##4차 과제: 202340339 이하늘

###HW: [1] 교재 p. 157, # 3.4(a)(b)

Identify the following models as ARMA(p, q) models (watch out for parameter redundancy), and determine whether they are causal and/or invertible:

(a) $x_t = 0.80x_{t-1} - 0.15x_{t-2} + w_t - 0.30w_{t-1}$

ARMA(2, 1) 모델,

$x_t - 0.80x_{t-1} + 0.15x_{t-2} = w_t - 0.30w_{t-1}$

$(1 - 0.8B + 0.15B^2)x_t = (1 - 0.30\mathrm{B})\mathrm{w_t}$

$\phi(B) = (1 - 0.8B + 0.15B^2), \theta(B) = (1 - 0.30\mathrm{B})$

1. Causal?

$$x_t = \frac{\theta(B)}{\phi(B)}w_t = \frac{(1 - 0.30\mathrm{B})}{(1 - 0.8B + 0.15B^2)}w_t$$

$$= \frac{(1 - 0.30\mathrm{B})}{(1 - 0.30B)(1 - 0.5B)}w_t = \frac{1}{1 - 0.5B}w_t$$

$$x_t = (1 + 0.5B + 0.5^2B^2 + \cdots)w_t$$

$$= w_t + 0.5w_{t-1} + 0.5^2w_{t-2} + \cdots$$

$$= \sum_{j=0}^{\infty}(0.5)^j w_{t-j}$$

➔ w_t의 무한 급수로 표현되기 때문에, causal

2. Invertible?

$$w_t = \frac{\phi(B)}{\theta(B)}x_t = \frac{(1 - 0.8B + 0.15B^2)}{(1 - 0.30\mathrm{B})}x_t$$

$$= \frac{(1 - 0.30B)(1 - 0.5B)}{(1 - 0.30\mathrm{B})}x_t = (1 - 0.5B)x_t$$

$$w_t = (1 - 0.5B)x_t$$

$$= x_t - 0.5x_{t-1}$$

➔ x_t의 linear combination으로 표현되기 때문에, invertible

(b) $x_t = x_{t-1} - 0.50x_{t-2} + w_t - w_{t-1}$

ARMA(2, 1) 모델,

$x_t - x_{t-1} + 0.50x_{t-2} = w_t - w_{t-1}$

$(1 - B + 0.50B^2)x_t = (1 - \mathrm{B})\mathrm{w_t}$

$\phi(B) = (1 - B + 0.50B^2), \theta(B) = (1 - \mathrm{B})$

1. Causal?

   $\phi(B) = (1 - B + 0.50B^2)$, which has root $1 \pm 1\mathrm{i}$

   $|\mathrm{z}|^2 = \sqrt{1^2 + 1^2} = \sqrt{2} > 1$

   ➔ causal

2. Invertible?

   $\theta(B) = (1 - \mathrm{B})$, which has roots 1

   ➔ Not invertible

###HW: [2] For AR(2) model, $x_t = -0.9x_{t-2} + w_t$,

(a) Find the roots of the autoregressive polynomial.

$x_t + 0.9x_{t-2} = w_t$

$(1 + 0.9B^2)x_t = \mathrm{w_t}$

$\phi(B) = 1 + 0.9B^2$

$1 + 0.9B^2 = 0$

$D = \phi_1^2 + 4\phi_2 = -3.6 < 0 ==> complex\ numbers$

$a, b = \frac{\phi_1}{2} \pm \frac{i}{2}\sqrt{-(\phi_1^2 + 4\phi_2)} = \frac{i}{2} \pm \sqrt{3.6}$

(b) Determine whether this model is causal and/or invertible.

