ZGY-G6Summit

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> *Remark.* I decided to do the problems on the older versions because... why not...

🌲0 [32♣] References

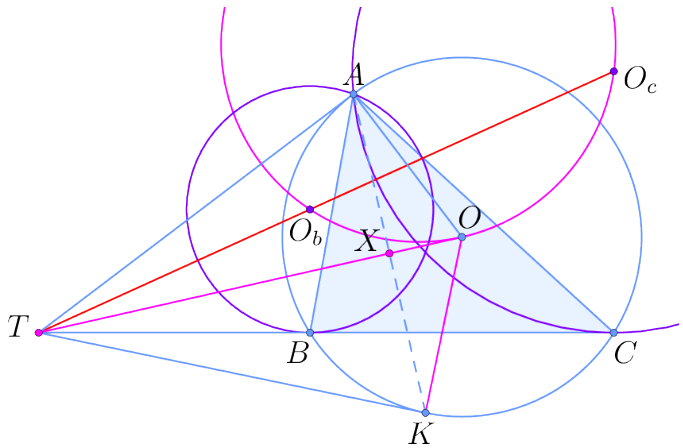
I again “stole” problems; they’re found on **my paper** oops. . .
[9♣] **09SLG6**, [9♣] **21SLG8**, [9♣] **21AMO6**, [5♣] **14APMO5**;

Contents

🌲1 [32♣] Baroque

🌲1.1 [2♣] Besant

Let k be a parabola with focus F. Let B and C be points on k, and suppose the tangents to k at B and C meet at a point A. Denote by O the circumcenter of $\triangle ABC$. Prove that $AF \perp FO$.



After getting rid of the parabola, we get a tamer triangle geometry problem:

Besant equivalent

In triangle ABC with circumcenter O, circles ω_b, ω_c with centers O_b, O_c pass through A and touch $\overline{BC}$ at B, C respectively. If P is the circumcenter of $\triangle OO_bO_c$, prove that $\angle OAP = 90°$.

We actually claim that A is in fact the O-Dumpty point in $\triangle OO_bO_c$. Let X be that of A in $\triangle ABC$.

Claim – $O_bOO_cA \overset{+}{\sim} BKCX$.

Proof. First, we contend that $\measuredangle(\overline{O_bO_c}, \overline{BC}) = -\measuredangle OTA = \measuredangle OTK$. Indeed, $\overline{O_bO_c}$, $\overline{OT}$ are respectively perpendicular to the A-median and symmedian, so the "$\measuredangle(\overline{O_bO_c}, \overline{BC}) = -\measuredangle OTA$" reduces to a simple isogonality.

By above, $\triangle TAO'$, TBO_b, TCO_c, TKO are all similar right triangles, so there exists a spiral similarity at T mapping the two quadrilaterals. $\square$

This implies the result…

🌲 1.2 [3♣] 95SLG8

In cyclic quadrilateral $ABCD$, $E = AC \cap BD$ and $F = AB \cap CD$. Prove that F is collinear with the orthocenters of $\triangle EAD, EBC$.

Obviously, F is on the radical axis of (AB), (CD), so we are done by Gauss-Bodenmiller on $ADBC$.

🌲 1.3 [3♣] 12JanTST2

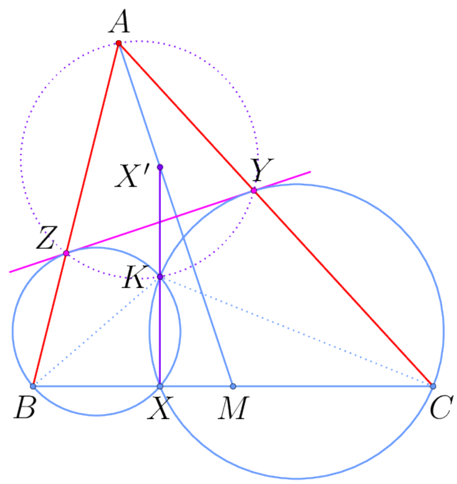
Let $ABCD$ be a cyclic quadrilateral whose diagonals meet at P and let E and F be the feet of perpendiculars from P to $\overline{AB}$ and $\overline{CD}$, respectively. Segments $\overline{BF}$ and $\overline{CE}$ meet at Q. Prove that $\overline{PQ}$ is perpendicular to $\overline{EF}$.

> Evan really likes Gauss-Bodenmiller, apparently.
>
> CyclicISLscelesTrapezoid

Let $K = \overline{AB} \cap \overline{CD}$. Apply second isogonality lemma to $\angle(\overline{AC}, \overline{BD})$, so that $\overline{PQ}, \overline{PK}$ are isogonal wrt it. As the former is a diameter of $(PEFK)$, the latter is perpendicular to $\overline{EF}$.

1.4 [3♣] Gergonne reverse eng

In a triangle, the Gergonne point and two of the tangency points of the incircle are marked. Reconstruct the triangle using straightedge and compass.



Clearly, this is equivalent to

Change POV to contact triangle

In triangle ABC with A missing, but given the symmedian point K and the other vertices, construct A.

We first establish some results about this configuration:

Theorem

In triangle ABC with symmedian point K and associated petal triangle XYZ, let M be the midpoint of $\overline{BC}$.

- $\overline{AM} \perp \overline{YZ}$;
- K is the centroid of $\triangle XYZ$;

Proof. The first item holds since in $\triangle AYZ$, $\overline{AK}$ passes through the circumcenter, so $\overline{AM}$ is an altitude by isogonals. To prove the second item, suffices to prove that $\overline{KX}$ bisects $\overline{YZ}$:

$$(Y, Z; \overline{KX} \cap \overline{YZ}, \infty_{YZ}) = (\infty_{KY}\infty_{KZ}; \infty_{KX}\infty_{YZ}) = (\infty_{\perp AC}\infty_{\perp AB}; \infty_{\perp AM}\infty_{\perp BC}) \stackrel{A}{=} (BC; M\infty_{BC}) = -1. \qquad \square$$

We'll say that $\overline{BC}$ is 'horizontal'. Then, we may construct the A-median by y-scaling the Schwatt line by 2. Next, construct $\overline{YZ}$ through $\frac{1}{2}(3K - X)$ perpendicular to the A-median. Then, intersect that with (BK), (CK) to find Y, Z, from which A easily follows.

Note. Config issue here... are there two possibilities though?

1.5 [3♣] 73SL1

Let a tetrahedron $ABCD$ be inscribed in a sphere S. Find the locus of points P inside the sphere S for which the equality

$$\frac{AP}{PA_1} + \frac{BP}{PB_1} + \frac{CP}{PC_1} + \frac{DP}{PD_1} = 4$$

holds, where A_1, B_1, C_1, and D_1 are the intersection points of S with the lines AP, BP, CP, and DP, respectively.

(Here, capital letters are shorthand for the vectors to those points.)

Let $p = \operatorname{Pow}(P, (ABCD))$, and assume that S is the unit sphere. By power of a point at P, the given equation becomes $\sum_{\text{cyc}} AP^2 = 4p = 4(1 - P \cdot P)$. Since $AP^2 = A \cdot A + P \cdot P - 2A \cdot P = 1 + P \cdot P - 2A \cdot P$, this in turn becomes

$$4(1 - P \cdot P) = \sum_{\text{cyc}} (1 + P \cdot P - 2A \cdot P) \iff 8P \cdot \left(P - \frac{A+B+C+D}{4}\right) = 0.$$

Let $G = \frac{A+B+C+D}{4}$ be the centroid; then, the last equation means that $\overline{OP} \perp \overline{GP}$ unless $P \in \{O, G\}$. In other words, **P lies on the sphere with diameter OG**.

