Linear Algebra Recitation

Week 9

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Orthogonal matrices

- (Definition) A matrix Q is called orthogonal if:

$$Q^T = Q^{-1} \iff Q^TQ = QQ^T = I$$

(Q is square & invertible)

- Show that if Q is orthogonal, then its columns are orthonormal:

$$Q = [q_1 | \cdots | q_n], \quad Q^TQ = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} [q_1 | \cdots | q_n] = \begin{bmatrix} q_1^Tq_1 & \cdots & q_1^Tq_n \\ \vdots & q_i^Tq_i & \\ q_n^Tq_1 & \cdots & q_n^Tq_n \end{bmatrix} = I$$

$$\Rightarrow q_i \cdot q_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

If columns are orthonormal & square matrix $\Rightarrow$ matrix is orthogonal.

- Show that if Q is orthogonal, then its rows are orthonormal:

$$QQ^T = I$$

Let $v_1, \cdots, v_n \in \mathbb{R}^m$ $n \le m$, $v_1, \cdots, v_n$ are orthonormal.

Are $v_1, \cdots, v_n$ lin. independent?

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}$$

① $c_1 v_1^T v_1 + c_2 \cancelto{0}{v_1^T v_2} + \cdots + c_n \cancelto{0}{v_1^T v_n} = 0$

$\Rightarrow c_1 \underbrace{\|v_1\|^2}_{1} = 0 \Rightarrow c_1 = 0$

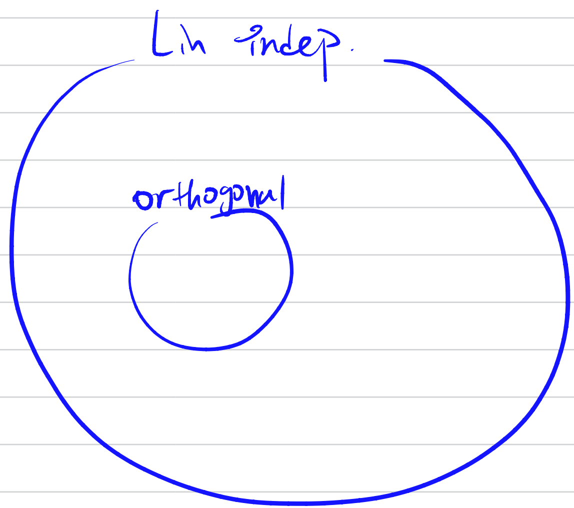
② $\Rightarrow c_2 \|v_2\|^2 = 0 \Rightarrow c_2 = 0$

$\vdots$

ⓝ 〃 $\Rightarrow c_n = 0$

Orthogonal $\Rightarrow$ Lin. indep.

Lin. indep $\not\Rightarrow$ Orthogonal



Orthogonal matrices

$u \cdot v = u^T v = v^T u$

$u \cdot u = \|u\|^2$

- Show that if Q is orthogonal, then $Qx \cdot Qy = x \cdot y$ (preserves dot product).

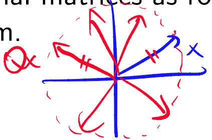
$$Qx \cdot Qy = ((Qx))^T Qy = x^T Q^T Q y = x^T y$$

- Show that if Q is orthogonal, then $\|Qx\| = \|x\|$ (preserves norm). Hint: use the above fact.

$$\underset{\|Qx\|^2}{\underset{\|}{Qx \cdot Qx}} = x^T Q^T Q x = x^T x = \|x\|^2$$

$$\Rightarrow \|Qx\| = \|x\|$$

- You can think of orthogonal matrices as rotations or reflections, or some combination of them.



Determinants

The formulas

- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find $\det(A)$. $= ad - bc$

- Let $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Find $\det(B)$. $= aei + bfg + cdh$

  $- (ceg + bdi + afh)$

Determinants

$$A=\begin{bmatrix}1&1\\2&-1\end{bmatrix}\sim\begin{bmatrix}1&1\\0&-3\end{bmatrix}\underset{\times -\frac{1}{3}}{\sim}\begin{bmatrix}1&1\\0&1\end{bmatrix}\sim\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$\det(A)$ $\quad$ $\det(A)$ $\quad$ $-\frac{1}{3}\det(A)$ $\quad$ $-\frac{1}{3}\det(A) = 1$

$$\det\begin{bmatrix}2&-1\\1&1\end{bmatrix}=-\det(A)$$

$\det(A)=-3$

The defining properties of determinants:

- $\det(I_n) = 1$ for all $n \in \mathbb{N}$.
- Multiplying a scalar λ to a column/row scales the determinant by λ.
- Adding a scalar multiple of one column to another column preserves determinant.

$$\det(C_1|C_2)=\det(C_1|C_2+\lambda C_1),\quad \det\left(\frac{r_1}{r_2}\right)=\det\left(\frac{r_1}{r_2+\lambda r_1}\right)$$

Other properties related to determinants:

- $\det(A) \neq 0 \iff$ the columns are lin. independent $\iff$ invertible.
- $\det(AB) = \det(A)\det(B)$ (doesn't hold for sums!).
- $\det(A^T) = \det(A)$
- Switching two columns or rows will reverse the sign of the determinant.
- The determinant of A is the **signed** volume of the parallelopiped spanned by the columns (rows) of A.

Determinants

- If A is orthogonal, what is $\det(A)$?.

$A^TA = I$

$\det(A^TA) = 1$

$\det(A^T)\det(A) = 1$

$(\det(A))^2 = 1$

$\Rightarrow \det(A) = \pm 1$

- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $\det(A) = 1$. What is $\det \begin{pmatrix} 2a & b \\ 2c & d \end{pmatrix}$?

$= 2\det(A) = 2$

Determinants

$B \in \mathbb{R}^{3\times 3}$

- Let $\det(B) = 1$. What is $\det(3B)$?

$$\det\begin{pmatrix} 3b_1 \\ \hline 3b_2 \\ \hline 3b_3 \end{pmatrix} = 3^3 \det(B) = 27$$

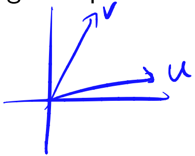
- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What is A^{-1}? $= \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Determinants

Let $C \in \mathbb{R}^{2\times 2}$ whose columns are the vectors u, v.

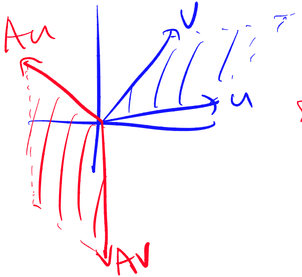
- What is the area of the parallelogram spanned by u and v?

$|\det(C)|$



$\det(u|v)$
$= -\det(v|u)$

- For any matrix A, what is the area of the parallelogram spanned by Au and Av? $\det(C) = 2$



$\Rightarrow \det(Au|Av) = \det(AC)$
$= \det(A) \cdot 2$

$* \; AC = A(u|v)$
$= (Au|Av)$