$\phi(B) = (1 + 0.9B^2)$, which has root $\frac{i}{2} \pm \sqrt{3.6}$

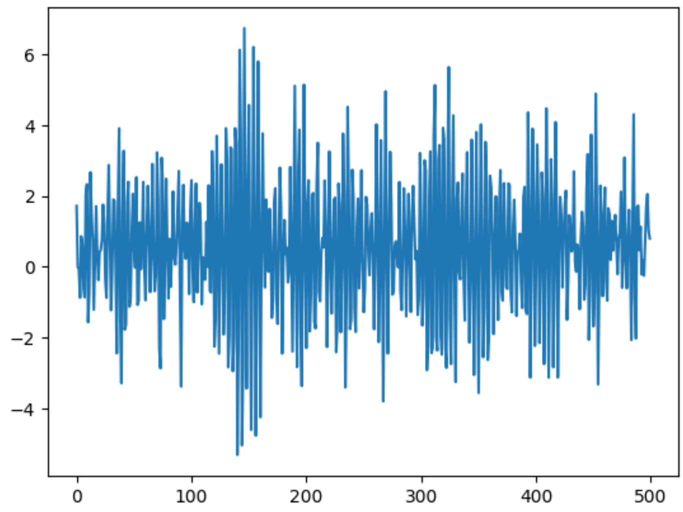
$\phi_2 = -0.9 > -1, \quad \phi_2 - \phi_1 = -0.9 < 1, \quad \phi_2 + \phi_1 = -0.9 < 1$

➔ causal

(c) 이 모형의 가상의 데이터를 생성하여 SACF를 계산하고, 그래프를 그리시오. 이론의 ACF 그래프와 유사한지 확인하시오.

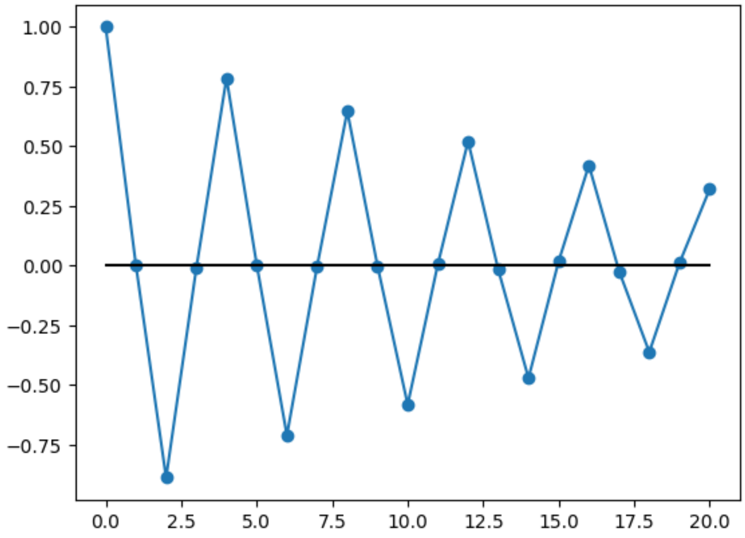
```
#코드
import numpy as np
import matplotlib.pyplot as plt

#SACF
def ACF(D):
    x = D
    n = len(D)
    mu = np.mean(D)
    L = []
    for h in range(21): #h=0,1,...,20 - lag 20까지
        Lh = []
        for t in range(0, n-h):
            ac = (x[t+h] - mu)*(x[t] - mu)
            Lh.append(ac)
        autocov_h = sum(Lh)/n
        L.append(autocov_h)
    AutoCov = np.array(L)
    Sacf = AutoCov/AutoCov[0]
    plt.plot(Sacf, marker = 'o')
    y = [0 for t in range(21)]
    plt.plot(y, 'black')
    plt.show()
    return Sacf

def ARMA_22(phi0, phi1, phi2, th1, th2, ss, n):
    L = []
    w = np.random.normal(0,ss**0.5,n+2) #표준편차 #w[0],...,w[n+1] 총 n+2개
    x00,x0 = 0,0 #초기값
    for t in range(2, n+2): #t=2,3,...,n+1 총 n개
```

```
        xt = phi0 + phi1*x0 + phi2*x00 + w[t] + th1*w[t-1] + th2*w[t-2]

        L.append(xt)

        x00 = x0

        x0 = xt

    plt.plot(L)

    plt.show()

    return L


Ar2 = ARMA_22(1,0,-0.9,0,0,1,500)
```

