Gauge theories in particle physics(3rd, Aitchison and Hey)

Chapter 6, Quantum field theory II : Interacting scalar fields

Problems 6.3, P_{171}.

Let $\hat{\phi}(x,t)$ be a real scalar KG field one space dimension, sataisfying

$$(\Box_x + m^2)\hat{\phi}(x,t) \equiv (\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2)\hat{\phi}(x,t) = 0$$

(i) Explain why

$$T(\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)) = \theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)$$

Solution of (i):

$t_1 > t_2 \Rightarrow \theta(t_1-t_2) = 1, \theta(t_2-t_1) = 0 \Rightarrow T(\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)) = \theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)$

$t_1 < t_2 \Rightarrow \theta(t_1-t_2) = 0, \theta(t_2-t_1) = 1 \Rightarrow T(\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)) = \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)$

Combining the above two lines proves (i) immediately.

(ii)Using equation $(E.46)$, show that

$$\frac{d}{dx}\theta(x-a) = \delta(x-a)$$

Solutions of (ii) :

$(E.46)$ says,

$$\int_{-\infty}^{x} \delta(x'-a)dx' = \theta(x-a)$$

Simply derivate both side of $(E.46)$ one could prove (ii).

(iii)Using the result of (ii) with appropriate changes of variable, and equation (5.105), show that

$$\frac{\partial}{\partial t_1}\{T[\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)]\} = \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)$$

Solutions of (iii):

$$\begin{aligned}\frac{\partial}{\partial t_1}\{T[\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)]\} &= \frac{\partial}{\partial t_1}[\theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)] \\ &= \frac{\partial}{\partial t_1}[\theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)] + \frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)]\end{aligned}$$

Calculating the first term:

$$\begin{aligned}\frac{\partial}{\partial t_1}[\theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)] &= \{\frac{d}{dt_1}[\theta(t_1-t_2)]\hat{\phi}(x_1,t_1) + \theta(t_1-t_2)\frac{d}{dt_1}[\hat{\phi}(x_1,t_1)]\}\hat{\phi}(x_2,t_2) \\ &= [\delta(t_1-t_2)\hat{\phi}(x_1,t_1) + \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)]\hat{\phi}(x_2,t_2) \\ &= \delta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2) \qquad (1)\end{aligned}$$

Calculating the second term:

$$\frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)] = \hat{\phi}(x_2,t_2)\frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_1,t_1)]$$

(because $\hat{\phi}(x_2,t_2)$ is constant to $\frac{\partial}{\partial t_1}$, so it could be moved in front of the expression)

$$= \hat{\phi}(x_2,t_2)\{\frac{d}{dt_1}[\theta(t_2-t_1)]\hat{\phi}(x_1,t_1) + \theta(t_2-t_1)\frac{d}{dt_1}[\hat{\phi}(x_1,t_1)]\}$$

$$= \hat{\phi}(x_2,t_2)[-\delta(t_2-t_1)\hat{\phi}(x_1,t_1) + \theta(t_2-t_1)\dot{\hat{\phi}}(x_1,t_1)]$$

$$= -\hat{\phi}(x_2,t_2)\delta(t_2-t_1)\hat{\phi}(x_1,t_1) + \hat{\phi}(x_2,t_2)\theta(t_2-t_1)\dot{\hat{\phi}}(x_1,t_1) \qquad (2)$$

Add up (1) and (2),

$$\delta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2) - \hat{\phi}(x_2,t_2)\delta(t_2-t_1)\hat{\phi}(x_1,t_1) + \hat{\phi}(x_2,t_2)\theta(t_2-t_1)\dot{\hat{\phi}}(x_1,t_1)$$

$$= \delta(t_1-t_2)[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) - \hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)] + \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)$$

(because $\delta(x)$ is even function, so $\delta(t_1-t_2) = \delta(t_2-t_1)$)

$$= \delta(t_1-t_2)[\hat{\phi}(x_1,t_1), \hat{\phi}(x_2,t_2)] + \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)$$

