Linear Algebra Recitation

Week 14

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Dec 10, 2021

Singular Value Decomposition $A = WDW^T$ (if A is sym)

True or false: for any matrix A, the matrices A^TA, AA^T are symmetric.

$\in \mathbb{R}^{m \times n}$ $\quad \in \mathbb{R}^{n \times n}$ $\quad \in \mathbb{R}^{m \times m}$

$$(A^TA)^T = A^T(A^T)^T = A^TA$$

$$(AA^T)^T = (A^T)^TA^T = AA^T$$

True or false: for any matrix A, the matrices A^TA, AA^T are diagonalizable.

spectral theory of sym. matrices.

Singular Value Decomposition

True or false: for any matrix A, the eigenvalues of $A^T A$, AA^T are non-negative. $A^TAx = \lambda x$, $\|x\| = 1$

$$\lambda = \lambda \|x\|^2 = \lambda (x \cdot x) = (x \cdot \lambda x) = (x \cdot A^TAx) = (Ax \cdot Ax)$$

$$= \|Ax\|^2$$

True or false: for any matrix A, the non-negative eigenvalues of $A^T A$, AA^T are the same. cf) last week's slides & quiz

$A^TAx = \lambda x$

$AA^T\boxed{Ax} = \lambda \boxed{Ax}$, $AA^Ty = \lambda y$

What do you call the square roots of the eigenvalues of $A^T A$ or AA^T?

Singular Values of A.

Singular Value Decomposition

E.val decomp.
$A = WDW^{-1}$

The SVD of a matrix $A \in \mathbb{R}^{m\times n}$ is

$$A = USV^{*T},$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are **orthogonal** matrices, and $S \in \mathbb{R}^{m\times n}$ contains the singular values on its diagonal. The columns of U are called the "left singular vectors" of A, and the columns of V are called the "right singular vectors" of A.

$\in \mathbb{R}^n$

Question: The eigenvectors of A^TA will become the (left/right) singular vectors of A. The eigenvectors of AA^T will become the (left/right) singular vectors of A.

$\in \mathbb{R}^m$

This is the reason we can find **orthogonal** matrices U, V.

Singular Value Decomposition

How to compute ~~the~~ a SVD $A = USV^{\cancel{*}\,T}$.

1. Find e.val decomposition of A^TA or AA^T. (Choose the one that is smaller or easier to compute) $\implies WDW^T$.
2. Set S to be the "square root" matrix of D. ⚠ S is not square.
3. Set W as either left or right singular vector (i.e., W = U or V).
4. if $W = V$, find U via the identity $Av_i = \sigma_i u_i$. $AV = US$
5. if $W = U$, find V via $u_i^TA = \sigma_i v_i^T$. $U^TA = SV^T$

Note 1: once you get U by finding the eigenvectors of AA^T , you **don't** want to compute V by finding eigenvectors of A^TA. Instead, use the identity above.

Note 2: if there are singular values that are zero, this means the corresponding singular vectors can be any vector that make U and V orthogonal. Choosing the right vectors in this case require some practice.

Exercises

Easy.

Find ~~the~~ two SVDs of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$A^TA = AA^T = I \Rightarrow (A^TA - I) = 0$

$\Downarrow$ Nullspace: $\mathbb{R}^2$

$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = USV^T = UIV^T$

① $U = I$, $V = I \Rightarrow A = III = I$: Valid

② $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A = UIV^T = I$: Valid

③ $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$: Valid

④ $U = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = U^{-1}$: $A = UIV^T$ $\Rightarrow$ not an SVD (U, V not orth) but is an e.val decomp.

Exercises

Obvious: $A = I \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} I$ is a valid SVD.

Easy.

Find the SVD of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

① $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

e.vec of $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\sigma_1^2 = 1 \to \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\sigma_2^2 = 0 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

② $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$u_i^T A = \sigma_i v_i^T$

$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = v_1^T$

$= \begin{bmatrix} 1 & 0 \end{bmatrix}$

$u_2^T A = \underset{\overset{\|}{0}}{\sigma_2} v_2^T$

What to do?

③ $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ //

v_2: chosen to make V orthogonal.

$A = USV^T$

When A is diagonalizable, the SVD gives the e.val Decomposition.

↳ Singular values σ_i = e.vals λ_i

Exercises

Requires some calculation.

Find the SVD of $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

$A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad \sigma_1^2 = 3$, $\sigma_2^2 = 1$

$(AA^T - 3I) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$(AA^T - I) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$S = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$\dfrac{u_i^T A}{\sigma_i} = v_i^T$

$V = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -1/\sqrt{2} & -1/\sqrt{3} \\ \frac{1}{\sqrt{6}} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$

$\uparrow v_1 \quad \uparrow v_2 \quad \uparrow v_3$

$\left(1/\sqrt{2} \;\; 1/\sqrt{2} \right) / \sqrt{3}\, A = \left(\frac{2}{\sqrt{6}} \;\; \frac{1}{\sqrt{6}} \;\; \frac{1}{6} \right) = v_1^T$

$\left[1/\sqrt{2} \;\; -1/\sqrt{2} \right] A = \left[0 \;\; -1/\sqrt{2} \;\; 1/\sqrt{2} \right] = v_2^T$

$A = \underset{2\times 2}{U}\, \underset{2\times 3}{S}\, \underset{3\times 3}{V^T}$

$\rightsquigarrow v_3$? Choose any vector that makes V orthogonal: $v_3^T = \left[\frac{1}{\sqrt{3}} \;\; \frac{-1}{\sqrt{3}} \;\; \frac{-1}{\sqrt{3}} \right]$

Exercises

Requires some calculation.

$A^T = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

Find the SVD of $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

$AA^T = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \leftarrow$ Choose this one!

$\sigma_1^2 = 8 \Rightarrow AA^T - 8I = \begin{bmatrix} 0 & 0 \\ 0 & -6 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\sigma_2^2 = 2 \Rightarrow AA^T - 2I = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ①, $S = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ ②

Use $u_i^T A = \sigma_i v_i^T$

$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \frac{1}{2\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = v_1^T$

$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = v_2^T$

$\Rightarrow V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ ③

$\quad\quad\uparrow v_1 \quad\quad \uparrow v_2$

$A = USV^T$

Exercises

Conceptual.

If A is an orthogonal matrix, what are the singular values of A?

$A^T A = I$!

$\Rightarrow \sigma_i^2 = 1$ for all i

we choose the positive ones $\sigma_i = 1$ $\forall_i$ (convention)

Exercises

Conceptual.

If $A = USV^{\cancel{*}T}$, what is the SVD of A^T?

$$A^T = (USV^T)^T = VS^TU^T = VSU^T$$

Furthermore, if A is invertible, what is the SVD of A^{-1}?

$$A^{-1} = (USV^T)^{-1} = VS^{-1}U^T$$

$$S^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{bmatrix}$$