Gauge theories in particle physics(3rd, Aitchison and Hey)

Chapter 7, Quantum field theory III : Complex scalar fields, Dirac and maxwell fields; introduction of electromagnetic interactions

Problems 7.1, P_{200}.

Verify that the Lagranian $\hat{\mathcal{L}}$ of (7.1) is invariant (i.e. $\hat{\mathcal{L}}(\hat{\phi}_1, \hat{\phi}_2) = \hat{\mathcal{L}}(\hat{\phi}'_1, \hat{\phi}'_2)$ under the ransformation (7.2) of the fields $(\hat{\phi}_1, \hat{\phi}_2) \to (\hat{\phi}'_1, \hat{\phi}'_2)$).

Solutions of *Problem 7.1*:

Equation (7.1) is a Lagrangian for two free fields $\hat{\phi}_1, \hat{\phi}_2$ having the same mass M, as following:

$$\hat{\mathcal{L}} = \frac{1}{2}\partial_\mu \hat{\phi}_1 \partial^\mu \hat{\phi}_1 - \frac{1}{2}M^2\hat{\phi}_1^2 + \frac{1}{2}\partial_\mu \hat{\phi}_2 \partial^\mu \hat{\phi}_2 - \frac{1}{2}M^2\hat{\phi}_2^2$$

Because $\hat{\phi}_1, \hat{\phi}_2$ are composed of create/destroy operators multiplying exponential function, like $\hat{a}_k e^{-ik\cdot x} + \hat{a}_k^\dagger e^{ik\cdot x}$ (cf. equation (6.52) for instance). And according to the definition of $\hat{a}$ and $\hat{a}^\dagger$((5.61), (5.62)), $\hat{a} = \frac{1}{\sqrt{2}}(\sqrt{m\omega}\hat{q} + \frac{i}{\sqrt{m\omega}}\hat{p})$, $\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\sqrt{m\omega}\hat{q} - \frac{i}{\sqrt{m\omega}}\hat{p})$. At the same time, contravariant vector $\partial^\mu = (\partial/\partial t, -\nabla)$ and covariant $\partial_\mu = (\partial/\partial t, \nabla)$ could act on the $\hat{q}, \hat{p}$. So, these two kinds of "operators" are not "independent". As a result, $\hat{\phi}_1, \hat{\phi}_2$ could **NOT** move forward or backward among the expressions.

Although the following solution looks pretty simple, it's **wrong**:

$$\begin{aligned}
\hat{\mathcal{L}} &= \frac{1}{2}\partial_\mu\partial^\mu(\hat{\phi}_1)^2 - \frac{1}{2}M^2\hat{\phi}_1^2 + \frac{1}{2}\partial_\mu\partial^\mu(\hat{\phi}_2)^2 - \frac{1}{2}M^2\hat{\phi}_2^2 \\
&= \frac{1}{2}(\partial_\mu\partial^\mu - M^2)\hat{\phi}_1^2 + \frac{1}{2}(\partial_\mu\partial^\mu - M^2)\hat{\phi}_2^2 \\
&= \frac{1}{2}(\partial_\mu\partial^\mu - M^2)(\hat{\phi}_1^2 + \hat{\phi}_2^2) \qquad (1)
\end{aligned}$$

And the equation of (7.2) is :

$$\begin{aligned}
\hat{\phi}'_1 &= (cos\alpha)\hat{\phi}_1 - (sin\alpha)\hat{\phi}_2 \\
\hat{\phi}'_2 &= (sin\alpha)\hat{\phi}_1 + (cos\alpha)\hat{\phi}_2
\end{aligned}$$

So, do the replacing as $(\hat{\phi}_1 \to \hat{\phi}'_1, \hat{\phi}_2 \to \hat{\phi}'_2)$ to (1) will obtain,

$$\begin{aligned}
\hat{\mathcal{L}} &= \frac{1}{2}(\partial_\mu\partial^\mu - M^2)(\hat{\phi}_1'^2 + \hat{\phi}_2'^2) \\
&= \frac{1}{2}(\partial_\mu\partial^\mu - M^2)\{[(cos\alpha)\hat{\phi}_1 - (sin\alpha)\hat{\phi}_2]^2 + [(sin\alpha)\hat{\phi}_1 + (cos\alpha)\hat{\phi}_2]^2\} \\
&= \frac{1}{2}(\partial_\mu\partial^\mu - M^2)(cos^2\alpha\hat{\phi}_1^2 - 2cos\alpha sin\alpha\hat{\phi}_1\hat{\phi}_2 + sin^2\alpha\hat{\phi}_2^2 + sin^2\alpha\hat{\phi}_1^2 + 2sin\alpha cos\alpha\hat{\phi}_1\hat{\phi}_2 + cos^2\alpha\hat{\phi}_2^2) \\
&= \frac{1}{2}(\partial_\mu\partial^\mu - M^2)[(cos^2\alpha + sin\alpha^2)\hat{\phi}_1^2 + (sin\alpha^2 + cos^2\alpha)\hat{\phi}_2^2] \\
&= \frac{1}{2}(\partial_\mu\partial^\mu - M^2)(\hat{\phi}_1^2 + \hat{\phi}_2^2)
\end{aligned}$$

The **correct solution** might replace $\hat{\phi}_1$ and $\hat{\phi}_2$ with the expression of $\hat{\phi}'_1$ and $\hat{\phi}'_2$ (7.2) in the Lagrangian (7.1). Details like this :

$$\begin{aligned}
\hat{\mathcal{L}} &= \frac{1}{2}\partial_\mu\hat{\phi}_1\partial^\mu\hat{\phi}_1 - \frac{1}{2}M^2\hat{\phi}_1^2 + \frac{1}{2}\partial_\mu\hat{\phi}_2\partial^\mu\hat{\phi}_2 - \frac{1}{2}M^2\hat{\phi}_2^2 \\
&= \frac{1}{2}\partial_\mu\hat{\phi}'_1\partial^\mu\hat{\phi}'_1 - \frac{1}{2}M^2\hat{\phi}_1^{'2} + \frac{1}{2}\partial_\mu\hat{\phi}'_2\partial^\mu\hat{\phi}'_2 - \frac{1}{2}M^2\hat{\phi}_2^{'2} \\
&= \frac{1}{2}\partial_\mu[(cos\alpha)\hat{\phi}_1 - (sin\alpha)\hat{\phi}_2]\partial^\mu[(cos\alpha)\hat{\phi}_1 - (sin\alpha)\hat{\phi}_2] - \frac{1}{2}M^2[(cos\alpha)\hat{\phi}_1 - (sin\alpha)\hat{\phi}_2]^2 \\
&+ \frac{1}{2}\partial_\mu[(sin\alpha)\hat{\phi}_1 + (cos\alpha)\hat{\phi}_2]\partial^\mu[(sin\alpha)\hat{\phi}_1 + (cos\alpha)\hat{\phi}_2] - \frac{1}{2}M^2[(sin\alpha)\hat{\phi}_1 + (cos\alpha)\hat{\phi}_2]^2 \\
&= \frac{1}{2}[(cos\alpha)\partial_\mu\hat{\phi}_1 - (sin\alpha)\partial_\mu\hat{\phi}_2][(cos\alpha)\partial^\mu\hat{\phi}_1 - (sin\alpha)\partial^\mu\hat{\phi}_2] - \frac{1}{2}M^2(cos^2\alpha\hat{\phi}_1^2 - 2cos\alpha sin\alpha\hat{\phi}_1\hat{\phi}_2 + sin^2\alpha\hat{\phi}_2^2) \\
&+ \frac{1}{2}[(sin\alpha)\partial_\mu\hat{\phi}_1 + (cos\alpha)\partial_\mu\hat{\phi}_2][(sin\alpha)\partial^\mu\hat{\phi}_1 + (cos\alpha)\partial^\mu\hat{\phi}_2] - \frac{1}{2}M^2(sin^2\alpha\hat{\phi}_1^2 + 2sin\alpha cos\alpha\hat{\phi}_1\hat{\phi}_2 + cos^2\alpha\hat{\phi}_2^2) \\
&= \frac{1}{2}[(cos^2\alpha)\partial_\mu\hat{\phi}_1\partial^\mu\hat{\phi}_1 - (cos\alpha sin\alpha)\partial_\mu\hat{\phi}_1\partial^\mu\hat{\phi}_2 - (sin\alpha cos\alpha)\partial_\mu\hat{\phi}_2\partial^\mu\hat{\phi}_1 + (sin^2\alpha)\partial_\mu\hat{\phi}_2\partial^\mu\hat{\phi}_2] \\
&+ \frac{1}{2}[(sin^2\alpha)\partial_\mu\hat{\phi}_1\partial^\mu\hat{\phi}_1 + (sin\alpha cos\alpha)\partial_\mu\hat{\phi}_1\partial^\mu\hat{\phi}_2 + (cos\alpha sin\alpha)\partial_\mu\hat{\phi}_2\partial^\mu\hat{\phi}_1 + (cos^2\alpha)\partial_\mu\hat{\phi}_2\partial^\mu\hat{\phi}_2] - \frac{1}{2}M^2(\hat{\phi}_1^2 + \hat{\phi}_2^2) \\
&= \frac{1}{2}(\partial_\mu\hat{\phi}_1\partial^\mu\hat{\phi}_1 + \partial_\mu\hat{\phi}_2\partial^\mu\hat{\phi}_2) - \frac{1}{2}M^2(\hat{\phi}_1^2 + \hat{\phi}_2^2) \\
&= \frac{1}{2}\partial_\mu\hat{\phi}_1\partial^\mu\hat{\phi}_1 - \frac{1}{2}M^2\hat{\phi}_1^2 + \frac{1}{2}\partial_\mu\hat{\phi}_2\partial^\mu\hat{\phi}_2 - \frac{1}{2}M^2\hat{\phi}_2^2 \qquad \text{(proof is done)}
\end{aligned}$$

Problems 7.2, P_{200}.

