

Select geometry favorites

People

September 24, 2022

This is a collection of problems we've done and found particularly cute/beautiful/hard, inspired by chapter 11 of *Euclidean Geometry in Math Olympiads* and Eric Shen's paper *Geometry At Its Best*. Have fun!

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🌲 O Problems

Remark. Some attempt has been made to deviate from the aforementioned two famous geometry papers.

:)))

Problem 1 (SL 1998/G4). Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

Problem 2 (SL 2009/G3). Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y , respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram.

Prove that $GR = GS$.

Problem 3 (SL 2015/G4). Let ABC be an acute triangle and let M be the midpoint of AC . A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC . Determine all possible values of $\frac{BT}{BM}$.

Problem 4 (SL 2016/G7). Let I be the incentre of a non-equilateral triangle ABC , I_A be the A -excentre, I'_A be the reflection of I_A in BC , and l_A be the reflection of line AI'_A in AI . Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B .

- (a) Prove that P lies on line OI where O is the circumcentre of triangle ABC .
- (b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y . Show that $\angle XIY = 120^\circ$.

Problem 5 (EGMO 2015/6). Let H be the orthocentre and G be the centroid of acute-angled triangle ABC with $AB \neq AC$. The line AG intersects the circumcircle of ABC at A and P . Let P' be the reflection of P in the line BC . Prove that $\angle CAB = 60$ if and only if $HG = GP'$.

Problem 6 (Iran TST 2018/1/4). Let ABC be a triangle ($\angle A \neq 90^\circ$). $\overline{BE}, \overline{CF}$ are the altitudes of the triangle. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N . Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that \overline{AP} bisects \overline{BC} .

Problem 7 (Eric Shen). In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. Points X, Y are the projections of E, F onto \overline{AD} respectively. Let $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{TZ}$, and Q lies on \overline{EF} such that $\overline{PQ} \parallel \overline{BC}$, prove that \overline{AQ} is a median.

Problem 8 (China TST 2015/2/3). Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G . Let D be the midpoint of \overline{BC} , and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{AD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω .

Problem 9 (IMO 2019/6). Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC , CA , and AB at D , E , and F , respectively. The line through D perpendicular to EF meets ω at R . Line AR meets ω again at P . The circumcircles of triangle PCE and PBF meet again at Q .

Prove that lines DI and PQ meet on the line through A perpendicular to AI .

Problem 10 (MOP 2019 & USA TST 2019/6). Let ABC be a triangle with incenter I , and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to \overline{BC} at A_1 . Define points B_1 on \overline{CA} and C_1 on \overline{AB} analogously, using the excircles opposite B and C , respectively.

- (MOP 2019) Let E, F be the feet of the altitudes from B, C respectively. Prove that if \overline{EF} touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.
- (USA TST 2019/6) Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then \overline{AD} is tangent to the circumcircle of $\triangle DB_1C_1$.

Problem 11 (USAMO 2016/3). Let $\triangle ABC$ be an acute triangle, and let I_B, I_C , and O denote its B -excenter, C -excenter, and circumcenter, respectively. Points E and Y are selected on \overline{AC} such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on \overline{AB} such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $\overleftrightarrow{I_B F}$ and $\overleftrightarrow{I_C E}$ meet at P . Prove that \overline{PO} and \overline{YZ} are perpendicular.

Problem 12 (TSTST 2018/3). Let ABC be an acute triangle with incenter I , circumcenter O , and circumcircle Γ . Let M be the midpoint of \overline{AB} . Ray AI meets \overline{BC} at D . Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y , while line CO meets γ at C and Q . Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D . Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ .

Problem 13 (IMO 2021/3). Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD , respectively. Prove that the lines BC, EF , and O_1O_2 are concurrent.

Problem 14 (USAMO 2021/6). Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}, \overline{BC} \parallel \overline{EF}, \overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y , and Z be the midpoints of $\overline{AD}, \overline{BE}$, and \overline{CF} . Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

Problem 15 (SL 2021/G8). Let ABC be a triangle with circumcircle ω and let Ω_A be the A -excicle. Let X and Y be the intersection points of ω and Ω_A . Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R . Prove that $\overline{AR} \perp \overline{BC}$.

