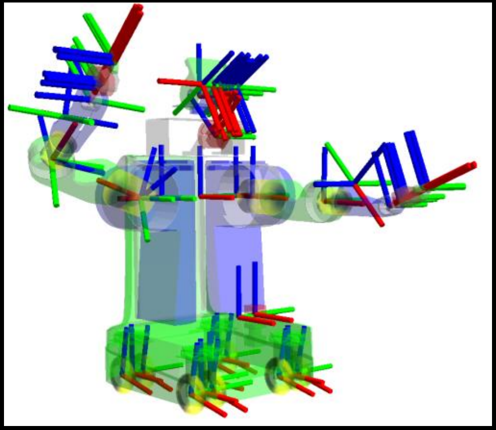
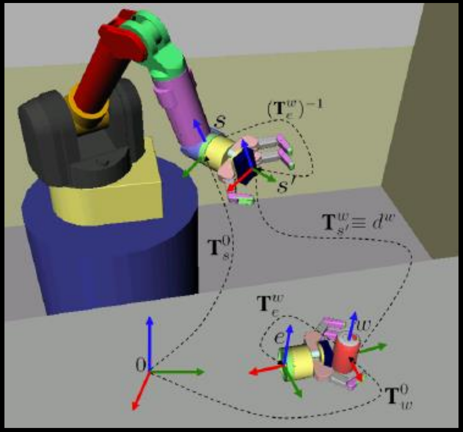
Transformations

This lecture is being recorded

Last time…

- We saw how to manipulate vectors and matrices
- Today: we will look at representations of rigid-body transformations based on matrices and vectors
- An understanding of 2D and 3D rigid-body transformations is key to motion planning (and robotics in general)





- There are many representations, none is the “best”
  - Each representation is useful in a different way

Outline

- Homogenous Transforms
- Euler angles

Outline

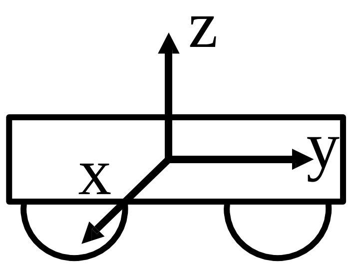
- Homogenous Transforms
- Euler angles

Homogenous Transforms Outline

- <u>Notational Conventions</u>
- Definitions
- Homogeneous Transforms
- Semantics and Interpretations
- Summary

Objects and Embedded Coordinate Frames

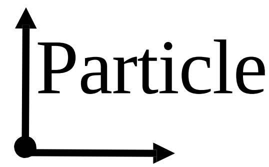
- Objects of interest are real: wheels, sensors, obstacles.
- Abstract them by sets of axes fixed to the body.



- These axes:
  - Have a state of motion
  - Can be used to express vectors.
- Call them coordinate frames.

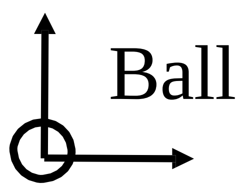
Coordinate Frames

- <u>Points</u> possess position but not orientation:



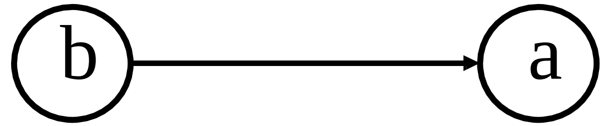
(orientation undefined)

- <u>Rigid Bodies</u> possess position and orientation:



(can rotate)

Relations

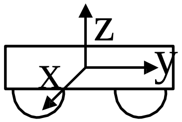


- Mechanics is about relations between objects.
- a is "r-related" to b is written: $\boxed{r_a^b}$
- Example velocity (v) of robot (r) relative to earth (e): $\boxed{v_r^e}$
- Relationship is directional and asymmetric. $\boxed{r_a^b \neq r_b^a}$

Properties

- "r' is not a property of a.
  - "the" velocity of an object is not defined.
- It's a property of a *relative* to b.
- a and b are real objects.
- a can be a point but b <u>must be</u> a rigid body.

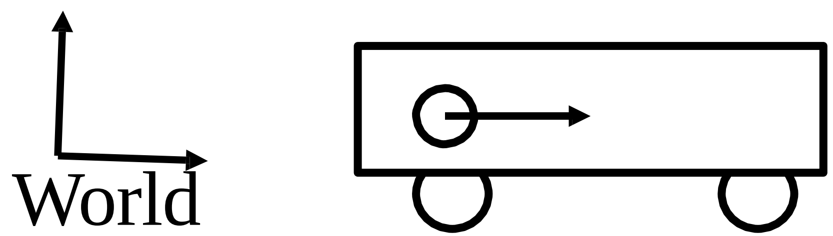
$$r_a^b$$



Sub/Super Scripts – Physics Vectors

- subscripts denote the frame/ object possessing the vector quantity: $\vec{v}_{ball}$
- superscripts denote the frame/object with respect to which the quantity is measured (i.e. the datum):

$$\vec{v}_{ball}^{world} = \vec{v}_{ball}^{train} + \vec{v}_{train}^{world}$$



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Notational Conventions

- Vectors:

$$p = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

- Also <u>sometimes</u> as $\underline{p}$ or as $\vec{p}$ to emphasize it is a vector.
- Matrices:

$$T = \begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}$$

Converting Coordinates

$$\boxed{p^b = T_a^b p^a}$$

- We will see later that T_a^b notation satisfies our conventions where it means the <u>'T' property of 'object' a w.r.t 'object' b</u>.

