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Nonparametric estimation of the trend in reflected fractional SDE

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ABSTRACT

This paper deals with the consistency, a rate of convergence and the asymptotic distribution of a nonparametric estimator of the trend in the Skorokhod reflection problem defined by a fractional SDE and a Moreau sweeping process.

1. Introduction

Consider $T > 0$ and the Skorokhod reflection problem

$$\begin{cases} X_\varepsilon(t) & = & \displaystyle\int_0^t b(X_\varepsilon(s))ds + \varepsilon B(t) + Y_\varepsilon(t) \\ -\dot{Y}_\varepsilon(t) & \in & \mathcal{N}_{C(t)}(X_\varepsilon(t))|DY_\varepsilon|\text{-a.e. with } Y_\varepsilon(0) = x_0 \end{cases} ; t \in [0, T], \qquad (1)$$

where $b : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, $\varepsilon > 0$, B is a fractional Brownian motion of Hurst index $H \in]1/2, 1[$, $\dot{Y}_\varepsilon$ is the Radon–Nikodym derivative of the differential measure DY_ε of Y_ε with respect to its variation measure $|DY_\varepsilon|$, the multifunction $C : [0, T] \rightrightarrows \mathbb{R}$ is Lipschitz continuous for the Hausdorff distance, $x_0 \in C(0)$ and $\mathcal{N}_{C(t)}(X_\varepsilon(t))$ is the normal cone of $C(t)$ at point $X_\varepsilon(t)$. The definition of the normal cone is stated later.

A solution to Problem (1), if it exists, is a couple $(X_\varepsilon, Y_\varepsilon)$ of continuous functions from $[0, T]$ into $\mathbb{R}$ such that $X_\varepsilon(t) \in C(t)$ for every $t \in [0, T]$. Roughly speaking, X_ε coincides with the solution to $dX_\varepsilon^*(t) = b(X_\varepsilon^*(t))dt + \varepsilon dB(t)$, except when X_ε hits the frontier of C. Each time this situation occurs, X_ε is pushed inside of C with a minimal force by Y_ε. The differential inclusion defining the process Y_ε in Problem (1) is equivalent to a (Moreau) sweeping process. Several authors studied Problem (1) when $H = 1/2$. For instance, the reader can refer to Bernicot and Venel (2011), Slominski and Wojciechowski (2013) or Castaing et al. (2016a). When $H \neq 1/2$, the reader can refer to Falkowski and Slominski (2015) or Castaing et al. Castaing et al. (2017). In this last paper, the authors proved the existence, uniqueness and the convergence of an approximation scheme of the solution to Problem (1) under a *nonempty interior* condition on C (see Assumption 2.2). In fact, in all these papers, the authors studied the Skorokhod reflection problem defined by a SDE and a sweeping process for a multiplicative and/or multidimensional noise.

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Let $K : \mathbb{R} \to \mathbb{R}_+$ be a kernel. The paper deals with the consistency, a rate of convergence and the asymptotic distribution of the nonparametric estimator

$$\widehat{\tau}_{\varepsilon}(t) := \frac{1}{h_{\varepsilon}} \int_0^t \int_0^T K\left(\frac{s-u}{h_{\varepsilon}}\right) dX_{\varepsilon}(s)du;\ t \in [0,T]$$

of the trend

$$\tau(.) := \int_0^{.} b(x(u))du + y(.) - x_0$$

of Problem (1), where

$$\begin{cases} x(t) & = & \displaystyle\int_0^t b(x(s))ds + y(t) \\ -\dot{y}(t) & \in & \mathcal{N}_{C(t)}(x(t))|Dy|\text{-a.e. with } y(0) = x_0 \end{cases} \tag{2}$$

and $h_{\varepsilon} > 0$ goes to zero when $\varepsilon \to 0$.

Along the last two decades, many authors studied statistical inference in stochastic differential equations driven by the fractional Brownian motion. Most references on the estimation of the trend component in fractional SDE deals with parametric estimators (see Kleptsyna and Le Breton (2001), Tudor and Viens (2007), Hu and Nualart (2010), Chronopoulou and Tindel (2013), Neuenkirch and Tindel (2014), Mishura and Ralchenko (2014), Hu et al. (0000), etc.). On the nonparametric estimation of the trend component in fractional SDE, there are only few references. Saussereau (2014) and Comte and Marie (2019) study the consistency of some Nadaraya–Watson's-type estimators of the drift function in a fractional SDE. In Mishra and Prakasa Rao (2011), Mishra and Prakasa Rao established the consistency and a rate of convergence of a nonparametric estimator of the whole trend of the solution to a fractional SDE. Our paper generalizes their results to the Skorokhod reflection Problem (1). On the nonparametric estimation in Itô's calculus framework, the reader can refer to Kutoyants (1994, 2004). Up to our knowledge, there is no reference on the nonparametric estimation of the trend in reflected fractional SDE.

