Solutions of chapter 8 of Gauge theories in particle physics(3rd, Aitchison and Hey)

Chapter 8, Elementary processes in scalar and spinor electrodynamics

Problems 8.4, P_{245}. Verify for a simple choice of Lorentz transformation (e.g. one along the x' - axis) that $\bar{\psi}'\gamma^\mu\psi$ is a 4-vector, where ψ' and ψ are four-component Dirac spinors transforming as in section 4.4.

Solution of problem 8.4

First of all, the notation in this problem is confusing : in the expression of $\bar{\psi}'\gamma^\mu\psi$, the " $'$ " means a final status; while in the section of 4.4, the " $'$ " means a new frame. To cancel this confusion, in this solution, $\bar{\psi}'\gamma^\mu\psi$ will change to $\bar{\psi}\gamma^\mu\psi$.

Refer to P_{90} of textbook, we know $\bar{\psi} \equiv \psi^\dagger\gamma^0$. So, $\bar{\psi}\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\psi$.

Supposing the Lorentz transformation along X-axis, then,

$$
\begin{aligned}
\bar{\psi}'\gamma^\mu\psi' &= {\psi'}^\dagger\gamma^0\gamma^\mu\psi' = {\psi'}^\dagger\alpha_x\psi' = \boldsymbol{j}_x && (\gamma^0\gamma^\mu = \alpha_x)\\
&= (e^{\alpha_x\vartheta/2}\psi)^\dagger\gamma^0\gamma^\mu(e^{\alpha_x\vartheta/2}\psi)\\
&= \psi^\dagger e^{\alpha_x\vartheta/2}\gamma^0\gamma^\mu e^{\alpha_x\vartheta/2}\psi\\
&= \psi^\dagger\gamma^0\gamma^\mu e^{\alpha_x\vartheta}\psi\\
&= \psi^\dagger\gamma^0\gamma^\mu(cosh\vartheta + \alpha_x sinh\vartheta)\psi && \text{(c.f. (4.91))}\\
&= \psi^\dagger\gamma^0\gamma^\mu cosh\vartheta\psi + \psi^\dagger\gamma^0\gamma^\mu\alpha_x sinh\vartheta\psi\\
&= \psi^\dagger\alpha_x cosh\vartheta\psi + \psi^\dagger sinh\vartheta\psi && (\gamma^0\gamma^\mu\alpha_x = (\alpha_x)^2 = \mathbf{1})\\
&= cosh\vartheta\psi^\dagger\alpha_x\psi + sinh\vartheta\psi^\dagger\psi\\
&= cosh\vartheta\boldsymbol{j}_x + sinh\vartheta\rho && \text{(c.f. (4.50) and (4.56))}
\end{aligned}
$$

This is very analogous to the equation (4.85) in P_{88}, therefore $\bar{\psi}\gamma^\mu\psi$ is a 4-vector.

Proof is done for problem 8.4

Problems 8.5, P_{245}. (a) Using the u-spinors normalized as in (4.105), the $\phi^{1,2}$ of (8.47), and the result for $\boldsymbol{\sigma}\cdot\boldsymbol{A}\boldsymbol{\sigma}\cdot\boldsymbol{B}$ from problem 4.4, show that

$$u^\dagger(k',s'=1)u(k,s=1)=(E+m)\boldsymbol{\{}1+\frac{\boldsymbol{k}'\cdot\boldsymbol{k}}{(E+m)^2}+\frac{i\phi^{1\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{k}'\times\boldsymbol{k}\phi^1}{(E+m)^2}\boldsymbol{\}}$$

(b)For any vector $\boldsymbol{A}=(A^1,A^2,A^3)$, show that $\phi^{1\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^1=A^3$. Find similar expressions for $\phi^{1\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^2$, $\phi^{2\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^1$, $\phi^{2\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^2$

(c) Show that the S of (8.46) is equal to

$$S=(E+m)^2\boldsymbol{\{}\ [\ 1+\frac{\boldsymbol{k}'\cdot\boldsymbol{k}}{(E+m)^2}\ ]^2+\frac{(\boldsymbol{k}'\times\boldsymbol{k})^2}{(E+m)^4}\boldsymbol{\}}$$

(d)Using $cos\theta=\boldsymbol{k}\cdot\boldsymbol{k}'/(|\boldsymbol{k}||\boldsymbol{k}'|), |\boldsymbol{k}|=|\boldsymbol{k}'|$ and $v=|\boldsymbol{k}|/E$, show that.

$$S=(2E)^2(1-v^2sin^2\theta/2)$$

Solution of problem 8.5

From (8.47),

$$\phi^1=\begin{pmatrix}1\\0\end{pmatrix}\quad\phi^2=\begin{pmatrix}0\\1\end{pmatrix}$$

From (4.105) one gets,

$$u(k,s=1)=(E+m)^{1/2}\begin{pmatrix}\phi^1\\\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\phi^1\end{pmatrix}=(E+m)^{1/2}\begin{pmatrix}1\\\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\end{pmatrix}\phi^1$$

$$u^\dagger(k',s'=1)=(E+m)^{1/2}\phi^{1\dagger}(1\quad\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}'}{E+m})$$