(" $[\hat{\phi}(x_1,t_1), \hat{\phi}(x_2,t_2)] = 0$ " is a typical commutation relation of $\hat{\phi}(x)$, cf (5.105))

$$= \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)$$

(iv)Using (5.109) and (5.104) show that

$$\frac{\partial^2}{\partial t_1^2}\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\} = -i\delta(x_1-x_2)\delta(t_1-t_2) + T[\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)]$$

and hence show that

$$(\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_1^2} + m^2)\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\} = -i\delta(x_1-x_2)\delta(t_1-t_2)$$

This shows that $T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]$ is a Green function(See appendix G, equation $(G.25)$ – the i is included here conventionally) for the KG operator

$$\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_1^2} + m^2$$

The four-dimensional generalization is immediate.

Solutions of (iv):

$$\frac{\partial^2}{\partial t_1^2}\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\}$$

$$= \frac{\partial}{\partial t_1}[\theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)]$$

$$= \frac{\partial}{\partial t_1}[\theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2)] + \frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)]$$

The first term is :

$$\frac{\partial}{\partial t_1}[\theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)]$$

$$= \{\frac{d}{dt_1}[\theta(t_1-t_2)]\dot{\hat{\phi}}(x_1,t_1) + \theta(t_1-t_2)\frac{d}{dt_1}[\dot{\hat{\phi}}(x_1,t_1)]\}\hat{\phi}(x_2,t_2)$$

$$= [\delta(t_1-t_2)]\dot{\hat{\phi}}(x_1,t_1) + \theta(t_1-t_2)\ddot{\hat{\phi}}(x_1,t_1)]\hat{\phi}(x_2,t_2)$$

$$= \delta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_1-t_2)\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)$$

Similarly, the second term is :

$$\begin{aligned}
&\frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)] \\
&= \hat{\phi}(x_2,t_2)\{\frac{d}{dt_1}[\theta(t_2-t_1)]\dot{\hat{\phi}}(x_1,t_1)+\theta(t_2-t_1)\frac{d}{dt_1}[\dot{\hat{\phi}}(x_1,t_1)]\} \\
&= \hat{\phi}(x_2,t_2)[-\delta(t_2-t_1)]\dot{\hat{\phi}}(x_1,t_1)+\theta(t_2-t_1)\ddot{\hat{\phi}}(x_1,t_1)] \\
&= -\hat{\phi}(x_2,t_2)\delta(t_2-t_1)\dot{\hat{\phi}}(x_1,t_1)+\hat{\phi}(x_2,t_2)\theta(t_2-t_1)\ddot{\hat{\phi}}(x_1,t_1) \\
&= -\delta(t_1-t_2)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\ddot{\hat{\phi}}(x_1,t_1)
\end{aligned}$$

Adding up the two terms :

$$\begin{aligned}
&\frac{\partial^2}{\partial t_1^2}\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\} \\
&= \frac{\partial}{\partial t_1}[\theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2)]+\frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)] \\
&= \delta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)+\theta(t_1-t_2)\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)-\delta(t_1-t_2)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\ddot{\hat{\phi}}(x_1,t_1) \\
&= \delta(t_1-t_2)[\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)-\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)]+\theta(t_1-t_2)\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\ddot{\hat{\phi}}(x_1,t_1) \\
&\qquad (\text{“ } \dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)-\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)=i\delta(x_1-x_2) \text{ ” , cf (5.104) and (5.109))} \\
&= -i\delta(x_1-x_2)\delta(t_1-t_2)+T[\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)] \qquad (3)
\end{aligned}$$

Next, calculating $\frac{\partial^2}{\partial x_1^2}\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\}$. The first step is calculating $\frac{\partial}{\partial x_1}\{T[\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)]\}$.

