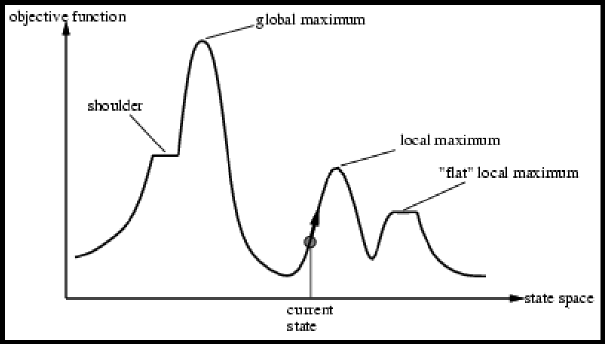
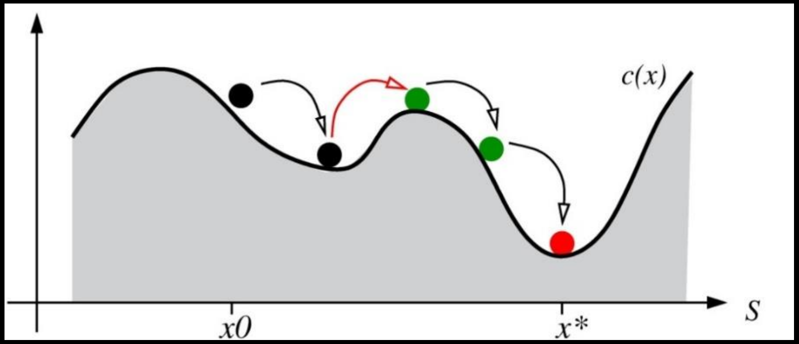
Searching for a Path

Last time…

- We saw how to frame optimization as a graph search problem





- But what if we need *a path* from a start to a goal that is feasible/optimal?
- Motion planning for robots requires solving this problem!

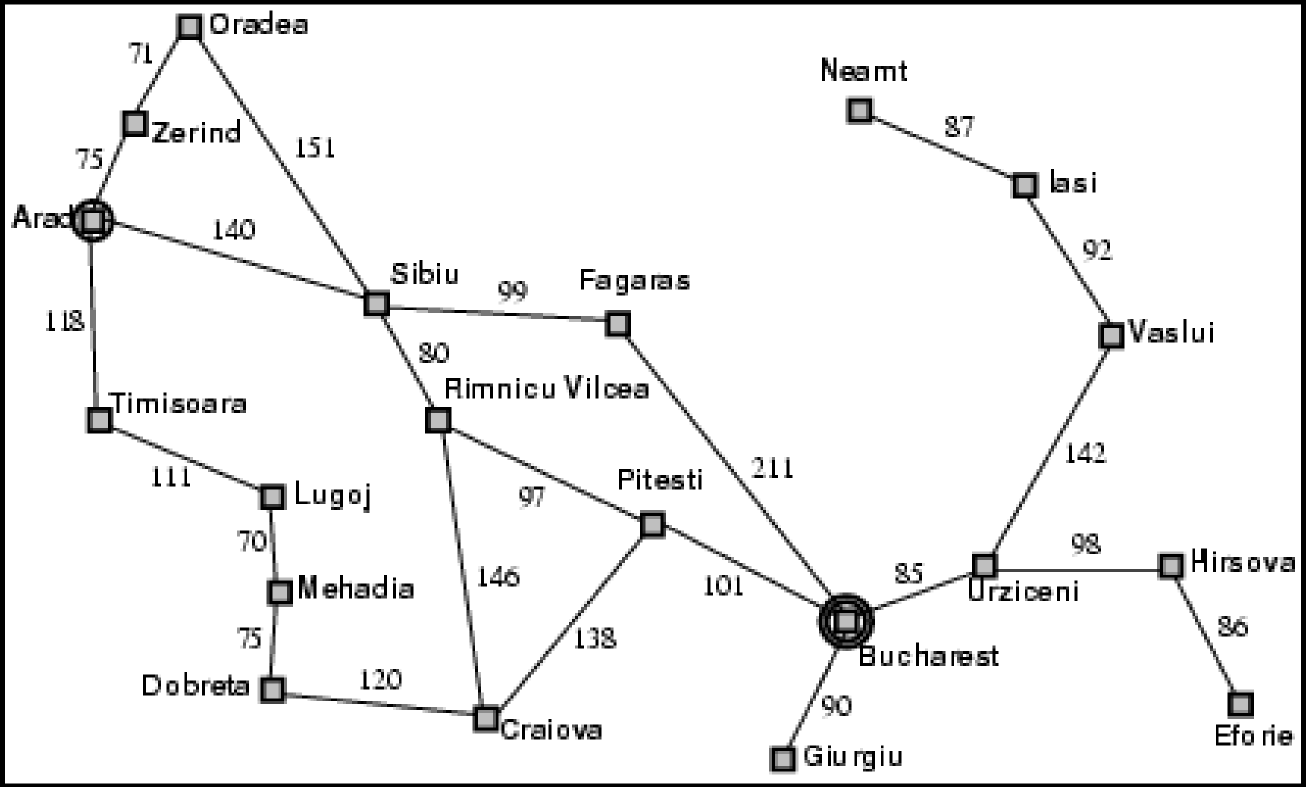
Outline

- Formulating a path search problem
  - Examples
  - General considerations for robotics
- Tree search algorithms
  - Uninformed search
  - Informed search

