

ZGY-ConfigGeo

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Remark. ’cause every night I lie in bed, synthetic geo fills my head... geometry is keeping me awake...

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♣ 1 [50♣] External references

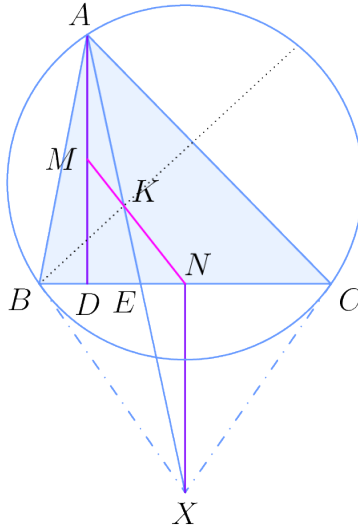
As I really enjoyed quite a few problems on this unit, some of them may be found on [a geo paper](#) I wrote, namely: [3♣] **TwCh0061**, [5♣] **18SLG5**, [5♣] **12RMM6**, [5♣] **20FakeUSMO3**, [5♣] **20MOP1Z**, [9♣] **20IGOA4**, [9♣] **20DeuXMOII3**, and [9♣] **20USEMO3**.

Remark. Oops, I ‘stole’ almost all the [9♣] problemsd...also, I should really set up von someday.

♣ 2 [13♣] Configs

♣ 2.1 [3♣] Schwatt

Let ABC be a triangle with altitude \overline{AD} . Let M and N denote the midpoints of \overline{AD} and \overline{BC} . Show that line MN passes through the symmedian point K of $\triangle ABC$ (this line is called the A -Schwatt line).



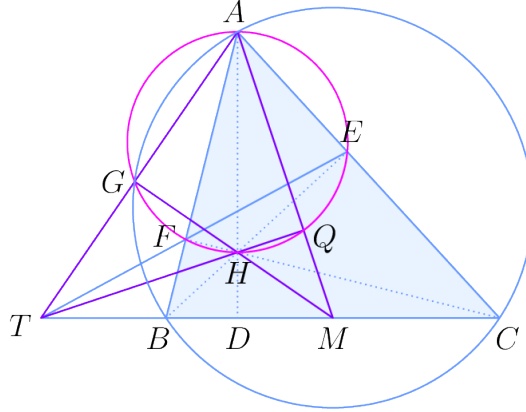
Let the A -symmedian meet \overline{BC} at E , ω be the circumcircle of $\triangle ABC$, and $X \in \overline{AE}$ be the pole of \overline{BC} wrt ω . Clearly, $\overline{AD}, \overline{NX} \perp \overline{BC} \Rightarrow \overline{AD} \parallel \overline{NX}$. Then

$$-1 = (A, C; B, \overline{BK}) \stackrel{B}{=} (AE; KX) \stackrel{N}{=} (A, D; \overline{NK} \cap \overline{AD}, \infty_{AD})$$

so \overline{NK} bisects \overline{AD} .

Remark. Instead of using this config to solve TSTST 2016/6, I actually discovered it from that problem...

♠ 2.2 [2♣] + [2♣] Humpty Dumpty



Although defined in the next problem, we invoke the notation early: G as A -orthocenter Miquel point, and $T = \overline{AG} \cap \overline{EF} \cap \overline{BC}$ (which exists by radical axis).

By isogonal / antiparallel lines, \overline{AM} is a symmedian in $\triangle AEF$; since $Q \in (AH)$, this is $(AQ; EF) = -1$. First,

Lemma (3b) – T, Q, H collinear.

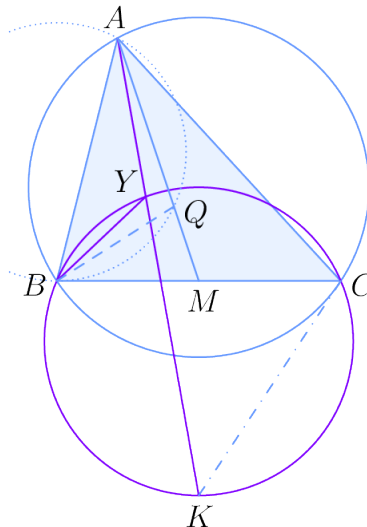
Proof. Harmonics: $(D, \overline{HQ} \cap \overline{BC}; B, C) \stackrel{H}{=} (AQ; EF) = -1$ implies $T = \overline{HQ} \cap \overline{BC}$. □

Second, $\angle MDH = \angle MQH = 90^\circ$ means $MDQH$ cyclic, or in PoP terms, $TQ \cdot TH = TM \cdot TD$. By midpoints of harmonics bundles lemma on $(TD; BC)$, the last product equals $TB \cdot TC$, so **2a: BCHQ cyclic** by PoP converse.

Remark. Iran TST 2018/1/4 should be in the pset...or is it only in the D version?

Third, from the above lemma and $\overline{AHD} \perp \overline{TM}$, **3c: H is orthocenter of triangle ATM**. If A' is the antipode of A on (ABC) , then $\angle HGA = \angle A'GA = 90^\circ$ implies **3a: A', M, H, G collinear**.

Fourth, using midpoints of harmonics on $(TD; BC)$ again, we can obtain $MQ \cdot MA = MD \cdot MT = MB^2 = MC^2$, implying **2b: line BC touches (ABQ), (ACQ)**.



Let O be the circumcenter of $\triangle ABC$, K be the pole of \overline{BC} wrt (ABC) , $X \neq A$ be the intersection of the A -symmedian with (ABC) and $Y = (A + X)/2$ the A -Dumpty point, so that $Y \in (OBCK)$.

Lemma (2c) – Q, Y isogonal.

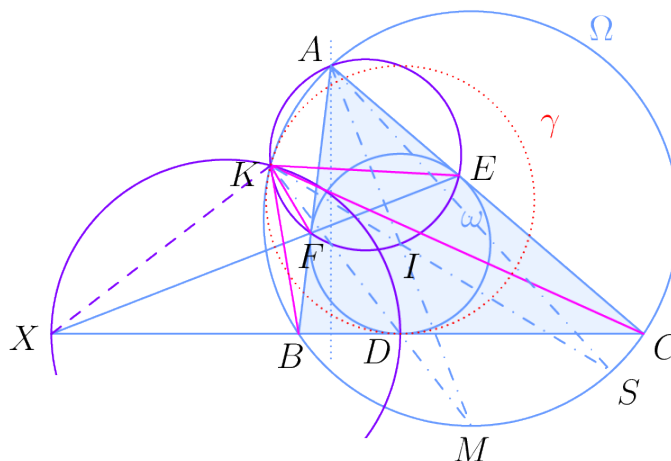
Proof. By symmetry, sufficient to prove $\angle ABY = \angle QBC$. Indeed, observing that $\overline{AY}, \overline{AM}$ isogonal, we have

$$\angle KYB = \angle KCB = \angle CAB \Rightarrow \angle ABY = \angle BAY + \angle KYB = \angle CA Y = \angle MAB \stackrel{\text{tangency}}{=} -\angle MAY. \quad \square$$

2.3 [3♣] Sharky-Devil

A scary fish and a fiend

20TSTST2 AoPS thread title

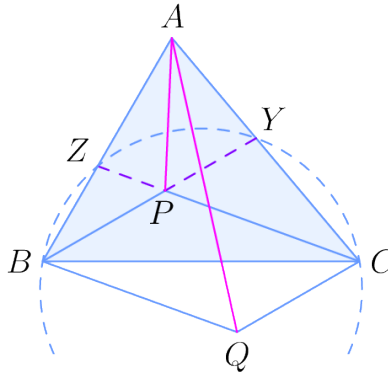


Let M be the midpoint of arc BC excluding A , K be the A -SD point, S be the antipode of A on (ABC) . Let Ω, ω denote the circumcircle and incircle respectively.