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\{T[\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)]\} \\
&= \frac{\partial}{\partial x_1}[\theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)] \\
&= \frac{\partial}{\partial x_1}[\theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]+\frac{\partial}{\partial x_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)] \\
&= \theta(t_1-t_2)\frac{\partial}{\partial x_1}[\hat{\phi}(x_1,t_1)]\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\frac{\partial}{\partial x_1}[\hat{\phi}(x_1,t_1)]
\end{aligned}$$

Then,

$$\begin{aligned}
&\frac{\partial^2}{\partial x_1^2}\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\} \\
&= \frac{\partial}{\partial x_1}\{\theta(t_1-t_2)\frac{\partial}{\partial x_1}[\hat{\phi}(x_1,t_1)]\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\frac{\partial}{\partial x_1}[\hat{\phi}(x_1,t_1)]\} \\
&= \theta(t_1-t_2)\frac{\partial^2}{\partial x_1^2}[\hat{\phi}(x_1,t_1)]\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\frac{\partial^2}{\partial x_1^2}[\hat{\phi}(x_1,t_1)]
\end{aligned}$$

According to the assumption of this problem,

$$(\Box_x+m^2)\hat{\phi}(x,t)\equiv(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+m^2)\hat{\phi}(x,t)=0$$

One can easily get, $\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+m^2=0$, or $\frac{\partial^2}{\partial x^2}-m^2=\frac{\partial^2}{\partial t^2}$. So,

$$\begin{aligned}
&(\frac{\partial^2}{\partial x_1^2}-m^2)\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\} \\
&= \theta(t_1-t_2)\frac{\partial^2}{\partial t_1^2}[\hat{\phi}(x_1,t_1)]\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\frac{\partial^2}{\partial t_1^2}[\hat{\phi}(x_1,t_1)] \\
&= T[\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)] \qquad (4)
\end{aligned}$$

Now, substituting the expression of (3) and (4) into the following term

$$
\begin{aligned}
&(\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_1^2} + m^2)\{T[\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2)]\} \\
&= \frac{\partial^2}{\partial t_1^2}\{T[\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2)]\} - (\frac{\partial^2}{\partial x_1^2} - m^2)\{T[\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2)]\} \\
&= -i\delta(x_1 - x_2)\delta(t_1 - t_2) + T[\ddot{\hat{\phi}}(x_1, t_1)\hat{\phi}(x_2, t_2)] - T[\ddot{\hat{\phi}}(x_1, t_1)\hat{\phi}(x_2, t_2)] \\
&= -i\delta(x_1 - x_2)\delta(t_1 - t_2)
\end{aligned}
$$

Problems 6.4, P_{172}.Verify (6.90)

$$\langle 0|\hat{a}_A(p'_A)\hat{\phi}_A(x_1)|0\rangle = \frac{1}{\sqrt{2E_A}}e^{ip'_A\cdot x_1}$$

Solution of *problem 6.4*:

$$\begin{aligned}
&\langle 0|\hat{a}_A(p'_A)\hat{\phi}_A(x_1)|0\rangle \\
&= \langle 0|\hat{a}_A(p'_A)\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3\sqrt{2E_k}}[\hat{a}_A(k)e^{-ik\cdot x_1} + \hat{a}^\dagger_A(k)e^{ik\cdot x_1}]|0\rangle \\
&= \langle 0|\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3\sqrt{2E_k}}[\hat{a}_A(p'_A)\hat{a}_A(k)e^{-ik\cdot x_1} + \hat{a}_A(p'_A)\hat{a}^\dagger_A(k)e^{ik\cdot x_1}]|0\rangle \\
&= \int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3\sqrt{2E_k}}\langle 0|[\hat{a}_A(p'_A)\hat{a}_A(k)]|0\rangle e^{-ik\cdot x_1} + \int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3\sqrt{2E_k}}\langle 0|[\hat{a}_A(p'_A)\hat{a}^\dagger_A(k)]|0\rangle e^{ik\cdot x_1} \\
&= \int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3\sqrt{2E_k}}\langle 0|[\hat{a}_A(p'_A)\hat{a}^\dagger_A(k)]|0\rangle e^{ik\cdot x_1}. \qquad (\hat{a}|0\rangle = 0 \text{ , cf } (6.70)) \\
&= \int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3\sqrt{2E_k}}\langle 0|\{[\hat{a}_a(p'_A),\hat{a_A}^\dagger(k)] + \hat{a}^\dagger_A(k)\hat{a}_A(p'_A)]\}|0\rangle e^{ik\cdot x_1} \\
&= \int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3\sqrt{2E_k}}\langle 0|[\hat{a}_a(p'_A),\hat{a_A}^\dagger(k)]|0\rangle e^{ik\cdot x_1}. \qquad (\hat{a}|0\rangle = 0 \text{ , cf } (6.70)) \\
&= \int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3\sqrt{2E_k}}(2\pi)^3\delta^3(p'_A - k)e^{ik\cdot x_1} \\
&= \frac{1}{\sqrt{2E_A}}e^{ip'_A\cdot x_1}
\end{aligned}$$

Problems 6.5, P_{172}.Verify (6.92)

$$\langle 0|T(\hat{\phi}_c(x_1)\hat{\phi}_c(x_2))|0\rangle = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}}[\theta(t_1-t_2)e^{-i\omega_k(t_1-t_2)+i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)} + \theta(t_2-t_1)e^{-i\omega_k(t_2-t_1)+i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}]$$

Solution of *problem 6.5*:

$$\begin{aligned}
&\langle 0|T(\hat{\phi}_c(x_1)\hat{\phi}_c(x_2))|0\rangle \\
&= \langle 0|\theta(t_1-t_2)\hat{\phi}_c(x_1,t_1)\hat{\phi}_c(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}_c(x_2,t_2)\hat{\phi}_c(x_1,t_1)|0\rangle \\
&= \langle 0|\theta(t_1-t_2)\hat{\phi}_c(x_1,t_1)\hat{\phi}_c(x_2,t_2)|0\rangle + \langle 0|\theta(t_2-t_1)\hat{\phi}_c(x_2,t_2)\hat{\phi}_c(x_1,t_1)|0\rangle
\end{aligned}$$

Calculating the first term, $\langle 0|\theta(t_1-t_2)\hat{\phi}_c(x_1,t_1)\hat{\phi}_c(x_2,t_2)|0\rangle$. Substituting the expression of $\hat{\phi}_c(x_1)$ and $\hat{\phi}_c(x_2)$ into it, cf. (6.52).