Section 2 deals with some preliminaries on the Skorokhod reflection problem defined by a fractional SDE and a sweeping process. Section 3 deals with the consistency, a rate of convergence and the asymptotic distribution of the estimator $\widehat{\tau}_{\varepsilon}(t)$.

Notations and basic properties:

(1) For every $h > 0$, $K_h := 1/hK(\cdot/h)$.

(2) Consider a Hilbert space $(E, \langle ., . \rangle)$. For every closed convex subset K of E and every $x \in E$, $\mathcal{N}_K(x)$ is the normal cone of K at x:

$$\mathcal{N}_K(x) := \{y \in E : \forall z \in K,\ \langle y, z - x\rangle \leqslant 0\}.$$

In particular, for $E = \mathbb{R}$ and $K = [l, u]$ with $l, u \in \mathbb{R}$ such that $l \leqslant u$,

$$\mathcal{N}_K(x) = \begin{cases} \mathbb{R}_- & \text{if} \quad x = l \\ \mathbb{R}_+ & \text{if} \quad x = u \\ \{0\} & \text{if} \quad x \in]l, u[\end{cases}.$$

(3) For every $t \in]0, T]$, $\Delta_t := \{(u, v) \in [0, t]^2 : u < v\}$.

(4) For every function f from $[0, T]$ into $\mathbb{R}$ and $(s, t) \in \Delta_T$, $f(s, t) := f(t) - f(s)$.

(5) Consider $(s, t) \in \Delta_T$. The vector space of continuous functions from $[s, t]$ into $\mathbb{R}$ is denoted by $C^0([s, t], \mathbb{R})$ and equipped with the uniform norm $\|.\|_{\infty,s,t}$ defined by

$$\|f\|_{\infty,s,t} := \sup_{u\in[s,t]} |f(u)|;\ \forall f \in C^0([s, t], \mathbb{R}),$$

or the semi-norm $\|.\|_{0,s,t}$ defined by

$$\|f\|_{0,s,t} := \sup_{u,v\in[s,t]} |f(v) - f(u)|;\ \forall f \in C^0([s, t], \mathbb{R}).$$

Moreover, $\|.\|_{\infty,T} := \|.\|_{\infty,0,T}$ and $\|.\|_{0,T} := \|.\|_{0,0,T}$.

(6) Consider $(s, t) \in \Delta_T$. The set of all dissections of $[s, t]$ is denoted by $\mathfrak{D}_{[s,t]}$.

(7) Consider $(s, t) \in \Delta_T$. A function $f : [s, t] \to \mathbb{R}$ is of finite 1-variation if and only if,

$$\|f\|_{1\text{-var},s,t} := \sup\left\{\sum_{k=1}^{n-1} |f(t_k, t_{k+1})|;\ n \in \mathbb{N}^* \text{ and } (t_k)_{k\in[\![1,n]\!]} \in \mathfrak{D}_{[s,t]}\right\} < \infty.$$

Consider the vector space

$$C^{1\text{-var}}([s, t], \mathbb{R}) := \{f \in C^0([s, t], \mathbb{R}) : \|f\|_{1\text{-var},s,t} < \infty\}.$$

The map $\|.\|_{1\text{-var},s,t}$ is a semi-norm on $C^{1\text{-var}}([s, t], \mathbb{R})$. Moreover, $\|.\|_{1\text{-var},T} := \|.\|_{1\text{-var},0,T}$.

(8) The vector space of Lipschitz continuous functions from a closed interval $I \subset \mathbb{R}$ into $\mathbb{R}$ is denoted by $\text{Lip}(I)$ and equipped with the Lipschitz semi-norm $\|.\|_{\text{Lip},I}$ defined by

$$\|f\|_{\text{Lip},I} := \sup\left\{\frac{|f(t)-f(s)|}{|t-s|}; s,t \in I \text{ and } s \neq t\right\}$$

for every $f \in \text{Lip}(I)$. Moreover, $\|.\|_{\text{Lip}} := \|.\|_{\text{Lip},\mathbb{R}}$ and $\|.\|_{\text{Lip},T} := \|.\|_{\text{Lip},[0,T]}$.

(9) For every $L > 0$,

$$\Theta_0(L) := \{f \in \text{Lip}(\mathbb{R}) : |f(0)| + \|f\|_{\text{Lip}} \leqslant L\}.$$

2. Preliminaries

This section deals with some preliminaries on the Skorokhod reflection problem defined by a fractional SDE and a sweeping process.