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Homogenous Transforms Outline

- Notational Conventions
- Definitions
- Homogeneous Transforms
- Semantics and Interpretations
- Summary

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Affine Transformation

- Most general linear transformation

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

- r’s and t’s are the transform constants
- Can be used to effect translation, rotation, scale, reflections, and shear.
- Preserves linearity but not distance (hence, not areas or angles).

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Homogeneous Transformation

- Set t1 = t2 = 0:

Homogeneous

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + 0$$

- r's are the transform constants
- Can be used to effect rotation, scale, reflections, and shear (<u>not translation</u>).
- Preserves linearity but not distance (hence, not areas or angles).

Orthogonal Transformation

- Looks the same …
  but:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$\begin{aligned} r_{11}r_{12} + r_{21}r_{22} &= 0 \\ r_{11}r_{11} + r_{21}r_{21} &= 1 \\ r_{12}r_{12} + r_{22}r_{22} &= 1 \end{aligned}$$

- Can be used to effect rotation, reflections.
- Preserves linearity <u>AND distance</u> (hence, areas and angles).

Rotation Matrix

- Looks the same …
  but:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$r_{11} r_{12} + r_{21} r_{22} = 0$$
$$r_{11} r_{11} + r_{21} r_{21} = 1$$
$$r_{12} r_{12} + r_{22} r_{22} = 1$$

Determinant(**R**)= 1

- Can be used to effect rotation.
- Preserves linearity AND distance (hence, areas and angles).

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Definitions

- Heading = angle of path tangent.
- Yaw = rotation about vertical axis
- Pitch = rotation about level sideways axis
- Roll = rotation about “forward” axis.
- Attitude = pitch & roll
- Azimuth = yaw (for a pointing device)
- Elevation = pitch (for a pointing device)

Definitions

- Orientation = attitude & yaw.
- Pose = position & orientation

2D: $\begin{bmatrix} x & y & \psi \end{bmatrix}^T$ 3D: $\begin{bmatrix} x & y & z & \theta & \varphi & \psi \end{bmatrix}^T$

- Posture = Pose plus some configuration

$$\begin{bmatrix} x & y & z & \theta & \varphi & \psi & q \end{bmatrix}^T$$

- Motion = movement of the whole body through space.

Homogenous Transforms Outline

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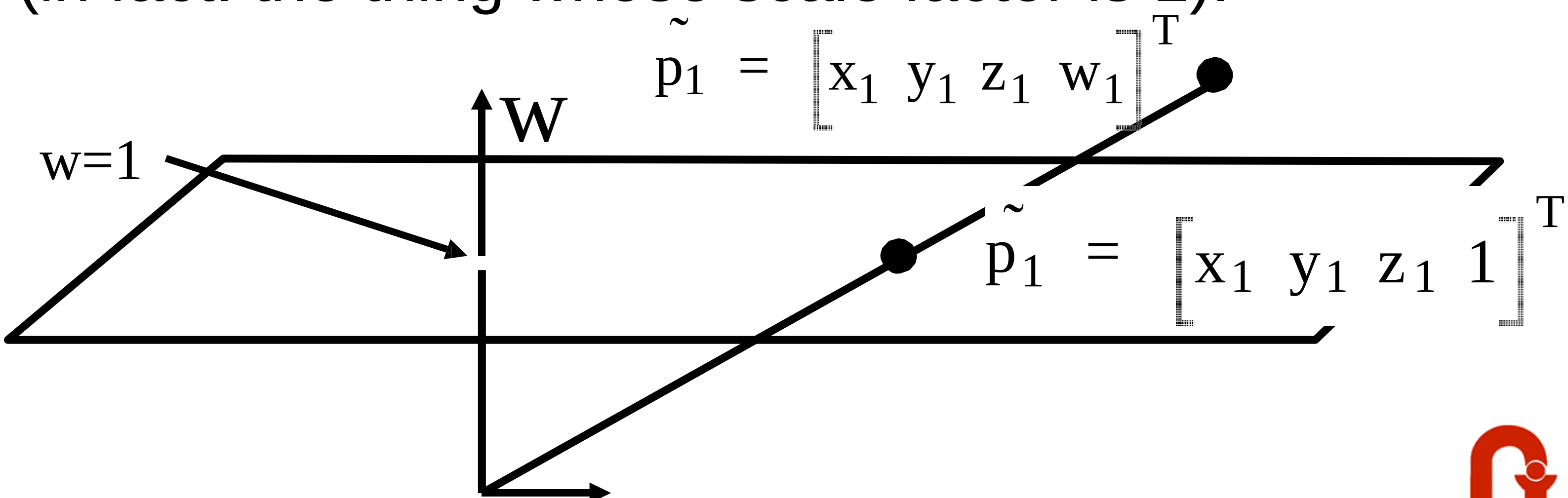
Homogeneous Coordinates

- Coordinates which are <u>unique up to a scale factor</u>. i.e

$$\underline{x} = 6\underline{x} = -12\underline{x} = 3.14\underline{x} = \text{same thing}$$

- The numbers in the vectors are not the same but we interpret them to mean the same thing (in fact. the thing whose scale factor is 1).



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Pure Directions

- Its also possible to represent pure directions
  - Pure in the sense they “are everywhere” (i.e. have no position and cannot be moved).
- We use a scale factor of zero to get a pure direction:

$$d_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 0 \end{bmatrix}$$

- It will shortly be clear why this works.

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Why Bother?