(a) Verify that, for $\hat{N}_\phi^\mu$ given by (7.23), the corresponding $\hat{N}_\phi$ of (7.14) reduces to the form (7.24); and that, with $\hat{H}$ given by (7.21), $[\hat{N}_\phi, \hat{H}] = 0$.

Solutions of problem 7.2(a):

The equation (7.23) is,

$$\hat{N}_\phi^\mu = i(\hat{\phi}^\dagger \partial^\mu \hat{\phi} - \hat{\phi}\partial^\mu \hat{\phi}^\dagger) \tag{7.23}$$

The equation (7.14) reads,

$$\hat{N}_\phi = \int \hat{N}_\phi^0 d^3\mathbf{x} \tag{7.14}$$

The equation (7.24) says,

$$\hat{N}_\phi = \int \frac{d^3\mathbf{k}}{(2\pi)^3}[\hat{a}^\dagger(k)\hat{a}(k) - \hat{b}^\dagger(k)\hat{b}(k)] \tag{7.24}$$

From (7.23),

$$\begin{aligned}\hat{N}_\phi^0 &= i(\hat{\phi}^\dagger \partial^0 \hat{\phi} - \hat{\phi}\partial^0 \hat{\phi}^\dagger) \\ &= i(\hat{\phi}^\dagger \frac{\partial}{\partial t}\hat{\phi} - \hat{\phi}\frac{\partial}{\partial t}\hat{\phi}^\dagger)\end{aligned}$$

From equation (7.16),

$$\hat{\phi} = \int \frac{d^3\mathbf{k}}{(2\pi^3)\sqrt{2\omega}}[\hat{a}(k)e^{-ik\cdot x} + \hat{b}^\dagger(k)e^{ik\cdot x}]$$

Accordingly,

$$\hat{\phi}^\dagger = \int \frac{d^3\mathbf{k}'}{(2\pi^3)\sqrt{2\omega}}[\hat{a}^\dagger(k')e^{ik'\cdot x} + \hat{b}(k')e^{-ik'\cdot x}]$$

And $\partial\hat{\phi}/\partial t$ is ,

$$\begin{aligned}\frac{\partial\hat{\phi}}{\partial t} &= \frac{\partial}{\partial t}\int \frac{d^3\mathbf{k}}{(2\pi^3)\sqrt{2\omega}}[\hat{a}(k)e^{-ik\cdot x} + \hat{b}^\dagger(k)e^{ik\cdot x}] \\ &= \int \frac{d^3\mathbf{k}}{(2\pi^3)\sqrt{2\omega}}\frac{\partial}{\partial t}[\hat{a}(k)e^{-ik\cdot x} + \hat{b}^\dagger(k)e^{ik\cdot x}] \\ &= \int \frac{d^3\mathbf{k}}{(2\pi^3)\sqrt{2\omega}}[\hat{a}(k)\frac{\partial}{\partial t}(e^{-ik\cdot x}) + \hat{b}^\dagger(k)\frac{\partial}{\partial t}(e^{ik\cdot x})] \\ &= \int \frac{d^3\mathbf{k}}{(2\pi^3)\sqrt{2\omega}}[-ik_0\hat{a}(k)e^{-ik\cdot x} + ik_0\hat{b}^\dagger(k)e^{ik\cdot x}] \\ &= -ik_0\int \frac{d^3\mathbf{k}}{(2\pi^3)\sqrt{2\omega}}[\hat{a}(k)e^{-ik\cdot x} - \hat{b}^\dagger(k)e^{ik\cdot x}]\end{aligned}$$

Similarly, $\partial\hat{\phi}^\dagger/\partial t$ is ,

$$\begin{aligned}\frac{\partial\hat{\phi}^\dagger}{\partial t} &= \frac{\partial}{\partial t}\int \frac{d^3\mathbf{k}'}{(2\pi^3)\sqrt{2\omega}}[\hat{a}^\dagger(k')e^{ik'\cdot x} + \hat{b}(k')e^{-ik'\cdot x}] \\ &= \int \frac{d^3\mathbf{k}'}{(2\pi^3)\sqrt{2\omega}}\frac{\partial}{\partial t}[\hat{a}^\dagger(k')e^{ik'\cdot x} + \hat{b}(k')e^{-ik'\cdot x}] \\ &= \int \frac{d^3\mathbf{k}'}{(2\pi^3)\sqrt{2\omega}}[\hat{a}^\dagger(k')\frac{\partial}{\partial t}(e^{ik'\cdot x}) + \hat{b}(k')\frac{\partial}{\partial t}(e^{-ik'\cdot x})] \\ &= \int \frac{d^3\mathbf{k}'}{(2\pi^3)\sqrt{2\omega}}[ik_0'\hat{a}^\dagger(k')e^{ik'\cdot x} - ik_0'\hat{b}(k')e^{-ik'\cdot x}] \\ &= ik_0'\int \frac{d^3\mathbf{k}'}{(2\pi^3)\sqrt{2\omega}}[\hat{a}^\dagger(k')e^{ik'\cdot x} - \hat{b}(k')e^{-ik'\cdot x}]\end{aligned}$$

Substituting the expressions of $\phi, \phi^\dagger, \partial\hat{\phi}/\partial t$ and $\partial\hat{\phi}^\dagger/\partial t$ into $\hat{N}^0_\phi = i(\hat{\phi}^\dagger \frac{\partial}{\partial t}\hat{\phi} - \hat{\phi}\frac{\partial}{\partial t}\hat{\phi}^\dagger)$, one gets,

$$
\begin{aligned}
\hat{N}^0_\phi &= i(\hat{\phi}^\dagger\partial^0\hat{\phi} - \hat{\phi}\partial^0\hat{\phi}^\dagger) \\
&= i(\{\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}[\hat{a}^\dagger(k')e^{ik'\cdot x} + \hat{b}(k')e^{-ik'\cdot x}]\}\{-ik_0\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{a}(k)e^{-ik\cdot x} - \hat{b}^\dagger(k)e^{ik\cdot x}]\} \\
&\quad - \{\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{a}(k)e^{-ik\cdot x} + \hat{b}^\dagger(k)e^{ik\cdot x}]\}\{ik'_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}[\hat{a}^\dagger(k')e^{ik'\cdot x} - \hat{b}(k')e^{-ik'\cdot x}]\}) \\
&= k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{a}^\dagger(k')e^{ik'\cdot x} + \hat{b}(k')e^{-ik'\cdot x}][\hat{a}(k)e^{-ik\cdot x} - \hat{b}^\dagger(k)e^{ik\cdot x}] \\
&\quad + k'_0\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}[\hat{a}(k)e^{-ik\cdot x} + \hat{b}^\dagger(k)e^{ik\cdot x}][\hat{a}^\dagger(k')e^{ik'\cdot x} - \hat{b}(k')e^{-ik'\cdot x}] \\
&= k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{a}^\dagger(k')\hat{a}(k)e^{i(k'-k)\cdot x} - \hat{a}^\dagger(k')\hat{b}^\dagger(k)e^{i(k'+k)\cdot x} + \hat{b}(k')\hat{a}(k)e^{-i(k'+k)\cdot x} - \hat{b}(k')\hat{b}^\dagger(k)e^{i(k-k')\cdot x}] \\
&\quad + k'_0\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}[\hat{a}(k)\hat{a}^\dagger(k')e^{i(k'-k)\cdot x} - \hat{a}(k)\hat{b}(k')e^{-i(k'+k)\cdot x} + \hat{b}^\dagger(k)\hat{a}^\dagger(k')e^{i(k'+k)\cdot x} - \hat{b}^\dagger(k)\hat{b}(k')e^{i(k-k')\cdot x}]
\end{aligned}
$$

