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[1]

$$Y_t \mid \mathcal{F}_{t-1} \sim N(0, \sigma^2)$$

$$\sigma^2 = 0.7 + 0.2 Y_{t-1}^2$$

$$Y_t \mid Y_{t-1} = y, \mathcal{F}_{t-1} \sim N(0, 0.7 + 0.2y^2)$$

cdf

$$P(Y_t \le x \mid Y_{t-1} = y, \mathcal{F}_{t-1})$$
$$= P(\sigma_t \varepsilon_t \le x \mid Y_{t-1} = y)$$
$$= P(\sqrt{0.7 + 0.2Y_{t-1}^2}\, \varepsilon_t \le x \mid Y_{t-1} = y)$$
$$= P\left(\varepsilon_t \le \frac{x}{\sqrt{0.7+0.2y^2}}\right) = \Phi\left(\frac{x}{\sqrt{0.7+0.2y^2}}\right)$$

$\Phi(x)$ = cdf of standard normal distribution

[2]

$$Y_t = \varepsilon_t \sqrt{0.3 + 0.8 Y_{t-1}^2} \qquad \varepsilon_t \sim N(0,1)$$

$$Y_t \mid \mathcal{F}_{t-1} \sim N(0, \sigma^2)$$
$$\sigma^2 = 0.3 + 0.8 Y_{t-1}^2$$

(a)

$$E(Y_t) = E(E(Y_t \mid \mathcal{F}_{t-1})) = 0$$

$$V(Y_t) = E(V(Y_t \mid \mathcal{F}_{t-1})) + V(E(Y_t \mid \mathcal{F}_{t-1}))$$
$$= E(0.3 + 0.8 Y_{t-1}^2) + 0$$
$$= 0.3 + 0.8 E(Y_{t-1}^2)$$
$$= 0.3 + 0.8 V(Y_{t-1})$$

$$E(Y_{t-1}^2) = V(Y_{t-1}) + \underbrace{E(Y_{t-1})^2}_{=0}$$

Stationary한 분포이기 때문에 $V(Y_t) = V(Y_{t-1})$

$$\therefore V(Y_t) = 0.3 + 0.8 V(Y_t)$$
$$(1-0.8)V(Y_t) = 0.3$$
$$V(Y_t) = \frac{0.3}{1-0.8} = 1.5$$

(b)

Let $\nu_t = Y_t^2 - \sigma_t^2$.

then ν_t is white noise

proof

$$E(\nu_t) = E(Y_t^2 - \sigma_t^2)$$
$$= E(\sigma_t^2 \varepsilon_t^2 - \sigma_t^2)$$
$$= E(\sigma_t^2(\varepsilon_t^2 - 1))$$
$$= E(\sigma_t^2) E(\varepsilon_t^2 - 1)$$
$$= E(\sigma_t^2)(E(\varepsilon_t^2) - 1)$$
$$= E(\sigma_t^2)(V(\varepsilon_t) - 1) \qquad \because E(\varepsilon_t) = 0$$
$$= E(\sigma_t^2)(1-1) = 0$$

$$Cov(\nu_t, \nu_{t+h}) = E(\nu_t \nu_{t+h})$$
$$= E(\sigma_t^2(\varepsilon_t^2 - 1)\sigma_{t+h}^2(\varepsilon_{t+h}^2 - 1))$$
$$= E(\sigma_t^2 \sigma_{t+h}^2) E(\varepsilon_t^2 \varepsilon_{t+h}^2 - \varepsilon_t^2 - \varepsilon_{t+h}^2 + 1)$$
$$= E(\sigma_t^2 \sigma_{t+h}^2)(E(\varepsilon_t^2)E(\varepsilon_{t+h}^2) - E(\varepsilon_t^2) - E(\varepsilon_{t+h}^2) + 1)$$
$$= E(\sigma_t^2 \sigma_{t+h}^2)(1 - 1 - 1 + 1) = 0$$

$$V(\nu_t) = E(\nu_t^2)$$
$$= E(\sigma_t^4(\varepsilon_t^2 - 1)^2)$$
$$= E(\sigma_t^4) E((\varepsilon_t - 1)^2) \quad \text{by stationarity}$$
$$= \sigma_\nu^2 \quad \text{not depending on } t$$