- Points in 3D can be rotated, reflected, scaled, and sheared with 3 X 3 matrices….

$$p_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = Tp_1 = \begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} t_{xx}x_1 + t_{xy}y_1 + t_{xz}z_1 \\ t_{yx}x_1 + t_{yy}y_1 + t_{yz}z_1 \\ t_{zx}x_1 + t_{zy}y_1 + t_{zz}z_1 \end{bmatrix}$$

- But not translated.

$$p_2 = p_1 + p_k = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} \neq \mathrm{Trans}(p_k)p_1$$

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Trick: Move to 4D

$$p_2 = p_1 + p_k = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} + \begin{bmatrix} x_k \\ y_k \\ z_k \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_k \\ 0 & 1 & 0 & y_k \\ 0 & 0 & 1 & z_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \mathrm{Trans}(p_k)p_1$$

$$\boxed{\begin{aligned} x_2 &= 1 \times x_1 + x_k \\ y_2 &= 1 \times y_1 + y_k \\ z_2 &= 1 \times z_1 + z_k \end{aligned}}$$

- The scale factor in the <u>vector</u> is used to add a scaled amount of the 4th <u>matrix</u> column.

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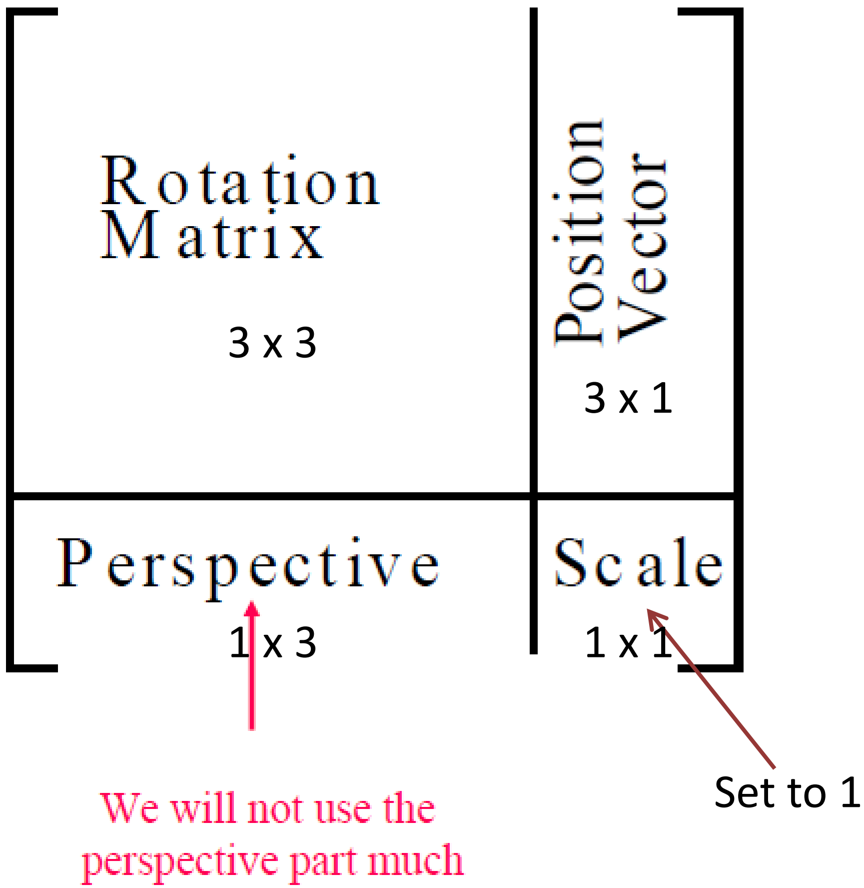
Format of Homogeneous Transforms (HTs)



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Inverse of a HT

$$\left[\begin{array}{ccc|c} & R & & \underline{p} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]^{-1} = \left[\begin{array}{ccc|c} & R^T & & -R^T\underline{p} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

- Of course standard matrix inverse also works, but this is faster

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Homogenous Transforms Outline

