Gauge theories in particle physics(3rd, Aitchison and Hey)

Chapter 4, Relativistic quantum mechanics

This chapter with a topic of *relativistic quantum mechanics*. There're 6 sectors in this chapter. The topics including *KG equation*, *Dirac equation*, *Spin introduction*, *Dirac spinor's Lorentz transformation properties*, *negative-energy solutions* and *electromagnetic interaction via the gauge principle*.

Sector 4.1. KG equation

This sector introduces *Klein-Gordon equation* which is transited from (non-relativisitic)*Schrödinger equation*. Then positive and negative solutions, and probability current are presented.
Like *Schrödinger equation*, *Klein-Gordon equation* results from operator replacements, $E \to i\partial/\partial t$, and $\boldsymbol{p} \to -i\nabla$. The difference is the energy expression changed to $E^2 = \boldsymbol{p}^2 + m^2$. After substituting the operators, the *Klein-Gordon equation* shows as,

$$-\frac{\partial^2 \phi}{\partial t^2} = (-\nabla^2 + m^2)\phi \qquad (\textit{Klein-Gordon equation})$$

The plane-wave solution of *Klein-Gordon equation*, $(\partial^2/\partial t^2 - \nabla^2 + m^2)\phi = 0$, is

$$\phi(\boldsymbol{x}, t) = Ne^{-iEt+i\boldsymbol{p}\cdot\boldsymbol{x}} = Ne^{-ip\cdot x} \qquad (p, x \text{ in } e^{-ip\cdot x} \text{ are both four-momenta})$$

It's just at this stage, Schrödinger and others found that for the plane-wave solution, same $\boldsymbol{p}$ has two energy solutions : $E = \pm(\boldsymbol{p} + m)^{1/2}$. The negative energy is an astonished one at that time.
Similar problem arises from *probability density* calculation.

$$\begin{aligned}\rho &= i[\phi^* \frac{\partial \phi}{\partial t} - (\frac{\partial \phi^*}{\partial t})\phi] \\ &= 2|N|^2 E \qquad (\text{By substituting } \phi(\boldsymbol{x}, t) = Ne^{-ip\cdot x})\end{aligned}$$

Since E can be positive or negative, so be ρ. But obviously, *probability density* shouldn't be negative.
The contradictions of both E and ρ could be negative lead to one of the main topics in this sector, including the discovery of Dirac equation.

Sector 4.2. Dirac equation

The *Dirac equation* doesn't solve the negative energy, while it indeed solves the problem of negative probability density.
It's clear for Dirac that the negative probability density due to the *KG equation* contains $\partial^2/\partial t^2$ term, which leads to a continuity equation with a "probability density" containing $\partial/\partial t$, and hence to negative probabilities.
In order to obtain a positive-definite probability density $\rho \geqslant 0$, he required an equation linear in $\partial/\partial t$. For relativistic covariance, the equation must also be linear in ∇. He postulated the equation,

$$i\frac{\partial \phi(\boldsymbol{x}, t)}{\partial t} = [-i(\alpha_1 \frac{\partial}{\partial x^1} \alpha_2 \frac{\partial}{\partial x^2} + \alpha_3 \frac{\partial}{\partial x^3}) + \beta m]\phi(\boldsymbol{x}, t) = (-i\boldsymbol{\alpha} \cdot \nabla + \beta m)\phi(\boldsymbol{x}, t)$$

To find the conditions of the α's and β, Dirac (unbelievably) demanded that his wave function ϕ has to satisfy a *KG equation* ! This demand is unbelievable in the sense he proposed new equation to solve the negative energy and probability density of *KG equation*, in turn, he required his new wave function to satisfy the defective equation.

This "apparent stupid" action turns out to be one of the triumphs of theoretical physics : the discovery of *Dirac matrix* and its native ability to describe the *spin* !

To make Dirac wave function satisfy *KG equation*, the matrix α_i and β has to be matrices and follow *anti-commutation relations*: $\{\alpha_i, \beta\} = 0, \{\alpha_i, \beta_j\} = 2\delta_{ij}\mathbf{1}, \beta^2 = 1$.

