Projective geo
(and popular culture?)

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> If I were to take an all-geometry USAMO, I'd make Black MOP and you (two) would too...
>
> Tiger Zhang

We assume a basic knowledge of harmonics and homography. (refer to, say, chapter 9 of)

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1 Projective warmups

Some familiar examples, the middle three of which are from AIME. (Some are trivialized by a single perspectivity.)

Problem 1.1 (AoPS Olympiad Geometry 12/1???). Let circle ω touch line ℓ at a point B, which has antipode A on that circle. Let P be an arbitrary point outside ω, and let the tangents from P to ω touch it at X, Y. Then, $\overline{AP} \cap \ell$ is the midpoint of the segment with endpoints $\overline{AX} \cap \ell, \overline{AY} \cap \ell$.

Problem 1.2 (2016 II/10). Triangle ABC is inscribed in circle ω. Points P and Q are on side $\overline{AB}$ with $AP < AQ$. Rays CP and CQ meet ω again at S and T (other than C), respectively. If $AP = 4$, $PQ = 3$, $QB = 6$, $BT = 5$, and $AS = 7$, then what is ST?

Problem 1.3 (2018 II/14). The incircle ω of triangle ABC is tangent to $\overline{BC}$ at X. Let $Y \neq X$ be the other intersection of $\overline{AX}$ with ω. Points P and Q lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{PQ}$ is tangent to ω at Y. Assume that $AP = 3$, $PB = 4$, $AC = 8$. What is AQ?

Problem 1.4 (2019 I/15). Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$. What is PQ^2?

2 Some other projective favorites

Harmonics fun.

Problem 2.1 (Shortlist 2015 G3, due to Vivian Loh). Let ABC be a triangle with $\angle C = 90°$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that $\overline{CH}$ bisects $\overline{AD}$. Let P be the intersection point of the lines $\overline{BD}$ and $\overline{CH}$. Let ω be the semicircle with diameter $\overline{BD}$ that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines $\overline{CQ}$ and $\overline{AD}$ meet on ω.

Problem 2.2 (China Southeast 2018/5). In the isosceles triangle ABC with $AB = AC$, the center of a circle ω is the midpoint of the side BC, and $\overline{AB}, \overline{AC}$ are tangent to the circle at points E, F respectively. Point G is on ω with $\angle AGE = 90°$. A tangent line of ω passes through G, and meets $\overline{AC}$ at K. Prove that line BK bisects $\overline{EF}$.

3 Setup

At the end we'll get to "one of the latest fads in olympiad geometry" (Evan Chen, OTIS homography)…

Recall the following:

Definition (involution)
An involution (or involutive pairing) on a line is a map that is its own inverse, and preserves cross-ratio.

Thus, we can project involutions on a line to those on a pencil of lines through a point, and vice versa.

Theorem (characterization)
All non-identical involutions on a line are inversions (possibly with negative power) about some point, possibly at infinity (in which case the inversion is a simple reflection about some point on the line.)

Proof. Routine. □

4 Desargues involution

4.1 The original

Theorem (Desargues's involution theorem aka DIT)
Let quadrilateral $ABCD$ have circumconic γ, and let ℓ be any line. Then, there is an involution on ℓ swapping $(\overline{AB} \cap \ell, \overline{CD} \cap \ell)$, $(\overline{AD} \cap \ell, \overline{BC} \cap \ell)$, $(\overline{AC} \cap \ell, \overline{BD} \cap \ell)$, and the two points $\gamma \cap \ell$.

I think this is proved using homography, but none of the people who taught me it bothered to prove it. Perhaps it isn't that enlightening. From my olympiad experience I'd say this theorem itseif isn't that useful.

Problem 4.1 (Butterfly theorem). In circle ω, M lies on the chords $\overline{AC}$, $\overline{BD}$, $\overline{PQ}$, and is the midpoint of the last. Then M is equidistant from $\overline{AB} \cap \overline{PQ}$, $\overline{CD} \cap \overline{PQ}$.

