Linear Algebra Recitation

Week 2

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Linear independence

- To show linear independence of a set of vectors, always recall the definition. In fact, until you learn Gaussian elimination, which is a general method of solving systems of linear equations, the quiz problems will be simple enough so that you can tell linear independence by just remembering its definition.
  Definition: A set of vectors is linearly independent iff

$v_1, \cdots, v_n \in V$, $c_1, \cdots, c_n \in \mathbb{R}$.

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

Linear independence

- Why does showing linear independence involve solving a system of linear equations? **Exercise:** write the matrix-vector multiplication

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}}_{\mathbf{x}} = 2\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

as a linear combination of the columns of A.

question: $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ Linearly Independent?

Sps $c_1\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \vec{0}$

$$\begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0} \qquad \begin{cases} c_1 + 2c_2 = 0 \\ -2c_1 + c_2 + 2c_3 = 0 \\ -c_1 + 3c_2 + 2c_3 = 0 \end{cases}$$

must solve system of linear equations!

Linear independence: tips

☆ If a vector is a linear comb. of other vectors $\Rightarrow$ L.D $\because \left(\begin{array}{l} V_1 = aV_2 + bV_3 \\ V_1 - aV_2 - bV_3 = \vec{0} \end{array} \right)$

- If you see that one of the vectors in the set is the zero vector, the set is linearly dependent. Why?

  $\{u_1, u_2, \vec{0}\}$ $\qquad 0 \cdot u_1 + 0 \cdot u_2 + 3\,\vec{0} = 0$

  can be nonzero.

- For a set of vectors in $\mathbb{R}^n$, if the set contains more than n vectors, the set is linearly dependent. Why?

  $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

- For a set of vectors in $\mathbb{R}^n$, if the set contains one vector, the set is linearly dependent if and only if the vector is a zero vector. Why?

  $\text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ $\qquad \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ $\qquad c_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$ $\Rightarrow c_1 \neq 0$

- For a set of vectors in $\mathbb{R}^n$, if the set contains two vectors, the set is linearly dependent if and only if one vector is a scalar multiple of the other. Why?

  $c_1 V_1 + c_2 V_2 = 0$ $\qquad V_1 = \lambda V_2$
  $c_1 V_1 = -c_2 V_2$ $\qquad V_1 - \lambda V_2 = 0$
  $V_1 = -\frac{c_2}{c_1} V_2$ $\qquad \Rightarrow$ L.D

Linear independence: exercises

- A problem might ask: Is $u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ linearly independent from $v_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$? $\implies$ is $\{u, v_1, v_2\}$ linearly independent?

no

$V_1 + V_2 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} = 2 \cdot u$

$u = \frac{1}{2} v_1 + \frac{1}{2} v_2$

- Let u be defined as above, and $w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Is $\underbrace{\{u, w_1, w_2, w_3\}}$ linearly independent? Hint: try using one of the tips mentioned in the previous page. (DIY)

$\subset \mathbb{R}^3$

Basis & dimension of a *vector space*

- A set of vectors form a basis for a vector space if it (i) spans the vector space, and (ii) it is a linearly independent set.
- the dimension of a vector space is the number of vectors in its basis. It is often denoted as $\dim(V)$.
- Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\underbrace{\text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)}$

- Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?
- Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Basis & dimension of a *vector space*

- Is $\left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

- Find a basis for span $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \cancel{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \}$$

- What is the dimension of the vector space above?

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Column/row space and rank of a *matrix*

- The column/row space of a matrix is the vector space spanned by the columns/rows of the matrix.

- The column space is also called the range of a matrix, and the column space of a matrix A is often denoted as col(A) or $\mathcal{R}(A)$.

$f(x) = y \in$ Range of f $\quad A = [v_1 \mid v_2 \mid v_3] \quad x = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad Ax = c_1 v_1 + c_2 v_2 + c_3 v_3 \in Col(A)$

- The row space of matrix A is the column space of A^T.

- The rank of A, denoted as rank(A) or rk(A), is the dimension of col(A). $= \dim Col\, A^T$

- In fact, the dimension of the column space is equal to the dimension of the row space, thus rank = column rank = row rank.

Basis, dimension, column/row space, rank recap

- $\text{rank}(A) = \dim(\text{col}(A)) = \dim(\text{col}(A^T))$.
- by the definition of dimension, $\text{rank}(A)$ is the number of vectors in the basis of the column space of $A \implies$ number of linearly independent columns of A.
- **Exercise:** What is the column space and rank of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$?

$\text{rank}(A) = \dim(\text{col}(A))$

$\therefore \text{rank}(A) = 2$

vector space : $\text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \text{Col}(A)$

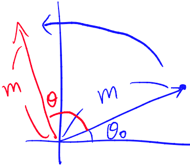
$\Rightarrow$ basis : $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\Rightarrow$ dimension : 2

- Take some time to internalize these concepts. These will appear time and time again in the future.

Some famous matricies

- Rotation in 2-d. The following matrix rotates a vector counterclockwise by θ radians.



$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} m\cos\theta_0 \\ m\sin\theta_0 \end{bmatrix}$ $\Rightarrow$ Compute Rx, and use some trig identities.

You should get $Rx = \begin{bmatrix} m\cos(\theta_1+\theta_0) \\ m\sin(\theta_1+\theta_0) \end{bmatrix}$

- Reflection in 2-d. The following matrix reflects a vector about a line that makes a θ-radian angle with the x axis.

$$F = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$