Linear Algebra Recitation

Week 3

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The subspace of a vector space

- **Definition.** A set V is a vector space iff it is equipped with two operations, vec. add & scalar mul, $u, v, w \in V$, $c_1, c_2 \in \mathbb{R}$

- V is closed under vec. add: $\forall\, v, u \in V$, $v + u \in V$; scalar mul: $\forall\, c \in \mathbb{R}$, $cv \in V$
- $\vec{0} \in V$

- **Definition.** A set S is a subspace of a vector space V iff
  - S is a subset of V.
  - S is a vector space (with same operations as V)

- Note that the operations of S are "inherited" from V.

① $v+u=u+v$

② $u+(v+w)=(u+v)+w$

③ Vector addition must have unique identity element (zero vector)

④ every vector v in vector space must have unique inverse element (v^{-1})

⑤ $1\cdot v=v$

$v+\underbrace{v^{-1}}_{-v}=e$

⑥ $c_1(c_2v)=(c_1c_2)v$.

⑦ $c(v+u)=cv+cu$ ⑧ $(c_1+c_2)v = c_1v+c_2v$

$c\cdot(x,y)=(cx,cy)$

$c*(x,y)=(cx,y)$

$1*(x,y)=(x,y)$

$c_2*(c_1*(x,y))=c_2*(c_1x,y)$

$=(c_2c_1x,y)$

$=(c_2c_1)*(x,y)$

① $(x_1,y_1)*(x_2,y_2)=(x_1x_2,y_1y_2)$

$(x_2,y_2)*(x_1,y_1)=(x_2x_1,y_2y_1)$

③ $(x,y)*e=(x,y)$

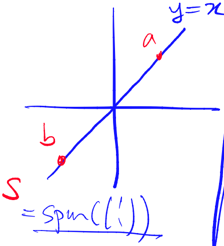
$e=(1,1)$

④ $(x,y)*\underbrace{(x,y)^{-1}}_{(\frac{1}{x},\frac{1}{y})}=e=(1,1)$

The subspace of a vector space: Exercises

- Is $S_1 = \{(x, y) \in \mathbb{R}^2 : x - y = 0\}$ a subspace of $\mathbb{R}^2$? $ca+b \in S$

(cx_1+x_2, cy_1+y_2)



① ✓ $(0,0) \in S$

② $a = (x_1, y_1)$ $\quad a+b \in S$

$b = (x_2, y_2)$ $\quad a+b = (x_1+x_2, y_1+y_2)$

$x_1 - y_1 = 0$ $\quad x_1+x_2-(y_1+y_2) = 0$?

$x_2 - y_2 = 0$

$x_1 - y_1 + x_2 - y_2 = 0$

③ $c \in \mathbb{R}$

$ca = (cx_1, cy_1) \in S$ $\quad cx_1 - cy_1 = 0$

- Is $S_2 = \{(x, y) \in \mathbb{R}^2 : x - y = 1\}$ a subspace of $\mathbb{R}^2$? $c(x_1 - y_1) = 0$

The subspace of a vector space: Exercises

- Is $\mathcal{P}_n$, the set of real polynomials of degree at most n, a subspace of $\mathbb{R}[x]$, the vector space of all real polynomials? First, show that $\mathbb{R}[x]$ is a vector space (infinite dimensional).

$\mathbb{R}[x]$ a vector space?

① $f(x) \equiv 0 \in$ zero vector

① $p(x), q(x) \in \mathbb{R}[x]$

$p(x) + q(x) = r(x) \in \mathbb{R}[x]$

② $c\,p(x) \in \mathbb{R}[x]$

P_n ① $f(x) \equiv 0 \in P_n$

② $s(x), w(x) \in P_n$

$s(x) + w(x) \in P_n$

③ $c\,s(x) \in P_n$

- Is $S = \{p(x) \in \mathcal{P}_n : p(1) = 0\}$ a subspace of $\mathcal{P}_n$?

$p(x), q(x) \in P_n$

① $r(x) = p(x) + q(x)$

$r(1) = p(1) + q(1) = 0$

② $c\,p(x)$, $c\,p(1) = 0$

③ $f(x) \equiv 0 \in S$

The subspace of a vector space: Exercises

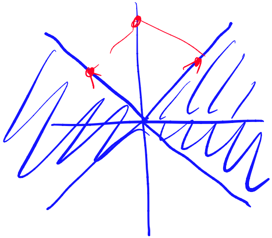
- Is $S = \{p(x) \in \mathcal{P}_n : p(1) = 1\}$ a subspace of $\mathcal{P}_n$?

