

Select geometry favorites

People

November 17, 2022

This is a collection of problems we've done and found particularly cute/beautiful/hard, inspired by chapter 11 of *Euclidean Geometry in Math Olympiads* and Eric Shen's paper *Geometry At Its Best*. Have fun!

(Note: Here, ∞_{XY} denotes the point at infinity along line XY .)

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🌲 O Problems

Remark. Some attempt has been made to deviate from the aforementioned two famous geometry papers.

:)))

Problem 1 (SL 1998/G4). Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

Problem 2 (SL 2009/G3). Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y , respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram.

Prove that $GR = GS$.

Problem 3 (SL 2015/G4). Let ABC be an acute triangle and let M be the midpoint of AC . A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC . Determine all possible values of $\frac{BT}{BM}$.

Problem 4 (SL 2016/G7). Let I be the incentre of a non-equilateral triangle ABC , I_A be the A -excentre, I'_A be the reflection of I_A in BC , and l_A be the reflection of line AI'_A in AI . Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B .

- (a) Prove that P lies on line OI where O is the circumcentre of triangle ABC .
- (b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y . Show that $\angle XIY = 120^\circ$.

Problem 5 (EGMO 2015/6). Let H be the orthocentre and G be the centroid of acute-angled triangle ABC with $AB \neq AC$. The line AG intersects the circumcircle of ABC at A and P . Let P' be the reflection of P in the line BC . Prove that $\angle CAB = 60$ if and only if $HG = GP'$.

Problem 6 (Iran TST 2018/1/4). Let ABC be a triangle ($\angle A \neq 90^\circ$). $\overline{BE}, \overline{CF}$ are the altitudes of the triangle. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N . Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that \overline{AP} bisects \overline{BC} .

Problem 7 (Eric Shen). In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. Points X, Y are the projections of E, F onto \overline{AD} respectively. Let $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{TZ}$, and Q lies on \overline{EF} such that $\overline{PQ} \parallel \overline{BC}$, prove that \overline{AQ} is a median.

Problem 8 (China TST 2015/2/3). Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G . Let D be the midpoint of \overline{BC} , and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{AD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω .

Problem 9 (IMO 2019/6). Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC , CA , and AB at D , E , and F , respectively. The line through D perpendicular to EF meets ω at R . Line AR meets ω again at P . The circumcircles of triangle PCE and PBF meet again at Q .

Prove that lines DI and PQ meet on the line through A perpendicular to AI .

Problem 10 (MOP 2019 & USA TST 2019/6). Let ABC be a triangle with incenter I , and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to \overline{BC} at A_1 . Define points B_1 on \overline{CA} and C_1 on \overline{AB} analogously, using the excircles opposite B and C , respectively.

- (a) (MOP 2019) Let E, F be the feet of the altitudes from B, C respectively. Prove that if \overline{EF} touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.
- (b) (USA TST 2019/6) Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then \overline{AD} is tangent to the circumcircle of $\triangle DB_1C_1$.

Problem 11 (USAMO 2016/3). Let $\triangle ABC$ be an acute triangle, and let I_B, I_C , and O denote its B -excenter, C -excenter, and circumcenter, respectively. Points E and Y are selected on \overline{AC} such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on \overline{AB} such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $\overleftrightarrow{I_B F}$ and $\overleftrightarrow{I_C E}$ meet at P . Prove that \overline{PO} and \overline{YZ} are perpendicular.

Problem 12 (TSTST 2018/3). Let ABC be an acute triangle with incenter I , circumcenter O , and circumcircle Γ . Let M be the midpoint of \overline{AB} . Ray AI meets \overline{BC} at D . Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y , while line CO meets ω at C and Q . Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D . Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ .

Problem 13 (IMO 2021/3). Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD , respectively. Prove that the lines BC, EF , and O_1O_2 are concurrent.

Problem 14 (RMM 2012/6 & Brazil 2013/6). In triangle ABC with incenter I and circumcenter O , let the incircle ω touch $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F respectively.

- (RMM 2012/6) Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC ; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A ; the points B' and C' are defined similarly.
- (Brazil 2013/6) Let P be the Gergonne point of $\triangle ABC$, and its reflections in $\overline{EF}, \overline{FD}$ and \overline{DE} be P_a, P_b, P_c , respectively.

Prove that $P_a \in \overline{AA'}$, and that $\overline{AP_aA'}, \overline{BP_bB'}, \overline{CP_cC'}, \overline{IO}$ are concurrent.

Problem 15 (USAMO 2021/6). Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}, \overline{BC} \parallel \overline{EF}, \overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y , and Z be the midpoints of $\overline{AD}, \overline{BE}$, and \overline{CF} . Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

Problem 16 (SL 2021/G8). Let ABC be a triangle with circumcircle ω and let Ω_A be the A -excircle. Let X and Y be the intersection points of ω and Ω_A . Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R . Prove that $\overline{AR} \perp \overline{BC}$.

🌲 1 Solutions

🌲 1.1 SL 1998/G4

Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

1.2 SL 2015/G4

Let ABC be an acute triangle and let M be the midpoint of AC . A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC . Determine all possible values of $\frac{BT}{BM}$.

Solution by **CyclicSLscalesTrapezoid**.

The answer is $\sqrt{2}$ only. Let the $X \neq B$ be defined as $(ABC) \cap (BPMQ)$, and let N be the midpoint of \overline{BT} .

Claim 1 – $XNMT$ is cyclic.

Proof. Since N is also the midpoint of \overline{PQ} , there is a spiral similarity at X sending PNQ to AMC . Thus, we have

$$\angle XMN = \angle XAP = \angle XTB,$$

so $XNMT$ is cyclic. □

Claim 2 – \overline{BM} is tangent to the circumcircle of $XNMT$.