Formulating a path search problem

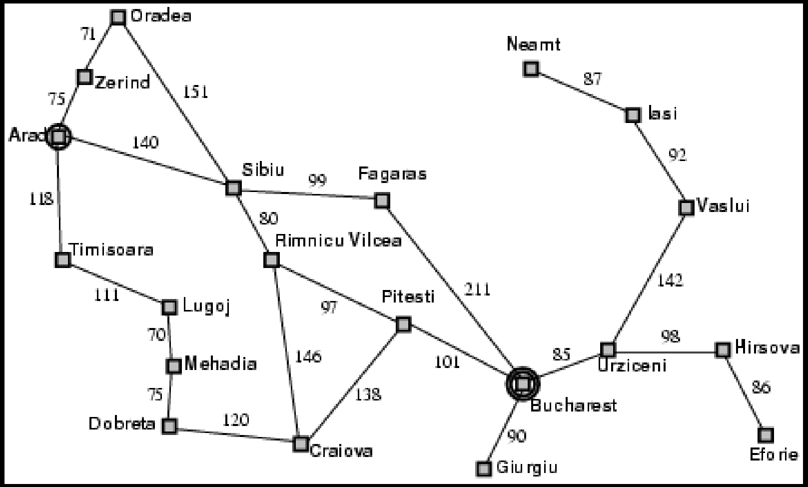
1. State Space
2. Successor Function
3. Actions
4. Action Cost
5. Goal Test

Example: Romanian roadmap

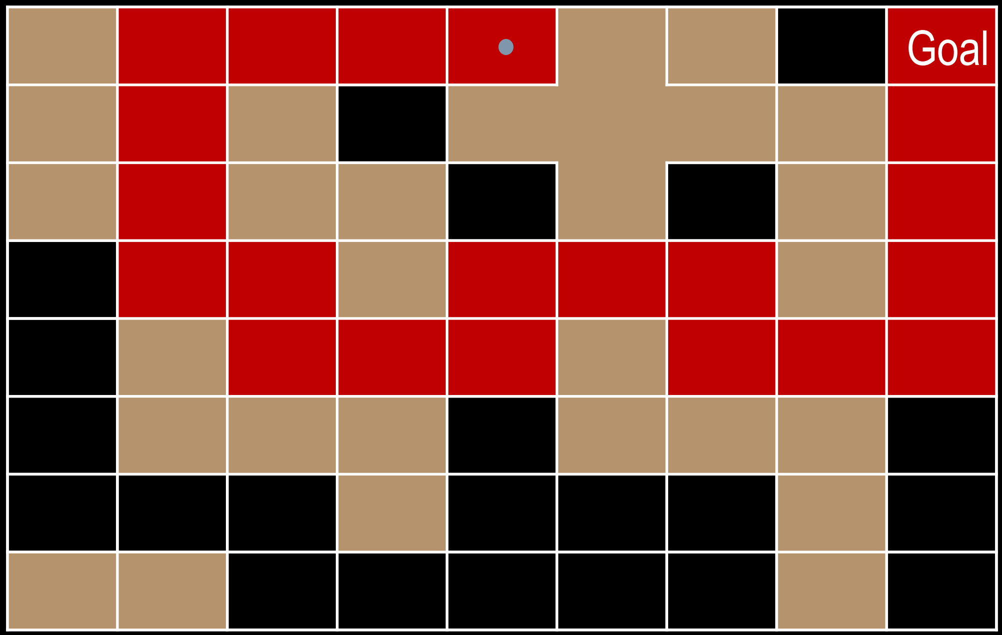


Example: A Romanian roadmap

1. State Space
   - The set of cities
2. Successor Function
   - A city's successors are cities directly connected to it (its neighbors)
3. Actions
   - Move to a neighboring city
4. Action Cost
   - Distance between the cities
5. Goal Test
   - Check if at goal city



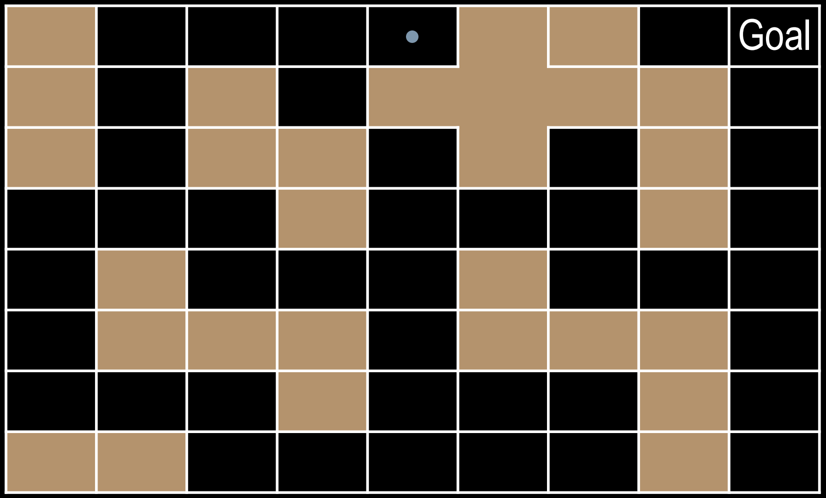
Example : Point robot in a maze:

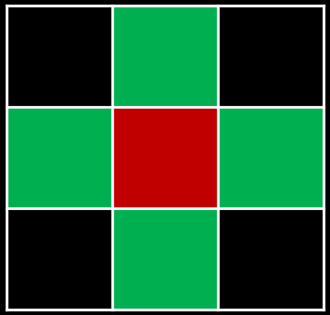


- Find a sequence of free cells that goes from start to goal

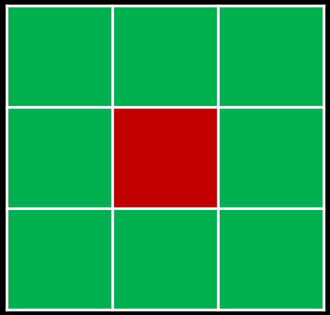
Point Robot Example

1. State Space
   - The space of cells, usually in x,y coordinates
2. Successor Function
   - A cell's successors are its free neighbors
   - 4-connected vs. 8-connected
3. Actions
   - Move to a neighboring cell
4. Action Cost
   - Distance between cells traversed
   - Are costs the same for 4 vs 8 connected?
5. Goal Test
   - Check if at goal cell
   - Multiple cells can be marked as goals





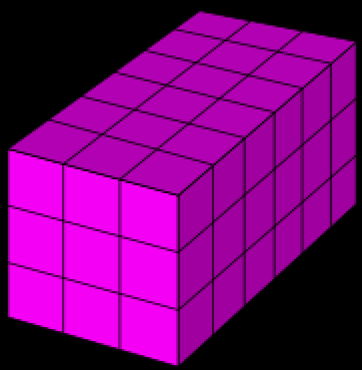
4-connected



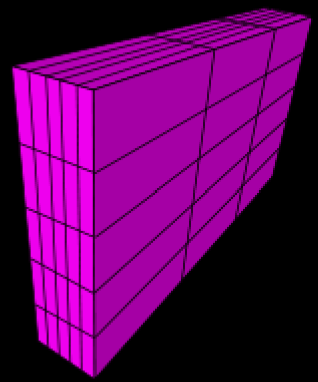
8-connected

Formulating the problem: State space

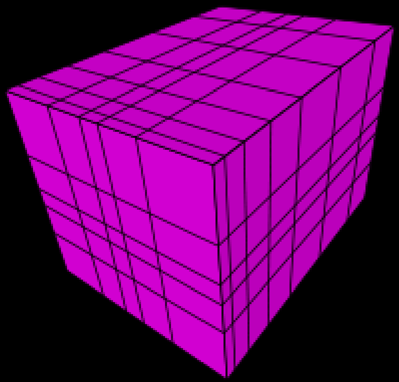
- For motion planning in robotics, state space is often a grid
- There are many kinds of grids!



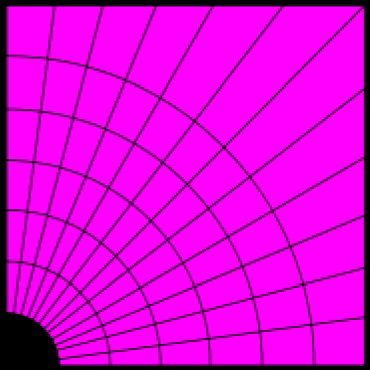
Cartesian Grid



Regular Grid



Rectilinear Grid

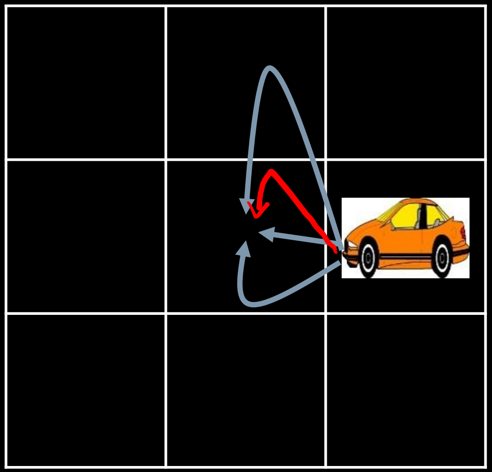


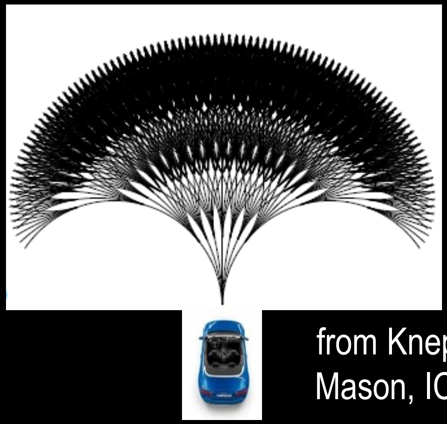
Curvilinear Grid

- The choice of grid (i.e. state space) is crucial to performance and accuracy
- Remember, the world is really continuous; these are all approximations