🌲 1 Solutions

🌲 1.1 SL 1998/G4, by

Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

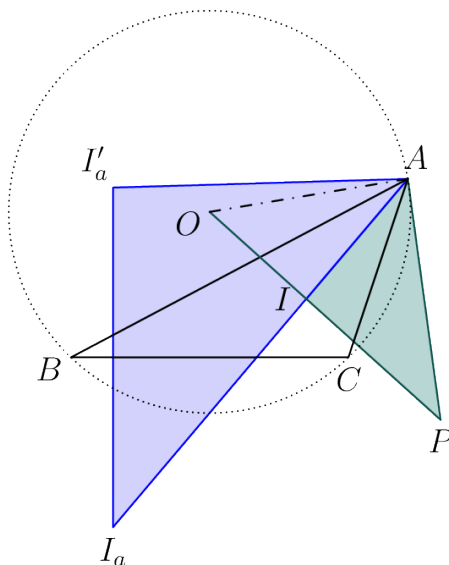
Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

1.2 SL 2016/G7

Let I be the incentre of a non-equilateral triangle ABC , I_A be the A -excentre, I'_A be the reflection of I_A in BC , and l_A be the reflection of line AI'_A in AI . Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B .

- Prove that P lies on line OI where O is the circumcentre of triangle ABC .
- Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y . Show that $\angle XIY = 120^\circ$.



Redefine P as the inverse of I ; it's clear via Poncelet spam that this point satisfies the second part. For the first part we assert more strongly that:

Claim – $\triangle AI_A I'_A \stackrel{+}{\sim} \triangle API$.

Proof. One of the few uses of SAS similarity? By angle chasing, $\angle I_A = \angle P$ follows easily. To finish, we show $I_A I'_A / I_A A = IP / AP$; indeed, the first ratio equals $2 \cos \angle BI_A C = 2 \sin \frac{A}{2}$ because of similar triangles; thus, we're left to length chase IP / AP ; this becomes

$$\begin{aligned} \frac{OP}{AP} - \frac{OI}{OA} \frac{OA}{AP} &= \frac{OI}{AI} - \frac{OI^2}{OA \cdot AI} = \frac{R}{AI} - \frac{R^2 - 2rR}{R \cdot AI} \\ &= \frac{R - (R - 2r)}{AI} = 2 \sin \frac{A}{2}, \end{aligned}$$

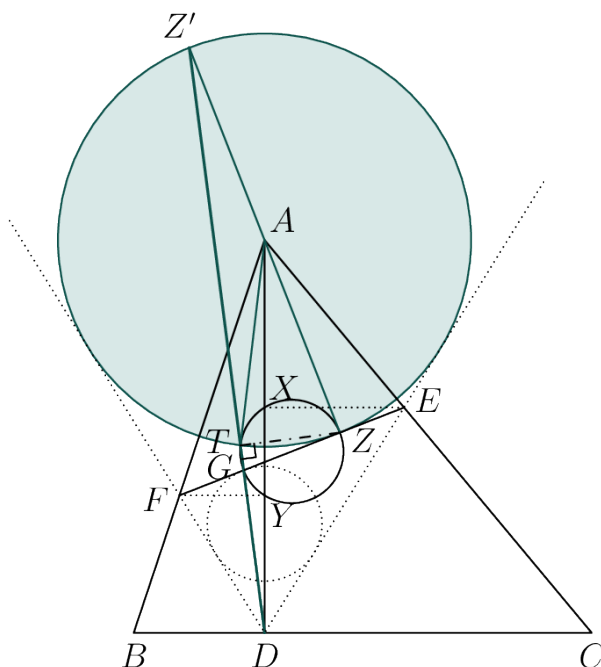
so the ratios are equal, as needed. □

The claim clearly implies the isogonality.

Remark. Surprising how people found the inverse but not the similar triangles...

1.3 Mock AIME 2019/15', by Eric Shen

In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AZ} \cap \overline{QT}$, and Q lies on \overline{EF} such that $\overline{PQ} \parallel \overline{BC}$, prove that \overline{AQ} is a median.



Construct points X, Y as the projections of E, F onto \overline{AD} respectively. *

After drawing a diagram on Geogebra, we obtain:

Characterization of T

T is the harmonic conjugate of Z wrt XY – i.e. it lies on $\omega = (XYZ)$ so that the resulting quadrilateral is harmonic.