And the matrices should act on a wave function which has several components arranged as a column vector, usually called as *(Dirac) spinor*, in the form of.

$$\phi = u(p,s)e^{-ip_+ \cdot x} = (E+m)^{1/2}\begin{pmatrix} \phi^s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\phi^s \end{pmatrix} e^{-ip_+ \cdot x} \qquad (s = 1,2; \phi^1 = \binom{0}{1}, \phi^2 = \binom{1}{0})$$

$$\phi = v(p,s)e^{-i(-p_+ \cdot x)} = (E+m)^{1/2}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^s \\ \chi^s \end{pmatrix} e^{ip_+ \cdot x} \qquad (s = 1,2; \chi^1 = \binom{1}{0}, \chi^2 = \binom{0}{1})$$

The *probability current* in Dirac equation is good in terms of the ρ is a scalar density which is explicitly positive-definitive, after the new definition $\rho = \phi^\dagger(x)\phi(x)$, and $\rho = \sum_{a=1}^{4} |\phi_a|^2 > 0$. To satisfy the conservation law $\partial\rho/\partial t + \nabla \cdot \boldsymbol{j} = 0$, the $\boldsymbol{j}$ has to be defined as, $\boldsymbol{j} = \phi(x)^\dagger \boldsymbol{\alpha}\phi(x)$.

The reason why the definition of ρ in *KG equation* and *Driac equation* are different is not so clear.

Sector 4.3. Spin

In the rest frame of a particle($\boldsymbol{p} = 0$), it's straightforward to interpret $\frac{1}{2}\boldsymbol{\Sigma}$ as *spin* $-\frac{1}{2}$ operators, where $\boldsymbol{\Sigma}$ is

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}$$

So we may say that *Dirac equation* describes a particle with *spin* $-\frac{1}{2}$.

The reasons are in two aspects : (1) the operators $\frac{1}{2}\boldsymbol{\Sigma}$ satisfying $[\frac{1}{2}\Sigma_x, \frac{1}{2}\Sigma_y] = i\frac{1}{2}\Sigma_z$. This is consistent with *angular momentum*'s three Hermitian operators $(\hat{J}_x, \hat{J}_y, \hat{J}_z)$ which satisfies $[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$.

(2) $(\frac{1}{2}\boldsymbol{\Sigma})^2 = \frac{3}{4}\boldsymbol{I}$ where I is the unit 4×4 matrix. This is consistent with *angular momentum*'s " eigenvalues of $\hat{\boldsymbol{J}}^2$ are (with $\hbar = 1$) $j(j+1)$ for $j = 0, \frac{1}{2}, 1 \ldots$", for this case is $j = \frac{1}{2}$.

In the case of $\boldsymbol{p} \neq 0$, $\frac{1}{2}\boldsymbol{\Sigma}$ is no longer a suitable spin operator, since it fails to commute with the energy operator, which is now $(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)$ for a plane-wave solution with momentum $\boldsymbol{p}$. Yet there are still just two independent states for a given 4-momentum. Hence there must be some operator which does commute with $(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)$, and whose eigenvalues can be used to distinguish the two states. One of the most useful is the *helicity operator* $h(\boldsymbol{p})$.

$$h(\mathbf{p}) = \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} & \mathbf{0} \\ \mathbf{0} & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \end{pmatrix}$$

It's not difficult to check that $h(\boldsymbol{p})$ satisfy the following equation therefore it's a appropriate operator for four-component spinor which acts as eigenstate under this context.

$$\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} & \mathbf{0} \\ \mathbf{0} & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \end{pmatrix} \begin{pmatrix} \phi \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\phi \end{pmatrix} = \begin{pmatrix} \phi \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\phi \end{pmatrix}$$

Sector 4.4. Lorentz transformation properties of spinors

At first, it's necessary to state why *Lorentz transformation* is so important to check. Because in particle physics, it's no way to set a fix reference point to any physics system due to it's so tiny. And if the system could transform following the law of *Lorentz transformation*, we don't need to think about setting a fix reference point, since we can change the system freely and keep the system "well behaved" only if it(the system) transforms according to the rule of *Lorentz transformation*.

The key point for this sector are Dirac spinors are *Lorentz transformation*; and $j^\mu = (\rho, \boldsymbol{J})$ is also *Lorentz transformation* if make some small change. For instance, one needs to introduce $\bar{\psi} = \psi^\dagger\beta$ first. By this $j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$ (for *Dirac equation*)is *Lorentz transformation* also.

The reason why need four components instead of two for *Dirac spinors* is because the spinors also played a role to distinguish parity of ($spin - \frac{1}{2}$) particles.

Sector 4.5. The negative-energy solutions

The positive-energy spinors and negative-energy spinors as the last two equations shown in sector 4.2.

Dirac's brilliant interpretation of the negative-energy solutions of the Dirac equation is taking it as anti-particle. Which is one of the triumphs of theoretical physics. .

Although Dirac's interpretation was proved by Carl Anderson who discovered the positron first, his interpretation cannot be applied to *spin-0* particles since bosons are not subject to the exclusion principle.

