Linear Algebra Recitation

Week 5

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Matrix Inversion

- $AB = I$ $\quad B = [u|v|w]$, $I = [e_1|e_2|e_3]$
- $BA = I$ $\quad Ax = b \quad x = A^{-1}b$

$\in \mathbb{R}^{3\times 3}$

- Let A be invertible. Then finding the inverse A^{-1} is solving three system of equations: $B = [u|v|w] = [A^{-1}e_1|A^{-1}e_2|A^{-1}e_3]$

  $= A^{-1}[e_1|e_2|e_3]$

  $[A|e_1]$ $[A|e_2]$ $[A|e_3]$

  $$Au = e_1, \quad Av = e_2 \quad Aw = e_3,$$

  $u = A^{-1}e_1$ $\quad v = A^{-1}e_2$ $\quad w = A^{-1}e_3$ $\qquad B = A^{-1}I = A^{-1}$

  where e_k is the k^{th} standard unit vector. If we can find such vectors u, v, w, then

  $[A|e_1|e_2|e_3] = [A|I] \sim [\underbrace{RREF}_{=I} | \underbrace{u|v|w}_{B = A^{-1}}]$

  $$A[u|v|w] = [u|v|w]A = I$$

  and we call $A^{-1} = [u|v|w]$.
- It is clear that u, v, w can be obtained from Gauss elimination.
- Note that the inverse of a matrix is only relevant to a square matrix.

Matrix Inversion

$$\begin{bmatrix}1&2&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$x_3=b_3$

$x_2+x_3=b_2 \Rightarrow x_2=b_2-b_3$

$x_1+2x_2+x_3=b_1$

- Basic example: Let $A=\begin{bmatrix}1&2&1\\1&1&2\\2&1&3\end{bmatrix}$. Find A^{-1} using Gauss elimination.

$$\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\1&1&2&0&1&0\\2&1&3&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&-1&1&-1&1&0\\0&-3&1&-2&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&-1&1&-1&0\\0&0&-2&1&-3&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&1&1&-1&0\\0&0&1&-1/2&3/2&-1/2\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&2&0&3/2&-3/2&1/2\\0&1&0&3/2&-5/2&1/2\\0&0&1&-1/2&3/2&-1/2\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1&0&0&-1/2&7/2&-1/2\\0&1&0&3/2&-5/2&1/2\\0&0&1&-1/2&3/2&-1/2\end{array}\right]$$

RREF $=I$ u   v   w $=B=A^{-1}$

check if this holds!

$AB=I$

$B=A^{-1}$

The four fundamental subspaces $A : \mathbb{R}^p \to \mathbb{R}^q$, $q\overset{p}{\boxed{A}}$ $p\overset{q}{\boxed{A^T}}$

- Recall the definition of orthogonal complement from last week.
- (Important) review the properties of orthogonal complements.
  - $\text{col}(A^T)^\perp = \mathcal{N}(A)$
  - $\text{col}(A)^\perp = \mathcal{N}(A^T)$

$v \in \text{Col}(A^T)$, $u \in N(A)$

$v \in \mathbb{R}^p$ $\quad u \in \mathbb{R}^p$

$v \cdot u = 0$

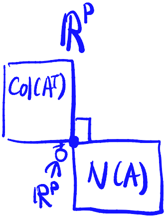
$\text{Col}(A^T) \cap N(A) = \{\vec{0}\}$

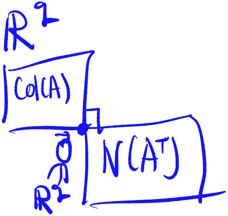
$\dim \text{Col}(A^T) + \dim N(A) = \dim \mathbb{R}^p$

$w \in \text{Col}(A)$ $\quad z \in N(A^T)$

$w \in \mathbb{R}^q$ $\quad z \in \mathbb{R}^q$

$\dim \text{Col}(A) + \dim N(A^T) = \dim \mathbb{R}^q$





Review: rank-nullity theorem (a.k.a. the FTLA)

domain

$q \begin{array}{c} p \\ \boxed{A} \end{array}$

- rank-nullity theorem:

$\cdot \dim Col(A^T) + \dim N(A) = \dim \mathbb{R}^p = p$

$rank(A) = rank(A^T)$

$\cdot \dim Col(A) + \dim N(A^T) = \dim \mathbb{R}^q = q$

Codomain

- recall the idea of proof:
  the two spaces are orthogonal complements of each other +
  dimension theorem of orthogonal complements.

Injective, Surjective, Bijective

$\mathbb{R}^n \to \mathbb{R}^m$ $y \in \mathbb{R}^m$

Let $A \in \mathbb{R}^{m\times n}$. $\exists x \in \mathbb{R}^n : Ax = y = x_1 a_1 + \cdots + x_n a_n$

$m<n$ $m>n$

- Rank-Nullity theorem is important (as always). If you cannot prove the " $\iff$ "s in each statement, take some time to think about it.
- A is injective $\iff$ one to one $\iff$ $\mathcal{N}(A)$ trivial $\iff$ $\dim(\text{col}(A^T)) = n$ (full column rank)

  $Ax_1 = b$, $Ax_2 = b$ $\Rightarrow A(x_1 - x_2) = 0$, $x_1 - x_2 = 0$, $x_1 = x_2$
- A^T is injective (rank A) $\iff$ $\mathcal{N}(A^T)$ trivial $\iff$ $\dim(\text{col}(A)) = m$
- A is surjective $\iff$ $\dim(\text{col}(A^T)) = m$ (full row rank) (rank A)
- A^T is surjective $\iff$ $\dim(\text{col}(A)) = n$
- Could you relate the injectivity/surjectivity of A and A^T?
- $m < n$, $m > n$, $m = n$. What happens in each case?

$m<n$: A cannot be injective; surjective when $\text{rank}(A)=m$

$m>n$: A cannot be surjective; injective when $\text{rank}(A)=n$

$m=n$: injective $\Leftrightarrow$ surjective $\Rightarrow$ A is bijective

A bijective $\Leftrightarrow$ injective + surjective $\Leftrightarrow$ A invertible

System of linear equations: revisited

$A(x_p + x_n) = Ax_p + 0$

- c.f. last week's slides. $X = x_p + \underline{N(A)}$
- In terms of injectivity/surjectivity/bijectivity,
- $Ax = b$ has a unique solution $\iff$ $\begin{cases} N(A) \text{ is trivial} \Leftrightarrow A \text{ injective} \\ b \in Col(A) \end{cases}$
- $Ax = b$ has infinitely many solutions $\iff$ $\begin{cases} N(A) \text{ is not trivial} \\ b \in Col(A) \end{cases}$ $\Leftrightarrow A$ not injective
- $Ax = b$ has no solutions $\iff$ $b \notin Col(A)$

$\Rightarrow A$ is not surjective

not iff!