In American style, we show that this choice of T lies on ω_a (defined as the circle @ A thru Z) and (DZ) ,

Verification

For $AZ = AT$, we use PoP / length chase to get $AZ^2 = AX \cdot AY$ whence \overline{AZ} touches ω . Hence, by harmonics \overline{AT} is also tangent to ω , so this property follows.

$\angle DTZ = 90^\circ$ is much less straightforward. We define $Z' = 2A - Z$ and $G = E + F - Z$ as the antipodes of Z on the circle at A through Z . By a well-known lemma, D, Z', G collinear (along the cevian through the intouch point in $\triangle DEF$).

But also at the same time, T is on $\omega, \omega_a \Rightarrow \angle ZTG = \angle ZTZ' = \pi/2$ due to antipodes. Hence, $\angle DTZ = \pi/2$, completing the verification.

* Eric Shen originally included these points in the problem statement (as seen in the 2019 version of “Geometry At Its Best”), but I guess the problem’s made harder by deleting them. For me, thinking about their relevance/use was important in making nonzero progress on the problem.

By this definition, we clearly have $(AP; XY) = -1$. From here (the chase is best discovered backwards), harmonic chasing suffices. Define $K = \overline{EF} \cap \overline{A\infty_{BC}}$. Then

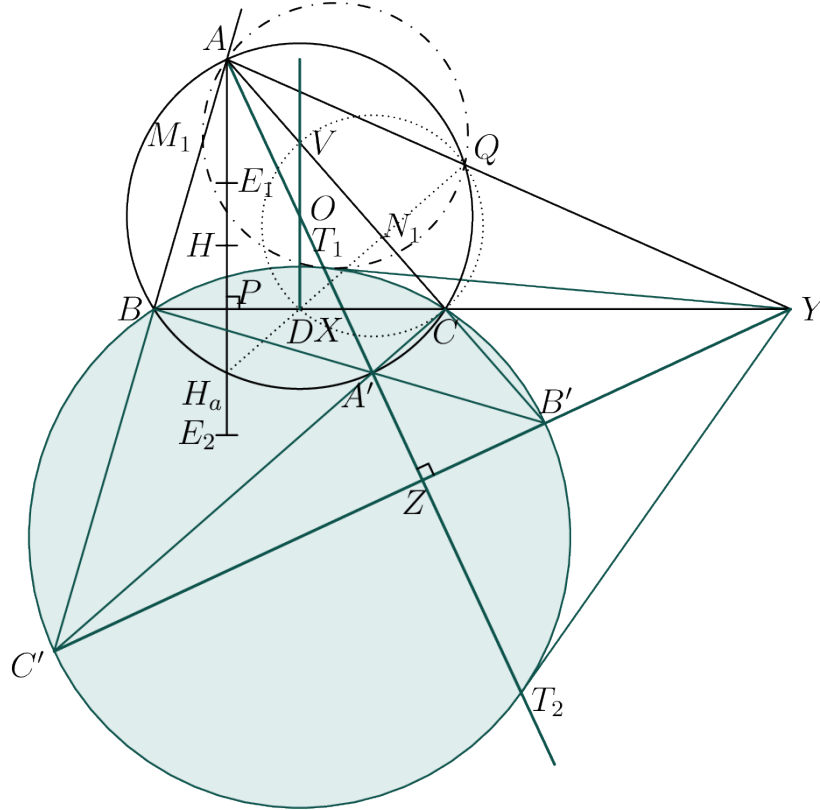
$$(\overline{AQ} \cap \overline{BC}, \infty_{BC}; B, C) \stackrel{A}{=} (QK; FE) \stackrel{\infty_{BC}}{=} (PA; YX) = -1,$$

the end!

1.4 China TST 2015/2/3

Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G . Let D be the midpoint of \overline{BC} , and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{AD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω .



Solution by **nryan**. As much of the parallel/perpendicular lines aren't even relevant (just to give us some equal ratios), we simplify as follows:

Problem reworded

In acute $\triangle ABC$ with circumcenter O and orthocenter H , D is the midpoint of \overline{BC} , and the altitude from A meets (BC) at E (either one works). Let $U, V = \overline{OD} \cap \overline{AB}, \overline{AC}$, respectively; define $M, N \in \overline{AB}, \overline{AC}$ with (lengths directed)

$$UM/MB = VN/NC = AE/EH.$$

Let ω be the circle tangent to segments OB, OC at B, C respectively. Prove that (AMN) is tangent to ω .