- (a) Angle bisector theorem on $KB/KC \stackrel{\text{spiral}}{=} BF/EC = BD/DC$ implies that \overline{KD} bisects $\angle BKC \Rightarrow K, D, M$ collinear.
- (b) Due to the previous part (and the circles-inscribed-in-segments lemma from EGMO book) we may construct a circle γ tangent to Ω, \overline{BC} at K, D respectively. (This circle is tangent internally to Ω, ω at K, D respectively: $\omega \subseteq \gamma \subseteq \Omega$.) Now, by Monge on (ω, γ, Ω) , \overline{KD} passes through the exsimilicenter of (Ω, ω) .
- (c) By angle bisector theorem, equivalent to prove $EP/PF = EK/KF$; by spiral, $EK/KF = EC/FB = CD/DB$. Construct $X = \overline{EF} \cap \overline{BC}$ so that $(XD; BC) = -1$. Since \overline{KD} bisects $\angle BKC$, we have $\angle DKX = 90^\circ = \angle DPX$, or $XDKP$ cyclic. In other words, $(XBF), (XCE), (XPD)$ concur at $K \neq X$, so $CD/DB = EP/PF$ follows from spiral.
- (d) By radical axis on $((AI), \Omega, (BIC)), \overline{AK}, \overline{BC}$, and the line through I perpendicular to \overline{AI} concur.
- (e) Let $U = \overline{KD} \cap (AI) \ (\neq K)$. Then, $\overline{UI} \parallel \overline{BC}$ by spiral at A , so $\overline{AU} \perp \overline{UI} \parallel \overline{BC}$.

♣ 2.4 [3♣] First isogonality lemma

In a triangle ABC , let P be an interior point with $\angle ABP = \angle PCA$, and $Q = B + C - P$. Then $\overline{AP}, \overline{AQ}$ isogonal wrt $\angle A$.



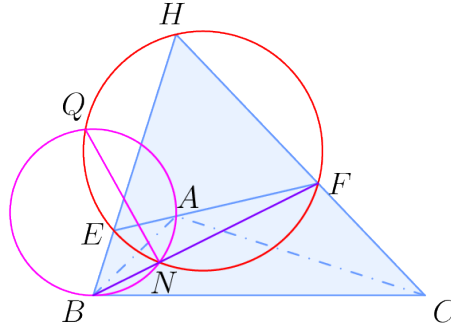
Let $B' = \overline{PB'} \cap \overline{AC}$, $C' = \overline{PC'} \cap \overline{AB}$. Then, consider the composition h of a homothety at A with scale factor $AB/AB' = AC/AC'$ with a reflection in the bisector of $\angle A$. As $(B', C', P) \xrightarrow{h} (B, C, Q)$ by linearity / etc, the result follows.

♣ 3 [65♣] Contest probs

Note. Split into two sections for accessibility reasons.

♣ 3.1 [5♣] 21EGMO3

Let ABC be a triangle with an obtuse angle at A . Let E and F be the intersections of the external bisector of angle A with the altitudes of ABC through B and C respectively. Let M and N be the points on the segments EC and FB respectively such that $\angle EMA = \angle BCA$ and $\angle ANF = \angle ABC$. Prove that the points E, F, N, M lie on a circle.



The problem becomes a lot simpler if we consider problem wrt $\triangle HBC$ where H is the orthocenter (of $\triangle ABC$.) Define:

- Q as the H -orthocenter Miquel point in $\triangle HBC$, aka the A -Humpty point in $\triangle ABC$;
- ω_b as the circle through A touching \overline{BC} at B ; it's well-known that $Q \in \omega_b$, while the given angle condition implies $N \in \omega_b$ as well.

Lemma (source?) – $HQEF$ cyclic.

Proof. By angle chasing, $\triangle BEA \sim \triangle CFA$; thus, if $A' \in (HBC)$ is the reflection of A in \overline{BC} (so that $(QA'; BC) = -1$),

$$\Rightarrow \frac{BE}{CF} = \frac{BA}{CA} = \frac{BA'}{CA'} \stackrel{\text{harmonics}}{=} \frac{BQ}{CQ},$$

proving the lemma via spiral. □

To finish, all we need is:

Claim – $NQHF$ cyclic.

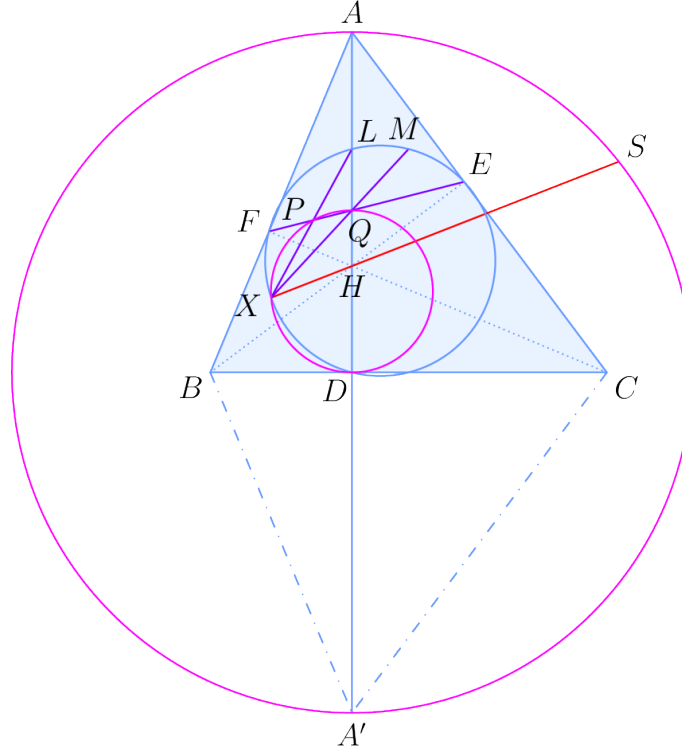
Proof. $\angle QNF = \angle QNB = \angle QBC = \angle QHC = \angle QHF$. □

From above claim, $M, N \in (AQEF)$ completing the proof.

Remark. Everyone's sol is the same...