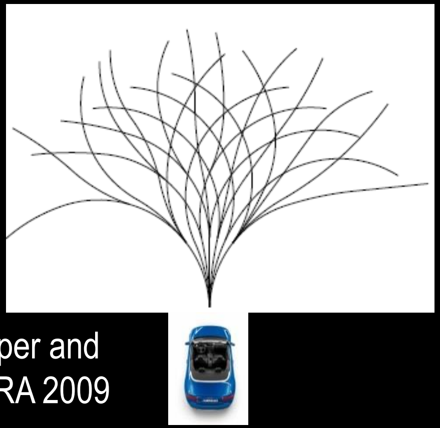
Formulating the problem: Actions

- Actions in motion planning are also often continuous
- For example, many ways to move between neighboring cells
- Usually pick a discrete action set *a priori*
- What are the tradeoffs in picking action sets?
- This will be a major issue in non-holonomic motion planning



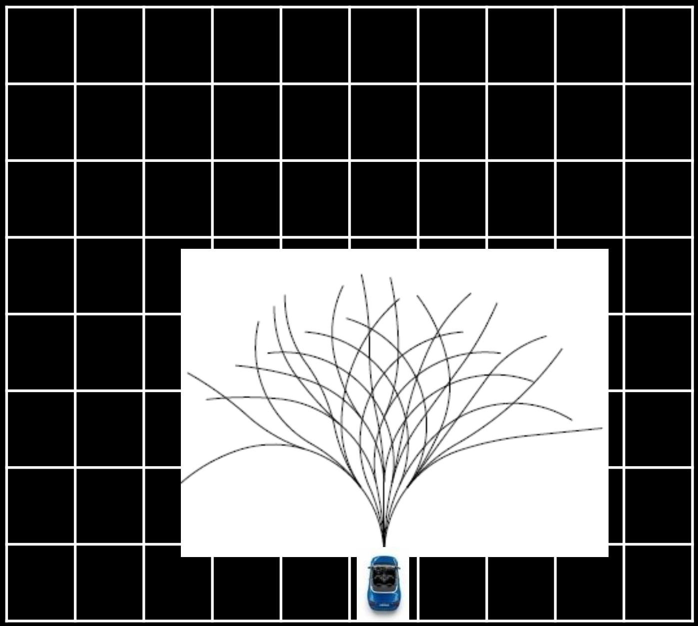


from Knepper and Mason, ICRA 2009



Formulating the problem: Successors

- These are largely determined by the action set
- Successors may not be known a priori
    - I.e. you have to try each action in your action set to see which cell you end in



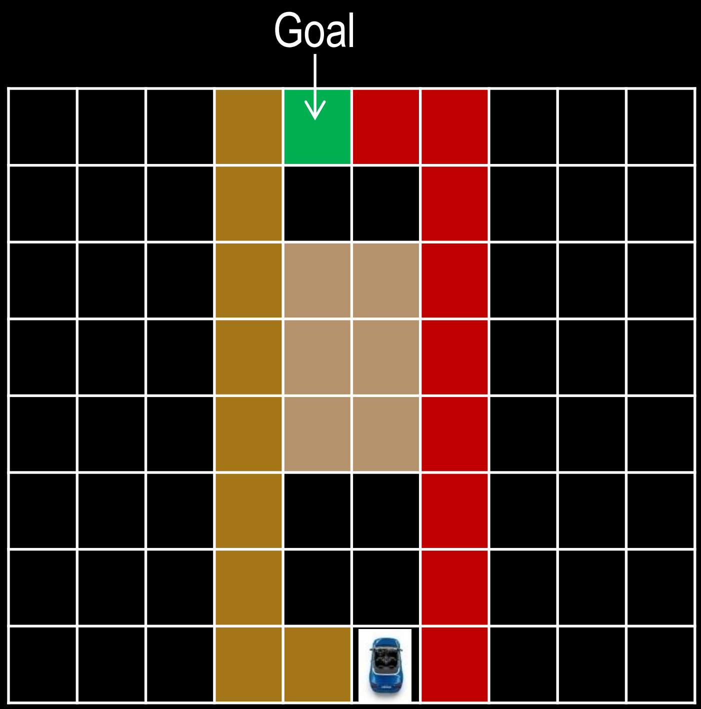
Formulating the Problem: Action Cost

- Depends on what you’re trying to optimize
  - Minimum Path Length: Cost is distance traversed when executing action
  - What is action cost for path smoothness?
- Sometimes we consider more than one criterion
  - Linear combination of cost functions (most common):

