CHAPTER 17 선형대수학

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(i) $L(u+v)=L(u)+L(v)$

(ii) $L(cu)=cL(u)$

ⓐ ① $L(x_1+y_1,\ x_2+y_2)$

$=(x_1+y_1+x_2+y_2,\ 2x_2+2y_2)$

$=L(x_1,x_2)+L(y_1,y_2)$

$=(x_1+x_2,\ 2x_2)+(y_1+y_2,\ 2y_2)$

$=(x_1+x_2+y_1+y_2,\ 2x_2+2y_2)$

② $L(cx_1,cx_2)=(cx_1+cx_2,\ 2cx_2)$

$=cL(x_1,x_2)=c(x_1+x_2,\ 2x_2)$

$=(cx_1+cx_2,\ 2cx_2)$

∴ ⓐ는 선형변환

ⓑ ① $L(x_1+y_1,\ x_2+y_2)$

$=(x_1+y_1,\ x_1+y_1+x_2+y_2,\ 3x_2+3y_2)$

$=L(x_1,x_2)+L(y_1,y_2)$

$=(x_1,\ x_1+x_2,\ 3x_2)+(y_1,\ y_1+y_2,\ 3y_2)$

$=(x_1+y_1,\ x_1+x_2+y_1+y_2,\ 3x_2+3y_2)$

② $L(cx_1)=(cx_1,\ cx_1+cx_2,\ 3cx_2)$

$=cL(x_1)=c(x_1,\ x_1+x_2,\ 3x_2)$

∴ ⓑ는 선형 변환

ⓒ ① $L(x_1+y_1,\ x_2+y_2,\ x_3+y_3)$

$=(2x_1+2y_1+x_2+y_2,\ x_2+y_2+3x_3+3y_3,\ x_1+y_1+4x_3+4y_3)$

$=L(x_1,x_2,x_3)+L(y_1,y_2,y_3)$

$=(2x_1+2y_1+x_2+y_2,\ x_2+y_2+3x_3+3y_3,\ x_1+y_1+4x_3+4y_3)$

② $L(cx_1,cx_2,cx_3)=(2cx_1+cx_2,\ cx_2+3cx_3,\ cx_1+4cx_3)$

$=L(x_1,x_2,x_3)\times c=c(2x_1+x_2,\ x_2+3x_3,\ x_1+4x_3)$

∴ ⓒ는 선형변환

ⓓ ① $L(x_1+y_1,\ x_2+y_2)$

$=(x_1+y_1,\ 2(x_1+y_1)(x_2+y_2))$

$L(x_1,x_2)+L(y_1,y_2)$

$=(x_1+y_1,\ 2x_1x_2+2y_1y_2)$

이때 $L(x_1+y_1,\ x_2+y_2)\neq L(x_1,x_2)+L(y_1,y_2)$

∴ ⓓ는 선형변환이 아니다.

ⓔ ① $L\left(\begin{bmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\end{bmatrix}\right)=\begin{bmatrix}1&2&1\\3&1&0\end{bmatrix}\begin{bmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\end{bmatrix}$

$=\begin{bmatrix}x_1+y_1+2x_2+2y_2+x_3+y_3\\3x_1+3y_1+x_2+y_2\end{bmatrix}$

$=L\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)+L\left(\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\right)$

$=\begin{bmatrix}x_1+2x_2+x_3\\3x_1+x_2\end{bmatrix}+\begin{bmatrix}y_1+2y_2+y_3\\3y_1+y_2\end{bmatrix}$

② $L\left(\begin{bmatrix}cx_1\\cx_2\\cx_3\end{bmatrix}\right)=\begin{bmatrix}1&2&1\\3&1&0\end{bmatrix}\begin{bmatrix}x_1\times c\\x_2\times c\\x_3\times c\end{bmatrix}$

$=cL\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=c\begin{bmatrix}1&2&1\\3&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

∴ ⓔ는 선형 변환

ⓕ ① $L\left(\begin{bmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\end{bmatrix}\right)=\begin{bmatrix}x_1+y_1+x_2+y_2\\2x_2+2y_2\\3x_3+3y_3\end{bmatrix}$

$=L\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)+L\left(\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\right)=\begin{bmatrix}x_1+x_2\\2x_2\\3x_3\end{bmatrix}+\begin{bmatrix}y_1+y_2\\2y_2\\3y_3\end{bmatrix}$

