

# Desargues and his meme

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We are lazy, so here is a problem compilation, one of the shortest handouts of all time.

## Very brief blurb

Desargues was a sinner, a mastermind spinner,  
Mathematical genius, his thoughts got much bigger.  
With lines and planes, he played his wicked game,  
Geometry was his realm, and he conquered the terrain.

He twisted and turned, in his mathematical maze,  
Proving the theorems that left others amazed.  
His mind was a canvas, where concepts would collide,  
Creating new dimensions, in which truths would reside.

From perspective, he derived his duality,  
Projective geometry, his art with clarity.  
He saw the world in a different light,  
Unveiling hidden symmetries, day and night.

Desargues danced with angels and demons alike,  
Challenging the norms, never afraid to strike.  
His sins were his passion, his rebellion was clear,  
In a world of shapes and numbers, he had no fear.

So raise a toast to Desargues, the sinner with a vision,  
Whose mathematical legacy defies all derision.  
For in his wickedness, he found the truth,  
And left us with a geometric marvel, in our youth.  
– ChatGPT '23

## 🌲 Acknowledgement

Eric Shen for teaching me this black magic and its very interesting applications. Thanks so much Eric!  
(Also, sorry about making you take RELMO...)

– Neal

## 1 Opening examples

### Example 1 (OMMC Main 2023/24 by Tiger)

Define acute  $\triangle ABC$  with circumcenter  $O$ . The circumcircle of  $\triangle ABO$  meets segment  $BC$  at  $D \neq B$ , segment  $AC$  at  $F \neq A$ , and the Euler line of  $\triangle ABC$  at  $P \neq O$ . The circumcircle of  $\triangle ACO$  meets segment  $BC$  at  $E \neq C$ . Let  $\overline{BC}$  and  $\overline{FP}$  intersect at  $X$ , with  $C$  between  $B$  and  $X$ . If  $BD = 13$ ,  $EC = 8$ , and  $CX = 27$ , find  $DE$ .

## 2 Problems

Approximately increasing difficulty...

**Problem 1 (USA TST 2004/4).** Let  $ABC$  be a triangle. Choose a point  $D$  in its interior. Let  $\omega_1$  be a circle passing through  $B$  and  $D$  and  $\omega_2$  be a circle passing through  $C$  and  $D$  so that the other point of intersection of the two circles lies on  $AD$ . Let  $\omega_1$  and  $\omega_2$  intersect side  $BC$  at  $E$  and  $F$ , respectively. Denote by  $X$  the intersection of  $DF$ ,  $AB$  and  $Y$  the intersection of  $DE$ ,  $AC$ . Show that  $XY \parallel BC$ .

**Problem 2 (CJMO 2021/1).** Let  $ABC$  be an acute triangle, and let the feet of the altitudes from  $A$ ,  $B$ ,  $C$  to  $\overline{BC}$ ,  $\overline{CA}$ ,  $\overline{AB}$  be  $D$ ,  $E$ ,  $F$ , respectively. Points  $X \neq F$  and  $Y \neq E$  lie on lines  $CF$  and  $BE$  respectively such that  $\angle XAD = \angle DAB$  and  $\angle YAD = \angle DAC$ . Prove that  $X$ ,  $D$ ,  $Y$  are collinear.

**Problem 3 (IGO 2018/I5).** \* Suppose that  $ABCD$  is a parallelogram such that  $\angle DAC = 90^\circ$ . Let  $H$  be the foot of perpendicular from  $A$  to  $DC$ , also let  $P$  be a point along the line  $AC$  such that the line  $PD$  is tangent to the circumcircle of the triangle  $ABD$ . Prove that  $\angle PBA = \angle DBH$ .

**Problem 4 (Serbia 2017/6).** Let  $k$  be the circumcircle of  $\triangle ABC$  and let  $k_a$  be  $A$ -excircle. Let the two common tangents of  $k$ ,  $k_a$  cut  $BC$  in  $P$ ,  $Q$ . Prove that  $\angle PAB = \angle CAQ$ .

**Problem 5 (Taiwan TST 2014/3/3).** Let  $ABC$  be a triangle with circumcircle  $\Gamma$  and let  $M$  be an arbitrary point on  $\Gamma$ . Suppose the tangents from  $M$  to the incircle of  $\triangle ABC$  intersect  $\overline{BC}$  at two distinct points  $X_1$  and  $X_2$ . Prove that the circumcircle of triangle  $MX_1X_2$  passes through the tangency point of the  $A$ -mixtilinear incircle with  $\Gamma$ .

