Solutions of the exercises from "Introduction to high energy physics" (4th edition, Donald H. Perkins)

Problem 1.3, P_{33}. The values of mc^2 for the π^+ and muon μ^+ are $139.57MeV$ and $105.66MeV$ respectively. Find the kinetic energy of the muon in the decay $\pi^+ \to \mu^+ + \nu_\mu$ assuming that the neutrino is massless. For a neutrino of finite but very small mass m_ν show that , compared with the case of a massless neutrino, the muon momentum would be reduced by the fraction

$$\frac{\Delta p}{p} = -\frac{m_\nu^2(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2} \approx -\frac{4m_\nu^2}{10^4}$$

where $m_n u$ is in MeV.

Solution version 1 of problem 1.3 :

Obviously, in this kind of decay case, the initial particle π^+ is at rest. To the two final particles, if we set the momentum of μ^+ as p_μ and ν_μ as p_ν, then $p_\mu = -p_\nu = p$.

And according to energy conservation,

$$\begin{aligned}
& E_\pi = E_\mu + E_\nu \\
\Rightarrow & \sqrt{p_\pi^2 + m_\pi^2} = \sqrt{p_\mu^2 + m_\mu^2} + \sqrt{p_\nu^2 + m_\nu^2} && \text{(here, } c = \hbar = 1\text{, same in following.)} \\
\Rightarrow & m_\pi = \sqrt{p^2 + m_\mu^2} + \sqrt{p^2 + m_\nu^2} && (p_\pi = 0, p_\mu = -p_\nu = p) \\
\Rightarrow & m_\pi - \sqrt{p^2 + m_\nu^2} = \sqrt{p^2 + m_\mu^2} \\
\Rightarrow & m_\pi^2 - 2m_\pi\sqrt{p^2 + m_\nu^2} + m_\nu^2 = m_\mu^2 && \text{(square both sides and the } p^2 \text{ are cancelled)} \\
\Rightarrow & \sqrt{p^2 + m_\nu^2} = \frac{m_\pi^2 + m_\nu^2 - m_\mu^2}{2m_\pi} \\
\Rightarrow & p^2 = (\frac{m_\pi^2 + m_\nu^2 - m_\mu^2}{2m_\pi})^2 - m_\nu^2 \\
\Rightarrow & p^2 = \frac{(m_\pi^2 + m_\nu^2 - m_\mu^2)^2 - 4m_\pi^2 m_\nu^2}{(2m_\pi)^2} \\
\Rightarrow & p^2 = \frac{m_\pi^4 + m_\nu^4 + m_\mu^4 - 2m_\pi^2 m_\nu^2 - 2m_\pi^2 m_\mu^2 - 2m_\nu^2 m_\mu^2}{(2m_\pi)^2} \\
\Rightarrow & p = \frac{\sqrt{m_\pi^4 + m_\mu^4 - 2m_\pi^2 m_\nu^2 - 2m_\pi^2 m_\mu^2 - 2m_\nu^2 m_\mu^2}}{2m_\pi}
\end{aligned}$$

(Considering the tiny mass of neutrino, m_ν^4 could be discarded due to the existence of m_ν^2 terms !!!)

$$\begin{aligned}
\Rightarrow & p = \frac{\sqrt{(m_\pi^2 - m_\mu^2)^2 - 2m_\nu^2(m_\pi^2 + m_\mu^2)}}{2m_\pi} \\
\Rightarrow & p = \frac{\sqrt{(m_\pi^2 - m_\mu^2)^2(1 - \frac{2m_\nu^2(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2})}}{2m_\pi} \\
\Rightarrow & p = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}\sqrt{(1 - (\frac{2m_\nu^2(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2})} \\
\Rightarrow & p = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}(1 - \frac{1}{2}\frac{2m_\nu^2(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2}) && (\sqrt{1-x} = 1 - \tfrac{1}{2}x\text{, when } x \text{ is small enough}) \\
\Rightarrow & p = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}(1 - \frac{m_\nu^2(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2}) && (1)
\end{aligned}$$

So, in the case of neutrino is massless, its energy is

$$E_{kinetic} = pc = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = \frac{139.57^2 - 105.66^2}{2 \times 139.57} \approxeq 30MeV$$

And from equation 1, $\Delta p = -\frac{m_\nu^2(m_\pi^2+m_\mu^2)}{(m_\pi^2-m_\mu^2)^2}(while, p = \frac{m_\pi^2-m_\mu^2}{2m_\pi})$. So,

$$\frac{\Delta p}{p} = -\frac{m_\nu^2(m_\pi^2+m_\mu^2)}{(m_\pi^2-m_\mu^2)^2}$$

.... *End of solution version 1 of problem 1.3* ....

