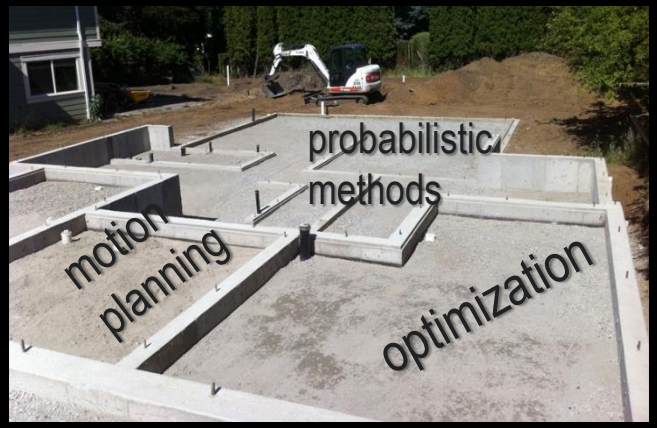
Putting it All Together

WHAT IS ALGORITHMIC ROBOTICS?

What we learned in this course

- Built a **foundation** for more advanced robotics courses/concepts



- Let's review some of the most important concepts

Fundamentals

Using Matrix Inversion

- Probably the most common problem in linear algebra: Given a matrix $\mathbf{A}$ and vector b, and the following equation

$$\mathbf{A}x = b$$

solve for the vector x

- Use this to solve a system of linear equations. For example:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$\begin{array}{ccccc} \mathbf{A} & & x & & b \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{array}$$

Solving $\mathbf{A}x = b$

- If $\mathbf{A}$ is <u>n x n</u> and b is n x 1
  - Check rank or determinant of A to see if it is invertible.
  - If so, use the matrix inverse:

$$\mathbf{A}^{-1}\mathbf{A}x = \mathbf{A}^{-1}b$$
$$\mathbf{I}x = \mathbf{A}^{-1}b$$
$$x = \mathbf{A}^{-1}b$$

  - If not, no solution

- What if $\mathbf{A}$ is not square?

The Pseudo-inverse

- The Moore-Penrose Pseudo-inverse is defined as
  - Left pseudo-inverse:

$$\mathbf{A}^{+} = (\mathbf{A}^{T}\mathbf{A})^{-\mathbf{1}}\mathbf{A}^{T}$$

    - Has some of the properties of the inverse, most importantly:

$$\mathbf{A}^{+}\mathbf{A} = \mathbf{I}$$

  - Derivation

$$\mathbf{I} = (\mathbf{A}^{T}\mathbf{A})^{-\mathbf{1}}(\mathbf{A}^{T}\mathbf{A})$$
$$\mathbf{I} = [(\mathbf{A}^{T}\mathbf{A})^{-\mathbf{1}}\mathbf{A}^{T}]\mathbf{A}$$
$$\mathbf{I} = \mathbf{A}^{+}\mathbf{A}$$

- The right pseudo-inverse is derived similarly to get $\mathbf{A}\mathbf{A}^{+} = \mathbf{I}$

The Pseudo-inverse

$$\mathbf{A}^{+} = (\mathbf{A}^{T}\mathbf{A})^{-1}\mathbf{A}^{T}$$

- Works even when $\mathbf{A}$ is not square
- What about $(\mathbf{A}^{T}\mathbf{A})^{-1}$?
  - $(\mathbf{A}^{T}\mathbf{A})$ is automatically square
  - But we need to check if $(\mathbf{A}^{T}\mathbf{A})$ is invertable

- If $\mathbf{A}$ is square and invertable, then $\mathbf{A}^{+} = \mathbf{A}^{-1}$
  - We don't loose any generality by always using the pseudo-inverse

The Pseudo-inverse

- Can use the pseudo-inverse like an inverse to solve $\mathbf{A}x = b$ when $\mathbf{A}$ is m x n and b is m x 1 :

$$\mathbf{A}^{+}\mathbf{A}x = \mathbf{A}^{+}b$$
$$\mathbf{I}x = \mathbf{A}^{+}b$$
$$x = \mathbf{A}^{+}b$$

- This is known as the least-squares solution
- Remember that $(\mathbf{A}^{T}\mathbf{A})$ must be invertable

Rotation Matrix

- Looks the same …
  but:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$r_{11} r_{12} + r_{21} r_{22} = 0$$
$$r_{11} r_{11} + r_{21} r_{21} = 1$$
$$r_{12} r_{12} + r_{22} r_{22} = 1$$

Determinant(**R**)= 1

- Can be used to effect rotation.
- Preserves linearity <u>AND distance</u> (hence, areas and angles).

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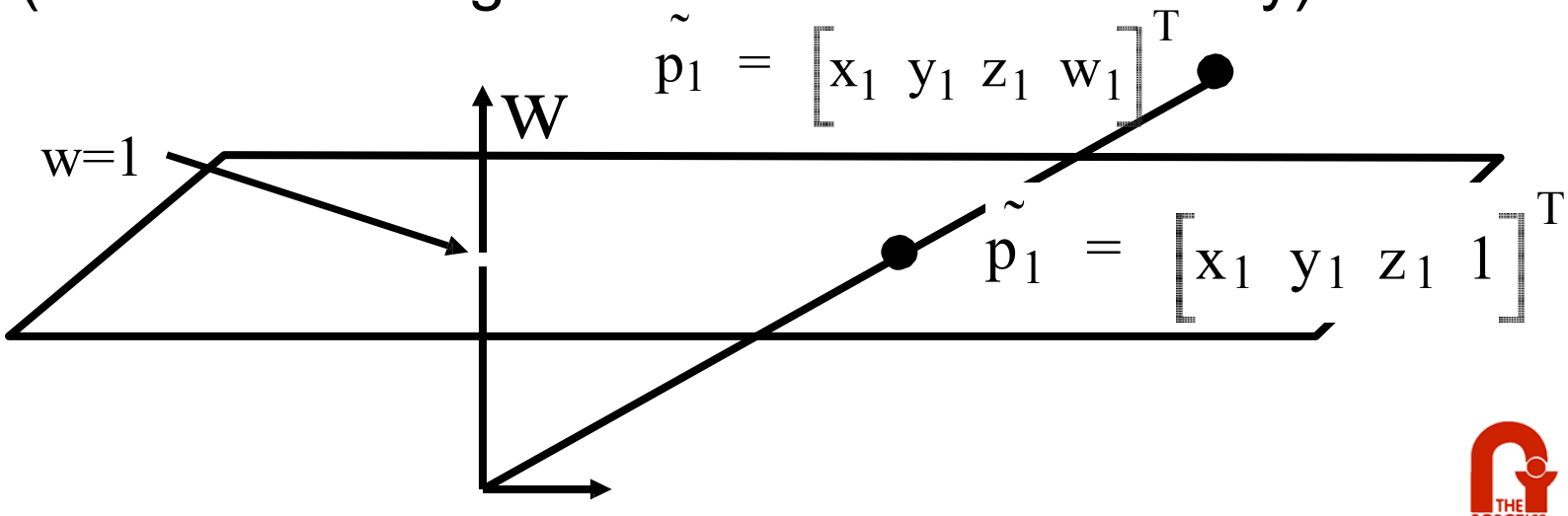
Homogeneous Coordinates

- Coordinates which are <u>unique up to a scale factor</u>. i.e

$$\underline{x} = 6\underline{x} = -12\underline{x} = 3.14\underline{x} = \text{same thing}$$

- The numbers in the vectors are not the same but we interpret them to mean the same thing (in fact. the thing whose scale factor is unity).



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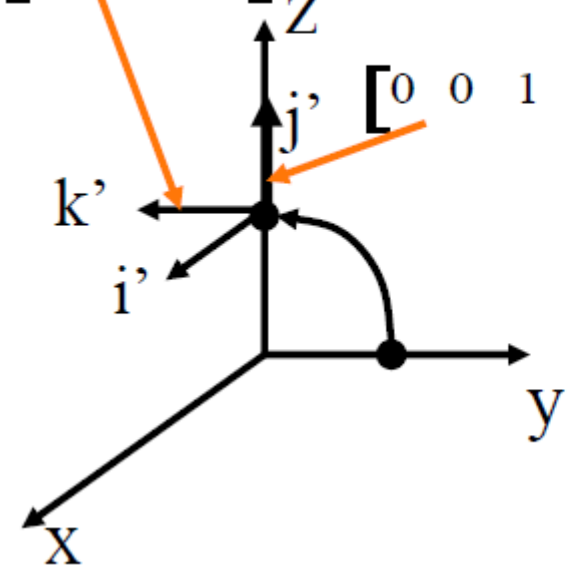
Example: Operating on a Frame

- Each column of this result is the transformation of the corresponding column in the original identity matrix

$$[0\ -1\ 0\ 0]^T \qquad I' = Rotx(\pi/2)Trans(0, v, 0)I$$



$$[0\ 0\ 1\ 0]^T$$

$$I' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & v \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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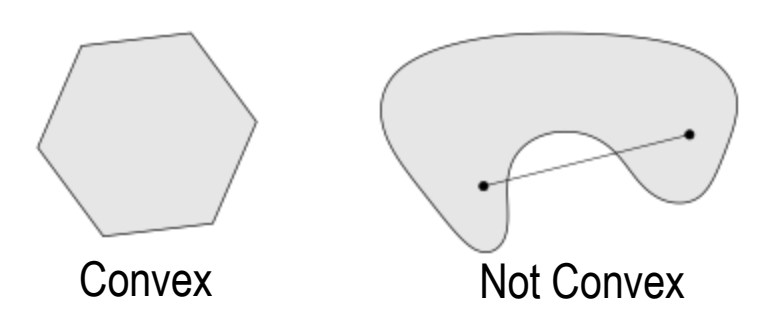
Optimization

Convex Sets

- Convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

- Examples:



Convex Functions

The domain of the function

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$

- I.e. the line segment between (x, f(x)) and (y, f(y)) lies above the graph of f

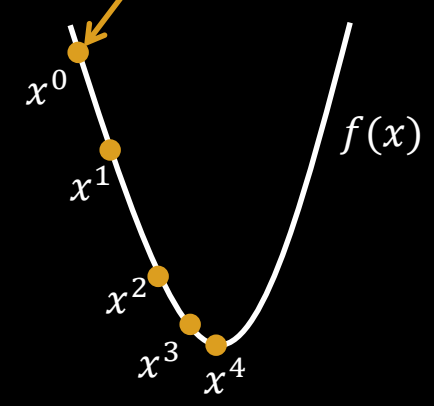
General descent algorithm

Many ways to do these

given a starting point $x \in \mathbf{dom}\, f$.
repeat
1. Determine a descent direction Δx.
2. *Line search.* Choose a step size $t > 0$.
3. *Update.* $x := x + t\Delta x$.

until stopping criterion is satisfied.