We define a load of new points as follows.

- $A' = 2O - A$;
- E_1, E_2 be the choices of E with $AE/EH > 0$ and $AE/EH < 0$ respectively. We will only consider M_1, N_1 , because the negative case is identically handled;
- $T_1, T_2 = \overline{AO} \cap \omega, X = \overline{AO} \cap \overline{BC}$, corresponding to E_1, E_2 from earlier;

- $Y = \overline{T_1 T_1} \cap \overline{T_2 T_2} \cap \overline{BC}$ (which exists since $(BC; T_1 T_2) = -1$);
- Q as the harmonic conjugate of A' wrt BC .

Claim 1 – Q is the Miquel point of $ABCDUV$.

Proof. First, note that if H_a is the reflection of H in \overline{BC} , then H_a, D, Q collinear because

$$-1 = (D \infty_{BC}; BC) \stackrel{H_a}{=} (\overline{H_a D} \cap (ABC), A'; B, C)$$

whence $Q \in \overline{H_a D}$. Now the result follows by Reim because $AH_a QC$ cyclic and $\overline{DV} \parallel \overline{AH_a}$. \square

Claim 2 – (AQT_1) touches ω , $\overline{YT_1}$ at T_1 .

Proof. Sufficient to show $Q \in \overline{AY}$, so that the claim will follow by PoP @Y. Indeed,

$$-1 = (BC; XY) \stackrel{A}{=} (B, C; A', \overline{AY} \cap (ABC))$$

so we're done. \square

Claim 3 – $AE_1/E_1H = AT_1/T_1A'$.

Proof. Construct $P = \frac{E_1+E_2}{2}$, $Z = \frac{T_1+T_2}{2}$ as the foot of A onto \overline{BC} and the foot of O' onto \overline{AO} respectively (O' is the center of ω , and the pole of \overline{BC} wrt (ABC)). Then, sufficient to show $AH/AP = AA'/AZ$.

It's not too hard to show via midpoints of harmonics/etc that Z is the inverse of X wrt (ABC) . Define $B', C' = \overline{A'B} \cap \overline{AC}, \overline{A'C} \cap \overline{AB}$, so that $\overline{ZB'C'}$ is the polar of X wrt (ABC) by Brokard. Meanwhile, note that Z, A' are the orthocenter and foot of A -altitude in $\triangle AB'C'$ (because the polar of $X \perp \overline{AOA'}$ by definition).

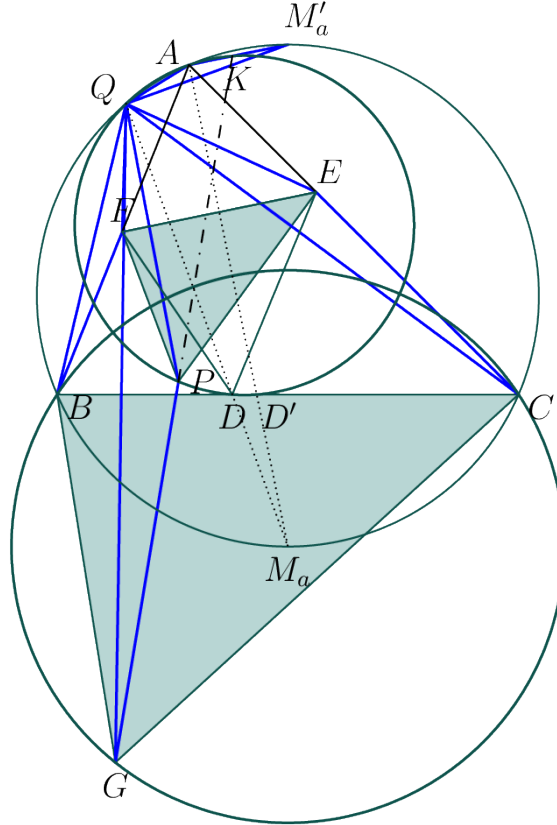
But also recall that $\triangle ABC \sim \triangle AB'C'$! Thus the mentioned points correspond in their respective triangles which completes the claim. \square

We're actually finished now! $M_1, N_1 \in (AQT_1)$ follows by spiral similarity at Q , hence done.