♣ 3.2 [3♣] 19IndTST8

Let ABC be an acute triangle with circumcircle Γ and altitudes \overline{AD} , \overline{BE} , \overline{CF} meeting at H . Let ω be the circumcircle of $\triangle DEF$. Point $S \neq A$ lies on Γ such that $DS = DA$. Line \overline{AD} meets \overline{EF} at Q , and meets ω at $L \neq D$. Point M is chosen such that \overline{DM} is a diameter of ω . Point P lies on \overline{EF} with $\overline{DP} \perp \overline{EF}$. Prove that lines SH , MQ , PL are concurrent.



Obviously, L is the midpoint of minor arc BC , M is the antipode of D (on ω).

Construct $X = (DQ) \cap \omega$ ($\neq D$). I claim this is the desired concurrency point. Two of the three desired lines are easy to deal with, in American fashion:

Claim 1 - $X \in \overline{LP}$, \overline{MQ} .

Proof. $\angle DPQ = 90^\circ = \angle DXQ$ implies $DQPX$ cyclic. If $X' = \overline{LP} \cap \omega$, then $LP \cdot LX' = LQ \cdot LA = LE^2 = LF^2$, or $DQPX'$ cyclic. Hence, $X' = X$.

To see that $X \in \overline{MQ}$, simply observe that $\angle DXQ = \angle DXM = 90^\circ$ by construction. \square

Define $A' = 2D - A$ as the reflection of A in \overline{BC} , allowing us to define S more naturally as $(AA') \cap (ABC)$ ($\neq A$). Since $\angle BAQ = -\angle BAD = -\angle QEB$, $BA'EQ$ cyclic. For the last line, we can actually show:

Claim 2 - If i denotes the inversion at H with (negative) power $p = HA \cdot HD = HB \cdot HE = HC \cdot HF$, then $S \xrightarrow{i} X$.

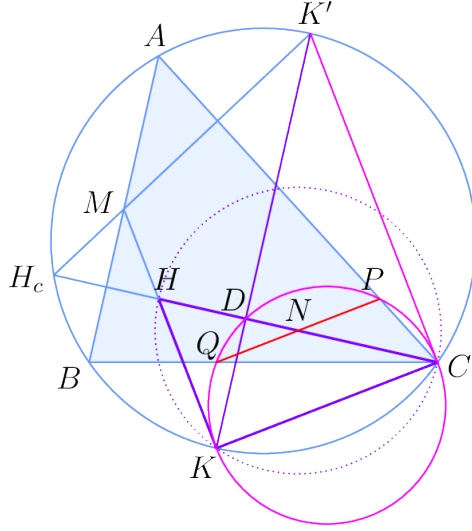
Proof. It's well-known that i swaps (ABC) , ω .

I claim that i swaps (Q, A') . Indeed applying power of a point converse to claim 2 implies $HA' \cdot HQ = HB \cdot HE = p$.

As H, A, D, Q, A' all collinear, it follows that i swaps $((AA'), (DQ))$ and thus (S, X) as well. \square

♠ 3.3 [3♣] 19Shrg20

Let O be the circumcenter of triangle ABC , H be its orthocenter, and M be the midpoint of AB . The line MH meets the line passing through O and parallel to AB at point K lying on the circumcircle of ABC . Let P be the projection of K onto AC . Prove that $PH \parallel BC$.



Let H_c be the reflection of H in \overline{AB} , $D = \overline{CH_c} \cap (CH)$, $Q = \overline{BC} \cap (CH)$, and $N = \overline{PQ} \cap \overline{CH}$; s denote the spiral at K mapping $P, Q, D \rightarrow A, B, H_c$. Finally let K' be the reflection of K in the perpendicular bisector of \overline{AB} . By the given condition this is also the antipode of K in the $\Omega = (ABC)$, so that $\angle K'CK = 90^\circ$ whence $\overline{CK'}$ touches (CK) .

Claim 1 – N is the midpoint of \overline{PQ} .

Proof. It's well-known that $\overline{H_cK'}$ passes through M , i.e. it bisects \overline{AB} ; applying s^{-1} means \overline{CD} bisects \overline{PQ} as desired. \square

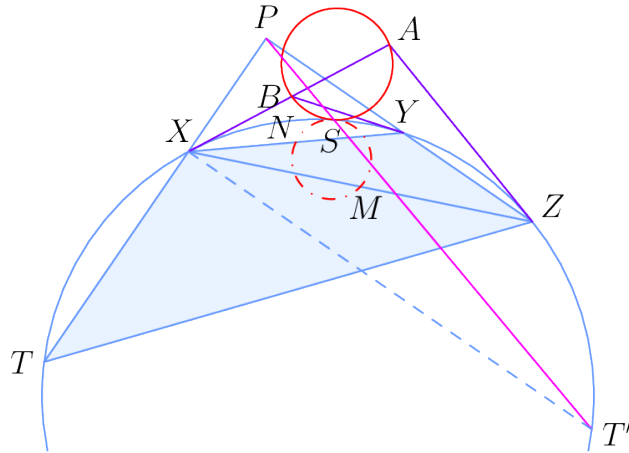
Claim 2 – N is also the midpoint of \overline{CH} .

Proof. Applying s^{-1} to " $\overline{KK'} \parallel \overline{AB}$ " means that $\overline{KC} \parallel \overline{PQ}$, so $CPQK$ is a cyclic isosceles trapezoid. Thus $N = \frac{P+Q}{2}$ is on the common perpendicular bisector of \overline{CK} , \overline{PQ} . But in right $\triangle HKC$, since it's on a perpendicular bisector of a side and the hypotenuse, it must be the circumcenter, hence $NC = NH$ as required. \square

It follows that $CPHQ$ is a parallelogram, completing the proof.

♠ 3.4 [5♣] 16ChnTST26

The diagonals of a cyclic quadrilateral $ABCD$ intersect at P , and there exists a circle Γ tangent to the extensions of \overline{AB} , \overline{BC} , \overline{AD} , \overline{DC} at X , Y , Z , T respectively. Circle Ω passes through points A , B , and is externally tangent to circle Γ at S . Prove that $\overline{SP} \perp \overline{ST}$.

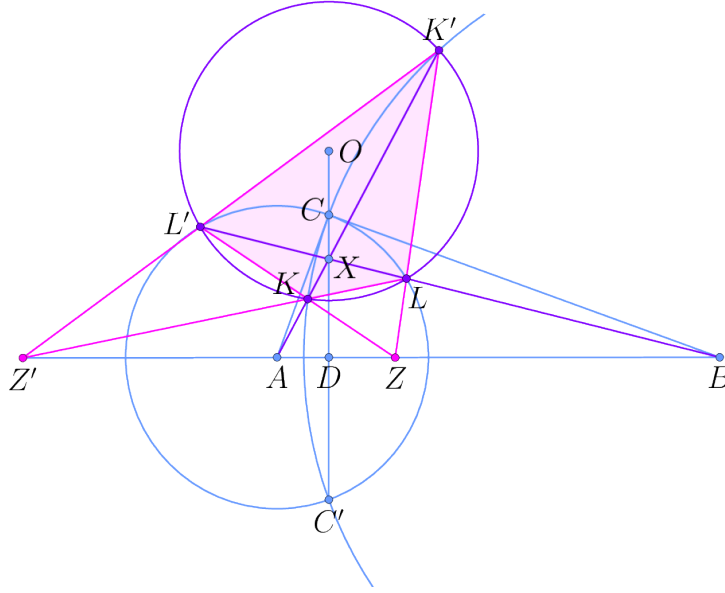


Solution outline with **CyclicISLscalesTrapezoid** and **v4913**. (*AoPS post link*)

By Brianchon's on $AXBCTD$ and $AZDCYB$, \overline{XY} and \overline{YZ} intersect at P . By angle chasing, \overline{XT} and \overline{YT} are perpendicular. Let M and N be the midpoints of \overline{XZ} and \overline{XY} , respectively, let T' be the antipode of T with respect to Γ , and redefine S as the second intersection of $\overline{PT'}$ with Γ . By inversion about Γ , it suffices to show that Γ is tangent to the circumcircle of SMN at S . By angle chasing, $XYZT'$ is a cyclic isosceles trapezoid, so we are done by **SL 2011/G4**.

