Select geometry favorites

People

September 21, 2022

This is a collection of problems we've done and found particularly cute/beautiful/hard, inspired by chapter 11 of *Euclidean Geometry in Math Olympiads* and Eric Shen's paper *Geometry At Its Best*. Have fun!

Contents

0 Problems

Remark. Some attempt has been made to deviate from the aformentioned two famous geometry papers.

:)))

Problem 1 (SL 1998/G4). Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

Problem 2 (SL 2009/G3). Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram.
Prove that $GR = GS$.

Problem 3 (SL 2015/G4). Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

Problem 4 (SL 2016/G7). Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120°$.

Problem 5 (EGMO 2015/6). Let H be the orthocentre and G be the centroid of acute-angled triangle ABC with $AB \neq AC$. The line AG intersects the circumcircle of ABC at A and P. Let P' be the reflection of P in the line BC. Prove that $\angle CAB = 60$ if and only if $HG = GP'$.

Problem 6 (Iran TST 2018/1/4). Let ABC be a triangle ($\angle A \neq 90°$). $\overline{BE}, \overline{CF}$ are the altitudes of the triangle. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N. Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that $\overline{AP}$ bisects $\overline{BC}$.

Problem 7 (Eric Shen). In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. Points X, Y are the projections of E, F onto $\overline{AD}$ respectively. Let $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90°$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{TZ}$, and Q lies on $\overline{EF}$ such that $\overline{PQ} \parallel \overline{BC}$, prove that $\overline{AQ}$ is a median.

Problem 8 (China TST 2015/2/3). Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G. Let D be the midpoint of $\overline{BC}$, and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{AD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω.

Problem 9 (MOP 2019 & USA TST 2019/6). Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90°$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

(a) (MOP 2019) Let E, F be the feet of the altitudes from B, C respectively. Prove that if $\overline{EF}$ touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.

(b) (USA TST 2019/6) Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.

Problem 10 (USAMO 2016/3). Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $\overleftrightarrow{I_BF}$ and $\overleftrightarrow{I_CE}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

Problem 11 (TSTST 2018/3). Let ABC be an acute triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of $\overline{AB}$. Ray AI meets $\overline{BC}$ at D. Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ.

Problem 12 (IMO 2021/3). Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Problem 13 (USAMO 2021/6). Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

Problem 14 (SL 2021/G8). Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.

1 Solutions

1.1 SL 1998/G4, by

Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

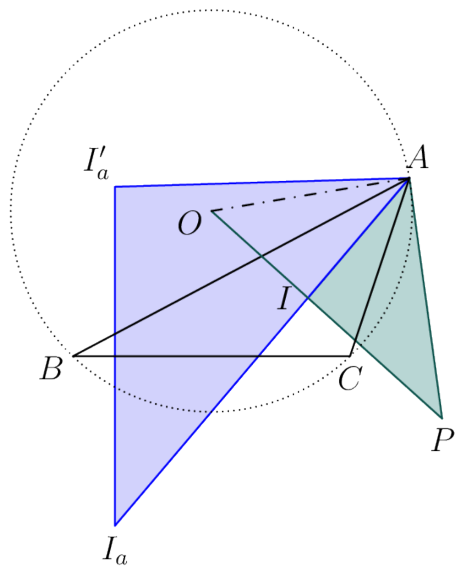
$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

1.2 SL 2016/G7

Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120°$.



Redefine P as the inverse of I; it's clear via Poncelet spam that this point satisfies the second part. For the first part we assert more strongly that:

> **Claim –** $\triangle AI_aI'_a \overset{+}{\sim} \triangle API$.

Proof. One of the few uses of SAS similarity? By angle chasing, $\angle I_a = \angle P$ follows easily. To finish, we show $I_aI'_a/I_aA = IP/AP$; indeed, the first ratio equals $2\cos\angle BI_aC = 2\sin\frac{A}{2}$ because of similar triangles; thus, we're left to length chase IP/AP; this becomes

$$\begin{aligned}\frac{OP}{AP} - \frac{OI}{OA}\frac{OA}{AP} &= \frac{OI}{AI} - \frac{OI^2}{OA\cdot AI} = \frac{R}{AI} - \frac{R^2-2rR}{R\cdot AI} \\ &= \frac{R-(R-2r)}{AI} = 2\sin\frac{A}{2},\end{aligned}$$

