Gauge theories in particle physics(3rd, Aitchison and Hey)

Chapter 5, Quantum field theory I : The free scalar field

This chapter gives an elementary introduction to quantum field theory, which is the established "language" of the Standard Model of particle physics. There are three sectors in this chapter. The first two sectors focus on formalism introduction and the third sector just show the "4D(3D of dimension(x, y, z) + 1D dimension of time(t)) " version results.

Sector 5.1

The first sector has three stages: Stage 1 introduces mode coordinate based on classic mechanics; Stage 2 applies mode coordinate to quantum mechanics; Stage 3 combines quantum mechanics to continuous mode.

Stage 5.1.1, Why introduce mode coordinate?

Because QFT needs mode coordinates, ⌣̈. The key point actually is once people utilizes this mode coordinates instead of original coordinates, like x, y, z and t, the energy of (multi-atoms) systems could be expressed very neatly as if it's only consisted of N separate and free harmonic oscillators, the other coupling items like $q_1 q_2 \dots q_n$ etc, are all disappeared.
Under the original coordinate, the expression of energy is :

$$E = \sum_{i=1}^{N} \frac{1}{2} m \dot{q}_r^2 + V(q_1, \dots, q_r)$$

Attention, the $V(q_1, \dots, q_r)$ includes couplings items, like $-q_1 q_2$ in $V(q_1 q_2) = k(q_1^2 + q_2^2 - q_1 q_2)$.
While under the mode coordinates, the energy could be expressed as a sum of N separate terms,

$$E = \sum_{i=1}^{N} [\frac{1}{2} m \dot{Q}_r^2 + \frac{1}{2} m \omega_r^2 Q_r^2]$$

Obviously, under this mode coordinate, the total energy contains no couplings terms of the form of $q_1 q_2 \dots q_n$; the energy has the remarkable form of a simple sum of "N" independent uncoupled oscillators, each with characteristic frequency ω_r.

In the case of two dimension, it's easy to explain and understand in details.
In the original coordinate, $V = k(q_1^2 + q_2^2 - q_1 q_2)$. So energy E could be expressed as

$$E = T + V = \frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 + V(q_1, q_2) = \frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 + k(q_1^2 + q_2^2 - q_1 q_2)$$

While in the (normal)mode coordinate, there are two mathematic tricks.

$$Q_1 = (q_1 + q_2)/\sqrt{2}, \quad Q_2 = (q_1 - q_2)/\sqrt{2}$$

Equivalently,

$$q_1(t) = [Q_1(t) + Q_2(t)]/\sqrt{2}, \quad q_2(t) = [Q_1(t) - Q_2(t)]/\sqrt{2}$$

After this substitution, or in the new proposed mode coordinate, the potential V could be expressed without any coupling items, like $q_1 q_2$.

$$\begin{aligned} V &= k(q_1^2 + q_2^2 - q_1 q_2) \\ &= k[(\frac{Q1+Q2}{\sqrt{2}})^2 + (\frac{Q1-Q2}{\sqrt{2}})^2 - (\frac{Q1+Q2}{\sqrt{2}})(\frac{Q1-Q2}{\sqrt{2}})] \\ &= \frac{k}{2}(Q_1^2 + 3Q_2^2) \\ &= \frac{1}{2}m(\sqrt{\frac{k}{m}})^2 Q_1^2 + \frac{1}{2}m(\sqrt{\frac{3k}{m}})^2 Q_1^2 \\ &= \frac{1}{2}m\omega_1^2 Q_1^2 + \frac{1}{2}m\omega_2^2 Q_2^2 \end{aligned}$$

Be careful, the last equation $\omega_1 = \sqrt{\frac{k}{m}}$, $\omega_2 = \sqrt{\frac{3k}{m}}$ comes from the content above equation(5.12) in page 112.

So, the total energy is therefore(BTW, $\frac{1}{2}m\dot{Q}_1^2 + \frac{1}{2}m\dot{Q}_2^2 = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}m\dot{q}_2^2$),

$$\begin{aligned} E &= T + V \\ &= \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}m\dot{q}_2^2 + V(q_1, q_2) \\ &= \frac{1}{2}m\dot{Q}_1^2 + \frac{1}{2}m\dot{Q}_2^2 + V(q_1, q_2) \\ &= \frac{1}{2}m\dot{Q}_1^2 + \frac{1}{2}m\dot{Q}_2^2 + \frac{1}{2}m\omega_1^2 Q_1^2 + \frac{1}{2}m\omega_2^2 Q_2^2 \end{aligned} \tag{1}$$

The equation (1)has exactly the form appropriate to a system of two *non-nteracting* 'things', each executing simple harmonic motion: the two 'things' are actually the two modes. So, the remarkable thing is, in mode coordinate, no interaction; while in original coordinate, atoms do interact.

