Linear Algebra Recitation

Week 4

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Gaussian elimination

$PA = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} A = \begin{bmatrix} P_1 A \\ P_2 A \\ P_3 A \end{bmatrix} = \begin{bmatrix} a_2 \\ a_1 \\ a_3 \end{bmatrix} \Rightarrow \begin{matrix} P_1 = [0\ 1\ 0] \\ P_2 = [1\ 0\ 0] \\ P_3 = [0\ 0\ 1] \end{matrix}$

- Elementary row operations do not change the solution of a linear system of equations.

$\underset{3\times3}{A} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $\quad \underset{3\times3}{P}A = \begin{bmatrix} a_2 \\ a_1 \\ a_3 \end{bmatrix}$ $\quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- (i) Row permutations (ii) multiplyting a row by a scalar (iii) adding a scalar multiple of a row to another row

$\underset{3\times3}{M}A = \begin{bmatrix} a_1 \\ 2a_2 \\ a_3 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Basic example: Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Solve $Ax = b$.

$\underset{3\times3}{R}A = \begin{bmatrix} a_1 \\ 2a_1 + a_2 \\ a_2 + 3a_3 \end{bmatrix}$ $\quad R = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Find pivot columns.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & -1 \\ 2 & 1 & 3 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & -3 & 1 & 0 \end{array}\right] \sim \underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]}_{\text{REF}}$$

$$\underbrace{\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right] \sim \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right]}_{\text{RREF}}}_{\text{Jordan}}$$

⊛ # of Pivot Columns = # L.I Cols of A

⊛ Pivot cols span Col(A)

Now that we know Gaussian elimination,

- Let's revisit some old topics that we can now fully understand.
- Determine whether Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a linearly independent set.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 3 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ -1 & -1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -1 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

REF

Now that we know Gaussian elimination,

- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,1,2) = (3,-1,1)$, $T(2,1,1) = (0,1,-2)$, and $T(1,2,3) = (-1,0,1)$. Find $T(1,-1,2)$.

$u, v \in V$, $\lambda \in \mathbb{R}$, $T(\lambda u + v) = \lambda T(u) + T(v)$

$T\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}3\\-1\\1\end{pmatrix}$, $T\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\-2\end{pmatrix}$, $T\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$

Find $T\begin{pmatrix}1\\-1\\2\end{pmatrix}$.

$$b = \begin{pmatrix}1\\-1\\2\end{pmatrix} = x_1\begin{pmatrix}1\\1\\2\end{pmatrix} + x_2\begin{pmatrix}2\\1\\1\end{pmatrix} + x_3\begin{pmatrix}1\\2\\3\end{pmatrix}, \quad x_1, x_2, x_3 \in \mathbb{R}$$

$$T\begin{pmatrix}1\\-1\\2\end{pmatrix} = T\left(x_1\begin{pmatrix}1\\1\\2\end{pmatrix} + x_2\begin{pmatrix}2\\1\\1\end{pmatrix} + x_3\begin{pmatrix}1\\2\\3\end{pmatrix}\right) = x_1 T\begin{pmatrix}1\\1\\2\end{pmatrix} + x_2 T\begin{pmatrix}2\\1\\1\end{pmatrix} + x_3 T\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\overset{A}{\begin{bmatrix}1&2&1\\1&1&2\\2&1&3\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = b \quad \left[\begin{array}{ccc|c}1&2&1&1\\1&1&2&-1\\2&1&3&2\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&1&1\\0&-1&1&-2\\0&-3&1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&1&1\\0&1&-1&2\\0&0&-2&6\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1&2&1&1\\0&1&-1&2\\0&0&1&-3\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&-1\\0&0&1&-3\end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{} = 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= x_1\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + x_3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_B$$

Fundamental Theorem of Linear Algebra

$u \perp v \Leftrightarrow u \cdot v = 0$

- Recall the definition of orthogonal ~~subspaces~~ complement ("perp")

$W \subseteq V \quad W^{\perp} = \{ v \in V : v \cdot w = 0,\ w \in W \}$

$W \cap W^{\perp} = \{ \vec{0} \}$

$u \quad u \cdot u = 0 \Leftrightarrow u = \vec{0}$

- Show that $\mathcal{N}(A) \subseteq col(A^T)^{\perp}$.

$x \in N(A)$

$\Rightarrow Ax = 0$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} x = \begin{bmatrix} a_1 \cdot x \\ a_2 \cdot x \\ \vdots \\ a_n \cdot x \end{bmatrix} = \vec{0}$$

$a_1 \cdot x = 0$
$a_2 \cdot x = 0$
$\vdots$
$a_n \cdot x = 0$

$x \in (col(A^T))^{\perp}$

$x \in (col(A^T))^{\perp}$

$$Ax = \begin{bmatrix} a_1 \cdot x \\ a_2 \cdot x \\ \vdots \\ a_3 \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$$

$\Rightarrow x \in N(A)$

- Show that for a subspace $S \subseteq V$, $\dim S + \dim S^{\perp} = \dim V$.
  (Try it at home. Here is a proof: `https://textbooks.math.gatech.edu/ila/orthogonal-complements.html`.)

Fundamental Theorem of Linear Algebra $A: \mathbb{R}^n \to \mathbb{R}^m$

- Obtain the fundamental theorem of linear algebra from the above results

$$\dim(\mathrm{Col}(A^T)) + \dim(\underset{\substack{\| \\ (\mathrm{Col}(A^T))^\perp}}{N(A)}) = n = \dim(\mathbb{R}^n)$$

$(= \mathrm{Rank}(A))$

- The four fundamental subspaces: $\to$ Next week.

Solution of $Ax = b$

- Always think in terms of the FTLA. Also, always view a matrix-vector product Ax as a linear combination of the columns of A.
- If a solution exists, then $x = x_p + x_h$, where $x_h \in \mathcal{N}(A)$. $Ax_h = 0$
- Thus the solution set is $X = x_p + \mathcal{N}(A)$.
- x unique $\iff$ solution set X contains only one element $\iff$ $\mathcal{N}(A)$ is trivial $\underset{\text{FTLA}}{\iff}$ $\dim(\text{col}(A)) = \text{rank}(A) = n$.
- Thus, you can tell if the system has a unique solution by checking the number of linearly independent columns/rows ($= \text{rank}(A)$). also $b \in \text{Col}(A)$
- When does $Ax = b$ have infinitely many solutions?

$\begin{cases} N(A) \text{ is not trivial} \iff \text{rank}(A) < n \\ \text{and } b \in \text{Col}(A) \end{cases}$

- When does $Ax = b$ have no solutions?

$b \notin \text{Col}(A)$

$$\overset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$