(According to P_{161}, k_0, k'_0 is unrestricted, and $k_0 = k'_0 = z$. But $k' \neq k$, that's why we reached this step.)

$$
\begin{aligned}
&= k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}\{[\hat{a}^\dagger(k')\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k')]e^{i(k'-k)\cdot x} - [\hat{b}(k')\hat{b}^\dagger(k) + \hat{b}^\dagger(k)\hat{b}(k')]e^{i(k-k')\cdot x}\} \\
&\quad + k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}\{[\hat{b}^\dagger(k)\hat{a}^\dagger(k') - \hat{a}^\dagger(k')\hat{b}^\dagger(k)]e^{i(k'+k)\cdot x} + [\hat{b}(k')\hat{a}(k) - \hat{a}(k)\hat{b}(k')]e^{-i(k'+k)\cdot x}\}
\end{aligned}
$$

(the second term after "+" = 0 due to commutator relations)

$$
\begin{aligned}
&= k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}\{[\hat{a}^\dagger(k')\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k')]e^{i(k'-k)\cdot x} - [\hat{b}(k')\hat{b}^\dagger(k) + \hat{b}^\dagger(k)\hat{b}(k')]e^{i(k-k')\cdot x}\} \\
&= k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}\{[2\hat{a}^\dagger(k')\hat{a}(k) - [\hat{a}^\dagger(k'), \hat{a}(k)]]e^{i(k'-k)\cdot x} - [2\hat{b}^\dagger(k)\hat{b}(k') - [\hat{b}^\dagger(k), \hat{b}(k')]]e^{i(k-k')\cdot x}\} \\
&= k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[2\hat{a}^\dagger(k')\hat{a}(k)]e^{i(k'-k)\cdot x} - k_0\int \frac{d^3\mathbf{k}'}{(2\pi^3)}\int \frac{d^3\mathbf{k}}{(2\pi^3)}[\hat{a}^\dagger(k'), \hat{a}(k)]e^{i(k'-k)\cdot x} \\
&\quad - k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[2\hat{b}^\dagger(k)\hat{b}(k')]e^{i(k-k')\cdot x} + k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{b}^\dagger(k), \hat{b}(k')]e^{i(k'-k)\cdot x}
\end{aligned}
$$

(the second and fourth terms in above expression will cancel out, so only the first and third ones survived.)

$$
\begin{aligned}
&= 2k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{a}^\dagger(k')\hat{a}(k)]e^{i(k'-k)\cdot x} - 2k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{b}^\dagger(k)\hat{b}(k')]e^{i(k-k')\cdot x} \\
&= 2k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{a}^\dagger(k')\hat{a}(k)]e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} - 2k_0\int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{b}^\dagger(k)\hat{b}(k')]e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}.
\end{aligned}
$$

($k_0 = k'_0$, so $e^{i(k'_0-k_0)x_0} = e^{i(k_0-k'_0)x_0} = 1$. Only in 3 dimension form, $\int \frac{d^3\mathbf{k}'}{(2\pi)^3}e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} = \delta(\mathbf{x})$, Cf. (E.27))

Substituting this expression into equation (7.14), $\hat{N}_\phi = \int \hat{N}^0_\phi d^3\mathbf{x}$. One gets,

$$
\begin{aligned}
\hat{N}_\phi &= \int \hat{N}^0_\phi d^3\mathbf{x} \qquad (7.14)\\
&= \int \{2k_0 \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} [\hat{a}^\dagger(k')\hat{a}(k)] e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} - 2k_0 \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} [\hat{b}^\dagger(k)\hat{b}(k')] e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}]\} d^3\mathbf{x}\\
&= \{2k_0 \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} [\hat{a}^\dagger(k')\hat{a}(k)] \int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} d^3\mathbf{x}\}\\
&\quad - \{2k_0 \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} [\hat{b}^\dagger(k)\hat{b}(k')]\} \int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} d^3\mathbf{x}\\
&= \{2k_0 \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} [\hat{a}^\dagger(k')\hat{a}(k)]\delta(\mathbf{k}-\mathbf{k}')\} - \{2k_0 \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} [\hat{b}^\dagger(k)\hat{b}(k')]\}\delta(\mathbf{k}'-\mathbf{k}) \qquad (E.26)\\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3} [\hat{a}^\dagger(k)\hat{a}(k) - \hat{b}^\dagger(k)\hat{b}(k)]
\end{aligned}
$$

(This is just the (7.24), by considering $\mathbf{k} = \mathbf{k}'$, and by assuming $2k_0 = 2\omega$.)

Proof is almost robust yet there is one flaw need to be wiped off, that's the value of k_0, k'_0. According to P_{161}, authors emphasized that k_0 is not $(k^2 + m_C^2)^{1/2}$. But at the last step, to cancel the terms of $\sqrt{\omega}$, one has to set $k_0 = \omega$. It's sounds not so plausible...

Not really ! The point is in the page of P_{161}, it's interaction field where mass off-shell happens which means the mass is not conservative, so $k_0 \neq \sqrt{(k^2 + m_C^2)}$; while in this exercise the field is free field which essentially come from (7.1) and no any interaction at all, so it's mass on-shell therefore $k_0 = \sqrt{(k^2 + m_C^2)}$.

Part 2 of *problem* 7.2(*a*): it's to check that $[\hat{N}_\phi, \hat{H}] = 0$, with $\hat{H}$ given by (7.21).

The equation (7.21) reads,

$$
\hat{H} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} [\hat{a}^\dagger(k)\hat{a}(k) + \hat{b}^\dagger(k)\hat{b}(k)]\omega \qquad (7.21)
$$

So, $[\hat{N}_\phi, \hat{H}] = \hat{N}_\phi \hat{H} - \hat{H}\hat{N}_\phi$.

$$
\begin{aligned}
&\hat{N}_\phi \hat{H}\\
&= \int \frac{d^3\mathbf{k}'}{(2\pi)^3} [\hat{a}^\dagger(k')\hat{a}(k') - \hat{b}^\dagger(k')\hat{b}(k')] \int \frac{d^3\mathbf{k}}{(2\pi)^3} [\hat{a}^\dagger(k)\hat{a}(k) + \hat{b}^\dagger(k)\hat{b}(k)]\\
&= \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{(2\pi)^3} [\hat{a}^\dagger(k')\hat{a}(k') - \hat{b}^\dagger(k')\hat{b}(k')][\hat{a}^\dagger(k)\hat{a}(k) + \hat{b}^\dagger(k)\hat{b}(k)]\\
&= \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{(2\pi)^3} [\hat{a}^\dagger(k')\hat{a}(k')\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}^\dagger(k')\hat{a}(k')\hat{b}^\dagger(k)\hat{b}(k) - \hat{b}^\dagger(k')\hat{b}(k')\hat{a}^\dagger(k)\hat{a}(k) - \hat{b}^\dagger(k')\hat{b}(k')\hat{b}^\dagger(k)\hat{b}(k)] \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
&\hat{H}\hat{N}_\phi\\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3} [\hat{a}^\dagger(k)\hat{a}(k) + \hat{b}^\dagger(k)\hat{b}(k)] \int \frac{d^3\mathbf{k}'}{(2\pi)^3} [\hat{a}^\dagger(k')\hat{a}(k') - \hat{b}^\dagger(k')\hat{b}(k')]\\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} [\hat{a}^\dagger(k)\hat{a}(k) + \hat{b}^\dagger(k)\hat{b}(k)][\hat{a}^\dagger(k')\hat{a}(k') - \hat{b}^\dagger(k')\hat{b}(k')]\\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} [\hat{a}^\dagger(k)\hat{a}(k)\hat{a}^\dagger(k')\hat{a}(k') - \hat{a}^\dagger(k)\hat{a}(k)\hat{b}^\dagger(k')\hat{b}(k') + \hat{b}^\dagger(k)\hat{b}(k)\hat{a}^\dagger(k')\hat{a}(k') - \hat{b}^\dagger(k)\hat{b}(k)\hat{b}^\dagger(k')\hat{b}(k')] \qquad (3)
\end{aligned}
$$

From the comparison of (2) and (3), one can see the terms in (3) "corresponds" to the $1^{st}, 2^{nd}, 3^{rd}$ and 4^{th} of (2) are $1^{st}, 3^{rd}, 2^{nd}$ and 4^{th}. Taking the 1^{st} terms in both (2) and (3) as a example to prove, the other "matched" terms are similar.

