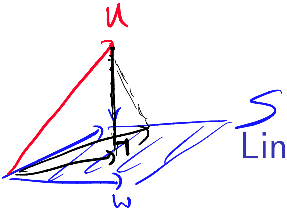

Linear Algebra Recitation

Week 6

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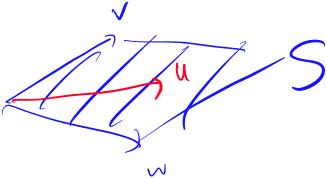
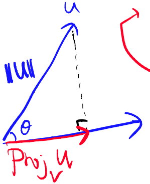


October 15, 2021

(Orthogonal) Projection matrices

- $\{u_1, \ldots, u_n\}$ is an orthogonal set of vectors if: $u_i \cdot u_j = 0 \quad i \neq j$
- $\{u_1, \ldots, u_n\}$ is an orthonormal set of vectors if: $+ \ \|u_i\| = 1$
- V is a finite dimensional vector space, and let $u, v \in V$. Recall the orthogonal projection of a vector v to a vector u.



$\therefore \ u \cdot v = \|u\| \, \|v\| \cos\theta$

$u \cdot \frac{v}{\|v\|} = \|u\| \cos\theta$

$= \|\mathrm{Proj}_v u\|$

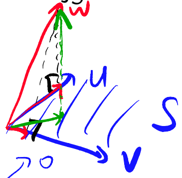
$\|\mathrm{Proj}_v u\| = u \cdot \frac{v}{\|v\|}$

$$\mathrm{Proj}_v u = \overbrace{\left(u \cdot \frac{v}{\|v\|}\right)}^{\text{(magnitude) component}} \overbrace{\frac{v}{\|v\|}}^{\text{direction}}$$

- In general, a real square matrix P is an (orthogonal) projection matrix iff $P^2 = P$ and $P^T = P$. $PPx = Px$
- Note: We are mostly concerned about orthogonal projections in this course, thus a "projection" will refer to an orthogonal projection.

(Orthogonal) Projection matrices

- Projection of a vector to higher dimensions.
- Let $u, v, w \in \mathbb{R}^3$, and u, v are orthogonal. Define $S = \mathrm{span}(u, v)$ (hence u, v form an orthogonal basis for S). Find $\mathrm{Proj}_S w$. Also, find the matrix P such that $Pw = \mathrm{Proj}_S w$.



$$\mathrm{Proj}_S w = c_1 u + c_2 v$$

$$= \left(w \cdot \frac{u}{\|u\|}\right) \frac{u}{\|u\|} + \left(w \cdot \frac{v}{\|v\|}\right) \frac{v}{\|v\|}$$

$$(w - \mathrm{Proj}_S w) \cdot u = w \cdot u - \left(w \cdot \frac{u}{\|u\|}\right) \frac{\overbrace{u \cdot u}^{\|u\|^2}}{\|u\|} - \cancel{\left(w \cdot \frac{v}{\|v\|}\right) \frac{\overbrace{v \cdot u}^{\to 0}}{\|v\|}}$$

$$= w \cdot u - \left(w \cdot \frac{u}{\cancel{\|u\|}}\right) \cancel{\|u\|} = w \cdot u - w \cdot u = 0$$

- An orthogonal projection means that $w - \mathrm{Proj}_S w$ is orthogonal to every vector in S, or equivalently, $w - \mathrm{Proj}_S w \in S^\perp$.

(Orthogonal) Projection matrices

- Generalization: Let $w, u_1, \ldots, u_k \in V$, where $u_1, \ldots, u_k$ are orthogonal. Define $S = \mathrm{span}(u_1, \ldots, u_k)$. Find P such that $Pw = \mathrm{Proj}_S w$.