Input to the algorithm



Numerical differentiation

1. Pick a small h
2. Use one of two common Finite Difference methods:
   a) Newton's Difference Quotient

$$Df(x) \approx \frac{f(x+h) - f(x)}{h}$$

   b) Symmetric Difference Quotient

$$Df(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

- There are other numerical methods which can give better estimates but use more function evaluations

$$\underset{x}{\text{minimize}}\ f(x)$$

- assume f is convex
- assume x is unconstrained

Is f <u>twice</u> continuously differentiable and is second derivative fast to compute?

- yes → Newton's method
- no → Is f <u>once</u> continuously differentiable?
  - yes → Can the gradient be computed quickly?
    - yes → Gradient Descent
    - no → Stochastic Gradient Descent (for $f(x) = f_1(x) + f_2(x) + \cdots$)
  - no → Can a <u>subgradient</u> be computed quickly?
    - yes → Subgradient Method
    - no → Gradient Descent with Numerical Differentiation ☹ (we will cover more methods for this later)

*Many more methods not covered!

The standard form general optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

$$p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}$$

- $p^\star = \infty$ if problem is infeasible (no x satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

Duality

The Primal Problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

The Dual Problem

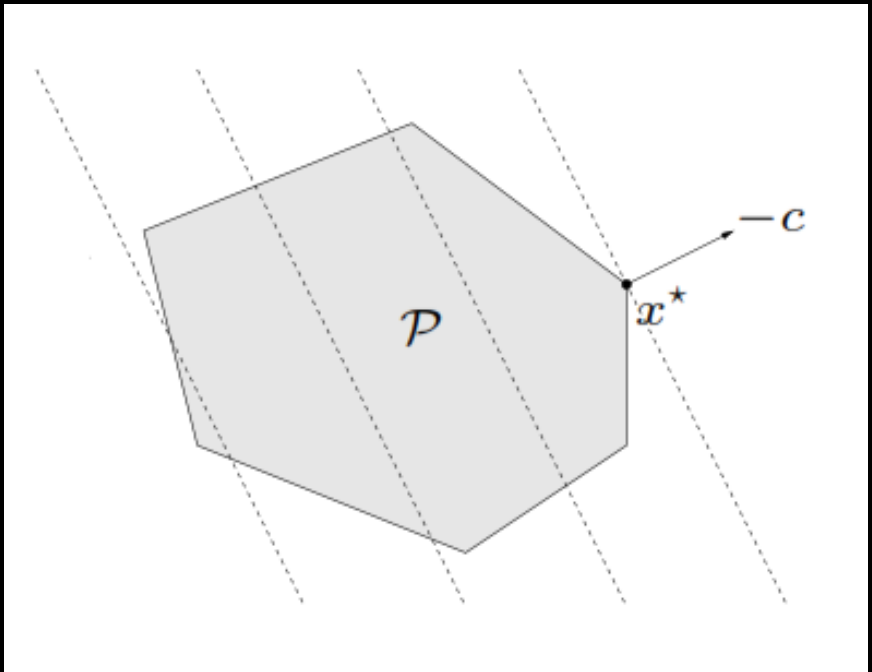
$$\begin{array}{ll} \underline{\text{maximize}} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- Solution to dual problem is a lower-bound on solution to primal problem p^* (more on this soon)
- *The Dual problem is always convex*, regardless of convexity of primal
  - We can use local methods to solve it!
- λ, ν are dual feasible if $\lambda \geq 0, (\lambda, \nu) \in \mathbf{dom}\ g$

Linear Programming

- The feasible set is a polyhedron

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

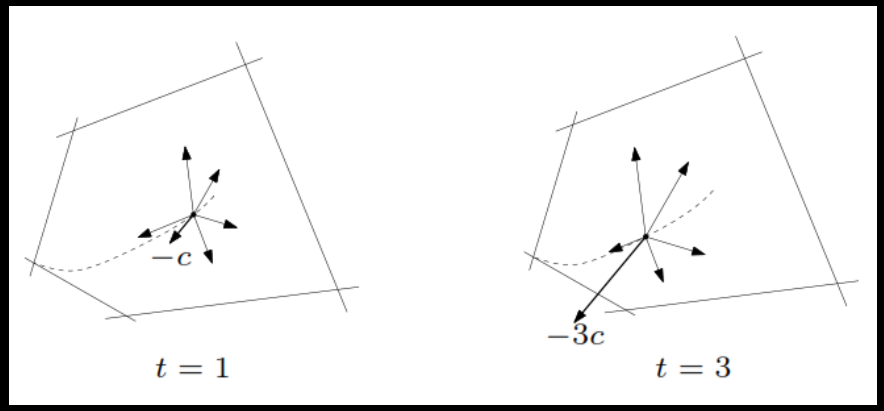


Barrier Method

- Idea: Trace out the central path by increasing t until you get to optimum
- Example for LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, 6 \end{array}$$

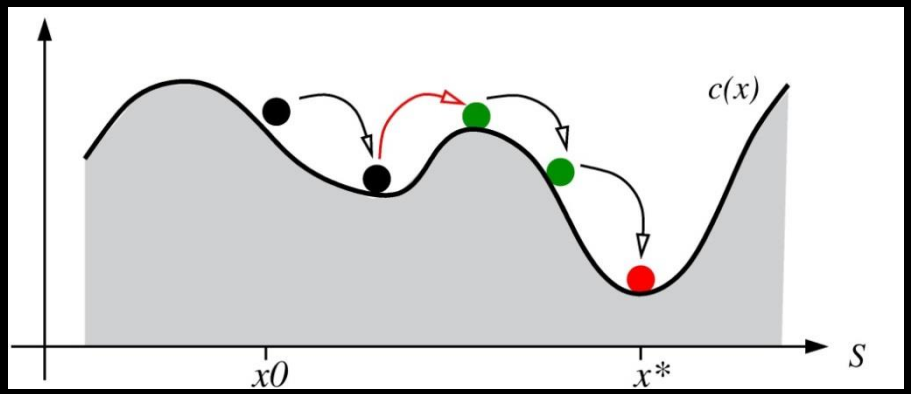
$$x^*(t) = \min_{feasible\ x} tc^T x - \sum_{i=1}^{m} \log(-a_i^T x + b_i)$$



Simulated annealing

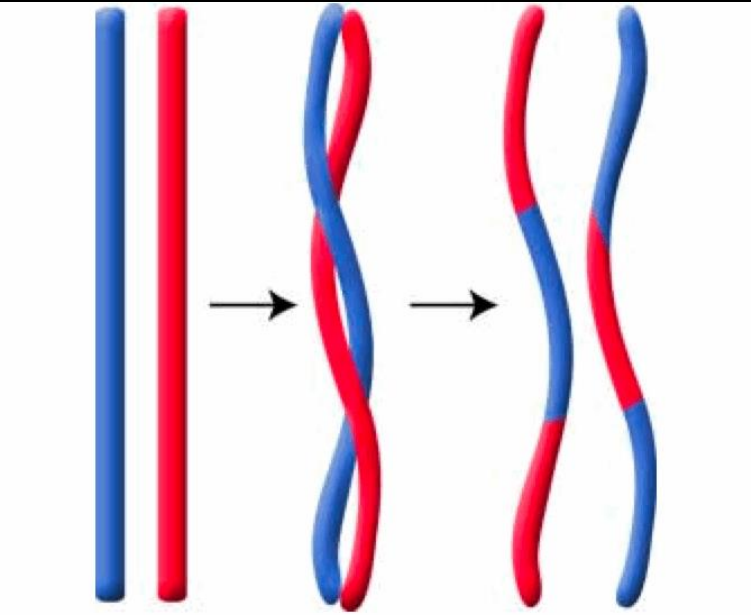
- Hill climbing problems
  - Gets stuck on plateaus
  - Returns sub-optimal solutions (local minima)

- Simulated annealing main idea: explicitly inject variability into the search process



Genetic algorithms

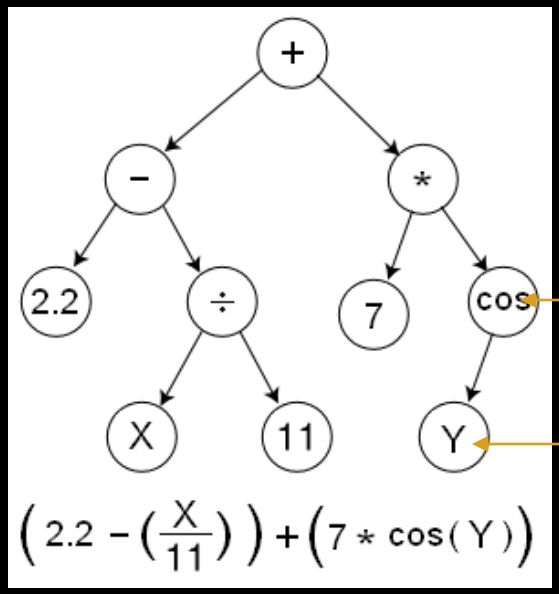
- Inspired (loosely) by the process of evolution in nature
- Operators
  - Crossover: New states generated from two *parent* states.
  - Mutation: Randomly change a component of the state



http://www.genomenewsnetwork.org/resources/whats_a_genome/Chp3_2.shtml

Genetic Programming

- Evolve *functions* instead of vectors of numbers
- Individuals represented as trees:



Non-leaf nodes are operators

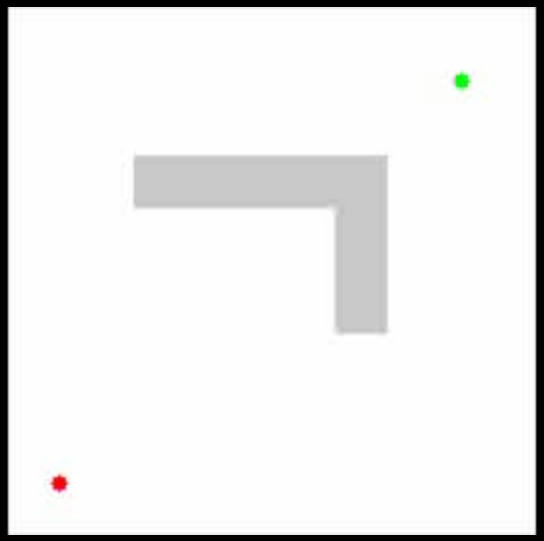
Leaf nodes are operands
(input variables or constants)