So,

$$\begin{aligned}u^\dagger(k',s'=1)u(k,s=1)&=(E+m)^{1/2}\phi^{1\dagger}(1\quad\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}'}{E+m})(E+m)^{1/2}\begin{pmatrix}1\\\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{E+m}\end{pmatrix}\phi^1\\&=(E+m)\phi^{1\dagger}[1+\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k}')(\boldsymbol{\sigma}\cdot\boldsymbol{k})}{(E+m)^2}]\phi^1\\&=(E+m)\phi^{1\dagger}[1+\frac{\boldsymbol{k}'\cdot\boldsymbol{k}}{(E+m)^2}+i\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}'\times\boldsymbol{k}}{(E+m)^2}]\phi^1\\&\qquad\text{(from problem 4.4, }(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B})=\boldsymbol{A}\cdot\boldsymbol{B}\boldsymbol{1}+i\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{B}.)\\&=(E+m)\boldsymbol{\{}1+\frac{\boldsymbol{k}'\cdot\boldsymbol{k}}{(E+m)^2}+\frac{i\phi^{1\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{k}'\times\boldsymbol{k}\phi^1}{(E+m)^2}\boldsymbol{\}}\end{aligned}$$

(b)

$$\begin{aligned}\phi^{1\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^1&=(1\quad0)\begin{pmatrix}0&1\\1&0\end{pmatrix}A_1\begin{pmatrix}1\\0\end{pmatrix}+(1\quad0)\begin{pmatrix}0&-i\\i&0\end{pmatrix}A_2\begin{pmatrix}1\\0\end{pmatrix}+(1\quad0)\begin{pmatrix}1&0\\0&-1\end{pmatrix}A_3\begin{pmatrix}1\\0\end{pmatrix}\\&=(0\quad1)\begin{pmatrix}A_1\\0\end{pmatrix}+(0\quad-i)\begin{pmatrix}A_2\\0\end{pmatrix}+(1\quad0)\begin{pmatrix}A_3\\0\end{pmatrix}=A_3\end{aligned}$$

Similarly,

$$\phi^{1\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^2=(1\quad0)\begin{pmatrix}0&1\\1&0\end{pmatrix}A_1\begin{pmatrix}0\\1\end{pmatrix}+(1\quad0)\begin{pmatrix}0&-i\\i&0\end{pmatrix}A_2\begin{pmatrix}0\\1\end{pmatrix}+(1\quad0)\begin{pmatrix}1&0\\0&-1\end{pmatrix}A_3\begin{pmatrix}0\\1\end{pmatrix}$$

$$= (0 \quad 1)\begin{pmatrix} 0 \\ A_1 \end{pmatrix} + (0 \quad -i)\begin{pmatrix} 0 \\ A_2 \end{pmatrix} + (1 \quad 0)\begin{pmatrix} 0 \\ A_3 \end{pmatrix} = A_1 - iA_2$$

Similarly,

$$\phi^{2\dagger}\boldsymbol{\sigma} \cdot \boldsymbol{A}\phi^1 = (0 \quad 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (0 \quad 1)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} A_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (0 \quad 1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (1 \quad 0)\begin{pmatrix} A_1 \\ 0 \end{pmatrix} + (i \quad 0)\begin{pmatrix} A_2 \\ 0 \end{pmatrix} + (0 \quad -1)\begin{pmatrix} A_3 \\ 0 \end{pmatrix} = A_1 + iA_2$$

Similarly,

$$\phi^{2\dagger}\boldsymbol{\sigma} \cdot \boldsymbol{A}\phi^2 = (0 \quad 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (0 \quad 1)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} A_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (0 \quad 1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (1 \quad 0)\begin{pmatrix} 0 \\ A_1 \end{pmatrix} + (i \quad 0)\begin{pmatrix} 0 \\ A_2 \end{pmatrix} + (0 \quad -1)\begin{pmatrix} 0 \\ A_3 \end{pmatrix} = -A_3$$

In summary,

$$\phi^{1\dagger}\boldsymbol{\sigma} \cdot \boldsymbol{A}\phi^1 = A_3 \qquad \phi^{1\dagger}\boldsymbol{\sigma} \cdot \boldsymbol{A}\phi^2 = A_1 - iA_2 \qquad \phi^{2\dagger}\boldsymbol{\sigma} \cdot \boldsymbol{A}\phi^1 = A_1 + iA_2 \qquad \phi^{2\dagger}\boldsymbol{\sigma} \cdot \boldsymbol{A}\phi^2 = -A_3$$