    Cost = $a_1C_1 + a_2C_2 + a_3C_3 \ldots.$

Formulating the Problem: Goal Test

- Goals are most commonly specific cells you want to get to
- But they can be more abstract, too!
- Example Goals:
  - A state where X is visible
  - A state where the robot is contacting X
  - Topological goals



A topological goal example: go **right** around the obstacle (need whole path to evaluate if goal reached)

Finding a Path:
Tree Search Algorithms

Tree Search Algorithms

```
function Tree-Search(problem, strategy)
    Root of search tree <- Initial state of the problem
    While 1
        If no nodes to expand
                return failure
        Choose a node n to expand according to strategy
        If n is a goal state
                return solution path  //back-track from goal to
                                      //start in the tree to get path
        Else
                NewNodes <- expand n
                Add NewNodes as children of n in the search tree
```

Strategies are evaluated in terms of

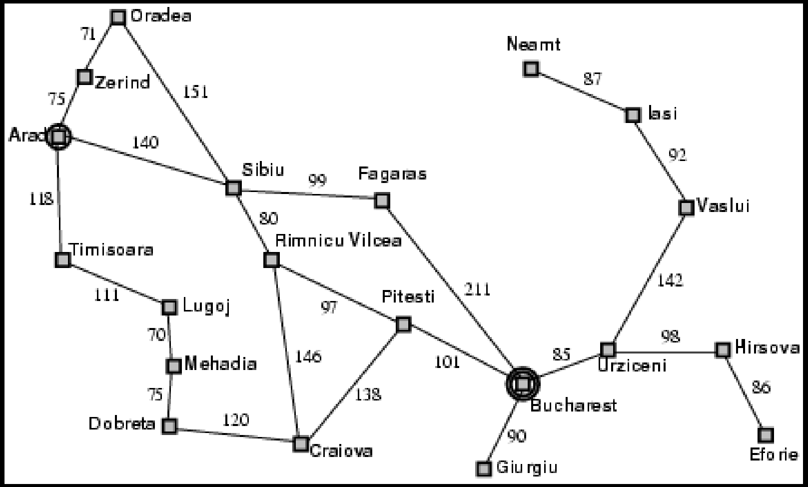
- completeness: does it always find a solution if one exists?
- optimality: does it always find a least-cost solution?

- Two types of complexity
  - time complexity: number of nodes visited
  - space complexity: maximum number of nodes in memory

Time and space complexity are measured in terms of

- *b:* maximum branching factor of the search tree (may → ∞)
- *d:* depth of the least-cost solution
- *m*: maximum depth of the state space (may be ∞)

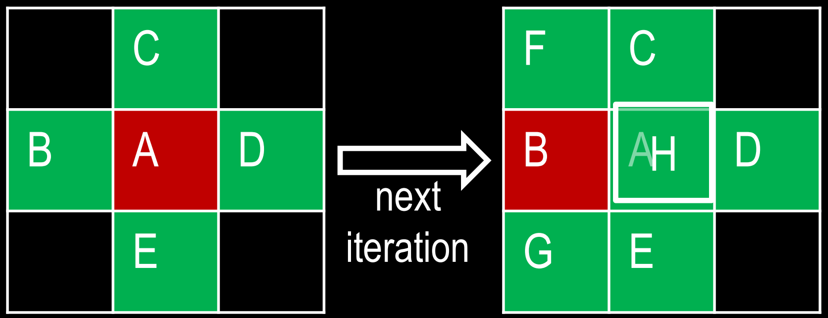
  - What is *b*? (worst case)
  - What is *d*?
  - What is *m*?



Tree Search Algorithm Implementation

- Open list: List of nodes we know about but haven’t expanded yet
  - Implement as a priority queue

- Need to avoid re-expanding the same state



- Solution: A *closed list* to track which nodes are already explored

Uninformed Search Strategies

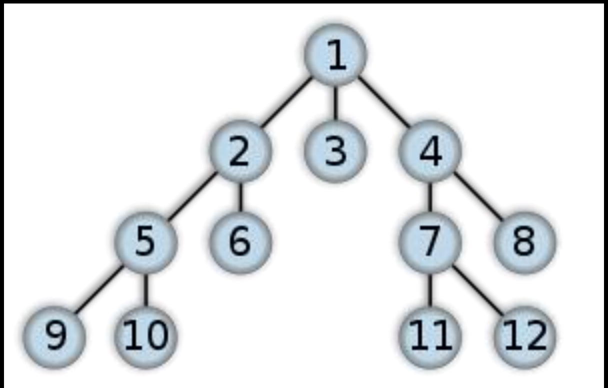
Uninformed search strategies

- Uninformed search strategies use only the information available in the problem definition

- What does it mean to be uninformed?
  - You only know which states are connected by which actions. No additional information.
  - Later we'll talk about informed search, in which you can *estimate* which actions are likely to be better than others.