- Useful for regression problems

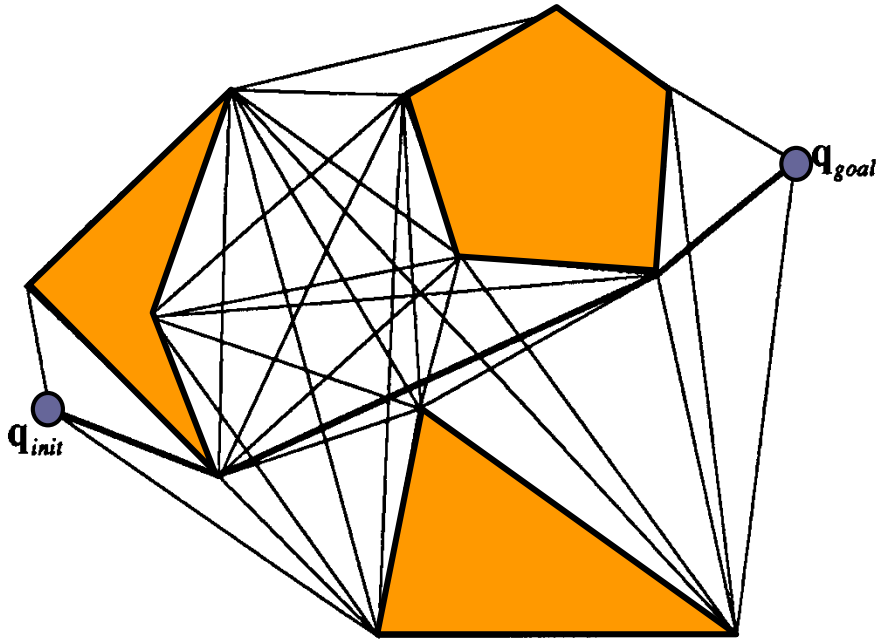
Motion Planning

A* Search

- Open list is a *priority queue*, nodes are sorted according to f(n)
- g(n) is sum of edge costs from root node to n

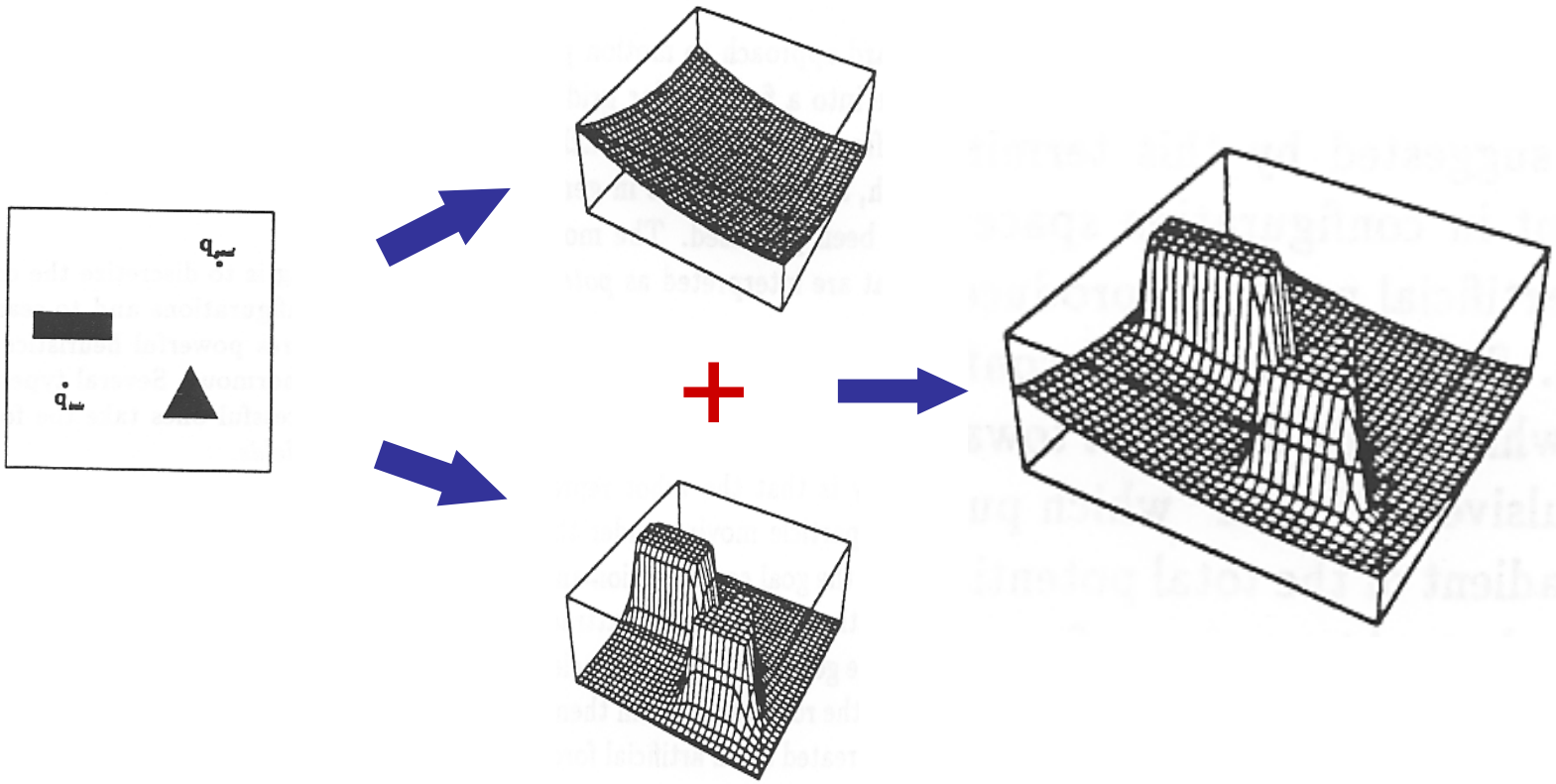


Visibility graph

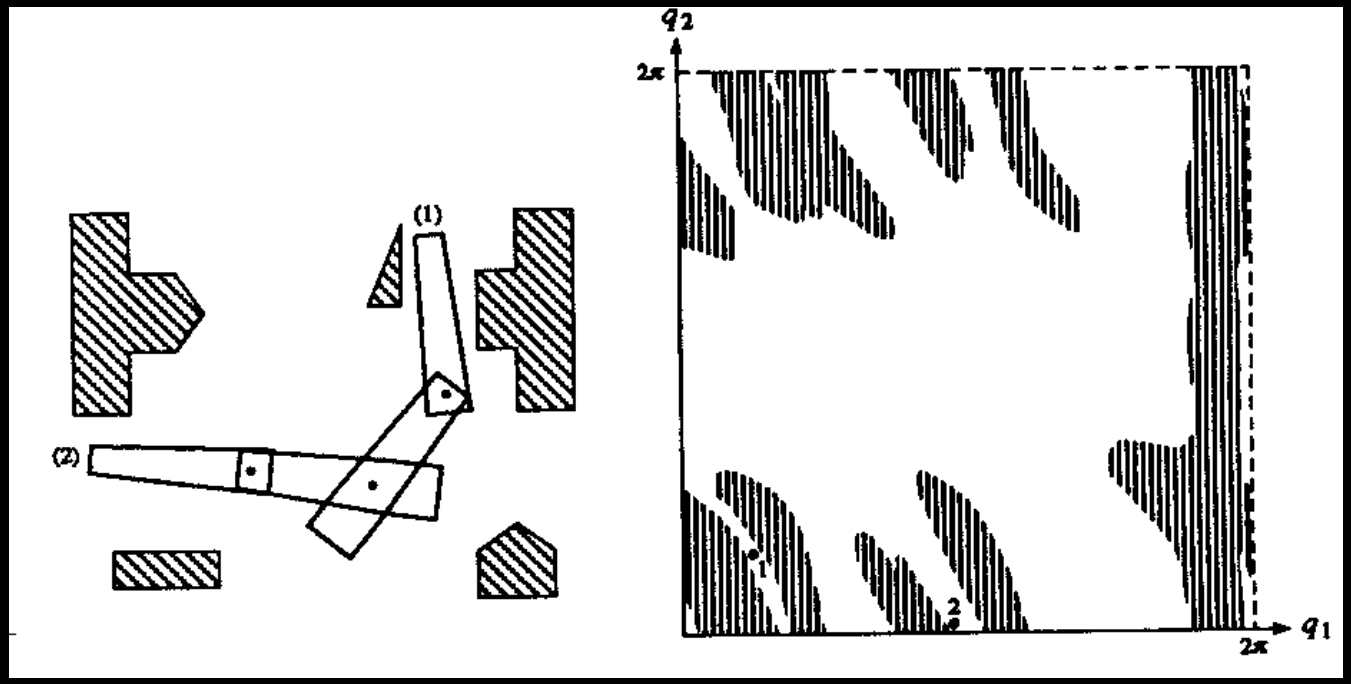


- A **visibility graph** is a graph such that
  - Nodes: q_{init}, q_{goal}, or an obstacle vertex.
  - Edges: An edge exists between nodes u and v if the line segment between u and v is an obstacle edge or it does not intersect the obstacles.