(c) From (8.46),

$$S = \frac{1}{2}\sum_{s',s} |u'^{\dagger}u|^2$$

$$\begin{aligned}
|u^{\dagger}(k', s'=1)u(k, s=1)|^2 &= |(E+m)\boldsymbol{\{}1 + \frac{\boldsymbol{k}' \cdot \boldsymbol{k}}{(E+m)^2} + \frac{i\phi^{1\dagger}\boldsymbol{\sigma} \cdot \boldsymbol{k}' \times \boldsymbol{k}\phi^1}{(E+m)^2}\boldsymbol{\}}|^2 \\
&= (E+m)^2\boldsymbol{\{}[1 + \frac{\boldsymbol{k}' \cdot \boldsymbol{k}}{(E+m)^2}]^2 + [\frac{\phi^{1\dagger}\boldsymbol{\sigma} \cdot \boldsymbol{k}' \times \boldsymbol{k}\phi^1}{(E+m)^2}]^2\boldsymbol{\}} \\
&= (E+m)^2\boldsymbol{\{}[1 + \frac{\boldsymbol{k}' \cdot \boldsymbol{k}}{(E+m)^2}]^2 + \frac{[(\boldsymbol{k}' \times \boldsymbol{k})_3]^2}{(E+m)^4}\boldsymbol{\}} \qquad (\phi^{1\dagger}\boldsymbol{\sigma} \cdot \boldsymbol{A}\phi^1 = A^3) \\
&= C^2[(1 + \frac{B}{C^2})^2 + (\frac{A_3}{C^2})^2] \quad (\text{Substitute}: \ E+m = C, \boldsymbol{k}' \cdot \boldsymbol{k} = B, (\boldsymbol{k}' \times \boldsymbol{k})_3 = A_3)
\end{aligned}$$

And in similar,

$$\begin{aligned}
|u^{\dagger}(k', s'=1)u(k, s=2)|^2 &= |(E+m)\boldsymbol{\{}\phi^{1\dagger}[1 + \frac{\boldsymbol{k}' \cdot \boldsymbol{k}}{(E+m)^2}]\phi^2 + \frac{i\phi^{1\dagger}\boldsymbol{\sigma} \cdot \boldsymbol{k}' \times \boldsymbol{k}\phi^2}{(E+m)^2}\boldsymbol{\}}|^2 \\
&= |(E+m)[\frac{i\phi^{1\dagger}\boldsymbol{\sigma} \cdot \boldsymbol{k}' \times \boldsymbol{k}\phi^2}{(E+m)^2}]|^2 \\
&\qquad (\text{Pay extreme attention}: \text{The term in [ ] was cancelled due to } \phi^{1\dagger}[\ ]\phi^2 = 0) \\
&= |C[\frac{i(A_1 - iA_2)}{C^2}]|^2 \\
&= |C(\frac{A_2}{C^2} + \frac{iA_1}{C^2})|^2 \\
&= C^2[(\frac{A_2}{C^2})^2 + (\frac{A_1}{C^2})^2]
\end{aligned}$$

And in similar,

$$
\begin{aligned}
|u^\dagger(k', s'=2)u(k, s=1)|^2 &= |(E+m)\boldsymbol{\{}\phi^{2\dagger}[1+\frac{\boldsymbol{k}'\cdot\boldsymbol{k}}{(E+m)^2}]\phi^1 + \frac{i\phi^{2\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{k}'\times\boldsymbol{k}\phi^1}{(E+m)^2}\boldsymbol{\}}|^2 \\
&= |(E+m)[\frac{i\phi^{2\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{k}'\times\boldsymbol{k}\phi^1}{(E+m)^2}]|^2
\end{aligned}
$$

(Pay extreme attention : The term in [ ] was cancelled due to $\phi^{2\dagger}[\quad]\phi^1 = 0$)

$$
\begin{aligned}
&= |C[\frac{i(A_1+iA_2)}{C^2}]|^2 \\
&= |C(-\frac{A_2}{C^2} + i\frac{A_1}{C^2})|^2 \\
&= C^2[(\frac{A_2}{C^2})^2 + (\frac{A_1}{C^2})^2]
\end{aligned}
$$

And in similar,

$$
\begin{aligned}
|u^\dagger(k', s'=2)u(k, s=2)|^2 &= |(E+m)\boldsymbol{\{}1+\frac{\boldsymbol{k}'\cdot\boldsymbol{k}}{(E+m)^2} + \frac{i\phi^{2\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{k}'\times\boldsymbol{k}\phi^2}{(E+m)^2}\boldsymbol{\}}|^2 \\
&= |C\boldsymbol{\{}1+\frac{B}{C^2}+\frac{i(-iA_3)}{C^2}\boldsymbol{\}}|^2 \\
&= |C\boldsymbol{\{}1+\frac{B}{C^2}+\frac{iA_3}{C^2}\boldsymbol{\}}|^2 \\
&= C^2[(1+\frac{B}{C^2})^2 + (\frac{A_3}{C^2})^2]
\end{aligned}
$$