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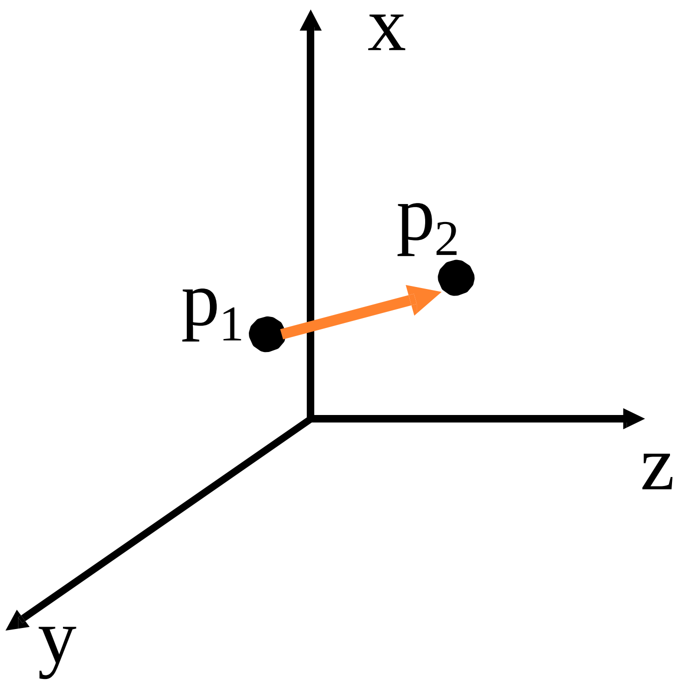
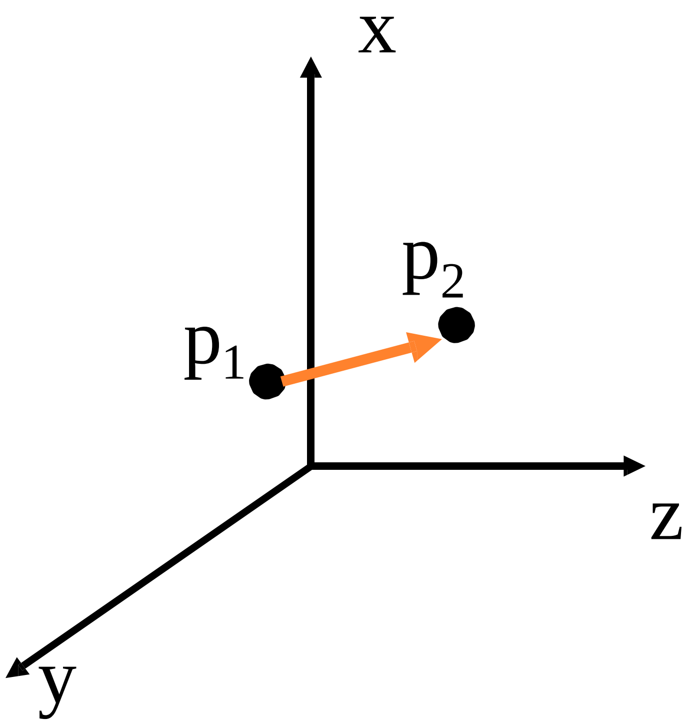
Operators

- Mapping:
  - Point1 -> Point2 (both expressed in same coordinates)



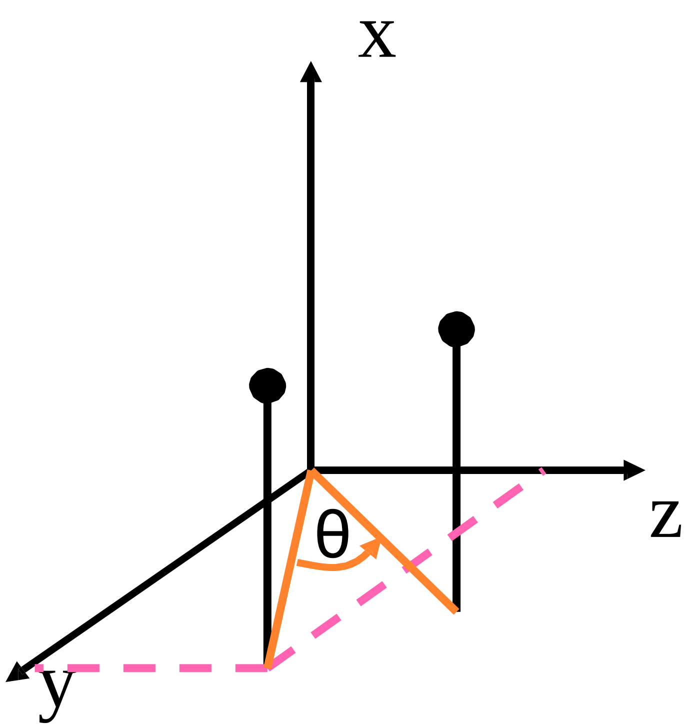
Operators



$$\mathrm{Trans}(u, v, w) = \begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operators

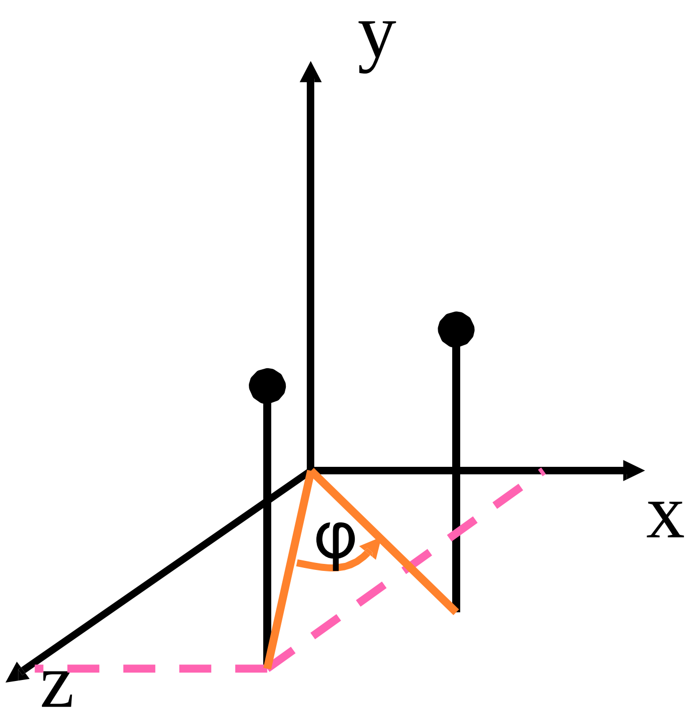
$s\theta = \sin(\theta)$
$c\theta = \cos(\theta)$