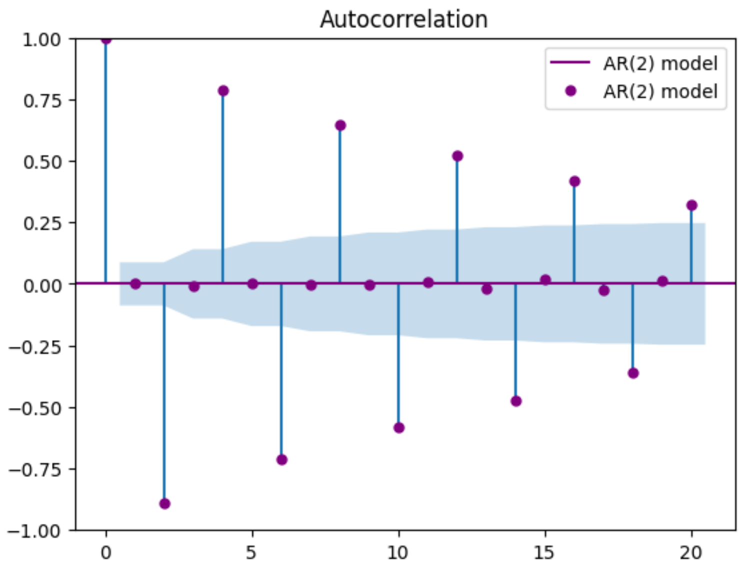



```
from statsmodels.tsa.stattools import acf, pacf

import statsmodels.api as sm


#이론적 ACF

def ACF_python(D, COLOR, TITLE):

    sm.graphics.tsa.plot_acf(np.array(D), lags=20, color = COLOR, label = TITLE)

    plt.legend()

    plt.show()
```

sacf_20 = ACF(Ar2)



ACF_python(Ar2, 'purple','AR(2) model')



→ 유사한 추세를 보임

###HW: [3] For MA(2) model, $x_t = w_t + 0.5w_{t-1} - 0.2w_{t-2}$,

(a) Find the moving average polynomial.

$$x_t = w_t + 0.5w_{t-1} - 0.2w_{t-2}$$

$$x_t = (1 + 0.5B - 0.2B^2)\mathrm{w_t}$$

$$\theta(B) = (1 + 0.5B - 0.2B^2)$$

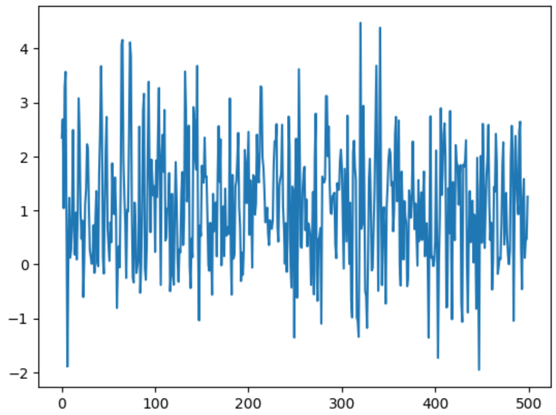
(b) 이 모형의 가상의 데이터를 생성하여 SACF를 계산하고 그래프를 그리시오. 이론의 ACF 그래프와 유사한지 확인하시오

#코드

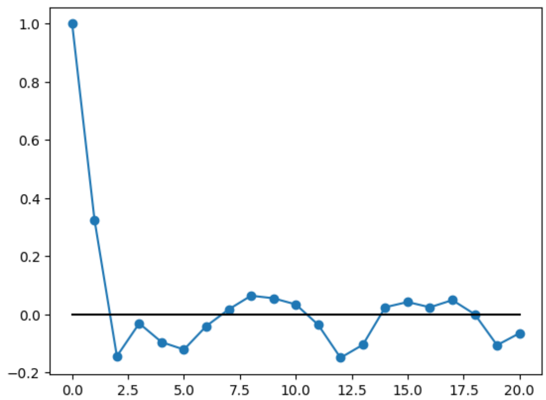
import numpy as np

import matplotlib.pyplot as plt

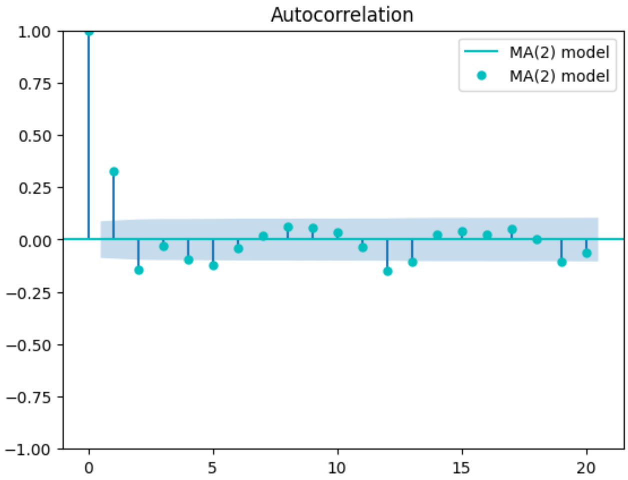
MA2 = ARMA_22(1,0,0,0.5,-0.2,1,500)



sacf_02 = ACF(MA2)