So,

$$
\begin{aligned}
S &= \frac{1}{2}\sum_{s',s}|u^{'\dagger}u|^2 \\
&= \frac{C^2}{2}\{[(1+\frac{B}{C^2})^2+(\frac{A_3}{C^2})^2] + [(\frac{A_2}{C^2})^2+(\frac{A_1}{C^2})^2] + [(\frac{A_2}{C^2})^2+(\frac{A_1}{C^2})^2] + [(1+\frac{B}{C^2})^2+(\frac{A_3}{C^2})^2]\} \\
&= \frac{C^2}{2}\{2(1+\frac{B}{C^2})^2 + 2[(\frac{A_3}{C^2})^2+(\frac{A_2}{C^2})^2+(\frac{A_1}{C^2})^2]\} \\
&= C^2[(1+\frac{B}{C^2})^2 + (\frac{\boldsymbol{A}^2}{C^2})^2] \\
&= (E+m)^2\boldsymbol{\{}[1+\frac{\boldsymbol{k}'\cdot\boldsymbol{k}}{(E+m)^2}]^2 + \frac{(\boldsymbol{k}'\times\boldsymbol{k})^2}{(E+m)^4}\boldsymbol{\}} \quad \text{(Substitute back : } E+m=C, \boldsymbol{k}'\cdot\boldsymbol{k}=B, (\boldsymbol{k}'\times\boldsymbol{k})=\boldsymbol{A})
\end{aligned}
$$

(d) *The key point are : to make m expressed by v and introduce x.*
$m^2 = E^2 - \boldsymbol{k}^2 = E^2 - (Ev)^2 = E^2(1-v^2) \Rightarrow m = E\sqrt{1-v^2}, E+m = E(1+\sqrt{1-v^2})$.
Set $x = |\boldsymbol{k}|^2/(E+m)^2 = (Ev)^2/(E+E\sqrt{1-v^2})^2 = v^2/(1+\sqrt{1-v^2})^2 = (1-\sqrt{1-v^2})^2/v^2$.
And from above definition, $cos\theta = \boldsymbol{k}\cdot\boldsymbol{k}'/(|\boldsymbol{k}||\boldsymbol{k}'|)$. So,

$$
\begin{aligned}
&(E+m)^2\boldsymbol{\{}[1+\frac{\boldsymbol{k}'\cdot\boldsymbol{k}}{(E+m)^2}]^2 + \frac{(\boldsymbol{k}'\times\boldsymbol{k})^2}{(E+m)^4}\boldsymbol{\}} \\
&= E^2(1+\sqrt{1-v^2})^2[(1+xcos\theta)^2 + x^2sin^2\theta] \\
&= E^2(1+\sqrt{1-v^2})^2(1+2xcos\theta+x^2) \\
&= E^2[(1+\sqrt{1-v^2})^2 + 2(1+\sqrt{1-v^2})^2\frac{(1-\sqrt{1-v^2})^2}{v^2}cos\theta + (\frac{(1-\sqrt{1-v^2})^2}{v^2})^2] \\
&= E^2[2+2(1-v^2)+2v^2cos\theta] \\
&= (2E)^2(1-v^2sin^2\frac{\theta}{2})
\end{aligned}
$$

Problems 8.8, P_{246}. Verify equation (8.78) for the lepton tensor $L^{\mu\nu}$.

Solution of problem 8.8 :

Equation (8.78) reads,

$$L^{\mu\nu} = \frac{1}{2}Tr[(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu] = 2[k'^\mu k^\nu + k'^\nu k^\mu - (k' \cdot k)g^{\mu\nu}] + 2m^2 g^{\mu\nu}$$

From (8.76), we know,

$$L^{\mu\nu} = \frac{1}{2}Tr[(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu] = \frac{1}{2}(Tr[\not{k}'\gamma^\mu\not{k}\gamma^\nu] + m^2 Tr(\gamma^\mu\gamma^\nu))$$

$$\begin{aligned}
&Tr[\not{k}'\gamma^\mu\not{k}\gamma^\nu] = Tr[\gamma^\sigma k'_\sigma\gamma^\mu\gamma^\nu\gamma^\rho k_\rho] \\
&= Tr[\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho]k'_\sigma k_\rho && \text{(Cf. (8.77))} \\
&= 4(g^{\sigma\mu}g^{\rho\nu} - g^{\sigma\rho}g^{\mu\nu} + g^{\sigma\nu}g^{\mu\rho})k'_\sigma k_\rho && \text{(Cf wikipedia : “gamma matrices”)} \\
&= 4g^{\sigma\mu}g^{\rho\nu}k'_\sigma k_\rho - g^{\sigma\rho}g^{\mu\nu}k'_\sigma k_\rho + g^{\sigma\nu}g^{\mu\rho}k'_\sigma k_\rho \\
&= 4(k'^\mu k^\nu - k'^\rho g^{\mu\nu}k_\rho + k'^\nu k^\mu) \\
&= 4(k'^\mu k^\nu + k'^\nu k^\mu - (k' \cdot k)g^{\mu\nu})
\end{aligned}$$

And,

$$Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \qquad \text{(Cf wikipedia : “gamma matrices”)}$$

So,

$$\begin{aligned}
L^{\mu\nu} &= \frac{1}{2}Tr[(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu] \\
&= \frac{1}{2}(Tr[\not{k}'\gamma^\mu\not{k}\gamma^\nu] + m^2 Tr(\gamma^\mu\gamma^\nu)) \\
&= \frac{1}{2}[4(k'^\mu k^\nu + k'^\nu k^\mu - (k' \cdot k)g^{\mu\nu}) + m^2 4g^{\mu\nu}] \\
&= 2[k'^\mu k^\nu + k'^\nu k^\mu - (k' \cdot k)g^{\mu\nu}] + 2m^2 g^{\mu\nu} && \text{(This is exactly equation (8.78))}
\end{aligned}$$

Verification is done.