Proof. We have

$$\angle XNM = \angle XPA = \angle XPB = \angle XMB.$$

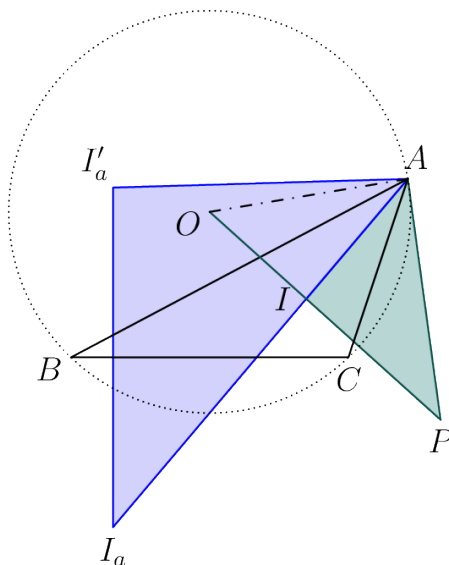
□

By Power of a Point, $BM^2 = BN \cdot BT = \frac{BT^2}{2}$, so $\frac{BT}{BM} = \sqrt{2}$.

1.3 SL 2016/G7

Let I be the incentre of a non-equilateral triangle ABC , I_A be the A -excentre, I'_A be the reflection of I_A in BC , and l_A be the reflection of line AI'_A in AI . Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B .

- Prove that P lies on line OI where O is the circumcentre of triangle ABC .
- Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y . Show that $\angle XIY = 120^\circ$.



Redefine P as the inverse of I ; it's clear via Poncelet spam that this point satisfies the second part. For the first part we assert more strongly that:

Claim - $\triangle AI_A I'_A \sim \triangle API$.

Proof. One of the few uses of SAS similarity? By angle chasing, $\angle I_A = \angle P$ follows easily. To finish, we show $I_A I'_A / I_A A = IP / AP$; indeed, the first ratio equals $2 \cos \angle BI_A C = 2 \sin \frac{A}{2}$ because of similar triangles; thus, we're left to length chase IP / AP ; this becomes

$$\begin{aligned} \frac{OP}{AP} - \frac{OI}{OA} \frac{OA}{AP} &= \frac{OI}{AI} - \frac{OI^2}{OA \cdot AI} = \frac{R}{AI} - \frac{R^2 - 2rR}{R \cdot AI} \\ &= \frac{R - (R - 2r)}{AI} = 2 \sin \frac{A}{2}, \end{aligned}$$

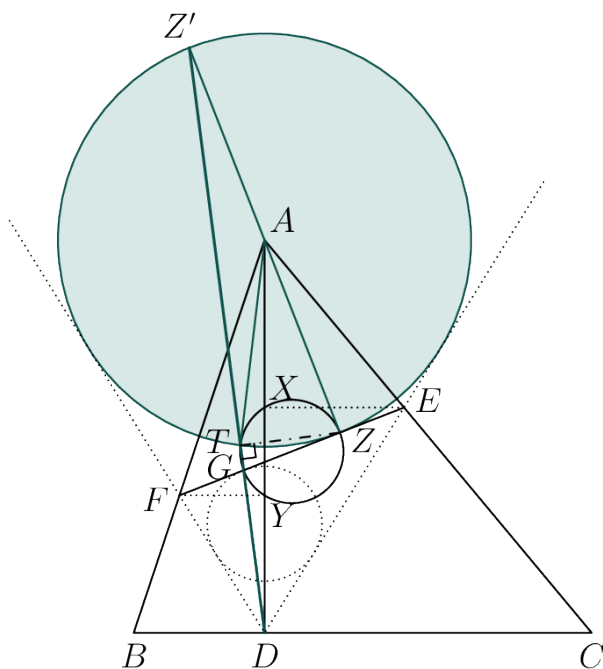
so the ratios are equal, as needed. □

The claim clearly implies the isogonality.

Remark. Surprising how people found the inverse but not the similar triangles...

1.4 Mock AIME 2019/15', by Eric Shen & Raymond Feng

In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AZ} \cap \overline{QT}$, and Q lies on \overline{EF} such that $\overline{PQ} \parallel \overline{BC}$, prove that \overline{AQ} bisects \overline{BC} .



Construct points X, Y as the projections of E, F onto \overline{AD} respectively. *

After drawing a diagram on Geogebra, we obtain:

Characterization of T

T is the harmonic conjugate of Z wrt XY – i.e. it lies on $\omega = (XYZ)$ so that the resulting quadrilateral is harmonic.

In American style, we show that this choice of T lies on ω_a (defined as the circle at A thru Z) and (DZ) ,

Verification (inspired by USA TST 2015/1)

For $AZ = AT$, we use power of a point / length chase to get $AZ^2 = AX \cdot AY$ whence \overline{AZ} touches ω . Hence, by harmonics \overline{AT} is also tangent to ω , so this property follows.

$\angle DTZ = 90^\circ$ is much less straightforward. We define $Z' = 2A - Z$ and $G = E + F - Z$ as the antipodes of Z on the circle at A through Z . By a well-known lemma, D, Z', G collinear (along the cevian through the intouch point in $\triangle DEF$).

But also at the same time, T is on ω , $\omega_a \Rightarrow \angle ZTG = \angle ZTZ' = \pi/2$ due to antipodes. Hence, $\angle DTZ = \pi/2$, completing the verification.

*Eric Shen originally included these points in the problem statement (as seen in the 2019 version of “Geometry At Its Best”), but I guess the problem’s made harder by deleting them. For me, thinking about their relevance/use was important in making nonzero progress on the problem.