♣ 3.5 [3♣] 13DecTST3

Let ABC be a scalene triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C . Let X be a point in the interior of the segment CD . Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. The circumcircle of triangle DKL intersects segment AB at a second point T (other than D). Prove that $\angle ACT = \angle BCT$.



Let ω_a, ω_b be the circles A, B through C , $K' = \overline{AX} \cap \omega_b$ ($\neq K$), and similarly for L' . Let $C' \in \omega_a \cap \omega_b$ be the reflection of C in \overline{AB} .

Claim 1 – $KLK'L'$ is harmonic.

Proof. The quadrilateral is cyclic by power of a point at X : $XK \cdot XK' = XC \cdot XC' = XL \cdot XL'$.

Call its circumcircle Ω . Meanwhile, power of a point at A means it's harmonic too:

$$AK \cdot AK' = AC^2 = AL^2 = AL'^2 \Rightarrow \overline{AL}, \overline{AL'} \text{ touch } \Omega.$$

□

Let O be the center of Ω , and $Z = \overline{KL} \cap \overline{K'L'}$, $Z' = \overline{KL} \cap \overline{K'L'}$ which both lie on \overline{AB} by Brokard. As \overline{AB} is the polar of $X = \overline{KK'} \cap \overline{LL'}$ wrt Ω , D is the Miquel point of $KLK'L'$, whence $Z \in (DKL)$ and $Z = T$.

Claim 2 – $\angle ZCZ' = 90^\circ$.

Proof. Equivalent to prove $DC^2 = DZ \cdot DZ'$. O is the orthocenter of $\triangle XZZ'$ by Brokard, while it's also the orthocenter of $\triangle XAB$ because $\overline{AO} \perp \overline{LL'}$, $\overline{BO} \perp \overline{KK'}$. Recall that in a triangle ABC with orthocenter H and D the foot of the A -altitude, $DB \cdot DC = DH \cdot DA$. Thus, applying the result to $\triangle XZZ'$, XAB , we obtain

$$DZ \cdot DZ' = DO \cdot DX = DA \cdot DB = DC^2$$

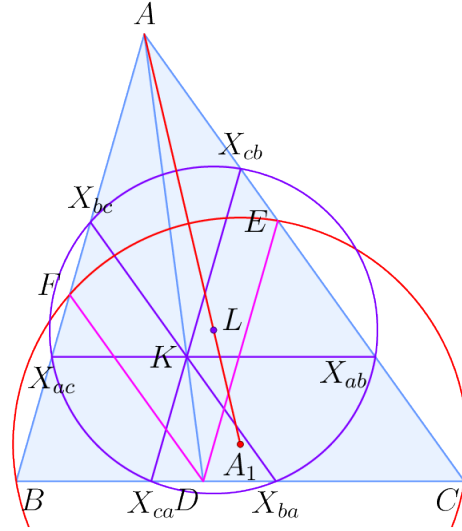
as needed. □

Finally, since $(AB; ZZ') = -1$ (Ceva-Menelaus) and $\angle ZCZ' = \angle ACB = 90^\circ$, \overline{CZ} bisects $\angle ACB$ by a well-known result.

♣ 3.6 [3♣] 05ChnTST

Let ω be the circumcircle of acute triangle ABC . The tangents to ω at B and C intersect at P , and $D = \overline{AP} \cap \overline{BC}$. Points E, F are on \overline{AC} and \overline{AB} , respectively, such that $\overline{DE} \parallel \overline{BA}$ and $\overline{DF} \parallel \overline{CA}$.

- Prove that points F, B, C , and E are concyclic.
- Let A_1 denote the circumcenter of cyclic quadrilateral $FBCE$. Points B_1 and C_1 are defined similarly. Prove that $\overline{AA_1}, \overline{BB_1}$, and $\overline{CC_1}$ are concurrent.



For part (a), since \overline{AD} is a symmedian in $\triangle ABC$ and a median in $\triangle AEF$, $\overline{BC}, \overline{EF}$ are antiparallel wrt $\angle A$.

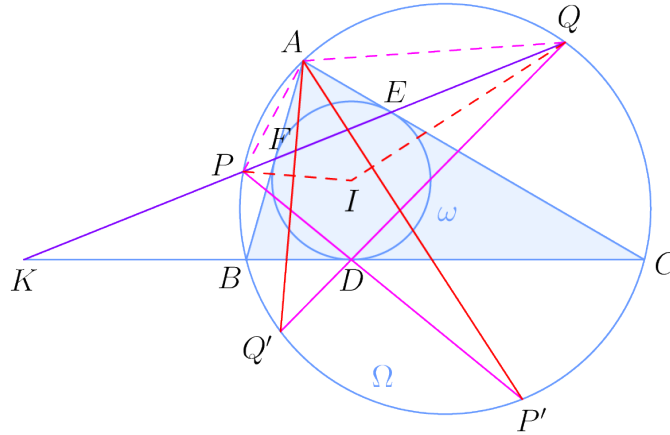
For part (b), we'll show that $\overline{AA_1}$ passes through the center L of the Lemoine circle ω . Define K as the symmedian point, and $X_{bc} = \overline{K\infty AC} \cap \overline{AB}$ and its five other variants. Consider homothety h at A mapping $D \rightarrow K$. By parallel lines, this homothety also maps $BCEF \rightarrow X_{ac}X_{ab}X_{cb}X_{bc}$ and thus their circumcenters $A_1 \rightarrow L$ as well. Hence $L \in \overline{AA_1}$ as required.

Remark. ω is the lemoine circle

♣ 3.7 [5♣] 19SLG6

Let $\triangle ABC$ be a triangle with incenter I whose incircle touches sides BC, CA, AB at D, E, F . Line EF meets the circumcircle of $\triangle ABC$ at two points P and Q . Prove that

$$\angle DPA + \angle AQD = \angle QIP.$$



If Ω, ω are the circumcircle and incircle respectively, define $P' = \overline{PD} \cap \omega$ ($\neq P$), and Q' similarly.

Claim – $\overline{AQ'}$ is the polar of P wrt ω . Thus, $\overline{AQ'} \perp \overline{PI}$.