🌲 1.6 [3♣] 95SLG6

Let $A_1A_2A_3A_4$ be a tetrahedron, G its centroid, and A_1', A_2', A_3', and A_4' the points where the circumsphere of $A_1A_2A_3A_4$ intersects GA_1, GA_2, GA_3 and GA_4, respectively. Prove that

$$GA_1 \cdot GA_2 \cdot GA_3 \cdot GA_4 \le GA_1' \cdot GA_2' \cdot GA_3' \cdot GA_4'$$

and

$$\frac{1}{GA_1'} + \frac{1}{GA_2'} + \frac{1}{GA_3'} + \frac{1}{GA_4'} \le \frac{1}{GA_1} + \frac{1}{GA_2} + \frac{1}{GA_3} + \frac{1}{GA_4}.$$

The product Rewrite as

$$\sqrt[4]{\prod_k GA_k} \le p.$$

Recalling from **SL 1973/1** that $\sum_k GA_k^2 = 4p$, this is just QM-GM applied to to the 4 values GA_k^2.

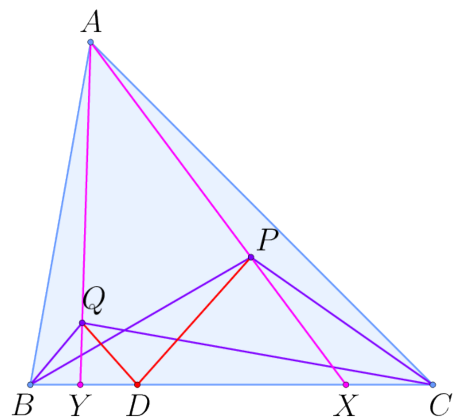
The sum Rewrite as

$$\sum_k GA_k \le p \sum_k \frac{1}{GA_k} = \frac{1}{4} \left(\sum_k GA_k^2 \right) \left(\sum_k \frac{1}{GA_k} \right),$$

which is just rearrangement inequality, since $[a^2, b^2, c^2, d^2]$ and $[\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}]$ are in opposite orders for any positive reals a, b, c, d.

🌲 1.7 [3♣] 10USATST7

In triangle ABC, let P and Q be two interior points such that $\angle ABP = \angle QBC$ and $\angle ACP = \angle QCB$. Point D lies on segment BC. Prove that $\angle APB + \angle DPC = 180^\circ$ if and only if $\angle AQC + \angle DQB = 180^\circ$.



Solution 1, by trig bash Define D_p, D_q as the respective points D satisfying the angle conditions, and we wish to show they coincide (by $\frac{BD_p}{D_pC} = \frac{BD_q}{D_qC}$). Reinterpret the angle condition as "$\overline{AP}$, $\overline{D_pP}$ isogonal wrt $\angle BPC$" and the like. Let $X = \overline{AP} \cap \overline{BC}$, $Y = \overline{AQ} \cap \overline{BC}$, so that

$$\frac{BD_p}{D_pC} \overset{\text{isogonals}}{=} \frac{(BP/PC)^2}{BX/XC} = \frac{BP^2/CP^2}{[ABP]/[ACP]} = \frac{BP/(BA \sin \angle ABP)}{CP/(CA \sin \angle ACP)}$$

and similarly for BD_q/D_qC, reducing the desired equality to

$$\frac{BP/CP}{\sin \angle ABP / \sin \angle ACP} = \frac{BQ/CQ}{\sin \angle ABQ / \sin \angle ACQ}$$

which is just law of sines:

$$\frac{BP/CP}{BQ/CQ} = \frac{\sin \angle PCB / \sin \angle PBC}{\sin \angle QCB / \sin \angle QBC} \overset{\text{isogonals}}{=} \frac{\sin \angle ACP / \sin \angle ABP}{\sin \angle ACQ / \sin \angle ABQ}.$$

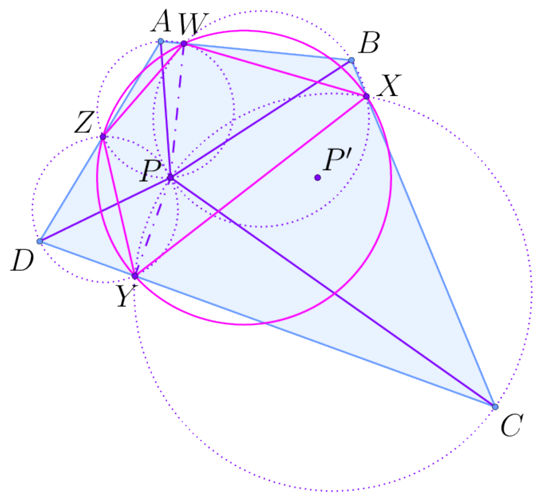
Solution 2, by synthetics We prove the more general lemma (the problem follows from applying it to A and quadrilateral $BPCQ$):

> **Lemma** (key to IMO 2018/6) **–** A point P has an isogonal conjugate wrt quadrilateral $ABCD$ iff $\measuredangle DPA + \measuredangle BPC = 0^\circ$.

We'll prove one direction; the other follows from this and phantom points. We require the following sublemma:

> **Sublemma –** A point P has an isogonal conjugate P' wrt a (not necessarily convex) polygon $\mathcal{A}$ iff its pedal polygon wrt $\mathcal{A}$ is cyclic. Also, the pedal circle (if it exists) has center $\frac{1}{2}(P + P')$.

Proof. Proceed by induction on sides, adding them in one at a time. The key observation is that by isogonals there exists an in-ellipse with foci P, P'. Then, let $P_1, \dots, P_n$ be the reflections of P in the sides of $\mathcal{A}$, so that the $P'P_k$ are all equal, by definition of ellipse. Then the pedal circle is found by applying a homothety with scale $\frac{1}{2}$ at P to the circle through P' through the P_k.
(The converse is omitted and is some ugly phantom points.) $\square$



Proof of key 18IMO6 lemma (mostly by Eric Shen) This is just an angle chase: clearly, $AWPZ$ and cyclic variants are cyclic so that

$$\begin{aligned} \measuredangle APB + \measuredangle CPD &= \measuredangle APW + \measuredangle WPB + \measuredangle CPY + \measuredangle YPD \\ &= \measuredangle AZW + \measuredangle WXB + \measuredangle CXY + \measuredangle YZD = \measuredangle WXY + \measuredangle YZW = 0^\circ. \quad \square \end{aligned}$$

> *Remark.* Isogonal conjugates can be thought of as wrt a set of lines instead of a polygon…

🌲 1.8 [9♣] 96IMO5

Let $ABCDEF$ be a hexagon whose opposite sides are parallel and denote its perimeter by p.Let R_A, R_C, R_E denote the circumradius of triangles FAB, BCD, DEF, respectively. Prove that

$$R_A + R_C + R_E \geq \frac{p}{2}.$$

(Official solution?) Obviously, opposite angles are equal. The soul of the problem derives from:

> **Claim –** $BF \geq AB \sin B + AF \sin F = AB \sin E + AF \sin C$.

Proof. $BF \geq \operatorname{dist}(\overline{BC}, \overline{EF}) = \operatorname{dist}(A, \overline{BC}) + \operatorname{dist}(A, \overline{EF}) = AB \sin B + AF \sin F = AB \sin E + AF \sin C.$ □

Similarly $BF \geq DC \sin C + DE \sin E$, so summing these two variants and dividing by $\sin A$ gives

$$4R_a = \frac{2BF}{\sin A} \geq (CD + AF)\frac{\sin C}{\sin A} + (AB + DE)\frac{\sin E}{\sin A}.$$

Finally, we obtain the problem by cyclically summing the last equation (adding 3 things, not 6), then spamming AM-GM to cancel out the sine ratios.

(Equality holds whenever $\overline{BF} \perp \overline{BC}, \overline{EF}$ and cyclic variants, or in other words, $ABCDEF$ regular.)

> *Remark.* definitely one of the problems of all time

🌲 1.9 [3♣] 99IMO1

A set S of points from the space will be called completely symmetric if it has at least three elements and fulfills the condition that for every two distinct points A and B from S, the perpendicular bisector plane of the segment AB is a plane of symmetry for S. Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

First, we tackle the 2-dimensional case: for any set of points $\{A_1, \ldots, A_n\}$, since no two of the $\overline{A_1A_k}$ are parallel, the perpendicular bisectors of those segments are all distinct. Since the only planar n-point set with at least $n-1$ lines of symmetry is the regular n-gon, the 2-dimensional case is resolved. (Think of this in terms of rotational symmetry: reflecting in two such perp bisectors constitutes a rotation.)