The first term of (2) is,

$$
\begin{aligned}
&\hat{a}^{\dagger}(k')\hat{a}(k')\hat{a}^{\dagger}(k)\hat{a}(k) \\
&= \hat{a}^{\dagger}(k')\{[\hat{a}(k'), \hat{a}^{\dagger}(k)] + \hat{a}^{\dagger}(k)\hat{a}(k')\}\hat{a}(k) \\
&= \hat{a}^{\dagger}(k')[\hat{a}(k'), \hat{a}^{\dagger}(k)]\hat{a}(k) + \hat{a}^{\dagger}(k')\hat{a}^{\dagger}(k)\hat{a}(k')\hat{a}(k) \\
&= \hat{a}^{\dagger}(k')[\hat{a}(k'), \hat{a}^{\dagger}(k)]\hat{a}(k) + \hat{a}^{\dagger}(k)\hat{a}^{\dagger}(k')\hat{a}(k)\hat{a}(k') \\
&= \hat{a}^{\dagger}(k')[\hat{a}(k'), \hat{a}^{\dagger}(k)]\hat{a}(k) + \hat{a}^{\dagger}(k)\{[\hat{a}^{\dagger}(k'), \hat{a}(k)] + \hat{a}(k)\hat{a}^{\dagger}(k')\}\hat{a}(k') \\
&= \hat{a}^{\dagger}(k')[\hat{a}(k'), \hat{a}^{\dagger}(k)]\hat{a}(k) + \hat{a}^{\dagger}(k)[\hat{a}^{\dagger}(k'), \hat{a}(k)]\hat{a}(k') + \hat{a}^{\dagger}(k)\hat{a}(k)\hat{a}^{\dagger}(k')\hat{a}(k') \\
&= \hat{a}^{\dagger}(k')[\hat{a}(k'), \hat{a}^{\dagger}(k)]\hat{a}(k) - \hat{a}^{\dagger}(k)[\hat{a}(k), \hat{a}^{\dagger}(k')]\hat{a}(k') + \hat{a}^{\dagger}(k)\hat{a}(k)\hat{a}^{\dagger}(k')\hat{a}(k')
\end{aligned}
$$

Let integral acts on it,

$$
\begin{aligned}
&\int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \{\hat{a}^{\dagger}(k')[\hat{a}(k'), \hat{a}^{\dagger}(k)]\hat{a}(k) - \hat{a}^{\dagger}(k)[\hat{a}(k), \hat{a}^{\dagger}(k')]\hat{a}(k') + \hat{a}^{\dagger}(k)\hat{a}(k)\hat{a}^{\dagger}(k')\hat{a}(k')\} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \{\hat{a}^{\dagger}(k')[\hat{a}(k'), \hat{a}^{\dagger}(k)]\hat{a}(k)\} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \{\hat{a}^{\dagger}(k)[\hat{a}(k), \hat{a}^{\dagger}(k')]\hat{a}(k')\} \\
&\quad + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \{\hat{a}^{\dagger}(k)\hat{a}(k)\hat{a}^{\dagger}(k')\hat{a}(k')\} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \{\hat{a}^{\dagger}(k')[(2\pi)^3\delta^3(\mathbf{k}' - \mathbf{k})]\hat{a}(k)\} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \{\hat{a}^{\dagger}(k)[(2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}')]\hat{a}(k')\} \\
&\quad + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \{\hat{a}^{\dagger}(k)\hat{a}(k)\hat{a}^{\dagger}(k')\hat{a}(k')\} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \{\hat{a}^{\dagger}(k)\hat{a}(k)\} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \{\hat{a}^{\dagger}(k)\hat{a}(k)\} + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \{\hat{a}^{\dagger}(k)\hat{a}(k)\hat{a}^{\dagger}(k')\hat{a}(k')\} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \{\hat{a}^{\dagger}(k)\hat{a}(k)\hat{a}^{\dagger}(k')\hat{a}(k')\}
\end{aligned}
$$

This is exactly the first term in (3). And other "matched " terms could be proved in the similar way. As a result, $\hat{N}_\phi \hat{H} = \hat{H}\hat{N}_\phi$. And finally, $[\hat{N}_\phi, \hat{H}] = \hat{N}_\phi \hat{H} - \hat{H}\hat{N}_\phi = 0$.

proof of problem 7.2.(a) is done.

Problem 7.3 Show that $[\hat{\phi}(x_1), \hat{\phi}^\dagger(x_2)] = 0 \quad for\ (x_1 - x_2)^2 < 0.$

[Hint: insert expression (7.16) for the $\hat{\phi}$'s and use the commutation relations (7.18) to express the commutator as the difference of two integrals; in the second integral, $x_1 - x_2$ can be transformed to $-(x_1 - x_2)$ by a Lorentz transformation - the time-ordering of space-like separated events is frame-depement !].

Solution of Problem 7.3:

$$[\hat{\phi}(x_1), \hat{\phi}^\dagger(x_2)] = \hat{\phi}(x_1)\hat{\phi}^\dagger(x_2) - \hat{\phi}^\dagger(x_2)\hat{\phi}(x_1)$$

And from equation (7.16),

$$\hat{\phi}(x_1) = \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{a}(k)e^{-ik\cdot x_1} + \hat{b}^\dagger(k)e^{ik\cdot x_1}]$$

$$\hat{\phi}^\dagger(x_2) = \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}}[\hat{a}^\dagger(k')e^{ik'\cdot x_2} + \hat{b}(k')e^{-ik'\cdot x_2}]$$

So,

$$\begin{aligned}
\hat{\phi}(x_1)\hat{\phi}^\dagger(x_2) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{a}(k)e^{-ik\cdot x_1} + \hat{b}^\dagger(k)e^{ik\cdot x_1}] \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}}[\hat{a}^\dagger(k')e^{ik'\cdot x_2} + \hat{b}(k')e^{-ik'\cdot x_2}] \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}}[\hat{a}(k)e^{-ik\cdot x_1} + \hat{b}^\dagger(k)e^{ik\cdot x_1}][\hat{a}^\dagger(k')e^{ik'\cdot x_2} + \hat{b}(k')e^{-ik'\cdot x_2}] \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}}[\hat{a}(k)\hat{a}^\dagger(k')e^{i(-k\cdot x_1 + k'\cdot x_2)} + \hat{a}(k)\hat{b}(k')e^{-i(k\cdot x_1 + k'\cdot x_2)} \\
&\qquad + \hat{b}^\dagger(k)\hat{a}^\dagger(k')e^{i(k\cdot x_1 + k'\cdot x_2)} + \hat{b}^\dagger(k)\hat{b}(k')e^{i(k\cdot x_1 - k'\cdot x_2)}]
\end{aligned}$$

$$\begin{aligned}
\hat{\phi}^\dagger(x_2)\hat{\phi}(x_1) &= \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}}[\hat{a}^\dagger(k')e^{ik'\cdot x_2} + \hat{b}(k')e^{-ik'\cdot x_2}] \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{a}(k)e^{-ik\cdot x_1} + \hat{b}^\dagger(k)e^{ik\cdot x_1}] \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}}[\hat{a}^\dagger(k')e^{ik'\cdot x_2} + \hat{b}(k')e^{-ik'\cdot x_2}][\hat{a}(k)e^{-ik\cdot x_1} + \hat{b}^\dagger(k)e^{ik\cdot x_1}] \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}}[\hat{a}^\dagger(k')\hat{a}(k)e^{i(k'\cdot x_2 - k\cdot x_1)} + \hat{a}^\dagger(k')\hat{b}^\dagger(k)e^{i(k'\cdot x_2 + k\cdot x_1)} \\
&\qquad + \hat{b}(k')\hat{a}(k)e^{-i(k'\cdot x_2 + k\cdot x_1)} + \hat{b}(k')\hat{b}^\dagger(k)e^{i(-k'\cdot x_2 + k\cdot x_1)}]
\end{aligned}$$

So,

$$\begin{aligned}
&[\hat{\phi}(x_1), \hat{\phi}^\dagger(x_2)] \\
&= \hat{\phi}(x_1)\hat{\phi}^\dagger(x_2) - \hat{\phi}^\dagger(x_2)\hat{\phi}(x_1) \\
&= \int\int [\hat{a}(k)\hat{a}^\dagger(k')e^{i(-k\cdot x_1 + k'\cdot x_2)} + \hat{a}(k)\hat{b}(k')e^{-i(k\cdot x_1 + k'\cdot x_2)} + \hat{b}^\dagger(k)\hat{a}^\dagger(k')e^{i(k\cdot x_1 + k'\cdot x_2)} + \hat{b}^\dagger(k)\hat{b}(k')e^{i(k\cdot x_1 - k'\cdot x_2)}] \\
&- \int\int [\hat{a}^\dagger(k')\hat{a}(k)e^{i(k'\cdot x_2 - k\cdot x_1)} + \hat{a}^\dagger(k')\hat{b}^\dagger(k)e^{i(k'\cdot x_2 + k\cdot x_1)} + \hat{b}(k')\hat{a}(k)e^{-i(k'\cdot x_2 + k\cdot x_1)} + \hat{b}(k')\hat{b}^\dagger(k)e^{i(-k'\cdot x_2 + k\cdot x_1)}] \\
&= \int\int [\hat{a}(k)\hat{a}^\dagger(k') - \hat{a}^\dagger(k')\hat{a}(k)]e^{i(-k\cdot x_1 + k'\cdot x_2)} - \int\int [\hat{b}(k')\hat{b}^\dagger(k) - \hat{b}^\dagger(k)\hat{b}(k')]e^{i(k\cdot x_1 - k'\cdot x_2)} \\
&= \int\int [\hat{a}(k), \hat{a}^\dagger(k')]e^{i(-k\cdot x_1 + k'\cdot x_2)} - \int\int [\hat{b}(k'), \hat{b}^\dagger(k)]e^{i(k\cdot x_1 - k'\cdot x_2)} \\
&= \int\int (2\pi)^3\delta^3(k - k')e^{i(-k\cdot x_1 + k'\cdot x_2)} - \int\int (2\pi)^3\delta^3(k - k')e^{i(k\cdot x_1 - k'\cdot x_2)} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}e^{ik\cdot(-x_1 + x_2)} - \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}e^{ik\cdot(x_1 - x_2)}
\end{aligned}$$