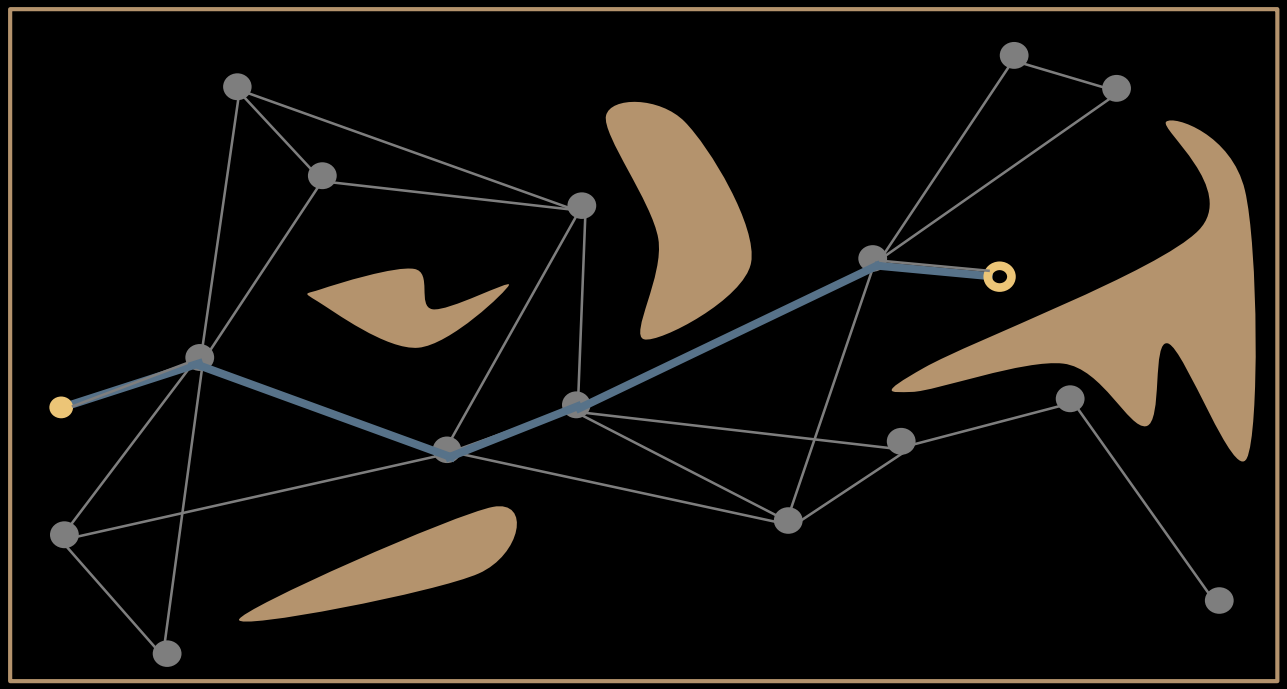
Potential Fields



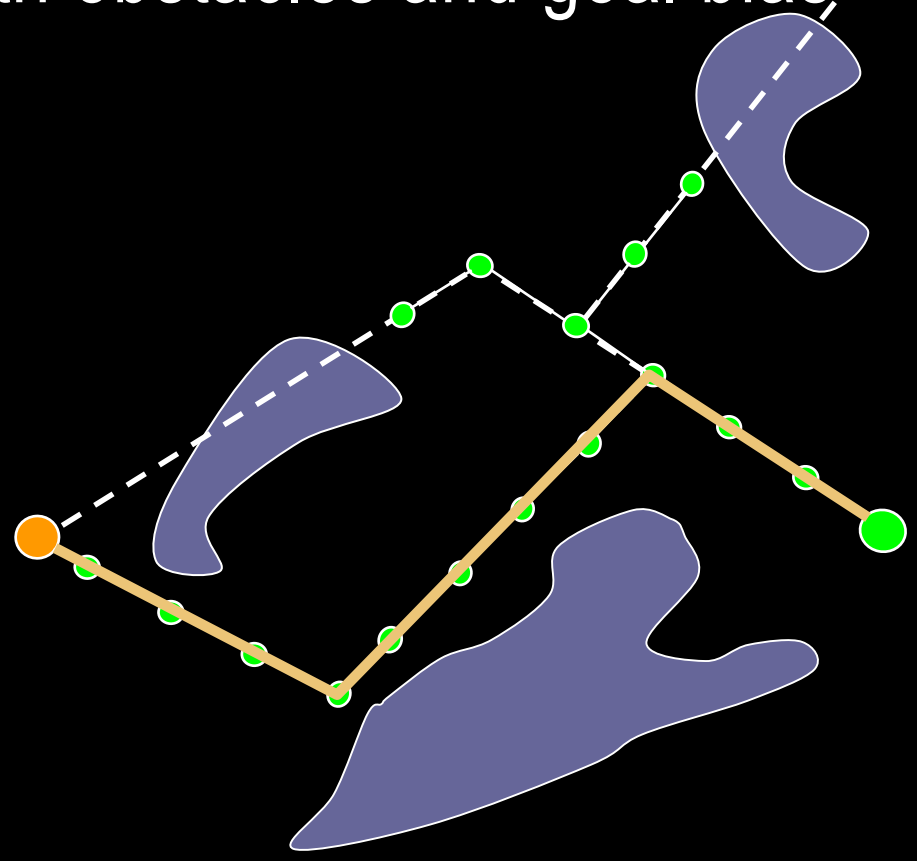
Configuration Space for Articulated Robots



PRM Example



RRT with obstacles and goal bias

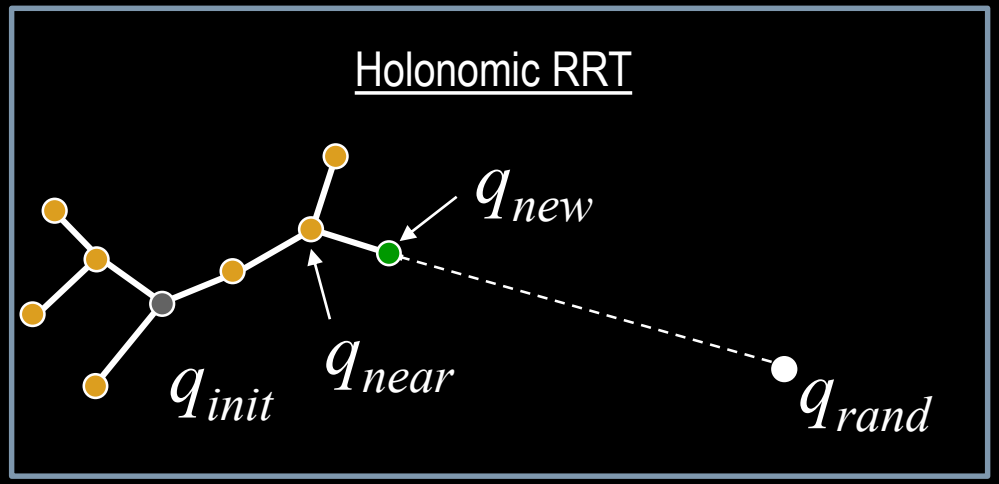


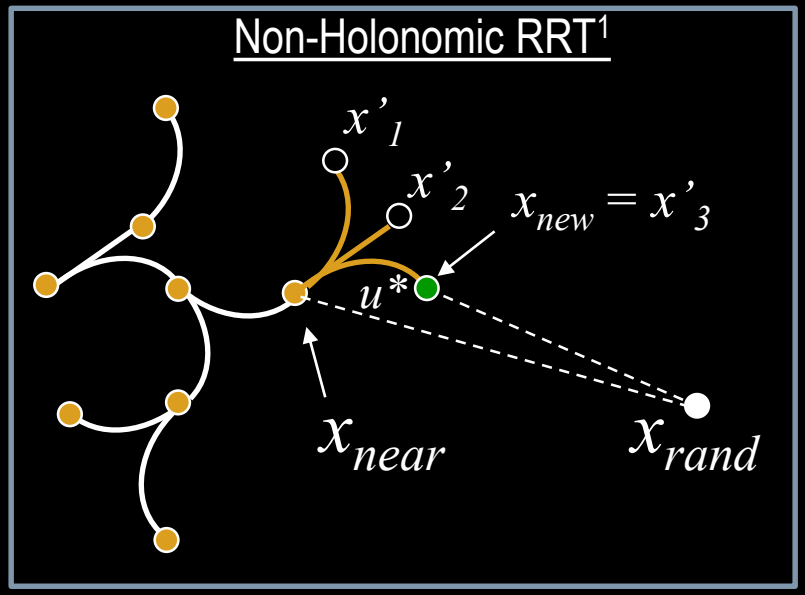
RRTs for Non-Holonomic Systems

- Apply motion primitives (i.e. simple actions) at q_{near}

$$x' = f(x,u) \text{ --- use action } u \text{ from } x \text{ to arrive at } x'$$

$$\text{chose } u_* = \arg\min_{u_i} d(f(x,u_i), x_{rand})$$



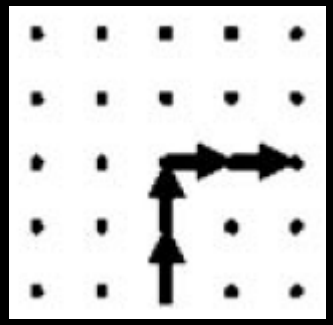


- You probably won't reach x_{rand} by doing this
  - Key point: No problem, you're still exploring!

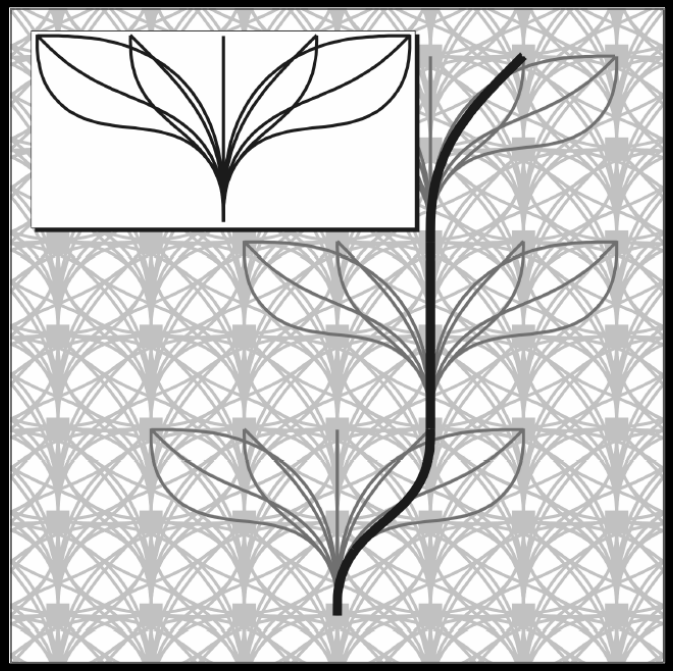
[1]often called *Kinodynamic RRT*

State Lattice for Non-holonomic planning

- Pre-compute state lattice
- Two methods to get lattice:
  - Forward: For certain systems, can sequence primitives to make lattice
  - Inverse: Discretize space, use BVP solvers to find trajectories between states
- Impose continuity constraints at graph vertices
- Search state lattice like any graph (i.e. A*)



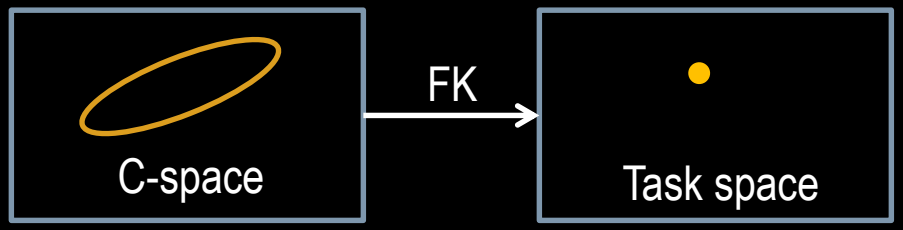
Traditional lattice yields discontinuous motion



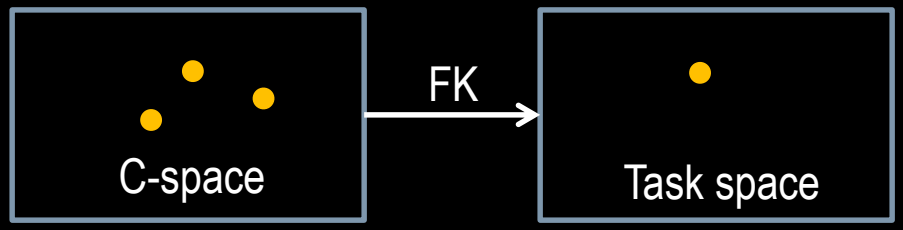
Pivtoraiko et al. 2009

C-Space and Task Space

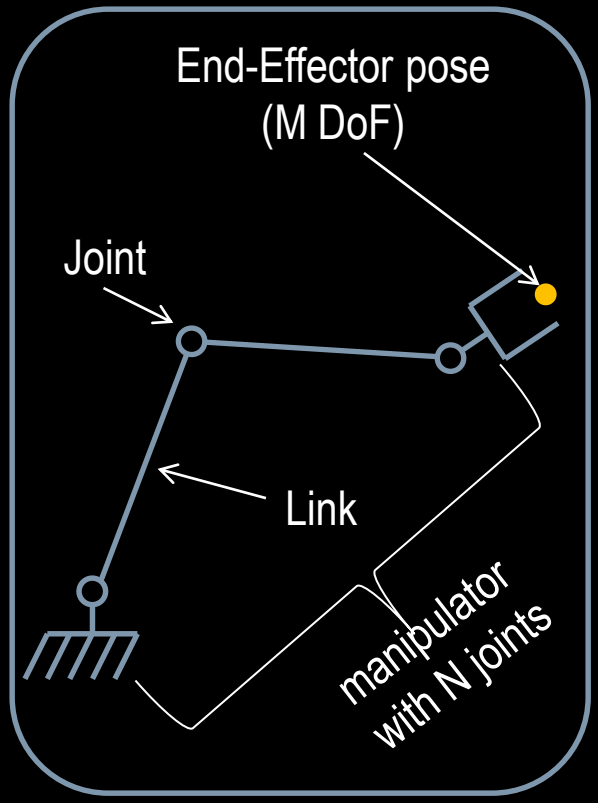
- If N > M, FK maps *a continuum* of configurations to *one* end-effector pose:



- If N = M, FK maps a *finite number* of configurations to one end-effector pose:



- If N < M, you're in trouble (may not be able to reach a target pose)



Iterative Jacobian Pseudo-Inverse Inverse Kinematics

While true

$x_{current} = FK(q_{current})$

$\dot{x} = (x_{target} - x_{current})$

error = $||\dot{x}||$

If error < threshold

return $q_{current}$

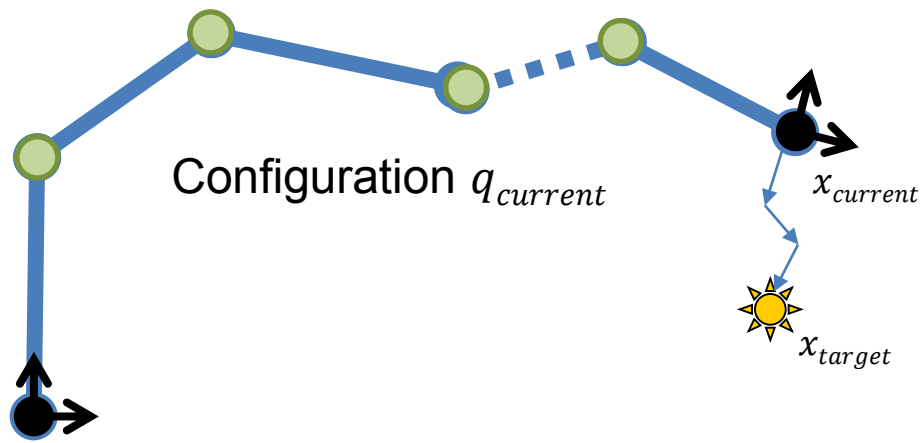
$\dot{q} = J(q)^{+}\,\dot{x}$

If($||\dot{q}|| > \alpha$)

$\dot{q} = \alpha(\dot{q} / ||\dot{q}||)$

$q_{current} = q_{current} + \dot{q}$

end



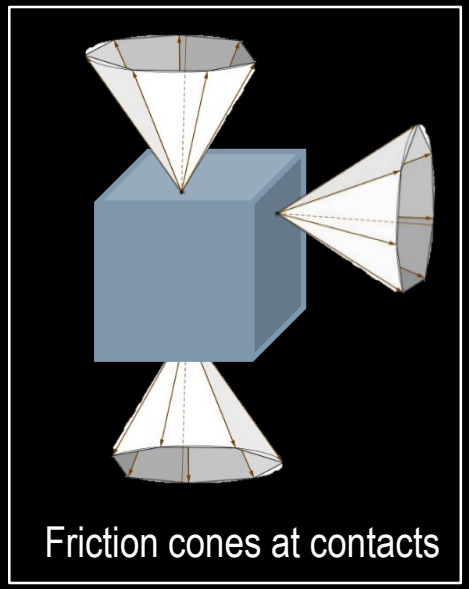
- This is a local method, it will get stuck in local minima (i.e. joint limits)!!!
- α is the step size
- Numerical error handling not shown
- A correction matrix has to be applied to the angular velocity components to map them into the target frame (not shown)

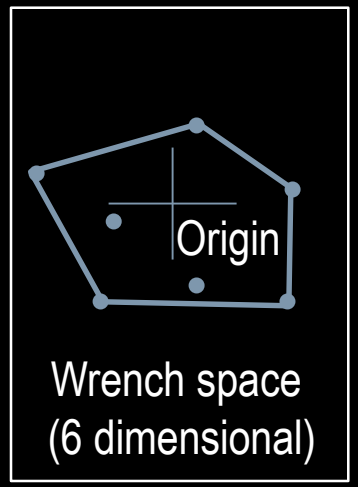
Testing for Force Closure Grasps

- Many algorithms exist to test for force closure, here is one:

Input: Contact locations
Output: Is the grasp in Force-Closure? (Yes or No)

1. Approximate the friction cone at each contact with a set of wrenches
2. Combine wrenches from all cones into a set of points *S* in wrench space
3. Compute the *convex hull* of *S*
4. If the origin is inside the convex hull, return YES. If not, return NO.



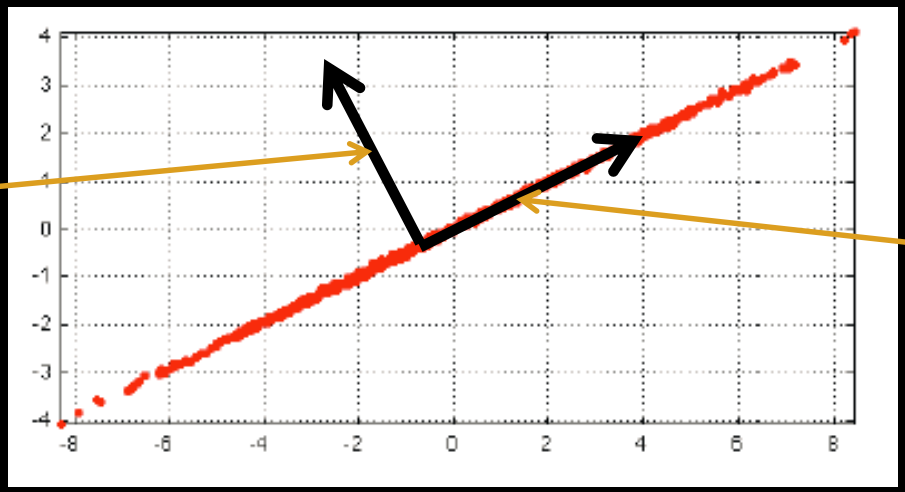


Point Cloud Processing

The main idea of PCA

- We care about the variance of the data
- High-variance implies high importance
  - Why does this make sense?

Data does not vary much in this direction



Data varies a lot in this direction

- But how do we compute the high-variance directions?

SVD

- SVD decomposes any matrix M into the following form:

M	=	U	*	Σ	*	V^T
m x n	=	m x m		m x n		n x n

Columns of U are eigenvectors of MM^T

$$\Sigma \equiv \begin{bmatrix} \sigma_{\tilde{1}} & & & & & \\ & \ddots & & & 0 & \\ & & \sigma_{\tilde{r}} & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

Columns of V are eigenvectors of M^TM

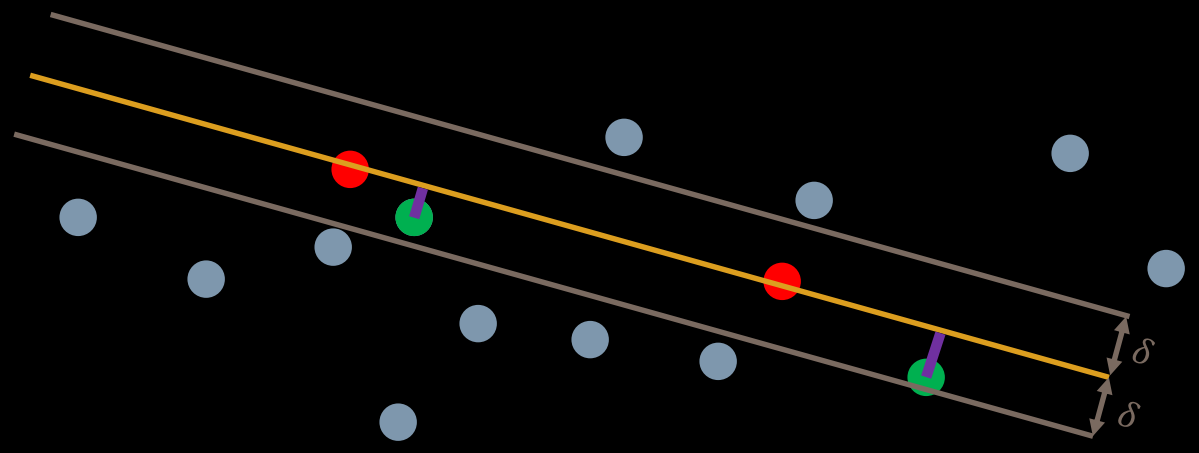
σs are $\sqrt{\text{eigenvalues}}$ of M^TM and MM^T in decreasing order of magnitude

PCA algorithm to remove rotation *and reduce dimension*

1. Given a dataset X
2. Compute the mean of X, μ
3. $X = X - \mu$ (subtract μ from every point in X)
4. Compute the covariance of X, $Q = \frac{XX^T}{(n-1)}$
5. SVD$(Q) = U\Sigma V^T$
   - Each column of V is a principle component
6. Compute the variance of each principle component: $s = \mathrm{diag}(\Sigma)^2$
7. Remove all columns of V whose corresponding entry in s is less than some small threshold, call this new matrix V_s
8. Compute $X_{new} = V_s^T X$

(Note: this may cause reflection)

RANSAC Line fitting example



Pick R (hypothetical inliers)
Fit Model to R
Find C (consensus set)
Compute Error of Model on $C \cup R$

Point set registration: Known Correspondences

- In summary, to solve

$$P = \{p_1, p_2, \dots p_n\}$$
$$Q = \{q_1, q_2, \dots, q_n\}$$

$$\underset{R \in SO(3), t \in \boldsymbol{R}^3}{\operatorname{argmin}} \sum_{i=1}^{n} \|(Rp_i + t) - q_i\|^2$$