Since reflection in any perpendicular bisector maps the set to itself, it also maps the centroid to itself, so the centroid is equidistant from all the vertices, and is thus also the circumcenter.

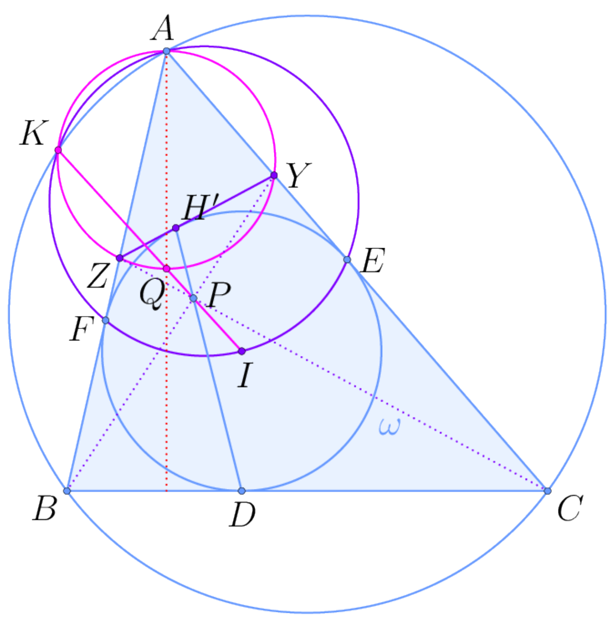
Applying the 2-dimensional case, the points (if not all coplanar) form a polyhedron whose faces are all regular polygons, so all edges have equal length... now consider a vertex V and 3 neighbors P_1, P_2, P_3 determining adjacent faces $[\ldots P_1VP_2\ldots]$ and $[\ldots P_2VP_3\ldots]$, and the reflection in perpendicular bisector of $\overline{P_1P_3}$, which must map P_2 to itself. It follows that $\angle P_1VP_2 = \angle P_2VP_3$; by this logic, all faces have the same number of sides, and the polyhedron is regular.
It's routine to show that the cube, dodecahedron, and icosahedron all fail, so we're left with the regular tetrahedron and octahedron.

> *Remark.* oops rip, apparently the IMO psc dropped the 3-dimensional case from the prob on the real contest...

2 [21♣] Classical

2.1 [3♣] 16HMMTT10

Let ABC be a triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F. Point P lies on $\overline{EF}$ such that $\overline{DP} \perp \overline{EF}$. Ray BP meets $\overline{AC}$ at Y and ray CP meets $\overline{AB}$ at Z. Point Q is selected on the circumcircle of $\triangle AYZ$ so that $\overline{AQ} \perp \overline{BC}$. Prove that P, I, Q are collinear.



Claim – $\overline{YZ}$ touches the incircle ω and is antiparallel to $\overline{BC}$ wrt $\angle A$.

Proof. Let $H' \in \omega$ be the reflection of the orthocenter of $\triangle DEF$ in $\overline{EF}$,, and Y', Z' the poles of $\overline{H'E}, \overline{H'F}$ wrt ω. By Brianchon, $\overline{BY'}, \overline{CZ'}, \overline{DH'}, \overline{EF}$ concur (at P), so $Y = Y', Z = Z'$. Now, a bisector of the angle formed by $\overline{YZ}, \overline{BC}$ is $\overline{AI}$ so the antiparallel part follows. □

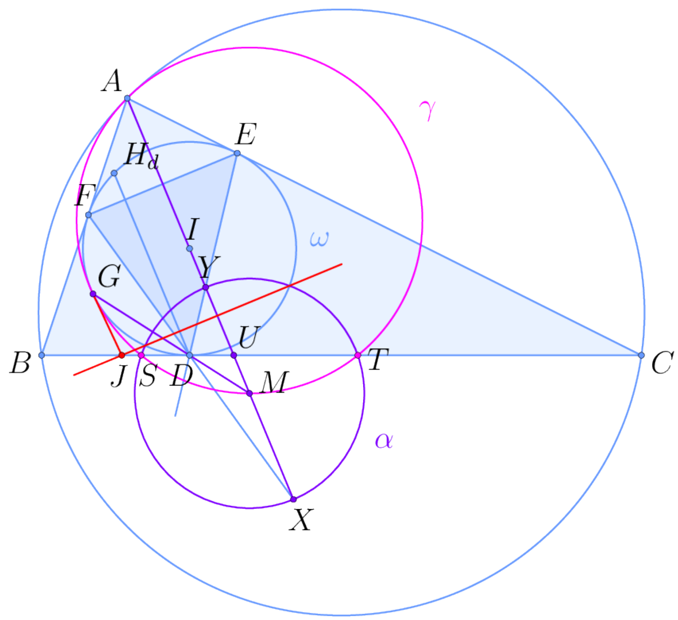
Let K be the Miquel point of $BCEF$, and $Q' = \overline{IPK} \cap (AYZ)$. Since $\angle Q'KA = 90°$, Q' is the antipode of A on (AYZ). Because $\overline{YZ}$ is antiparallel to $\overline{BC}$ wrt $\angle A$,$\overline{AQ'} \perp \overline{BC}$, so $Q = Q'$.

2.2 [5♣] 16ELMO6

Elmo is now learning olympiad geometry. In triangle ABC with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\angle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\angle XSY = \angle XTY = 90°$. Finally, let γ be the circumcircle of $\triangle AST$.

(a) Help Elmo show that γ is tangent to the circumcircle of $\triangle ABC$.

(b) Help Elmo show that γ is tangent to the incircle of $\triangle ABC$.



Let ω be the incircle of $\triangle ABC$, $U = \overline{AXY} \cap \overline{BC}$, and $M = \frac{X+Y}{2}$ be the center of $\alpha = (XYST)$.

Solution to part (a) It's equivalent to show $\overline{AS}, \overline{AT}$ isogonal. In fact, we assert X, Y are incenter and A-excenter of $\triangle AST$ in some order. Indeed, the given angle condition combined with

$$(AU;XY) \stackrel{D}{=} (\overline{AD} \cap \omega, D; E, F) = -1$$

means that $\overline{YS}, \overline{XS}$ are the bisectors of $\angle ASU$; similarly, $\overline{YT}, \overline{XT}$ are the bisectors of $\angle ATU$, so X, Y are indeed the claimed incenter and A-excenter of $\triangle AST$.

Solution to part (b) Let G be the D-orthocenter Miquel point in $\triangle DEF$, and $H_d \in \omega$ be the reflection of the orthocenter of $\triangle DEF$ in $\overline{EF}$, so that $(H_dG;EF) = -1$ and $\overline{DH_d} \parallel \overline{AXY}$. We assert G is the desired contact point.

> **Claim 1 –** α is orthogonal to ω.

Proof. Clearly, this is equivalent (in turn) to 'X, Y inverses wrt ω', $ID^2 = IX \cdot IY$ and thus '$\overline{ID}$ tangent to (XYD)'. The last statement is seen by

$$\measuredangle IDE = \measuredangle FDH_d = \measuredangle DXI. \qquad \square$$

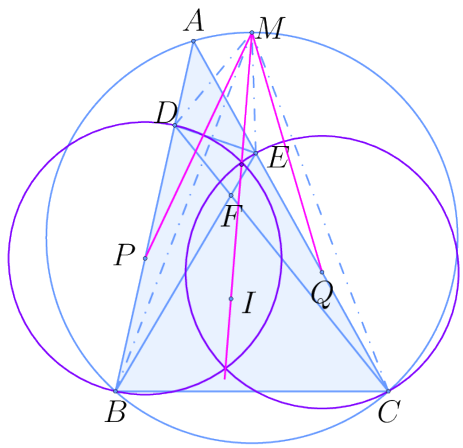
> **Claim 2 –** G, D, M collinear, so the former two are inverses in α.