ACF_python(MA2, 'c','MA(2) model')



→ 유사한 추세를 보임

###HW: [4] 위의 [1]번 문제 (a) 모형에서 가상의 데이터를 생성하여 SACF를 계산하고, 그래프를 그리시오. 이론의 ACF 그래프와 유사한지 확인하시오. (a). $x_t = 0.80x_{t-1} - 0.15x_{t-2} + w_t - 0.30w_{t-1}$

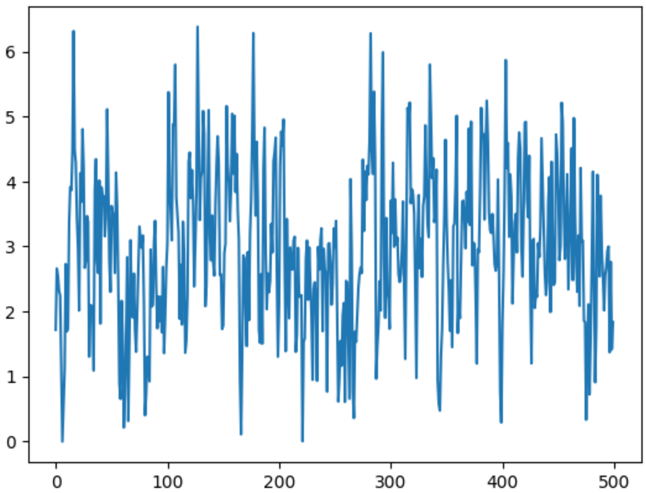
➔ ARMA(2, 1) 모델

#코드

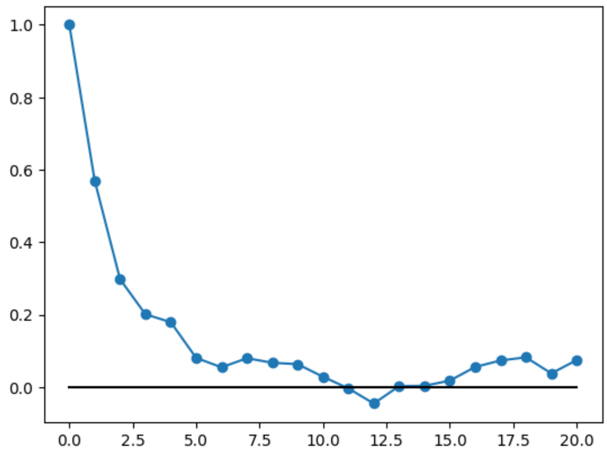
import numpy as np

import matplotlib.pyplot as plt

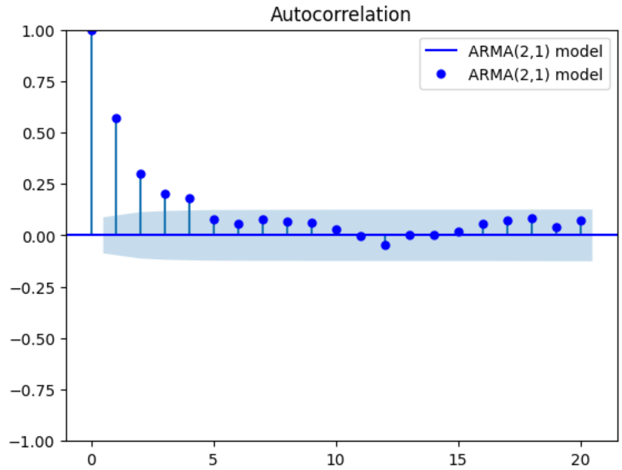
Arma21 = ARMA_22(1,0.8,-0.15,-0.3,0,1,500)



sacf_21 = ACF(Arma21)