□

so $\nu_t \sim N(0, \sigma_\nu^2)$

$$Y_t^2 = \sigma_t^2 + \nu_t$$
$$= 0.3 + 0.8 Y_{t-1}^2 + \nu_t$$

$\rightarrow AR(1)$

[3]

$Y_t = \sigma_t \varepsilon_t \qquad \sigma_t^2 = 0.2 + 0.3Y_{t-1}^2 + 0.5Y_{t-2}^2 + 0.1Y_{t-3}^2$

(a)

$0.3+0.5+0.1 = 0.9 < 1$

$\therefore$ Y_t is stationary.

(b)

$Y_t = 10, \; Y_{t-1} = 12 \quad Y_{t-2} = 9 \quad Y_{t-3} = 14$

$$\hat{\sigma}_{t+1}^2 = E(\sigma_{t+1}^2 \mid Y_t, Y_{t-1}, \cdots)$$
$$= E(0.2 + 0.3Y_t^2 + 0.5Y_{t-1}^2 + 0.1Y_{t-2}^2 \mid Y_t, \cdots)$$
$$= 0.2 + 0.3\times 10^2 + 0.5\times 12^2 + 0.1\times 9^2$$
$$= 0.2 + 30 + 72 + 8.1 = 110.3$$

$$\hat{\sigma}_{t+2}^2 = E(\sigma_{t+2}^2 \mid Y_t, Y_{t-1}, \cdots)$$
$$= 0.2 + 0.3\hat{\sigma}_{t+1}^2 + 0.5Y_t^2 + 0.1Y_{t-1}^2$$
$$= 0.2 + 0.3\times 110.3 + 0.5\times 10^2 + 0.1\times 12^2$$
$$= 0.2 + 33.09 + 50 + 14.4$$
$$= 97.69$$

$$\hat{\sigma}_{t+3}^2 = E(\sigma_{t+3}^2 \mid Y_t, \cdots)$$
$$= 0.2 + 0.3\times 97.69 + 0.5\times 110.3 + 0.1\times 10^2$$
$$= 0.2 + 29.307 + 55.15 + 10$$
$$= 94.657$$

[4]

$Y_t = \sigma_t \varepsilon_t \qquad \varepsilon_t \sim N(0, 1)$

$\sigma_t^2 = 0.2 + 0.3Y_{t-1}^2 + 0.4\sigma_{t-1}^2$

(a) $0.3 + 0.4 = 0.7 < 1$

Y_t is stationary.

(b)

$$E(Y_t) = E(E(Y_t \mid \mathcal{F}_{t-1}))$$
$$= 0 \qquad \because Y_t \mid \mathcal{F}_{t-1} \sim N(0, \sigma_t^2)$$

Let $\nu_t = Y_t^2 - \sigma_t^2$ then ν_t is white noise

$$Y_t^2 = \sigma_t^2 + \nu_t$$
$$= 0.2 + 0.3Y_{t-1}^2 + 0.4\sigma_{t-1}^2 + \nu_t$$
$$= 0.2 + 0.3Y_{t-1}^2 + 0.4(Y_{t-1}^2 - \nu_{t-1}) + \nu_t \quad \because \sigma_t^2 = Y_t^2 - \nu_t$$
$$= 0.2 + (0.3+0.4)Y_{t-1}^2 + \nu_t - 0.4\nu_{t-1}$$

$$V(Y_t) = E(Y_t^2)$$
$$= E(0.2 + (0.3+0.4)Y_{t-1}^2 + \nu_t - 0.4\nu_{t-1})$$
$$= 0.2 + 0.7E(Y_{t-1}^2) + 0$$

by stationarity $E(Y_t^2) = E(Y_{t-1}^2)$

$$\therefore V(Y_t) = \frac{0.2}{1-0.7} = \frac{2}{3}$$

(c)

Let $v_t = Y_t^2 - \sigma_t^2$ $\quad$ v_t is white noise.