By this definition, we clearly have $(AP; XY) = -1$. From here (the chase is best discovered backwards), harmonic chasing suffices. Define $K = \overline{EF} \cap \overline{A\infty_{BC}}$. Then

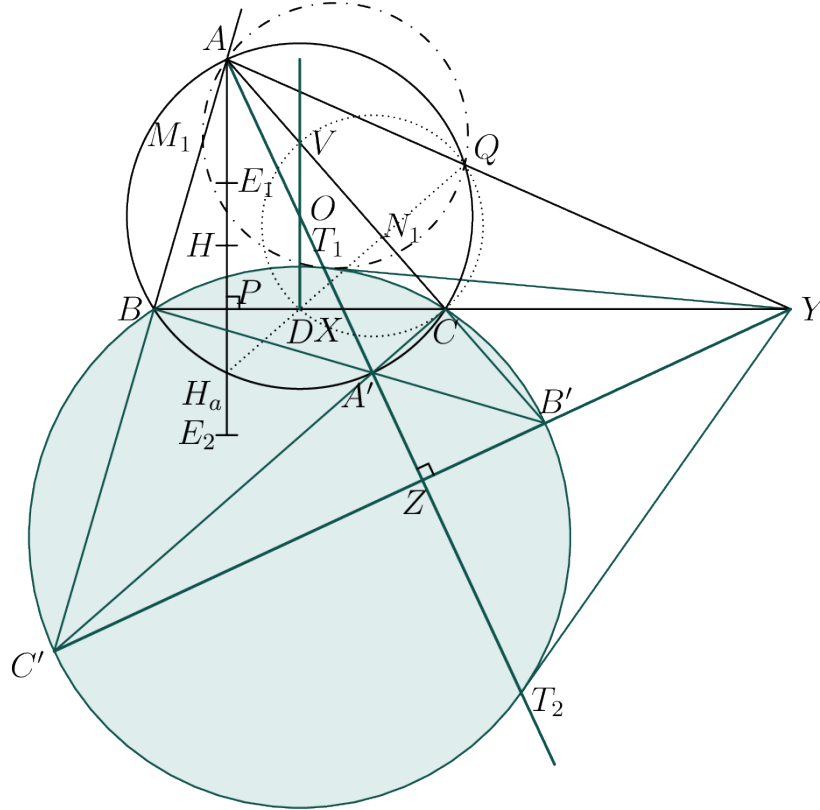
$$(\overline{AQ} \cap \overline{BC}, \infty_{BC}; B, C) \stackrel{A}{=} (QK; FE) \stackrel{\infty_{BC}}{=} (PA; YX) = -1,$$

the end!

1.5 China TST 2015/2/3

Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G . Let D be the midpoint of \overline{BC} , and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{AD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω .



As much of the parallel/perpendicular lines aren't even relevant (just to give us some equal ratios), we simplify as follows:

Problem reworded

In acute $\triangle ABC$ with circumcenter O and orthocenter H , D is the midpoint of \overline{BC} , and the altitude from A meets (BC) at E (either one works). Let $U, V = \overline{OD} \cap \overline{AB}, \overline{AC}$, respectively; define $M, N \in \overline{AB}, \overline{AC}$ with (lengths directed)

$$UM/MB = VN/NC = AE/EH.$$

Let ω be the circle tangent to segments OB, OC at B, C respectively. Prove that (AMN) is tangent to ω .

We define a load of new points as follows.

- $A' = 2O - A$;
- E_1, E_2 be the choices of E with $AE/EH > 0$ and $AE/EH < 0$ respectively. We will only consider M_1, N_1 , because the negative case is identically handled;
- $T_1, T_2 = \overline{AO} \cap \omega, X = \overline{AO} \cap \overline{BC}$, corresponding to E_1, E_2 from earlier;

- $Y = \overline{T_1 T_1} \cap \overline{T_2 T_2} \cap \overline{BC}$ (which exists since $(BC; T_1 T_2) = -1$);
- Q as the harmonic conjugate of A' wrt BC , or equivalently, the reflection of the A -orthocenter Miquel point Q_a in the perpendicular bisector of \overline{BC} , \overline{DUV} .

Claim 1 – Q is the Miquel point of $ABCDUV$.

Proof. Observe that $Q \in \overline{H_a D}$, which follows by $Q_a \in \overline{A' PH}$ reflected in \overline{DUV} . The result holds by Reim because $AH_a QC$ cyclic and $\overline{DV} \parallel \overline{AH_a}$. \square

Claim 2 – (AQT_1) touches ω , $\overline{YT_1}$ at T_1 .

Proof. Sufficient to show $Q \in \overline{AY}$, so that the claim will follow by power of a point @Y. Indeed,

$$-1 = (BC; XY) \stackrel{A}{=} (B, C; A', \overline{AY} \cap (ABC))$$

so we're done. \square

Claim 3 – $AE_1/E_1H = AT_1/T_1A'$.