Proof. Since A is obviously on that polar, it suffices to prove $(P, \overline{AQ} \cap \overline{EF}; E, F) = -1$. Indeed,

$$(P, \overline{AQ} \cap \overline{EF}; E, F) \stackrel{A}{=} (PQ'; BC) \stackrel{Q}{=} (KD; BC) = -1. \quad \square$$

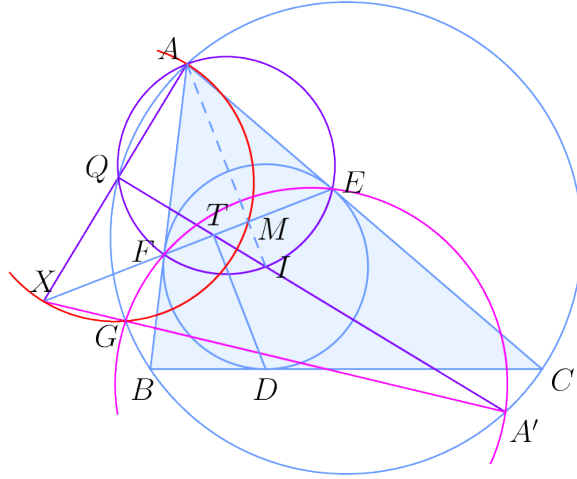
Now the problem is a simple angle chase: the claim implies $\angle QIP = \angle P'AQ'$, while (arcs directed mod 360°)

$$\angle DPA + \angle AQD = \frac{1}{2}(\widehat{P'A} + \widehat{AQ'}) = \angle P'AQ'$$

as well, as required.

♣ 3.8 [3♣] 19ESLG3

Let $\triangle ABC$ be an acute triangle with incenter I and circumcenter O . The incircle touches sides BC , CA , and AB at D , E , and F respectively, and A' is the reflection of A over O . The circumcircles of ABC and $A'EF$ meet at G , and the circumcircles of AMG and $A'EF$ meet at a point $H \neq G$, where M is the midpoint of EF . Prove that if GH and EF meet at T , then $DT \perp EF$.



Redefine T as the foot from D to \overline{EF} , so that we want T on the radical axis of (AMG) , $(A'EF)$. Construct Q as the A -SD point.

By radical axis on (AI) , $(A'EFG)$, (ABC) , there exists a point $X = \overline{AQ} \cap \overline{EF} \cap \overline{AG}$.

Claim – $AGMX$, $IMQX$ cyclic.

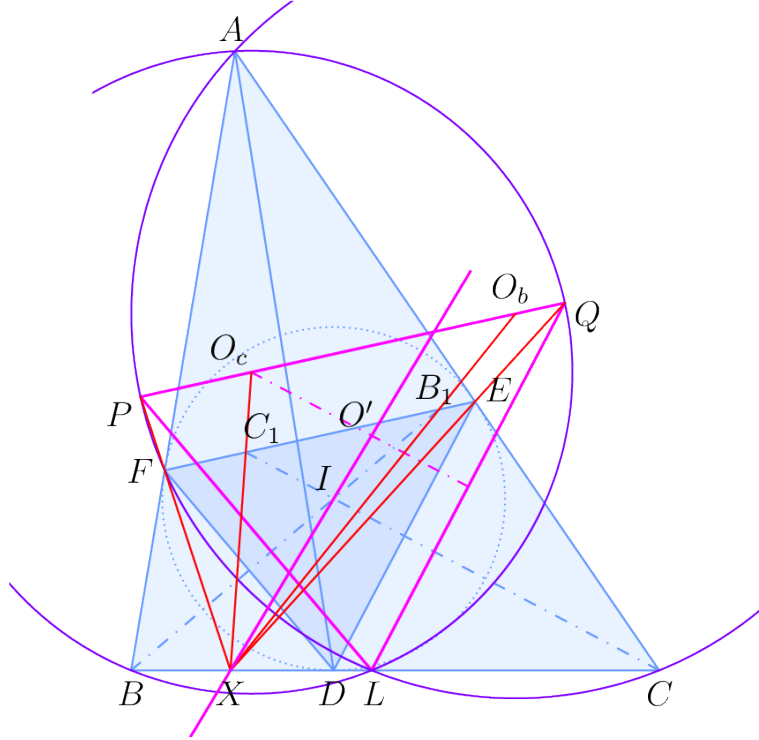
Proof. The first follows from $\angle AGX = \angle AGA' = 90^\circ = \angle AMX$, while the second from $\angle IQX = 90^\circ = \angle IMX$. \square

Finish by power of a point at T : $\text{Pow}(T, (A'EF)) = TE \cdot TF = TQ \cdot TI = TM \cdot TX = \text{Pow}(T, (AMGX))$, as required.

♣ 3.9 [5♣] 13Shrg19

Let ABC be a triangle with circumcenter O and incenter I . The incircle is tangent to sides \overline{BC} , \overline{CA} , \overline{AB} at A_0 , B_0 , C_0 . Point L lies on \overline{BC} so that $\angle BAL = \angle CAL$. The perpendicular bisector of \overline{AL} meets BI and CI at Q and P , respectively. Let C_1 and B_1 denote the projections of B and C onto lines CI and BI . Let O_1 and O_2 denote the circumcenters of triangles ABL and ACL .

Prove that the six lines BC , PC_0 , QB_0 , C_1O_1 , B_1O_2 , and OI are concurrent.



Rename A_0, B_0, C_0, O_1, O_2 to D, E, F, O_c, O_b respectively.

Claim 1 – $\triangle LQP, \triangle DEF$ are homothetic.

Proof. Observe that P, Q are midpoints of \widehat{AL} on (ACL) , (ABL) respectively, so that $\angle ALQ = \frac{B}{2}$; thus

$$\angle QLC = \frac{B}{2} + \angle ABL + \angle LAB = \frac{A+B}{2} = \frac{\pi - C}{2} = \angle ELC \Rightarrow \overline{LQ} \parallel \overline{DE}$$

and its cyclic variant, $\overline{LP} \parallel \overline{DF}$. Additionally $\overline{PQ}, \overline{EF} \perp \overline{AI}$ (by design) implies $\overline{PQ} \parallel \overline{EF}$; as the triangles have parallel sides, they're indeed homothetic. \square

Let $X = \overline{BC} \cap \overline{QE} \cap \overline{PF}$ be the similicenter of these two triangles. I is the orthocenter of $\triangle LPQ$ because $\overline{CP} \perp \overline{DE} \parallel \overline{LQ}$ and $\overline{BQ} \perp \overline{LP}$ analogously.

Claim 2 – \overline{OI} is the common Euler line of $\triangle DEF$, $\triangle LPQ$, and passes through X .

Proof. It's well-known that \overline{OI} is the Euler line of $\triangle DEF$. By homothety, the Euler line of $\triangle DEF$ is parallel to that of $\triangle LPQ$.