Time for its more useful and/or famous counterpart...

4.2 The dual

Theorem (Dual of ... aka DDIT)
Let quadrilateral $ABCD$ with inconic γ have $E, F = \overline{AB} \cap \overline{CD}, \overline{AD} \cap \overline{BC}$, and let P be any point in the plane. Then there exists an involutive pairing on the pencil of lines through P, swapping $(\overline{PA}, \overline{PC})$, $(\overline{PB}, \overline{PD})$, $(\overline{PE}, \overline{PF})$, and the (two) tangents from P to γ.

Proof. Reciprocate (do pole/polar thing) wrt γ. □

So how is this used? One of two things happens usually:

(a) Used in a angle-reflective manner: isogonals, for instance;

(b) Project the involutive pairs of lines onto some other line.

Perhaps this is best seen by example.

5 Standard examples

We pull the two most famous examples from USAMO, as noted in lectures by Evan Chen and others.

Problem 5.1 (AMO 2012/5). Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

Problem 5.2 (AMO 2018/5). Let $ABCD$ be a convex cyclic quadrilateral with $E = \overline{AC} \cap \overline{BD}$, $F = \overline{AB} \cap \overline{CD}$, $G = \overline{DA} \cap \overline{BC}$. The circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q. Assume C, B, P, G and C, Q, D, F are collinear in that order. Let $M = \overline{FP} \cap \overline{GQ}$. Prove that $\angle MAC = 90°$.

Remark. I thought for so long that Desargues involution was somehow related to the more famous theorem (at least in old culture).
Even after hearing that phrase I didn't get it. I remember thinking, "don't they say this problem is nuked by ...?" Didn't make sense of the solutions probably because they were the official ones. Key missing word: **dual**.

6 Pset

An olympiad geometer's oblivion, especially after the agony of USAMO 2022. Some of these are also standard. Sadly, it's very hard to make DDIT proposals...

Also, this is nowhere near a complete list. Thanks to Eric Shen (CA) for teaching DDIT to me, and recommending some of the below problems.

Problem 6.1 (CAMO 2021/1). Let ABC be an acute triangle, and let the feet of the altitudes from A, B, C to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ be D, E, F, respectively. Points $X \neq F$ and $Y \neq E$ lie on lines CF and BE respectively such that $\angle XAD = \angle DAB$ and $\angle YAD = \angle DAC$. Prove that X, D, Y are collinear.

Problem 6.2 (Shortlist 2007 G3). Let $ABCD$ be a trapezoid whose diagonals meet at P. Point Q lies between parallel lines BC and AD, and line CD separates points P and Q. Given that $\angle AQD = \angle CQB$, prove that $\angle BQP = \angle DAQ$.

Problem 6.3 (Taiwan TST 2014/3/3). Let M be any point on the circumcircle of triangle ABC. Suppose the tangents from M to the incircle meet $\overline{BC}$ at two points X_1 and X_2. Prove that the circumcircle of triangle MX_1X_2 intersects the circumcircle of ABC again at the tangency point of the A-mixtilinear incircle.

Problem 6.4 (Serbia 2017/6). Let ABC be a triangle. Suppose the two common tangents to its circumcircle and A-excircle meet line BC at two points P and Q. Prove that $\angle PAB = \angle CAQ$.

Problem 6.5 (USA TST 2018/5). Let $ABCD$ be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at H. Denote by M and N the midpoints of $\overline{BC}$ and $\overline{CD}$. Rays MH and NH meet $\overline{AD}$ and $\overline{AB}$ at S and T, respectively. Prove that there exists a point E, lying outside quadrilateral $ABCD$, such that

- ray EH bisects both angles $\angle BES$, $\angle TED$, and
- $\angle BEN = \angle MED$.

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Problem 6.6 (Shortlist 2012 G8). Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Problem 6.7 (Shortlist 2021 G8 (sic), due to Tiger Zhang). Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.