$\|u_i\| = 1$ $\forall i = 1, \ldots, k$ $u_i \in \mathbb{R}^n$

$$= \left\|\mathrm{Proj}_{u_1}^{w}\right\| \frac{u_1}{\|u_1\|} + \cdots + \left\|\mathrm{Proj}_{u_k}^{w}\right\| \frac{u_k}{\|u_k\|}$$

$$= (w \cdot u_1)\, u_1 + \cdots + (w \cdot u_k)\, u_k$$

$$UU^T w = [u_1 | \cdots | u_k] \begin{bmatrix} u_1 \cdot w \\ \vdots \\ u_k \cdot w \end{bmatrix}$$

$U = [u_1 | \cdots | u_k]$ $\quad P = UU^T$

$$= \begin{bmatrix} u_{11}(u_1 \cdot w) + u_{21}(u_2 \cdot w) + \cdots + u_{k1}(u_k \cdot w) \\ \vdots \\ u_{1n}(u_1 \cdot w) + \cdots + u_{kn}(u_k \cdot w) \end{bmatrix}$$

$$U^T = \begin{bmatrix} u_1^T \\ \vdots \\ u_k^T \end{bmatrix}$$

$$U^T w = \begin{bmatrix} u_1^T w \\ \vdots \\ u_k^T w \end{bmatrix} = \begin{bmatrix} u_1 \cdot w \\ \vdots \\ u_k \cdot w \end{bmatrix}$$

$$= (u_1 \cdot w) \underbrace{\begin{bmatrix} u_{11} \\ \vdots \\ u_{1n} \end{bmatrix}}_{= u_1} + \cdots + (u_k \cdot w) \underbrace{\begin{bmatrix} u_{k1} \\ \vdots \\ u_{kn} \end{bmatrix}}_{= u_k} = (u_1 \cdot w) u_1 + \cdots + (u_k \cdot w) u_k$$

- We have only considered projecting on a space spanned by an orthogonal basis. This is because any basis can be turned into an orthonormal basis via Gram-Schmidt process.

When $\{u_1, \dots, u_k\}$ orthonormal

$$\hookrightarrow \underbrace{P = UU^T}$$

If not Orthonormal, but still orthogonal

$$\text{Proj}_S^w = \left(w \cdot \frac{u_1}{\|u_1\|}\right) \frac{u_1}{\|u_1\|} + \cdots + \left(w \cdot \frac{u_k}{\|u_k\|}\right) \frac{u_k}{\|u_k\|}$$

$$U = [u_1 | \cdots | u_k], \quad U^T = \begin{bmatrix} u_1^T \\ \vdots \\ u_k^T \end{bmatrix}$$

$$U^TU = \begin{bmatrix} u_1^Tu_1 & & 0 \\ & u_2^Tu_2 & \\ 0 & & \ddots \\ & & & u_k^Tu_k \end{bmatrix}$$

$$\underbrace{(U^TU)^{-1}} = \begin{bmatrix} \frac{1}{\|u_1\|^2} & & \\ & \ddots & \\ & & \frac{1}{\|u_k\|^2} \end{bmatrix}$$

$$\underbrace{P = U(U^TU)^{-1}U^T}$$

$$P^2 = U(\cancel{U^TU})^{-1}(\cancel{U^TU})(U^TU)^{-1}U^T$$
$$= U(U^TU)^{-1}U^T = P$$

if $\{u_1, \dots, u_k\}$ not even orthogonal

$\Rightarrow$ the dot product formula does not work, and deriving the formula directly is more complicated.

instead,

We want to find $z = \overbrace{c_1u_1 + \cdots + c_ku_k}^{\text{some projection onto } S} = Uc$ such that $\|w - z\| = \|w - Uc\|$ is minimized (orthogonal projection)

this is called a least-squares problem, and can be solved by solving $U^TUc = U^Tw$ (normal equation).

If $\{u_1, \dots, u_k\}$ are at least linearly independent, then $c = (U^TU)^{-1}U^Tw$ is the unique minimizer of $\|w - Uc\|$. (can be shown via simple calculus).