It's important to realize that the modes are non-interacting by virtue of the fact that we ignored higher quadratic terms in $V(q_1, q_2)$. If the higher terms are included, that will be the basis of the quantum field theory description of particle interactions !

Stage 5.1.2, Apply mode coordinates in quantum mechanics

As mentioned in stage 1, once apply mode coordinates in classic mechanics, the energy of multi-atoms system could be expressed as independent harmonic oscillators. While in quantum mechanics, these oscillators obey the laws of quantum mechanics, so that each mode oscillator exists only in certain definite states, whose energy eigenvalues are quantized !

For each mode of frequency of ω_r, the allowed energy values are $\epsilon_r = (n_r + \frac{1}{2})\hbar\omega_r$, where n_r is a positive integer or zero. Which means to individual oscillator, $\hbar\omega_r$ is the quantum of vibrational energy, or more simple, is the amount of "unit energy" under this ω_r; but ϵ_r, or the energy of an allowed mode state, it is uniquely determined by n_r – the number of ϵ_r.

If ω_r was interpreted as wavelength, which means the "unit energy" is determined by $\hbar\omega_r = \hbar\nu$; and n_r could be interpreted as the number of photo "embraced". As a result, the total energy is $\epsilon_r = n_r\hbar\omega_r = n_r\hbar\nu$.

To the whole multi-atoms system, the total energy is the sum of all individual oscillators, so has the

form of,

$$E = \sum_{i=1}^{N} (n_r + \frac{1}{2})\hbar\omega_r$$

The frequencies ω_r are determined by the interatomic forces and are common to both classical and quantum descriptions.

Note particularly that although the number of modes N is fixed, the values of the n_r's are unrestricted, except insofar as the total energy is fixed. Which means to a multi-atoms system, although the total energy and the types of different wavelength photons are fixed, how many photons in each types is flexible.

In the case of real solid, these quanta of vibrational energy are called *photons*. Or, *a photon is an elementary quantum of vibrational excitation.*

Stage 5.1.3, Classic and quantum mechanics in continuous mode

Stage 1 and stage 2 are both in discrete mode, while continuous mode might more close to real life. In this mode, the displacement field, $\phi(x,t)$, is introduced.

In the continuous problem the analogue of the small-displacement assumption, which limited the potential energy in the discrete case to quadratic powers, implies that $\phi(x,t)$ obeys the wave equation:

$$\frac{1}{c^2}\frac{\partial^2\phi(x,t)}{\partial t^2} = \frac{\partial^2\phi(x,t)}{\partial x^2}$$

where c is the wave propagation velocity.

For simplicity, suppose the string is stretched between $x = 0$ and $x = L$, this constrains $\phi(x,t)$ vanish at these end points. A suitable form for $\phi(x,t)$ which does this is

$$\phi_r(x,t) = A_r(t)sin(\frac{r\pi x}{L})$$

where r = 1, 2, 3, ..., which expresses the fact that an exact number of half-wavelengths must fit onto the interval $(0, L)$. Here, $\lambda = \frac{2\pi c}{\omega} = \frac{2\pi c}{r\pi/L}$; half-length meant for $sin(x)$, $x = i * \frac{\lambda}{2}$ will result $sin(x) = 0$.

Substitute $\phi_r(x,t)$ into wave equation will obtain

$$\ddot{A}_r = -\omega_r^2 A_r$$

where, $\omega_r^2 = r^2\pi^2c^2/L^2$ is the frequency of A_r.

As a result, $\phi_r(x,t)$ equation indicates each motion of the string has a definite wavelength and frequency. Just as discrete case, the general motion of the string is a superposition of modes,

$$\phi_r(x,t) = \sum_{i=1}^{\infty} A_r(t)sin(\frac{r\pi x}{L})$$

This is actually a *Fourier series*.

BTW, this $\phi_r(x,t)$ is composed of discrete number($A_r(t)$) and continuous functions($sin(\frac{r\pi x}{L})$). This is very similar to signals' FFT analysis to decompose into discrete coefficients and their continuous

frequency components.

