Solutions of 11.1,11.2 and 11.6 “Introduction to high energy physics” (4th edition, Donald H. Perkins)

Problem 11.1.

11.1 The average number $\bar{n}$ of ionising collisions suffered by a fast particle of charge ze in traversing dx (g cm^{-2}) of a medium, resulting in energy transfers in the range $E' \to E' + dE'$, is

$$\bar{n} = f(E')dE'dx = \frac{2\pi z^2 e^4 N_0 Z}{mv^2 A} \frac{dE'}{(E')^2} \left(1 - \frac{v^2}{c^2} \frac{E'}{E'_{\max}}\right) dx$$

where the symbols are as in subsection 11.5.1 and the maximum transferable energy is $E'_{\max} = 2mv^2/(1 - \beta^2)$, with $\beta = v/c$. For individual particles, the distribution in the number n of collisions follows the Poisson law, so that $\langle (n - \bar{n})^2 \rangle = \bar{n}$. If we multiply the above equation by $(E')^2$ and integrate, we obtain the mean squared deviation in energy loss, $\epsilon^2 = \langle (\Delta E - \bar{\Delta} \bar{E})^2 \rangle$, about the mean value $\bar{\Delta}\bar{E}$. Show that

$$\epsilon^2 = 0.6(mc^2)^2 \gamma^2 \frac{Z}{A} \left(1 - \frac{\beta^2}{2}\right) \delta x$$

Calculate the fractional rms deviation in energy loss, $\epsilon/\bar{\Delta}\bar{E}$, for protons of kinetic energy 500 MeV traversing (a) 0.1, (b) 1.0 and (c) 10 g cm^{-2} of plastic scintillator $(Z/A = 1/2)$. Take dE/dx as 3 MeV g^{-1} cm^2.

Solution of problem 11.1.

At first, one has to be clear that in this problem, the “the average number $\bar{n}$ of ionising collisions” indicates the times of collisions is $\bar{n}$. And in each collision, the transferable energy ranges from 0 to E'_{max}. For individual particles, the distribution of $\bar{n}$ follows Poisson law. But if integrate the “energy density ” in above range, i.e. $\int_0^{E'_{max}} E'^2 f(E')dE'$, one can obtain the mean squared deviation in energy loss, as this problem intends to show.

Ignoring the constant parts firstly in $\bar{n} = f(E')dE'dx$, one can get,

$$\begin{aligned}
&\frac{1}{mv^2}\int_0^{E'_{max}} (1 - \frac{v^2}{c^2}\frac{E'}{E'_{max}})dE'dx \\
&= \frac{1}{mv^2}[E' - \frac{(E')^2}{E'_{max}}\frac{\beta^2}{2}]_0^{E'_{max}} \\
&= 2\gamma^2(1 - \frac{\beta^2}{2}) \qquad \text{(by substituting the value of } E'_{max}.)
\end{aligned}$$

So,

$$
\begin{aligned}
&\int_0^{E'_{max}} \bar{n}(E')^2 \\
&= \int_0^{E'_{max}} \frac{2\pi z^2 e^4 N_0 Z}{mv^2 A} dE'(1 - \frac{v^2}{c^2}\frac{E'}{E'_{max}})dx \\
&= \int_0^{E'_{max}} \frac{2\pi z^2 r_e^2 N_0 (m_e c^2)^2}{mv^2}\frac{Z}{A} dE'(1 - \frac{v^2}{c^2}\frac{E'}{E'_{max}})dx \\
&= 2\pi z^2 r_e^2 N_0 (m_e c^2)^2 \frac{Z}{A} 2\gamma^2 (1 - \frac{\beta^2}{2})\delta x \\
&= \frac{4\pi z^2 r_e^2 N_0 m_e c^2}{m_e c^2}(m_e c^2)^2 \frac{Z}{A}\gamma^2(1 - \frac{\beta^2}{2})\delta x \\
&= 0.6(m_e c^2)^2 \frac{Z}{A}\gamma^2 (1 - \frac{\beta^2}{2})\delta x \qquad (4\pi z^2 r_e^2 N_0 m_e c^2 = 0.3, m_e c^2 = 0.5, \text{ units are ignored})
\end{aligned}
$$

Next, calculating the $\epsilon/\bar{\Delta}\bar{E}$ under different situations.

