Linear Algebra Recitation

Week 10

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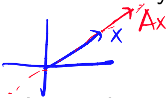
Eigenvalues and eigenvectors

Definition (Eigenvalue, eigenvector) $A \in \mathbb{R}^{n \times n}$

(λ, x) is an eigenpair (eigenvalue, eigenvector) of the matrix A if:

$x \neq \vec{0}$, $Ax = \lambda x$

- An eigenvector x is the "sweet spot" of of the matrix A: Ax does not rotate x. It only scales x by λ.



- If x is **an** eigenvector corresponding to λ, what else is an eigenvector?

$Ax = \lambda x$

$cAx = c\lambda x$

$A(cx) = \lambda(cx)$

☆ infinitely many e.vecs.

How many Lin. indep e.vecs for A?

Exercises

- If x is an eigenvector of A, how would you compute $A^{100}x$? Multiplying a moderate-sized ($n \sim 10^4$) matrix 100 times will even take a computer a very long time!

$$Ax = \lambda x$$

$$A^2x = \lambda \underbrace{Ax} = \lambda^2 x$$

$$\vdots$$

$$A^{100}x = \lambda^{100}x$$

$$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} w_0 \\ d_0 \end{bmatrix} = \begin{bmatrix} w_1 \\ d_1 \end{bmatrix}$$

$$\begin{bmatrix} w_{100} \\ d_{100} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}^{100}\begin{bmatrix} w_0 \\ d_0 \end{bmatrix}$$

if we have 2 lin. indep e.vecs

$$\begin{bmatrix} w_0 \\ d_0 \end{bmatrix} = c_1 x_1 + c_2 x_2$$

$$\begin{bmatrix} w_{100} \\ d_{100} \end{bmatrix} = c_1\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}^{100} x_1 + c_2\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}^{100} x_2 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2$$

How to find eigenvalue λ?

- (Definition) $Ax = \lambda x \implies \overbrace{(A - \lambda I)x = 0}^{Bx=0}$.
- When does a nonzero x exist?

$N(A-\lambda I)$ is non-trivial $\iff$ not injective $\iff \det(A-\lambda I) = 0$

- What is $\det(A - \lambda I)$ called?

Characteristic polynomial

$$\begin{bmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{bmatrix}$$

- This is often denoted as $p_A(\lambda) = \det(A - \lambda I)$.
- For $A \in \mathbb{R}^{n \times n}$, what is the degree of $p_A(\lambda)$?

n

- How can we find the eigenvalues from $p_A(\lambda)$?

e.vals are the roots of $p_A(\lambda)$

How to find eigenvector x corresponding to λ?

- Once we know an eigenvalue λ of A, how can we find the associated eigenvectors?
- Recall, an eigenpair must satisfy $(A - \lambda I)x = 0$. What can be said about the null space of $A - \lambda I$? $Bx = 0$

$(A - \lambda I)x = 0$

find basis for $N(A - \lambda I)$

if $N(A - \lambda I) = \text{Span}\{\underline{x_1, x_2}\}$

Thm) $Ax_3 = \lambda_1 x_3$, $Ax_4 = \lambda_2 x_4$

$\lambda_1 \neq \lambda_2 \Rightarrow x_1, x_2$ lin. indep.

Def) if λ has multiplicity 1 in $P_A(\lambda)$: non-degenerate e. val.

Exercises

- Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. FInd the eigenvalues of A, and associated eigenvectors.

① form $P_A(\lambda)$ $\Rightarrow$ find e. vals

② find e. vecs.

$$P_A(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1$$

$$= \lambda^2 - 2\lambda$$

$$= \lambda(\lambda - 2)$$

i) $\lambda_1 = 0 \Rightarrow (A - \lambda_1 I)x = 0$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} x = 0$$

$x_1 - x_2 = 0$

$N(A - \lambda_1 I) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$x^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

ii) $\lambda_2 = 2 \Rightarrow (A - 2I)x = 0$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} x = 0$$

$x_1 + x_2 = 0$

$N(A - \lambda_2 I) = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$x^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Exercises

Diagonal / U. triangular / L. triangular $\Rightarrow$ diagonal entries are e.vals, and det is the product of diag.

- Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. FInd the eigenvalues of A, and associated eigenvectors.

$p_A(\lambda) = \det(A - \lambda I) = (1-\lambda)^2$

$(A - I)x = 0$

$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} x = 0$

$x_2 = 0 \Rightarrow N(A - I) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

- It is generally easy to tell if a given vector is an eigenvector or not:
- Without prior knowledge, are the vectors $x_1 = (1, 1)^T$ and $x_2 = (-2, 0)^T$ eigenvectors of A above? If so, what are the associated eigenvalues?

$Ax_1 = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Ax_2 = A\begin{bmatrix} -2 \\ 0 \end{bmatrix} = 1\begin{bmatrix} -2 \\ 0 \end{bmatrix}$

A special case:

- A has a zero eigenvalue $\iff$ $Ax = 0$

$\iff N(A)$ non trivial

$\iff \det(A) = 0$

Similar matrices $AX = XB \Rightarrow [Ax_1 | \cdots | Ax_n] = [b_1 x_1 | \cdots | b_n x_n]$

$X = [x_1 | \cdots | x_n]$ $\quad B = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{bmatrix}$

Definition

A and B are similar matrices if: there exists an invertible X

$$A = XBX^{-1}$$

- Similarity preserves characteristic polynomial

- Therefore, similar matrices must have same eigenvalues.

Exercises

- Are $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$ similar? Answer in 3 seconds.

- In general, it is not trivial to tell whether two matrices are similar or not, especially when two matrices have the same eigenvalues.
- Similar matrices must have same eigenvalues, but matrices with same eigenvalues need not be similar!
- e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are not similar because they do not have the same Jordan forms (a topic that may be covered later).

Diagonalization and eigen (spectral) decomposition

A is diagonalizable if there is an invertible matrix C and a diagonal matrix B such that

$$A = CBC^{-1}$$ ← A must be square matrix

$$AC = CB$$

$$A[c_1|\cdots|c_n] = CB = C\begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{bmatrix}$$

$$[Ac_1|\cdots|Ac_n] = [b_1c_1|\cdots|b_nc_n]$$

$$Ac_k = b_kc_k \Rightarrow (b_k, c_k) \text{ are e. pairs.}$$

Not diagonalizable when A does not have n Lin. indep e. vecs.

$A = CBC^{-1}$: spectral / eigen decomp. $B = C^{-1}AC$: diagonalization

Diagonalization and eigen (spectral) decomposition

- When is a matrix Diagonalizable? When A has n lin. indep evecs.

  (to make C invertible)

Some Sufficient Conditions for A tobe diagon-

① A has n distinct e.vals.

Lemma. e.Vecs corresponding to distinct e.vals are lin. indep.

② A is symmetric.

Diagonalization and eigen (spectral) decomposition

a) Compute the eigenvalues of the matrix

(1) form $P_A(\lambda) = \det(A - \lambda I)$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$= \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1$

$= \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$

$\lambda_1 = 3, \quad \lambda_2 = 1$

b) Find a matrix C such that

$$B = C^{-1}AC$$

where B has only diagonal entries.

(1) $\lambda_1 = 3 \Rightarrow (A - 3I)x = 0$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x = 0$

$\Rightarrow x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(2) $\lambda_2 = 1 \Rightarrow (A - I)x = 0$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x = 0$

$\Rightarrow x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$C = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$