선형대수학 과제 8

8-5 proof

Let $\{v_1, v_2 \cdots v_r\}$에서

v_i는 v_i를 제외한 $\{v_1, v_2 \cdots v_{r-1}\}$이 있을때 $\{v_1, v_2 \cdots v_{r-1}\}$은 선형독립이라 하고 $\{v_1, v_2 \cdots v_r\}$는 선형종속이라 가정하자

㉠ $v_i = c_1 v_1 + c_2 v_2 \cdots c_{r-1} v_{r-1}$ 을 만족한다

$Av_i = c_1 Av_1 + c_2 Av_2 \cdots c_{r-1} Av_{r-1}$

ⓐ $\lambda_i v_i = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 \cdots c_{r-1} \lambda_{r-1} v_{r-1}$

㉠에서 λ_i를 곱하면

ⓑ $\Rightarrow \lambda_i v_i = c_1 \lambda_i v_1 + c_2 \lambda_i v_2 \cdots c_{r-1} \lambda_i v_{r-1}$

ⓐ = ⓑ 이므로

$0 = c_1(\lambda_1 - \lambda_i) v_1 + c_2(\lambda_2 - \lambda_i) v_2$

$\cdots c_{r-1}(\lambda_{r-1} - \lambda_i) v_{r-1}$

이때 $v_1, v_2 \cdots v_{r-1}$은 선형독립이라고 가정했으므로 $c_i = 0$ 이다 ($\forall i$)

($\because$ λ들은 모두 다르다고 가정)

이때 $v_i = c_1 v_1 + c_2 v_2 \cdots c_{r-1} v_{r-1} = 0$

이므로 v_i는 고유벡터가 될 수 없다.

$\therefore$ 가정의 결과가 모순이므로

$\{v_1, v_2 \cdots v_r\}$은 선형독립이다.

8-8 proof

$p(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$

$p(0) = \det(-A) = (-1)^n \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n$

$\det(-A) = (-1)^n \det(A)$ 이므로

$\therefore \det(A) = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n$

$p(\lambda)$에서 λ^{n-1}항의 계수는 $\sum_{i=1}^{n}(-1)\lambda_i = -\sum_{i=1}^{n}\lambda_i$

$\det(\lambda I - A)$에서 λ^{n-1} 계수는 $\lambda I - A$의 주대각성분 곱에서 나타난다

$$\lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ \vdots & \lambda - a_{22} & & \vdots \\ & & \ddots & \vdots \\ -a_{n1} & \cdots & & \lambda - a_{nn} \end{bmatrix}$$

λ^{n-1} 계수 $= \sum(-1)a_{ii} = -\sum a_{ii}$

$p(\lambda)$에서 λ^{n-1} 계수와 비교하면

$\sum \lambda_i = \sum a_{ii} = tr(A)$

$\therefore tr(A)$는 고유값의 합이다

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(a) $\det(\lambda I - A)$

$= \det\begin{bmatrix} \lambda-2 & -1 \\ 1 & \lambda \end{bmatrix}$

$= (\lambda-2)\lambda - (-1)$

$= \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$

(b)

$\det(\lambda I - A)$

$= \begin{vmatrix} \lambda-2 & -27 & 0 \\ 0 & \lambda-4 & -40 \\ 0 & -3 & \lambda-30 \end{vmatrix}$

$= (\lambda-2)(\lambda-4)(\lambda-30)$
$\quad - (\lambda-2)\times 120$

$= (\lambda-2)(\lambda^2 - 34\lambda + 120 - 120)$

$= (\lambda-2)\lambda(\lambda-34)$

$= \lambda^3 - 36\lambda^2 + 68\lambda$

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(a) $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = 0$

$= \lambda^2 + 3\lambda - (-2) = (\lambda+2)(\lambda+1) = 0$

$\lambda = -1, -2$

① $\lambda = -1$일 때

$-I - A = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$

$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} x = 0$

$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Let $x_2 = s$ $\quad x_1 + x_2 = 0$

$x_1 + s = 0, \; x_1 = -s$

고유벡터 $[-1 \;\; 1]^T$

② $\lambda = -2$

$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} x = 0$

$\left[\begin{array}{cc|c} -2 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = s \quad 2x_1 + s = 0, \; x_1 = -\frac{1}{2}s$

고유벡터 $[-\frac{1}{2} \;\; 1]^T$

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 1 \\ -2 & \lambda-4 \end{vmatrix} = 0$

$= (\lambda-1)(\lambda-4) + 2 = \lambda^2 - 5\lambda + 6$

$= (\lambda-2)(\lambda-3) = 0$

① $\lambda = 2$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$v_2 = [-1 \;\; 1]^T$