1. Compute centroids and centered vectors:

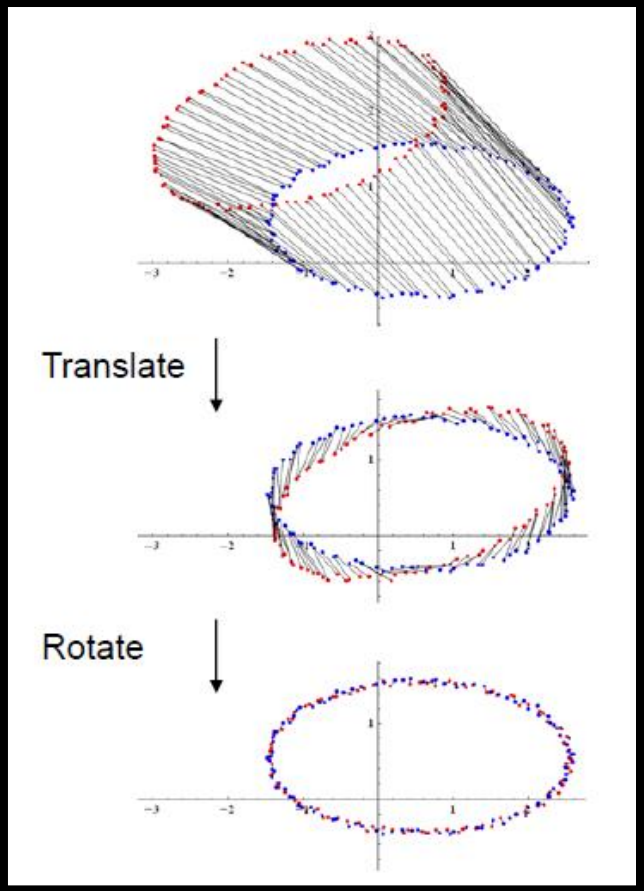
$$\bar{p} = \frac{\sum_{i=1}^{n} p_i}{n} \qquad \bar{q} = \frac{\sum_{i=1}^{n} q_i}{n} \qquad \begin{aligned} x_i &= p_i - \bar{p} \\ y_i &= q_i - \bar{q} \end{aligned}$$

2. Compute SVD of covariance matrix of centered vectors:

$$S = XY^T \qquad \mathrm{SVD}(S) = U\Sigma V^T$$

3. Compute R and t:

$$R = V \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(VU^T) \end{bmatrix} U^T \qquad t = \bar{q} - R\bar{p}$$



Iterative Closest Point (ICP)

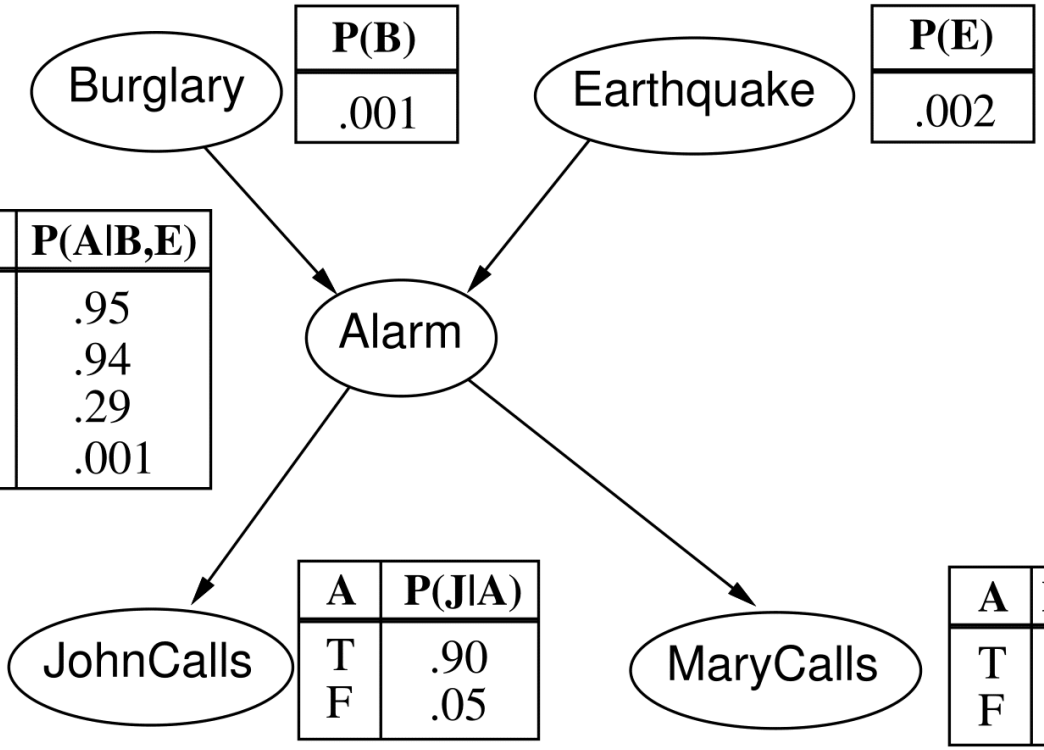
- We assumed we knew the correspondences between points
  - In practice this is rarely true
- ICP iteratively computes correspondences and registers point sets
- There are many variants of ICP, here is a simple one
- Won't always succeed!
  - Need to add a way to terminate based on
    - time
    - number of iterations
    - lack of progress
- Need to tune ϵ

```
Input: P and Q (not necessarily same size)
Output: P aligned to Q
```

```
Set P to some initial pose
While not Done
   //Compute Correspondences
   C = ∅
   For each p_i ∈ P
      find the closest q_i
      C = C ∪ {p_i, q_i}

   //Compute Transform
   //(see previous slide)
   R, t ← GetTransform(C_p, C_q)

   If Σ_{i=1}^{n} ‖(R C_{p_i} + t) − C_{q_i}‖² < ϵ
      return P

   // Update all P
   For each p_i ∈ P
      p_i = R p_i + t
```

Probabilistic Reasoning

Example contd.



P(B)
.001

P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

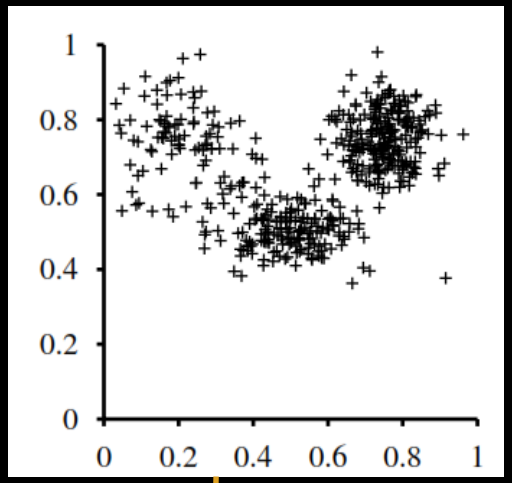
A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

Learning a Gaussian Mixture Model (GMM)

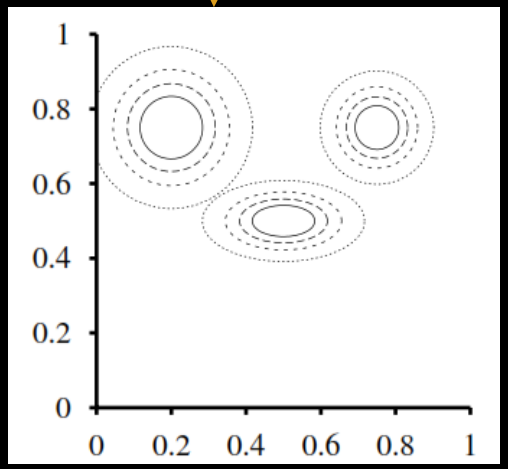
- A GMM is simply a set of Gaussians
  - The set jointly defines a probability distribution over the space

- Lets say we want to model a set of data points with a GMM
- Problem: we don't know which data point came from which Gaussian
  - i.e. we don't know how to partition the data



data

Expectation Maximization



GMM

Hidden Markov Models (HMMs)

- An HMM is a temporal probabilistic model in which the state is described by a **single discreet random variable**.

$\mathbf{X}_t$ is a single, discrete variable (usually $\mathbf{E}_t$ is too)
Domain of X_t is $\{1, \ldots, S\}$

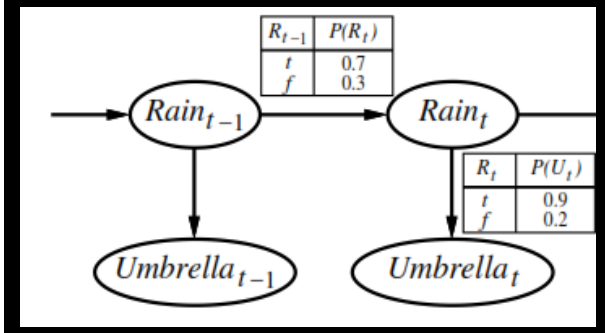
Transition matrix $\mathbf{T}_{ij} = P(X_t = j | X_{t-1} = i)$, e.g., $\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$

Sensor matrix $\mathbf{O}_t$ for each time step, diagonal elements $P(e_t | X_t = i)$
e.g., with $U_1 = true$, $\mathbf{O}_1 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix}$

Forward and backward messages as column vectors:

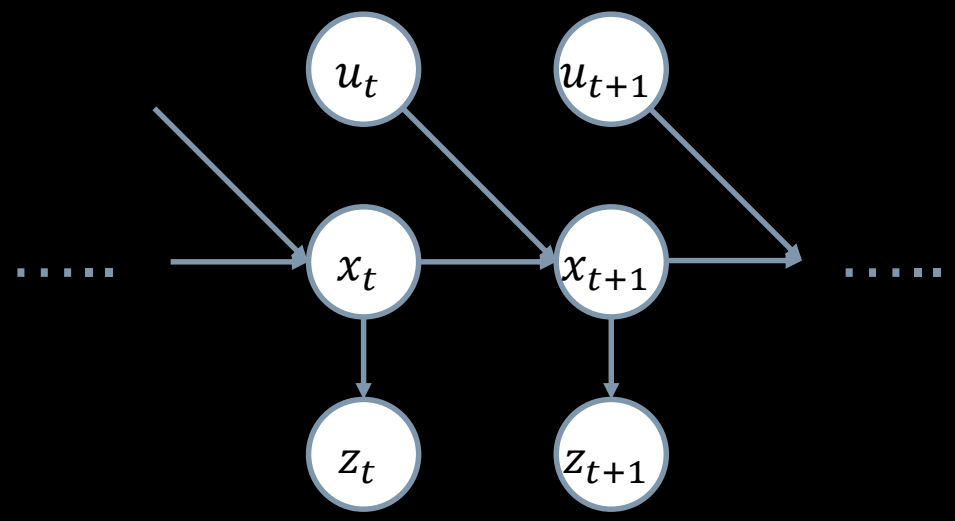
$$\mathbf{f}_{1:t+1} = \alpha \mathbf{O}_{t+1} \mathbf{T}^\top \mathbf{f}_{1:t}$$
$$\mathbf{b}_{k+1:t} = \mathbf{T} \mathbf{O}_{k+1} \mathbf{b}_{k+2:t}$$

Forward-backward algorithm needs time $O(S^2 t)$ and space $O(St)$



Kalman Filters

- The real world is not discrete! Need to consider **continuous** variables



- Kalman filters are used to track state of robots, chemical plants, planets, etc.
- Key Idea: Arbitrary continuous models are intractable, so represent everything with *Gaussians*
  - Gaussian prior, linear Gaussian transition model and sensor model

Prediction-Correction Cycle for Kalman Filters

Prediction

$$\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$$
$$\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