$$\begin{aligned}
&\langle 0|\theta(t_1-t_2)\hat{\phi}_c(x_1,t_1)\hat{\phi}_c(x_2,t_2)|0\rangle \\
&= \langle 0|\theta(t_1-t_2)\int \frac{d^3k_1}{(2\pi)^3\sqrt{2E_{k1}}}[\hat{a}_c(k_1)e^{-ik_1\cdot x_1} + \hat{a}_c^\dagger(k_1)e^{ik_1\cdot x_1}]\cdot \int \frac{d^3k_2}{(2\pi)^3\sqrt{2E_{k_2}}}[\hat{a}_c(k_2)e^{-ik_2\cdot x_2} + \hat{a}_c^\dagger(k_2)e^{ik_2\cdot x_2}]|0\rangle \\
&= \theta(t_1-t_2)\int \frac{d^3k_1}{(2\pi)^3\sqrt{2E_{k1}}}[\langle 0|\hat{a}_c(k_1)e^{-ik_1\cdot x_1} + \langle 0|\hat{a}_c^\dagger(k_1)e^{ik_1\cdot x_1}]\cdot \int \frac{d^3k_2}{(2\pi)^3\sqrt{2E_{k_2}}}[\hat{a}_c(k_2)e^{-ik_2\cdot x_2}|0\rangle + \hat{a}_c^\dagger(k_2)e^{ik_2\cdot x_2}|0\rangle] \\
&= \theta(t_1-t_2)\int \frac{d^3k_1}{(2\pi)^3\sqrt{2E_{k1}}}[\langle 0|\hat{a}_c(k_1)e^{-ik_1\cdot x_1}]\cdot \int \frac{d^3k_2}{(2\pi)^3\sqrt{2E_{k_2}}}[\hat{a}_c^\dagger(k_2)e^{ik_2\cdot x_2}|0\rangle] \\
&= \langle 0|\ \{\theta(t_1-t_2)\int \frac{d^3k_1}{(2\pi)^3\sqrt{2E_{k1}}}[\hat{a}_c(k_1)e^{-ik_1\cdot x_1}]\cdot \int \frac{d^3k_2}{(2\pi)^3\sqrt{2E_{k_2}}}[\hat{a}_c^\dagger(k_2)e^{ik_2\cdot x_2}]\}\ |0\rangle \\
&= \langle 0|\ \{\theta(t_1-t_2)\int \frac{d^3k_1}{(2\pi)^3\sqrt{2E_{k1}}}\int \frac{d^3k_2}{(2\pi)^3\sqrt{2E_{k_2}}}[\hat{a}_c(k_1)e^{-ik_1\cdot x_1}]\cdot[\hat{a}_c^\dagger(k_2)e^{ik_2\cdot x_2}]\}\ |0\rangle \\
&\qquad\qquad (\textstyle\int xdx\cdot\int ydy = \int[\int(x\cdot y)dy]dx) \\
&= \langle 0|\ \{\theta(t_1-t_2)\int \frac{d^3k_1}{(2\pi)^3\sqrt{2E_{k1}}}\int \frac{d^3k_2}{(2\pi)^3\sqrt{2E_{k_2}}}[\hat{a}_c(k_1)\hat{a}_c^\dagger(k_2)]e^{-ik_1\cdot x_1+ik_2\cdot x_2}\}\ |0\rangle \\
&= \langle 0|\ \{\theta(t_1-t_2)\int \frac{d^3k_1}{(2\pi)^3\sqrt{2E_{k1}}}\int \frac{d^3k_2}{(2\pi)^3\sqrt{2E_{k_2}}}\{[\hat{a}_c(k_1),\hat{a}_c^\dagger(k_2)] - \hat{a}_c^\dagger(k_2)\hat{a}_c(k_1)\}e^{-ik_1\cdot x_1+ik_2\cdot x_2}\}\ |0\rangle \\
&= \langle 0|\ \{\theta(t_1-t_2)\int \frac{d^3k_1}{(2\pi)^3\sqrt{2E_{k1}}}\int \frac{d^3k_2}{(2\pi)^3\sqrt{2E_{k_2}}}[\hat{a}_c(k_1),\hat{a}_c^\dagger(k_2)]e^{-ik_1\cdot x_1+ik_2\cdot x_2}\}\ |0\rangle. \quad (\hat{a}|0\rangle = 0\ ,\ \text{cf (6.70)}) \\
&= \theta(t_1-t_2)\int \frac{d^3k_1}{(2\pi)^3\sqrt{2E_{k1}}}\langle 0|\ \{\int \frac{d^3k_2}{(2\pi)^3\sqrt{2E_{k_2}}}[(2\pi)^3\delta^3(k_1-k_2)]e^{-ik_1\cdot x_1+ik_2\cdot x_2}\}\ |0\rangle \\
&= \theta(t_1-t_2)\int \frac{d^3k}{(2\pi)^3 2E_k}e^{-ik\cdot x_1+ik\cdot x_2} \qquad \text{(after integral, } k_1=k_2=k) \\
&= \theta(t_1-t_2)\int \frac{d^3k}{(2\pi)^3 2\omega_k}e^{-i(\omega_k t_1-\mathbf{k}\cdot\mathbf{x}_1)+i(\omega_k t_2-\mathbf{k}\cdot\mathbf{x}_2)} \\
&= \theta(t_1-t_2)\int \frac{d^3k}{(2\pi)^3 2\omega_k}e^{-i\omega_k(t_1-t_2)+i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}
\end{aligned}$$