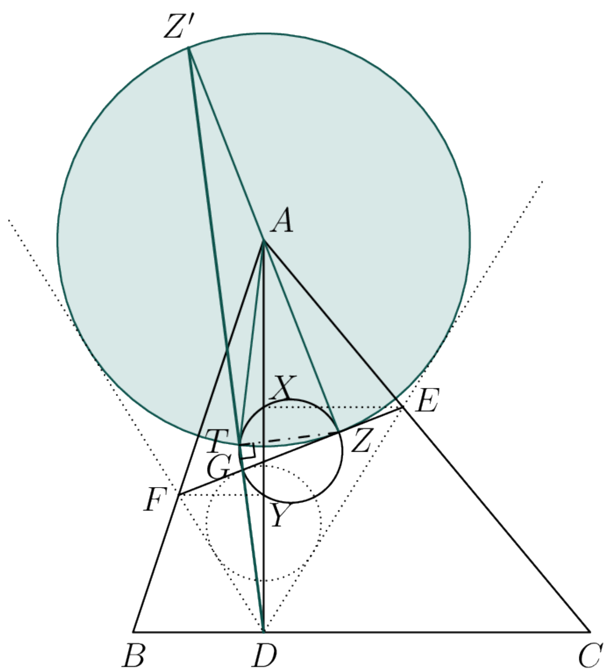
so the ratios are equal, as needed. $\square$

The claim clearly implies the isogonality.

> *Remark.* Surprising how people found the inverse but not the similar triangles…

1.3 Mock AIME 2019/15', by Eric Shen

In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. There exists a point T such that $\angle DTZ = 90°$ and $AZ = AT$. If $P = \overline{AZ} \cap \overline{QT}$, and Q lies on $\overline{EF}$ such that $\overline{PQ} \parallel \overline{BC}$, prove that $\overline{AQ}$ is a median.



Construct points X, Y as the projections of E, F onto $\overline{AD}$ respectively. *

After drawing a diagram on Geogebra, we obtain:

Characterization of T

T is the harmonic conjugate of Z wrt XY– i.e. it lies on $\omega = (XYZ)$ so that the resulting quadrilateral is harmonic.

In American style, we show that this choice of T lies on ω_a (defined as the circle @A thru Z) and (DZ),

Verification

For $AZ = AT$, we use PoP / length chase to get $AZ^2 = AX \cdot AY$ whence $\overline{AZ}$ touches ω. Hence, by harmonics $\overline{AT}$ is also tangent to ω, so this property follows.

$\angle DTZ = 90°$ is much less straightforward. We define $Z' = 2A - Z$ and $G = E + F - Z$ as the antipodes of Z on the circle at A through Z. By a well-known lemma, D, Z', G collinear (along the cevian through the intouch point in $\triangle DEF$).

But also at the same time, T is on $\omega, \omega_a \implies \angle ZTG = \angle ZTZ' = \pi/2$ due to antipodes. Hence, $\angle DTZ = \pi/2$, completing the verification.

* Eric Shen originally included these points in the problem statement (as seen in the 2019 version of "Geometry At Its Best"), but I guess the problem's made harder by deleting them. For me, thinking about their relevance/use was important in making nonzero progress on the problem.

By this definition, we clearly have $(AP; XY) = -1$. From here (the chase is best discovered backwards), harmonic chasing suffices. Define $K = \overline{EF} \cap \overline{A\infty_{BC}}$. Then

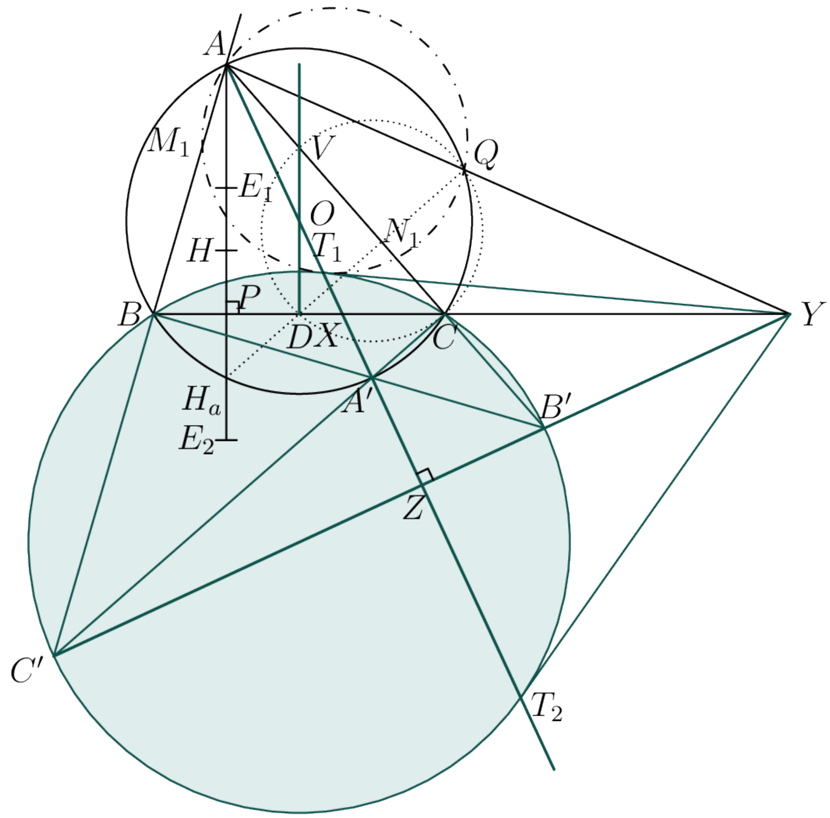
$$(\overline{AQ} \cap \overline{BC}, \infty_{BC}; B, C) \stackrel{A}{=} (QK; FE) \stackrel{\infty_{BC}}{=} (PA; YX) = -1,$$

the end!