Solution version 2 of problem 1.3 :

Set $p_{4\pi} = (m_\pi, 0)$ as four-momentum of π. Similarly, $p_{4\mu} = (E_\mu, p)$, $p_{4\nu} = (E_\nu, -p)$. Then, we have,

$$\begin{aligned}
&p_{4\pi} = p_{4\mu} + p_{4\nu} \\
\Rightarrow\ &p_{4\pi} - p_{4\mu} = p_{4\nu} \\
\Rightarrow\ &(p_{4\pi} - p_{4\mu})^2 = p_{4\nu}^2 \\
\Rightarrow\ &p_{4\pi}^2 - 2p_{4\pi}p_{4\mu} + p_{4\mu}^2 = p_{4\nu}^2 \\
\Rightarrow\ &m_\pi^2 - 2m_\pi E_\mu + m_\mu^2 = m_\nu^2 \\
\Rightarrow\ &E_\mu = \frac{m_\pi^2 + m_\mu^2 - m_\nu^2}{2m_\pi} \\
\Rightarrow\ &E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} - \frac{m_\nu^2}{2m_\pi}
\end{aligned}$$

For particle μ, $p_\mu^2 = E_\mu^2 - m_\mu^2$(here, p_μ is three-momentum. In convention, it should be the shape of $\boldsymbol{P}_\mu$. But to be consistence with the problem saying, keep it as p_μ). So, in the case of neutrino is massless, $m_\nu = 0$,

$$\begin{aligned}
p_\mu^2 &= E_\mu^2 - m_\mu^2 \\
&= (\frac{m_\pi^2 + m_\mu^2}{2m_\pi})^2 - m_\mu^2 \\
&= (\frac{m_\pi^2 - m_\mu^2}{2m_\pi})^2 \\
\Rightarrow\ p_\mu &= \frac{m_\pi^2 - m_\mu^2}{2m_\pi}
\end{aligned}$$

The calculation of $\frac{\Delta p}{p}$ as following.

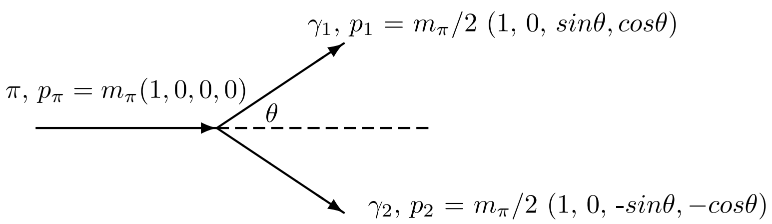
$$\begin{aligned}
\frac{\Delta p_\mu}{p_\mu} &= \frac{\Delta(p_\mu^2)}{2p_\mu^2} && (\Delta(p^2) = 2p\Delta p) \\
&= \frac{\Delta(E_\mu^2 - m_\mu^2)}{2(E_\mu^2 - m_\mu^2)} = \frac{\Delta(E_\mu^2)}{2(E_\mu^2 - m_\mu^2)} && (m_\mu^2 \text{ is constant}) \\
&= \frac{\Delta(E_\mu)E_\mu}{E_\mu^2 - m_\mu^2} \\
&= -[\frac{m_\nu^2}{2m_\pi}\frac{(m_\pi^2 + m_\mu^2)}{2m_\pi}]/(\frac{m_\pi^2 - m_\mu^2}{2m_\pi})^2 \\
&= -\frac{m_\nu^2(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2}
\end{aligned}$$

.... *End of solution version 2 of problem 1.3* ....

Problem 1.4. Deduce an expression for the energy of a γ-ray from the decay of the neutral pion, $\pi^0 \to 2\gamma$, in terms of the mass m, energy E and velocity βc of the pion and the angle of emission θ in the pion rest frame. Show that if the pion has spin zero, so that the angular distribution is isotropic, the laboratory energy spectrum of the γ-rays will be flat, extending from $E(1+\beta)/2$ to $E(1-\beta)/2$. Find an expression for the disparity D(the ratio of energies) of the γ-rays and show the $D > 3$ in half the decays and $D > 7$ in one quarter of them.

Solution of problem 1.4:

Set the four-momentum of pion in its reset frame. Define $p_\gamma = m_\pi/2(1, 0, sin\theta, cos\theta)$(set the π's original traveling direction as Z axis, "up" direction as Y, "inside" the paper as X), as following picture shows.