Proof. $(\overline{GD} \cap \overline{XY}, \infty XY; X, Y) \overset{D}{=} (H_d G; EF) = -1 \Rightarrow M \in \overline{GD}$. $\square$

Since γ is sent to $\overline{BSTC}$ by inversion in α, we must have $G \in \gamma$.
Finally, let J be the pole of $\overline{DG}$ wrt ω, so that we want $\overline{JG}$ also tangent to γ at G, which is equivalent to J being the radical center of α, γ, ω.
As $\alpha \perp \omega$, their radical axis is the polar of M wrt ω. J is on this line by la Hire applied to the fact that $M \in \overline{GD}$, the polar of J wrt ω, so it's indeed the needed radical center, concluding the proof.

🌲 2.3 [5♣] 14EGMO2

Let D and E be points in the interiors of sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$. Let the lines CD and BE meet at F. Prove that the incenter I of triangle ABC, the orthocenter H of triangle DEF and the midpoint M of the arc BAC of the circumcircle of triangle ABC are collinear.

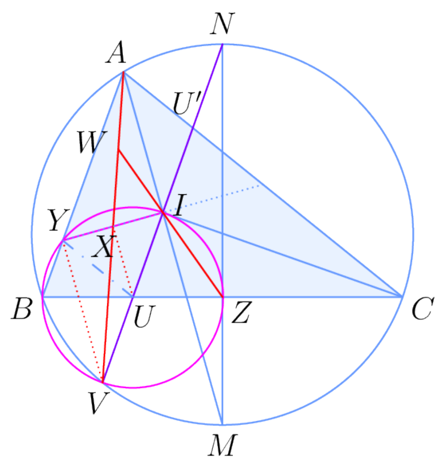


Since $\overline{BI} \perp \overline{CD}$ and $\overline{CI} \perp \overline{BE}$, I is the orthocenter of $\triangle BCF$. By Gauss-Bodenmiller on $BCDE$, the problem reduces to showing that $\operatorname{Pow}(I, (BD)) = \operatorname{Pow}(I, (CE))$.

Let P, Q be the midpoints of $\overline{BD}, \overline{CE}$. As $(BD), (CE)$ have the same radius $(BC/2)$, this is in turn equivalent to $MP = MQ$. Check that $\triangle MBD \cong MCE$, which implies that their medians MP, MQ are indeed equal.

2.4 [3♣] 14SLG7

Let ABC be a triangle with circumcircle Ω and incenter I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V, respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y. Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X and Y are collinear, then the points I, W, and Z are also collinear.



Let M, N be the midpoints of $\widehat{BVC}, \widehat{BAC}$, and $U' \in \overline{AC}$ be the reflection of U in I.

> **Claim 1 –** $\triangle YVU$ and $\triangle AIU'$ are homothetic, so $\overline{YU} \parallel \overline{AC}$.

Proof. It's given that two pairs of corresponding sides are parallel, so it suffices to show that the triangles are similar. By the similar triangles induced parallel lines AI, UX, VY, we obtain

$$\frac{UX}{VY} = \frac{IU}{IV}, \frac{UX}{AI} = \frac{VU}{VI} \Rightarrow \frac{VY}{AI} = \frac{VU}{UI} = \frac{VU}{IU'},$$

so the triangles are similar by SAS. □

Since a reflection in I swaps $(\overline{UY}, \overline{AC})$, $2Y - I \in \overline{AC}$. As there is only one point with this characteristic (namely, the contact point of the A-mixtilinear incircle with $\overline{AB}$), we have $\overline{IY} \perp \overline{AI}, \overline{VY}$.

> **Claim 2 –** B, I, V, Y lie on a circle tangent to $\overline{CI}$, and I, N, V collinear.

Proof. The concyclicity / tangency is an angle chase; check that

$$\measuredangle BYI = 90^\circ + \measuredangle BAI = \measuredangle BIC \text{ and } \measuredangle VYI = 90^\circ = \measuredangle VIC.$$

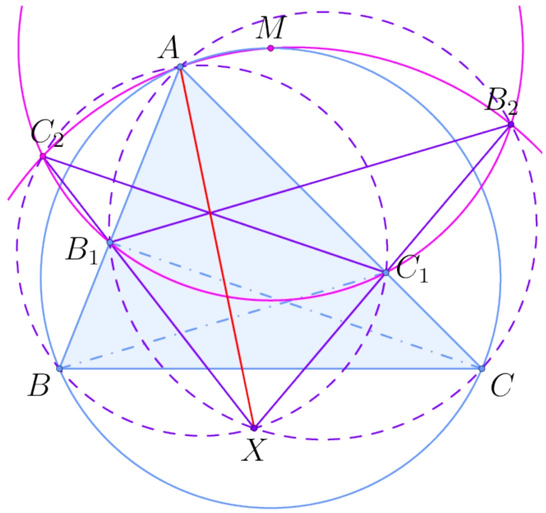
If $N' = \overline{IV} \cap (ABC)$, then $\overline{N'A} \parallel \overline{IY}$ by Reim, so $N = N'$, proving the collinearity. □

Now $WX = WA = WI$ due to $\angle AIX = 90^\circ$. Meanwhile, $MB^2 = MC^2 = MI^2 = MZ \cdot MN$, so $\triangle MIZ \overset{-}{\sim} \triangle MNI$, whence

$$\measuredangle WIA = -\measuredangle VAM = -\measuredangle INM = \measuredangle ZIM \Rightarrow I, W, Z \text{ collinear.}$$

2.5 [5♣] 20GOTEEM5

Let ABC be a scalene triangle and let B_1 and C_1 be variable points on sides $\overline{BA}$ and $\overline{CA}$, respectively, such that $BB_1 = CC_1$. Let $B_2 \neq B_1$ denote the point on $\odot(ACB_1)$ such that BC_1 is parallel to B_1B_2, and let $C_2 \neq C_1$ denote the point on $\odot(ABC_1)$ such that CB_1 is parallel to C_1C_2. Prove that as B_1, C_1 vary, the circumcircle of $\triangle AB_2C_2$ passes through a fixed point, other than A.



Let X be the Miquel point of BB_1CC_1, and M be the midpoint of $\widehat{BAC}$ on (ABC), aka the Miquel point of BB_1C_1C.

Claim 1 – X lies on the bisector of $\angle A$.

Proof. The spiral similarity at X sending $B, B_1 \to C_1, C$ is a rotation due to $BB_1 = CC_1$; as a result, $XB_1 = XC$ and $\angle B_1AX = \angle XAC$. □

Claim 2 – X, B_1, C_2 are collinear. Analogously, so are X, B_2, C_1.

Proof. Let $C_2' = \overline{XB_1} \cap (ABC_1)$ $(\neq X)$; check that $\measuredangle XC_2C_1 = \measuredangle XAC = \measuredangle XB_1C \implies \overline{C_1C_2'} \parallel \overline{B_1C}$. □

We assert M is the desired fixed point. To see this, we in fact have:

Claim 3 – M, A are the circumcenter and Miquel point of $B_1B_2C_1C_2$.

Proof. By spiral similarity again, $MB_1 = MC_1$; the first point follows from

$$2\angle B_1C_2C_1 \overset{\text{claim 1}}{=} \angle A = \angle B_1MC_1$$

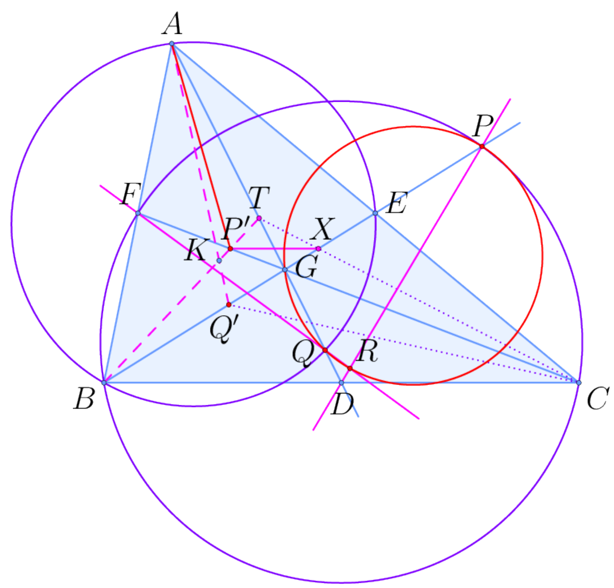
while the second from $A = (XB_1B_2) \cap (XC_1C_2)$ $(\neq X)$. □

Remark. why does this look like Pascal fsr

3 [35♣] Romantic

3.1 [5♣] 12RMM2

Given a non-isosceles triangle ABC, let D, E, and F denote the midpoints of the sides BC, CA, and AB respectively. The circle (BCF) and the line BE meet again at P, and the circle (ABE) and the line AD meet again at Q. Finally, $R = \overline{DP} \cap \overline{FQ}$. Prove that the centroid G lies on (PQR).