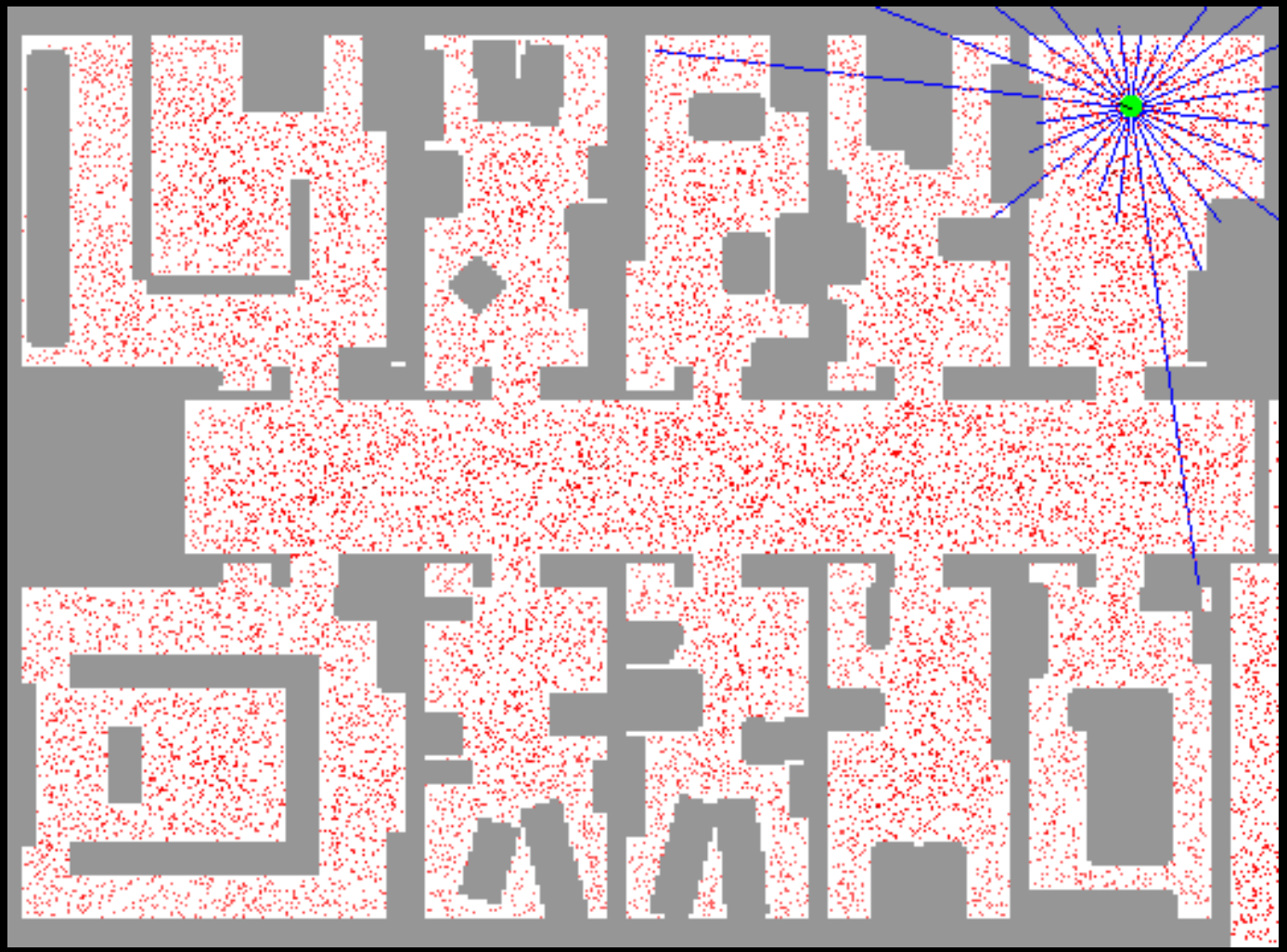
$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$$
$$\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$$

Correction

KF Variants:

- Extended Kalman Filter
- Unscented Kalman Filter

Particle filter



Fox et al.: Mobile robot localization with 24 sonar sensors

http://www.cs.washington.edu/ai/Mobile_Robotics/mcl/

The Markov Zoo

- Markov process + partial observability = HMM
- Markov process + actions = MDP
- Markov process + partial observability + actions = HMM + actions = MDP + partial observability = **POMDP**

	full observability	*partial observability*
no actions	**Markov process**	**HMM**
actions	**MDP**	**POMDP**

Value Iteration vs. Policy Iteration for MDPs

- Value iteration update:

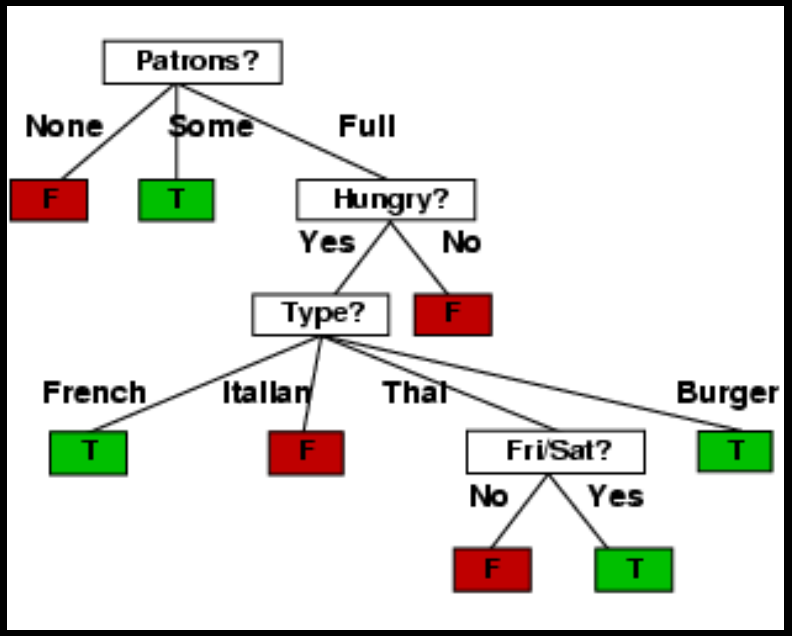
$$U_{i+1}(s) = R(s) + \gamma * \max_{a} \sum_{s'} P(s' \mid s,a)\, U_i(s')$$

- Policy iteration update:

$$U_{i+1}(s) = R(s) + \gamma * \sum_{s'} P(s' \mid s, \pi_i(s))\, U_i(s')$$

Learning

Supervised Learning: Decision Trees

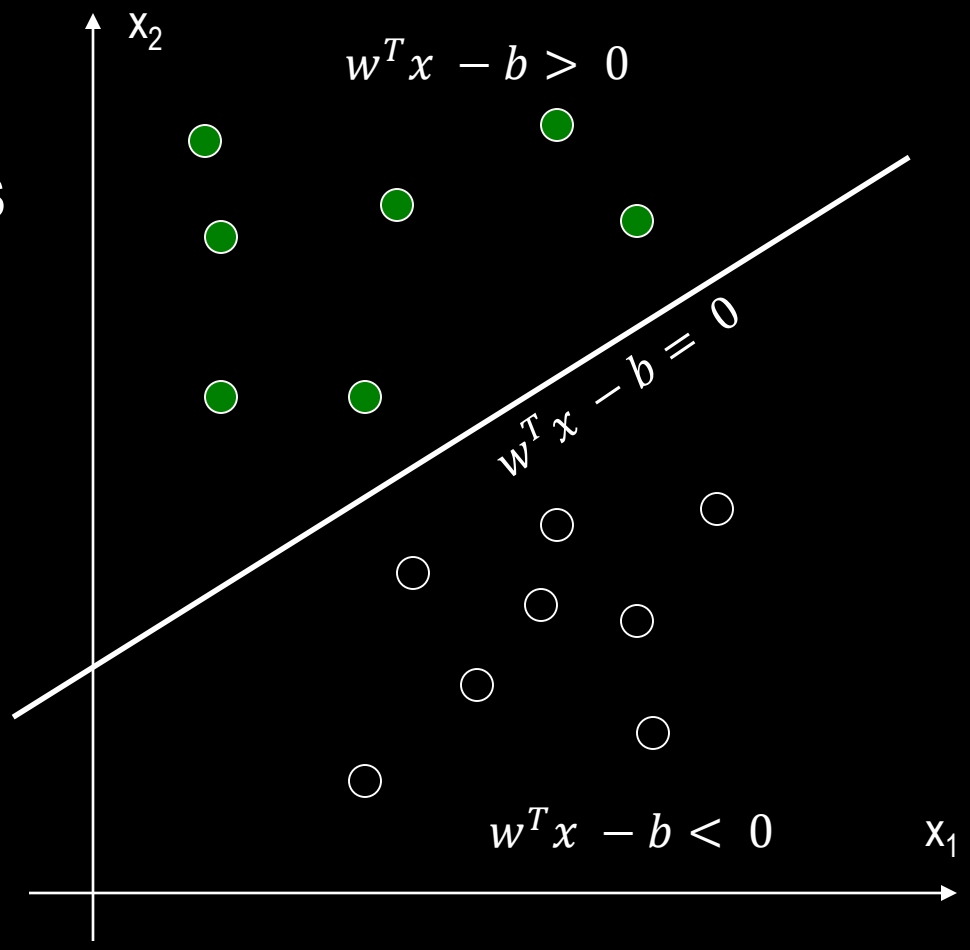


- Split on variables that give highest information gain at each level
  - Based on computing entropy of data for each branch

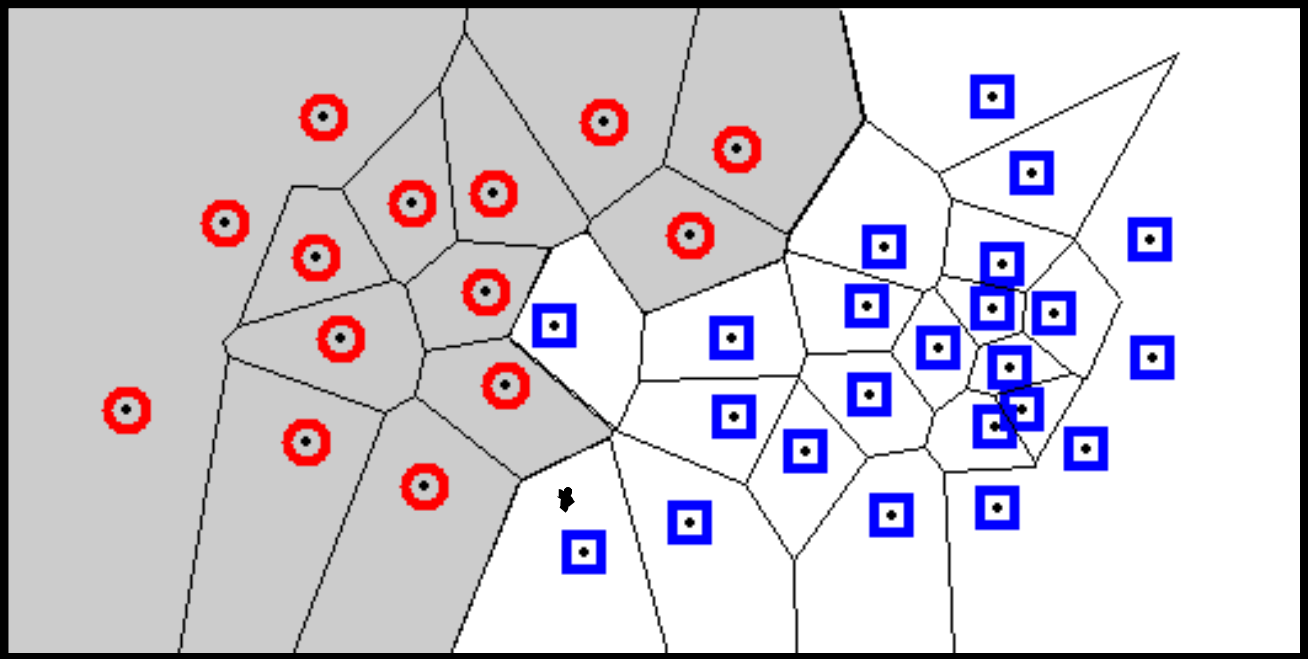
Support Vector Machines (SVM)