According to the hint, " in the second integral, $x_1 - x_2$ can be transformed to $-(x_1 - x_2)$ by a Lorentz transformation - the time-ordering of space-like separated events is frame-depedent ! " As a result,

$$\begin{aligned}
[\hat{\phi}(x_1), \hat{\phi}^\dagger(x_2)] &= \hat{\phi}(x_1)\hat{\phi}^\dagger(x_2) - \hat{\phi}^\dagger(x_2)\hat{\phi}(x_1) \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} e^{ik\cdot(-x_1+x_2)} - \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} e^{ik\cdot(x_1-x_2)} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} e^{ik\cdot(-x_1+x_2)} - \int \frac{d^3(-\mathbf{k})}{(2\pi)^3\sqrt{2\omega}} e^{i(-k)\cdot[-(x_1-x_2)]} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} e^{ik\cdot(-x_1+x_2)} - \int \frac{d^3(\mathbf{k}_c)}{(2\pi)^3\sqrt{2\omega}} e^{i(k_c)\cdot[-(x_1-x_2)]} \qquad \text{(Change } -k \text{ to } k_c\text{)} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} e^{ik\cdot(-x_1+x_2)} - \int \frac{d^3(\mathbf{k}_c)}{(2\pi)^3\sqrt{2\omega}} e^{i(k_c)\cdot(-x_1+x_2)} \\
&= 0 \qquad \text{(Two terms are same. So, \textit{proof is done.})}
\end{aligned}$$

Mathematically, there is another way to prove that $[\hat{\phi}(x_1), \hat{\phi}^\dagger(x_2)] = 0$ at this stage.

$$\begin{aligned}
[\hat{\phi}(x_1), \hat{\phi}^\dagger(x_2)] &= \hat{\phi}(x_1)\hat{\phi}^\dagger(x_2) - \hat{\phi}^\dagger(x_2)\hat{\phi}(x_1) \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} e^{ik\cdot(-x_1+x_2)} - \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} e^{ik\cdot(x_1-x_2)} \\
&= \frac{\delta(x_1-x_2)}{\sqrt{2\omega}} - \frac{\delta(x_2-x_1)}{\sqrt{2\omega}} \qquad \text{(Cf. (E.27))} \\
&= \frac{\delta(x_1-x_2)}{\sqrt{2\omega}} - \frac{\delta(x_1-x_2)}{\sqrt{2\omega}}. \qquad (\delta \text{ function is even function. Cf. (E.32))} \\
&= 0 \qquad \text{(\textit{proof is done})}
\end{aligned}$$

Actually, in the process of proof, all of the k in the expression of $ik \cdot (x_1 - x_2)$ are actually $\mathbf{k}$ because the integral is $\int d^3\mathbf{k}$, not $\int d^4\mathbf{k}$(under this context, it's k of course because this k represents four-momentum).

Be attention what this problem wants to express : In QFT, "causality" requires that every particle has to have a corresponding anti-particle, with the same mass and opposite quantum numbers. Cf. P_{181}

However, one has to be ware also, this conclusion rooted from the definition of complex field equation (7.15) and the followed expression of $\hat{\phi}, \hat{\phi}^\dagger$.

Problem 7.4 Verify that varing $\psi^\dagger$ in the action principle with Lagrangian (7.33) gives the Dirac equation.

Proof of *Problem 7.4*:

Equation (7.33) reads, $\mathcal{L}_D = i\psi^\dagger\dot{\psi} + i\psi^\dagger\alpha\cdot\nabla\psi - m\psi^\dagger\beta\psi$.

So(refer to equation (5.87) in P_{127} and the comments above it),

$$\frac{\partial(\mathcal{L}_D)}{\partial\psi} = -m\psi^\dagger\beta$$

$$\frac{\partial(\mathcal{L}_D)}{\partial(\nabla\psi)} = i\psi^\dagger\alpha$$

$$\frac{\partial(\mathcal{L}_D)}{\partial\dot{\psi}} = i\psi^\dagger$$

From equation (5.89) in P_{128},

$$\frac{\partial\mathcal{L}}{\partial\phi} - \nabla\cdot(\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}) - \frac{\partial}{\partial t}(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}) = 0$$

So, in this case,

$$\begin{aligned}
&\frac{\partial(\mathcal{L}_D)}{\partial\psi} - \nabla\cdot(\frac{\partial(\mathcal{L}_D)}{\partial(\nabla\psi)}) - \frac{\partial}{\partial t}(\frac{\partial(\mathcal{L}_D)}{\partial\dot{\psi}}) = 0 \\
&\Longrightarrow -m\psi^\dagger\beta - \nabla\cdot(i\psi^\dagger\alpha) - \frac{\partial}{\partial t}(i\psi^\dagger) = 0 \\
&\Longrightarrow -i\frac{\partial\psi^\dagger}{\partial t} = \psi^\dagger m\beta + i\nabla\cdot(\psi^\dagger\alpha) \\
&\Longrightarrow -i\frac{\partial\psi^\dagger}{\partial t} = \psi^\dagger(\beta m + i\alpha\cdot\overleftarrow{\nabla}). \qquad \text{(It's Hermitian conjugate is } i\partial\psi/\partial t = (-i\alpha\cdot\nabla + \beta m)\psi\text{)}
\end{aligned}$$

That's exactly Dirac equation. *Proof is done.*

Problem 7.5 Verify equation (7.44) .

Proof of *Problem 7.5*: Equation (7.44) reads, $\{\hat{\psi}_\alpha(x,t), \hat{\psi}^\dagger_\beta(y,t)\} = \delta(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}$

From equation (7.35),

$$\hat{\psi}_\alpha(x,t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{c}_\alpha(k)u(k,\alpha)e^{-ik\cdot x} + \hat{d}^\dagger_\alpha(k)v(k,\alpha)e^{ik\cdot x}]$$

$$\hat{\psi}^\dagger_\beta(y,t) = \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}}[\hat{c}^\dagger_\beta(k')u^\dagger(k',\beta)e^{ik'\cdot y} + \hat{d}_\beta(k')v^\dagger(k',\beta)e^{-ik'\cdot y}]$$

And $\{\hat{\psi}_\alpha(x,t), \hat{\psi}^\dagger_\beta(y,t)\} = \hat{\psi}_\alpha(x,t)\hat{\psi}^\dagger_\beta(y,t) + \hat{\psi}^\dagger_\beta(y,t)\hat{\psi}_\alpha(x,t)$

