Proposal Compilation– Version Public

nya!

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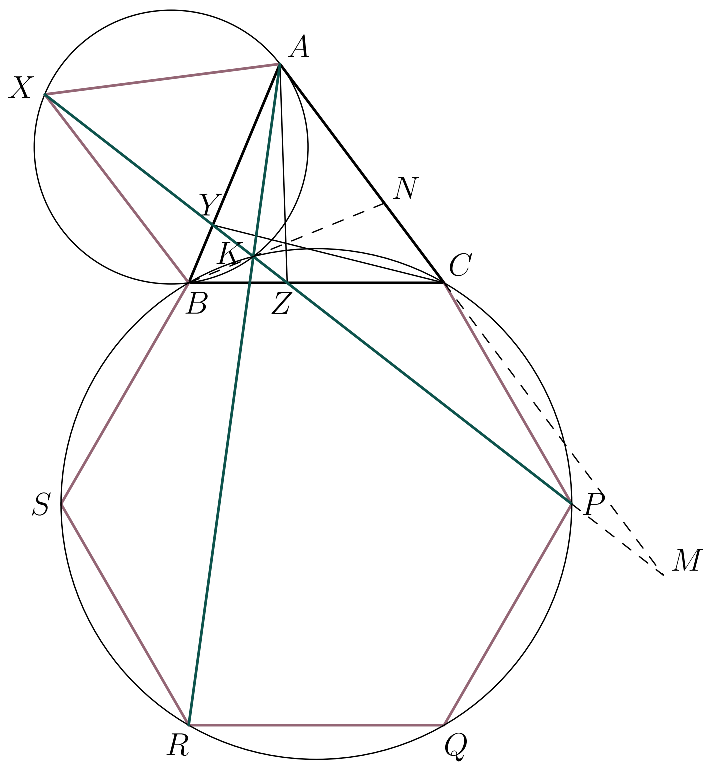
> *Remark.* Version Private… you may have seen some of those.
> Public proposals aren't that great because they're from 2021, when I didn't have that many math ideas, sorry…

Contents

1 External regular polygons

In acute triangle ABC, equilateral triangle ABX and regular hexagon $BCPQRS$ are externally constructed on their respective sides. Let line XP intersect $\overline{AB}$, $\overline{BC}$, and $\overline{AR}$ at Y, Z, K, respectively. Prove that $\overline{AZ}$, $\overline{BK}$, $\overline{CY}$ are concurrent.



Extend lines XP, BK to meet line AC at points M, N, respectively. Then, we have,

Claim – $(AC; MN) = -1$.

Proof. First, because rotations are spiral similarities, and $\overline{AR}$, $\overline{XP}$ are related by a $\pi/3$ one, K is also the second intersection of (ABX) and (BPR) distinct from B, that is, the second intersection of the circumcircles of the two regular polygons.
Due to this we have $\angle AKC = 2\pi - \angle AKB - \angle BKC = \pi/2$.
Now we obtain $\angle NKC = \pi - \angle BKC = \pi/6$, and $\angle MKC = \angle PKC = \pi/6$, and $\overline{CK}$ bisects $\angle MKN$. By a well-known lemma the claim is proven. □

Now Ceva-Menelaus in reverse trivializes the problem.

2 3-dimensional fake geo

Tetrahedron $ABCD$ has the property that $\overline{DA}, \overline{DB}, \overline{DC}$ are mutually perpendicular. Define the **A-exsphere** to be the sphere tangent to face BCD and to the extensions of the other faces. We also define the other exspheres similarly. Let the radius of such a sphere be called an exradius. It is known that $DA^{-2}+DB^{-2}+DC^{-2} = 9409$. If the A-,B-, C-exradii are $1/138, 1/168, 1/152$, respectively, then the D-exradius may be expressed as a/b for coprime positive integers a, b. Find $a+b$.

Let the 'legs' be $DA, DB, DC = a, b, c$, and we may let the vertices be $A = (a, 0, 0)$, $B = (0, b, 0)$, $C = (0, 0, c)$, $D = (0, 0, 0)$. Hence plane BCD has equation $x/a + y/b + z/c = 1$. Let the exradii be r_a, etc.

Claim – $1/r_a = -1/a + 1/b + 1/c + \sqrt{\sum_{\text{cyc}} a^{-2}}$.