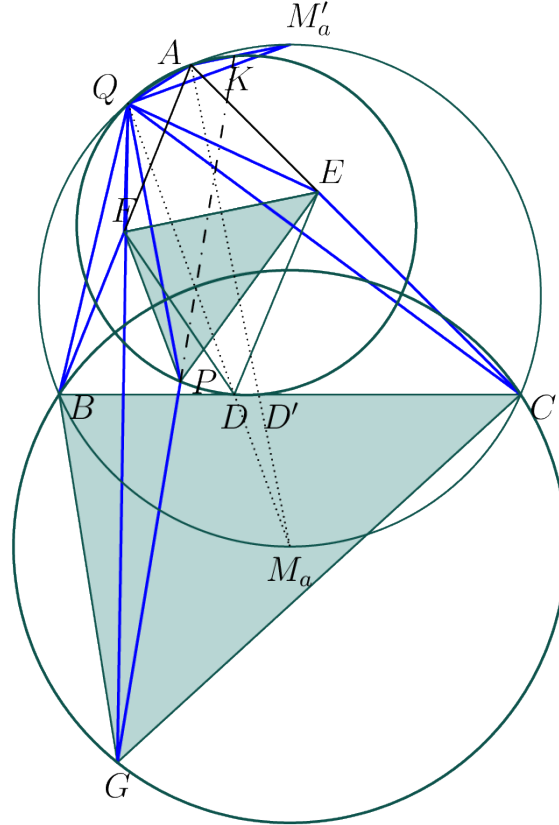
Proof. Define $B' = \overline{A'B} \cap \overline{AC}$, $C' = \overline{A'C} \cap \overline{AB}$. Using the logic of **USA TST 2007/5**, we know that $\triangle ABC \sim \triangle AB'C'$, and that Q is the A -orthocenter Miquel point in $\triangle AB'C'$. Next, let P, Z be the foot from A to $\overline{BC}, \overline{B'C'}$ respectively. If r denotes the reflection + homothety at A that maps $B, C \Rightarrow B', C'$, then observing that $(E_1, P, E_2) \xrightarrow{r} (T_1, Z, T_2)$ wins. \square

To finish the problem, observe $M_1, N_1 \in (AQT_1)$ follows by spiral similarity at Q , completing the proof.

1.6 IMO 2019/6, by Anant Mudgal

Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC , CA , and AB at D , E , and F , respectively. The line through D perpendicular to \overline{EF} meets ω at R . Line AR meets ω again at P . The circumcircles of triangle PCE and PBF meet again at Q .

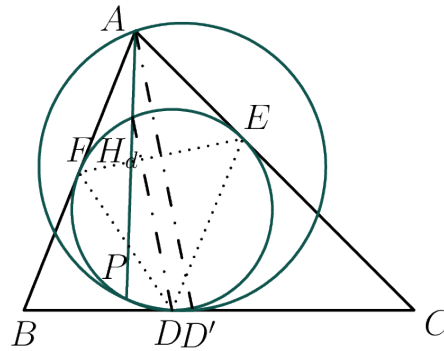
Prove that lines DI and PQ meet on the line through A perpendicular to \overline{AI} .



Observe that P is the D -orthocenter Miquel in $\triangle DEF$. Define K as the intersection of the A -external bisector with \overline{AD} . We make the following definitions...

- Let ω, ω_a denote the incircle and (BIC) respectively;
- Define X as intersection of *segment* PK with ω . Let Q instead denote the A -SD point;
- G be the harmonic conjugate of I wrt BC , D' as the foot of the A -angle bisector; M_a as the midpoint of arc BC exc. A ; M'_a as the antipode of M_a on (ABC) ;
- H as orthocenter of $\triangle DEF$, and H_d its reflection over \overline{EF} .

\Rightarrow because $MB^2 = MD \cdot MQ = MD' \cdot MA, Q \in (ADD'K)$.



Claim 1 - $P \in (ADD'KQ)$.

Proof. Observe that $(PH_d; EF) = -1$ whence A, P, H_d collinear. Then because $\overline{DH_d} \parallel \overline{AI}$ because both perpendicular to \overline{EF} . Hence result by degenerate Reim. \square

Claim 2 - $\Delta PFE \stackrel{+}{\sim} \Delta GBC$.

Proof. Proceed by spiral at Q . Observe that $\triangle H_d E F \stackrel{+}{\sim} \triangle ICB$ by angle chase. Because $(H_d P; EF) = (IG; BC) = -1$, the needed similarity follows. \square

Claim 3 – K, G, P collinear.

Proof. An angle chase, using the previous two claims:

$$\angle QPK^{\text{claim 1}} = \angle QAK = \angle QAM_a^{\text{spiral}} = \angle QPG,$$

the end.

Using last two claims, we may chase:

$$\angle PXB = \angle GXB = \angle GCB^{\text{spiral}} = \angle PEF = \angle PFB$$

whence $PXFB$ (and also $PXEC$ by symmetry) cyclic.

This completes the proof. \square

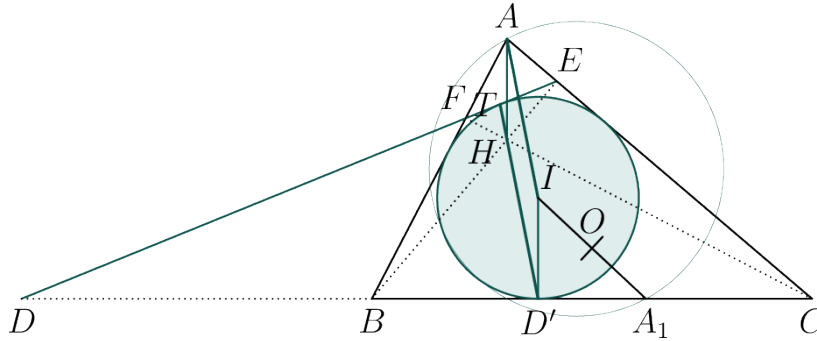
Remark. ggb way too op

1.7 MOP + USA TST, by Ankan Bhattacharya

Let ABC be a triangle with incenter I , and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to \overline{BC} at A_1 . Define points B_1 on \overline{CA} and C_1 on \overline{AB} analogously, using the excircles opposite B and C , respectively.

1.7.1 MOP

Let E, F be the feet of the altitudes from B, C respectively. Prove that if \overline{EF} touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.



Call the incircle ω .

Claim 1 – D, E, F are collinear.