$$\begin{aligned}
&\hat{\psi}_\alpha(x,t)\hat{\psi}^\dagger_\beta(y,t)\\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{c}_\alpha(k)u(k,\alpha)e^{-ik\cdot x} + \hat{d}^\dagger_\alpha(k)v(k,\alpha)e^{ik\cdot x}] \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}}[\hat{c}^\dagger_\beta(k')u^\dagger(k',\beta)e^{ik'\cdot y} + \hat{d}_\beta(k')v^\dagger(k',\beta)e^{-ik'\cdot y}]\\
&= \int\int [\hat{c}_\alpha(k)u(k,\alpha)e^{-ik\cdot x} + \hat{d}^\dagger_\alpha(k)v(k,\alpha)e^{ik\cdot x}][\hat{c}^\dagger_\beta(k')u^\dagger(k',\beta)e^{ik'\cdot y} + \hat{d}_\beta(k')v^\dagger(k',\beta)e^{-ik'\cdot y}]\\
&= \int\int [\hat{c}_\alpha(k)u(k,\alpha)e^{-ik\cdot x}\hat{c}^\dagger_\beta(k')u^\dagger(k',\beta)e^{ik'\cdot y} + \hat{c}_\alpha(k)u(k,\alpha)e^{-ik\cdot x}\hat{d}_\beta(k')v^\dagger(k',\beta)e^{-ik'\cdot y}]\\
&\qquad + [\hat{d}^\dagger_\alpha(k)v(k,\alpha)e^{ik\cdot x}\hat{c}^\dagger_\beta(k')u^\dagger(k',\beta)e^{ik'\cdot y} + \hat{d}^\dagger_\alpha(k)v(k,\alpha)e^{ik\cdot x}\hat{d}_\beta(k')v^\dagger(k',\beta)e^{-ik'\cdot y}]\\
&= \int\int [\hat{c}_\alpha(k)\hat{c}^\dagger_\beta(k')u(k,\alpha)u^\dagger(k',\beta)e^{i(-k\cdot x+k'\cdot y)} + \hat{c}_\alpha(k)\hat{d}_\beta(k')u(k,\alpha)v^\dagger(k',\beta)e^{-i(k\cdot x+k'\cdot y)}]\\
&\qquad + [\hat{d}^\dagger_\alpha(k)\hat{c}^\dagger_\beta(k')v(k,\alpha)u^\dagger(k',\beta)e^{i(k\cdot x+k'\cdot y)} + \hat{d}^\dagger_\alpha(k)\hat{d}_\beta(k')v(k,\alpha)v^\dagger(k',\beta)e^{i(k\cdot x-k'\cdot y)}]
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\hat{\psi}^\dagger_\beta(y,t)\hat{\psi}_\alpha(x,t)\\
&= \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}}[\hat{c}^\dagger_\beta(k')u^\dagger(k',\beta)e^{ik'\cdot y} + \hat{d}_\beta(k')v^\dagger(k',\beta)e^{-ik'\cdot y}] \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}}[\hat{c}_\alpha(k)u(k,\alpha)e^{-ik\cdot x} + \hat{d}^\dagger_\alpha(k)v(k,\alpha)e^{ik\cdot x}]\\
&= \int\int [\hat{c}^\dagger_\beta(k')u^\dagger(k',\beta)e^{ik'\cdot y} + \hat{d}_\beta(k')v^\dagger(k',\beta)e^{-ik'\cdot y}][\hat{c}_\alpha(k)u(k,\alpha)e^{-ik\cdot x} + \hat{d}^\dagger_\alpha(k)v(k,\alpha)e^{ik\cdot x}]\\
&= \int\int [\hat{c}^\dagger_\beta(k')u^\dagger(k',\beta)e^{ik'\cdot y}\hat{c}_\alpha(k)u(k,\alpha)e^{-ik\cdot x} + \hat{c}^\dagger_\beta(k')u^\dagger(k',\beta)e^{ik'\cdot y}\hat{d}^\dagger_\alpha(k)v(k,\alpha)e^{ik\cdot x}]\\
&\qquad + [\hat{d}_\beta(k')v^\dagger(k',\beta)e^{-ik'\cdot y}\hat{c}_\alpha(k)u(k,\alpha)e^{-ik\cdot x} + \hat{d}_\beta(k')v^\dagger(k',\beta)e^{-ik'\cdot y}\hat{d}^\dagger_\alpha(k)v(k,\alpha)e^{ik\cdot x}]\\
&= \int\int [\hat{c}^\dagger_\beta(k')\hat{c}_\alpha(k)u^\dagger(k',\beta)u(k,\alpha)e^{i(k'\cdot y-k\cdot x)} + \hat{c}^\dagger_\beta(k')\hat{d}^\dagger_\alpha(k)u^\dagger(k',\beta)v(k,\alpha)e^{i(k'\cdot y+k\cdot x)}]\\
&\qquad + [\hat{d}_\beta(k')\hat{c}_\alpha(k)v^\dagger(k',\beta)u(k,\alpha)e^{-i(k'\cdot y+k\cdot x)} + \hat{d}_\beta(k')\hat{d}^\dagger_\alpha(k)v^\dagger(k',\beta)v(k,\alpha)e^{i(-k'\cdot y+k\cdot x)})]
\end{aligned}$$

So,

$$\begin{aligned}
&\{\hat{\psi}_\alpha(x,t), \hat{\psi}^\dagger_\beta(y,t)\}\\
&= \hat{\psi}_\alpha(x,t)\hat{\psi}^\dagger_\beta(y,t) + \hat{\psi}^\dagger_\beta(y,t)\hat{\psi}_\alpha(x,t)\\
&= \int\int [\hat{c}_\alpha(k)\hat{c}^\dagger_\beta(k')u(k,\alpha)u^\dagger(k',\beta) + \hat{c}^\dagger_\beta(k')\hat{c}_\alpha(k)u^\dagger(k',\beta)u(k,\alpha)]e^{i(-k\cdot x+k'\cdot y)}\\
&\qquad + [\hat{c}_\alpha(k)\hat{d}_\beta(k')u(k,\alpha)v^\dagger(k',\beta) + \hat{d}_\beta(k')\hat{c}_\alpha(k)v^\dagger(k',\beta)u(k,\alpha)]e^{-i(k\cdot x+k'\cdot y)}\\
&\qquad + [\hat{d}^\dagger_\alpha(k)\hat{c}^\dagger_\beta(k')v(k,\alpha)u^\dagger(k',\beta) + \hat{c}^\dagger_\beta(k')\hat{d}^\dagger_\alpha(k)u^\dagger(k',\beta)v(k,\alpha)]e^{i(k\cdot x+k'\cdot y)}\\
&\qquad + [\hat{d}^\dagger_\alpha(k)\hat{d}_\beta(k')v(k,\alpha)v^\dagger(k',\beta) + \hat{d}_\beta(k')\hat{d}^\dagger_\alpha(k)v^\dagger(k',\beta)v(k,\alpha)]e^{i(-k'\cdot y+k\cdot x)}
\end{aligned}$$

As spinors, u and v, $u \cdot v^\dagger = v^\dagger \cdot u = E_{uv}$; $u \cdot u^\dagger = u^\dagger \cdot u = E_u$; $v \cdot v^\dagger = v^\dagger \cdot v = E_v$. So, above equation could simplified as

$$\begin{aligned}
&= \int\int [\hat{c}_\alpha(k)\hat{c}^\dagger_\beta(k') + \hat{c}^\dagger_\beta(k')\hat{c}_\alpha(k)]E_u e^{i(-k\cdot x+k'\cdot y)} + [\hat{c}_\alpha(k)\hat{d}_\beta(k') + \hat{d}_\beta(k')\hat{c}_\alpha(k)]E_{uv} e^{-i(k\cdot x+k'\cdot y)} \\
&\quad + [\hat{d}^\dagger_\alpha(k)\hat{c}^\dagger_\beta(k') + \hat{c}^\dagger_\beta(k')\hat{d}^\dagger_\alpha(k)]E_{uv} e^{i(k\cdot x+k'\cdot y)} + [\hat{d}^\dagger_\alpha(k)\hat{d}_\beta(k') + \hat{d}_\beta(k')\hat{d}^\dagger_\alpha(k)]E_v e^{i(-k'\cdot y+k\cdot x)} \\
&= \int\int [\hat{c}_\alpha(k)\hat{c}^\dagger_\beta(k') + \hat{c}^\dagger_\beta(k')\hat{c}_\alpha(k)]E_u e^{i(-k\cdot x+k'\cdot y)} + [\hat{d}^\dagger_\alpha(k)\hat{d}_\beta(k') + \hat{d}_\beta(k')\hat{d}^\dagger_\alpha(k)]E_v e^{i(-k'\cdot y+k\cdot x)} \\
&+ \int\int [\hat{c}_\alpha(k)\hat{d}_\beta(k') + \hat{d}_\beta(k')\hat{c}_\alpha(k)]E_{uv} e^{-i(k\cdot x+k'\cdot y)} + [\hat{d}^\dagger_\alpha(k)\hat{c}^\dagger_\beta(k') + \hat{c}^\dagger_\beta(k')\hat{d}^\dagger_\alpha(k)]E_{uv} e^{i(k\cdot x+k'\cdot y)} \\
&= \int\int \{\hat{c}_\alpha(k), \hat{c}^\dagger_\beta(k')\}E_u e^{i(-k\cdot x+k'\cdot y)} + \{\hat{d}^\dagger_\alpha(k), \hat{d}_\beta(k')\}E_v e^{i(-k'\cdot y+k\cdot x} \\
&+ \int\int \{\hat{c}_\alpha(k), \hat{d}_\beta(k')\}E_{uv} e^{-i(k\cdot x+k'\cdot y)} + \{\hat{d}^\dagger_\alpha(k), \hat{c}^\dagger_\beta(k')\}E_{uv} e^{i(k\cdot x+k'\cdot y)} \qquad \text{(Discard this line due to (7.41).)} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}} \{\hat{c}_\alpha(k), \hat{c}^\dagger_\beta(k')\}E_u e^{i(-k\cdot x+k'\cdot y)} + \{\hat{d}^\dagger_\alpha(k), \hat{d}_\beta(k')\}E_v e^{i(-k'\cdot y+k\cdot x} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} \int \frac{d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega'}} [(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\alpha\beta}E_u e^{i(-k\cdot x+k'\cdot y)} + (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\alpha\beta}E_v e^{i(-k'\cdot y+k\cdot x)}] \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega} 2 \times \delta_{\alpha\beta} \times E_u \times e^{ik\cdot(-x+y)} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\delta_{\alpha\beta} \times e^{ik\cdot(-x+y)} \qquad (\omega = E_u = E_v) \\
&= \delta(-\mathbf{x}+\mathbf{y})\delta_{\alpha\beta} \qquad \text{(E.27)} \\
&= \delta(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta} \qquad (\delta \text{ function is even function. Cf. (E.32))}
\end{aligned}$$

Proof is done.