First, the following theorem states a sufficient condition of existence and uniqueness of the solution to the unperturbed sweeping process defined by

$$\begin{cases} -\dot{y}(t) & \in & \mathcal{N}_{C(t)}(y(t))|Dy|\text{-a.e.} \\ y(0) & = & y_0 \end{cases} ; t \in [0,T], \tag{3}$$

where $y_0 \in C(0)$.

Theorem 2.1. *Assume that for every $t \in [0,T]$, $C(t)$ is a compact interval of $\mathbb{R}$. Moreover, assume that there exist $r > 0$ and $a \in \mathbb{R}$ such that*

$$[a-r, a+r] \subset \text{int}(C(t)); \forall t \in [0,T].$$

Then, Problem (3) has a unique continuous solution of finite 1-variation $y : [0,T] \to \mathbb{R}$ such that

$$\|y\|_{1\text{-var},T} \leqslant \max\{0, \|y_0 - a\| - r\}.$$

See Monteiro Marques (1984) for a proof.

In the sequel, the multifunction C fulfills the following assumption.

Assumption 2.2. For every $t \in [0,T]$, $C(t)$ is a compact interval of $\mathbb{R}$. Moreover, there exist $r > 0$ and a continuous selection $\gamma : [0,T] \to \mathbb{R}$ such that

$$[\gamma(t) - r, \gamma(t) + r] \subset \text{int}(C(t)); \forall t \in [0,T].$$

Let $\varphi : [0,T] \to \mathbb{R}$ be a continuous function such that $\varphi(0) = 0$ and consider the (generic) Skorokhod reflection problem

$$\begin{cases} v_\varphi(t) & = & \varphi(t) + w_\varphi(t) \\ -\dot{w}_\varphi(t) & \in & \mathcal{N}_{C_\varphi(t)}(w_\varphi(t))|Dw_\varphi|\text{-a.e. with } w_\varphi(0) = x_0, \end{cases} \tag{4}$$

where

$$C_\varphi(t) := \{v - \varphi(t); v \in C(t)\}; \forall t \in [0,T],$$

$v_\varphi : [0,T] \to \mathbb{R}$ is a continuous function and $w_\varphi : [0,T] \to \mathbb{R}$ is a continuous function of finite 1-variation. Under Assumption 2.2, by Theorem 2.1 together with Castaing et al. (2016b), Lemma 2.2, Problem (4) has a unique solution. Moreover, the following proposition provides a suitable control of $w_\varphi - w_\psi$ for any continuous functions $\varphi, \psi : [0,T] \to \mathbb{R}$ such that $\varphi(0) = \psi(0) = 0$.

Proposition 2.3. *Under Assumption 2.2, for every continuous functions $\varphi, \psi : [0,T] \to \mathbb{R}$ such that $\varphi(0) = \psi(0) = 0$,*

$$\|w_\varphi - w_\psi\|_{\infty,T} \leqslant \|\varphi - \psi\|_{\infty,T}.$$

See Slominski and Wojciechowski (2013), Proposition 2.3 for a proof.

Under Assumption 2.2, note that there exist $R > 0$, $N \in \mathbb{N}^*$ and $(t_0, \dots, t_N) \in \mathfrak{D}_{[0,T]}$ such that

$$[\gamma(t_k) - R, \gamma(t_k) + R] \subset C(t)$$

for every $k \in [\![0, N-1]\!]$ and $t \in [t_k, t_{k+1}]$.

Proposition 2.4. *Consider $(s,t) \in \Delta_T$ and $\rho \in]0, R/2]$. Under Assumption 2.2, if $\|\varphi\|_{0,s,t} \leqslant \rho$, then*

$$\|w_\varphi\|_{1\text{-var},s,t} \leqslant N \sup_{u \in [0,T]} \sup_{v,w \in C(u)} |w - v|.$$

The proof of Proposition 2.4 is the same as the proof of Castaing et al. (2017), Proposition 2.5 but with the upper bound for the 1-variation norm of the 1-dimensional unperturbed sweeping process provided in Theorem 2.1 instead of the corresponding upper bound in the multidimensional case provided in Castaing et al. (2017), Proposition 2.1.

For any $t \in [0, T]$,

$$\mathcal{N}_{C_\varphi(t)}(w_\varphi(t)) = \mathcal{N}_{C(t)-\varphi(t)}(v_\varphi(t) - \varphi(t)) = \mathcal{N}_{C(t)}(v_\varphi(t)).$$

Then Problem (4) is equivalent to

$$\begin{cases} v_\varphi(t) & = & \varphi(t) + w_\varphi(t) \\ -\dot{w}_\varphi(t) & \in & \mathcal{N}_{C(t)}(v_\varphi(t))|Dw_\varphi|\text{-a.e. with } w_\varphi(0) = x_0. \end{cases}$$

So, one can use the previous results of this section in order to establish the existence and uniqueness of the solution to Problems (1) and (2).