$$Y_t^2 = \sigma_t^2 + v_t$$
$$= 0.2 + 0.3Y_{t-1}^2 + 0.4\sigma_{t-1}^2 + v_t$$
$$= 0.2 + 0.3Y_{t-1}^2 + 0.4(Y_{t-1}^2 - v_{t-1}) + v_t$$
$$= 0.2 + 0.7Y_{t-1}^2 + v_t - 0.4v_t$$

ARMA(1.1)

(d)

$$\hat{\sigma}_{t+1}^2 = E(\sigma_{t+1}^2 \mid Y_t, Y_{t-1}, \cdots)$$
$$= E(0.2 + 0.3Y_t^2 + 0.4\sigma_t^2 \mid Y_t, \cdots)$$
$$= 0.2 + 0.3 \times 5^2 + 0.4\sigma_t^2$$
$$= 0.2 + 7.5 + 0.4\sigma_t^2$$
$$= 7.7 + 0.4\sigma_t^2$$

$$\hat{\sigma}_{t+2}^2 = E(\sigma_{t+2}^2 \mid Y_t, \cdots)$$

$E(Y_{t+1} \mid Y_t, \cdots) = E(\sigma_{t+1}^2\varepsilon_{t+1}^2 \mid Y_t, \cdots) = \hat{\sigma}_{t+1}^2$

$$= E(0.2 + 0.3Y_{t+1}^2 + 0.4\sigma_{t+1}^2 \mid Y_t, \cdots)$$
$$= 0.2 + (0.3 + 0.4)\hat{\sigma}_{t+1}^2$$
$$= 0.2 + 0.7(7.7 + 0.4\sigma_t^2)$$
$$= 5.59 + 0.28\sigma_t^2$$

$$\hat{\sigma}_{t+3}^2 = E(\sigma_{t+3}^2 \mid Y_t, \cdots)$$
$$= E(0.2 + 0.3Y_{t+2}^2 + 0.4\sigma_{t+2}^2 \mid Y_t, \cdots)$$
$$= 0.2 + (0.3 + 0.4)\hat{\sigma}_{t+2}^2$$
$$= 0.2 + 0.7(5.59 + 0.28\sigma_t^2)$$
$$= 4.113 + 0.196\sigma_t^2$$

[5]

$Y_t = \sigma_t\varepsilon_t$ $\quad$ $\varepsilon_t \sim N(0.1)$

$$\sigma_t^2 = 0.2 + 0.1Y_{t-1}^2 + 0.2Y_{t-2}^2 + 0.15Y_{t-3}^2 + 0.3\sigma_{t-1}^2 + 0.11\sigma_{t-2}^2$$

(a)

$$0.1 + 0.2 + 0.15 + 0.3 + 0.11 = 0.86 < 1$$

Y_t is stationary.

GARCH(3.2)

(b)

$E(Y_t) = E(E(Y_t \mid F_{t-1})) = 0$ $\quad$ $Y_t \mid F_{t-1} \sim N(0.\sigma_t^2)$

$$V(Y_t) = \frac{w}{1 - \Sigma\alpha - \Sigma\beta}$$
$$= \frac{0.2}{1 - 0.45 - 0.41}$$
$$= \frac{0.2}{0.86} = \frac{20}{86} = 0.2326$$

(c)

$v_t = Y_t^2 - \sigma_t^2$ $\quad$ v_t is white noise

$$Y_t^2 = \sigma_t^2 + v_t$$
$$= 0.2 + 0.1Y_{t-1}^2 + 0.2Y_{t-2}^2 + 0.15Y_{t-3}^2 + 0.3\sigma_{t-1}^2 + 0.11\sigma_{t-2}^2 + v_t$$
$$= 0.2 + 0.1Y_{t-1}^2 + 0.2Y_{t-2}^2 + 0.15Y_{t-3}^2 + 0.3(Y_{t-1}^2 - v_{t-1}) + 0.11(Y_{t-2}^2 - v_{t-2}) + v_t$$
$$= 0.2 + 0.4Y_{t-1}^2 + 0.31Y_{t-2}^2 + 0.15Y_{t-3}^2 + v_t - 0.3v_{t-1} - 0.11v_{t-2}$$