Of course, what we more cared is energy expression. Similar to *discrete and original* case,

$$E = \sum_{i=1}^{N} \frac{1}{2} m \dot{q}_r^2 + V(q_1, \ldots, q_r)$$

In *continuous and original* case, energy could be expressed as,

$$E = \int_0^L [\frac{1}{2}\rho(\frac{\partial \phi}{\partial t})^2 + \frac{1}{2}\rho c^2 (\frac{\partial \phi}{\partial t})^2] dx$$

where ρ is the mass per unit length of the string, assumed constant.

While in *continuous and mode* case, the energy expression is(The proof of this equation is available at the bottom of page 117),

$$E = \sum_{i=1}^{N} [\frac{1}{2}\rho \dot{A}_r^2 + \frac{1}{2}\rho \omega_r^2 A_r^2]$$

BTW, this equation is almost same as the *discrete and mode* expression,

$$E = \sum_{i=1}^{N} [\frac{1}{2} m \dot{Q}_r^2 + \frac{1}{2} m \omega_r^2 Q_r^2]$$

Applying quantum mechanics to the classical continuous field system is same as discrete case – to quantize all of mode oscillators, except that now the sum extend to infinity.

$$E = \sum_{i=1}^{\infty} (n_r + \frac{1}{2}) \hbar \omega_r$$

Summary of sector 5.1.: The essential idea of this sector is *quantizing independent modes.*

Sector 5.2 Lagrange-Hamilton formulation

In principle, this sector actually is another version of sector 1. In sector 1, the mechanics system is *newtonian*; instead, in this sector 2, the mechanics system is based on: Hamilton's principle of *least action*, with the action defined of a *Lagrangian*.

Stage 5.2.1 The action principle : Lagrangian particle mechanics

In the Newtonian approach, equations of motion are postulated which involve forces as the essential physical input; from these, the trajectories of the particle can be calculated.

In the least action approach, equations of motion are not postulated as basic, and the primacy of forces yields to that of *potentials*. The path by which a particle actually travels is determined by the postulate(or principle) that it has to follow that particular path, out of infinitely many ones, for which a certain quantity – the action – minimized. The action S is defined by

$$S = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))dt$$

where $q(t)$ is the position of the particle as a function of time, $\dot{q}(t)$ is its velocity; L is the Lagrangian which defined as $L = T - V$, the difference of kinetic and potential energies.

So, this case, the function of S is not a simple function of t – rather it's a function of the entire set of points $q(t)$. In simple, there're many $q_i(t)$, according to least action, make S be smallest is the right one. Mathematically, $\delta S = 0$. This deduce the celebrated *Euler-Lagrange equation of motion*:

$$\frac{\partial L}{\partial q(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} = 0$$

Stage 5.2.2 Quantum mechanics under Heisenberg – Lagrange – Hamilton

In the Schrödinger picture, the dynamical variables such as position x are independent of time, and the time dependence is carried by wavefunction. While in Heisenberg picture, the wavefunction is fixed(in the sense of time) and the dynamical variables change with time.

Besides, there is one fundamental difference between quantum mechanics and classical mechanics : in the former, the dynamical variables are operators which in general do not commute. That is to say, in quantum mechanics, $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0$.

In this Heisenberg picture, there're a few important equations.

$$[\hat{q}(t), \hat{p}(t)] = i\hbar = i$$

where $\hat{q}(t)$ is position operator, $\hat{p}(t) \equiv \partial\hat{L}/\partial\dot{\hat{q}}$.

Instead of Schrödinger equation in the picture of Schrödinger, $i\hbar\frac{\partial\phi}{\partial t} = \hat{H}\phi$, the equation of motion in Heisenberg picture is,

$$\dot{\hat{A}} = -i[\hat{A}, \hat{H}]$$

where $\hat{A}$ is any dynamical observable, the *Hamiltonian* operator $\hat{H}$ is defined in terms of the Lagrangian operator by

$$\hat{H} = \hat{p}\dot{\hat{q}} - \hat{L}$$

For example, in the oscillator case,

$$\hat{L} = \frac{1}{2}m\dot{\hat{q}}^2 - \frac{1}{2}m\omega^2\hat{q}^2$$

$$\hat{p} = m\dot{\hat{q}}$$

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2$$

In Heisenberg picture, the operator evolves with time. Take the $\hat{A} \equiv \hat{p}$ as a example, according to $\dot{\hat{A}} = -i[\hat{A}, \hat{H}]$, the Heisenberg equation of motion leads to(proof of this equation in P_{122})

$$\dot{\hat{p}} = -m\omega^2\hat{q}$$

Substituting $\hat{p} = m\dot{\hat{q}}$, above equation could be shown in the way of familiar classical equation of motion for the position of the oscillator.

$$\ddot{\hat{q}} = -\omega^2\hat{q}$$

Although this formula is very similar to equation (5.10) and (5.11) in P_{112} which indicates the classical newtonian case, one has to be careful, above equation solely stating how an *operator* evolves with time, not the corresponding particle will actually be found.

