Youth EUCLID MO

Year: 2022

Day: 1/1

Have fun :)

Problem 1. Determine all pairs of rational numbers $x, y > 0$ satisfying

$$3x^2 + 2xy + 3y^2 = \frac{1}{x^2} + \frac{1}{y^2}.$$

Problem 2. Let a, b, c, d, e, f be positive integers. Evan is building with a large supply of three types of blocks:

1. Blocks with width a, length 1, height 1
2. Blocks with width 1, length b, height 1
3. Blocks with width 1, length 1, height c

If Evan can place blocks to form a rectangular prism with width d, length e, height f, show he could build a prism with identical dimensions and orientation with blocks of just one type.

(Evan cannot change a block's orientation, so he cannot rotate a block or flip it on a side during building.)

Problem 3. Variable triangles ABC and DEF share a fixed incircle ω and circumcircle Ω. Let ω_a be the A-mixtilinear incircle in $\triangle ABC$, and similarly for ω_d. Determine (as the triangles vary) the locus of the intersection of the common external tangents to these two circles.

(The X-mixtilinear incircle of a triangle XYZ is the circle tangent to segments XY, XZ as well as the circumcircle internally.)

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.