Therefore, the orthogonal projection of w onto S is

$z = Uc = U(U^TU)^{-1}U^Tw$, and $P = U(U^TU)^{-1}U^T$

(Orthogonal) Projection matrices

G-S

- Exercise: Let $A = \begin{array}{cc} v_1 & v_2 \\ \end{array}$ $\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Find $\mathrm{Proj}_{\mathrm{col}(A)} b$.

$\{u_1, u_2\}$

$$\mathrm{Proj}_{\underset{\underset{\mathrm{Span}(u_1, u_2)}{\parallel}}{\mathrm{Col}(A)}} b = (b \cdot u_1) u_1 + (b \cdot u_2) u_2$$

or

$$U = [u_1 \mid u_2]$$

$$P = U U^T$$

$$\mathrm{Proj}_{\mathrm{Col}(A)} b = Pb$$

Orthonormal Basis of a Vector Space

- An orthonormal basis is (i) a basis, (ii) that is an orthonormal set.
- Demo of why a orthonormal basis is desirable:
  Let $u, v, w \in \mathbb{R}^3$, and u, v form an orthonormal set. Define $S = \mathrm{span}(u, v)$ (hence u, v form a orthonormal basis for S).
  Compute $\underbrace{\|\text{Proj}_u w\|}_{= w \cdot \frac{u}{\|u\|} = w \cdot u} u + \|\text{Proj}_v w\| v = \text{Proj}_u w + \text{Proj}_v w$

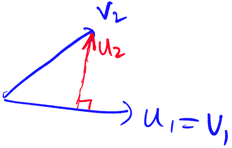
  $\left[w - (\text{Proj}_u w + \text{Proj}_v w)\right] \in S^{\perp}$

  (only because u, v orthogonal)

- What if $\{u, v\}$ is a basis of S that is not orthonormal?
  $\implies \|\text{Proj}_u w\| \frac{u}{\|u\|} + \|\text{Proj}_v w\| \frac{v}{\|v\|}$ may **not** be an orthogonal projection of w to S.

The Gram-Schmidt process

- Every basis can be turned into an orthonormal basis.
- Input: a basis $\{v_1, \dots, v_n\}$ of $V \longrightarrow$ Output: an **orthonormal** basis $\{u_1, \dots, u_n\}$ of V.
- Step 1: Start with $u_1 = v_1$. $u_1 = \frac{u_1}{\|u_1\|}$



$$u_2 = v_2 - \text{Proj}_{u_1} v_2 \quad , \quad u_2 = \frac{u_2}{\|u_2\|}$$

$$u_3 = v_3 - \text{Proj}_{\text{span}(u_1,u_2)} v_3 = v_3 - \left[\|\text{Proj}_{u_1} v_3\| u_1 + \|\text{Proj}_{u_2} v_3\| u_2\right]$$

$$u_4 = v_4 - \text{Proj}_{\text{span}(u_1,u_2,u_3)} v_4$$

$$\vdots$$

$$u_n = v_n - \text{Proj}_{\text{span}(u_1,\dots,u_{n-1})} v_n$$

The Gram-Schmidt process

$\|v_1\| = \sqrt{10}$

$\|v_2\| = \sqrt{15}$

$\|v_3\| = \sqrt{11}$

- Exercise: $A = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 2 & 3 & 0 \\ 1 & -1 & 1 \end{bmatrix} \end{array}$. Find an orthonormal basis for col(A).

$u_1 = v_1 / \|v_1\| = v_1/\sqrt{10}$

$u_2 = v_2 - (v_2 \cdot u_1)u_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix} - \frac{1\,\cancel{5}}{2\cancel{\sqrt{10}}} \frac{1}{\cancel{\sqrt{10}}} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \begin{bmatrix} 3/2 \\ -2 \\ 2 \\ -3/2 \end{bmatrix} \rightsquigarrow$ normalize

$u_3 = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2 = \text{DIY}$