To find the position of particle, one has to sandwiching above equation between wavefunctions, one then can see that the *average* position of the particle will follow the classical trajectory. But *fluctuations* about this trajectory will certainly occur : a quantum particle does not follow a ray-like classical trajectory, neither does a photon ! Moreover, in quantum mechanics, due to the fluctuated trajectories of particles, unique paths of action are forbidden, as a result, we should in principle include all possible paths. That's the central idea of Feynman etc. His way to weight each paths is $exp(iS/\hbar)$. So the quantum mechanical amplitude to go from $q(t_1)$ to $q(t_1)$ is proportional to

$$\sum_{all\_paths\_q(t)} exp[\frac{i}{\hbar}\int_{t_1}^{t_2} L(q(t), \dot{q}(t))dt]$$

Stage 5.2.3 Quantum oscillator expressed in the language of operator

For the oscillator Hamiltonian,

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2$$

By introducing *create oscillator quanta* $\hat{a}_r^\dagger$ and *destroy oscillator quanta* $\hat{a}_r$(definition in P_{124}), and by using $[\hat{a}, \hat{a}^\dagger] = 1$, the oscillator Hamiltonian could be expressed as (proof in P_{124})

$$\hat{H} = \frac{1}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)\omega = (\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega$$

Moreover, one could prove(see P_{125})

$$[\hat{H}, \hat{a}] = -\omega\hat{a}$$

$$[\hat{H}, \hat{a}^\dagger] = \omega \hat{a}^\dagger$$

Consider now a state $|n\rangle$ which is an eigenstate of $\hat{H}$ with energy E_n: $\hat{H}|n\rangle = E_n|n\rangle$ Finally, one gets (see proof in P_{125})

$$\hat{H}(\hat{a}^\dagger|n\rangle) = (E_n + \omega)(\hat{a}^\dagger|n\rangle)$$

$$\hat{H}(\hat{a}|n\rangle) = (E_n - \omega)(\hat{a}|n\rangle)$$

At this moment, it's clear why $\hat{a}^\dagger$ and $\hat{a}$ are called as create and destroy oscillator quanta separately, since they raise and lower the energy of $|n\rangle$ by one unit of $\omega(\hbar = 1)$.
Now, since $\hat{H} \sim \hat{p}^2 + \hat{q}^2$ with $\hat{p}$ and $\hat{q}$ Hermitian, we can prove that $\langle\phi|\hat{H}|\phi\rangle$ is positive-definite for any state $|\phi\rangle$. Thus the operator $\hat{a}$ cannot lower the energy indefinitely : there must exist a lowest state $|0\rangle$ such that

$$\hat{a}|0\rangle = 0$$

This defines the lowest-energy state of the system; its energy is the 'zero-point energy' of quantum oscillator(Under this context, the following equation comes from above equation, $\hat{H}|n\rangle = E_n|n\rangle$),

$$\hat{H}|0\rangle = \frac{1}{2}\omega|0\rangle$$

Accordingly, the first excited state is, $|1\rangle = \hat{a}^\dagger|0\rangle$, with energy $(1 + \frac{1}{2})\omega$. The nth state has energy $(n + \frac{1}{2})\omega$ and is proportional to $(\hat{a}^\dagger)^n|0\rangle$. Therefore, we have,

$$\hat{H}|n\rangle = (\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega|n\rangle = E_n|n\rangle = (n + \frac{1}{2})\omega|n\rangle$$

It's easy to generalize to a system whose Lagrangian is a sum of N independent oscillators,

$$\begin{aligned}\hat{H} &= \sum_{r=1}^{N}[\frac{1}{2m}\hat{p}_r^2 + \frac{1}{2}m\omega_r^2\hat{q}_r^2] \\ &= \sum_{r=1}^{N}(\hat{a}_r^\dagger\hat{a}_r + \frac{1}{2})\omega_r\end{aligned}$$

Since the eigenvalues of each number operator $\hat{n}_r = \hat{a}_r^\dagger\hat{a}_r$ are n_r(refer to equation (5.75) in P_{126} for details), so the eigenvalues of $\hat{H}$ indeed have the form :

$$E = \sum_{r=1}^{N}(n_r + \frac{1}{2})\omega_r$$

The corresponding eigenstates are products $|n_1\rangle|n_2\rangle \dots |n_N\rangle$ of N individual oscillator eigenstates, where $|n_r\rangle$ contains quanta of excitation, of frequency ω_r; the product state is usually abbreviated to $|n_1, n_2, \dots, n_N\rangle$.