Theorem 2.5. *Under Assumption 2.2,*

(1) Problem (1) has a unique solution $(X_\varepsilon, Y_\varepsilon)$. Moreover, its paths belong to

$$C^{p\text{-var}}([0, T], \mathbb{R}) \times C^{1\text{-var}}([0, T], \mathbb{R})$$

for every $p > 1/H$.

(2) Problem (2) has a unique solution (x, y). Moreover, it is a Lipschitz continuous map from $[0, T]$ into $\mathbb{R}^2$ such that

$$\|y\|_{\mathrm{Lip},T} \leqslant \|b\|_{\mathrm{Lip}} + \|C\|_{\mathrm{Lip},T}$$

and

$$\|x\|_{\mathrm{Lip},T} \leqslant 2\|b\|_{\mathrm{Lip}} + \|C\|_{\mathrm{Lip},T}.$$

The proof of the existence of solutions to Problem (1) in Theorem 2.5 is the same as the proof of Castaing et al. (2017), Theorem 3.1 but with the upper bound for the 1-variation norm of w_φ in Problem (4) provided in Proposition 2.4 instead of the corresponding upper bound in the multidimensional case provided in Castaing et al. (2017), Proposition 2.5. Castaing et al. (2017), Proposition 4.1 give the uniqueness of the solution to Problem (1). Castaing et al. (2016b), Theorem 4.2 give the existence, uniqueness and the regularity of the solution to Problem (2).

3. Convergence of the trend estimator

This section deals with the consistency, a rate of convergence and the asymptotic distribution of the estimator $\widehat{\tau}_\varepsilon(t)$. First, the following lemma deals with the convergence of X_ε and Y_ε when $\varepsilon \to 0$.

Lemma 3.1. *Under Assumption 2.2, if $b \in \Theta_0(L)$ with $L > 0$, then there exists a deterministic constant $\mathfrak{c}_{H,L,T} > 0$, depending only on H, L and T, such that*

$$\mathbb{E}(\|X_\varepsilon - x\|_{\infty,T}^2) + \mathbb{E}(\|Y_\varepsilon - y\|_{\infty,T}^2) \leqslant \mathfrak{c}_{H,L,T}\varepsilon^2.$$

Proof. Consider $H_\varepsilon := X_\varepsilon - Y_\varepsilon$ and $h := x - y$. By Proposition 2.3, for any $t \in [0, T]$,

$$\|Y_\varepsilon - y\|_{\infty,t} \leqslant \|H_\varepsilon - y\|_{\infty,t}.$$

Then,

$$\begin{aligned} |X_\varepsilon(t) - x(t)| &\leqslant \|H_\varepsilon - h\|_{\infty,t} + \|Y_\varepsilon - y\|_{\infty,t} \leqslant 2\|H_\varepsilon - h\|_{\infty,t} \\ &\leqslant 2L\int_0^t |X_\varepsilon(s) - x(s)|ds + 2\varepsilon\|B\|_{\infty,t}. \end{aligned}$$

By Gronwall's lemma,

$$|X_\varepsilon(t) - x(t)| \leqslant 2\varepsilon\|B\|_{\infty,T}e^{2LT}.$$

Moreover,

$$\begin{aligned} |Y_\varepsilon(t) - y(t)| &\leqslant |H_\varepsilon(t) - h(t)| + |X_\varepsilon(t) - x(t)| \\ &\leqslant (TL + 1)\|X_\varepsilon - x\|_{\infty,T} + \varepsilon\|B\|_{\infty,T} \\ &\leqslant \varepsilon\|B\|_{\infty,T}(2e^{2LT}(TL + 1) + 1). \end{aligned}$$

This concludes the proof because $\mathbb{E}(\|B\|_{\infty,T}^2) < \infty$. $\square$

In the sequel, the bandwidth h_ε and the kernel K fulfill the following assumptions.

Assumption 3.2. The bandwidth h_ε satisfies $\varepsilon = o(h_\varepsilon^{1-H})$.

Assumption 3.3. The kernel K is bounded and $K^{-1}(\{0\})^c =]A, B[$ with $A < B$.

For instance, the triangular kernel

$$u \in \mathbb{R} \longmapsto (1 - |u|)\mathbf{1}_{|u|\leqslant 1}$$

or the parabolic kernel

$$u \in \mathbb{R} \longmapsto \frac{3}{4}(1 - u^2)\mathbf{1}_{|u|\leqslant 1}$$

fulfill Assumption 3.3.

Let us now establish the consistency and a rate of convergence for the estimator $\widehat{\tau}_\varepsilon$ of the trend τ of Problem (1).