$$Rotx(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operators

$s\theta = \sin(\theta)$
$c\theta = \cos(\theta)$



$$\mathrm{Roty}(\phi) = \begin{bmatrix} c\phi & 0 & s\phi & 0 \\ 0 & 1 & 0 & 0 \\ -s\phi & 0 & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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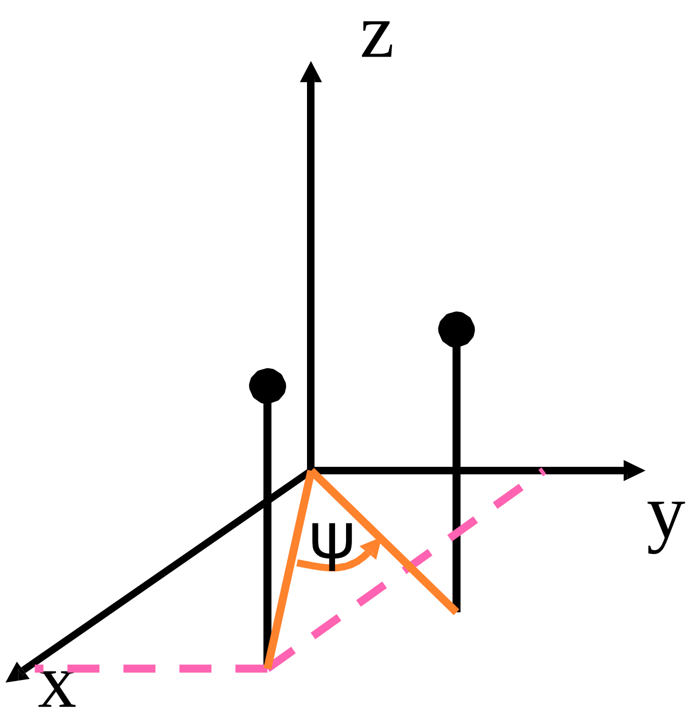
Operators

sθ = sin(θ)
cθ = cos(θ)

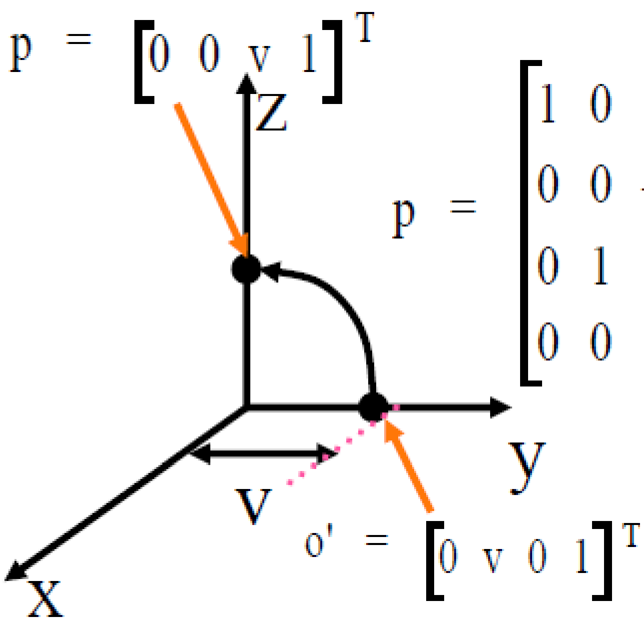


$$Rotz(\psi) = \begin{bmatrix} c\psi & -s\psi & 0 & 0 \\ s\psi & c\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Operating on a Point

- A point at the origin is translated along the y axis by ‘v’ units and then the resulting point is rotated by 90 degrees around the x axis.

$$p = Rotx(\pi/2)Trans(0,v,0)o$$



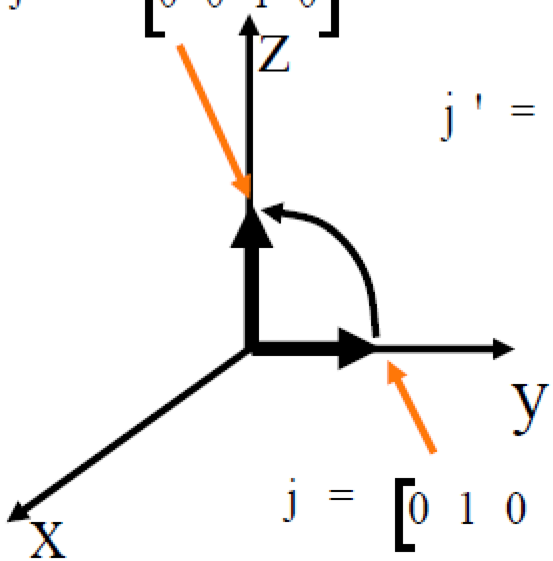
$$p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ v \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ v \\ 1 \end{bmatrix}$$

Example: Operating on a Direction

- The y axis unit vector is translated along the y axis by v units and then rotated by 90 degrees around the x axis.

$$j' = Rotx(\pi/2)Trans(0,v,0)\ j$$

$$j' = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}^T$$



$$j = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}^T$$

$$j' = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & -1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & v\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & -1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

- Having a zero scale factor <u>disables translation</u>.