ARMA(3.2)

[6]

$$Y_t = \sigma_t \varepsilon_t \qquad \varepsilon_t \sim N(0,1)$$

$$\sigma_t^2 = \omega + \alpha Y_{t-1}^2 + \beta \sigma_{t-1}^2$$

Show that

$$\sigma_t^2 = \frac{\omega}{1-\beta} + \alpha \sum_{i=1}^{\infty} \beta^{i-1} Y_{t-i}^2$$

$$\sigma_t^2 = \omega + \alpha Y_{t-1}^2 + \beta \sigma_{t-1}^2$$

$$= \omega + \alpha Y_{t-1}^2 + \beta(\omega + \alpha Y_{t-2}^2 + \beta \sigma_{t-2}^2)$$

$$= \omega + \beta\omega + \alpha Y_{t-1}^2 + \alpha\beta Y_{t-2}^2 + \beta^2(\omega + \alpha Y_{t-3}^2 + \beta \sigma_{t-3}^2)$$

$\vdots$

$$= \omega + \beta\omega + \beta^2\omega + \cdots + \beta^{n-1}\omega$$
$$+ \alpha Y_{t-1}^2 + \alpha\beta Y_{t-2}^2 + \alpha\beta^2 Y_{t-3}^2 + \cdots + \alpha\beta^{n-1} Y_{t-n}^2$$
$$+ \beta^n \sigma_{t-n}^2$$

$\downarrow$ $n \to \infty$

$$= \omega(1 + \beta + \beta^2 + \cdots)$$
$$+ \alpha \sum_{i=1}^{\infty} \beta^{i-1} Y_{t-i}^2$$
$$+ 0$$

$\because$ $0 < \beta < 1$ if Y_t is stationary

$$= \frac{\omega}{1-\beta} + \alpha \sum_{i=1}^{\infty} \beta^{i-1} Y_{t-i}^2$$

[7]

$$X_t \mid \mathcal{F}_{t-1} \sim \text{Poisson}(\lambda_t)$$

$$\lambda_t = \omega + \alpha X_{t-1} + \beta \lambda_{t-1}$$

Show that

$$\lambda_t = \frac{\omega}{1-\beta} + \alpha \sum_{i=1}^{t} \beta^{i-1} X_{t-i} + \beta^t\left(\lambda_0 - \frac{\omega}{1-\beta}\right)$$

$$\lambda_t = \omega + \alpha Y_{t-1} + \beta \lambda_{t-1}$$

$$= \omega + \alpha Y_{t-1} + \beta(\omega + \alpha Y_{t-2} + \beta \lambda_{t-2})$$

$$= \omega + \beta\omega + \alpha Y_{t-1} + \alpha\beta Y_{t-2} + \beta^2 \lambda_{t-2}$$

$$= \omega + \beta\omega + \alpha Y_{t-1} + \alpha\beta Y_{t-2} + \beta^2(\omega + \alpha Y_{t-3} + \beta \lambda_{t-3})$$

$$= \omega + \beta\omega + \beta^2\omega + \alpha Y_{t-1} + \alpha\beta Y_{t-2} + \alpha\beta^2 Y_{t-3} + \beta^3 \lambda_{t-3}$$

$\vdots$

$$= \omega + \beta\omega + \beta^2\omega + \cdots + \beta^{t-1}\omega$$
$$+ \alpha Y_{t-1} + \alpha\beta Y_{t-2} + \cdots + \alpha\beta^{t-1} Y_0$$
$$+ \beta^t \lambda_0$$

$$= \omega\left(\frac{1-\beta^t}{1-\beta}\right) + \alpha \sum_{i=1}^{t} \beta^{i-1} Y_{t-i} + \beta^t \lambda_0$$

$$= \frac{\omega}{1-\beta} + \alpha \sum_{i=1}^{t} \beta^{i-1} Y_{t-i} + \beta^t\left(\lambda_0 - \frac{\omega}{1-\beta}\right)$$