Theorem 3.4. *Under Assumptions 2.2 and 3.3, if $b \in \Theta_0(L)$ with $L > 0$, then there exists a deterministic constant $\mathfrak{c}_{C,H,K,L,T} > 0$, depending only on C, H, K, L and T, such that*

$$\sup_{t\in[0,T]} \mathbb{E}(|\widehat{\tau}_\varepsilon(t) - \tau(t)|^2) \leqslant \mathfrak{c}_{C,H,K,L,T}(\varepsilon^2 + h_\varepsilon^2 + \varepsilon^2 h_\varepsilon^{2H-2}).$$

In particular, under Assumption 3.2, the estimator $\widehat{\tau}_\varepsilon$ is consistent.

Proof. First of all, for any $t \in [0, T]$,

$$\begin{aligned}
\widehat{\tau}_\varepsilon(t) - \tau(t) &= \int_0^t \int_0^T K_{h_\varepsilon}(s-u)dX_\varepsilon(s)du - \int_0^t b(x(u))du - y(t) + x_0\\
&= \alpha_\varepsilon(t) + \beta_\varepsilon(t) + \gamma_\varepsilon(t) + \zeta_\varepsilon(t) + \eta_\varepsilon(t),
\end{aligned}$$

where

$$\begin{aligned}
\alpha_\varepsilon(t) &:= \int_0^t \int_0^T K_{h_\varepsilon}(s-u)(b(X_\varepsilon(s)) - b(x(s)))dsdu,\\
\beta_\varepsilon(t) &:= \int_0^t \int_0^T K_{h_\varepsilon}(s-u)b(x(s))dsdu - \int_0^t b(x(u))du,\\
\gamma_\varepsilon(t) &:= \varepsilon \int_0^t \int_0^T K_{h_\varepsilon}(s-u)dB(s)du,\\
\zeta_\varepsilon(t) &:= \int_0^t \int_0^T K_{h_\varepsilon}(s-u)d(Y_\varepsilon - y)(s)du \text{ and}\\
\eta_\varepsilon(t) &:= \int_0^t \int_0^T K_{h_\varepsilon}(s-u)dy(s)du - y(t) + x_0.
\end{aligned}$$

Let us find suitable controls of the supremum on $[0, T]$ of the second order moment of all these components.

- Note that

$$\begin{aligned}
|\alpha_\varepsilon(t)| &= \left|\int_0^t \int_{-u/h_\varepsilon}^{(T-u)/h_\varepsilon} K(s)(b(X_\varepsilon(h_\varepsilon s + u)) - b(x(h_\varepsilon s + u)))dsdu\right|\\
&\leqslant \|b\|_{\mathrm{Lip}} \int_0^t \sup_{0\leqslant h_\varepsilon s+u\leqslant T} |X_\varepsilon(h_\varepsilon s + u) - x(h_\varepsilon s + u)|du \leqslant LT\|X_\varepsilon - x\|_{\infty,T}.
\end{aligned}$$

  Then, by Lemma 3.1,

$$\sup_{t\in[0,T]} \mathbb{E}(\alpha_\varepsilon(t)^2) \leqslant L^2T^2\mathfrak{c}_{H,L,T}\varepsilon^2.$$

- Since C is a Lipschitz continuous and compact-valued multifunction, x is bounded by a deterministic constant $M > 0$ depending only on C (not on b). Then,

$$\begin{aligned}
|\beta_\varepsilon(t)| &= \left|\int_0^T b(x(s)) \int_{(s-t)/h_\varepsilon}^{s/h_\varepsilon} K(u)duds - \int_0^t b(x(u))du\right|\\
&= \left|\int_{-\infty}^{\infty} K(u) \int_0^T b(x(s))\mathbf{1}_{[h_\varepsilon u, h_\varepsilon u+t]}(s)dsdu - \int_0^t b(x(s))ds\right|\\
&= \left|\int_A^B K(u)\left(\int_{0\vee(h_\varepsilon u)}^{T\wedge(h_\varepsilon u+t)} b(x(s))ds - \int_0^t b(x(s))ds\right)du\right|\\
&\leqslant 2h_\varepsilon \sup_{z\in[-M,M]} |b(z)| \int_A^B K(u)|u|du.
\end{aligned}$$