The non-filed-theoretic interpretation of the negative-energy solutions which works for both bosons and fermions is due to *Feynman* : in essence, the idea is that the negative four-monentum solutions will be used to describe anti-particles, for both bosons and fermions.

Feynman's interpretation resulted from the equivalence of negative-energy solution:

$$j^\mu_{em}(\pi^-) = (-e)2|N|^2[(m^2 + \boldsymbol{p}^2)^{1/2}, \boldsymbol{p}] = (+e)2|N|^2[-(m^2 + \boldsymbol{p}^2)^{1/2}, -\boldsymbol{p}]$$

Feynman's hypothesis for boson : The emission(absorption) of an anti-particle of 4-momentum p^μ is physically equivalent to the absorption(emission) of a particle of 4-momentum $-p^\mu$.

Feynman's hypothesis for fermions : The invariant amplitude for the emission(absorption) of an anti-fermion of 4-momentum p^μ and spin projection s_z in the rest frame is equal to the amplitude(minus the amplitude) for the absorption(emission) of a fermion of 4-momentum $-p^\mu$ and spin projection $-s_z$ in the rest frame.

Problem 4.4.

(a)Using the explicit forms for the 2×2 Pauli matrices, verify the commutation(square brackets) and anti-commutation(braces) relation [note the summation convention for repeated indices : $\epsilon_{ijk} \equiv \sum_{k=1}^{3}\epsilon_{ijk}\sigma_k$]: $[\sigma_i,\sigma_j]=2i\epsilon_{ijk}\sigma_k$, $(\sigma_i,\sigma_j)=2\sigma_{ij}\mathbf{1}$, where ϵ_{ijk} is the usual antisymmetric tensor.

$$\epsilon_{ijk}=\begin{cases} +1 & \text{for an even permutation of } 1,2,3 \\ -1 & \text{for an odd permutation of } 1,2,3 \\ 0 & \text{if two or more indices are the same,} \end{cases}$$

σ_{ij} is the usual Kronecker delta, and $\mathbf{1}$ is the 2×2 matrix. Hence show that $\sigma_i\sigma_j=\delta_{ij}\mathbf{1}+i\epsilon_{ijk}\sigma_k$.

(b)Use this last identity to prove the result $(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})=\boldsymbol{a}\cdot\boldsymbol{b}\mathbf{1}+i\boldsymbol{\sigma}\cdot\boldsymbol{a}\times\boldsymbol{b}$. Using the explicit 2×2 form for

$$\boldsymbol{\sigma}\cdot\mathbf{p}=\begin{pmatrix} p_z & p_x-ip_y \\ p_x+ip_y & -p_z \end{pmatrix}$$

show that $(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2=\boldsymbol{p}^2\mathbf{1}$

Solution of problem 4.4:

$$[\sigma_1,\sigma_2]=\sigma_1\sigma_2-\sigma_2\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}-\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$=\begin{pmatrix}i&0\\0&-i\end{pmatrix}-\begin{pmatrix}-i&0\\0&i\end{pmatrix}=\begin{pmatrix}2i&0\\0&-2i\end{pmatrix}$$

On the other hand,

$$\begin{aligned} 2i\epsilon_{ijk}\sigma_k &= 2i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k \\ &= 2i(\epsilon_{121}\sigma_1+\epsilon_{122}\sigma_2+\epsilon_{123}\sigma_3) \\ &= 2i\epsilon_{123}\sigma_3 && (\epsilon_{121}=\epsilon_{122}=0) \\ &= 2i\sigma_3 && (\epsilon_{123}=1) \\ &= \begin{pmatrix}2i&0\\0&-2i\end{pmatrix} \end{aligned}$$

So, $[\sigma_i,\sigma_j]=2i\epsilon_{ijk}\sigma_k$. Using explicit forms also could prove that $\{\sigma_i,\sigma_j\}=2\sigma_{ij}\mathbf{1}$

$$[\sigma_i,\sigma_j]=\sigma_i\sigma_j-\sigma_j\sigma_i,\ \{\sigma_i,\sigma_j\}=\sigma_i\sigma_j+\sigma_j\sigma_i\ \Rightarrow\sigma_i\sigma_j=\tfrac{1}{2}([\sigma_i,\sigma_j]+\{\sigma_i,\sigma_j\})=\delta_{ij}\mathbf{1}+i\epsilon_{ijk}\sigma_k$$

Problem 4.8.