At first, calculating the γ and β values.

$$
\begin{aligned}
\gamma &= \frac{E}{m} \qquad (\text{c} = 1) \\
&= \frac{938 + 500}{938} \\
&\Rightarrow \gamma^2 = 0.43, \beta^2 = 0.57
\end{aligned}
$$

Substituting the value into ϵ equation, one gets,

$$
\begin{aligned}
\epsilon^2 &= 0.6 \times (0.5)^2 \times 11 - 0.43 \times 12 \times (1 - \frac{0.43}{2})\,\delta x \\
&= 0.126\,\delta x
\end{aligned}
$$

(1). For $0.1\,g\,cm^{-2}$ plastic scintillator, $\epsilon = 0.11$; $\bar{\Delta}\bar{E} = 3 \times 0.1 = 0.3$. So, $\epsilon/\bar{\Delta}\bar{E} = 0.11/0.3 = 0.37$.

(2). For $1.0\,g\,cm^{-2}$ plastic scintillator, $\epsilon = 0.35$; $\bar{\Delta}\bar{E} = 3 \times 1 = 3$. So, $\epsilon/\bar{\Delta}\bar{E} = 0.35/3 = 0.12$.

(3). For $10.0\,g\,cm^{-2}$ plastic scintillator, $\epsilon = 1.12$; $\bar{\Delta}\bar{E} = 3 \times 10 = 30$. So, $\epsilon/\bar{\Delta}\bar{E} = 1.12/30 = 0.037$.

Problem 11.2, P_{375}.

11.2 A narrow pencil beam of singly charged particles of very high momentum p, travelling along the x-axis, traverses a slab of material s radiation lengths in thickness. If ionisation loss in the slab may be neglected, calculate the rms lateral spread of the beam in the y-direction, as it emerges from the slab.

(*Hint*: Consider an element of slab of thickness dx at depth x, and find the contribution $(dy)^2$ that this element makes to the mean squared lateral deflection, then integrate over the slab thickness.)

Use the formula you derive to compute the rms lateral spread of a beam of 10 GeV/c muons in traversing a 100 m pipe filled with (a) air and (b) helium, at NTP.

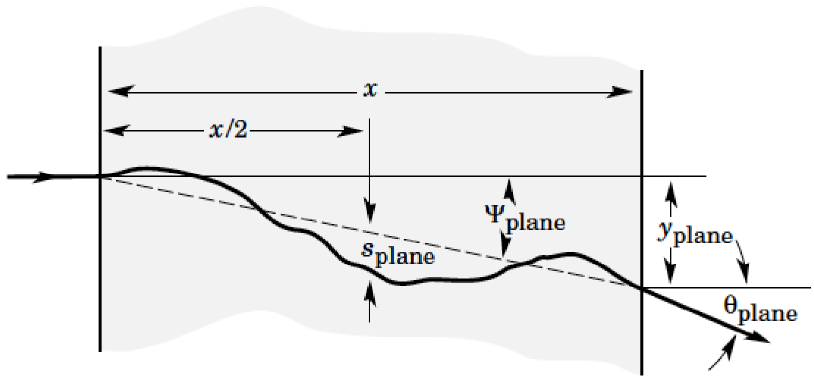

Figure 27.10: Quantities used to describe multiple Coulomb scattering. The particle is incident in the plane of the figure.

Solution of problem 11.2.

The physics involves in this exercise is the so-called *multiple Coulomb scattering*. This phenomena happens when a charged particle traversing a medium is deflected by many small-angle scatters. Most of this deflection is due to Coulomb scattering from nuclei(However, for hadronic projectiles, the strong interactions also contribute to multiple scattering).

The Coulomb scattering distribution is well represented by the theory of *Moliére*. It's roughly Gaussian for small deflection angles, but at larger angles(greater thatn a few $\theta_0 = \phi_{rms}$(Cf. (11.14)) of textbook), it behaves like Rutherford scattering, with large tails than does a Gaussian distribution.

There're a few formulas to describe the $\theta_0 = \phi_{rms}$. Equation (11.14) is an earlier and easier one which is not so accurate comparing to later developed ones(For details, see chapter of "passage of particles through matter" in PDG).