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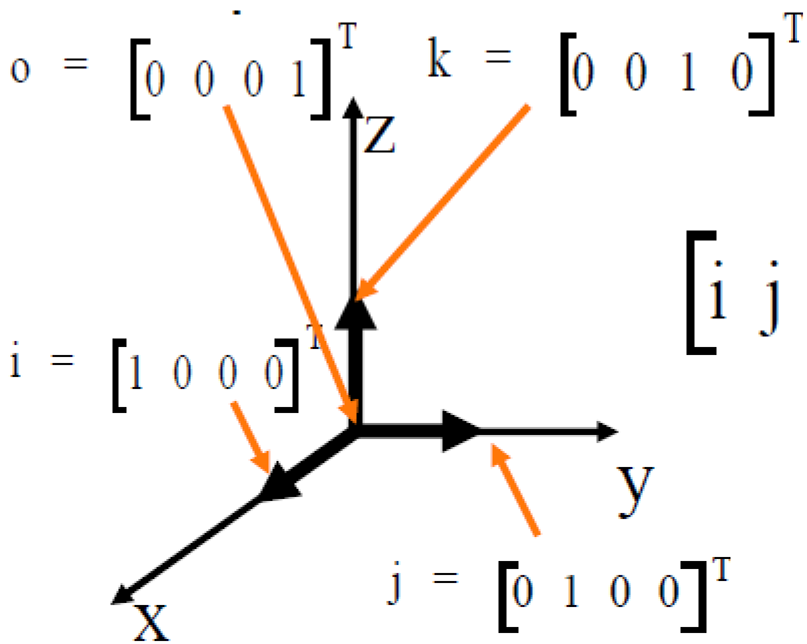
HTs as Coordinate Frames

- The columns of the identity HT can be considered to represent 3 directions and a point – the coordinate frame itself.



$$\begin{bmatrix} i & j & k & o \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

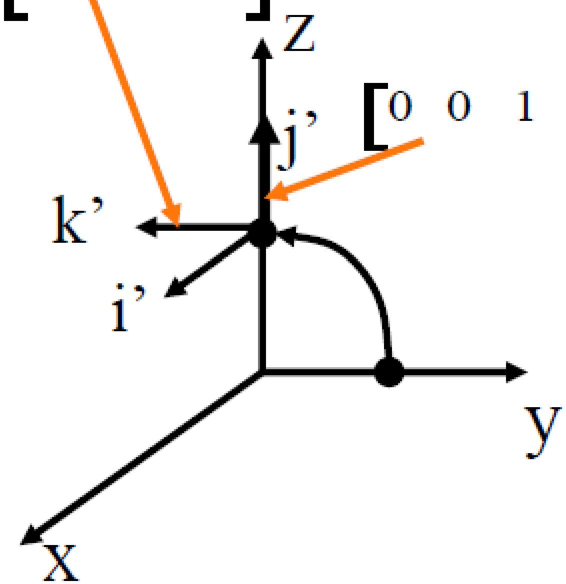
Example: Operating on a Frame

- Each resulting column of this result is the transformation of the corresponding column in the original identity matrix

$$[0 \;\; -1 \;\; 0 \;\; 0]^T$$

$$I' = Rotx(\pi/2)\,Trans(0, v, 0)\,I$$

$$[0 \;\; 0 \;\; 1 \;\; 0]^T$$



$$I' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \boxed{0} & \boxed{0} & 0 \\ 0 & \boxed{0} & \boxed{-1} & 0 \\ 0 & \boxed{1} & \boxed{0} & v \\ 0 & \boxed{0} & \boxed{0} & 1 \end{bmatrix}$$

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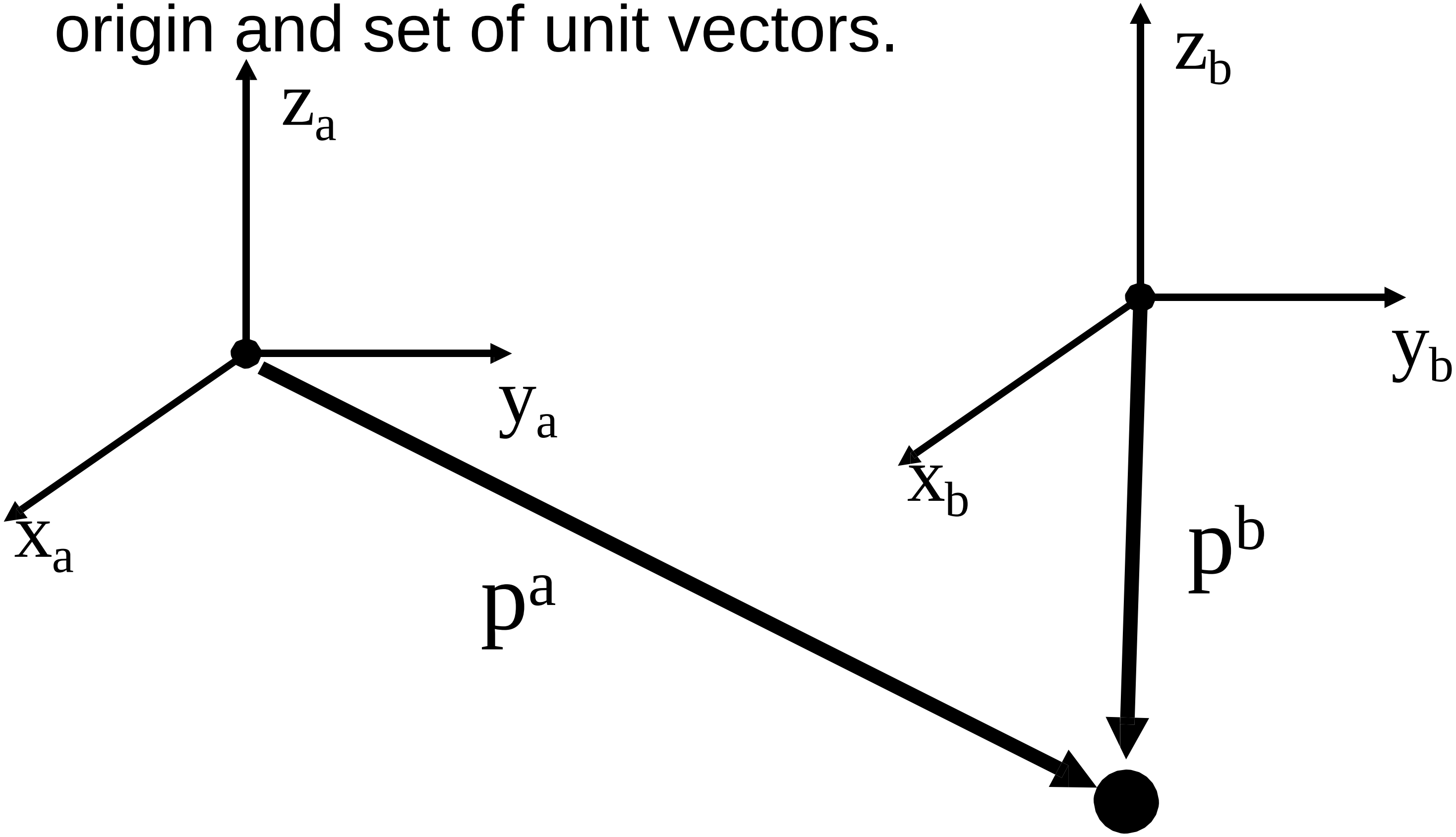
Huh?