Problem 7.7 Verify equation (7.59).

Equation (7.59) reads,

$$(\not{k} - m)(\not{k} + m) = (k^2 - m^2)$$

where, $\not{k} = \gamma^\mu k_\mu$, and $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$.

Proof of problem 7.7:

$$\begin{aligned}
&(\not{k} - m)(\not{k} + m)\\
&= (\gamma^\mu k_\mu - m)(\gamma^\mu k_\mu + m)\\
&= (\gamma^0 k_0 - \boldsymbol{\gamma}\mathbf{k} - m)(\gamma^0 k_0 - \boldsymbol{\gamma}\mathbf{k} + m)\\
&= [(\gamma^0 k_0 - m) - \boldsymbol{\gamma}\mathbf{k}][(\gamma^0 k_0 + m) - \boldsymbol{\gamma}\mathbf{k}]
\end{aligned}$$

$$= (\begin{pmatrix} k^0 & 0 \\ 0 & -k^0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} - \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{k} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 \end{pmatrix})(\begin{pmatrix} k^0 & 0 \\ 0 & -k^0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} - \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{k} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 \end{pmatrix})$$

$$= \begin{pmatrix} k^0 - m & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & -k^0 - m \end{pmatrix}\begin{pmatrix} k^0 + m & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & -k^0 + m \end{pmatrix}$$

$$= \begin{pmatrix} (k^0 - m)(k^0 + m) - (\boldsymbol{\sigma}\cdot\boldsymbol{k})^2 & -(k^0 - m)\boldsymbol{\sigma}\cdot\boldsymbol{k} - \boldsymbol{\sigma}\cdot\boldsymbol{k}(-k^0 + m) \\ \boldsymbol{\sigma}\cdot\boldsymbol{k}(k^0 + m) + (-k^0 - m)\boldsymbol{\sigma}\cdot\boldsymbol{k} & -(\boldsymbol{\sigma}\cdot\boldsymbol{k})^2 + (-k^0 - m)(-k^0 + m) \end{pmatrix}$$

$$= \begin{pmatrix} (k^0)^2 - m^2 - (\boldsymbol{\sigma}\cdot\boldsymbol{k})^2 & 0 \\ 0 & (k^0)^2 - m^2 - (\boldsymbol{\sigma}\cdot\boldsymbol{k})^2 \end{pmatrix} = \begin{pmatrix} k^2 - m^2 & 0 \\ 0 & k^2 - m^2 \end{pmatrix} = (k^2 - m^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= k^2 - m^2$$

proof is done

Problem 7.8 Verify the expression given in (7.61) for $\sum_s u(k,s)\bar{u}(k,s)$.[Hint: first, note that u is a four-component Dirac spinor arranged as a column, while $\bar{u}$ is another four-component spinor but this time arranged as a row because of the transpose in the † symbol. So, '$u\bar{u}$' has the form

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \begin{pmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \end{pmatrix} = \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 & \dots \\ u_2\bar{u}_1 & u_2\bar{u}_2 & \dots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

and is therefore a 4×4 matrix. use the expression (4.105) for the u's, and take

$$\phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}]$$

Verify that

$$\phi^1\phi^{1\dagger} + \phi^2\phi^{2\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly, verify that the expression for $\sum_s v(k,s)\bar{v}(k,s)$.

Solution of problem 7.8:

$$\phi^1\phi^{1\dagger} + \phi^2\phi^{2\dagger} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Next, proof equation (7.61) which based on the expression (4.105) as following shown,

$$u(k,s) = (E+m)^{1/2}\begin{pmatrix} \phi^s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^s \end{pmatrix} \quad s = 1, 2$$

So,

$$u(k,1) = (E+m)^{1/2}\begin{pmatrix} \phi^1 \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^1 \end{pmatrix} \qquad \bar{u}(k,1) = (E+m)^{1/2}\begin{pmatrix} \phi^{1\dagger} & -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^{1\dagger} \end{pmatrix}$$

$$u(k,2) = (E+m)^{1/2}\begin{pmatrix} \phi^2 \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^2 \end{pmatrix} \qquad \bar{u}(k,2) = (E+m)^{1/2}\begin{pmatrix} \phi^{2\dagger} & -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^{2\dagger} \end{pmatrix}$$

Then,

$$u(k,1)\bar{u}(k,1) = (E{+}m)\begin{pmatrix} \phi^1 \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^1 \end{pmatrix}\begin{pmatrix} \phi^{1\dagger} & -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^{1\dagger} \end{pmatrix} = (E{+}m)\begin{pmatrix} \phi^1\phi^{1\dagger} & -\phi^1\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^{1\dagger} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^1\phi^{1\dagger} & -(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m})\phi^1(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m})\phi^{1\dagger} \end{pmatrix}$$

$$= (E+m)\begin{pmatrix} \phi^1\phi^{1\dagger} & -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^1\phi^{1\dagger} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^1\phi^{1\dagger} & -(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m})^2\phi^1\phi^{1\dagger} \end{pmatrix} \qquad \text{(noticed } \phi_1 \text{ is a vector)}$$

$$u(k,2)\bar{u}(k,2) = (E{+}m)\begin{pmatrix} \phi^2 \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^2 \end{pmatrix}\begin{pmatrix} \phi^{2\dagger} & -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^{2\dagger} \end{pmatrix} = (E{+}m)\begin{pmatrix} \phi^2\phi^{2\dagger} & -\phi^2\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^{2\dagger} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^2\phi^{2\dagger} & -(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m})\phi^2(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m})\phi^{2\dagger} \end{pmatrix}$$

$$= (E+m)\begin{pmatrix} \phi^2\phi^{2\dagger} & -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^2\phi^{2\dagger} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^2\phi^{2\dagger} & -(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m})^2\phi^2\phi^{2\dagger} \end{pmatrix} \qquad \text{(noticed } \phi_2 \text{ is a vector)}$$

Substituting the expression of $u(k,1)\bar{u}(k,1)$ and $u(k,2)\bar{u}(k,2)$ into $\sum_s u(k,s)\bar{u}(k,s)$, one gets,

$$\sum_s u(k,s)\bar{u}(k,s) = u(k,1)\bar{u}(k,1) + u(k,2)\bar{u}(k,2)$$

$$= (E+m)\begin{pmatrix} \phi^1\phi^{1\dagger} + \phi^2\phi^{2\dagger} & -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}(\phi^1\phi^{1\dagger} + \phi^2\phi^{2\dagger}) \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}(\phi^1\phi^{1\dagger} + \phi^2\phi^{2\dagger}) & -(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m})^2(\phi^1\phi^{1\dagger} + \phi^2\phi^{2\dagger}) \end{pmatrix}$$

$$= \begin{pmatrix} (E+m)(\phi^1\phi^{1\dagger} + \phi^2\phi^{2\dagger}) & -\boldsymbol{\sigma}\cdot\boldsymbol{k}(\phi^1\phi^{1\dagger} + \phi^2\phi^{2\dagger}) \\ \boldsymbol{\sigma}\cdot\boldsymbol{k}(\phi^1\phi^{1\dagger} + \phi^2\phi^{2\dagger}) & -\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k})^2}{E+m}(\phi^1\phi^{1\dagger} + \phi^2\phi^{2\dagger}) \end{pmatrix} = \begin{pmatrix} E+m & 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 \\ 0 & E+m & 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 & -\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k})^2}{E+m} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 & -\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k})^2}{E+m} \end{pmatrix}$$

$$= \begin{pmatrix} E+m & 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 \\ 0 & E+m & 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 & -(E+m) & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 & -(E+m) \end{pmatrix} \qquad ((\boldsymbol{\sigma}\cdot\boldsymbol{k})^2 = \boldsymbol{k}^2 = E^2 - m^2.\text{Cf equation (4.47)})$$

$$= E\gamma^0 - \boldsymbol{k}\cdot\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} + m$$

$$= k^0\gamma^0 - \boldsymbol{k}\cdot\boldsymbol{\gamma} + m$$

$$= (\not{k} + m)$$

In the case of $v(k,s)$, the procedure is very similar, so it's omitted here.

