Linear Algebra Recitation

Week 1

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Notes on vector arithmetic

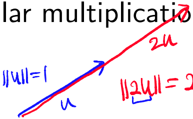
- Vector addition

$u, v \in V$ $\qquad u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$u + v \in V$ $\qquad \underline{u+v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$u + 2u = 3u$

- Scalar multiplication



$\|2u\| = 2\|u\| = 2.$

☆ Some vector fields (e.g. $\mathbb{R}^n$) have other operations

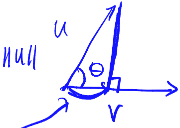
e.g.) Products too (dot product, cross product, ...)

Knowing the formula: Dot product

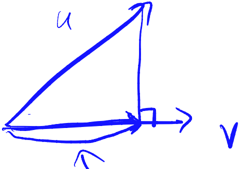
Let $u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. If θ is the angle the two vectors make, $\cos(\theta) = ?$

$$\boxed{\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}}$$

$$u \cdot \frac{v}{\|v\|}$$

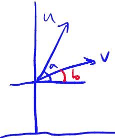


$$\cos\theta = \frac{u \cdot \frac{v}{\|v\|}}{\|u\|} = \frac{u \cdot v}{\|u\| \, \|v\|}$$



$$\|\mathrm{Proj}_v u\| = u \cdot \frac{v}{\|v\|}$$

In $\mathbb{R}^2$, $u = \begin{bmatrix} \|u\| \cos a \\ \|u\| \sin a \end{bmatrix}$, $v = \begin{bmatrix} \|v\| \cos b \\ \|v\| \sin b \end{bmatrix}$



$$u \cdot v = \|u\|\|v\| \cos a \cos b + \|u\|\|v\| \sin a \sin b$$
$$= \|u\| \, \|v\| (\cos a \cos b + \sin a \sin b)$$
$$= \|u\|\|v\| \cos(a-b)$$
$$= \|u\|\|v\| \cos(\theta)$$

Similar calculations in $\mathbb{R}^3$, but this formula can be considered an alternative definition for $u \cdot v$ for $\mathbb{R}^n$

$u \cdot v = 4$

$\|u\| = \sqrt{6} = \sqrt{1+4+1}$

$\|v\| = \sqrt{0+1+4} = \sqrt{5}$

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{4}{\sqrt{6}\sqrt{5}} = \frac{4}{\sqrt{30}}$$

Knowing the formula: Dot product

When is $\cos(\theta) = 0$? What can be said about $u \cdot v$ when $\cos(\theta) = 0$?

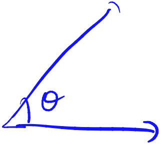
$0 \le \theta \le \pi$

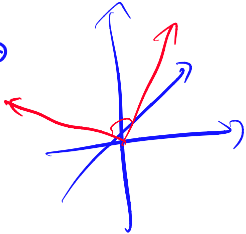
: when $\theta = \frac{\pi}{2}$.

$\rightarrow \cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$

$u \cdot v = \|u\|\|v\| \cos\theta$ $\rightarrow 0$

$= 0$





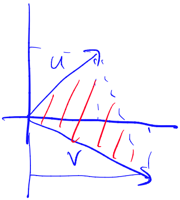
$$\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\|\|v\|}\right)$$

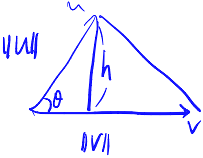
Knowing the formula: Dot product

Let $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Using the dot product formula, find the area of the triangle created by u, v.

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$



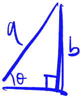


$A = \frac{1}{2} b h$

$h = \|u\| \sin\theta$

$= \|u\| \sqrt{1 - \cos^2\theta} = \sqrt{2}\sqrt{1 - \frac{1}{10}}$

$= \sqrt{2}\sqrt{\frac{9}{10}} = \frac{3}{\sqrt{5}}$



$\sin\theta = \frac{b}{a}$

$b = a \sin\theta$

$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{1}{\sqrt{2}\sqrt{5}}$

$= \frac{1}{\sqrt{10}}$

$\therefore A = \frac{1}{2}\|v\| h = \frac{1}{2}\sqrt{5}\frac{3}{\sqrt{5}} = \frac{3}{2}$

Dot-product, Norm, Distance: What's what?

in $\mathbb{R}^n$

- Dot product (inner product)

more general.

$u \cdot v$ (discussed above)

Dot product "induces" Euclidean norm

- Norm

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \qquad \|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = (u \cdot u)^{1/2}$$

$$\|u - 0\| = \sqrt{(u_1-0)^2 + (u_2-0)^2 + (u_3-0)^2}$$

Euclidean norm "induces" Euclidean distance.

- Distance

$a = (a_1, a_2, a_3)$

$b = (b_1, b_2, b_3)$

$$d(a,b) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2} = \|a - b\| = (a-b \cdot a-b)^{1/2}$$

Concepts to remember

- Linear combination

$u, v, w \in V \quad a, b, c \in \mathbb{R}$

$$\begin{bmatrix}3\\5\end{bmatrix} = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix}$$

$au + bv + cw$

$c_1 = 3, \ c_2 = 5$

- Span

$\{au + bv + cw : u, v, w \in V, a, b, c \in \mathbb{R}\}$

- Cauchy-Schwarz inequality

$\boxed{|u \cdot v| \le \|u\| \|v\|}$

$u = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}, \quad v = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$

$$|u_1v_1 + u_2v_2 + u_3v_3| \le \sqrt{u_1^2 + u_2^2 + u_3^2}\sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\left|\sum_{i=1}^{n} u_i v_i\right| \le \left(\sum_{i=1}^{n} u_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}}$$

- Triangle inequality

$\boxed{\|u + v\| \le \|u\| + \|v\|}$