- Columns of an input matrix are treated independently in multiplication.
- Every orthonormal matrix can be viewed as one set of axes located with respect to another set.
  - The “locations” can be read right from the the matrix – they’re just the columns.
- We can use this idea to track the position and orientation of rigid bodies….
  - Imagine embedding frames inside them somewhere and track their motions.

Converting Coordinates

- Converting coordinates is about expressing the <u>same physical point</u> with respect to a new origin and set of unit vectors.



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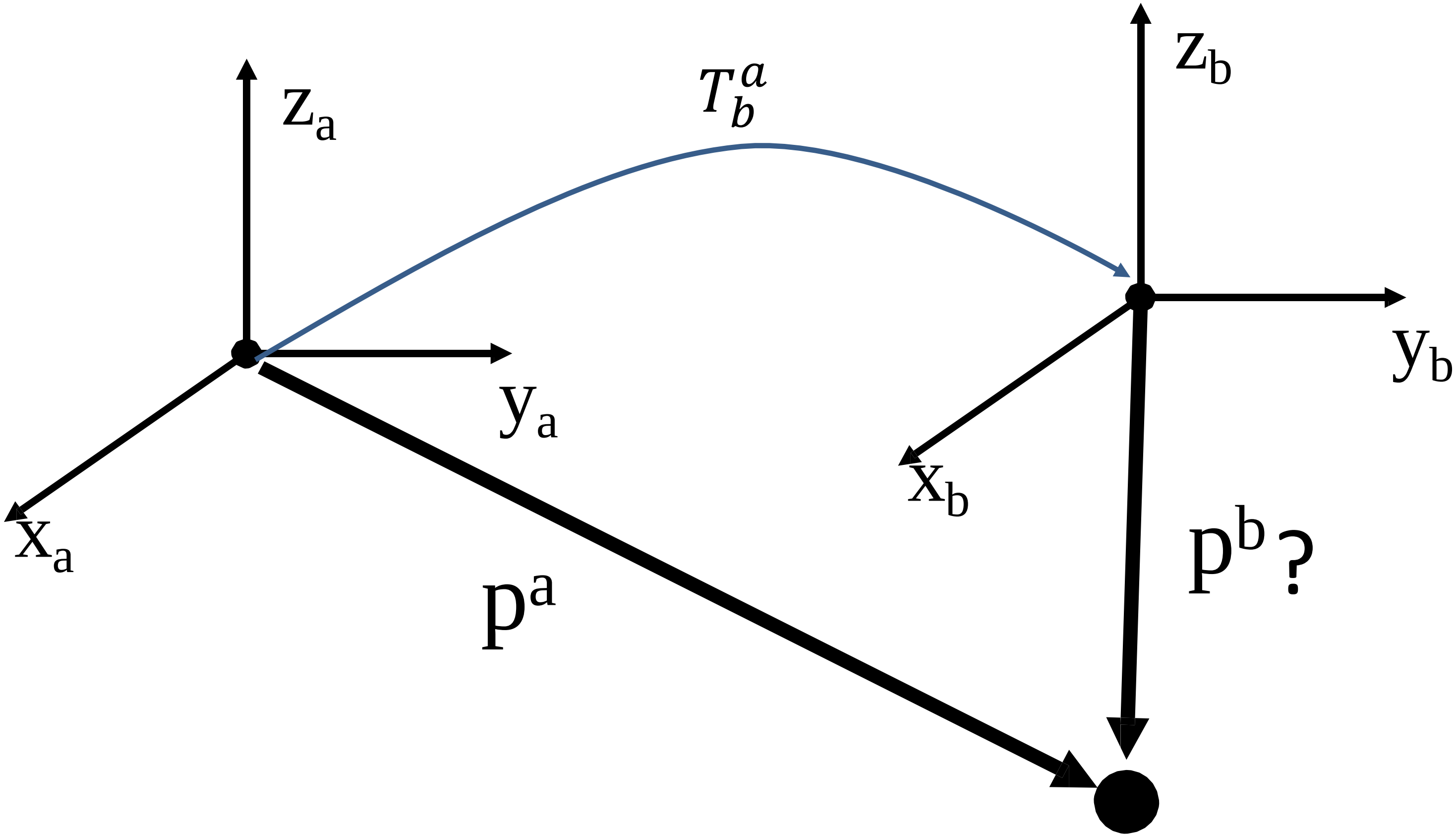

- Given T_b^a and p^a, how to compute p^b?