Solution partially by Anant Mudgal. Clearly, suffices to prove $\measuredangle AQF = \measuredangle BPD$. Let K be the symmedian point of $\triangle ABC$.

Let i_a denote the inversion at A with power $\frac{1}{2}AB \cdot AC$ composed with a reflection in $\angle BAC$, swapping (B, E) and (C, F), and its cyclic variants. Let $P' = i_b(P)$ and $Q' = i_a(Q)$. By design, the former is the intersection of the B-symmedian $\overline{BK}$ and the C-median $\overline{CGF}$, while the latter is that of the A-symmedian $\overline{AK}$ and B-median $\overline{BG}$. Finally, let $T = \overline{BK} \cap \overline{AM}$ be the isogonal conjugate of Q', and $X = \overline{CT} \cap \overline{BG}$.
First, we have

$$\measuredangle AQF \overset{i_a}{=} \measuredangle ACQ' \text{ and } \measuredangle BPD \overset{i_b}{=} \measuredangle BAP'.$$

Next, observe that because $\overline{TG}$ bisects $\overline{BC}$ and $P' = \overline{CG} \cap \overline{BT}$, $X = \overline{BG} \cap \overline{CT}$, we have (by Ceva-Menelaus, say) $\overline{XP'} \parallel \overline{BC}$. (1)

> **Claim –** $\triangle BP'A \overset{-}{\sim} BXC$, so $\measuredangle BAP' = \measuredangle XCB$.

Proof. Since isogonality gives $\measuredangle ABP' = \measuredangle XBC$, suffices to prove $BP'/BX = BA/BC$. Indeed, law of sines gives

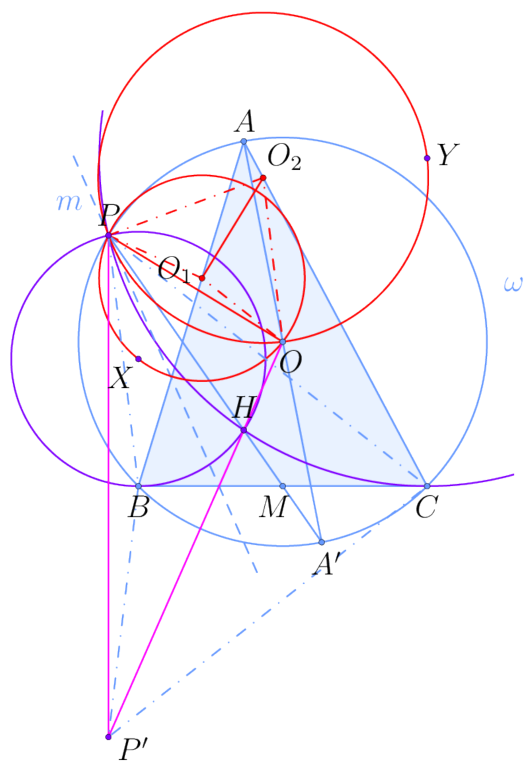
$$\frac{BP'}{BX} = \frac{\sin \angle BXP'}{\sin \angle BP'X} \overset{(1)}{=} \frac{\sin \angle XBC}{\sin \angle P'BC} \overset{\text{isogonal}}{=} \frac{\sin \angle EBC}{\sin \angle ABE} \overset{\text{ratio lemma}}{=} \frac{BA}{BC}. \qquad \square$$

Since T, Q' are isogonal conjugates, we may finally link our angle equalities and complete the proof:

$$\measuredangle AQF = \measuredangle ACQ' \overset{\text{isogonal}}{=} \measuredangle TCB \overset{\text{claim}}{=} \measuredangle BAP' = \measuredangle BPD.$$

3.2 [3♣] 19Mex6

Let ABC be a triangle such that $\angle BAC = 45°$. Let H and O be the orthocenter and circumcenter of ABC, respectively. Let ω be the circumcircle of ABC and P the point on ω such that the circumcircle of PBH is tangent to BC. Let X and Y be the circumcenters of PHB and PHC respectively. Let O_1, O_2 be the circumcenters of PXO and PYO respectively. Prove that O_1 and O_2 lie on AB and AC, respectively.



Let M be the midpoint of $\overline{BC}$ and A' the antipode of A on ω. By power of a point at the midpoint of $\overline{BC}$, P is the inverse of H wrt (BC) aka the A-queue point.

> **Claim –** There exist spiral similarities s_b, s_c at P mapping $O, X \xrightarrow{s_b} C, B$ and $O, Y \xrightarrow{s_c} B, C$. (sic)

Proof. Suffices to prove one of the two. Check that $PO = OC$, $PX = PB$ and

$$\measuredangle PXB = 2\measuredangle PBC = \measuredangle POC \implies \triangle POC \overset{+}{\sim} \triangle PXB. \qquad \square$$

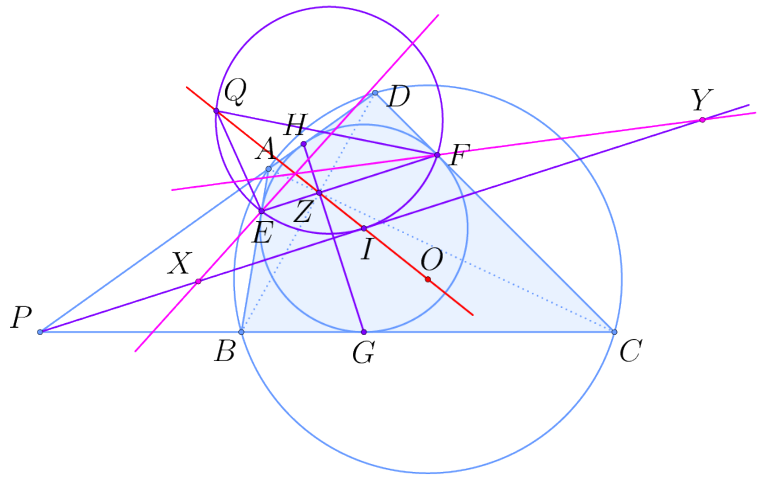
Clearly, this also means that $O_1 \xrightarrow{s_b} O$ and $O_2 \xrightarrow{s_c} O$. Let P' be the reflection of P in $\overline{BC}$, and consider the spiral similarity $t = s_b \circ s_c$ at P mapping $O_1, O_2 \longrightarrow B, C$. Since O, P' are the respective reflections of P in $\overline{O_1O_2}, \overline{BC}$, we also have $O \xrightarrow{t} P'$.

To finish the problem, let m denote the P-symmedian in $\triangle PBC$, which is just the reflection of $\overline{OHP'}$ in $\overline{BC}$ since $A = 45°$ means that $\overline{OB}, \overline{OC}$ touch $(BHCP')$. Then verify that

$$\measuredangle PBO_1 \overset{t}{=} \measuredangle PP'O \overset{\text{reflect}}{=} -\measuredangle(\overline{PP'}, m) \overset{\text{isogonal}}{=} \measuredangle OPM = \measuredangle PA'A = \measuredangle PBA \implies O_1 \in \overline{AB}.$$

3.3 [3♣] 20MOP7X

Convex quadrilateral $ABCD$ is inscribed in a circle Ω centered at O, and circumscribed about a circle ω with center I. Let ω be tangent to side AB at E and side CD at F. Suppose the exterior angle bisectors of $\angle DAB$ and $\angle ABC$ meet at a point X, while the exterior angle bisectors of $\angle BCD$ and $\angle CDA$ meet at a point Y. Prove that lines XE, YF, and OI are concurrent.