Problems 8.9, P_{246}. Verify L^{00} as equation (8.79).

Solution of problem 8.9 :

Equation (8.79) reads,

$$L^{00} = 4E^2(1 - v^2 sin^2\theta/2)$$

From problem 8.8, we know,

$$L^{\mu\nu} = 2[k^{'\mu}k^{\nu} + k^{'\nu}k^{\mu} - (k' \cdot k)g^{\mu\nu}] + 2m^2 g^{\mu\nu}$$

So,

$$\begin{aligned}
L^{00} &= 2[k^{'0}k^0 + k^{'0}k^0 - (k' \cdot k)g^{00}] + 2m^2 g^{00} \\
&= 2(2E^2 - k' \cdot k) + 2m^2 && (E' = E) \\
&= 2(2E^2 - E'E + \boldsymbol{k'}\boldsymbol{k}) + 2m^2 \\
&= 2(E^2 + \boldsymbol{k'}\boldsymbol{k} + m^2) && (E' = E) \\
&= 2(E^2 + m^2 + |\boldsymbol{k'}||\boldsymbol{k}|cos\theta) \\
&= 2[E^2 + E^2(1 - v^2) + cos\theta E^2 v^2] && \text{(Cf solution of problem 8.5)} \\
&= 2E^2[2 - v^2(1 - cos\theta)] \\
&= (2E)^2(1 - v^2 sin^2\frac{\theta}{2}) && \text{(This is just equation (8.79))}
\end{aligned}$$

Proof is done.

Problems 8.15, P_{246}.

Check the gauge invariance of $\mathcal{M}_{\gamma e^-}$ given by (8.161), by showing that if ϵ_μ is replaced by k_μ, or ϵ^*_ν by k'_ν, the result is zero.

Solution of problem 8.15 :

After replacing ϵ_μ by k_μ, (8.161) becomes,

$$-e^2\epsilon^*_\nu(k',\lambda')\bar{u}(p',s')\gamma^\nu\frac{(\not{p}+\not{k}+m)}{(p+k)^2-m^2}\not{k}u(p,s)-e^2\epsilon^*_\nu(k',\lambda')\bar{u}(p',s')\not{k}\frac{(\not{p}+\not{k}+m)}{(p-k')^2+m^2}\gamma^\nu u(p,s)$$

In the first term, replace $\not{k}u(p,s)$ by

$$\begin{aligned}\not{k}u(p,s) &= (\not{k}'+\not{p}'-\not{p})u(p,s) && \text{(4-momentum conservation)}\\ &= (\not{k}'+\not{p}'-m)u(p,s) && (\not{p}\psi=m\psi\text{, Dirac equation, (A.25) in }P_{455}\text{, Mandl and Shaw)}\\ &= (\not{k}+\not{p}-m)u(p,s) && \text{(4-momentum conservation)}\end{aligned}$$

The first term is then

$$-e^2\epsilon^*_\nu(k',\lambda')\bar{u}\gamma^\nu u$$

In the second term, replace $\bar{u}(p',s')\not{k}$ by

$$\begin{aligned}&\bar{u}(p',s')(\not{k}'+\not{p}'-\not{p})\\ &= \bar{u}(p',s')(\not{k}'-\not{p}+m) && (\bar{\psi}\not{p}=m\bar{\psi}\text{, Dirac equation, (A.26) in }P_{455}\text{, Mandl and Shaw)}\\ &= -\bar{u}(p',s')(\not{p}-\not{k}'-m)\end{aligned}$$

The second term becomes

$$e^2\epsilon^*_\nu(k',\lambda')\bar{u}\gamma^\nu u$$

Which cancels the first term.

Similarly when ϵ^*_ν is replaced by k'_ν.

This is another way to prove “Lorentz invariance”, saying, substitute ϵ^μ by k^μ. This way, the ϵ' in equation (8.162) will obtain same result as ϵ, since the k^μ part will give zero.

Problems 8.16, P_{246}. (a) The spin-averaged squared amplitude for lowest-order electron Compton scattering contains the interference term

$$\sum_{\lambda,\lambda',s,s'} \mathcal{M}^{(s)}_{\gamma e^-}\mathcal{M}^{(u)*}_{\gamma e^-}$$

where (s) and (u) refer to the $s-$ and $u-$ channel processes of figure 8.14(a) and (b) respectively. Obtain an expression analogous to (8.171) for this term, and prove that it is, in fact, zero. [Hint : use relations (J.4) and (J.5).]