- A discriminant function f(x) is a function that separates two sets of data according to their labels
- SVM learns a linear function:

$$f(x) = w^T x - b$$



K-Nearest Neighbor



Decision Regions for 1-NN Classification

ANN: Backpropagation Algorithm

1. Initialize the weights to some random values (or 0)
2. For each sample (x_j, y_j) in the training set
   a. Calculate the current output of the node, h_{x_j}
   b. For each output node k, update the weights

$$\Delta_k = (h_{x_k})(1 - h_{x_k})(y_k - h_{x_k})$$

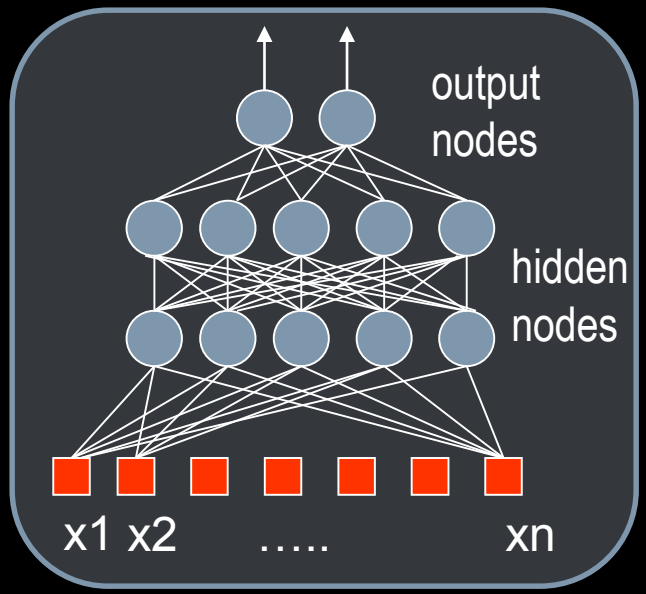
   c. For each hidden node j, update the weights

$$\Delta_j = (h_{x_j})(1 - h_{x_j}) \sum_k w_{j,k} \Delta_k$$

3. For all network weights do

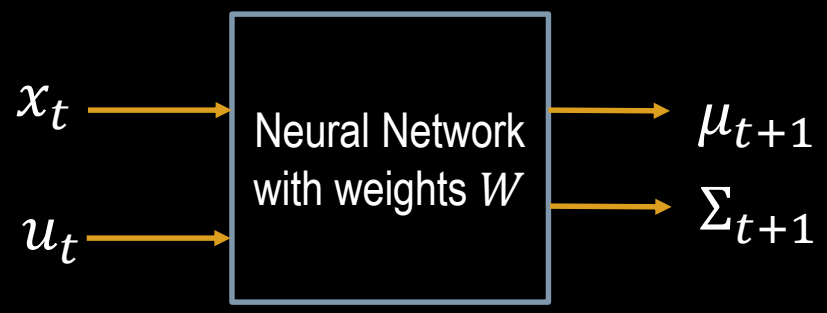
$$w_{i,j} = w_{i,j} + \alpha \Delta_j x_i$$

4. Repeat until weights converge or desired accuracy is achieved

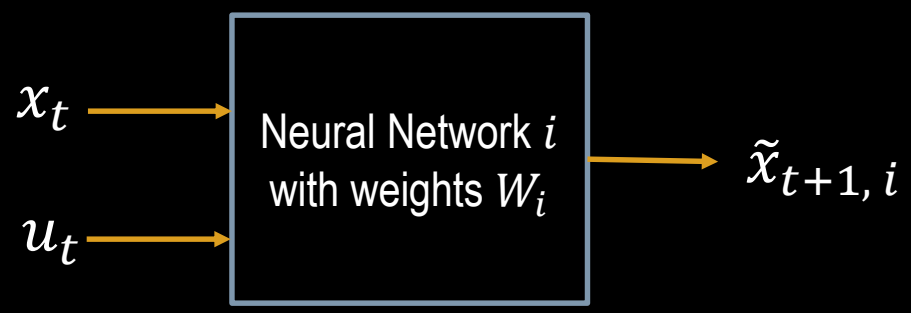


Uncertainty learning dynamics

- Predictive distributions for estimating *aleatoric* uncertainty



- Ensembles for estimating *epistemic* uncertainty



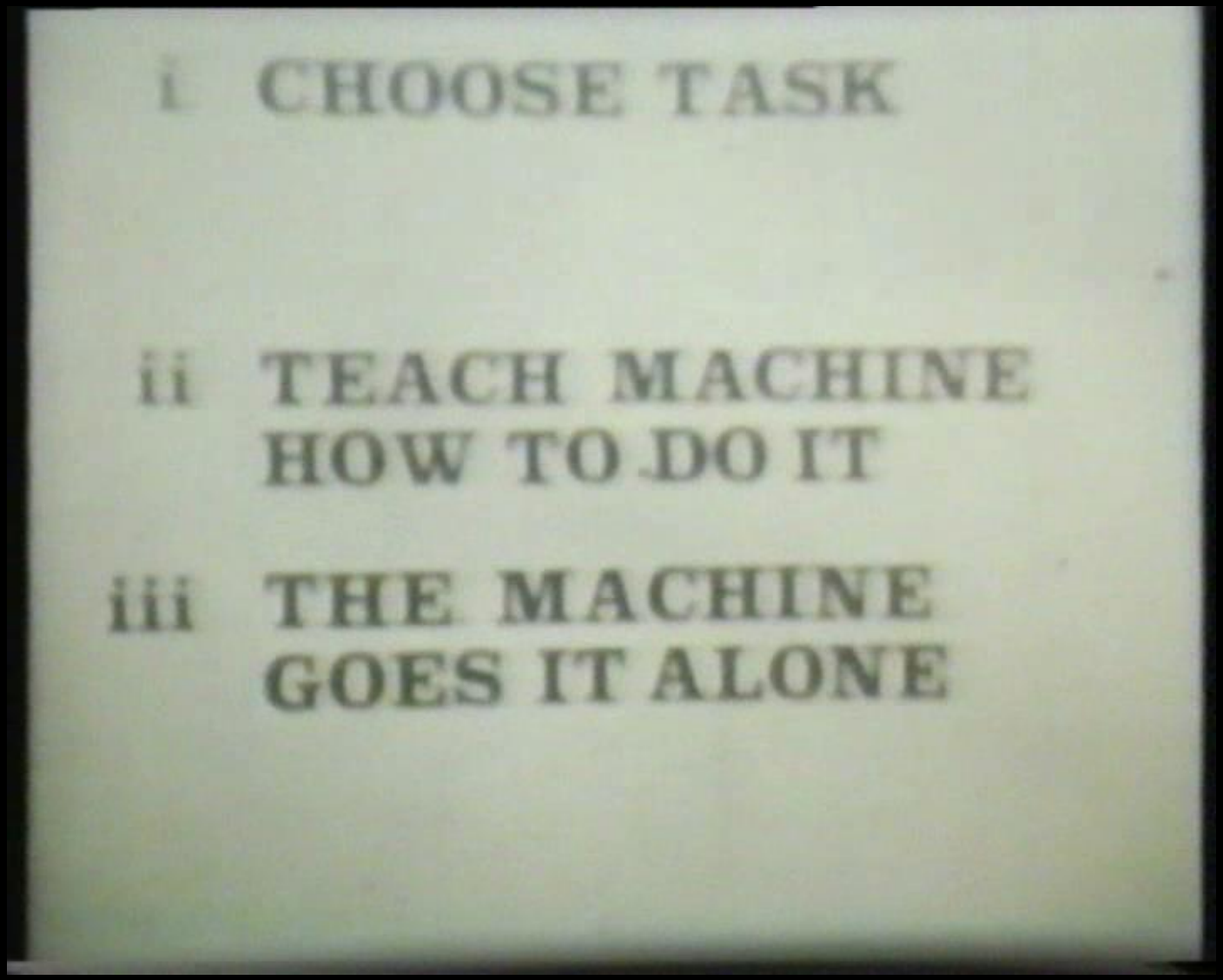
For $i = 1 \ldots K$

BREAK

Let’s play “How did they do that?”

Shakey, SRI, 1966-1972

Freddy, University of Edinburgh, 1973



Sensorless Parts Orienting, CMU, 1986

Navlab, CMU, 1986-2002

DARPA Urban Challenge, 2007

Team CMU

PR2 and ROS, Willow Garage, 2008-2013

KinectFusion, 2011

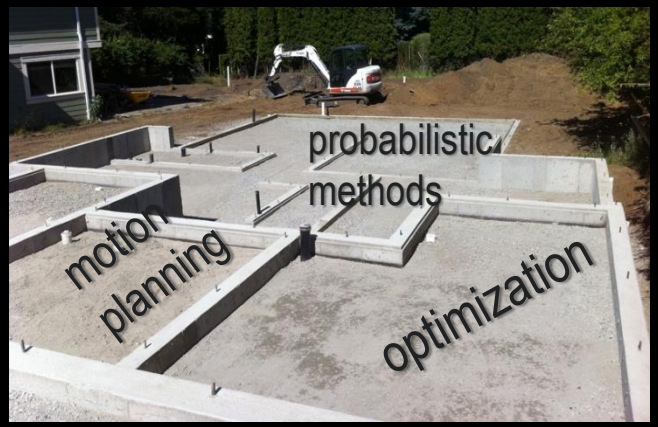
Kiva Robots, Amazon, 2014

DARPA Robotics Challenge, 2015

Team WPI-CMU

What we learned in this course

- Built a **foundation** for more advanced robotics courses/concepts



- The next step:

What to take next?

- Optimization (IOE 611)
- Artificial Intelligence (EECS 492/592)
- Motion Planning (EECS 598)
- Perception (vision in EECS 442/504/542, EECS 498: Self Driving Cars)
- Estimation and Mapping (EECS 467)
- Machine Learning (EECS 445)
- Kinematics/Dynamics (EECS 398)

- Consider getting involved in research!

Thank you for taking this course!