Check that the probability current $\boldsymbol{j} = \omega^\dagger \boldsymbol{\alpha} \omega$(for a plane-wave solution) transforms correctly under the x-axis rotation specified by (4.69) [*hint* : using $\omega' = exp(i\sum_x \alpha/2)\omega$ and (4.69) you need to show that $j'_x = \omega^\dagger \alpha'_x \omega = j_x$, $j'_y = \omega^\dagger \alpha'_y \omega = cos\alpha j_y + sin\alpha j_z$, $j'_z = \omega^\dagger \alpha'_z \omega = -sin\alpha j_y + cos\alpha j_z$]

Solution of problem 4.8:

$$
\begin{aligned}
j'_x &= \omega^\dagger \alpha'_x \omega \\
&= [exp(i\sum_x \alpha/2)\omega]^\dagger \alpha'_x [exp(i\sum_x \alpha/2)\omega] \\
&= [\omega^\dagger exp(-i\sum_x \alpha/2)]\alpha_x [exp(i\sum_x \alpha/2)\omega] \qquad (\alpha'_x = \alpha_x, \text{ c.f.(4.69)}) \\
&= \omega^\dagger \alpha_x \omega \\
&= j_x
\end{aligned}
$$

$$
\begin{aligned}
j'_y &= \omega^\dagger \alpha'_y \omega \\
&= [exp(i\sum_x \alpha/2)\omega]^\dagger \alpha'_y [exp(i\sum_x \alpha/2)\omega] \\
&= [\omega^\dagger exp(-i\sum_x \alpha/2)](\alpha_y cos\alpha + \alpha_z sin\alpha)[exp(i\sum_x \alpha/2)\omega] \qquad (\alpha'_y = \alpha_y cos\alpha + \alpha_z sin\alpha, \text{ c.f.(4.69)}) \\
&= \omega^\dagger (\alpha_y cos\alpha + \alpha_z sin\alpha)\omega \\
&= \omega^\dagger \alpha_y cos\alpha\omega + \omega^\dagger \alpha_z sin\alpha\omega \\
&= j_y cos\alpha + j_z sin\alpha
\end{aligned}
$$

Similarly, $j'_z = \omega^\dagger \alpha'_z \omega = -sin\alpha j_y + cos\alpha j_z$ can be proved also.

Proof is done

Problem 4.10.

Show that if we define $\bar{\psi} = \psi^\dagger\beta$, then the quantity $\bar{\psi}\psi$ is invariant under the transformation (4.83) and under the transformation (4.90).

Solution of problem 4.10:

According to equation(4.40), $\psi = \omega e^{-ip\cdot x}$ is a free particle solution for the Dirac equation, ω is a four-component spinor independent of x, and $e^{-ip\cdot x}$, with $p^\mu = (E, \boldsymbol{p})$, is the plane-wave solution corresponding to 4-momentum p^μ.

$$\begin{aligned}
\bar{\psi}\psi &= \psi^\dagger\beta\psi \\
&= [\omega e^{-ip\cdot x}]^\dagger\beta[\omega e^{-ip\cdot x}] \\
&= [\omega^\dagger e^{ip\cdot x}]\beta[\omega e^{-ip\cdot x}] \\
&= \omega^\dagger\beta\omega
\end{aligned}$$

The (4.83) in P_{87} reads,

$$\omega' = e^{i\sum\cdot\hat{\boldsymbol{n}}\theta/2}\omega$$

The (4.90) in P_{88} reads,

$$\omega' = e^{i\alpha_x\vartheta/2}\omega$$

Similar to the proof for ρ' in the equation of (4.84) in P_{88},

$$\begin{aligned}
(\bar{\psi}\psi)' &= (\omega^\dagger\beta\omega)' \\
&= \omega^{\dagger'}\beta'\omega' \\
&= \omega^\dagger exp(-\sum\cdot\hat{\boldsymbol{n}}\theta/2)\beta' exp(i\sum\cdot\hat{\boldsymbol{n}}\theta/2) \\
&= \omega^\dagger\beta\omega \qquad\qquad (\beta = \beta', \text{ since it doesn't change along with tranform})
\end{aligned}$$

In a very similar way, one can prove that $\bar{\psi}\psi$ is invariant under the transform under (4.90) easily.

Problem 4.11.

(a) Verify explicitly that the matrices $\boldsymbol{\alpha}$ and β of problem 4.3 satisfy the Dirac anti-commmutation relations. Defining the four 'γ matrices' $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$, where $\gamma^0 = \beta$ and $\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}$, show that the Dirac equation can be written in the form $(i\gamma^\mu\partial_\mu - m)\psi = 0$. find the anti-commutation relations of the γ matrices.