Breadth-first Search (BFS)

- Main idea: build search tree in layers
- Assumes all actions have equal cost
- Open list is a queue, new nodes are inserted at the **back**



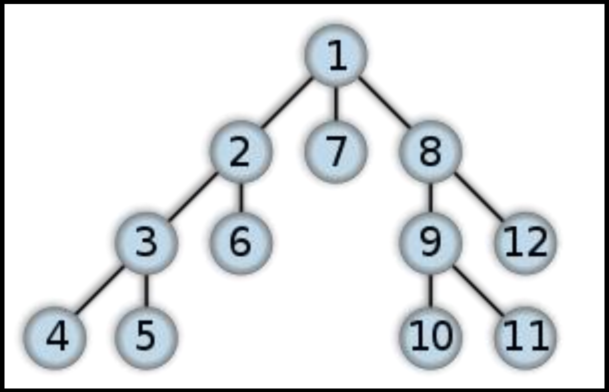
- Result: “Oldest” nodes are expanded first
- BFS finds the shortest path to the goal
- When would this strategy be very inefficient?

Properties of breadth-first search

- Complete?
  - Yes (if b is finite)
- Optimal?
  - Yes (**if cost = 1 per step**)
- Time?
  - $1+b+b^2+b^3+\ldots+b^d = O(b^d)$
- Space?
  - $O(b^d)$ (keeps every node in memory)

Depth-first Search (DFS)

- Main idea: Go as deep as possible as fast as possible
- Assumes all actions have equal cost
- Open list is a queue, new nodes are inserted at the **front**



- Result: “Newest” nodes are expanded first
- DFS does NOT necessarily find the shortest path to the goal
- When would this strategy be very inefficient?

Properties of depth-first search

- Complete?
  - No: fails in infinite-depth spaces
  - Yes: in finite spaces
- Optimal?
  - No
- Time?
  - $O(b^m)$: (m is max depth of state space)
  - terrible if *m* is much larger than *d*
  - but if solutions are plentiful, may be much faster than breadth-first
- Space?
  - $O(bm)$

Other uninformed searches

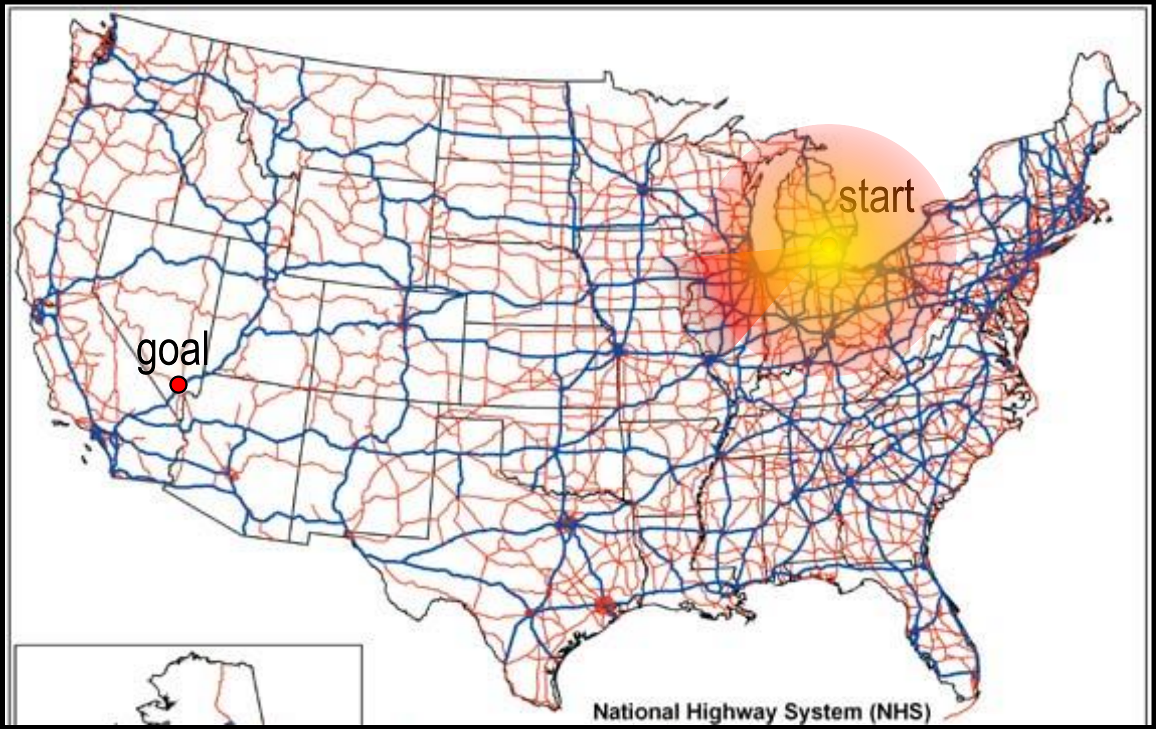
- Uniform-Cost: Same as BFS but expands *least-cost* unexpanded node
  - Implementation: Sort open list by cost-to-come to a node instead of depth
  - Dijkstra's algorithm: Uniform-cost search with no goal (get the least-cost path to all nodes)
- Depth-Limited: DFS with a fixed depth limit (l)
- Iterative Deepening: Run depth-limited search repeatedly with increasing (l)
- Bidirectional: Run BFS from start and goal simultaneously