ACF_python(Arma21, 'b', 'ARMA(2,1) model')



→ 유사한 추세를 보임

###HW: [5] In MA(1) model $x_t = w_t + \theta w_{t-1}$,

(a) find PACF ϕ_{33} in terms of θ.

$$\phi_{11} = \rho(1) = \frac{\theta}{1+\theta^2}, \qquad \rho(2) = 0 = \rho(3) = \cdots$$

$$\phi_{33} \Rightarrow \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{31} \\ \phi_{32} \\ \phi_{33} \end{bmatrix}$$

Use Cramer's Rule

$$\phi_{33} = \frac{det \begin{bmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{bmatrix}}{\det(A)}, A = \begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{bmatrix}$$

$$det\begin{bmatrix}1 & \rho_1 & \rho_2\\ \rho_1 & 1 & \rho_2\\ \rho_2 & \rho_1 & \rho_3\end{bmatrix} = \left[\begin{array}{ccc|cc}1 & \rho_1 & \rho_1 & 1 & \rho_1\\ \rho_1 & 1 & 0 & \rho_1 & 1\\ 0 & \rho_1 & 0 & 0 & \rho_1\end{array}\right] = 0+0+\rho_1^3-(0+0+0)=\rho_1^3$$

$$det\begin{bmatrix}1 & \rho_1 & \rho_2\\ \rho_1 & 1 & \rho_1\\ \rho_2 & \rho_1 & 1\end{bmatrix} = \left[\begin{array}{ccc|cc}1 & \rho_1 & 0 & 1 & \rho_1\\ \rho_1 & 1 & \rho_1 & \rho_1 & 1\\ 0 & \rho_1 & 1 & 0 & \rho_1\end{array}\right] = 1+0+0-(0+\rho_1^2+\rho_1^2)=1-2\rho_1^2$$

$$\phi_{33} = \frac{\rho_1^3}{1-2\rho_1^2} = \frac{\left(\frac{\theta}{1+\theta^2}\right)^3}{1-2\left(\frac{\theta}{1+\theta^2}\right)^2} = \frac{\frac{\theta^3}{(1+\theta^2)^3}}{\frac{(1+\theta^2)^2-2\theta^2}{(1+\theta^2)^2}} = \frac{\frac{\theta^3}{(1+\theta^2)^3}}{\frac{\theta^4+2\theta^2+1-2\theta^2}{(1+\theta^2)^2}} = \frac{\frac{\theta^3}{(1+\theta^2)^3}}{\frac{\theta^4+1}{(1+\theta^2)^2}} = \frac{\frac{\theta^3}{1+\theta^2}}{\frac{1+\theta^4}{1}}$$
$$= \frac{\theta^3}{(1+\theta^2)(1+\theta^4)} = \frac{\theta^3}{1+\theta^2+\theta^4+\theta^6}$$

(b) find PACF ϕ_{44} in terms of θ.

$$\phi_{11} = \rho(1) = \frac{\theta}{1+\theta^2}, \qquad \rho(2)=0=\rho(3)=\cdots$$

$$\phi_{44} => \begin{bmatrix}\rho_1\\ \rho_2\\ \rho_3\\ \rho_4\end{bmatrix} = \begin{bmatrix}1 & \rho_1 & \rho_2 & \rho_3\\ \rho_1 & 1 & \rho_1 & \rho_2\\ \rho_2 & \rho_1 & 1 & \rho_1\\ \rho_3 & \rho_2 & \rho_1 & 1\end{bmatrix}\begin{bmatrix}\phi_{31}\\ \phi_{32}\\ \phi_{33}\\ \phi_{44}\end{bmatrix}$$

Use Cramer's Rule

$$\phi_{44} = \frac{det\begin{bmatrix}1 & \rho_1 & \rho_2 & \rho_1\\ \rho_1 & 1 & \rho_1 & \rho_2\\ \rho_2 & \rho_1 & 1 & \rho_3\\ \rho_3 & \rho_2 & \rho_1 & \rho_4\end{bmatrix}}{\det(A)}, A = \begin{bmatrix}1 & \rho_1 & \rho_2 & \rho_3\\ \rho_1 & 1 & \rho_1 & \rho_2\\ \rho_2 & \rho_1 & 1 & \rho_1\\ \rho_3 & \rho_2 & \rho_1 & 1\end{bmatrix}$$