② $\lambda=3$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$V_3 = \left[-\frac{1}{2} \quad 1\right]^T$

ⓒ $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -2 & -2 \\ 0 & \lambda-2 & -1 \\ 1 & -2 & \lambda-2 \end{vmatrix}$

$= \left((\lambda-1)(\lambda-2)(\lambda-2) + 2\right)$

$\quad - (\lambda-2) \times -2 - (2)(\lambda-1)$

$= (\lambda^2 - 4\lambda + 4)(\lambda-1) + \cancel{2}$

$\quad + \cancel{2\lambda - 4} - \cancel{2\lambda + 2} = 0$

$\lambda = 1$ or 2

(c-1) $\lambda = 1$

$$\left[\begin{array}{ccc|c} 0 & -2 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$, $x_2 + x_3 = 0$, $x_2 = -s$

$x_1 - x_2 = 0$ $\quad x_1 = -s$

$V_1 = [-1 \ -1 \ 1]^T$

(c-2) $\lambda = 2$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = 0$, $x_1 = s$, $x_1 - 2x_2 = 0$

$x_2 = \frac{1}{2}s$

$V_2 = [1 \ \frac{1}{2} \ 0]^T$

ⓓ $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 & +1 \\ -1 & \lambda-2 & -1 \\ -2 & -2 & \lambda-3 \end{vmatrix}$

$= \left((\lambda-1)(\lambda-2)(\lambda-3) + 2\right)$

$\quad - \left(-2(\lambda-2) + 2(\lambda-1)\right)$

$= (\lambda-1)(\lambda-2)(\lambda-3)$

$\quad + 2 - (4-2) = (\lambda-1)(\lambda-2)(\lambda-3)$

(d-1) $\lambda = 1$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ -2 & -2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = 0$ $\quad x_1 = s$ $\quad x_2 = -s$

$V_1 = [1 \ -1 \ 0]^T$

(d-2) $\lambda = 2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ -2 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & +2 & 0 & 0 \end{array}\right]$$

$x_3 = s$ $\quad x_1 = -s$

$x_1 + 2x_2 = 0$ $\quad x_2 = \frac{1}{2}s$

$V_2 = [-1 \ \frac{1}{2} \ 1]^T$

(d-3) $\lambda = 3$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ -2 & -2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$ $\quad 2x_1 + x_3 = 0$ $\quad x_1 = -\frac{1}{2}s$

$x_2 = \frac{1}{2}s$

$V_3 = [-\frac{1}{2} \ \frac{1}{2} \ 1]^T$

(e) $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 3 & -3 \\ -3 & \lambda+5 & -3 \\ -6 & 6 & \lambda-4 \end{vmatrix}$

$\left((\lambda-1)(\lambda+5)(\lambda-4) + 54 + 54\right)$

$- \left(18(\lambda+5) - 9(\lambda-4) - 18(\lambda-1)\right)$

$= (\lambda-1)(\lambda+5)(\lambda-4) + 108$

$- \left(\cancel{18\lambda} + 90 - 9\lambda + 36 - \cancel{18\lambda} + 18\right)$

$= (\lambda-1)(\lambda^2+\lambda-20) + 108$

$+ 9\lambda - 144$

$= \lambda^3 + \cancel{\lambda^2} - 20\lambda - \cancel{\lambda^2} - \lambda + 20$

$+ 9\lambda - 144 + 108$

$= \lambda^3 - 12\lambda - 16$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -12 & -16 \\ & & -2 & 4 & 16 \\ \hline -2 & 1 & -2 & -8 & 0 \\ & & -2 & 8 & \\ \hline & 1 & -4 & 0 & \end{array}$$

$\Rightarrow \lambda^3 - 12\lambda - 16 = (\lambda+2)^2(\lambda-4)$

(e-1) $\lambda = 4$

$\left[\begin{array}{ccc|c} 3 & 3 & -3 & 0 \\ -3 & 9 & -3 & 0 \\ -6 & 6 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 3 & -3 & 0 \\ -6 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 = s$, $2x_2 - x_3 = 0$, $x_2 = \frac{1}{2}s$

$-x_1 + x_2 = 0$ $x_1 = \frac{1}{2}s$

$V_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^T$

(e-2) $\lambda = -2$

$\left[\begin{array}{ccc|c} -3 & 3 & -3 & 0 \\ -3 & 3 & -3 & 0 \\ -6 & 6 & -6 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 = s$, $x_3 = t$, $x_1 - s + t = 0$

$x_1 = s - t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s-t \\ s \\ t \end{bmatrix}$

$V_{-2} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(f) $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-1 \end{vmatrix}$