(b) Define the conjugate spinor, $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$ and use the previous result to find the equation satisfied by $\bar{\psi}$ and γ matrix notation.

(c) The Dirac probability current may be written as $j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$. Show that it satisfies the conservation law $\partial_\mu j^\mu = 0$.

Solution of problem 4.11:

(a).

From problem (4.3), the Dirac matrices are :

$$\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

$$\boldsymbol{\alpha}\beta = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}$$

$$\beta\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\alpha}\beta + \beta\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} = \mathbf{0}$$

From the Dirac equation, (4.23) in P_{77},

$$\begin{aligned}
& i\frac{\partial\psi}{\partial t} = (-i\boldsymbol{\alpha}\cdot\nabla + \beta m)\psi \\
& \Rightarrow (i\frac{\partial}{\partial t} + i\partial\cdot\nabla - \beta m)\psi = 0 \\
& \Rightarrow [i(\beta\frac{\partial}{\partial t} + i\beta\partial\cdot\nabla - \beta^2 m]\psi = 0 && \text{(left multiply } \beta \text{ on both sides)} \\
& \Rightarrow [i(\beta, \beta\partial)\cdot(\frac{\partial}{\partial t}, \nabla) - m]\psi = 0 && (\beta^2 = 1) \\
& \Rightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0
\end{aligned}$$

(b) Define the conjugate spinor, $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$ and use the previous result to find the equation satisfied by $\bar{\psi}$ and γ matrix notation.

$$\begin{aligned}
& (i\gamma^\mu\partial_\mu - m)\psi = 0 \\
& \Rightarrow [(i\gamma^\mu\partial_\mu - m)\psi]^\dagger = 0 \\
& \Rightarrow \psi^\dagger(-i\gamma^{\mu\dagger}\partial_\mu - m)^\dagger = 0 \\
& \Rightarrow \psi^\dagger(i\gamma^{\mu\dagger}\overleftarrow{\partial_\mu} + m) = 0 && (\overleftarrow{\partial_\mu} \text{ means that the derivative acts on what is to the left of it, i.e. on } \psi^\dagger) \\
& \Rightarrow \psi^\dagger(i\gamma^{\mu\dagger}\overleftarrow{\partial_\mu} + m)\gamma^0 = 0 \\
& \Rightarrow \bar{\psi}(i\overleftarrow{\not{\partial}}_\mu + m) = 0 && (\gamma^{\mu\dagger}\beta = \beta\gamma^\mu,\ i\not{\partial} = i\gamma^\mu\partial_\mu.)
\end{aligned}$$

(c) The Dirac probability current may be written as $j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$. Show that it satisfies the conservation law $\partial_\mu j^\mu = 0$.

$$\begin{aligned}
\partial_\mu j^\mu &= \partial_\mu(\bar{\psi}\gamma^\mu\psi) \\
&= [\partial_\mu(\bar{\psi})\gamma^\mu + \bar{\psi}\partial_\mu(\gamma^\mu)]\psi \\
&= \partial_\mu(\bar{\psi})\gamma^\mu\psi + \bar{\psi}\partial_\mu(\gamma^\mu)\psi \\
&= \bar{\psi}\overleftarrow{\not{\partial}}_\mu\psi + \bar{\psi}\not{\partial}\psi
\end{aligned} \tag{1}$$

From the last equation of (b), one gets,

$$\begin{aligned}
&\bar{\psi}(i\overleftarrow{\not{\partial}}_\mu + m) = 0 \\
&\Rightarrow \bar{\psi}(i\overleftarrow{\not{\partial}}_\mu + m)\psi = 0 \\
&\Rightarrow i\bar{\psi}\overleftarrow{\not{\partial}}_\mu\psi + m\bar{\psi}\psi = 0 \\
&\Rightarrow \bar{\psi}\overleftarrow{\not{\partial}}_\mu\psi = im\bar{\psi}\psi
\end{aligned} \tag{2}$$

From the last equation of (a), one gets,

$$\begin{aligned}
&(i\gamma^\mu\partial_\mu - m)\psi = 0 \\
&\Rightarrow \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = 0 \\
&\Rightarrow i\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}m\psi = 0 \\
&\Rightarrow \bar{\psi}\gamma^\mu\partial_\mu\psi = -i\bar{\psi}m\psi \\
&\Rightarrow \bar{\psi}\not{\partial}\psi = -i\bar{\psi}m\psi
\end{aligned} \tag{3}$$

Submitting (2) and (3) into (1), one get $\partial_\mu j^\mu = 0$.