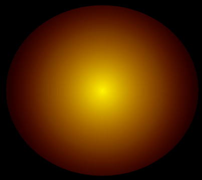
Criterion	Breadth-First	Uniform-Cost	Depth-First	Depth-Limited	Iterative Deepening	Bidirectional (if applicable)
Complete?	Yes^a	Yes^{a,b}	No	No	Yes^a	Yes^{a,d}
Time	$O(b^d)$	$O(b^{1+\lfloor C^*/\epsilon \rfloor})$	$O(b^m)$	$O(b^\ell)$	$O(b^d)$	$O(b^{d/2})$
Space	$O(b^d)$	$O(b^{1+\lfloor C^*/\epsilon \rfloor})$	$O(bm)$	$O(b\ell)$	$O(bd)$	$O(b^{d/2})$
Optimal?	Yes^c	Yes	No	No	Yes^c	Yes^{c,d}

Break

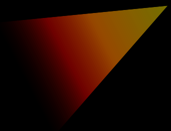
Informed Search Strategies

Search types





Uninformed search



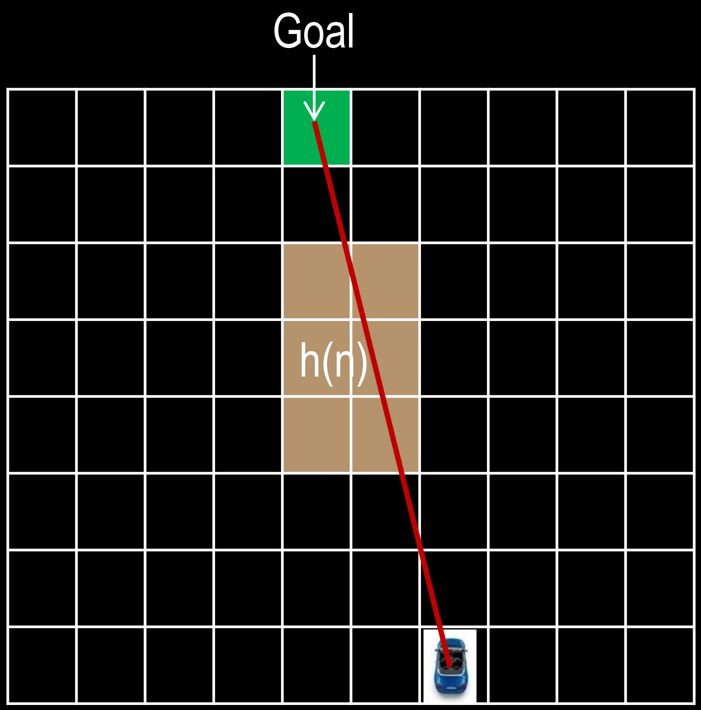
Informed search

Best-first Search

- Main idea: Use Heuristic function h(n) to estimate each node's distance to goal, expand node with minimum h(n)

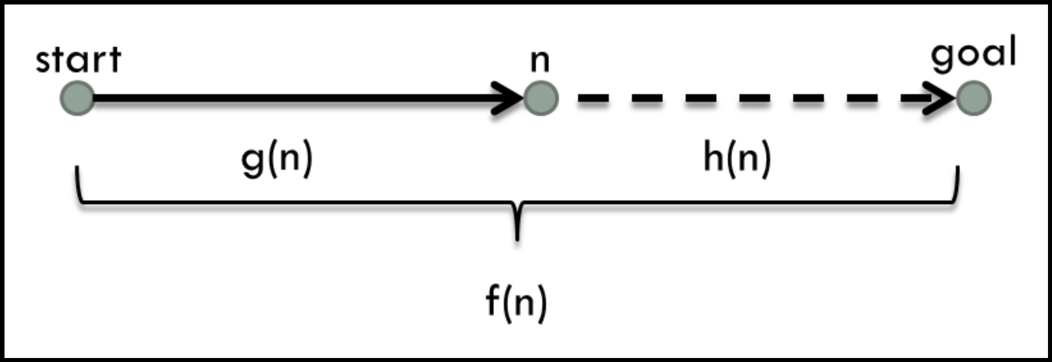
- Open list is a *priority queue*, nodes are sorted according to h(n)

- Result: Works great if heuristic is a good estimate
- Does not necessarily find least-cost path
- When would this strategy be inefficient?