$= (\lambda-1)^2(\lambda-2)$

$- \left((\lambda-1) + (\lambda-1)\right)$

$= (\lambda-1)(\lambda^2-3\lambda+2) - 2(\lambda-1)$

$= (\lambda-1)(\lambda^2-3\lambda) = (\lambda-1)\lambda(\lambda-3)$

A-1 $\lambda = 0$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

A-2 $\lambda = 1$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

A-3 $\lambda = +3$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = s \quad x_3 = s$

$x_2 + 2x_3 = 0 \quad x_2 = -\frac{1}{2}s$

$v_3 = [1 \quad -\frac{1}{2} \quad 1]^T$

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ⓐ 참

ⓑ 거짓 가역행렬은 $\det(A) \neq 0$이다

$\det(A) = 0 \times 2 \times 3 = 0$ 이므로

거짓

ⓒ 참 $(\lambda I - A)^T = \lambda I - A^T$

det는 같다

ⓓ 거짓 $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$의 고유값 4

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$의 고유값 2

$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$

$(\lambda - 2)\lambda$ 고유값 2 or 1

ⓔ 거짓 $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix}$

고유값 3 or 5

ⓕ 참

ⓖ 참

ⓗ 참

ⓘ 참

ⓙ 거짓

중근인경우 선형독립인 두 벡터가

같은 고유값을 가질수있다

ⓚ 참

ⓛ 거짓 0이 있으면 X

$\boxed{18}$

$\lambda_1 + \lambda_2 = 5$

$\lambda_1 \times \lambda_2 = -14$

$\lambda_1(5-\lambda_1) = -14$

$-\lambda_1^2 + \lambda_1 \times 5 + 14 = 0$

$\lambda_1^2 - 5\lambda_1 - 14 = 0$

$= (\lambda - 7)(\lambda + 2)$

7 or -2

$\boxed{20}$

$1 + a + b = 0$, $\boxed{a + b = -1}$

$\boxed{24}$ $\lambda I - A = \begin{bmatrix} \lambda-7 & -2 & 2 \\ 6 & \lambda+1 & -2 \\ -6 & -2 & \lambda+1 \end{bmatrix}$

$= \left((\lambda-7)(\lambda+1)^2 - 24 - 24\right)$

$- \left(-12(\lambda+1) - 12(\lambda+1) + 4(\lambda-7)\right)$

$= (\lambda-7)(\lambda^2+2\lambda+1) - 48$

$- (-24\lambda - 24 + 4\lambda - 28)$

$= \lambda^3 + 2\lambda^2 + \lambda - 7\lambda^2 - 14\lambda - 7 - 48$

$+ 24\lambda + 24 - 4\lambda + 28$

$= \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

$3I = A(A^2 - 5A + 7I)$

$I = A\left(\dfrac{A^2 - 5A + 7I}{3}\right)$

$A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$

$A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 49-12-12 & 14-2-4 & -14+4+2 \\ -42+6+12 & -12+1+4 & 12-2-2 \\ 42-12-6 & 12-2-2 & -12+4+1 \end{bmatrix}$

$= \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$

$-5A = \begin{bmatrix} -35 & -10 & 10 \\ 30 & 5 & -10 \\ -30 & -10 & +5 \end{bmatrix}$

$A^2 - 5A = \begin{bmatrix} -10 & -2 & 2 \\ 6 & -2 & -2 \\ -6 & -2 & -2 \end{bmatrix}$

$A^2 - 5A + 7I = \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$

$\boxed{27}$ $|\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & -1 \\ -2 & \lambda-2 & 0 \\ -8 & 0 & \lambda-3 \end{vmatrix}$

$= (\lambda-1)(\lambda-2)(\lambda-3)$

$- (8(\lambda-2))$

$= (\lambda-2)(\lambda^2 - 4\lambda + 3 - 8)$

$= (\lambda-2)(\lambda^2 - 4\lambda - 5) = (\lambda-2)(\lambda-5)(\lambda+1)$

A의 고유값 $2, 5, -1$

$\therefore 3A^3 - 2A^2 + A + 4I$의 고유값은

$3\times8 - 2\times4 + 2 + 4 = 22$

$3\times125 - 2\times25 + 5 + 4 = 334$

$-3 - 2 - 1 + 4 = -2$

$\therefore 22, 334, -2$

```python
import numpy as np

A = np.array([[1, -1], [2, 4]])
w1, V1 = np.linalg.eig(A) # A의 고윳값과 고유벡터 계산

print("A의 고윳값 = ", w1)
print("A의 고유벡터 = ", V1)
```

```
A의 고윳값 =  [2. 3.]
A의 고유벡터 =  [[-0.70710678  0.4472136 ]
 [ 0.70710678 -0.89442719]]
```