Let the incircle touch $\overline{BC}, \overline{AD}$ at G, H, and $Q = \overline{AD} \cap \overline{BC}$ (possibly at infinity). It's well-known that $\overline{AC}$, $\overline{BD}$, $\overline{OI}$, $\overline{EF}$,$\overline{GH}$ concur at a point Z.

We'll prove that (E, Z, F), (X, I, Y) are homothetic; since $\overline{XY}, \overline{EF} \perp \overline{GH}$, sufficient to show $EZ/ZF = XI/IY$.

To tackle the first ratio, let Q be the (known to be common) Miquel point of $ABCD$, $EGFH$, aka the inverse of Z in ω, and s (with scale factor r) the spiral similarity at Q mapping $A, E, B \to D, F, C$. Since $QEIF$ cyclic and $IE = IF$, we obtain

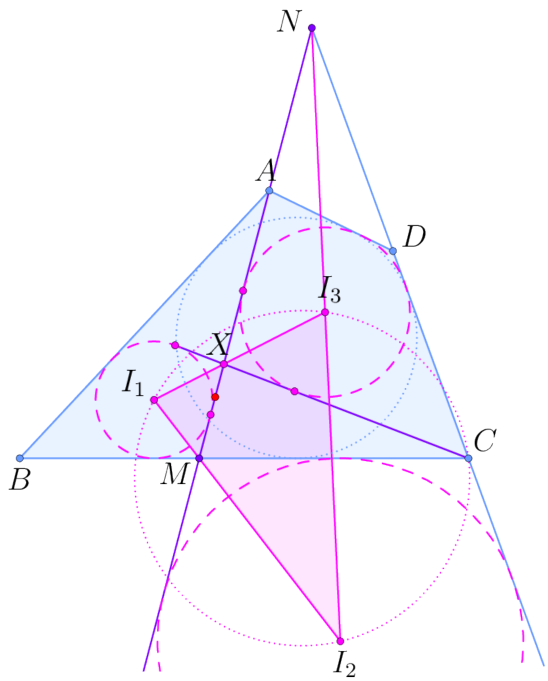
$$\frac{EZ}{ZF} \overset{\angle\text{ bisector thm.}}{=} \frac{EQ}{QF} \overset{s}{=} r.$$

The other length ratio is another computation:

$$\frac{XI}{IY} = \frac{IA \cdot IB/IE}{IC \cdot ID/IF} = \frac{2[AIB]/\sin\angle AIB}{[2[CID]/\sin\angle CID} = \frac{AB \cdot IE}{CD \cdot IF} = \frac{AB}{CD} \overset{s}{=} r.$$

🌲 3.4 [9♣] 09SLG8

Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2 and I_3 the incenters of $\triangle ABM$, $\triangle MNC$ and $\triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on g.



> *Note.* Actually, the incenters here may be excenters depending on configuration. Strictly speaking, the problem statement is false when ray MA meets ray BC, which occurs sometimes when M is too close to B.
>
> Fortunately, we can resolve this by defining the incenters as intersections of the bisectors of the angles formed by the sides of the quadrilateral and g, while stipulating that only I_2 is outside $ABCD$. The associated incircles are then defined as those centered at the I_k tangent to g.

Let ω_k be the incircle / excircle associated with I_k. We'll show that the reflections of g over the sides of $\triangle I_1I_2I_3$ are concurrent. Two of these reflections are just $\overline{CMB}$, $\overline{CND}$, so we're left to prove that C lies on the third reflection as well:

> **Claim –** C lies on the common internal tangent of ω_1, ω_3 distinct from $\overline{AMN}$.

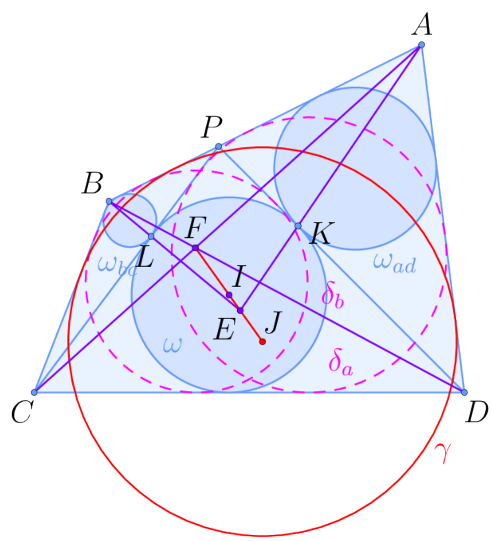
Proof. Construct $X \in \overline{AM}$ $(\neq M)$ so that $\overline{CX}$ touches ω_1. By Pitot on $ABCD$, $ABCX$, we have $AX - XC = AD - DC$, so $AXCD$ is cyclic by Pitot converse– in other words, $\overline{CX}$ touches ω_3 as well. □

As $g = \overline{AX}$ and $\overline{CX}$ are the common internal tangents, they're reflections in the line of centers, $\overline{I_1I_3}$, whence C lies on the third reflection as well.

3.5 [5♣] 07SLG8

Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD. Suppose that ω is tangent to the incircles of $\triangle ADP$ and $\triangle BPC$ at points K and L, respectively. Let $E = \overline{AC} \cap \overline{BD}$ and $F = \overline{AK} \cap \overline{BL}$. Prove that line EF passes through the center of ω.

> ...is literal Monge spam
>
> DottedCaculator



Let ω_{ad}, ω_{bc} be the respective incircles of triangles ADP, BCP, and construct circle γ tangent to segment AB and rays BC, AD. Since the incircle ω of $\triangle CPD$ touches the incircles ω_{ad} of $\triangle ADP$, ω_{bc} of $\triangle BCP$, $APCD$, $BPDC$ are both tangential, with incircles we'll call δ_a, δ_b respectively.

By Monge on $(\gamma, \omega, \delta_a)$, $(\gamma, \omega, \delta_b)$, $E = \overline{AC} \cap \overline{BD}$ is the exsimilicenter of γ, ω, while by Monge on $(\gamma, \omega, \omega_{ad})$, $(\gamma, \omega, \omega_{bc})$, $F = \overline{AK} \cap \overline{BL}$ is the insimilicenter of γ, ω.

As a result, E, F lie on the line of centers of γ, ω.

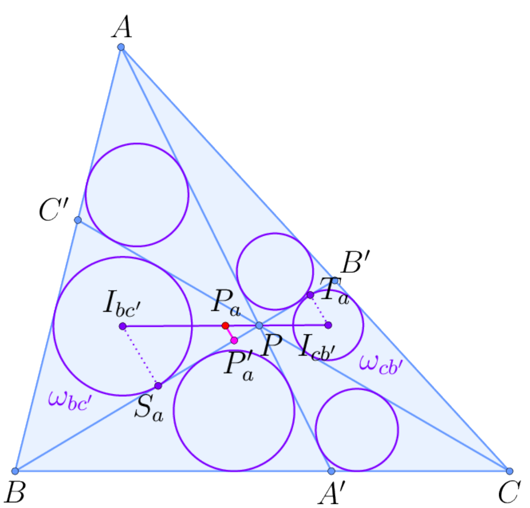
> *Remark.* if only this would help me solve 21ELMO6...

3.6 [5♣] 22USEMO3

Point P lies in the interior of a triangle ABC. Lines AP, BP, and CP meet the opposite sides of triangle ABC at points A', B', and C', respectively. Let P_A be the midpoint of the segment joining the incenters of triangles BPC' and CPB', and define points P_B and P_C analogously. Show that if

$$AB' + BC' + CA' = AC' + BA' + CB',$$

then points P, P_A, P_B, and P_C are concyclic.



Definition (directed length convention)
For any point X on lines PA, PB, or PC, PX is defined to be positive iff it's on the same side of P as A, B, or C.