② $L\left(\begin{bmatrix}cx_1\\cx_2\\cx_3\end{bmatrix}\right)=\begin{bmatrix}cx_1+cx_2\\2cx_2\\3cx_3\end{bmatrix}$

$=cL\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=c\begin{bmatrix}x_1+x_2\\2x_2\\3x_3\end{bmatrix}$

∴ ⓕ는 선형변환

$\boxed{5}$

(a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

$$= \begin{bmatrix} -3x_1 \\ 2x_2 \end{bmatrix}$$

$\therefore a = -3,\ b = 0$
$c = 0,\ d = 2$

$\therefore A = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a'x_1 + b'x_2 \\ c'x_1 + d'x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 \\ 3x_1 - 4x_2 \end{bmatrix}$$

$a' = 1,\ b' = 1,\ c' = 3,\ d' = -4$

$\therefore B = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}$

(c)
$$\begin{bmatrix} 2x_1 + x_2 + x_3 \\ 0x_1 + 2x_2 - 4x_3 \end{bmatrix}$$

$\therefore \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -4 \end{bmatrix}$

(d)
$$\begin{bmatrix} 2x_1 + 3x_2 + 0x_3 \\ 0x_1 + 2x_2 - x_3 \\ x_1 + 0x_2 + 2x_3 \\ 0x_1 + x_2 + 2x_3 \end{bmatrix} \therefore \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

(e)
$$\begin{bmatrix} x_1 + 2x_2 + 0x_3 + 0x_4 \\ x_1 \cdot 0 + x_2 0 + x_3 0 + 0 \cdot x_4 \\ 0x_1 + 2x_2 + 0x_3 + \cancel{0}x_4 \\ 0x_1 + 2x_2 - 0x_3 - x_4 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & -1 \end{bmatrix}$

(f)
$$\begin{bmatrix} x_1 + 2x_2 \\ 0x_1 + 0x_2 \\ x_1 - 3x_2 \\ 2x_1 + 0x_2 \end{bmatrix} \therefore \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & -3 \\ 2 & 0 \end{bmatrix}$$

(g)
$$\begin{bmatrix} x_1 - 5x_2 + 4x_3 \\ 0x_1 + x_2 - 6x_3 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$

(h) $\begin{bmatrix} 3x_1 + 0x_2 + 4x_3 - 5x_4 \end{bmatrix}$

$\therefore \begin{bmatrix} 3 & 0 & 4 & -5 \end{bmatrix}$

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L의 표준행렬을 A라 하자

$A \times v_{3\times1}$ 이 2×1 벡터이므로

A는 2x3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$ 라 하자

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} a \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{pmatrix} b \\ e \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{pmatrix} c \\ f \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

ⓐ $L(1,2,3)$

$$\begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+4+3 \\ 3+2-3 \end{bmatrix}$$

$$\therefore = \begin{bmatrix} 9 \\ 2 \end{bmatrix} = (9, 2)$$

ⓑ $$\begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-1 \\ 3+1 \end{bmatrix}$$

$$\therefore = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = (1, 4)$$

ⓒ $$\begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+6+1 \\ 6+3-1 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$\therefore (11, 8)$

ⓓ $$\begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+2y+z \\ 3x+y-z \end{bmatrix}$$

$\therefore = (2x+2y+z, 3x+y-z)$

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$L_2 \circ L_1$ 2.

방법 1

$L_2 \circ L_1 = L_2(2x+y, x+2y)$

$= (-2x-y+x+2y, 2x+4y)$

$\therefore = (-x+y, 2x+4y)$

방법 2

A는 L_1의 표준행렬

$$\begin{pmatrix} 2x+y \\ x+2y \end{pmatrix} \rightarrow A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

B는 L2 〃

$$\begin{pmatrix} -x+y \\ 0+2y \end{pmatrix} \rightarrow B\begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+1 & -1+2 \\ 0+2 & 0+4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 2 & 4 \end{bmatrix}$$

$\therefore (-x+y, 2x+4y)$

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$L_2 \circ L_1$의 표준행렬은 BA이다

$$BA = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2+2 & -3+8 \\ 6+1 & 9+4 \end{bmatrix}$$

$$\therefore = \begin{bmatrix} 0 & 5 \\ 7 & 13 \end{bmatrix}$$

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ⓐ 참 계수 + 퇴화차수 = 열개수(9)

ⓑ 참 행공간차원 = 열공간차원 = rank = 3

$3 + 1 = 4$

ⓒ 참 rank가 full → 선형독립

ⓓ 거짓 행공간과 열공간은 직교!