Apparently, $E_\gamma = m_\pi/2, \boldsymbol{p}_1^z = (m_\pi/2)cos\theta, \boldsymbol{p}_2^z = -(m_\pi/2)cos\theta$. Applying *Lorentz transformation* to transform from π's rest frame to lab frame, like $x' = \gamma(x - vt)$,

$$\begin{aligned}
E_{\gamma_1}^{lab} &= \frac{1}{\sqrt{(1-\beta^2}}(E_\gamma - \beta \boldsymbol{p}_1^z) \\
&= \frac{1}{\sqrt{1-\beta^2}}(\frac{m_\pi}{2} - \beta\frac{m_\pi}{2}cos\theta) \\
&= \frac{1}{\sqrt{1-\beta^2}}\frac{m_\pi}{2}(1-\beta cos\theta)
\end{aligned}$$

Similarly,

$$E_{\gamma_2}^{lab} = \frac{1}{\sqrt{1-\beta^2}}\frac{m_\pi}{2}(1+\beta cos\theta)$$

Taking the $E_{\gamma_2}^{lab}$ as "the energy of a γ-ray". (Taking $E_{\gamma_1}^{lab}$ is same, but $E_{\gamma_2}^{lab}$ is more applausible in the sense of being consistent with the answer.)

$$E_\gamma = E_{\gamma_2}^{lab} = \frac{1}{\sqrt{1-\beta^2}}\frac{m_\pi}{2}(1+\beta cos\theta) = \frac{E_\pi}{2}(1+\beta cos\theta)$$

If the pion has spin zero, then the γ has no preferred directions to project, as a result, the angular distribution of $\gamma-$rays is isotropic(4π cubic angle).

For the last questions, it's better to set the coordinates in spherical polar coordinates. The infinitesimal volume element is $d\Omega = r^2 sin\theta dr d\theta d\phi = dcos\theta d\phi$(set $r = 1$ and taking the absolute value of $d\Omega$.).

Notice that if the expression of $d\Omega = dcos\theta d\phi$ acts on E_γ, then the $d\Omega_{E_\gamma} = m_\pi\beta/2$ which is constant. This means the "γ energy density" in 3D dimension(sphere coordinates) is constant. As a result, the energy has a flat distribution; and under this context, $cos\theta$ ranges from -1 to 1, therefore E_γ has a flat distribution from $E_\pi(1+\beta)/2$ to $E_\pi(1-\beta)/2$.

If integrate $d\Omega$, one gets

$$N_{total} = \frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} sin\theta d\theta d\phi = \frac{1}{2}\int_{\theta=0}^{\pi} sin\theta d\theta = 1$$

In the case of $D > 3$, $D = (1+\beta cos\theta)/(1-\beta cos\theta) > 3 \Rightarrow \beta cos\theta > 1/2 \Rightarrow cos\theta > 1/2$ (set $\beta = 1$).
Attention, this $\theta > 1/2$ associates to, for instance, $E^{lab}_{\gamma_2} = \frac{m_\pi}{2}(1+\beta cos\theta)$; meanwhile, it indicates $\theta < -1/2$ in the case of $E^{lab}_{\gamma_1} = \frac{m_\pi}{2}(1-\beta cos\theta)$ since these two γ are just twins : if one γ in "up" half-sphere, correspondingly, another one must in "down" half-sphere. As a result, the "events number" is (notice $\int sin\theta = -\int cos\theta$, and taking the absolute value to make sure number is positive),

$$N_{D>3} = \frac{1}{2}\int_{cos\theta=0}^{1/2} dcos\theta + \frac{1}{2}\int_{cos\theta=-1}^{-1/2} dcos\theta = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$$

And, $N_{D>3}/N_{total} = (1/2)/1 = 1/2$. So in this case, there is half decays.
Similarly, In the case of $D > 7$, $D = (1+\beta cos\theta)/(1-\beta cos\theta) > 7 \Rightarrow \beta cos\theta > 1/4 \Rightarrow cos\theta > 1/4$ (set $\beta = 1$).

$$N_{D>7} = \frac{1}{2}\int_{cos\theta=0}^{1/4} dcos\theta + \frac{1}{2}\int_{cos\theta=-1}^{-3/4} dcos\theta = \frac{1}{8}+\frac{1}{8} = \frac{1}{4}$$

And, $N_{D>7}/N_{total} = (1/4)/1 = 1/4$. So in this case, there is one quarter decays.

