Gauge theories in particle physics(3rd, Aitchison and Hey)

Chapter 6, Quantum field theory II : Interacting scalar fields

Problems 6.3, P_{171}.

Let $\hat{\phi}(x,t)$ be a real scalar KG field one space dimension, sataisfying

$$(\Box_x + m^2)\hat{\phi}(x,t) \equiv (\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2)\hat{\phi}(x,t) = 0$$

(i) Explain why

$$T(\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)) = \theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)$$

Solution of (i):

$t_1 > t_2 \Rightarrow \theta(t_1-t_2) = 1, \theta(t_2-t_1) = 0 \Rightarrow T(\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)) = \theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)$

$t_1 < t_2 \Rightarrow \theta(t_1-t_2) = 0, \theta(t_2-t_1) = 1 \Rightarrow T(\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)) = \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)$

Combining the above two lines proves (i) immediately.

(ii)Using equation $(E.46)$, show that

$$\frac{d}{dx}\theta(x-a) = \delta(x-a)$$

Solutions of (ii) :

$(E.46)$ says,

$$\int_{-\infty}^{x} \delta(x'-a)dx' = \theta(x-a)$$

Simply derivate both side of $(E.46)$ one could prove (ii).

(iii)Using the result of (ii) with appropriate changes of variable, and equation (5.105), show that

$$\frac{\partial}{\partial t_1}\{T[\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)]\} = \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)$$

Solutions of (iii):

$$\begin{aligned}\frac{\partial}{\partial t_1}\{T[\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)]\} &= \frac{\partial}{\partial t_1}[\theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)] \\ &= \frac{\partial}{\partial t_1}[\theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)] + \frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)]\end{aligned}$$

Calculating the first term:

$$\begin{aligned}\frac{\partial}{\partial t_1}[\theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)] &= \{\frac{d}{dt_1}[\theta(t_1-t_2)]\hat{\phi}(x_1,t_1) + \theta(t_1-t_2)\frac{d}{dt_1}[\hat{\phi}(x_1,t_1)]\}\hat{\phi}(x_2,t_2) \\ &= [\delta(t_1-t_2)\hat{\phi}(x_1,t_1) + \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)]\hat{\phi}(x_2,t_2) \\ &= \delta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2) \qquad (1)\end{aligned}$$

Calculating the second term:

$$\frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)] = \hat{\phi}(x_2,t_2)\frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_1,t_1)]$$

(because $\hat{\phi}(x_2,t_2)$ is constant to $\frac{\partial}{\partial t_1}$, so it could be moved in front of the expression)

$$= \hat{\phi}(x_2,t_2)\{\frac{d}{dt_1}[\theta(t_2-t_1)]\hat{\phi}(x_1,t_1) + \theta(t_2-t_1)\frac{d}{dt_1}[\hat{\phi}(x_1,t_1)]\}$$

$$= \hat{\phi}(x_2,t_2)[-\delta(t_2-t_1)\hat{\phi}(x_1,t_1) + \theta(t_2-t_1)\dot{\hat{\phi}}(x_1,t_1)]$$

$$= -\hat{\phi}(x_2,t_2)\delta(t_2-t_1)\hat{\phi}(x_1,t_1) + \hat{\phi}(x_2,t_2)\theta(t_2-t_1)\dot{\hat{\phi}}(x_1,t_1) \quad (2)$$

Add up (1) and (2),

$$\delta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2) - \hat{\phi}(x_2,t_2)\delta(t_2-t_1)\hat{\phi}(x_1,t_1) + \hat{\phi}(x_2,t_2)\theta(t_2-t_1)\dot{\hat{\phi}}(x_1,t_1)$$

$$= \delta(t_1-t_2)[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) - \hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)] + \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)$$

(because $\delta(x)$ is even function, so $\delta(t_1-t_2) = \delta(t_2-t_1)$)

$$= \delta(t_1-t_2)[\hat{\phi}(x_1,t_1),\hat{\phi}(x_2,t_2)] + \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)$$

(“ $[\hat{\phi}(x_1,t_1),\hat{\phi}(x_2,t_2)] = 0$ ” is a typical commutation relation of $\hat{\phi}(x)$, cf (5.105))

$$= \theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)$$

(iv)Using (5.109) and (5.104) show that

$$\frac{\partial^2}{\partial t_1^2}\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\} = -i\delta(x_1-x_2)\delta(t_1-t_2) + T[\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)]$$

and hence show that

$$(\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_1^2} + m^2)\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\} = -i\delta(x_1-x_2)\delta(t_1-t_2)$$

This shows that $T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]$ is a Green function(See appendix G, equation $(G.25)$ – the i is included here conventionally) for the KG operator

$$\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_1^2} + m^2$$

The four-dimensional generalization is immediate.