From above figure which gets from chapter 27 of PDG 2012 version. Imagine there is an element "$dxdy$", $y \approxeq \phi x$ in the S plane, accordingly, we have,

$$dy_s = \alpha_s dx_s$$

Where α_s is the (tiny) scattering angle of the element "dxdy" around x in S plane.
The next step is to project the relation of $dy_s = \alpha_s dx_s$ from S plane to Ψ plane as above figure shown. This could be done by integrating the variance of above relation.

$$dy_s = \alpha_s dx_s$$

$$\Rightarrow \sigma^2_{dy\ s} = \sigma^2_{\alpha\ s}(dx_s)^2 \qquad \text{(Taking } dx_s \text{ as a constant)}$$

$$\Rightarrow \int_0^L \sigma^2_{dy\ s} = (dx_s)^2 \int_0^L \sigma^2_{\alpha\ s} \qquad \text{(L is the real path of incident particle)}$$

$$\Rightarrow \sigma^2_{dy\ \Psi} = \sigma^2_{\psi\ \Psi}(dx_\Psi)^2$$

(After integrating, σ^2 in S plane can be equivalent to Ψ plane which indicated by the broken line)

$$\Rightarrow \sigma_{dy\ \Psi} = \sigma_{\psi\ \Psi}(dx_\Psi)$$

$$\Rightarrow \sigma_Y = \int \sigma_{dy\ \Psi} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\ s\ \sigma_{\phi\ \theta}$$

(Now in θ plane, cf. (27.19) PDG 2012. $1/\sqrt{2}$ is the "project coefficient" , cf. P253 of textbook)

$$= \frac{1}{\sqrt{6}}\ s^{3/2}\ \frac{21}{p\beta}\frac{1}{\sqrt{X_0}} \qquad \text{(Substitute (11.14) of textbook as } \sigma_{\phi\ \theta})$$

Now, For 100 m pipe filled with air,

$$\sigma_Y = \frac{1}{\sqrt{6}}\frac{21}{p\beta}s^{3/2}\sqrt{\frac{1}{x_0}}$$

$$= \frac{1}{\sqrt{6}}\frac{21}{10^4}(10000)^{3/2}\sqrt{\frac{1}{3.039\times 10^4}} \qquad \text{(For air, } X_0 = 3.039\times 10^4 cm \text{ in NTP)}$$

$$= 4.88cm$$

Similarly, for 100 cm pipe filled with Helium,

$$\sigma_Y = \frac{1}{\sqrt{6}}\frac{21}{p\beta}s^{3/2}\sqrt{\frac{1}{x_0}}$$

$$= \frac{1}{\sqrt{6}}\frac{21}{10^4}(10000)^{3/2}\sqrt{\frac{1}{5.671\times 10^5}} \qquad \text{(For Helium air, } X_0 = 5.671\times 10^4 cm \text{ in NTP)}$$

$$= 1.13cm$$

Comment 0: It's hard to say the deduce of σ_Y in this solution is very "deep" level, although I spent a few days trying to figure out, as the hint indicates. If one wants to know how it come out in a very fundamental way, please refer this 83 page paper which looks rather a mathematical long paper than physical one, "The theory of small angle multiple scattering of fast charged particles", Volume 35, number 2, April 1963, Willlian T. Scott, Review of modern physics. Good luck ! ⌣̈
Comment 1(From "physicsforum"): The solid line is a typical actual path of a charged particle. The dashed line is a linear fit to that trajectory. The slope of the dashed is $tan\theta$.
Comment 2(From "physicsforum"): Each coulomb scattering is the same Rutherford scattering whether the force is attractive or repulsive. The Rutherford formula is the same for each. The individual scattering is not important, just the slop of the dashed line.

Problem 11.6.

11.6 **Extensive air showers in cosmic rays contain a 'soft' component of electrons and photons and a 'hard' component of muons. What is the origin of these components? Suppose that the central core of a shower, at sea level, contains a narrow, vertical, parallel beam of muons of energy 1000 GeV, which penetrates underground. Assume that the ionisation loss in rock is constant at 2 MeV g^{-1} cm^2. Find the depth in rock at which the muons come to rest, assuming a rock density of 3.0. Using the formula in Problem 11.2, estimate the radial spread in metres of the muons, taking account of their linear change in energy as they traverse the rock. Take the radiation length of rock as 25 g cm^{-2}.**

Solution of problem 11.6.