(b) Explain why the term

$$\sum_{\lambda,\lambda',s,s'} \mathcal{M}^{(s)}_{\gamma e^-}\mathcal{M}^{(u)*}_{\gamma e^-}$$

is given by (8.176) with s and u interchanged.

Solution of problem 8.16 (a) :

As in section 8.6.3, suppose the particle is massless(for u and u'). So,

$$\begin{aligned}
&\sum_{\lambda,\lambda',s,s'} \mathcal{M}^{(s)}_{\gamma e^-}\mathcal{M}^{(u)*}_{\gamma e^-} \\
&= \frac{e^4}{su}\sum_{\lambda,\lambda',s,s'} \epsilon^*_\nu(k',\lambda')\epsilon_\rho(k',\lambda')\epsilon_\mu(k,\lambda)\epsilon^*_\sigma(k,\lambda)\bar{u}(p',s')\gamma^\nu(\not{p}+\not{k})\gamma^\mu u(p,s)\bar{u}(p,s)\gamma^\rho(\not{p}-\not{k}')\gamma^\sigma u(p',s') \\
&= \frac{e^4}{su}g_{\nu\rho}g_{\mu\sigma}Tr\{\gamma^\nu(\not{p}+\not{k})\gamma^\mu\not{p}\gamma^\rho(\not{p}-\not{k}')\gamma^\sigma\not{p}'\} \qquad \text{(u, u' can move freely c.f. (8.62). } \textstyle\sum u\bar{u}=\not{p}\text{, cf (7.61))} \\
&= \frac{e^4}{su}Tr\{\gamma^\nu(\not{p}+\not{k})\gamma^\mu\not{p}\gamma_\nu(\not{p}-\not{k}')\gamma_\mu\not{p}'\} \qquad \text{(move } g_{\nu\rho}g_{\mu\sigma}\text{ inside)}
\end{aligned}$$

From (J.5), $\gamma^\mu\not{p}\gamma_\nu(\not{p}-\not{k}')\gamma_\nu = -2(\not{p}-\not{k}')\gamma_\nu\not{p}$.

From (J.4), $\gamma^\nu(\not{p}+\not{k})(\not{p}-\not{k}')\gamma_\nu = 4(p+k)\cdot(p-k')$

From (J.30), $Tr(\not{p}\not{p}') = 4p\cdot p'$.

$$\sum_{\lambda,\lambda',s,s'} \mathcal{M}^{(s)}_{\gamma e^-}\mathcal{M}^{(u)*}_{\gamma e^-} = -32\frac{e^4}{su}(p+k)\cdot(p-k')\cdot p'\cdot p$$

And,

$$\begin{aligned}
(p+k)\cdot(p-k') &= p^2 - p\cdot k' + k\cdot p - k\cdot k' \\
&= -p'\cdot k + k\cdot p - k\cdot k' && \text{(massless, so } p^2=0\text{)} \\
&= k\cdot(p-p'-k') && (p'\cdot k = k\cdot p'\text{, from squaring } p-k'=p'-k) \\
&= -k^2
\end{aligned}$$

This therefore vanishes when the photon is on-shell ($k^2=0$).

Problems 8.18, P_{246}.

(a) Derive an expression for the spin-averaged differential cross section for lower-order $e^-\mu^-$ scattering in the laboratory frame, defined by $p^\mu = (M, \mathbf{0})$ where M is now the muon mass, and shown that it may be written in the form

$$\frac{d\sigma}{d\Omega} = (\frac{d\sigma}{d\Omega})_{ns}[1 - (q^2/2M^2)tan^2(\theta/2)]$$

where the 'no-structure' cross section is that of e^-s^+ scattering and the electron mass has, as usual, been neglected.

(b) Neglecting all masses, evaluate the spin-averaged expression (8.183) in terms of s, u and t and use the result

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\frac{1}{4}\sum_{r,r';s,s'}|\mathcal{M}_{e^-\mu^-}(r,s;r',s')|^2$$

to show that the $e^-\mu^-$ cross section may be written in the form

$$\frac{d\sigma}{dt} = \frac{4\pi\alpha^2}{t^2}\frac{1}{2}(1+\frac{u^2}{s^2})$$

Show also that by introducing the variable y, defined in terms of laboratory variables by $y = (k-k')/k$, this reduces to the result

$$\frac{d\sigma}{dy} = \frac{4\pi\alpha^2}{t^2}s\frac{1}{2}[1+(1-y)^2].$$

Proof of problem (8.18)

(a)

We can calculate $L_{\mu\nu M^{\mu\nu}_{eff}}$ using (8.185) and (8.190).

