Linear Algebra Recitation

Week 13

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Spectral identities

If $A \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \dots \lambda_n$, what are the eigenvalues of cA^m, where $c \in \mathbb{R}$?

$Ax = \lambda x$

$A^2 x = \lambda A x = \lambda^2 x$

$\vdots$

$A^m x = \lambda^m x$

$c A^m x = c\lambda^m x$

$(c\lambda^m, x)$ is e. pair of cA^m.

$Ax = \lambda x$

$(A^3 + A)x$

$A^3 x + Ax$

$\lambda^3 x + \lambda x = (\lambda^3 + \lambda)x$

Generalization

$P(A) = C_m A^m + C_{m-1} A^{m-1} + \cdots + C_1 A + C_0 I$

$P(A)x = C_m A^m x + C_{m-1} A^{m-1} x + \cdots + C_0 x$

$= C_m \lambda^m x + C_{m-1}\lambda^{m-1} x + \cdots + C_0 x = (C_m \lambda^m + \cdots + C_1\lambda + C_0)x$

Spectral identities

If A is invertible, and has eigenpairs $(\lambda_i, x_i)_{i=1,\ldots,n}$, what are the eigenpairs of A^{-1}?

$$Ax = \lambda x$$

$$A^{-1}Ax = \lambda A^{-1}x$$

$$x = \lambda A^{-1}x$$

$$\frac{1}{\lambda}x = A^{-1}x$$

$\left(\frac{1}{\lambda}, x\right)$ are e.pairs of A^{-1}

Spectral identities $z = a+bi$, $\bar{z} = a-bi$, $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(Complex conjugates) Let $A \in \mathbb{R}^{n\times n}$ be a real matrix.

- if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (i.e., complex-valued with nonzero imaginary part) is an e.val of A, can a corresponding e.vec x be a real vector?

  Suppose $x \in \mathbb{R}^n$

  $$\underbrace{Ax}_{\in\, \mathbb{R}^n} = \underbrace{\lambda x}_{\in\, \mathbb{C}^n \setminus \mathbb{R}^n} \Rightarrow (\text{↯})$$

- In the same case as above, explain why $(\bar{\lambda}, \bar{x})$ is also an eigenpair of A.

  $$Ax = \lambda x \Rightarrow \overline{Ax} = \overline{\lambda x}$$

  $$\bar{A}\bar{x} = \bar{\lambda}\bar{x}$$

  $$A\bar{x} = \bar{\lambda}\bar{x}$$

  if $\lambda \in \mathbb{C} \setminus \mathbb{R}$

  $\bar{\lambda}$ is also an e.val

  !

Spectral identities

Let $A \in \mathbb{R}^{n \times n}$ where n is odd. Can all of the eigenvalues of A be in $\mathbb{C} \setminus \mathbb{R}$?

No

Spectral identities

(Conceptual) True or false: the eigenvalues of A are the diagonal entries on the row ~~reduced~~ echelon form (REF) obtained by elementary row operations, without dividing the rows to make the pivots equal to 1. Hint: consider e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. $= A$

$$p_A(\lambda) = \det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix} = \lambda^2 \underbrace{-5}_{-\mathrm{tr}(A)}\lambda \underbrace{-2}_{\det(A)}$$

$$\lambda = \frac{5 \pm \sqrt{33}}{2}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

not evals of A

Spectral identities Hint2: $\det(B) = \det(B^T)$

If $A \in \mathbb{R}^{n\times n}$ has eigenvalues $(\lambda_i)_{i=1,\dots,n}$, what are the eigenvalues of A^T?

Hint1: find Char. poly $P_{A^T}(\lambda)$ and compare with $P_A(\lambda)$

Hint3: $(C+D)^T = C^T + D^T$

$$\begin{aligned} P_{A^T}(\lambda) &= \det(A^T - \lambda I) \\ &= \det([A - \lambda I^T]^T) \\ &= \det(A - \lambda I^T) \\ &= \det(A - \lambda I) = P_A(\lambda) \end{aligned}$$

A^T, A have same e.vals.

In general, A and A^T do not have the same eigenvectors! Compare with the relationship between A and A^{-1}.

Spectral identities

How are the non-zero eigenvalues of $A^T A$ and AA^T related?

$A \in \mathbb{R}^{m \times n}$

$A^TA \in \mathbb{R}^{n \times n}$

$AA^T \in \mathbb{R}^{m \times m}$

$$A^TAx = \lambda x$$

$$AA^T\underbrace{Ax}_{y} = \lambda \underbrace{Ax}_{y}$$

$AA^Ty = \lambda y \Rightarrow (\lambda, y)$ is an e. pair of AA^T

SVD

σ: s. value $\Rightarrow$ σ^2 is e. val of A^TA or AA^T

e.g., $A \in \mathbb{R}^{2\times 4} \Rightarrow$ compute e. vals of AA^T

Spectral theory for symmetric matrices

- Can the eigenvalues of a symmetric matrix be in $\mathbb{C} \setminus \mathbb{R}$?

Let A be sym, and (λ, x) is e.pair of A

then $\lambda = \lambda \|x\|^2 = \lambda (x \cdot x) = (\lambda x \cdot x)$

$$= (Ax) \cdot x$$

$$= x \cdot (A^T x)$$

$$= x \cdot (Ax)$$

$$= x \cdot (\lambda x)$$

$$= \bar{\lambda} (x \cdot x) = \bar{\lambda} \|x\|^2$$

$$= \bar{\lambda}$$

$\lambda = \bar{\lambda}$

Spectral theory for symmetric matrices

- If A is symmetric, what can be said about two e.vecs x_1, x_2 corresponding to distict e.vals λ_1, λ_2?

$$\lambda_1 (x_1 \cdot x_2) = (Ax_1 \cdot x_2)$$
$$= x_1 \cdot A^T x_2$$
$$= x_1 \cdot A x_2$$
$$= x_1 \cdot \lambda_2 x_2$$
$$= \lambda_2 (x_1 \cdot x_2)$$

$$\Rightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{(x_1 \cdot x_2)}_{= 0} = 0$$

$$\Rightarrow x_1 \perp x_2$$

- All symmetric matrices are **orthogonally** diagonalizable. (Important)

$A = XDX^{-1} \Longrightarrow$ we can find Orthogonal X s.t. $A = XDX^{-1} = XDX^T$