An extensive air shower is an extensive (many km wide) cascade of ionized particles and electromagnetic radiation produced in the atmosphere when primary cosmic ray(one of the extraterrestrial origin) enters the atmosphere. After the primary cosmic particle has collided with the air molecule, the main part of the first interactions are pions. Also kaons and baryons may be created. Pions and kaons are not stable, thus they may decay into other particles.

The neutral pions π^0 decay into photons in a process, $\pi^0 \to \gamma + \gamma$. The photons produced form an electromagnetic cascade by creating more photons, electrons and positrons.

The charged pions $\pi^{\pm}$ preferentially decay into muons and neutrinos in the processes $\pi^+ \to \mu^+ + \nu$ and $\pi^- \to \mu^- + \nu$. This is how the muons and neutrinos are produced in the air shower.

Aslo kaon may be an origin of muons, which means the decay process is $K^{\pm} \to \mu^{\pm} + \nu$. In the other hand kaons can produce also pion via the decay mode, $K^{\pm} \to \mu^{\pm} + \pi^0$.

"Soft" particle usually means low energy particles, and "hard" particle indicates higher energy ones. For extensive air shower in cosmic rays, the electrons and photons are stable particles and usually in the "bottom level" of cascade which means their energy are relatively lower. For Pion and Kaon, they're not stable ones and usually in the "top level" of cascade, so their energy is higher and usually called 'hard' particles.

Assuming the depth of rock 1000 Gev moun penetrated is D, then,

$$\begin{aligned} D &= \frac{1.0 \times 10^6 (MeV)}{2\,(MeV\; g^{-1} cm^2) \times 3\,(g\; cm^{-3})} \\ &= \frac{10^6}{6} cm \\ &= \frac{10^4}{6}\, m \\ &= 1666.7\, m \end{aligned}$$

So, in the depth of 1666.7 meters, the muon is at rest.

As mentioned both mentioned in the PDG 2012 chapter 27 and this paper, "G.R. Lynch and O.I Dahl, Nucl. Instrum. Methods B58, 6 (1991)", if the layers materials are different, it's better find the whole length and radiation length then apply to equation (11.14) to get correct result, ϕ_{rms}(though for this exercise it's unnecessary).

Assuming the muon reaches depth of $h\ cm$ (based on sea level), at this position, the survived energy is $E_0 - 2\,MeV g^{-1} cm^2 \times 3\,g\,cm^{-3} \times h\,cm = (E_0 - 6h\,)MeV$.

In this case, the layer's material is same, rock. While the energy is not a constant as problem 11.2., actually, it changes from 1000 Gev to 0 Gev. However, as the problem says "taking account of their linear change in energy as they traverse the rock". So, we can find the equivalent constant energy deposited in the whole depth of rock, and then using this equivalent energy as a constant one to substitute in the formula.

$$\begin{aligned} p\beta = E_{equivalence} &= \frac{\int_0^D (E_0 - 6h)dh}{D} \\ &= \frac{\frac{1}{12}(E_0)^2}{D} \\ &= \frac{1}{2}E_0 \end{aligned}$$

So, we consider the muons penetrate into rock with D depth and constant energy : $\frac{1}{2}E_0$. Substitute these values into the formula in problem 11.2, one gets,

$$\begin{aligned} \sigma_Y &= \frac{1}{\sqrt{6}}\frac{21}{p\beta}s^{3/2}\sqrt{\frac{1}{X_0}} \\ &= \frac{21}{\sqrt{6}}\frac{2}{E_0}(\frac{E_0}{6})^{3/2}\sqrt{\frac{1}{X_0}} \qquad (x_0 \text{ with unit of cm}) \\ &= \frac{21}{\sqrt{6}}\frac{2}{10^6}(\frac{10^6}{6})^{3/2}\sqrt{\frac{1}{25/3}} \\ &= \frac{21}{\sqrt{6}}\frac{2\times 10^3}{(\sqrt{6})^3}\frac{\sqrt{3}}{5} \\ &= 4.1\,m \end{aligned}$$