$$\begin{aligned}
L_{\mu\nu M^{\mu\nu}_{eff}} &= 4[k'_\mu k_\nu + k'_\nu k_\mu + (q^2/2)g_{\mu\nu}][2p^\mu p^\nu + (q^2/2)g^{\mu\nu}]\\
&= 4[2k'_\mu k_\nu p^\mu p^\nu + q^2/2k'_\mu k_\nu g^{\mu\nu} + 2k'_\nu k_\mu p^\mu p^\nu + q^2/2k'_\nu k_\mu g^{\mu\nu} + q^2 g_{\mu\nu}p^\mu p^\nu + q^4]\\
&= 4[4k'\cdot pk\cdot p + q^2 k\cdot k' + q^2p^2 + q^4] && (?k'_\mu k_\nu p^\mu p^\nu = k'\cdot pk\cdot p?)\\
&= 4[4k'\cdot pk\cdot p + q^2(-q^2/2) + q^2M^2 + q^4] && (?\ k\cdot k' = -q^2?)\\
&= 8[2k'\cdot pk\cdot p + (q^2)^2/4 + q^2M^2/2]
\end{aligned}$$

In "laboratory" frame with $p^\mu = (M, 0)$, neglecting the electron mass so that $\omega = |\boldsymbol{k}| \equiv k, \omega' = |\boldsymbol{k}'| \equiv k'$, and using (8.219), the preceding expression becomes

$$\begin{aligned}
L_{\mu\nu M^{\mu\nu}_{eff}} &= 8[2kk'M^2 + 4k^2{k'}^2 sin^4\theta/2 - 2kk'M^2 sin^2\theta/2]\\
&= 8[2kk'M^2(1 - sin^2\theta/2) + 4k^2{k'}^2 sin^4\theta/2]\\
&= [16kk'M^2cos^2\theta/2][1 + 2\frac{kk'}{M^2}sin^2\theta/2tan^2\theta/2]\\
&= [16kk'M^2cos^2\theta/2][1 - (q^2/2M^2)tan^2\theta/2]\\
&= (\frac{d\sigma}{d\Omega})_{ns}[1 - (q^2/2M^2)tan^2(\theta/2)] && \text{(k.24)}
\end{aligned}$$

(b)

Considering all particles are massless, $s = 2k\cdot p, t = -2k\cdot k'$ and $u = -2p\cdot k'$. Hence, the expression becomes,

$$\begin{aligned} L_{\mu\nu M^{\mu\nu}_{eff}} &= 4[4k'\cdot pk\cdot p + q^2(-q^2/2) + q^2M^2 + q^4] \\ &= 4[-us - t^2/2 + t^2] \\ &= 4[-us + t^2/2] \end{aligned}$$

$d\sigma/dt$ is then given be

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{1}{16\pi}\frac{(4\pi\alpha)^2}{t^2}4(-\frac{u}{s} + \frac{t}{2s^2}) \\ &= \frac{4\pi\alpha^2}{t^2}\frac{1}{2}(1 + \frac{u^2}{s^2}) && \text{(massless} \Rightarrow s + t + u = 0 \Rightarrow t^2 = (u+s)^2) \\ &= \frac{4\pi\alpha^2}{t^2}s\frac{1}{2}(1 + (1-y)^2) && ((\text{K.41}) + (\text{K.46}) + (\text{K.50})) \Rightarrow dy = \tfrac{k'^2}{2\pi kM}2\pi\tfrac{dt}{2k'^2} \Rightarrow dt = sdy) \end{aligned}$$

Problems 8.19, P_{247}.

Consider the process $e^+e^- \to \mu^+\mu^-$ in the CM frame.

(a) Draw the lowest-order Feynman diagram and write down the corresponding amplitude.

(b) Show that the spin-averaged squared matrix element has the form

$$|\bar{\mathcal{M}}|^2 = \frac{(4\pi a)^2}{Q^4} L(e)_{\mu\nu} L(\mu)^{\mu\nu}$$

where Q^2 is the square of the total CM energy, and $L(e)$ depends on the e^- and e^+ momenta and $L(\mu)$ on those of the μ^+, μ^-.

(c) Evaluate the traces and the tensor contraction(neglecting lepton masses): (i) directly, using the trace theorems; and (ii) by using crossing symmetry and the results of section 8.7 for $e^-\mu^-$ scattering. hence show that

$$|\bar{\mathcal{M}}|^2 = (4\pi a)^2 (1 + cos^2\theta)^2$$

where θ is the CM scattering angle.

(d) Hence show that the total elastic cross section is (see equation (B.18) of appendix B)

$$\sigma = 4\pi\alpha^2/3Q^2$$

Figure 8.20 shows data (a) for σ in the $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \tau^+\tau^-$ and (b) for the angular distribution in $e^+e^- \to \mu^+\mu^-$. The data in (a) agree well with the prediction of part (d). The broken curve in figure 8.20(b) shows the pure QED prediction

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4Q^2}(1 + cos^2\theta)$$

It is clear that, while the distribution has the general $1 + cos^2\theta$ form as predicted, there is a small but definite forward-backward asymmetry. This arises because, in addition to the γ- exchange amplitude there is also a Z^0- exchange amplitude which we have neglected. Such asymmetries are an important test of the electroweak theory. They are too small to be visible in the total cross sections in (a).