ⓔ 참 $\mathrm{rank}(AB) \le \mathrm{rank}(B)$

ⓕ 참 $\mathrm{rank}(AB) = \mathrm{rank}(B)$.

B는 invertible

ⓖ 거짓 같다. 열공간차원 = 행공간차원

이고 행공간은 열공간의 Transpose

로 나타내므로

ⓗ 참 pivot이 없는 열이 존재하므로

해가 무수히 많다

ⓘ 참 열공간차원 2 이상이다.

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ⓐ

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 5 & 0 \\ 2 & 4 & -3 & 0 & 0 \\ -3 & 1 & 2 & -1 & 0 \\ 1 & 2 & -1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 5 & 0 \\ 0 & 0 & -5 & -10 & 0 \\ 0 & 7 & 5 & 14 & 0 \\ 0 & 0 & -2 & -4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 5 & 0 \\ 0 & 7 & 5 & 14 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 5 & 0 \\ 0 & 7 & 5 & 14 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 7 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3-\frac{8}{7} & 0 \\ 0 & 7 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rank는 3, Nullity는 1

ⓑ

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & -1 & 3 & 0 \\ 1 & 1 & 7 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rank 2, nullity 1

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A의 각 열이 선형독립이므로

A의 열공간 기저는

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} + z\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 3 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

solve x, y, z

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 2 & 3 & 1 & 3 \\ 3 & 6 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 0 & -3 & -7 & -1 \\ 0 & -3 & -10 & -5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & -3 & -7 & -1 \\ 0 & 0 & 3 & +4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & -3 & -7 & -1 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & -3 & 0 & -1+4\times\frac{7}{3} \\ 0 & 0 & 3 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & -3 & 0 & \frac{25}{3} \\ 0 & 0 & 3 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -25/9 \\ 0 & 0 & 1 & 4/3 \end{array}\right]$$

$$\therefore\ b = 5\begin{bmatrix}1\\2\\3\end{bmatrix} - 25/9\begin{bmatrix}3\\3\\6\end{bmatrix} + 4/3\begin{bmatrix}4\\1\\2\end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

A의 열공간의 기저 : $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

〃 차원 : 2

$$A \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$ 라하자 (free variable)

$x_4 = t$ 라하자 〃

$x_1 + 2s + t = 0$, $x_1 = -2s - t$

$x_2 + s = 0$, $x_2 = -s$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = s\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix} + t\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

영공간의 기저 : $\left\{\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\}$

〃 차원 : 2

행공간의 기저 : $\left\{\begin{bmatrix}1\\0\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$

〃 차원 : 2

$$A^T = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$ 라하자, $x_1 + s = 0$, $x_1 = -s$

$x_2 = 0$, $x_3 = s$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = s\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

좌영공간 기저 : $\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$

〃 차원 : 1

In [7]:

```python
import numpy as np

# 행렬 A를 출력하는 함수
def pprint(msg, A):
    print("---", msg, "---")
    (n,m) = A.shape
    for i in range(0, n):
        line = ""
        for j in range(0, m):
            line += "{0:.2f}".format(A[i,j]) + "₩t"
        print(line)
    print("")

a = np.array([[1,2,1,5],
              [2,4,-3,0],
              [-3,1,2,-1],
              [1,2,-1,1]]);
pprint("a", a)
print("rank(A) =", np.linalg.matrix_rank(a))
print("Nullity(A) =",a.shape[1]-np.linalg.matrix_rank(a))

b = np.array([[1,-2,1],
              [1,-1,3],
              [1,1,7]]);
pprint("b", b)
print("rank(B) =", np.linalg.matrix_rank(b))
print("Nullity(B) =",b.shape[1]-np.linalg.matrix_rank(b))
```

```
--- a ---
1.00	2.00	1.00	5.00
2.00	4.00	-3.00	0.00
-3.00	1.00	2.00	-1.00
1.00	2.00	-1.00	1.00

rank(A) = 3
Nullity(A) = 1
--- b ---
1.00	-2.00	1.00
1.00	-1.00	3.00
1.00	1.00	7.00

rank(B) = 2
Nullity(B) = 1
```