$$det\begin{bmatrix}1 & \rho_1 & \rho_2 & \rho_1\\ \rho_1 & 1 & \rho_1 & \rho_2\\ \rho_2 & \rho_1 & 1 & \rho_3\\ \rho_3 & \rho_2 & \rho_1 & \rho_4\end{bmatrix} = det\begin{bmatrix}1 & \rho_1 & 0 & \rho_1\\ \rho_1 & 1 & \rho_1 & 0\\ 0 & \rho_1 & 1 & 0\\ 0 & 0 & \rho_1 & 0\end{bmatrix}$$
$$= 1 * det\begin{bmatrix}1 & \rho_1 & 0\\ \rho_1 & 1 & 0\\ 0 & \rho_1 & 0\end{bmatrix} - (\rho_1) * det\begin{bmatrix}\rho_1 & \rho_1 & 0\\ 0 & 1 & 0\\ 0 & \rho_1 & 0\end{bmatrix} + 0 * det\begin{bmatrix}\rho_1 & 1 & 0\\ 0 & \rho_1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$-(\rho_1) * det\begin{bmatrix}\rho_1 & 1 & \rho_1\\ 0 & \rho_1 & 1\\ 0 & 0 & \rho_1\end{bmatrix} = 0-0+0-(\rho_1)*(\rho_1^3) = -\rho_1^4$$

$$det\begin{bmatrix}1 & \rho_1 & \rho_2 & \rho_3\\ \rho_1 & 1 & \rho_1 & \rho_2\\ \rho_2 & \rho_1 & 1 & \rho_1\\ \rho_3 & \rho_2 & \rho_1 & 1\end{bmatrix} = det\begin{bmatrix}1 & \rho_1 & 0 & 0\\ \rho_1 & 1 & \rho_1 & 0\\ 0 & \rho_1 & 1 & \rho_1\\ 0 & 0 & \rho_1 & 1\end{bmatrix}$$
$$= 1 * det\begin{bmatrix}1 & \rho_1 & 0\\ \rho_1 & 1 & \rho_1\\ 0 & \rho_1 & 1\end{bmatrix} - (\rho_1) * det\begin{bmatrix}\rho_1 & \rho_1 & 0\\ 0 & 1 & \rho_1\\ 0 & \rho_1 & 1\end{bmatrix} + 0 * det\begin{bmatrix}\rho_1 & 1 & 0\\ 0 & \rho_1 & \rho_1\\ 0 & 0 & 1\end{bmatrix}$$

$$-(0) * det\begin{bmatrix}\rho_1 & 1 & \rho_1\\ 0 & \rho_1 & 1\\ 0 & 0 & \rho_1\end{bmatrix} = (1-2\rho_1^2)-(\rho_1)*(\rho_1-\rho_1^3)+0-0 = (1-2\rho_1^2)-(\rho_1^2-\rho_1^4)$$
$$= (1-3\rho_1^2+\rho_1^4)$$

$$\phi_{44} = -\frac{\rho_1^4}{\rho_1^4 - 3\rho_1^2 + 1} = \frac{\left(\frac{\theta}{1+\theta^2}\right)^4}{\left(\frac{\theta}{1+\theta^2}\right)^4 - 3\left(\frac{\theta}{1+\theta^2}\right)^2 + 1} = \frac{\frac{\theta^4}{(1+\theta^2)^4}}{\frac{\theta^4 - 3\theta^2(1+\theta^2)^2 + (1+\theta^2)^4}{(1+\theta^2)^4}}$$

$$= \frac{\frac{\theta^4}{(1+\theta^2)^4}}{\frac{\theta^4 - 3\theta^2 - 3\theta^4 - 3\theta^6 + 1 + 4\theta^2 + 6\theta^4 + 4\theta^6 + \theta^8}{(1+\theta^2)^4}} = \frac{\theta^4}{\theta^8 + \theta^6 + 4\theta^4 + \theta^2 + 1}$$

###HW: [6] In MA(1) model $x_t = w_t + 0.3w_{t-1}$, we assume {x1, · · · , xn} are observed.