Solutions of (iv):

$$\frac{\partial^2}{\partial t_1^2}\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\}$$

$$= \frac{\partial}{\partial t_1}[\theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)]$$

$$= \frac{\partial}{\partial t_1}[\theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2)] + \frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)]$$

The first term is :

$$\frac{\partial}{\partial t_1}[\theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)]$$

$$= \{\frac{d}{dt_1}[\theta(t_1-t_2)]\dot{\hat{\phi}}(x_1,t_1) + \theta(t_1-t_2)\frac{d}{dt_1}[\dot{\hat{\phi}}(x_1,t_1)]\}\hat{\phi}(x_2,t_2)$$

$$= [\delta(t_1-t_2)]\dot{\hat{\phi}}(x_1,t_1) + \theta(t_1-t_2)\ddot{\hat{\phi}}(x_1,t_1)]\hat{\phi}(x_2,t_2)$$

$$= \delta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_1-t_2)\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)$$

Similarly, the second term is :

$$
\begin{aligned}
&\frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)] \\
&= \hat{\phi}(x_2,t_2)\{\frac{d}{dt_1}[\theta(t_2-t_1)]\dot{\hat{\phi}}(x_1,t_1)+\theta(t_2-t_1)\frac{d}{dt_1}[\dot{\hat{\phi}}(x_1,t_1)]\} \\
&= \hat{\phi}(x_2,t_2)[-\delta(t_2-t_1)]\dot{\hat{\phi}}(x_1,t_1)+\theta(t_2-t_1)\ddot{\hat{\phi}}(x_1,t_1)] \\
&= -\hat{\phi}(x_2,t_2)\delta(t_2-t_1)\dot{\hat{\phi}}(x_1,t_1)+\hat{\phi}(x_2,t_2)\theta(t_2-t_1)\ddot{\hat{\phi}}(x_1,t_1) \\
&= -\delta(t_1-t_2)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\ddot{\hat{\phi}}(x_1,t_1)
\end{aligned}
$$

Adding up the two terms :

$$
\begin{aligned}
&\frac{\partial^2}{\partial t_1^2}\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\} \\
&= \frac{\partial}{\partial t_1}[\theta(t_1-t_2)\dot{\hat{\phi}}(x_1,t)\hat{\phi}(x_2,t_2)]+\frac{\partial}{\partial t_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)] \\
&= \delta(t_1-t_2)\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)+\theta(t_1-t_2)\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)-\delta(t_1-t_2)\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\ddot{\hat{\phi}}(x_1,t_1) \\
&= \delta(t_1-t_2)[\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)-\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)]+\theta(t_1-t_2)\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\ddot{\hat{\phi}}(x_1,t_1) \\
&\qquad\qquad (\text{“ }\dot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)-\hat{\phi}(x_2,t_2)\dot{\hat{\phi}}(x_1,t_1)=i\delta(x_1-x_2)\text{ ” , cf (5.104) and (5.109)}) \\
&= -i\delta(x_1-x_2)\delta(t_1-t_2)+T[\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)] \qquad (3)
\end{aligned}
$$

Next, calculating $\frac{\partial^2}{\partial x_1^2}\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\}$. The first step is calculating $\frac{\partial}{\partial x_1}\{T[\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)]\}$.

$$
\begin{aligned}
&\frac{\partial}{\partial x_1}\{T[\hat{\phi}(x_1,x_2)\hat{\phi}(x_2,t_2)]\} \\
&= \frac{\partial}{\partial x_1}[\theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)] \\
&= \frac{\partial}{\partial x_1}[\theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]+\frac{\partial}{\partial x_1}[\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)] \\
&= \theta(t_1-t_2)\frac{\partial}{\partial x_1}[\hat{\phi}(x_1,t_1)]\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\frac{\partial}{\partial x_1}[\hat{\phi}(x_1,t_1)]
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\frac{\partial^2}{\partial x_1^2}\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\} \\
&= \frac{\partial}{\partial x_1}\{\theta(t_1-t_2)\frac{\partial}{\partial x_1}[\hat{\phi}(x_1,t_1)]\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\frac{\partial}{\partial x_1}[\hat{\phi}(x_1,t_1)]\} \\
&= \theta(t_1-t_2)\frac{\partial^2}{\partial x_1^2}[\hat{\phi}(x_1,t_1)]\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\frac{\partial^2}{\partial x_1^2}[\hat{\phi}(x_1,t_1)]
\end{aligned}
$$

Combining the field equation yielded from Euler-Lagrangian equation, $\frac{\partial^2\hat{\phi}}{\partial t^2}-\frac{\partial^2\hat{\phi}}{\partial x^2}=0$(set c = 1, also cf. (5.108)), one gets,

$$
\begin{aligned}
&\frac{\partial^2}{\partial x_1^2}\{T[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)]\} \\
&= \theta(t_1-t_2)\frac{\partial^2}{\partial t_1^2}[\hat{\phi}(x_1,t_1)]\hat{\phi}(x_2,t_2)+\theta(t_2-t_1)\hat{\phi}(x_2,t_2)\frac{\partial^2}{\partial t_1^2}[\hat{\phi}(x_1,t_1)] \\
&= T[\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)] \qquad (4)
\end{aligned}
$$

Now, substituting the expression of (3) and (4) into the following term

$$
\begin{aligned}
&(\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_1^2} + m^2)\{T[\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2)]\} \\
&= \frac{\partial^2}{\partial t_1^2}\{T[\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2)]\} - \frac{\partial^2}{\partial x_1^2}\{T[\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2)]\} + m^2\{T[\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2)]\} \\
&= -i\delta(x_1 - x_2)\delta(t_1 - t_2) + T[\ddot{\hat{\phi}}(x_1, t_1)\hat{\phi}(x_2, t_2)] - T[\ddot{\hat{\phi}}(x_1, t_1)\hat{\phi}(x_2, t_2)] + m^2\{T[\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2)]\} \\
&= -i\delta(x_1 - x_2)\delta(t_1 - t_2) + m^2\{T[\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2)]\} \\
&= -i\delta(x_1 - x_2)\delta(t_1 - t_2) + m^2[\theta(t_1 - t_2)\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2) + \theta(t_2 - t_1)\hat{\phi}(x_2, t_2)\hat{\phi}(x_1, t_1)]
\end{aligned}
$$