However, since these parallel lines share a point I (not at infinity), they must coincide. In order for a line to map to itself under a homothety, it must pass through the center— in other words, $X \in \overline{OI}$. \square

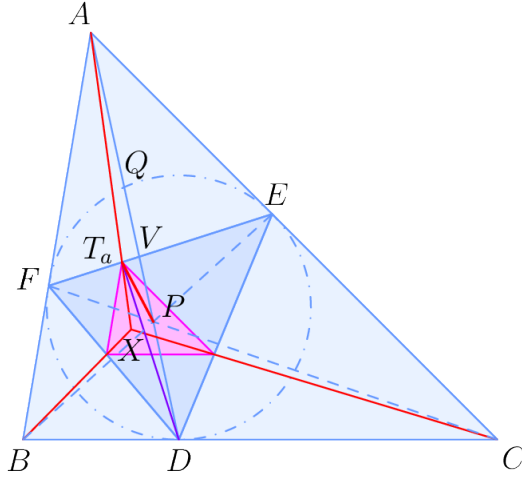
Let O' be the circumcenter of $\triangle LPQ$. It remains to prove that:

Claim 3 – O_c, C_1 correspond under the homothety.

Proof. Recall that \overline{CI} is the perpendicular bisector of \overline{DE} while $O_cL = O_cQ$ and $O'L = O'Q$ by design means $\overline{O_cO'}$ is the that of \overline{LQ} . By Iran lemma, $C_1 = \overline{CI} \cap \overline{EF}$, so it corresponds with $O_c = \overline{O_cO'} \cap \overline{PQ}$. \square

♠ 3.10 [3♣] 13Bra6

The incircle of triangle ABC touches sides BC , CA and AB at points D , E and F , respectively. Let P be the intersection of lines AD and BE . The reflections of P with respect to EF , FD and DE are X , Y and Z , respectively. Prove that lines AX , BY and CZ are concurrent at a point on line IO , where I and O are the incenter and circumcenter of triangle ABC .

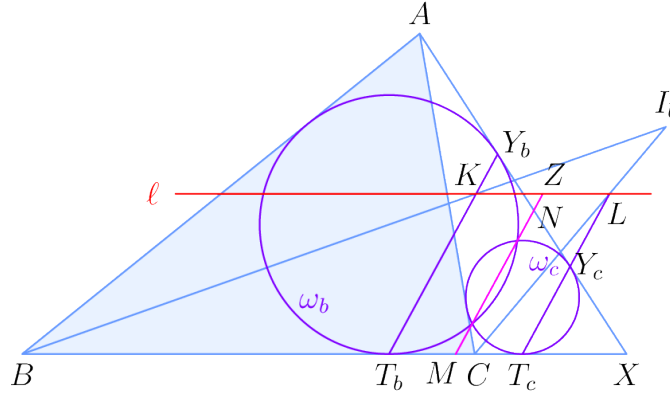


Let T_a be the foot from D to \overline{EF} , and X be the similicenter of homothetic triangles DEF , $I_a I_b I_c$. Clearly, it must also be the similicenter of their respective orthic triangles, so A , T_a , X collinear.

Next, let $V = \overline{AD} \cap \overline{EF}$, so that $(DV; AP) = -1$. Because $\angle DT_a V = 90^\circ$, \overline{EF} must bisect $\angle AT_a P$, whence $P_a \in \overline{AT_a A'}$. Considering triangles ABC , DEF , and the orthic triangle of $\triangle DEF$, the concurrency holds by cevian nest.

♠ 3.11 [5♣] 04SLG7

For a given triangle ABC , let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q . Prove that the line PQ passes through a point independent of X .



Let I_b be the B -excenter, and ℓ be the A -midline in $\triangle ABX, ACX$. Also, let ω_b, ω_c be the incircles of $\triangle ABX, ACX$ respectively, respectively touching \overline{BC} at T_b, T_c and \overline{AX} at Y_b, Y_c . Finally, let M, N be respective midpoints of $\overline{T_bT_c}, \overline{Y_bY_c}$, so that \overline{MNPQ} is the radical axis of ω_b, ω_c .

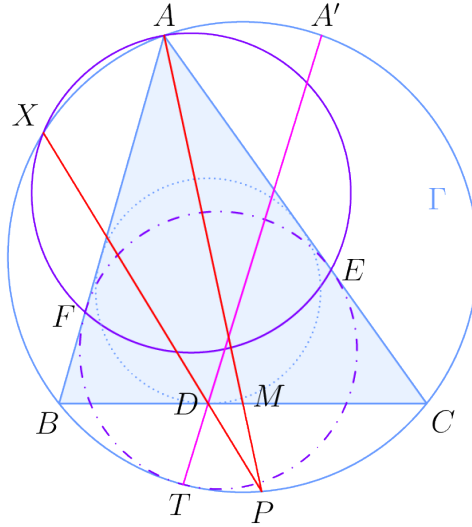
Recall the following result:

Lemma – In a triangle ABC , if the incircle touches $\overline{AB}, \overline{AC}$ at X, Y , then the B -midline, the C -angle bisector, and \overline{XY} are concurrent.

Consider the fixed points $K = \overline{BI_b} \cap \ell$ ($\in \overline{T_bY_b}$) and $L = \overline{CI_b} \cap \ell$ ($\in \overline{T_cY_c}$). It's routine to show that \overline{MN} is midway between the parallel lines $\overline{T_bY_b}, \overline{T_cY_c}$ and thus passes through $Z = \frac{K+L}{2}$, also a fixed point.

♠ 3.12 [5♣] 16SLG2

Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of \overline{BC} . Denote by D the foot of the perpendicular from I to \overline{BC} . The line through I perpendicular to \overline{AI} meets sides AB and AC at F and E respectively. Suppose the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A . Prove that lines XD and AM meet on Γ .



Let $A' \in \Gamma$ denote the reflection of A in the perpendicular bisector of \overline{BC} , and T denote the contact point of the A -mixtilinear incircle with Γ . Since \overline{TD} , \overline{TA} isogonal wrt $\angle BTC$, $A' \in \overline{TD}$. It's well-known that E, F lie on said mixtilinear incircle.

Claim – $(XT; BC) = -1$.

Proof. By a well-known lemma, \overline{TE} , \overline{TF} respectively bisect $\angle ATC$, $\angle ATB$, so

$$\frac{FB}{FT} \frac{FA}{TA} = \frac{EA}{TA} = \frac{EC}{TC} \Rightarrow \frac{XB}{XC} \stackrel{\text{spiral}}{=} \frac{FB}{EC} = \frac{TB}{TC}.$$

□

Now, let $P = \overline{XD} \cap \Gamma$ ($\neq X$). Then

$$(\overline{AP} \cap \overline{BC}, \infty_{BC}; B, C) \stackrel{A}{=} (PA'; BC) \stackrel{D}{=} (XT; BC) \stackrel{\text{claim}}{=} -1$$

means \overline{AP} bisects \overline{BC} . In other words, $P = \overline{XD} \cap \overline{AM} \cap \Gamma$ as required.