Proof. The A-excenter is of the form $(-r_a, r_a, r_a)$, and is r_a from face BCD, so applying the (directed) distance formula gives:

$$\frac{(-r_a)/a + r_a/b + r_a/c - 1}{\sqrt{\sum_{\text{cyc}} a^{-2}}} = -r_a.$$

Solving for $1/r_a$ gives $1/r_a = -1/a + 1/b + 1/c + \sqrt{\sum_{\text{cyc}} a^{-2}}$ as claimed. □

Note. A similar calculation gives

$$1/r_d = \sum_{\text{cyc}} a^{-1} - \sqrt{\sum_{\text{cyc}} a^{-2}}.$$

Summing the claim statement cyclically gives $\sum_{\text{cyc}}(1/r_a) = 1/a + 1/b + 1/c + 3\sqrt{\sum_{\text{cyc}} a^{-2}}$. As we are given $\sqrt{\sum_{\text{cyc}} a^{-2}} = \sqrt{9409} = 97$, we get

$$1/r_d = \sum_{\text{cyc}} 1/r_a - 4\sqrt{\sum_{\text{cyc}} a^{-2}} = 70;$$

$$r_d = 1/50 \implies \boxed{071}.$$

Remark. The original legs intended were $a, b, c = 1/63, 1/48, 1/56$.

3 Elaborate 69 joke

Let

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{2k}{k} \binom{n-k+2}{2} 2^{-n-3k}.$$

If $S^2 = a/b$ for coprime positive integers a, b, find $a+b$.

The sum can be rewritten as

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left((-1)^k \binom{2k}{k} 2^{-4k} \right) \left(2^{-(n-k)} \binom{n-k+2}{2} \right).$$

Seeing the convolution, we do a second rewrite:

$$S = \left(\sum_{a=0}^{\infty} (-1)^k \binom{2k}{k} 2^{-4k} \right) \left(\sum_{b=0}^{\infty} 2^{-b} \binom{b+2}{2} \right).$$

The first sum above evaluates as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} 2^{-4n} &= \sum_{n=0}^{\infty} (-1/4)^n \binom{2n}{n} (1/4)^n \\ &= \sum_{n=0}^{\infty} \binom{-1/2}{n} (1/4)^n \\ &= (1+1/4)^{-1/2} = (2/\sqrt{5}); \end{aligned}$$

(The last line follows by the binomial theorem.)

The second sum is easier:

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{n+2}{2} 2^{-n} &= \sum_{n=0}^{\infty} \binom{-2}{n} (-1/2)^n \\ &= (1-1/2)^{-2} = (4); \end{aligned}$$

Multiplying these sums together gives

$$S = (2/\sqrt{5})(4) = 8/\sqrt{5} \implies S^2 = 64/5 \implies 64 + 5 = \boxed{069}.$$

Remark. We use generalized binomial coefficients. Some alg-manip yields $\binom{-1/2}{n} = (-1/4)^n \binom{2n}{n}$ as used in the computation.

4 60deg antiproblem

In (convex) cyclic quadrilateral $ABCD$ with circumcenter O and diagonals $AC, BD = \sqrt{78}, 13$ respectively, we have $BC = CD$. Let the circumcenter P of $\triangle OAC$ lie on $\overline{BD}$. If the perpendicular from P to $\overline{AC}$ meets the circumcircle of $\triangle OBD$ at a point X on the opposite side of $\overline{AC}$ as P, then $BX/DX = (a - \sqrt{b})/c$ for some positive integers a, b, c with $\gcd(a, b, c) = 1$. Find $a + b + c$.

For brevity let ℓ be the perpendicular bisector of $\overline{AC}$ aka the perpendicular from P to $\overline{AC}$.

Claim – $\angle BOD = 120°$.

Proof. Observe that P lies on $\overline{BD}$ and the perpendicular bisector of $\overline{OC}$ which are supposed to be parallel. Thus the two mentioned lines are coincident which implies the result. □

Claim – X is the orthocenter of $\triangle ABD$.

Proof. Proceed by phantom points, letting H be the mentioned orthocenter. Then, assuming $ABCD$ is oriented clockwise, $\measuredangle BHD = 60° = \measuredangle BOD$ so $H \in (OBD)$;
Now we use the lemma that in a triangle with an $60°$ angle, that vertex is equidistant from. Now C, A are the circumcenter and orthocenter of $\triangle BHD$, so $HC = HA$ and $H \in \ell$ whence $H = X$ as needed. □

Now, to the answer extraction... in some horrible notation, let $BH = u < v = HD$.

Then using the fact that $OH^2 = 9R^2 - (a^2+b^2+c^2)$ in a general triangle ABC with $R = a/\sqrt{3}$ and $a^2 = b^2 - bc + c^2$, it follows that $AC = |u - v|$ in the actual problem. We get the following system (second equation by law of cosines):

$$\begin{cases} (u-v)^2 = AC^2 = 78; \\ u^2 - uv + v^2 = BD^2 = 169. \end{cases}$$

Homogenizing gives

$$7u^2 - 20uv + 7v^2 = 0 \implies \frac{u}{v} = \frac{10 \pm \sqrt{51}}{7} \implies \boxed{68.}$$