Stage 5.2.4 Lagrange-Hamilton classical field mechanics

This stage is very similar to **Stage 5.1.3**, but in another totally different approach – "Lagrange-Hamilton". In Stage 5.1.3, it's "Newtonian-energy". As expected, the Lagrange-Hamilton results

same expressions as Newtonian-energy.

The first step is $\{q_r(t); r = 1, 2, \ldots, N\} \xrightarrow{N \to \infty} \phi(x,t)$, where x now is a continuous variable labeling the displacement of the 'string'. At each point x we have an independent degree of freedom $\phi(x,t)$ – thus the field system has a 'continuous infinity' of degrees of freedom.

In this case, we formulate S in terms of Lagrangian density $\mathcal{L}$: $S = \int dt L$; $L = \int dx \mathcal{L}$. And $\mathcal{L}$ should be expressed as $\mathcal{L} = \mathcal{L}(\phi, \partial\phi/\partial x, \dot{\phi})$. The $\delta S = 0$ results Euler-Lagrange filed equation as following:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \nabla \cdot (\frac{\partial \mathcal{L}}{\partial(\nabla\phi)}) - \frac{\partial}{\partial t}(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}) = 0$$

Considering the Lagrange density $\mathcal{L}$ has the expected form of 'kinetic energy density minus potential energy density'(equation (5.48)), $\mathcal{L}_\rho = \frac{1}{2}\rho(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}\rho c^2(\frac{\partial\phi}{\partial x})^2$, inserting to above equation one can get,

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$$

This is exactly same result as in "Newtonian-energy" formularism.

As to the Hamiltonian(total energy) expression, first we define $\pi(x,t) \equiv \partial\mathcal{L}/\partial\dot{\phi}(x,t)$, then define Hamiltonian density $\mathcal{H}$ by $\mathcal{H}(\phi,\pi) \equiv \pi(x,t)\dot{\phi}(x,t) - \mathcal{L}$, one can get(detailed in P_{119})

$$\mathcal{H}_\rho = \frac{1}{2}[\frac{1}{\rho}\pi^2 + \rho c^2(\frac{\partial\phi}{\partial x})^2]$$

So, the Hamiltonian in Lagrange formularism is

$$H_\rho = \int [\frac{1}{2\rho}\pi^2(x,t) + \frac{1}{2}\rho c^2(\frac{\partial\phi(x,t)}{\partial x})^2]dx$$

This is exactly same as "Newtonian-energy" formularism(see equation(5.35)) also.

Stage 5.2.5 Heisenberg-Lagrange-Hamilton quantum field mechanics

In principle, this stage solely *quantized* the results in previous stage. In brief, promote ϕ and π to $\hat{\phi}$ and $\hat{\pi}$ in the picture of Heisenberg. The procedure is "find $\hat{H}$'s expression → find $\hat{H}$'s eigenvalues and eigenstates → describe system states by $\hat{a}^\dagger$ and $\hat{a}$ operators".

At first, with almost same procedure shown in stage 5.2.4, one can get this Hamiltonian expression as following.

$$\hat{H} = \int \hat{\mathcal{H}}dx = \int \frac{1}{2}[\hat{\pi}^2 + (\frac{\partial\hat{\phi}}{\partial x})^2]dx$$

Although it's not so clear how $\hat{\phi}$ and $\hat{\pi}$ come from in P_{131} (especially (5.112)), they're here,

$$\hat{\phi} = \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega}}[\hat{a}(k)e^{ikx-i\omega t} + \hat{a}^\dagger(k)e^{-ikx+i\omega t}]$$

$$\hat{\pi} = \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega}}(-i\omega)[\hat{a}(k)e^{ikx-i\omega t} - \hat{a}^\dagger(k)e^{-ikx+i\omega t}]$$

Substituting the expression of $\hat{\phi}$ and $\hat{\pi}$ to above equation one could get Heisenberg picture version of Lagrange–Hamilton,

$$\begin{aligned}\hat{H} &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\{\frac{1}{2}[\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k)]\omega\} \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\hat{a}^\dagger(k)\hat{a}(k)\omega + \int_{-\infty}^{\infty} \frac{dk}{2\pi}[\hat{a}(k), \hat{a}^\dagger(k)]\omega\end{aligned}$$

where factor $\frac{1}{2\pi}$ arises from normalization choice; the operator $\hat{a}^\dagger(k)$ creates, and $\hat{a}(k)$ destroys a quantum of the k mode.