For the last step, alternatively, there is another way to reach same conclusion.
According the Disparity definition(set $\beta = 1$).

$$\begin{aligned}
D &= \frac{1+cos\theta}{1-cos\theta} = -1 + \frac{2}{1-cos\theta} \\
\Rightarrow D+1 &= \frac{2}{1-cos\theta} \\
\Rightarrow 1-cos\theta &= \frac{2}{D+1} \\
\Rightarrow cos\theta &= 1-\frac{2}{D+1} \\
\Rightarrow d\Omega &\propto dcos\theta = d(1-\frac{2}{D+1}) \\
\Rightarrow \frac{d\Omega}{dD} &= \frac{2}{(D+1)^2} \\
\Rightarrow \int_{D>3}\frac{d\Omega}{dD}dD &= \int_{D=3}^{\infty}\frac{2}{(D+1)^2}dD = \frac{-2}{D+1}\,|_3^\infty = \frac{1}{2} \\
\Rightarrow \int_{D>7}\frac{d\Omega}{dD}dD &= \int_{D=7}^{\infty}\frac{2}{(D+1)^2}dD = \frac{-2}{D+1}\,|_7^\infty = \frac{1}{4}
\end{aligned}$$

Comment 1: The distribution of θ is isotropic, while the distribution of E_γ is flat (mathematically)due to $d\Omega$ contains $dcos\theta$ which "make" it flatten. It's remarkable.

Comment 2: Although the distribution of θ is isotropic, for bigger D, the two γ intends to locate near the "equator" of sphere. This is obvious since if D is bigger which means one γ gets dominates part of π's momentum thus almost along the same direction of π, so it will resident around the "equation". For smaller D, due to similar reason, γ have a preference to resident around two "poles" of their "earth".

Problem 11.3. Show that, in a head-on collision of a beam of relativistic particles of energy E_1 with one of energy E_2, the square of the energy in the center-of-momentum frame is $4E_1E_2$ and that for a crossing angle θ between the beams this is reduced by a factor $(1 + cos\theta)/2$. Show that the available kinetic energy in the head-on collision of two $25GeV$ protons is equal to that in the collision of a $1300GeV$ proton with a stationary nucleon.

Solution of problem 11.3:

Set the two relativistic particles' four momenta as $(E_1, \boldsymbol{P}_1)$ and $(E_2, \boldsymbol{P}_2)$. So,

$$\begin{aligned}
p^2 &= (E_1 + E_2)^2 - (\boldsymbol{P}_1 + \boldsymbol{P}_1)^2 \\
&= E_1^2 - \boldsymbol{P}_1^2 + E_2^2 - \boldsymbol{P}_2^2 + 2E_1E_2 - 2\boldsymbol{P}_1\boldsymbol{P}_2 \\
&= m_1^2 + m_2^2 + 2E_1E_2 + 2|\boldsymbol{P}_1||\boldsymbol{P}_2| && (\boldsymbol{P}_1 = -\boldsymbol{P}_2) \\
&= 4E_1E_2 && (m_1, m_2 \text{ are ignorable in the case of relativistic particles}) \\
&= 4E^2 && (\text{in the case of } E_1 = E_2 = E)
\end{aligned}$$

In the case of crossing angle θ existed, $\boldsymbol{P}_1$ should be replaced with $\boldsymbol{P}_1 cos\theta$. Under this context,

$$\begin{aligned}
p^2 &= m_1^2 + m_2^2 + 2E_1E_2 + 2cos\theta|\boldsymbol{P}_1|^2 \\
&= 2(1 + cos\theta)E_1E_2
\end{aligned}$$

The CMS energy of two head-on particles is $E^\star = 2E$, where E is the energy of two particles. While for a stationary nucleon case, the CMS energy is $E^\star = \sqrt{2mE}$.

So, two head-on protons with 25GeV will result CMS energy as $E^\star = 2E = 2 \times 25GeV = 50GeV$.

To reach $50GeV$ CMS energy, if one proton is stationary, another proton will need $E = (E^\star)^2/(2m) = (50)^2/(2 \times 0.938) = 1332GeV$.

Problem 11.4. A proton of momentum $\boldsymbol{p}$, large compared with its rest mass M, collides with a proton inside a target nucleus, with Fermi momentum $\boldsymbol{p}_f$. Find the available kinetic energy in the collision, as compared with that for a free-nucleon(stationary) target, when $\boldsymbol{p}$ and $\boldsymbol{p}_f$ are (a)parallel, (b)artiparallel, (c)orthogonal.