Let $\omega_{bc'}$ be the incircle of $\triangle PBC'$ with center $I_{bc'}$ and 5 other variants. Let S_a, T_a, P'_a be the projections of $I_{bc'}$, $I_{cb'}$, P_a to $\overline{BB'}$.

Claim – $\sum_{\text{cyc}} PP'_a = 0$.

Proof. Observe that $PS_a = PB + PC' - BC'$ $PT_a = PC + PB' - CB'$, and cyclic variants, whence

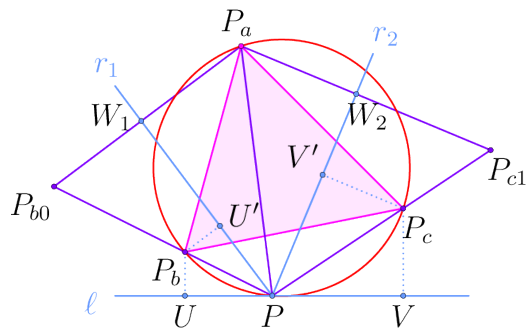
$$PP'_a = \frac{1}{2}\left(PB - PC + CB' - BC' + PC' - PB'\right),$$

which cyclically sums to zero by the given length condition. □

After this, A, B, C, A', B', C' are largely irrelevant. By symmetry, we're left with (possibly with points renamed):

Removing triangle ABC

Line ℓ and rays r_1, r_2 pass through point P, with the rays on the same side of ℓ. Points P_b, P_c lie on the bisectors of $\angle(\ell, r_1)$, $\angle(\ell, r_2$, with projections U, V onto ℓ, respectively. Let U', V' be the respective projections of P_b, P_c onto r_1, r_2, and P_a lies on the internal bisector of $\angle(r_1, r_2)$ with projections W_k onto r_k (for $k = 1, 2$), with $PW_1 = PW_2 = PU + PV$. Then $PP_aP_bP_c$ is concyclic.



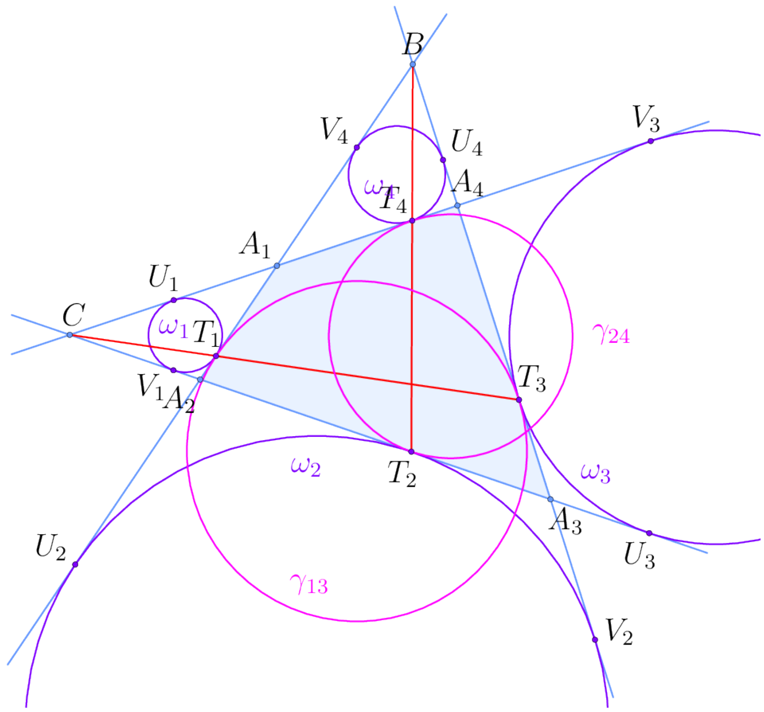
Let $P_{b0} = \overline{P_a W_1} \cap r_1$ and $P_{c1} = \overline{P_a W_2} \cap r_2$. By angle chase, $\triangle P_a P_{b0} \overset{+}{\sim} \triangle P_a P P_{c1}$, while by length chase, $P_{b0}P_b / P_b P = PP_c / P_c P_{c1}$, lengths directed normally. As a result, there exists a spiral similarity s at P_a mapping $P_{b0}, P_b, P \to P, P_c, P_{c1}$.

Finish by

$$\measuredangle P_b P_a P_c = \measuredangle(s) = \measuredangle P_{b0} P_a P = \measuredangle P_a P_{b0} P + \measuredangle P_{b0} P P_a = \measuredangle P_a P P_{c1} + \measuredangle P_a P_{b0} P = \measuredangle P_b P P_c.$$

🌲 3.7 [5♣] 10RMM3

Let $A_1A_2A_3A_4$ be a quadrilateral with no pair of parallel sides. For each $i = 1, 2, 3, 4$, define ω_i to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i$, A_iA_{i+1} and $A_{i+1}A_{i+2}$ (indices considered modulo 4). Let T_i be the point of tangency of ω_i with the side A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4 and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1 and T_1T_3 are concurrent.



Upon seeing only tangencies and collinearity/concurrence, the most reasonable first thought is definitely Monge…let $B = \overline{A_1A_2} \cap \overline{A_3A_4}$ and $C = \overline{A_1A_4} \cap \overline{A_2A_3}$.

Claim – B, T_2, T_4 are collinear iff $CT_2 = CT_4$.

Proof. If $CT_2 = CT_4$, then we may construct circle γ_{24} tangent to $\overline{CT_k}$ at T_k for $k = 2, 4$; applying Monge to it and ω_2, ω_4 yields the needed collinearity.
Conversely, assume that B, T_2, T_4 are indeed collinear. This time, construct γ_{24} as the circle tangent to $\overline{A_4A_1}$ at T_4 and $\overline{A_2A_3}$ at some unknown point. Applying Monge to the same three circles means that the insimilicenter of γ_{24} and ω_2 is $\overline{BT_4} \cap \overline{A_2A_3} = T_2$. Since this lies on one of the circles, it must lie on the other– in other words, those two circles touch $\overline{A_2A_3}$ at the same point. □

This reduces the problem to a length chase: $BT_1 = BT_3 \iff CT_2 = CT_4$. By symmetry, sufficient to prove one direction. For all k let ω_k touch $\overline{A_{k-1}A_k}$ and $\overline{A_{k+1}A_{k+2}}$ at U_k, V_k respectively, and assume $BT_1 = BT_3$ so that $T_3U_4 = U_1V_4$. As $(\overline{T_3U_4}, \overline{T_4V_3})$, $(\overline{T_4U_1}, \overline{T_1V_4})$ are the common internal tangents, this means $T_4U_1 = T_4V_3$ or $U_1T_4 = \frac{U_1V_3}{2}$. Similarly $V_1T_2 = \frac{V_1U_3}{2}$.
But $U_1V_3 = U_3V_1$ as the common external tangents to ω_1, ω_3, so we obtain

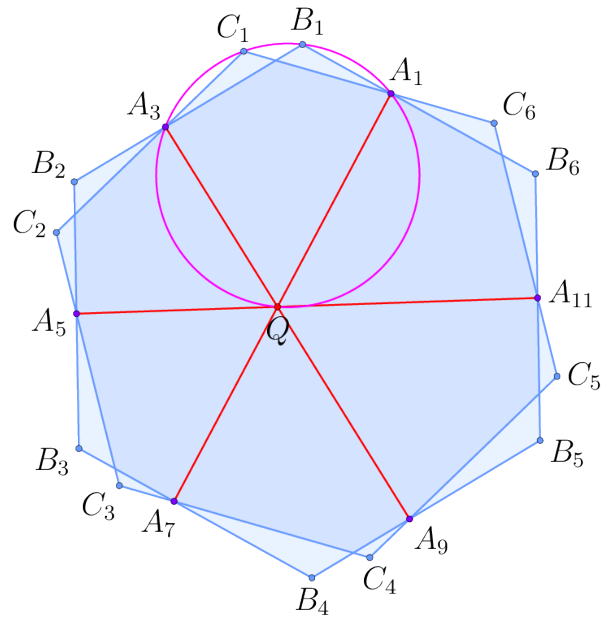
$$U_1T_4 = \frac{U_1V_3}{2} = \frac{U_3V_1}{2} = V_1T_2 \implies CT_2 = CT_4.$$

4 [14♣] Contemporary

4.1 [2♣] 20USEMO5

The sides of a convex 200-gon $A_1A_2 \dots A_{200}$ are colored red and blue in an alternating fashion. Suppose the extensions of the red sides determine a regular 100-gon, as do the extensions of the blue sides.