Moreover, since $|b(0)| + \|b\|_{\rm Lip} \leqslant L$,

$$|\beta_\varepsilon(t)| \leqslant 2(|A| \vee |B|)Lh_\varepsilon.$$

- By Memin et al. (2001), Theorem 1.1, there exists a deterministic constant $c_1 > 0$, only depending on H, such that

$$\begin{aligned}\mathbb{E}(\gamma_\varepsilon(t)^2) &\leqslant \varepsilon^2 t \int_0^t \mathbb{E}\left(\left|\int_0^T K_{h_\varepsilon}(s-u)dB(s)\right|^2\right)du \\ &\leqslant c_1\frac{\varepsilon^2 T}{h_\varepsilon^2}\int_0^T\left|\int_0^T K\left(\frac{s-u}{h_\varepsilon}\right)^{1/H}ds\right|^{2H}du \leqslant c_2\varepsilon^2 h_\varepsilon^{2H-2},\end{aligned}$$

where

$$c_2 := c_1 T^2\left|\int_A^B K(s)^{1/H}ds\right|^{2H}.$$

- Since the paths of $Y_\varepsilon - y$ are continuous and of finite 1-variation,

$$\begin{aligned}\zeta_\varepsilon(t) &= \int_0^T\int_0^t K_{h_\varepsilon}(s-u)dud(Y_\varepsilon - y)(s)\\ &= \int_0^T\int_{(s-t)/h_\varepsilon}^{s/h_\varepsilon} K(u)dud(Y_\varepsilon-y)(s)\\ &= \int_{-\infty}^{\infty}K(u)\int_0^T \mathbf{1}_{[h_\varepsilon u, h_\varepsilon u+t]}(s)d(Y_\varepsilon-y)(s)du\\ &= \int_A^B K(u)(Y_\varepsilon-y)(0\vee(h_\varepsilon u), T\wedge(h_\varepsilon u+t))du.\end{aligned}$$

Then, by Lemma 3.1,

$$\sup_{t\in[0,T]}\mathbb{E}(\zeta_\varepsilon(t)^2) \leqslant \mathbb{E}(\|Y_\varepsilon - y\|_{\infty,T}^2) \leqslant \mathfrak{c}_{H,L,T}\varepsilon^2.$$

- Since y is a Lipschitz continuous function (see Theorem 2.5.(2)),

$$\begin{aligned}|\eta_\varepsilon(t)| &= \left|\int_0^T\int_0^t K_{h_\varepsilon}(s-u)dudy(s) - y(t) + x_0\right|\\ &\leqslant \int_A^B K(u)|y(0\vee(h_\varepsilon u), T\wedge(h_\varepsilon u+t)) - y(0,t)|du \leqslant 2(|A|\vee|B|)\|y\|_{{\rm Lip},T}h_\varepsilon.\end{aligned}$$

Moreover, since $\|y\|_{{\rm Lip},T} \leqslant L + \|C\|_{{\rm Lip},T}$,

$$|\eta_\varepsilon(t)| \leqslant 2(|A|\vee|B|)(L+\|C\|_{{\rm Lip},T})h_\varepsilon. \quad \square$$

Theorem 3.4 says that the quadratic risk of the estimator $\widehat{\tau}_\varepsilon(t)$ involves a squared bias of order $\varepsilon^2 + h_\varepsilon^2$ and a variance term of order $\varepsilon^2 h_\varepsilon^{2H-2}$. The best possible rate $\varepsilon^{2/(2-H)}$ is reached for a bandwidth choice of order $\varepsilon^{1/(2-H)}$.

Corollary 3.5. *Under Assumptions 2.2 and 3.3, if $h_\varepsilon = \varepsilon^{1/(2-H)}$, then*

$$\lim_{\varepsilon\to 0}\varepsilon^{\alpha-2/(2-H)}\sup_{t\in[0,T]}\mathbb{E}(|\widehat{\tau}_\varepsilon(t) - \tau(t)|^2) = 0;\ \forall\alpha > 0.$$

Corollary 3.5 is a straightforward consequence of Theorem 3.4.

In the sequel, C fulfills the following assumption.

Assumption 3.6. There exist $\mathbf{l}, \mathbf{u} \in C^1([0,T],\mathbb{R})$ such that for every $t\in[0,T]$, $\mathbf{l}(t) < \mathbf{u}(t)$ and

$$C(t) = [\mathbf{l}(t), \mathbf{u}(t)].$$

Finally, Proposition 3.8 provides the asymptotic distribution of the estimator $\widehat{\tau}_\varepsilon(t)$ for every $t \in \mathcal{E} := \mathcal{E}_{\mathbf{l}} \cup \mathcal{E}_{\mathbf{u}} \cup \mathcal{E}_{{\rm int}(C)}$, where

$$\mathcal{E}_I := \{s\in[0,T] : \exists\varepsilon > 0, \forall r \in]s-\varepsilon, s+\varepsilon[, x(r) \in I(r)\}$$

for every multifunction $I : [0,T] \rightrightarrows \mathbb{R}$.