Proof. We will prove that the tangent line from D is antiparallel to \overline{BC} wrt $\angle A$. Indeed, this line is found by reflecting \overline{DBC} over \overline{DI} , a line perpendicular to \overline{AI} , so we win. \square

Let ω touch \overline{DEF} at a point T , and let D' denote the A -intouch point.

Claim 2 – $\overline{AI} \parallel \overline{HD'}$; hence $AID'H$ is a parallelogram and $AH = r$, the inradius of $\triangle ABC$.

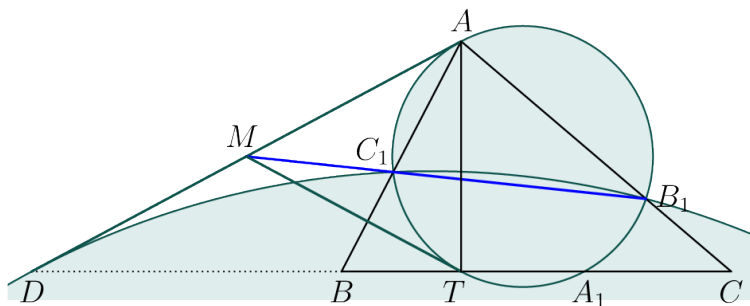
Proof. Because $BCEF$ is tangential, it follows by degenerate Brianchon that lines BE, CF, DT' concur, i.e. $H \in \overline{TD'}$. Observe that $DT = DD'$; then $\overline{THD'} \perp \overline{DI}$ by symmetry, while $\overline{AI} \perp \overline{DI}$ is given; the lines are thus parallel as claimed. \square

Now, let ω_a , etc denote (AB_1C_1) , etc, respectively. We observe that because the perpendicular from A_1 to \overline{BC} and its cyclic variants all concur at the point $2O - I$, it follows that all three circles must concur at this point by Miquel spam.

But because $r/2 = AH/2$ is the distance from O to \overline{BC} , we actually have $2O - I = A_1$ (also because of their feet onto \overline{BC}). Hence $A_1 \in \omega_a$ as desired.

1.7.2 USA TST 2019/6

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then \overline{AD} is tangent to the circumcircle of $\triangle DB_1C_1$.



We first make some observations from working backwards on the previous part.

First, $\overline{I_a A_1}$ is unconditionally the raxis of ω_b, ω_c , which is because $2O - I, A_1, I_a$ lie on the same line $\perp \overline{BC}$. Thus, if A_1 is to lie on ω_a , then by anglechase, ω_b, ω_c touch at A_1 whence $I_a A_1 \perp \overline{BC}$.

Also, by MOP 2019 converse (which follows by uniqueness wrt $\angle A$) we have D, E, F collinear. If T is the foot of A onto \overline{BC} , it follows that $(DT; BC) = -1$.

Claim 1 – The A -SD point coincides with the A - orthocenter Miquel.

Proof. Since $BF/CE = \cos B/\cos C = (s-c)/(s-b)$ from 19MOP, result follows by spiral.

Claim 2 – A_1 is the antipode of A on ω_a .

Proof. Angle chase, observing that ω_b, ω_c touch at A_1 / etc.

Claim 3 – \overline{AD} is tangent to ω_a .

Proof. Recall that \overline{ADQ} is perpendicular to $\overline{HIQ'}$; thus, equivalent to show $\overline{HQ} \parallel \overline{AA'}$ which is not hard.

By radical axis/etc, it suffices to show that the midpoint M of \overline{AD} lies on $\overline{B_1C_1}$. By symmetry, \overline{MA} , \overline{TA} touch ω_d .

Claim 4 - $(AT; B_1C_1) = -1$.

Proof. $(AT; B_1C_1) \stackrel{A}{=} (DT; BC) = -1$, the end.

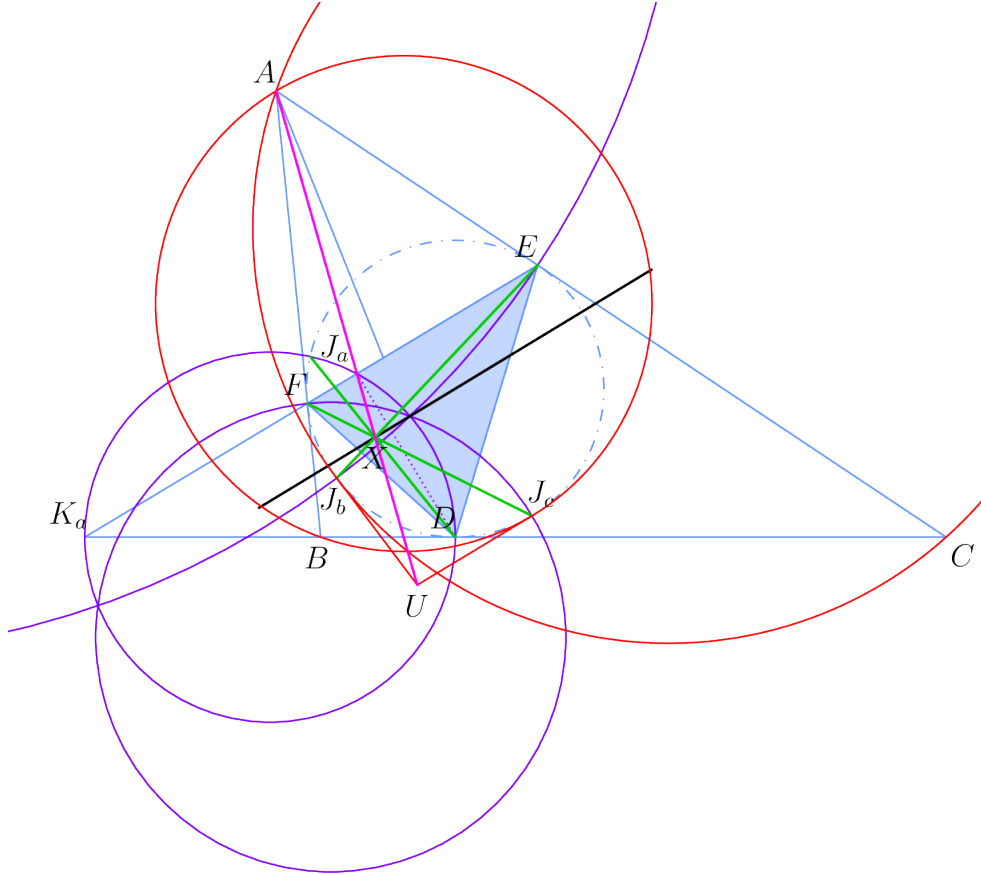
Result follows from power of a point converse on $MD^2 = MA^2 = MB_1 \cdot MC_1$.