Proof of problem (8.19)

(a)

We label the 4-momentum and spin of the ingoing e^- by k, s, of the ingoing e^+ by k_1, s_1, of the outgoing μ^- by p', r', and of the outgoing μ^+ by p_1, r_1. The Mandelstam variables are

$$s = Q^2 = (k + k_1)^2 = (p' + p_1)^2$$
$$t = (k_1 - p_1)^2 = (k - p')^2$$
$$u = (k - p_1)^2 = (k_1 - p')^2$$

The amplitude is,

$$ie\bar{v}(k_1, s_1)\gamma^\mu u(k, s)\frac{-ig_\mu\nu}{Q^2}ie\bar{u}(p', r')\gamma_\nu v(p_1, r_1)$$

Note that the $\bar{v}\gamma^\mu u$ factor depends only on the e^- and e^+ momenta, while the $\bar{u}\gamma_\nu v$ factor depends only on the momenta of the $\mu-$ and μ^+.

(b)

The spin-averaged squared cross matrix element is then

$$
\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{1}{4}(\frac{e^2}{Q^2})\sum_{s,s_1}\bar{v}(k_1,s_1)\gamma_\mu u(k,s)\bar{u}(k,s)\gamma_\nu v(k_1,s_1)\times\sum_{r',r_1}\bar{u}(p',r')\gamma^\mu v(p_1,r_1)\bar{v}(p_1,r_1)\gamma^\nu u(p',r') \\
&= \frac{(4\pi\alpha)^2}{Q^4}L(e)_{\mu\nu}L(\mu)^{\mu\nu}
\end{aligned}
$$

where

$$L(e)_{\mu\nu} = \frac{1}{2}Tr[(\not{k}_1 - m)\gamma_\mu(\not{k} + m)\gamma_\nu]$$

and

$$L(\mu)_{\mu\nu} = \frac{1}{2}Tr[(\not{p}' + M)\gamma^\mu(\not{p}_1 - M)\gamma^\nu]$$

using (7.61).

(c .1.)

Using the trace theorems as in(8.78), the lepton tensors are

$$L(e)_{\mu\nu} = 2[k_{1\mu}k_\nu + k_{1\nu}k_\mu - (k_1\cdot k)g_{\mu\nu}]$$

and

$$L(\mu)^{\mu\nu} = 2[p'^\mu p_1^\nu + p'^\nu p_1^\mu - (p'\cdot p_1)g^{\mu\nu}]$$

where now (in the massless limit) $Q^2/2 = p'\cdot p_1 = k_1\cdot k$. Hence

$$
\begin{aligned}
|\bar{\mathcal{M}}|^2 &= (\frac{e^2}{Q^2})^2 4[2p'\cdot k_1 p_1\cdot k + 2p'\cdot k p_1\cdot k_1 - Q^2 p'\cdot p_1 - Q^2 p'\cdot p_1 - Q^2 k_1\cdot k + (Q^2)^2] \\
&= (\frac{e^2}{Q^2})^2 4[2\frac{u^2}{4} + 2\frac{t^2}{4} - Q^2Q^2/2 - Q^2Q^2/2 + (Q^2)^2] \qquad \text{(square the } u \text{ and } t \text{ in (a))} \\
&= (\frac{e^2}{Q^2})^2 2[t^2 + u^2]
\end{aligned}
$$

In the massless limit, all 3-momenta have equal modulus which (slightly confusingly) we denote by k. Then

$$t = -2k^2(1 - cos\theta), u = -2k^2(1 + cos\theta) \qquad ((\text{K.16 - 18}))$$

So that,

$$t^2 + u^2 = 8k^4(1 + cos^2\theta)$$

Therefore,

$$
\begin{aligned}
|\bar{\mathcal{M}}|^2 &= (\frac{e^2}{Q^2})^2 2[t^2 + u^2] \\
&= (\frac{e^2}{4k^2})^2 2[8k^4(1 + cos^2\theta)] \qquad \text{(from (a), } Q^2 = 4k^2\text{)} \\
&= e^4(1 + cos^2\theta) \\
&= (4\pi\alpha)^2(1 + cos^2\theta)
\end{aligned}
$$

(c .2.)

Crossing symmetry(P_{230}-P_{233}) implies that the amplitude for

$$e^-(k,s) + e^+(k_1,s_1) \to \mu^-(p',r') + \mu^+(p_1,r_1)$$

is equal to (minus) the amplitude for

$$e^-(k,s) + \mu^-(-p_1,-r_1) \to \mu^-(p',r') + e^-(-k_1,-s_1) \qquad \text{(not consistent with the last line of } P_{232}\text{)}$$

(d)

Using the formula (6.129) for the differential cross section for elastic scattering in the centre of mass, and replacing $|\mathcal{M}_{fi}|^2$ by $|\bar{\mathcal{M}}|^2$, we obtain

$$\begin{aligned}\sigma &= \int \frac{d\sigma}{d\Omega} \\ &= \int_{-1}^{1} \frac{1}{64\pi^2 Q^2} 16\pi^2\alpha^2(1+cos^2\theta)2\pi dcos\theta \\ &= 4\pi\alpha^2/3Q^2\end{aligned}$$

This is exactly the result we need.