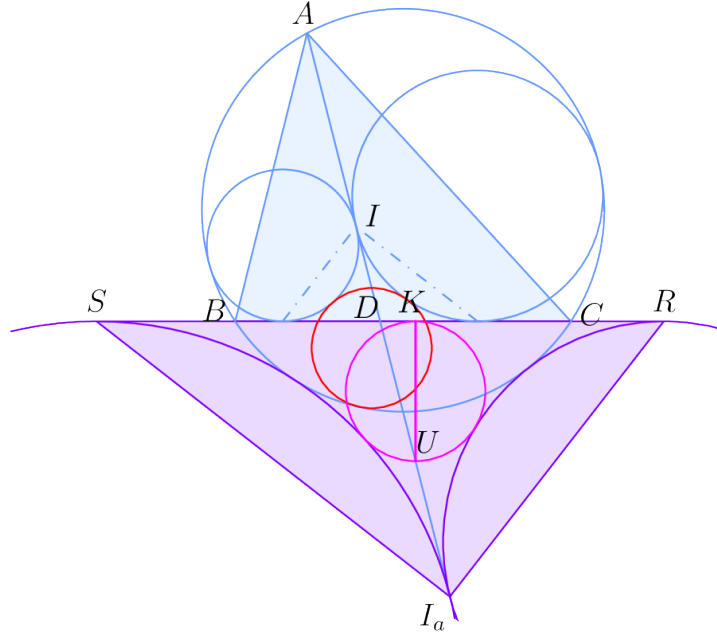
Remark. Should be [3♣] instead of [5♣]? No intention of casting aspersion, oops...

♠ 3.13 [9♣] Grant's Amerigeo

(in ARCH, H1746385.)

Convex pentagon $ABMCN$ is inscribed in circle Γ with diameter \overline{MN} , with $BM = CM$. Two distinct circles ω_B and ω_C are drawn, each tangent to segments AM and BC , and internally tangent to Γ . Finally, we draw a circle γ externally tangent to ω_B and ω_C , and internally tangent to Γ at a point W on arc \widehat{BMC} of Γ .

- Prove that \overline{AM} and \overline{WN} meet on γ .
- Prove that \overline{AM} passes through one of the intersections of γ and the A -mixtilinear incircle.



(asy'd without Geogebra conversion, despite lack of productivity in doing so...)

Let I, I_a, α be the incenter, A -excenter, and A -excircle respectively. Define $D = \overline{AI} \cap \overline{BC}$ and $T = \alpha \cap \overline{BC}$.

Claim 1 – ω_b, ω_c touch \overline{AD} at I .

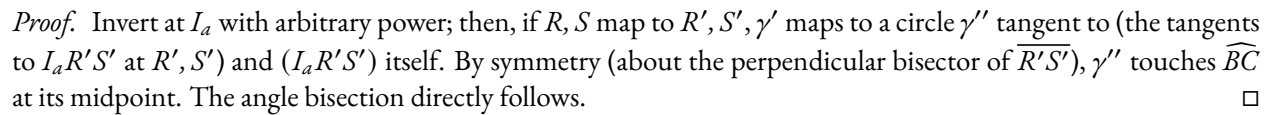
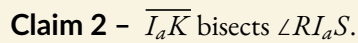
Proof. Recalling the properties of curvilinear incircles, the line through the tangency points of ω_b with $\overline{AD}, \overline{BD}$ passes through I . It follows that the former tangency point is simply I . \square

Now, let i denote the inversion about A with power $AB \cdot AC = AM \cdot AD = AI \cdot AI_a$ composed with a reflection in \overline{AM} . Let the images of ω_b, ω_c under i (which we call ω'_b, ω'_c) touch \overline{BC} at R, S , so that $DI_a = DR = DS$.

Lemma – If two circles α, β touch a segment AB at A, B respectively, and each other at P , then $\angle APB = 90^\circ$.

Proof. If M is the midpoint of \overline{AB} , then since it's on the radical axis of α, β , we have $MA = MB = MP$, which implies the result. \square

$\Rightarrow \triangle RI_aS$ right. Also, let γ map to a circle γ' tangent to ω'_b, ω'_c , and \overline{BC} at some point K .



Claim 3 – $U \in \overline{AI_aD}$.

$$UR \cdot UX = UK^2 = US \cdot UY$$

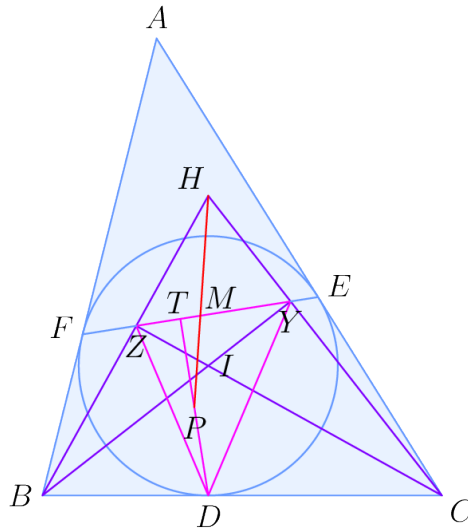
Now we can obtain **(a)**: $\overline{WN} \stackrel{i}{\Leftrightarrow} (AEK)$, so it's equivalent to show:

Proof. $\angle UKE = 90^\circ = \angle UAE$ using last claim.

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♣ 3.14 [3♣] 09IranTST9

In triangle ABC , D , E and F are the points of tangency of incircle with the center of I to BC , CA and AB respectively. Let M be the foot of the perpendicular from D to EF . P is on DM such that $DP = MP$. If H is the orthocenter of BIC , prove that PH bisects EF .



Recalling the lemma famously associated with this problem, let $Y, Z \in \overline{EF}$ be the feet from B, C to $\overline{CI}, \overline{BI}$ respectively. Then, we can get rid of triangle ABC as follows:

Iran TST 2009/9 reduced

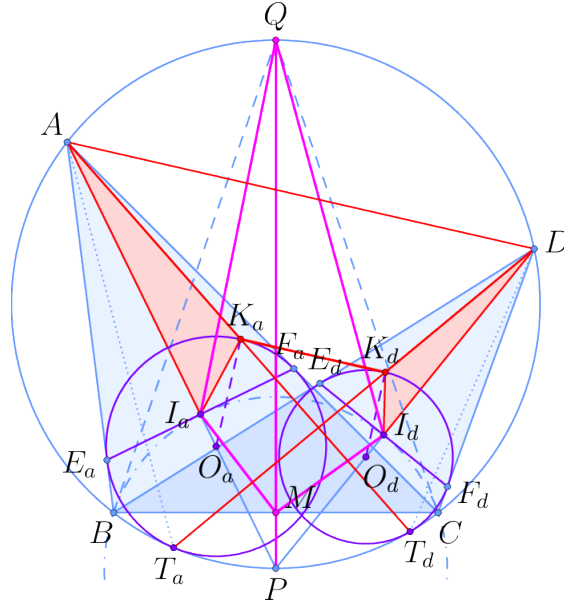
In triangle HBC , D, Y, Z are the respective feet of altitudes, and I is the orthocenter. If M is the foot from I to \overline{YZ} and P is the midpoint of the altitude from D to \overline{YZ} , then P, M, H are collinear

But this is just the below lemma from EGMO chapter 4, applied to $\triangle DYZ$, so we're done...

Lemma (EGMO Ch. 4) – In a triangle ABC , the midpoint of the A -altitude, the intouch point on \overline{BC} , and the A -excenter are collinear.

