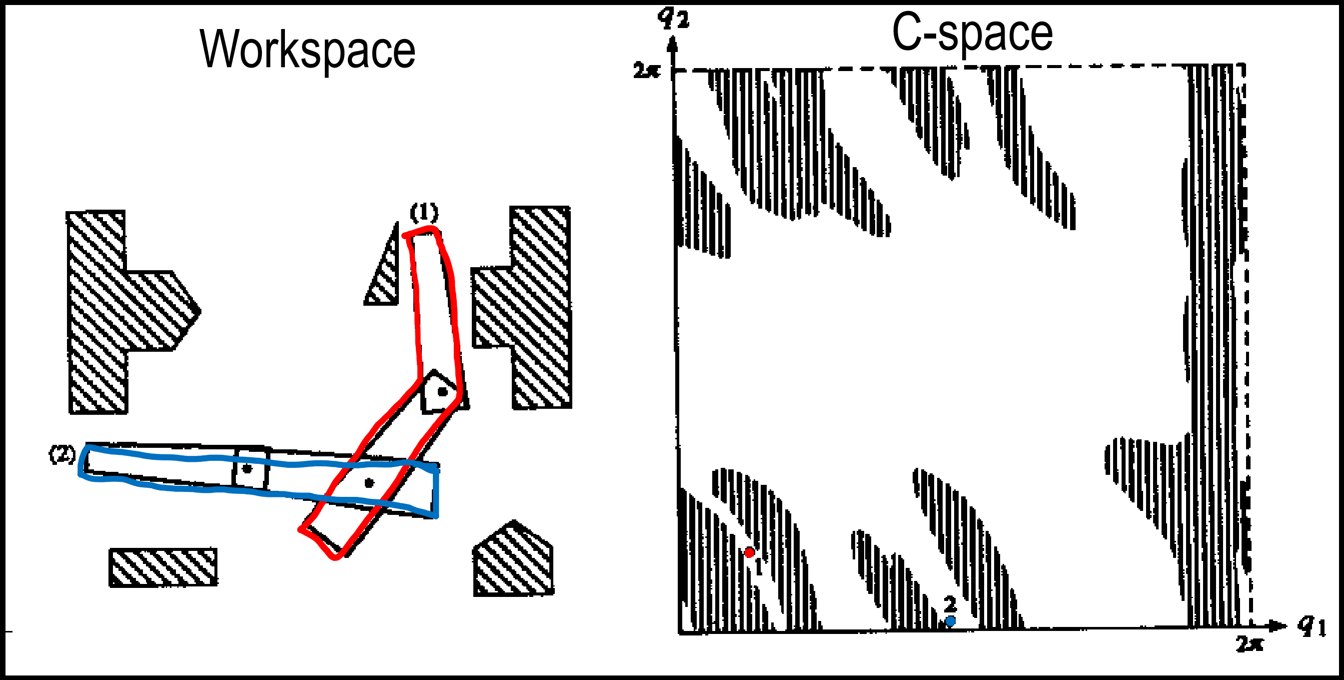
$$= \sqrt{(x-x')^2 + (y-y')^2 + \alpha \max_{a \in A} r_a}$$

$$= \sqrt{(x-x')^2 + (y-y')^2 + \alpha r_{\max}}$$



Distance Metric for Articulated Bodies

- Let's try to think of one



Discussion

- Do we need to have an explicit representation of C-obstacles to do path planning?
- Do we need a specialized distance metric in C-space to do path planning?
  - Can we use Euclidian distance in C-space?

Homework

- Read LaValle Ch. 5.5 - 5.6
- Read Probability Review
- HW 2 due on Wednesday!