Similarly, the second term $\langle 0|\theta(t_2-t_1)\hat{\phi}_c(x_2,t_2)\hat{\phi}_c(x_1,t_1)|0\rangle$ could result similar result.

$$\begin{aligned}
&\langle 0|\theta(t_2-t_1)\hat{\phi}_c(x_2,t_2)\hat{\phi}_c(x_1,t_1)|0\rangle \\
&= \theta(t_2-t_1)\int \frac{d^3k}{(2\pi)^3 2\omega_k}e^{-i\omega_k(t_2-t_1)+i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}
\end{aligned}$$

Combine the two terms, equation (6.92) was proved.

Problems 6.6, P_{172}.Verifying (6.99) and (6.100). After introduce new variables $x = x_1 + x_2, X = (x_1 + x_2)/2$, (6.91) could be expressed as (6.99) and (6.100), shown in following.

$$(-ig)^2(2\pi)^4\delta^4(P_A + P_B - P'_A - P'_B)\int d^4x e^{iq\cdot x}\int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\frac{i}{k^2 - m_c^2 + i\epsilon} \tag{6.99}$$

$$(-ig)^2(2\pi)^4\delta^4(P_A + P_B - P'_A - P'_B)\frac{i}{q^2 - m_c^2 + i\epsilon} \tag{6.100}$$

Proof of (6.99) (From (6.91)) :

$$\begin{aligned}
&(-ig)^2\int\int d^4x_1 d^4x_2 e^{i(P'_A - P_B)\cdot x_1}e^{i(P'_B - P_A)\cdot x_2}\langle 0|T(\hat{\phi}_c(x_1)\hat{\phi}_c(x_2))|0\rangle && (6.91)\\
&= (-ig)^2\int\int d^4x_1 d^4x_2 e^{i(P'_A - P_B)\cdot x_1 - i(P_A - P'_B)\cdot x_2}\int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x_1 - x_2)}\frac{i}{k^2 - m_c^2 + i\epsilon} && (\text{cf. } 6.98)\\
&= (-ig)^2\int\int d^4x_1 d^4x_2 e^{iq\cdot x_1 - iq\cdot x_2}\int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x_1 - x_2)}\frac{i}{k^2 - m_c^2 + i\epsilon}\\
&= (-ig)^2\int\int d^4x_1 d^4x_2 e^{iq\cdot x}\int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\frac{i}{k^2 - m_c^2 + i\epsilon} && (x \equiv x_1 - x_2)\\
&= (-ig)^2\int e^{iq\cdot x}d^4(x_1 - x_2)\int e^{i(P_A + P_B - P'_A - P'_B)\cdot x_2}d^4x_2\int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\frac{i}{k^2 - m_c^2 + i\epsilon}
\end{aligned}$$

(Four-dimension δ function: $\int_{-\infty}^{\infty} e^{i(k-k')\cdot x}d^4x = (2\pi)^4\delta^4(k' - k)$, for one-dimentsion δ function, Cf $(E.26)$)