First, recall that for any $f \in C^{1\text{-var}}([0,T],\mathbb{R})$, $\dot{f}$ is the Radon–Nikodym derivative of the differential measure Df of f with respect to its variation measure $|Df|$. In particular, if f is absolutely continuous, then

$$f(v)-f(u)=\int_u^v \dot{f}(s)ds;\ \forall(u,v)\in\Delta_T.$$

Lemma 3.7. *Under Assumption 3.6, $\dot{y}$ is continuous on $\mathcal{E}$.*

Proof. Since y is a Lipschitz continuous function, it is absolutely continuous. In other words, for every $(u,v)\in\Delta_T$,

$$y(v)-y(u)=\int_u^v \dot{y}(s)ds.$$

On the one hand, consider $t\in\mathcal{E}_{\mathrm{int}(C)}$. So, there exists $\varepsilon>0$ such that for any $s\in]t-\varepsilon,t+\varepsilon[$, $x(s)\in]\mathbf{l}(s),\mathbf{u}(s)[$ and then

$$\dot{y}(s)=0.$$

Therefore, $\dot{y}$ is continuous at time s. On the other hand, consider $t\in\mathcal{E}_{\mathbf{l}}$. So, there exists $\varepsilon>0$ such that for any $s\in]t-\varepsilon,t+\varepsilon[$, $x(s)=\mathbf{l}(s)$ and then

$$\dot{y}(s)=\dot{\mathbf{l}}(s)-b(\mathbf{l}(s)).$$

Therefore, since $\mathbf{l}\in \mathrm{C}^1([0,T],\mathbb{R})$, $\dot{y}$ is continuous at time s. The same idea gives the continuity of $\dot{y}$ on $\mathcal{E}_{\mathbf{u}}$. □

The previous lemma states that $\dot{y}$ is continuous when x stays a little time on the frontier or in the interior of C. Unfortunately, there is no reason for $\dot{y}$ to be continuous each time x enters or exits the frontier of C.

Proposition 3.8. *Under Assumptions 3.6 and 3.3, if $A\geqslant 0$, $t\in\mathcal{E}\cap[0,T[$ and $h_\varepsilon=\varepsilon^{1/(2-H)}$, then*

$$\varepsilon^{-1/(2-H)}(\widehat{\tau}_\varepsilon(t)-\tau(t)-\gamma_\varepsilon(t))\xrightarrow[\varepsilon\to 0]{\mathbb{L}^2}\mu(t)$$

and

$$\varepsilon^{-1/(2-H)}\dot{\gamma}_\varepsilon(t)\xrightarrow[\varepsilon\to 0]{\Delta}\mathcal{N}(0,\sigma_{H,K}^2),$$

where

$$\mu(t):=(b(x(t))-b(x(0))+\dot{y}(t)-\dot{y}(0))\int_A^B K(u)udu$$

and

$$\sigma_{H,K}^2:=H(2H-1)\int_A^B\int_A^B|u-v|^{2H-2}K(u)K(v)dudv.$$

Proof. Since

$$\sup_{t\in[0,T]}\mathbb{E}(\alpha_\varepsilon(t)^2)+\sup_{t\in[0,T]}\mathbb{E}(\zeta_\varepsilon(t)^2)=O(\varepsilon^2)$$

as established in the proof of Theorem 3.4,

$$\varepsilon^{-1/(2-H)}(\alpha_\varepsilon(t)+\zeta_\varepsilon(t))\xrightarrow[\varepsilon\to 0]{\mathbb{L}^2}0.$$

Let us study the behavior of $\varepsilon^{-1/(2-H)}(\beta_\varepsilon(t)+\eta_\varepsilon(t))$ when $\varepsilon\to 0$.

- Since $A\geqslant 0$ and $h_\varepsilon B+t<T$ for ε small enough,

$$\begin{aligned}\beta_\varepsilon(t)&=\int_A^B K(u)\left(\int_t^{h_\varepsilon u+t}b(x(s))ds-\int_0^{h_\varepsilon u}b(x(s))ds\right)du\\&=h_\varepsilon\int_A^B K(u)u\left(\int_0^1 b(x(sh_\varepsilon u+t))ds-\int_0^1 b(x(sh_\varepsilon u))ds\right)du.\end{aligned}$$

  Therefore, by Lebesgue's theorem,

$$\lim_{\varepsilon\to 0}\varepsilon^{-1/(2-H)}\beta_\varepsilon(t)=(b(x(t))-b(x(0)))\int_A^B K(u)udu.$$

- Since y is a Lipschitz continuous function, as recalled previously, $dy(s) = \dot{y}(s)ds$. Then,

$$\begin{aligned}\eta_\varepsilon(t) &= \int_A^B K(u)(y(t, h_\varepsilon u + t) - y(0, h_\varepsilon u))du \\ &= h_\varepsilon \int_A^B K(u)u\left(\int_0^1 \dot{y}(sh_\varepsilon u + t)ds - \int_0^1 \dot{y}(sh_\varepsilon u)ds\right)du.\end{aligned}$$