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Similarity Transforms

- Suppose you have a transform A^0 defined relative to frame 0, and you want to know what it is in frame 1. Assume you know T_1^0.

$$B = (T_1^0)^{-1} A^0 T_1^0$$

- B is transform A^0 represented in frame 1

Sequencing Transforms

- Any sequence of transforms can be represented by a single transform (Euler's rotation theorem)
- *How* you sequence transforms depends on if you are transforming w.r.t. the *fixed* or the *current* axes
- Transforming w.r.t. ***current*** axes: multiply *on the **right***

$$T_n^0 = T_1^0 T_2^1 \dots T_n^{n-1}$$

- Transforming w.r.t. ***fixed*** axes of frame 0: multiply *on the **left***

$$T_2^0 = T_1^0[(T_1^0)^{-1} A^0 T_1^0] = A^0 T_1^0$$

Similarity transform

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Homogenous Transforms Outline

- Notational Conventions
- Definitions
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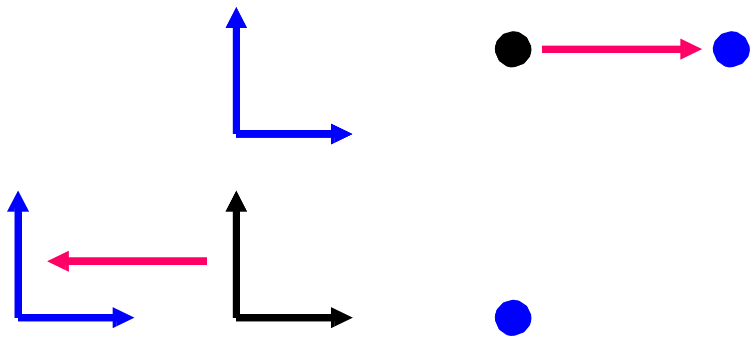
- <u>Summary</u>

Summary

- Everything is relative. There is no way to distinguish moving a point "forward" from moving the coordinate system "backward".



- In both cases, the resulting (blue) point has the same relationship to the blue frame.

Summary

- Homogeneous Transforms are:
  - Operators
  - Frames
- They can be both the things that operate on other things and the things operated upon.

Break

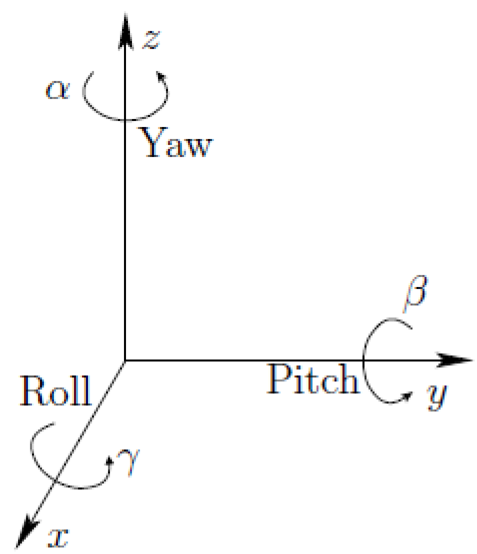
Euler Angles

- Can define rotation relative to the axes of a frame

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$R_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}$$



Rotating about more than one axis

- Great for rotating about a single axis
- What if we want to rotate about multiple axes?
  - Need to adopt a *convention*: for example roll-pitch-yaw

$$R(\alpha,\beta,\gamma) = R_z(\alpha)\,R_y(\beta)\,R_x(\gamma) =$$
$$\begin{pmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}$$

- There are 12 variants of ordering and axis selection for Euler angles

Problems with Euler Angles

- Single axis: no problem!
- Two or more axes
  - Results can be counter-intuitive
  - “Small” rotations are better than “large” ones
- Singularities
  - Many Euler Angles map to one rotation (Gimbal Lock)
  - Where singularities are depends on the convention
  - Small rotations near singularities can have big effects

Singularity Example

- Let's say this is our convention:

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Let's set $\beta = 0$

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Multiplying through, we get:

$$R = \begin{bmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\sin\gamma - \sin\alpha\cos\gamma & 0 \\ \sin\alpha\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\sin\gamma + \cos\alpha\cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Simplify:

$$R = \begin{bmatrix} \cos(\alpha+\gamma) & -\sin(\alpha+\gamma) & 0 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

α and γ do the same thing!
We have lost a degree
of freedom!

What to use?

- Euler angles are simplest but have singularity problems
  - Besides you usually convert to rotation matrices to actually use them

- My recommendation: Use Homogenous Transforms whenever possible
  - Easy to compose and manipulate (just linear algebra)
  - Rotation and translation in one consistent package
  - Not super-simple, but can “read” the coordinate axes by looking at columns

- Many software frameworks provide methods to convert between these representations (ROS, openrave)

- LaVallе Chapter 3 also discusses *quaternions* 四元法
  - We won’t cover this but it’s a very useful representation!

Reading and Homework

- Homework 1 is out
- Read chapter 13.1 from Boyd Linear Algebra Book
- Reading from optimization book
  - Introduction (Ch. 1)
  - Convex sets (Ch. 2.1-2.3, 2.5)
  - Convex functions (Ch. 3.1-3.2.5)