Solution of problem 11.4:

Set the two relativistic particles' four momenta as $(E_1, \boldsymbol{p})$ and $(E_2, \boldsymbol{p}_f)$. As mentioned in the solution of *problem 11.3* ,

$$\begin{aligned} E_{CMS}^2 = p^2 &= m_1^2 + m_2^2 + 2E_1E_2 + 2cos\theta|\boldsymbol{p}||\boldsymbol{p}_f| \\ &= 2M^2 + 2ME + 2cos\theta|\boldsymbol{p}||\boldsymbol{p}_f| \end{aligned}$$

So, in the case of (a) parallel, $\theta = \pi$, $E_{CMS}^2 = 2M^2 + 2ME - 2|\boldsymbol{p}||\boldsymbol{p}_f|$.
In the case of (b) anti-parallel, $\theta = 0$, $E_{CMS}^2 = 2M^2 + 2ME + 2|\boldsymbol{p}||\boldsymbol{p}_f|$.
In the case of (c) orthogonal, $\theta = \pi/2$, $E_{CMS}^2 = 2M^2 + 2ME$.

Problem 11.5. It was sometimes possible to differentiate between the tracks due to relativistic pions, protons and kaons(rest masses 140, 938 and 494 MeV respectively) in a bubble chamber by virtue of the high energy δ-rays produced. For a beam momentum of $5Gev/c$, what is the minimum δ-ray energy which must be observed to prove that it is produced by a pion rather than a kaon or proton ? What is the probability of observing such a knock-on electron in 1m of liquid hydrogen(density 0.06)?

Solution of problem 11.5:

The knock-out electrons are the source of δ-rays. The brief process is like this : incident (usually) relativistic particles produce primary ionization in collision the first stage, the knock-out electrons in this stage have a distribution in energy roughly like equation (11.9) in P_{350}; and this problem focus on this stage. Actually in the second stage after this one, some of high energy knock-out electrons could produce new ionization in traveling through the medium; and the resultant number of ion pairs in the second stage is $3 \sim 4$ times of the first stage, those electrons construct the "Landau tail".

In the first stage, the maximum energy could deliver to knock-out electrons is called T_{max}, which described in equation of (11.11).

$$\begin{aligned} T_{max} &= \frac{2mv^2E^2}{M^2+m^2+2mE} \\ &= \frac{2mv^2E^2}{M^2} \qquad (M^2 \gg m^2+2mE) \end{aligned}$$

From this equation, since for $5GeV$ pions, protons or kaons, their velocities are almost same, so what play a key to determine T_{max} is their rest masses, M^2. Moreover, the T_{max} is inverse proportional to their rest masses. Which means the electrons knocked out by pions have higher energy. And the "threshold" to discriminate the electrons were produced by pion or kaons is kaons' T_{max}, because above this energy, only pions could produce(protons have no chance to make a confusion due to it's heaviest among three particles, and thus has lowest T_{max}).

$$\begin{aligned} T_{max\ \pi} &= \frac{2m_ev^2E^2}{M^2} \\ &\approxeq \frac{2m_ec^2E^2}{M^2} \\ &= \frac{2\times 0.5\times (5\times 10^3)^2}{140^2} \\ &= 102MeV \end{aligned}$$

The result is slightly different from the answer of this problem in P_{411}. Possible reason is this solution takes a few approximate calculation, like ignoring the v^2 and m^2+2mE term.

Similarly, $T_{max\ k} = 1275MeV$.

The probability density of observing such a knock-out electron in 1m of liquid hydrogen could be calculated as (Cf. Problem 1.7(P_{75}) of "*Particle detector(Claus GRUPEN and Boris SHWARTZ, 2nd edition, Cambridge)*").

$$\phi(E)dE = K\frac{1}{\beta^2}\frac{Z}{A}\frac{x}{E^2}dE$$

where, $\phi(E)$ is the probability density; $K = 0.154Mev/(g/cm^2)$; Z and A are the atomic number and mass number of the atoms of the medium; x is absorber thickness in g/cm^2; $\beta = v/c$; E is the energy of incident particle.

So, the probability of observing such a electron is,

$$
\begin{aligned}
P &= \int_{T_{max\ \pi}}^{T_{max\ k}} \phi(E)dE \\
&= \int_{T_{max\ \pi}}^{T_{max\ k}} K\frac{1}{\beta^2}\frac{Z}{A}\frac{x}{E^2}dE \\
&= K\frac{1}{\beta^2}\frac{Z}{A}x(\frac{1}{T_{max\ \pi}} - \frac{1}{T_{max\ k}}) \\
&= K\frac{1}{\beta^2}\frac{Z}{A}x(\frac{1}{T_{max\ \pi}}) && (T_{max\ \pi} = 102MeV \ll T_{max\ k} = 1275MeV) \\
&= 0.154 \times \frac{1}{1^2} \times \frac{1}{1} \times (0.06 \times 100) \times \frac{1}{102} \\
&= 8.7 \times 10^{-3}
\end{aligned}
$$