  Therefore, since $\dot{y}$ is continuous on a neighborhood of t by Lemma 3.7, by Lebesgue's theorem,

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/(2-H)}\eta_\varepsilon(t) = (\dot{y}(t) - \dot{y}(0))\int_A^B K(u)udu.$$

Finally,

$$\dot{\gamma}_\varepsilon(t) = \varepsilon \int_0^T K_{h_\varepsilon}(s-t)dB(s) \rightsquigarrow \mathcal{N}(0, \sigma_\varepsilon(t)^2)$$

where

$$\begin{aligned}\sigma_\varepsilon(t)^2 &:= H(2H-1)\varepsilon^2 \int_0^T \int_0^T |s-r|^{2H-2}K_{h_\varepsilon}(r-t)K_{h_\varepsilon}(s-t)drds \\ &= \sigma_{H,K}\varepsilon^2 h_\varepsilon^{2H-2}.\end{aligned}$$

Therefore,

$$\varepsilon^{-1/(2-H)}\dot{\gamma}_\varepsilon(t) \xrightarrow[\varepsilon\to 0]{\Delta} \mathcal{N}(0, \sigma_{H,K}^2). \quad \square$$

Since x is Lipschitz continuous on $[0, T]$, the subset of times x enters or exists the frontier of C is countable. So, the Lebesgue measure of $\mathcal{E}$ is equal to T. Therefore, Proposition 3.8 is true for almost every t in $[0, T]$.

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References

Bernicot, F., Venel, J., 2011. Stochastic perturbation of sweeping process and a convergence result for an associated numerical scheme. J. Differential Equations 251 (4–5), 1195–1224.
Castaing, C., Marie, N., Raynaud de Fitte, P., 2017. Sweeping processes perturbed by rough signals. ArXiv e-prints.
Castaing, C., Monteiro Marques, M.D.P., Raynaud de Fitte, P., 2016a. A skorokhod problem governed by a closed convex moving set. J. Convex Anal. 23 (2), 387–423.
Castaing, C., Monteiro Marques, M.D.P., Raynaud de Fitte, P., 2016b. A skorokhod problem governed by a closed convex moving set. J. Convex Anal. 23 (2), 387–423.
Chronopoulou, A., Tindel, S., 2013. On inference for fractional differential equations. Stat. Inference Stoch. Process. 16 (1), 29–61.
Comte, F., Marie, N., 2019. Nonparametric estimation in fractional SDE. Stat. Inference Stoch. Process. http://dx.doi.org/10.1007/s11203-019-09196-y.
Falkowski, A., Slominski, L., 2015. Sweeping processes with stochastic perturbations generated by a fractional Brownian motion. ArXiv e-prints.
Hu, Y., Nualart, D., 2010. Parameter estimation for fractional Ornstein–Uhlenbeck processes. Statist. Probab. Lett. 80, 1030–1038.
Hu, Y., Nualart, D., Zhou, H., Drift parameter estimation for nonlinear stochastic differential equations driven by fractional Brownian motion, arXiv:1803.01032v1.
Kleptsyna, M.L., Le Breton, A., 2001. Some explicit statistical results about elementary fractional type models. Nonlinear Anal. 47, 4783–4794.
Kutoyants, Y., 1994. Identification of Dynamical Systems with Small Noise. Kluwer, Dordrecht.
Kutoyants, Y., 2004. Statistical Inference for Ergodic Diffusion Processes. Springer.
Memin, J., Mishura, Y., Valkeila, E., 2001. Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion. Statist. Probab. Lett. 51, 197–206.
Mishra, M.N., Prakasa Rao, B.L.S., 2011. Nonparameteric estimation of trend for stochastic differential equations driven by fractional Brownian motion. Stat. Inference Stoch. Process. 14 (2), 101–109.
Mishura, Y., Ralchenko, K., 2014. On drift parameter estimation in models with fractional brownian motion by discrete observations. Austrian J. Stat. 43 (3–4), 217–228.
Monteiro Marques, M.D.P., 1984. Rafle par un convexe semi-continu inférieurement d'intérieur non vide en dimension finie. In: Séminaire d'analyse convexe, vol. 16. 6, 24 pages.
Neuenkirch, A., Tindel, S., 2014. A least square-type procedure for parameter estimation in stochastic differential equations with additive fractional noise. Stat. Inference Stoch. Process. 17 (1), 99–120.
Saussereau, B., 2014. Nonparametric inference for fractional diffusion. Bernoulli 20 (2), 878–918.
Slominski, L., Wojciechowski, T., 2013. Stochastic differential equations with time-dependent reflecting barriers. Stochastics 85 (1), 27–47.
Tudor, C.A., Viens, F., 2007. Statistical aspects of the fractional stochastic calculus. Ann. Statist. 35 (3), 1183–1212.