1.5 IMO 2019/6

Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC , CA , and AB at D , E , and F , respectively. The line through D perpendicular to \overline{EF} meets ω at R . Line AR meets ω again at P . The circumcircles of triangle PCE and PBF meet again at Q .

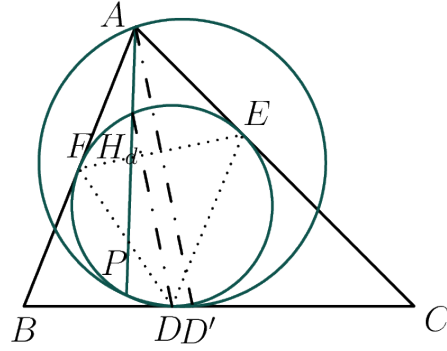
Prove that lines DI and PQ meet on the line through A perpendicular to \overline{AI} .



Observe that P is the D -orthocenter Miquel in $\triangle DEF$. Define K as the intersection of the A -external bisector with \overline{AD} . We make the following definitions...

- Let ω , ω_a denote the incircle and (BIC) respectively;
- Define X as intersection of segment PK with ω . Let Q instead denote the A -SD point;
- G be the harmonic conjugate of I wrt BC , D' as the foot of the A -angle bisector; M_a as the midpoint of arc BC exc. A ; M'_a as the antipode of M_a on (ABC) ;
- H as orthocenter of $\triangle DEF$, and H_d its reflection over \overline{EF} .

\Rightarrow because $MB^2 = MD \cdot MQ = MD' \cdot MA$, $Q \in (ADD'K)$.



Claim 1 - $P \in (ADD'KQ)$.

Proof. Observe that $(PH_d; EF) = -1$ whence A, P, H_d collinear. Then because $\overline{DH_d} \parallel \overline{AI}$ because both perpendicular to \overline{EF} . Hence result by degenerate Reim. \square

Claim 2 - $\triangle PFE \stackrel{+}{\sim} \triangle GBC$.

Proof. Proceed by spiral at Q . Observe that $\triangle H_d EF \stackrel{+}{\sim} \triangle ICB$ by angle chase. Because $(H_d P; EF) = (IG; BC) = -1$, the needed similarity follows. \square

Claim 3 - K, G, P collinear.

Proof. An angle chase, using the previous two claims:

$$\angle QPK \stackrel{\text{claim 1}}{=} \angle QAK = \angle QAM_a \stackrel{\text{spiral}}{=} \angle QPG,$$

the end. \square

Using last two claims, we may chase:

$$\angle PXB = \angle GXB = \angle GCB \stackrel{\text{spiral}}{=} \angle PEF = \angle PFB$$

whence $PXFB$ (and also $PXEC$ by symmetry) cyclic.

This completes the proof.

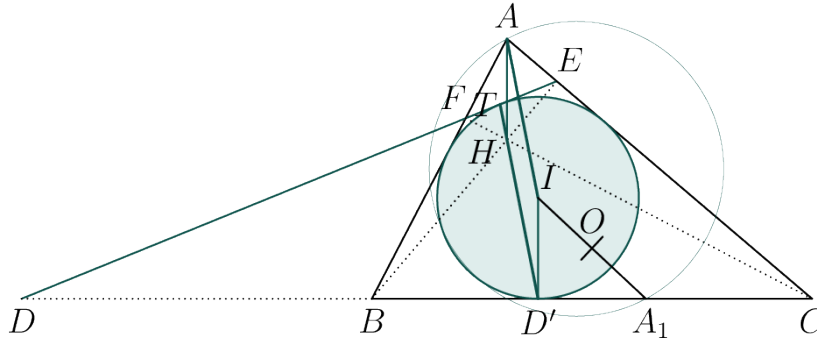
Remark. ggb way too op

1.6 MOP & USA TST 2019

Let ABC be a triangle with incenter I , and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to \overline{BC} at A_1 . Define points B_1 on \overline{CA} and C_1 on \overline{AB} analogously, using the excircles opposite B and C , respectively.

1.6.1 MOP

Let E, F be the feet of the altitudes from B, C respectively. Prove that if \overline{EF} touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.



Call the incircle ω .