In my opinion, the most funny thing is the second expression: it's obviously the continuum analogue of the zero-point energy $\frac{1}{2}\omega$ from this equation $\hat{H}|n\rangle = (\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega|n\rangle = E_n|n\rangle = (n + \frac{1}{2})\omega|n\rangle$; however, the second item of this expression is infinite: $\int_{-\infty}^{\infty} \frac{dk}{2\pi}[\hat{a}(k), \hat{a}^\dagger(k)]\omega$! How can imagine the zero-point energy is infinity !?

While, the conventional ploy is to argue that only energy *differences*, relative to a conveniently defined ground state, really matter – so that the infinite constant could be discarded ! Then the ground state $|0\rangle$ has energy zero, and the eigenvalues of $\hat{H}$ are of the form $\int \frac{dk}{2\pi}n(k)\omega$, where $n(k)$ is the number of quanta(counted by the number operator $\hat{a}^\dagger(k)\hat{a}(k)$) of energy $\omega = k$.

Personally, I don't think it's a good way to explain. But the conclusion should be verified by experiments(otherwise precedent physicists should take this "flaw" seriously), so the more acceptable explanation would be: This "infinity" actually is not "genuine infinity", in stead, it must be a *finite*. That's to say, the second term :$\int_{-\infty}^{\infty} \frac{dk}{2\pi}[\hat{a}(k), \hat{a}^\dagger(k)]\omega$ is a constant !

There are two reasons : (1), mathematically, it's impossible to make sure $\infty - \infty =$"finite" satisfies all systems; on the contrary, it's possible to make sure "$finite''$ – "$finite''$ = "$finite''$ to all systems. (2), Physically, it makes no sense to require the integrate range should range from " $-\infty''$ to "∞''. Because even to the whole Universe, there must exist an edge which means the "size" of Universe is NOT infinity. Supposing the "size" of our universe, $L = 10^n$ light year, then set the integrate range as $[-L, L]$ to replace $(-\infty, +\infty)$ sounds reasonable. Obviously, L is infinite. Under this context, the term $\int_{-\infty}^{\infty} \frac{dk}{2\pi}[\hat{a}(k), \hat{a}^\dagger(k)]\omega$ won't be "∞''.

Finally, we now can throw away the mechanical props and embrace the unadorned quantum field! We do not ask what is waving, we simply postulate a field – such as ϕ – and quantize it, like equation (5.116) shown. Its quanta of excitation are what we call particles.

Actually, any desired state in which excitation quanta are present can be formed by the appropriate application of $\hat{a}^\dagger(k)$ operators to the ground state $|0\rangle$. For instance, a two-quantum state containing one quantum of momentum k_1 and another of momentum k_2 may be written as $|k_1, k_2\rangle \propto \hat{a}^\dagger(k_1)\hat{a}^\dagger(k_2)|0\rangle$(A general state will contain an arbitrary number of quanta).

At the end of this stage, there're five very nice comments, each one is brilliant. Here we go.

comment (1) emphasizes that the Heisenberg opeator equations of motion are consistent with the Euler-Lagrange equations. That's to say, the equation (5.109) could equally get from (5.108) or

(5.126).

comment (2) The formalism of Heisenberg-Lagrange-Hamilton applied on quantum field mechanics encompasses both the wave and the particle aspects of matter and radiation. The former is evident from the plane-wave equation functions in the expansion of $\hat{\phi}$, (5.116), which in turn originate from the fact that $\hat{\phi}$ obeys the wave equation (5.108). The latter follows from the discrete nature of the energy spectrum and the associated operators, $\hat{a}, \hat{a}^\dagger$ which refer to individual quanta, i.e. *particles*.

comment (3) Points out the meaning of the ground state $|0\rangle$ for a quantum field – the state with no quanta in it , and hence without particle in it – actually is vacuum state.

comment (4) The equation (5.125) : $|k_1, k_2\rangle \propto \hat{a}^\dagger(k_1)\hat{a}^\dagger(k_2)|0\rangle$ actually indicates two identical bosons' state since this state is symmetric under the interchange $k_1 \leftrightarrow k_2$. This is inevitable feature of the formalism as so far developed – there is no possible way of distinguishing one quantum of energy form another, and we expect the two-quantum state to be indifferent to the order in which the quanta are put in it.

comment (5) Makes a connection between the quantum field theory formalism to ordinary 'wave-function' quantum mechanics : The vacuum to one-particle matrix elements of the field operators are just the familiar wavefunctions of single-particle quantum mechanics.