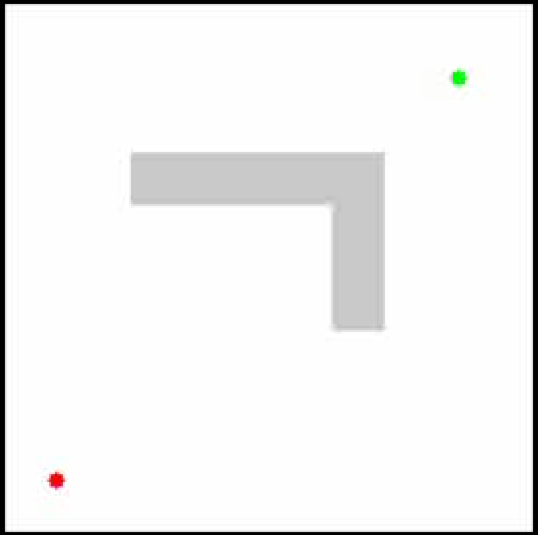
A * Evaluation functions

- Main idea: Select nodes based on estimated path to goal
- Evaluation function *f(n)* = *g(n)* + *h(n)*
  - *g(n)* = cost so far to reach *n* (cost-to-come)
  - *h(n)* = estimated cost from *n* to goal (cost-to-go)
  - *f(n)* = **estimated total cost** of path through *n* to goal



A* Search

- Open list is a *priority queue*, nodes are sorted according to f(n)
- g(n) is sum of edge costs from root node to n



A* Search

- IMPORTANT RESULT: If h(n) is *admissible*, A* will find the least-cost path!

- Admissibility: h(n) must *never overestimate* the true cost to reach the goal from n

  $h(n) \leq h^*(n)$, where $h^*(n)$ is the true cost to reach goal from n

  $h(n) \geq 0$ (so $h(G) = 0$ for goals G)

- "Inflating" the hueristic may give you faster search, but least-cost path is not guaranteed

Properties of A* (w/ admissible heuristic)

- <u>Complete?</u>
  - Yes (unless there are infinitely many nodes with f ≤ f(G))
- <u>Optimal?</u>
  - Yes
- <u>Time?</u>
  - Exponential, approximately $O(b^d)$ in the worst case
- <u>Space?</u>
  - $O(b^m)$ Keeps all nodes in memory

A* Optimality Proof

- Suppose A* expands a sub-optimal goal G_2 and adds it to the queue
- Let n be an unexpanded node on the least-cost path to an optimal goal G

$f(G_2) = g(G_2)$ since $h(G_2) = 0$

$g(G_2) > g(G)$ since G_2 is suboptimal

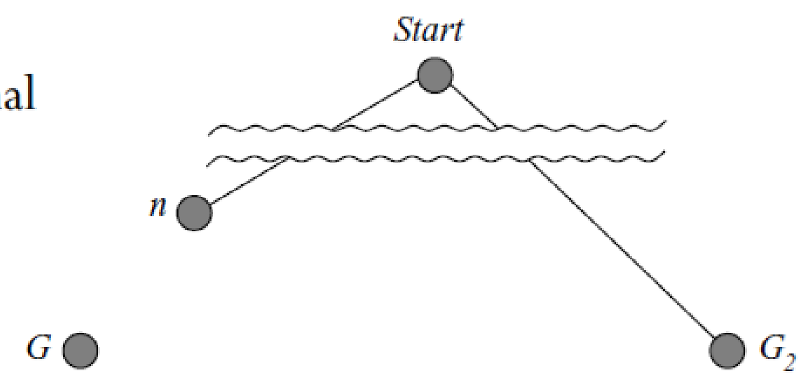
$f(G) = g(G)$ since $h(G) = 0$

$f(G_2) > f(G)$ from above

$h(n) \leq h^*(n)$ since h is admissible

$g(n) + h(n) \leq g(n) + h^*(n)$

$f(n) \leq f(G)$



Hence $f(G_2) > f(n)$, and A* will not have selected G_2 for expansion

- A* expands any node with smaller cost than the lowest-cost goal

Heuristic Consistency

- A heuristic is consistent (also known as monotonic) when it obeys

$$h(n_1) \leq C(n_1, n_2) + h(n_2)$$

for adjacent n_1, n_2 and with edge cost $C(n_1, n_2)$

- Ensures that it is impossible to decrease *f* by extending a path to include a neighboring node

- If a heuristic is consistent, we can use a closed list
- If not consistent, we cannot use a closed list and guarantee optimality

Questions

- If you set $g(x) = 0$ for all x, and $h(x)$ = depth of x, A* is equivalent to which search strategy?

- In the worst case, what percentage of nodes will A* explore?

Variants of A*

- There are many variants A*, some of the most popular in robotics:

- Dynamic A*, aka D* and Lifelong Planning A*
  - Fast updating of a plan when costs change in the environment
  - **D* lite** is used in practice, same as D* but easier to implement

- Anytime Repairing A* (ARA*)
  - Anytime A* with bounds on sub-optimality
  - *Anytime* means the longer you wait, the better the solution becomes but a solution is available at any time.

- Anytime Non-parametric A* (ANA*)
  - Like ARA* but with less parameters

Summary

- A search problem is defined by
  1. State Space
  2. Successor Function
  3. Actions
  4. Action Cost
  5. Goal Test

- Search types:
  - Uninformed: Uses only problem definition, no additional information
  - Informed: Relies on a heuristic to estimate cost to reach the goal

- A* is a strong approach
  - My default method for search problems

Homework

- LaVallе Ch. 1
- LaValle Ch. 6.1-6.2.4