(a) find three-step ahead forecast

$\widehat{X_{n+3}} = E[X_{n+3}|X_1, \ldots, X_n]$

$= E[\theta X_{n+2} - \theta^2 X_{n+1} + \theta^3 X_n - \theta^4 X_{n-1} + \cdots + w_{n+3}|X_1, \ldots, X_n]$

$= 0.3\widehat{X_{n+2}} - 0.3^2\widehat{X_{n+1}} + 0.3^3 X_n - 0.3^4 X_{n-1} + \cdots + (-1)^{n+1}0.3^{n+2}X_1$

$= 0.3\widehat{X_{n+2}} - 0.09\widehat{X_{n+1}} + \sum_{j=3}^{n+2}(-1)^{j-1}\theta^j X_{n+3-j}$

(b) find four-step ahead forecast.

$\widehat{X_{n+4}} = E[X_{n+4}|X_1, \ldots, X_n]$

$= E[\theta X_{n+3} - \theta^2 X_{n+2} + \theta^3 X_{n+1} - \theta^4 X_n + \cdots + w_{n+4}|X_1, \ldots, X_n]$

$= 0.3\widehat{X_{n+3}} - 0.3^2\widehat{X_{n+2}} + 0.3^3\widehat{X_{n+1}} - 0.3^4 X_n + \cdots + (-1)^{n+2}0.3^{n+3}X_1$

$= 0.3\widehat{X_{n+3}} - 0.09\widehat{X_{n+2}} + 0.027\widehat{X_{n+1}} + \sum_{j=4}^{n+3}(-1)^{j-1}\theta^j X_{n+4-j}$

###HW: [7] In ARMA(1) model $x_t = 0.5x_{t-1} + w_t + 0.2w_{t-1}$, we assume {x1, · · · , xn} are observed.

(a) find three-step ahead forecast

$\pi_j = (-1)^j(\phi\theta^{j-1} + \theta^j)$

$(1 - 0.5B)x_t = (1 + 0.2B)w_t$

$w_t = (1 - 0.5B)(1 - 0.2B + 0.2^2B^2 - \cdots)x_t$

$= x_t - 0.7x_{t-1} + (0.1 + 0.2^2)x_{t-2} - (0.1 + 0.2^3)x_{t-3} + \cdots$

$$x_t = 0.7x_{t-1} - (0.1 + 0.2^2)x_{t-2} + \cdots = \sum_{j=1}^{\infty}(-\pi_j)X_{t-j} + w_t, \qquad \pi_j = (-1)^j(0.5 * 0.2^{j-1} + 0.2^j)$$

$\widehat{X_{n+3}} = E[X_{n+3}|X_1, \ldots, X_n]$

$= E\left[\sum_{j=1}^{\infty}(-\pi_j)X_{t+3-j} + w_{n+3}\middle|X_1, \ldots, X_n\right]$

$$= 0.7\widehat{X_{n+2}} - (0.14)\widehat{X_{n+1}} + \sum_{j=3}^{n+2}(-\pi_j)X_{n+3-j}$$

n부터 1까지

(b) find four-step ahead forecast.

$$\pi_j = (-1)^j(\phi\theta^{j-1} + \theta^j)$$

$$(1 - 0.5B)x_t = (1 + 0.2B)w_t$$

$$w_t = (1 - 0.5B)(1 - 0.2B + 0.2^2B^2 - \cdots)x_t$$

$$= x_t - 0.7x_{t-1} + (0.1 + 0.2^2)x_{t-2} - (0.1 + 0.2^3)x_{t-3} + \cdots$$

$$x_t = 0.7x_{t-1} - (0.1 + 0.2^2)x_{t-2} + \cdots = \sum_{j=1}^{\infty}(-\pi_j)X_{t-j} + w_t, \qquad \pi_j = (-1)^j(0.5 * 0.2^{j-1} + 0.2^j)$$

$$\widehat{X_{n+4}} = E[X_{n+4}|X_1, \dots, X_n]$$

$$= E\left[\textstyle\sum_{j=1}^{\infty}(-\pi_j)X_{t+4-j} + w_{n+4}\middle|X_1, \dots, X_n\right]$$

$$= 0.7\widehat{X_{n+3}} - (0.14)\widehat{X_{n+2}} + 0.028\widehat{X_{n+1}} + \sum_{j=4}^{n+3}(-\pi_j)X_{n+4-j}$$

n부터 1까지