Sector 5.2 Generalizations: four dimensions, relativity and mass

Previous sectors in this chapter have shown how quantum mechanics may be married to field theory, but only consider one spatial dimension. To incorporate the demands of relativity, we must generalize to three dimension. This is very easy to do in the Lagrangian approach, for the scalar field $\phi(x,t)$(Here, scalar means that the field has only one independent component at each point (x,t) – unlike the electromagnetic field, for instance, for which the analogous quantity has four components, making up a 4-vector field $A^\mu(x,t) = (A_0(x,t), \mathbf{A}(x,t))$).

Massless three dimensions quanta non-relativity $\rightarrow$ Massless three dimensions quanta relativity $\rightarrow$ Mass three dimensions quanta relativity:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \nabla \cdot (\frac{\partial \mathcal{L}}{\partial(\nabla\phi)}) - \frac{\partial}{\partial t}(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}) = 0 \;\rightarrow\; \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}) = 0$$

$$\mathcal{L}_\rho = \frac{1}{2}\rho(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}\rho c^2(\frac{\partial\phi}{\partial x})^2 \;\rightarrow\; \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \;\rightarrow\; \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \;\rightarrow\; \partial_\mu\partial^\mu\phi = (\frac{\partial^2}{\partial t^2} - \nabla^2)\phi = 0 \;\rightarrow\; (\Box + m^2)\phi(x,t) = 0$$

The next step is to quantize the "mass, three-dimensions and relativity " field theory.

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega}}[\hat{a}(k)e^{-ik\cdot x} + \hat{a}^\dagger(k)e^{ik\cdot x}]$$

$$\hat{\pi}(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega}}(-i\omega)[\hat{a}(k)e^{-ik\cdot x} - \hat{a}^\dagger(k)e^{ik\cdot x}]$$

where, $k\cdot x$ is the four-dimensional dot product, $k\cdot x = \omega t - \mathbf{k}\cdot x$, and $\omega = \sqrt{\mathbf{K}^2 + m^2}$.

Similar to the equation (5.111), the Hamiltonian is found to be,

$$\hat{H}_{KG} = \int d^3\mathbf{x}\hat{\mathcal{H}}_{KG} = \int d^3\mathbf{x}\frac{1}{2}[\hat{\pi}^2 + \nabla\hat{\phi}\cdot\nabla\hat{\phi} + m^2\hat{\phi}^2]$$

Substituting the expression of $\hat{\phi}(x)$ and $\hat{\pi}(x)$ into above $\hat{H}_{KG}$ equation, one get

$$\begin{aligned}\hat{H}_{KG} &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{d^3\mathbf{k}}{(2\pi)^3}[\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k)]\omega \\ &= \int_{-\infty}^{\infty}\frac{d^3\mathbf{k}}{(2\pi)^3}\hat{a}^\dagger(k)\hat{a}(k)\omega\end{aligned}$$

The last $\hat{H}_{KG}$ expression supports the physical interpretation of the mode operators $\hat{a}^\dagger$ and $\hat{a}$ as creation and destruction operators for quanta of the field $\hat{\phi}$ as before, except that now the energy-momentum relation for these particles is the relativistic one, for particles of mass m.

Since $\hat{\phi}$ is real($\hat{\phi} = \hat{\phi}^\dagger$) and has no spin degrees of freedom, it is called a real scalar field. Only field quanta of one type enter – those created by $\hat{a}^\dagger$ and destroyed by $\hat{a}$. Thus $\hat{\phi}$ would correspond physically to a case where there was a unique particle state of a given mass m.

In SM, the Higgs field is a scalar field(though it contains several components with different charge). Right now, Higgs does exist. Still, it's not clear whether it's elementary or composite.

How (5.87) could get from (5.86), P_{127}

Equation (5.86) is,

$$\delta S = \int dt \int [\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}\delta(\frac{\partial \phi}{\partial x}) + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\delta\dot{\phi}]dx$$

Taking the second term of integral,

$$\begin{aligned}
&\int dt \int \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}\delta(\frac{\partial \phi}{\partial x})dx \\
&= \int dt\ [\int \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}d(\frac{\partial \phi}{\partial x})] \\
&= \int dt\ [\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}(\frac{\partial \phi}{\partial x}) - \int \frac{\partial \phi}{\partial x}d\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}] && \text{(discarding the 'surface' term)} \\
&= -\int dt \int \frac{\partial \phi}{\partial x}d\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)} \\
&= -\int dt \int \frac{\partial \phi}{\partial x}\frac{\partial}{\partial x}[\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}]dx \\
&= -\int dt \int \frac{\partial}{\partial x}[\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}](d\phi_x) && \text{(inverse transform from 4th to 5th line)} \\
&= -\int dt \int \frac{\partial}{\partial x}[\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}](\delta\phi_x dx) && \text{(inverse transform from 1st to 2nd line)}
\end{aligned}$$