Claim 1 – D, E, F are collinear.

Proof. We will prove that the tangent line from D is antiparallel to \overline{BC} wrt $\angle A$. Indeed, this line is found by reflecting \overline{DBC} over \overline{DI} , a line perpendicular to \overline{AI} , so we win. \square

Let ω touch \overline{DEF} at a point T , and let D' denote the A -intouch point.

Claim 2 – $\overline{AI} \parallel \overline{HD'}$; hence $AID'H$ is a parallelogram and $AH = r$, the inradius of $\triangle ABC$.

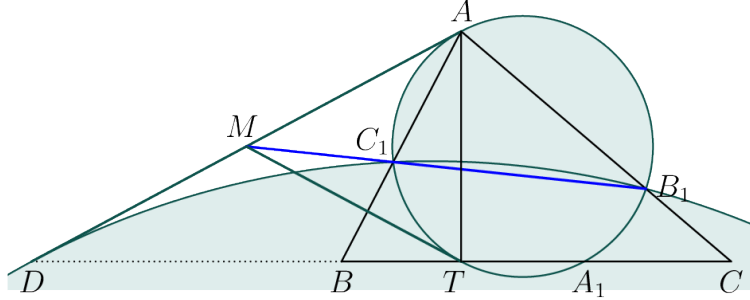
Proof. Because $BCEF$ is tangential, it follows by degenerate Brianchon that lines BE, CF, DT' concur, i.e. $H \in \overline{TD'}$. Observe that $DT = DD'$; then $\overline{THD'} \perp \overline{DI}$ by symmetry, while $\overline{AI} \perp \overline{DI}$ is given; the lines are thus parallel as claimed. \square

Now, let ω_a , etc denote (AB_1C_1) , etc, respectively. We observe that because the perpendicular from A_1 to \overline{BC} and its cyclic variants all concur at the point $2O - I$, it follows that all three circles must concur at this point by Miquel spam.

But because $r/2 = AH/2$ is the distance from O to \overline{BC} , we actually have $2O - I = A_1$ (also because of their feet onto \overline{BC}). Hence $A_1 \in \omega_a$ as desired.

1.6.2 USA TST 2019/6

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then \overline{AD} is tangent to the circumcircle of $\triangle DB_1C_1$.



We first make some observations from working backwards on the previous part.

First, $\overline{I_aA_1}$ is unconditionally the axis of ω_b, ω_c , which is because $2O - I, A_1, I_a$ lie on the same line $\perp \overline{BC}$. Thus, if A_1 is to lie on ω_a , then by anglechase, ω_b, ω_c touch at A_1 whence $I_aA_1 \perp \overline{BC}$.

Also, by MOP 2019 converse (which follows by uniqueness wrt $\angle A$) we have D, E, F collinear. If T is the foot of A onto \overline{BC} , it follows that $(DT; BC) = -1$.

Claim 1 – The A -SD point coincides with the A - orthocenter Miquel.

Proof. Since $BF/CE = \cos B/\cos C = (s-c)/(s-b)$ from 19MOP, result follows by spiral. □

Claim 2 – A_1 is the antipode of A on ω_a .

Proof. Angle chase, observing that ω_b, ω_c touch at A_1 / etc. □

Claim 3 – \overline{AD} is tangent to ω_a .

Proof. Recall that \overline{ADQ} is perpendicular to $\overline{HIQ'}$; thus, equivalent to show $\overline{HQ} \parallel \overline{AA'}$ which is not hard. □

By radical axis/etc, it suffices to show that the midpoint M of \overline{AD} lies on $\overline{B_1C_1}$. By symmetry, $\overline{MA}, \overline{TA}$ touch ω_a .

Claim 4 – $(AT; B_1C_1) = -1$.

Proof. $(AT; B_1C_1) \stackrel{A}{=} (DT; BC) = -1$, the end. □

Result follows from PoP converse on $MD^2 = MA^2 = MB_1 \cdot MC_1$.

🌲 1.7 IMO 2021/3

to be written later

🌲 1.8 SL 2021/G8

Let ABC be a triangle with circumcircle ω and let Ω_A be the A -excircle. Let X and Y be the intersection points of ω and Ω_A . Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R . Prove that $\overline{AR} \perp \overline{BC}$.

Remark. Brianchon exists?