🌲 1.8 RMM + Brazil

🌲 1.8.1 RMM 2012/6

Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC ; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A ; the points B' and C' are defined similarly. Prove that the lines AA' , BB' and CC' are concurrent at a point on the line IO .

Let $K_a = \overline{EF} \cap \overline{BC}$, $\gamma_a = (K_a D)$, $J_a = \omega_a \cap \gamma_a \cap \omega$ (and cyclic variants), and H and ℓ denote the orthocenter and Euler line of $\triangle DEF$, respectively. Also, let I_a, I_b, I_c be the excenters of $\triangle ABC$

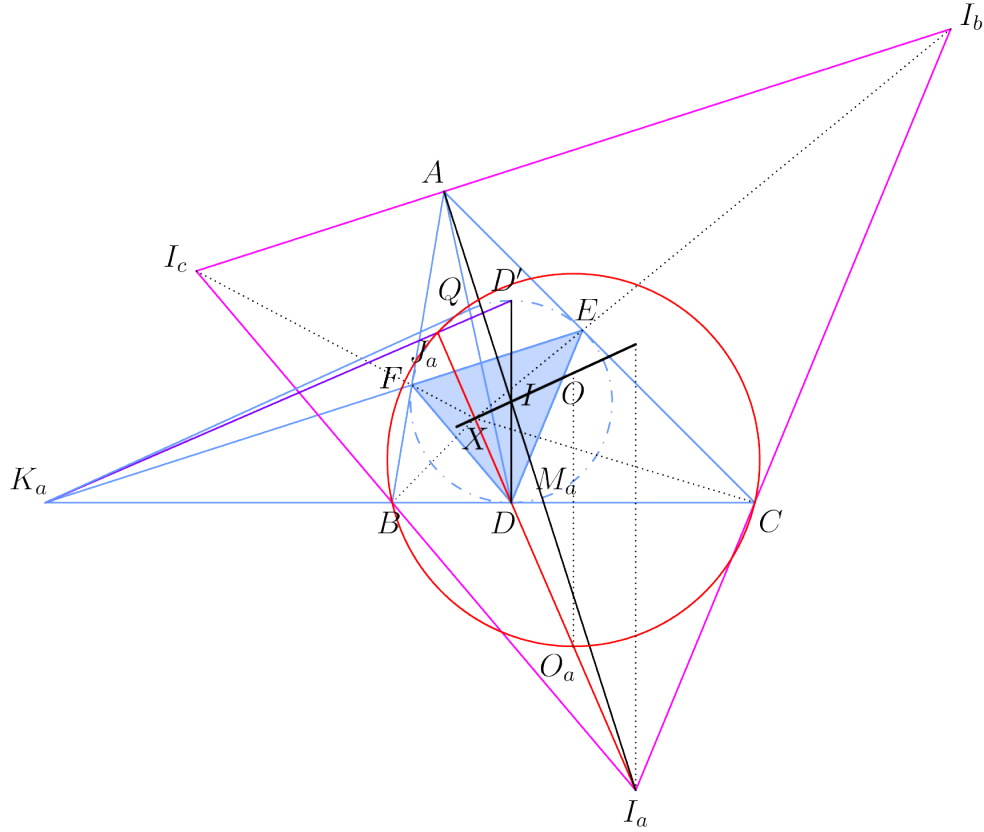


Solution 1, by radical axes Observe that ℓ is just \overline{OI} , and that γ_a , etc are coaxial Apollonian circles. Define X as the radical center of $\gamma_a, \gamma_b, \gamma_c, \omega$ (which exists since the former 3 circles are coaxial). We'll show this is the desired concurrency point in both problems. Clearly, $\overline{DJ_a}$ is the raxis of (γ_a, ω) , i.e. $X \in \overline{DJ_a}$.

Lemma 1 – ℓ is the raxis of γ_a and variants.

Proof. Let T_a denote the foot of D onto \overline{EF} , which is obviously on γ_a . Then H has power $HD \cdot HT_a$ (= variants) wrt the γ 's, hence on raxis; Meanwhile I has power r^2 wrt all circles by orthogonality, hence also on raxis, done. \square

Let tangents to ω at J_b, J_c meet at U ; then, \overline{AU} is the raxis of ω_b, ω_c . Clearly this is the polar of $\overline{J_b J_c} \cap \overline{EF}$. Recalling that $X = \overline{EJ_b} \cap \overline{FJ_c}$, follows by Brokard that $X \in \overline{AU}$, the end.



Solution 2, by homothety (v4913) Let D' be the antipode of D on ω , $Q = \overline{AD} \cap \omega$ ($\neq D$); then, because $(EF; DQ) = -1$, $\overline{K_a Q}$ touches ω as well. Also, because $\angle DJ_a D' = \angle DJ_a K_a = 90^\circ$, K_a, J_a, D' are collinear, whence $(DQ; J_a D') = -1$.