Taking the third term of integral,

$$\begin{aligned}
&\int dt \int [\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\delta\dot{\phi}]dx \\
&= \int dx \int [\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\delta\dot{\phi}]dt \\
&= \int dx \int [\frac{\partial \mathcal{L}}{\partial \dot{\phi}}d\dot{\phi}] \\
&= \int dx[(\frac{\partial \mathcal{L}}{\partial \dot{\phi}})\dot{\phi} - \int \dot{\phi}d(\frac{\partial \mathcal{L}}{\partial \dot{\phi}})] \\
&= -\int dx[\int \dot{\phi}d(\frac{\partial \mathcal{L}}{\partial \dot{\phi}})] \\
&= -\int dx \int \frac{\partial \phi}{\partial t}\frac{\partial}{\partial t}(\frac{\partial \mathcal{L}}{\partial \dot{\phi}})dt \\
&= -\int dx \int \frac{\partial}{\partial t}(\frac{\partial \mathcal{L}}{\partial \dot{\phi}})(\delta\phi_t dt) && \text{(refer to the second term case above)}
\end{aligned}$$

Substituting the two terms into (5.86), one could get (5.87).

$$\delta S = \int dt \int dx\ \delta\phi[\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x}(\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}) - \frac{\partial}{\partial t}(\frac{\partial \mathcal{L}}{\partial \dot{\phi}})] \qquad \text{(Proof is done)}$$

Comment 1, The "surface" terms could be discarded because in *quantum mechanics*, the integral usually ranged from $[-\infty, +\infty]$. Under this context, for ϕ or $\mathcal{L}$ related terms which contain the exponential expression like $e^{-ipx-i\omega t}$ will go to zero when x or t is ∞; similarly for the expression $e^{ipx+i\omega t}$ of $-\infty$. The terms like $e^{-ipx-i\omega t}$ at $-\infty$ are discarded directly due to have no physical meaning. So no matter under which situation, they're all discarded.

Comment 2, The two terms of $\delta\phi_x$ and $\delta\phi_t$ are unified as $\delta\phi$ under the double integrals of $\int dt \int dx$.

Basically, this chapter aims to introduce the formulism for *(relativistic)quantum field*, starts from the action and Euler-Lagrangian equation.

Actually, many terms are introduced in this chapter.

Newtonian mechanics $\longleftrightarrow$ Lagrangian mechanics.

Classical mechanics $\longleftrightarrow$ Quantum Mechanics.

Schrödinger picture $\longleftrightarrow$ Heisenberg picture.

Discrete degrees of freedom (mass points) $\longleftrightarrow$ a continuous degree of freedom(field).

For the action principle of Lagrangian particle mechanics, it actually is the unique way to solve the problems in quantum mechanics and QFT since in microcosm, one can't measure the trajectory and time of a particle accurately as in macrocosm.

Heisenberg picture, comparing to Schrödinger one, its dynamical variable change along the time while the waveform doesn't. This along-time-changed dynamical variable named as *operator*. The critical difference between classical mechanics and quantum mechanics operator is : classical operators usually commute while QM doesn't, i.e. for QM, $[A, B] = AB - BA \neq 0$.

Create operator $\hat{a}^\dagger$ and destroy operator $\hat{a}$ are introduced also(Thinking about the "†" as a " + " will help to discriminate which one is the create one).

Then the authors introduce the idea of field solely transit from finite number of points to infinite number of points. This transition arises a new feature of Lagrangian density because the ϕ(updated from $q_r(t)$) now is a continuous function of x, so $\mathcal{L}$ depends additionally on $\partial\phi/\partial x$, except on ϕ and $\dot{\phi}$. Also because of this new feature, the momentum named as "momentum canonically conjugate to ϕ" is defined as $\pi(x,t) = \partial\mathcal{L}/\partial\dot{\phi}(x,t)$(similar to the discrete style definition : $p = \partial\mathsf{L}/\partial\dot{q}$).

The comments in page $134 \sim 136$ are really nice and is deserved to be red again.

However, the paragraph above equation(5.124) are not so convinced and clear comparing to other parts of this chapter.