Proof is done.

Problem 7.10 Verify that if $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_{em}A_\mu$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the Euler-Lagrangian equations for A_μ yield the Maxwell form

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\nu_{em}.$$

[*Hint*: it is helpful to use antisymmetry of $F_{\mu\nu}$ to rewrite the '$F \cdot F$' term as $-\frac{1}{2}F_{\mu\nu}\partial^\mu A^\nu$.]

Solution:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_{em}A_\mu \\
&= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - j_{em}A_\mu \\
&= -\frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu) - j_{em}A_\mu \\
&= -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) - j_{em}A_\mu \qquad \text{(the two positive/negative terms are same)}
\end{aligned}$$

Substituting this Lagangian into the Euler-Lagrange equation of motion for a field :

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0$$

One gets,

$$\begin{aligned}
&\partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) - (-j_{em}) = 0. \\
\Rightarrow\ & -\partial_\nu \partial^\nu A^\mu + \partial_\nu \partial^\mu A^\nu - (-j_{em}) = 0 \\
\Rightarrow\ & \Box A^\mu - \partial_\nu \partial^\mu A^\nu = j_{em} \\
\Rightarrow\ & \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j_{em}
\end{aligned}$$

Proof is done

Problem 7.11

(a) Show that the Fourier transform of the free-field equation for A_μ(i.e. the one in the previous question with j^{em}_μ set to zero) is given by (7.87).

Solutions of problem 7.11 (a),

The equation (7.87) reads as,

$$(-k^2 g^{\nu\mu} + k^\nu k^\mu)\tilde{A}_\mu(k) \equiv M^{\nu\mu}\tilde{A}_\mu(k) = 0 \tag{7.87}$$

The free-field equation for A_μ is : $\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$.

The Fourier transform of "free-field equation for A_μ" is , $\int[\Box A^\nu - \partial^\nu(\partial_\mu A^\mu)]e^{ik\cdot x}d^4x = 0$.

$$\begin{aligned}
&\int[\Box A^\nu - \partial^\nu(\partial_\mu A^\mu)]e^{ik\cdot x}d^4x = 0\\
\Rightarrow&\int[\partial_\mu(\partial^\mu A^\nu) - \partial_\mu(\partial^\nu A^\mu)]e^{ik\cdot x}d^4x = 0\\
\Rightarrow&\int[\partial_\mu(\partial^\mu A^\nu)]e^{ik\cdot x}d^4x - \int[\partial_\mu(\partial^\nu A^\mu)]e^{ik\cdot x}d^4x = 0\\
\Rightarrow&\partial_\mu\int(\partial^\mu A^\nu)e^{ik\cdot x}d^4x - \partial_\mu\int(\partial^\nu A^\mu)e^{ik\cdot x}d^4x = 0\\
\Rightarrow&\partial_\mu\int e^{ik\cdot x}\delta^{(\mu)}A^\nu - \partial_\mu\int e^{ik\cdot x}\delta^{(\nu)}A^\mu = 0\\
\Rightarrow&-\partial_\mu\int A^\nu\delta^{(\mu)}e^{ik\cdot x} + \partial_\mu\int A^\mu\delta^{(\nu)}e^{ik\cdot x} = 0 && \text{(Discard surface terms)}\\
\Rightarrow&-\partial_\mu\int A^\nu(ik^\mu)e^{ik\cdot x}d^4x + \partial_\mu\int A^\mu(ik^\nu)e^{ik\cdot x}d^4x = 0\\
\Rightarrow&-(ik^\mu)\int\partial_\mu A^\nu e^{ik\cdot x}d^4x + (ik^\nu)\int\partial_\mu A^\mu e^{ik\cdot x}d^4x = 0\\
\Rightarrow&-(ik^\mu)\int e^{ik\cdot x}\delta_{(\mu)}A^\nu + (ik^\nu)\int e^{ik\cdot x}\delta_{(\mu)}A^\mu = 0\\
\Rightarrow&(ik^\mu)\int A^\nu(ik_\mu)e^{ik\cdot x}d^4x - (ik^\nu)\int A^\mu(ik_\mu)e^{ik\cdot x}d^4x = 0 && \text{(Discard surface terms)}\\
\Rightarrow&(-k^2)\int A^\nu e^{ik\cdot x}d^4x + k^\nu k_\mu\int A^\mu e^{ik\cdot x}d^4x = 0\\
\Rightarrow&(-k^2)\int g^{\nu\mu}A_\mu e^{ik\cdot x}d^4x + k^\nu k_\mu\int g^{\mu\nu}A_\nu e^{ik\cdot x}d^4x = 0\\
\Rightarrow&(-k^2)g^{\nu\mu}\int A_\mu e^{ik\cdot x}d^4x + k^\nu k_\mu g^{\mu\nu}\int A_\nu e^{ik\cdot x}d^4x = 0\\
\Rightarrow&(-k^2)g^{\nu\mu}\int A_\mu e^{ik\cdot x}d^4x + k^\nu k^\mu\int A_\nu e^{ik\cdot x}d^4x = 0 && (k^\mu = g^{\mu\nu}k_\mu = g^{\mu\nu}k_\nu = k_\mu g^{\mu\nu}.)\\
\Rightarrow&(-k^2 g^{\nu\mu} + k^\nu k^\mu)\tilde{A}_\mu(k) = 0 && \left(\textstyle\int A_\mu e^{ik\cdot x}d^4x = \int A_\nu e^{ik\cdot x}d^4x = \tilde{A}_\mu(k)\right)
\end{aligned}$$

The last equation is just the required equation (7.87).

Proof is done

Comment 0 : $g^{\mu\nu} = g_{\mu\nu}$ "flips" *contravariant* to *invariant*, and vice versa; solely changing the signs of the spatial components, the time component unchanged.

Comment 1: From the proof, it's easy to see that under this kind of equation $(-k^2 g^{\nu\mu} + k^\nu k^\mu)\tilde{A}_\mu(k) = 0$, "$k^\nu k^\mu$" could swap their positions as "$k^\mu k^\nu$". The reason is these "k^ν" or "k^μ" are actually result from the terms of " ∂^ν " and " ∂^μ " in the Maxwell equation: $\partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu = 0$. And "$\partial^\nu$" and "$\partial^\mu$" could swap in the Maxwell equation, so "k^ν" and "k^μ" can do also.

(b) Verify equation (7.91).

Solution of problem 7.11(b) :

Equation (7.91) reads,

$$-k^2A(k^2)g^\nu_\sigma + A(k^2)k^\nu k_\sigma = g^\nu_\sigma$$

And according to the textbook, this equation results from putting (7.89): $(M^{-1})^{\nu\mu} = A(k^2)g^{\nu\mu} + B(k^2)k^\nu k^\mu$ and (7.87): $(-k^2g^{\nu\mu} + k^\nu k^\mu)\tilde{A}_\mu(k) \equiv M^{\nu\mu}\tilde{A}_\mu(k) = 0$ into (7.90): $(M^{-1})^{\nu\mu}M_{\mu\sigma} = g^\nu_\sigma$.

From (7.87): $(-k^2g^{\nu\mu} + k^\nu k^\mu)\tilde{A}_\mu(k) \equiv M^{\nu\mu}\tilde{A}_\mu(k) = 0$, $M_{\mu\sigma} = (-k^2g_{\mu\sigma} + k_\mu k_\sigma)$.

So, substitute $(M^{-1})^{\nu\mu}$ and $M_{\mu\sigma}$ into equation (7.90) results,

$$(M^{-1})^{\nu\mu}M_{\mu\sigma} = g^\nu_\sigma$$

$$\Rightarrow (A(k^2)g^{\nu\mu} + B(k^2)k^\nu k^\mu)(-k^2g_{\mu\sigma} + k_\mu k_\sigma) = g^\nu_\sigma$$

$$\Rightarrow A(k^2)g^{\nu\mu}(-k^2g_{\mu\sigma}) + A(k^2)g^{\nu\mu}k_\mu k_\sigma + B(k^2)k^\nu k^\mu(-k^2g_{\mu\sigma}) + B(k^2)k^\nu k^\mu k_\mu k_\sigma = g^\nu_\sigma$$

$$\Rightarrow -k^2A(k^2)g^{\nu\mu}g_{\mu\sigma} + A(k^2)k^\nu k_\sigma - B(k^2)k^2k^\nu k_\sigma + B(k^2)k^2k^\nu k_\sigma = g^\nu_\sigma$$

$\Rightarrow -k^2A(k^2)g^\nu_\sigma + A(k^2)k^\nu k_\sigma = g^\nu_\sigma$ (This equation doesn't exist, but this conclusion just needed, Cf P_{190})