We start with X as the similicenter of homothetic triangles $DEF, I_a I_b I_c$. Let homothety h at X with scale factor r map $(D, E, F) \rightarrow (I_a, I_b, I_c)$. This must also map their circumcenters to each other, i.e. $I \xrightarrow{H} 2O - I$, whence $X \in \overline{OI}$.

Also, let M_a be the midpoint of \overline{BC} , $O_a \in \overline{DJ_a}$ be the midpoint of arc BC on ω_a not containing J_a (and variants).

Lemma 2 – J_a, D, I_a collinear.

Proof. Harmonics:

$$-1 = (DQ; D'J_a) \stackrel{D}{=} (\overline{BC} \cap \overline{AI}, \overline{J_a D} \cap \overline{AI}; I, A),$$

implying that $\overline{J_a D} \cap \overline{AI}$ is the A -excenter. □

Hence, $X = \overline{DJ_a} \cap \overline{EJ_b} \cap \overline{FJ_c}$.

Claim – O_a is the midpoint of $\overline{DI_a}$.

Proof. By symmetry, M_a is the foot of O_a onto \overline{BC} , while it's well-known that $2M - D$ is the foot of I_a onto \overline{BC} . M obviously being the midpoint of the segment with endpoints $D, 2M - D$ implies the claim by parallel lines. □

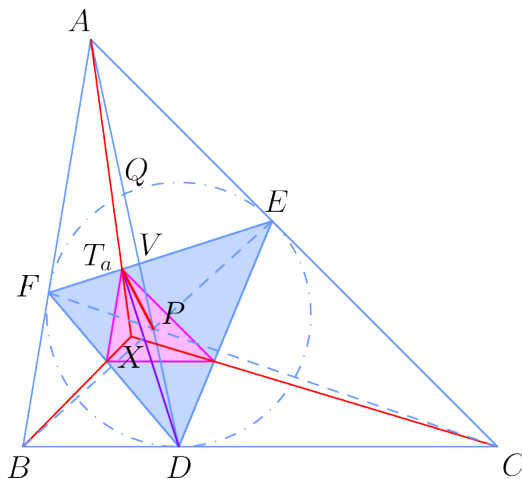
Therefore there must exist a homothety h' at X with scale factor $(1+r)/2$, mapping $(D, E, F) \rightarrow (O_a, O_b, O_c)$.
To show that our X is indeed the radical center of $\omega_a, \omega_b, \omega_c$, compute

$$\text{Pow}(X, \omega_a) = XJ_a \cdot XO_a \stackrel{h'}{=} \frac{1+r}{2} XJ_a \cdot XD = \frac{\text{Pow}(X, \omega)}{(r+1)/2},$$

a symmetric quantity wrt a, b, c .

1.8.2 Brazil 2013/6

The incircle of triangle ABC touches sides BC , CA and AB at points D , E and F , respectively. Let P be the intersection of lines AD and BE . The reflections of P with respect to EF , FD and DE are X , Y and Z , respectively. Prove that lines AX , BY and CZ are concurrent at a point on line IO , where I and O are the incenter and circumcenter of triangle ABC .



(We continue to use terminology from the previous subsection.) Let T_a be the projection of D onto \overline{EF} . As promised in the refactored statement in the problem section,

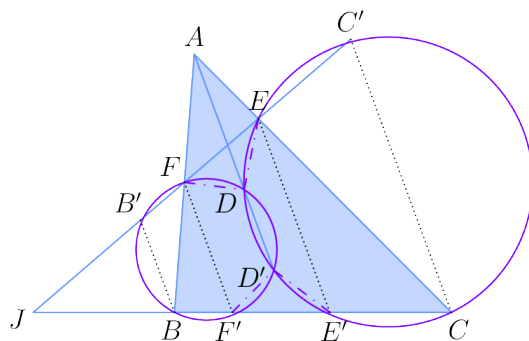
Claim – $T_a \in \overline{AXA'}$.

Proof. Because X is the similicenter of triangles DEF , $I_a I_b I_c$, it must also be similicenter of their orthic triangles. It follows that $T_a \in \overline{AX}$, as needed. \square

Next, let $V = \overline{AD} \cap \overline{EF}$, so that $(DV; AP) = -1$. Because $\angle DT_a V = 90^\circ$, \overline{EF} must bisect $\angle AT_a P$, whence $P_a \in \overline{AT_a A'}$. Considering triangles ABC , DEF , and the orthic triangle of $\triangle DEF$, the concurrency holds by cevian nest.

1.9 IMO 2021/3

Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD , respectively. Prove that the lines BC , EF , and O_1O_2 are concurrent.



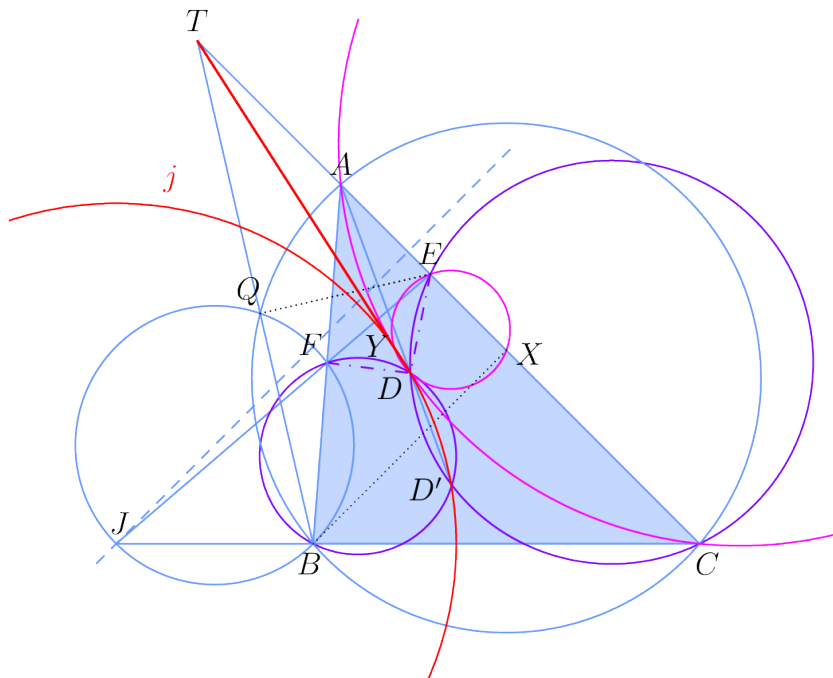
Solution by **v4913**.