♠ 3.15 [5♣] 20MOP1Z

Let $ABCD$ be a quadrilateral inscribed in circle Ω . Circles ω_A and ω_D are drawn internally tangent to Ω , such that ω_A is tangent to \overline{AB} and \overline{AC} while ω_D is tangent to \overline{DB} and \overline{DC} . Prove that we can draw a line parallel to \overline{AD} which is simultaneously tangent to both ω_A and ω_D .



Solution by **v4913**. Define...

- P, Q as the respective midpoints of $\widehat{BC}, \widehat{BAC}$, I_a, I_d as the respective incenters of ω_a, ω_d , and M as the midpoint of \overline{BC} ;
- O_a, O_d as respective centers of ω_a, ω_d , and $\gamma = (BI_a I_d C)$ (with center P), so that $\overline{QB}, \overline{QC}$ touch γ ;
- $E_a, F_a, T_a = \omega_a \cap \overline{AB}, \overline{AC}, \Omega$; K_a as the intersection of $\overline{AT_d}$ with ω_a closer to A , and their symmetric variants. It's well-known that Q, I_a, T_a collinear, and that I_a is the midpoint of $\overline{E_a F_a}$;
- s_a as the spiral similarity mapping $\gamma \rightarrow \omega_a$ and thus $Q, B, C, M \rightarrow A, E_a, F_a, I_a$. Since $\angle K_a A F_a = \frac{1}{2} \widehat{T_d C} = \angle I_d Q C$ by design, we also have $(K_a \xrightarrow{s_a} I_d)$.

We contend that $\overline{K_a K_d}$ is the desired tangent, using the following two parts:

Claim 1 – $\overline{O_a K_a}, \overline{O_d K_d} \perp \overline{AD}$.

Proof. We angle chase:

$$\angle(\overline{O_a K_a}, \overline{AD}) = \angle O_a K_a A + \angle K_a A D \stackrel{s_a}{=} \angle P I_d Q + \angle T_d Q D = \angle(\overline{P I_d}, \overline{Q D}) = \frac{1}{2} \widehat{P Q} = 90^\circ.$$

The claim follows by symmetry. □

Claim 2 – $\overline{K_a K_d} \parallel \overline{AD}$.

Proof. Let $X = \overline{AT_d} \cap \overline{DT_a}$, so that it'll suffice to prove $AK_a/AX = DK_d/DX$. Indeed, using s_a , $AK_a = QI_d \cdot \frac{AI_a}{QM}$ and similarly $DK_d = QI_a \cdot \frac{DI_d}{QM}$. We thus have:

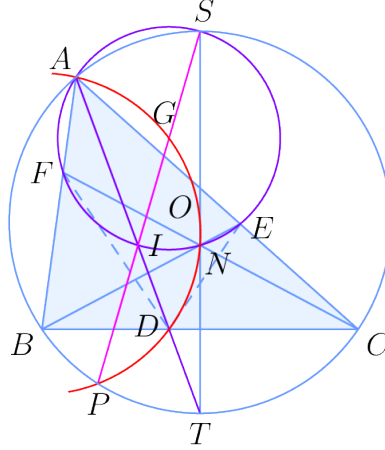
$$\frac{AK_a}{DK_d} = \frac{AI_a/QI_a}{DI_d/QI_d} = \frac{AT_a/QP}{DT_d/QP} = \frac{AX}{DX}. \quad \square$$

From the previous two claims, $\overline{O_a K_a}, \overline{O_d K_d} \perp \overline{K_a K_d}, \overline{AD}$ so $\overline{K_a K_d}$ touches both ω_a, ω_d while also parallel to \overline{AD} , as required.

♠ 3.16 [5♣] 21CHMMC6

Let ABC be a triangle with circumcenter O . The interior bisector of $\angle BAC$ intersects BC at D . Circle ω_A is tangent to segments AB and AC and internally tangent to the circumcircle at P . Let E and F be the points at which the B -excircle and C -excircle are tangent to AC and AB . Suppose that lines BE and CF pass through a common point N on the circumcircle of AEF .

- (a) Prove that the circumcircle of PDO passes through N .
 (b) Suppose that $PD/BC = 2/7$. Find, with proof, the value of $\cos \angle BAC$.



First part Let $\Omega = (ABC)$, $\omega = (AEF)$, the latter with center G . Also let S, T be the respective midpoints of $\widehat{BC}, \widehat{AC}$. Since $BF = CE$, $S \in \omega$ by spiral. Also, as customary in bash solutions, let $a = BC, b = CA, c = AB, s = \frac{a+b+c}{2}$.

Claim 1 – D is the Miquel point of $AENF$.

Proof. Let D' be the Miquel point of $AENF$ and thus $BFCE$ as well. Again, since $BF = CE$ and there exists a spiral similarity at D' mapping $B, F \rightarrow C, E$, that spiral similarity must in fact be a rotation. Thus, $D'B = D'E$, so AD' bisects $\angle BAC$. Additionally, by Brokard, D' lies on the line through $B = \overline{EN} \cap \overline{AF}$ and $C = \overline{FN} \cap \overline{AE}$, which pins down the position of D' . \square

Remark. This looks like one of **CyclicISLscalesTrapezoid**'s discarded problem ideas...

Claim 2 – $b + c = 2a$ and $I \in \omega$ as well.

Proof. Since $AFCD$ cyclic by definition of Miquel point, power of a point gives

$$a \frac{ac}{b+c} = BD \cdot BC = BF \cdot BA = c(s-a) \Rightarrow (2a-b-c)(a+b+c) = 0 \Rightarrow b+c = 2a.$$

Now if we let E', F' be the feet from I to $\overline{AC}, \overline{AB}$, then $\triangle FF'I \cong \triangle EE'I$ implies $\angle EIF = \angle E'IF' = \pi - A$, and $I \in \omega$ as claimed. \square

Claim 3 – N is the reflection of T in \overline{BC} .

Proof. Using the cyclic quads associated with Miquel points, we've $\angle NBD = \angle CFD = \angle CAD = \frac{A}{2}$; similarly, $\angle NCD = \frac{A}{2} = \angle NDC$. Noting that $T \neq N$ also satisfies these angle conditions, it follows that the two points are indeed reflections in \overline{BC} . \square

From the last claim, N must be the foot from I to \overline{ST} , while it's well-known that P is the foot from T to \overline{SI} . From the second claim, D is the midpoint of \overline{TI} . Obviously, G, O are the respective midpoints of $\overline{SI}, \overline{ST}$ while A is the foot from S to \overline{IT} .

To conclude, A, G, O, N, D, P lie on the nine-point circle of $\triangle IST$.

Second part Let $PD = 2, BC = 7$. Since $\angle IPT = \angle INT$ by the claims from the previous part, D is the center of $(INTP)$; we then obtain $TI = 4$ so $\triangle TBC$ has sides 4, 4, 7. We obtain

$$\cos A = -\cos \angle BTC = \frac{7^2 - 2 \cdot 4^2}{2 \cdot 4^2} = \boxed{\frac{17}{32}}.$$