$r(1) = p(1) + q(1) = 2 \neq 1$

$cP(1) = c \neq 1$

- Is $S = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \geq 0\}$ a subspace of $\mathbb{R}^2$?

$x^2 \geq y^2$



① $x \geq 0$, $y \geq 0$ — 1st quadrant

$x \geq y$

② $x \geq 0$, $y \leq 0$ — 4th quadrant

$x \geq -y$

③ $x \leq 0$, $y \geq 0$ — 2nd quadrant

$-x \geq y$

④ $x \leq 0$, $y \leq 0$ — 3rd quadrant

$-x \geq -y$

The subspace of a vector space: Exercises

- Is the set of functions $f : \mathbb{R} \to \mathbb{R}$ that have a discontinuity at $x = 1$ a vector space? NO.

$$f(x) = \begin{cases} 0 & x \leq 1 \\ 1 & x > 1 \end{cases}$$

$$g(x) = \begin{cases} 0 & x \leq 1 \\ -1 & x > 1 \end{cases}$$

$f(x)+g(x) = 0$ for all $x \in R$.

(Continuous at $x=1$)

- Is C^1, the set of once differentiable functions a vector space?

Yes. $f, g \in C^1$

$$\Rightarrow (f+g)' = f' + g' \Rightarrow f+g \in C^1$$

$$(cf)' = cf' \Rightarrow cf \in C^1$$

Subspaces of $\mathbb{R}^n$ related to a matrix

- The row and column space of a matrix. The dimension of either space is the **rank** of the matrix. $\dim(\text{col}(A)) = \dim(\text{row}(A)) = \text{rank}(A)$.

  $\dim(\text{row}(A)) = \dim(\text{col}(A^T))$

- The nullspace of a matrix (also called the kernel of a matrix). The dimension of the nullspace, $\dim(\mathcal{N}(A)$, is called the nullity of A.
  Definition. a vector $v \in V$ is in $\mathcal{N}(A)$ iff $Av = \vec{0}$

- Showing linear independence of vectors is related to finding the nullspace of a matrix. u, v, w, L.I ?

$c_1 u + c_2 v + c_3 w = \vec{0}$

if $c_1 = c_2 = c_3 = 0 \Rightarrow u, v, w$ are LI.

$\underbrace{[u \mid v \mid w]}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{v} = 0$ $\quad Av = 0$

$\mathcal{N}(A) = \{\vec{0}\}$
$\Rightarrow c_1 = c_2 = c_3 = 0$
$\Rightarrow u, v, w$ L.I.

Matrices as a linear transformation

- All matrices are linear transformations.
  Conversely, if $T : V \to W$ is a linear transformation and V, W are **finite dimensional** vector spaces, then there is a matrix representation for T.
  Definition: A transformation $T : V \to W$ is linear iff

$u, v \in V$

① $T(u+v) = T(u) + T(v)$
② $T(cu) = cT(u)$
$\Leftrightarrow T(cu+v) = cT(u) + T(v)$

- Consequences of linearity:
  if T is a linear transformation such that $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}-2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\-1\end{bmatrix}$, then find $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$.

$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)$

$$T\left(\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}-1\\-1\end{bmatrix}\right) = \begin{bmatrix}-2\\1\end{bmatrix} + \begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

Matrices as a linear transformation $Av = w$

- Let A be an $n \times m$ matrix. What is the domain and codomain of the matrix transform defined by $T(v) = Av$?

$\underbrace{[\;\;]}_{n \times m} [\;]_{m} \in \mathbb{R}^m \quad = \quad [\;] \in \mathbb{R}^n \qquad T: \mathbb{R}^m \to \mathbb{R}^n$ ✓

- Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be defined as $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ x_1 - x_2 \end{bmatrix}$. Is there a matrix representation for this transformation? YES

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$$

- Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined as $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_3 \\ \sqrt{x_2} \end{bmatrix}$. Is there a matrix representation for this transformation? NO

$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$

$A = [\;\;] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} cx_1 + dx_2 \\ \end{bmatrix}$

$(1+2)$

$1 \cdot 3 + 2 \cdot 4$?

$a = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$c = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$