**Problem 6 (USA TST 2018/5 (by Evan)).** Let  $ABCD$  be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at  $H$ . Denote by  $M$  and  $N$  the midpoints of  $\overline{BC}$  and  $\overline{CD}$ . Rays  $MH$  and  $NH$  meet  $\overline{AD}$  and  $\overline{AB}$  at  $S$  and  $T$ , respectively. Prove that there exists a point  $E$ , lying outside quadrilateral  $ABCD$ , such that

- ray  $EH$  bisects both angles  $\angle BES$ ,  $\angle TED$ , and
- $\angle BEN = \angle MED$ .

**Problem 7 (IMO 2019/2).** In triangle  $ABC$ , point  $A_1$  lies on side  $BC$  and point  $B_1$  lies on side  $AC$ . Let  $P$  and  $Q$  be points on segments  $AA_1$  and  $BB_1$ , respectively, such that  $PQ$  is parallel to  $AB$ . Let  $P_1$  be a point on line  $PB_1$ , such that  $B_1$  lies strictly between  $P$  and  $P_1$ , and  $\angle PP_1C = \angle BAC$ . Similarly, let  $Q_1$  be the point on line  $QA_1$ , such that  $A_1$  lies strictly between  $Q$  and  $Q_1$ , and  $\angle CQ_1Q = \angle CBA$ .

Prove that points  $P$ ,  $Q$ ,  $P_1$ , and  $Q_1$  are concyclic.

**Problem 8 (MOP HW #21).** In acute scalene  $\triangle ABC$  with circumcenter  $O$ , orthocenter  $H$ , Kosnita point  $X_{54} = K$ , define  $P = (HO) \cap (BOC)$ ,  $Q$  be the foot from line onto  $AO$ . Prove that  $P$ ,  $Q$ ,  $K$  are collinear. (The Kosnita point is the point at which the line through  $A$  and the circumcenter of  $\triangle BOC$  and the other two analogous lines concur; it is the isogonal conjugate of the nine-point center.

\*This was in fact group-solved using Desargue involution at MOP only a few days earlier!

**Problem 9 (Shortlist 2012/G8).** Let  $ABC$  be a triangle with circumcircle  $\omega$  and  $\ell$  a line without common points with  $\omega$ . Denote by  $P$  the foot of the perpendicular from the center of  $\omega$  to  $\ell$ . The side-lines  $BC, CA, AB$  intersect  $\ell$  at the points  $X, Y, Z$  different from  $P$ . Prove that the circumcircles of the triangles  $AXP, BYP$  and  $CZP$  have a common point different from  $P$  or are mutually tangent at  $P$ .

**Problem 10 (Shortlist 2021/G8).** Let  $ABC$  be a triangle with circumcircle  $\omega$  and let  $\Omega_A$  be the  $A$ -excircle. Let  $X$  and  $Y$  be the intersection points of  $\omega$  and  $\Omega_A$ . Let  $P$  and  $Q$  be the projections of  $A$  onto the tangent lines to  $\Omega_A$  at  $X$  and  $Y$  respectively. The tangent line at  $P$  to the circumcircle of the triangle  $APX$  intersects the tangent line at  $Q$  to the circumcircle of the triangle  $AQY$  at a point  $R$ . Prove that  $\overline{AR} \perp \overline{BC}$ .

## 🌲 2.1 Addendum

Shoutout to Eric again, for finding so many of these. Here are some more examples of DDIT, from a month later, oops!

**Problem 11 (USAMO 2012/5).** Let  $P$  be a point in the plane of  $\triangle ABC$ , and  $\gamma$  a line passing through  $P$ . Let  $A', B', C'$  be the points where the reflections of lines  $PA, PB, PC$  with respect to  $\gamma$  intersect lines  $BC, AC, AB$  respectively. Prove that  $A', B', C'$  are collinear.<sup>†</sup>

**Problem 12 (Shortlist (redacted)).** (redacted)

**Problem 13 (China 2020/2).** Let  $ABC$  be a triangle, and let the bisector of  $\angle A$  intersect  $\overline{BC}$  at  $D$ . Point  $P$  lies on line  $AD$  such that  $P, A, D$  are collinear in that order. Suppose  $\overline{PQ}$  is tangent to  $(ABD)$  at  $Q$ ,  $\overline{PR}$  is tangent to  $(ACD)$  at  $R$ , and  $Q$  and  $R$  lie on opposite sides of line  $AD$ . Let  $K = \overline{BR} \cap \overline{CQ}$ . Prove that if the line through  $K$  parallel to  $\overline{BC}$  intersects lines  $QD, AD, RD$  at  $E, L, F$ , respectively, then  $EL = KF$ .

**Problem 14 (redacted problem by me).** (redacted)

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<sup>†</sup>this is not a very good example