Prove that the 50 diagonals $\overline{A_1A_{101}}$,$\overline{A_3A_{103}}$, ..., $\overline{A_{99}A_{199}}$ are concurrent.



Replace 100 with $2n$. Then, label the $2n$-gons formed by extending alternate sides $B_1 \dots B_{2n}$ and $C_1 \dots C_{2n}$ (as shown for $n = 3$).

I claim the lines A_kA_{k+2n} (k ranges over all odds, so n lines) pass through the center of the spiral similarity mapping $B_k \to C_k \quad \forall k$. We'll call this point Q.

Clearly, $A_kB_kC_kA_{k+2}$ is cyclic . By definition of Miquel point, this circle also contains Q; in other words $\angle A_kQA_{k+2} = \frac{\pi}{n} \quad \forall k$. Finally

$$\angle A_kQA_{k+2n} = n\measuredangle = 180^\circ$$

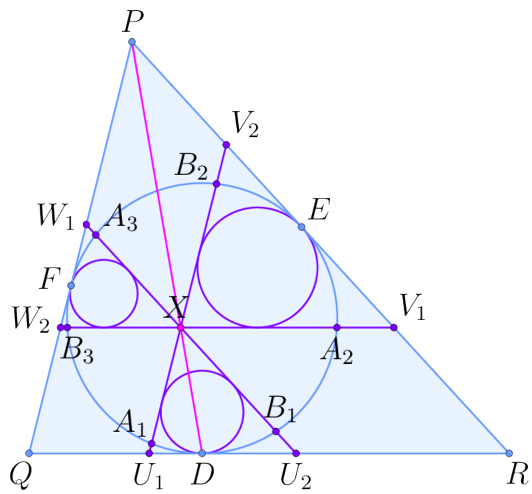
establishes the collinearity.

4.2 [5♣] 16RMM5

A convex hexagon $A_1B_1A_2B_2A_3B_3$ is inscribed in a circle Ω with radius R. The diagonals A_1B_2, A_2B_3, A_3B_1 are concurrent in X. For each $i = 1, 2, 3$ let ω_i tangent to the segments XA_i and XB_i and tangent to the arc A_iB_i of Ω that does not contain the other vertices of the hexagon; let r_i the radius of ω_i.

(a) Prove that $R \geq r_1 + r_2 + r_3$.

(b) If $R = r_1 + r_2 + r_3$, prove that the six points of tangency of the circumferences ω_i with the diagonals A_1B_2, A_2B_3, A_3B_1 are concyclic.



This problem is "troll" in a way. Draw triangle PQR with incircle Ω and sides parallel to the segments A_kB_{k+1}, as shown. Then, r_a is at most the inradius of $\triangle XP_1P_2$ as the circular "sector" is contained in that triangle. Since sum of perimeter/etc of triangles XP_1P_2 and cyclic variants sum to the that of $\triangle PQR$, so do their inradii. As a result $r_a + r_b + r_c \leq R$.

Now to the equality case: when equality holds, we must have ω_1, Ω, and $\overline{YZ}$ tangent at the same point. Let these points be D, E, F, also shown.

Claim – X is the Gergonne point of $\triangle PQR$.

Proof. The homothety at D sending $\omega_1 \to \Omega$ also sends $\triangle XP_1P_2 \to \triangle PQR$, so D, P, X collinear. Similarly so are (E, Q, X), (F, R, X), as needed. □

By the barycentric coordinates of X (where $a = YZ$ and cyclic variants, and $s = \frac{a+b+c}{2}$ as usual) , say, we may compute

$$\frac{DX}{DA} = \frac{1/(s-a)}{\sum_{\text{cyc}} 1/(s-a)}$$

The bolded segment is then (by similarity)

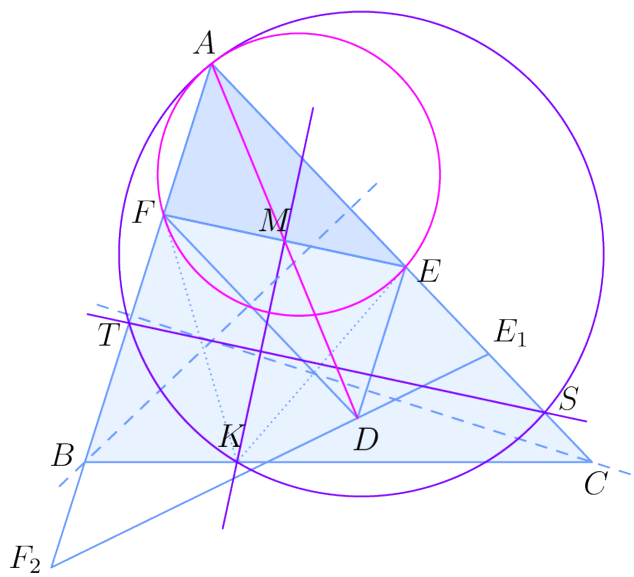
$$\frac{1}{\sum_{\text{cyc}} 1/(s-a)}, \text{ a symmetric quantity.}$$

4.3 [5♣] 14SLG6

Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of $\overline{EF}$. Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T, respectively. We call the pair (E, F) **interesting** if the quadrilateral $KSAT$ is cyclic.

Suppose that the pairs (E_1, F_1) and (E_2, F_2) are interesting. Prove that

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC}.$$



Let E_1, F_2 be the reflections of A in the B- and C-altitudes, respectively. It suffices to prove that for any interesting (E, F) we have $E_1E/EA = AF/FF_2$.

> **Claim 1 –** $\overline{AK}$ is a symmedian of $\triangle AST$ and thus $\triangle AEF$ as well.

Proof. Consider the orthocenter of $\triangle AST$, which is concyclic with S, T, M as M is the reflection of K in $\overline{ST}$. But since $\overline{AM}$ is a median of $\triangle AEF, AST$, M must in fact be the A-Humpty point in $\triangle AST$, so its reflection K in $\overline{ST}$ satisfies $(AK; ST) = -1$. □

Because K is also on the perpendicular bisector of $\overline{EF}$, it's the intersection of the tangents to (AEF) at E, F.

Now consider the transformation r consisting of a reflection in the bisector of $\angle A$ composed with a homothety with scale factor $2\cos A$, which maps $B, C \longrightarrow E_1, F_2$ as well as $K \longrightarrow N = E + F - A$. As $K \in \overline{BC} \xrightarrow{r} N \in \overline{E_1F_2}$, we are done by parallel lines:

$$\frac{E_1E}{EA} = \frac{E_1N}{NF_2} = \frac{AF}{FF_2}.$$

> *Remark.* wow, considering degen cases is actually op

🌲 4.4 [2♣] 08SLG6

Let $ABCD$ be a given convex quadrilateral. Prove that there exists a point P inside the quadrilateral such that

$$\begin{aligned} \angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA \\ = \angle PDA + \angle PCB = 90^\circ \end{aligned}$$

if and only if the diagonals AC and BD are perpendicular.

Recall the result from earlier:

> **Lemma –** A point P has an isogonal conjugate P' wrt a (not necessarily convex) polygon $\mathcal{A}$ iff its pedal polygon wrt $\mathcal{A}$ is cyclic. Also, the pedal circle (if it exists) has center $\frac{1}{2}(P + P')$.

The key observation is that the angle condition means that P has an isogonal conjugate P' with $\angle AP'B = 90^\circ$ and variants. Clearly, this point exists iff $\overline{AC} \perp \overline{BD}$, and is unique if so.