Let $J = \overline{EF} \cap \overline{BC}$, and $D' \in \overline{AD}$ be the isogonal conjugate of D wrt $\triangle ABC$. The given angle conditions imply that $BDD'F$, $CDD'E$ are cyclic, while power of a point at A implies $BCEF$ cyclic as well.

Claim 1 – J is the exsimilicenter of (EDC) , (FDB) ; hence, $JD = JD'$ by symmetry.

Proof. Construct $E_1 = (CDD'E) \cap \overline{BC}$ ($\neq C$), $F_1 = (BDD'F) \cap \overline{BC}$ ($\neq B$). By isogonality, $DF = D'F'$ and $DE = D'E'$ whence $DD'E'E$, $DD'F'F$ are both cyclic isosceles trapezoids. $\overline{DD'}$, $\overline{EE'}$, $\overline{FF'}$ share a perpendicular bisector j , and in fact, this is the bisector of $\angle J$, i.e. $JE = JE'$, $JF = JF'$.

Reflect B, C over j to obtain B', C' ; then, because $JB/JF' = JB/JF = JE/JC = JE'/JC$, there is a homothety at J mapping $(B, B', F, F') \rightarrow (E', E, C', C)$ and thus their circumcircles $(BB'DD') \rightarrow (CC'DD')$ as well. \square



Let $Y = (ADC) \cap (EXD)$ ($\neq D$), Q be the Miquel point of $ABCJEF$, and j the circle at J through D, D' . Observing that $\overline{O_1O_2}$ is the perpendicular bisector of \overline{DY} , it remains to prove $Y \in j$.

Claim 2 – $XQEB$ is cyclic.

Proof. This is a simple angle chase: using cyclic quadrilaterals $(ABCQ)$, $(JFBQ)$, $(ECJQ)$, and $(AEFQ)$, we obtain

$$\angle EQB = \angle EQA + \angle AQB = \angle ACB + \angle EFA = 2\angle ACB = \angle EXB$$

as desired. \square

Next, we characterize the radical axis of j , (JBF) – it's perpendicular to the line of centers and through A :

Claim 3 – The line through B and the center of (JBF) is perpendicular to \overline{AC} .

Proof. This is equivalent to “ t_b , the tangent to (JBF) at J , is parallel to \overline{AC} ”. Because $\angle(\overline{BJC}, t_b) = \angle BFJ = \angle JCE$, the result follows. \square

Because $\text{Pow}(A, j) = AD \cdot AD' = AQ \cdot AJ = \text{Pow}(A, (JBQF))$, A is on the radical axis of j , (JBF) . By the previous claim, it follows that \overline{AC} is the radical axis of j , (JBF) .

To finish, define $T = \overline{DY} \cap \overline{AC} \cap \overline{BQ}$ as the radical center of (JBF) , (ABC) , (EXD) , (ADC) , and the phantom point $Y' = \overline{TD} \cap j$ ($\neq D$). Because T is on \overline{AC} , the radical axis of j , (JBF) , we have

$$TY' \cdot TD = \text{Pow}(T, j) = \text{Pow}(T, (JBF)) = \text{Pow}(T, (ABCQ)) = TA \cdot TC = TY \cdot TD \Rightarrow Y = Y',$$

the end!

1.10 SL 2021/G8

Let ABC be a triangle with circumcircle ω and let Ω_A be the A -excircle. Let X and Y be the intersection points of ω and Ω_A . Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R . Prove that $\overline{AR} \perp \overline{BC}$.

Solution 1 by Brianchon from AoPS[†] Redefine R as intersection of tangent at D' and A -altitude and prove PR is tangent to ω_{XPA} . Let us denote some points: $D'D' \cap AB = B', D'D' \cap AC = C', D'D' \cap XX = B', XX \cap AB = X'$ and $CX \cap AP' = M$. apply Brianchon to the polar reciprocal $DD'XXB''C''$: Number 1 $DD \cap D'D' = \infty$ Number 2 $D'D' \cap XX = P'$ Number 3 $XX \cap XX = X$ Number 4 $XX \cap B''B'' = X'$ Number 5 $B''B'' \cap C''C'' = A$ Number 6 $C''C'' \cap DD = C'$ and lines 14, 25, 36 must be concurrent. Since $AP' \cap CX = M$ we can imply that $MX' \parallel BC$ By angle chase $\angle MX'A = \angle P'B'A = 180 - \angle AB'C' = 180 - \angle ABC = 180 - \angle AXC = \angle MXA$ so $MXX'A$ is concyclic. Again by angle chase $\angle MAX = \angle MX'P' = \angle X'P'B' = \angle PAR$ (since $P'APR$ is concyclic) thus $\angle XAP = \angle P'AR = \angle P'PR$ and we are done.

Solution 2 by DDIT (CyclicISLscalesTrapezoid)

[†]<https://artofproblemsolving.com/community/c6h2882551p25740378>