$$\begin{aligned}
&= (-ig)^2(2\pi)^4\delta^4(P_A + P_B - P'_A - P'_B)\int e^{iq\cdot x}d^4x\int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\frac{i}{k^2 - m_c^2 + i\epsilon} && (\text{This is}(6.99))\\
&= (-ig)^2(2\pi)^4\delta^4(P_A + P_B - P'_A - P'_B)\int (e^{-ik\cdot x}e^{iq\cdot x})d^4x\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m_c^2 + i\epsilon}\\
&= (-ig)^2(2\pi)^4\delta^4(P_A + P_B - P'_A - P'_B)(2\pi)^4\delta^4(k - q)\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m_c^2 + i\epsilon} && (\text{Cf. } (E.26))\\
&= (-ig)^2(2\pi)^4\delta^4(P_A + P_B - P'_A - P'_B)\int \delta^4(k - q)d^4k\frac{i}{k^2 - m_c^2 + i\epsilon}\\
&= (-ig)^2(2\pi)^4\delta^4(P_A + P_B - P'_A - P'_B)\frac{i}{q^2 - m_c^2 + i\epsilon} && (\text{This is}(6.100))
\end{aligned}$$

Problems 6.7

Show that the contribution of the contractions (6.88) to the S-matrix element (6.74) is given by (6.101).

Proof of *Problem 6.6*:

$$
\begin{aligned}
&\langle 0|\hat{a}_A(p'_A)\hat{\phi}_A(x_1)|0\rangle\ \langle 0|\hat{\phi}_A(x_2)\hat{a}^\dagger_A(p_A)|0\rangle\ \langle 0|\hat{a}_B(p'_B)\hat{\phi}_B(x_1)|0\rangle\ \langle 0|\hat{\phi}_B(x_2)\hat{a}^\dagger_B(p_B)|0\rangle\ \langle 0|T(\hat{\phi}_C(x_1)\hat{\phi}_C(x_2))|0\rangle \\
&= \frac{1}{\sqrt{2E_A}}e^{ip'_A\cdot x_1}\frac{1}{\sqrt{2E_A}}e^{-ip_A\cdot x_2}\frac{1}{\sqrt{2E_B}}e^{ip'_B\cdot x_1}\frac{1}{\sqrt{2E_B}}e^{-ip'_B\cdot x_2} \\
&= \frac{1}{(2E_A)(2E_B)}e^{i(p'_A+p'_B)\cdot x_1}e^{-i(p_A+p_B)\cdot x_2}
\end{aligned}
$$

Substituting this expression into (6.74), and through away the coefficients(it only relates to the factor of normalization), one get such an equation.

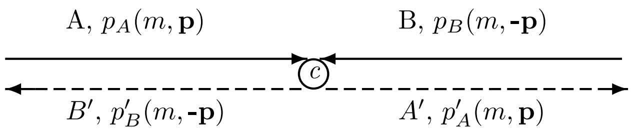
$$
(-ig)^2\int\int d^4x_1d^4x_2e^{i(P'_A+P'_B)\cdot x_1}e^{-i(P_A+P_B)\cdot x_2}\langle 0|T(\hat{\phi}_c(x_1)\hat{\phi}_c(x_2))|0\rangle \qquad \text{(Cf. to equation (6.91))}
$$

The following step is exactly shown as $P_{160} \sim P_{162}$ and cf. the proof of *6.5*, one could get, under this context, the contribution of the contractions (6.88) to the S-matrix element is :

$$
(-ig)^2(2\pi)^4\delta^4(P_A+P_B-P'_A-P'_B)\frac{i}{(p_A+p_B)^2-m_c^2+i\epsilon}
$$

Problems 6.8

Consider the case of equal masses $m_A = m_B = m_C$. Evaluate $u = (p_A - p'_B)^2$ in the CM frame , and show that $u \leq 0$, so that u can never equal m_C^2 in (6.100)(This result is generally true for such single particle 'exchange' processes).



This figure actually is the "CMS" version of figure 6.4 in P_{163}.

Obviously, from above figure, $p_A - p'_B = (m, \mathbf{p}) - (m, -\mathbf{p}) = (0, 2\mathbf{p})$. Thus,

$$
\begin{aligned}
u &= (p_A - p'_B)^2 \\
&= (p_A - p'_B)\cdot(p_A - p'_B) \\
&= -4\mathbf{p}^2 \leq 0
\end{aligned}
$$