1.4 China TST 2015/2/3

Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G. Let D be the midpoint of $\overline{BC}$, and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{AD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω.



As much of the parallel/perpendicular lines aren't even relevant (just to give us some equal ratios), we simplify as follows:

Problem reworded

In acute $\triangle ABC$ with circumcenter O and orthocenter H, D is the midpoint of $\overline{BC}$, and the altitude from A meets (BC) at E (either one works). Let $U, V = \overline{OD} \cap \overline{AB}, \overline{AC}$, respectively; define $M, N \in \overline{AB}, \overline{AC}$ with (lengths directed)

$$UM/MB = VN/NC = AE/EH.$$

Let ω be the circle tangent to segments OB, OC at B, C respectively. Prove that (AMN) is tangent to ω.

We define a load of new points as follows.

- $A' = 2O - A$;
- E_1, E_2 be the choices of E with $AE/EH > 0$ and $AE/EH < 0$ respectively. We will only consider M_1, N_1, because the negative case is identically handled;
- $T_1, T_2 = \overline{AO} \cap \omega, X = \overline{AO} \cap \overline{BC}$, corresponding to E_1, E_2 from earlier;

- $Y = \overline{T_1T_1} \cap \overline{T_2T_2} \cap \overline{BC}$ (which exists since $(BC; T_1T_2) = -1$);
- Q as the harmonic conjugate of A' wrt BC.

Claim 1 – Q is the Miquel point of $ABCDUV$.

Proof. First, note that if H_a is the reflection of H in $\overline{BC}$, then H_a, D, Q collinear because

$$-1 = (D\infty_{BC}; BC) \overset{H_a}{=} (\overline{H_aD} \cap (ABC), A'; B, C)$$

whence $Q \in \overline{H_aD}$. Now the result follows by Reim because AH_aQC cyclic and $\overline{DV} \parallel \overline{AH_a}$. □

Claim 2 – (AQT_1) touches ω, $\overline{YT_1}$ at T_1.

Proof. Sufficient to show $Q \in \overline{AY}$, so that the claim will follow by PoP @Y. Indeed,

$$-1 = (BC; XY) \overset{A}{=} (B, C; A', \overline{AY} \cap (ABC))$$

so we're done. □

Claim 3 – $AE_1/E_1H = AT_1/T_1A'$.

Proof. Construct $P = \frac{E_1+E_2}{2}$, $Z = \frac{T_1+T_2}{2}$ as the foot of A onto $\overline{BC}$ and the foot of O' onto $\overline{AO}$ respectively (O' is the center of ω, and the pole of $\overline{BC}$ wrt (ABC)). Then, sufficient to show $AH/AP = AA'/AZ$.

It's not too hard to show via midpoints of harmonics/etc that Z is the inverse of X wrt (ABC). Define $B', C' = \overline{A'B} \cap \overline{AC}, \overline{A'C} \cap \overline{AB}$, so that $\overline{ZB'C'}$ is the polar of X wrt (ABC) by Brokard. Meanwhile, note that Z, A' are the orthocenter and foot of A-altitude in $\triangle AB'C'$ (because the polar of $X \perp \overline{AOA'}$ by definition).

But also recall that $\triangle ABC \overset{-}{\sim} \triangle AB'C'$! Thus the mentioned points correspond in their respective triangles which completes the claim. □

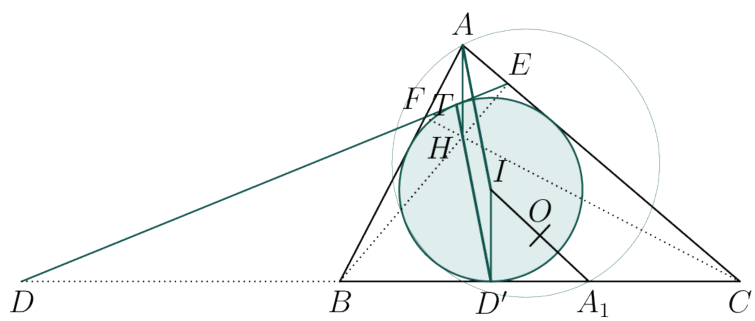
We're actually finished now! $M_1, N_1 \in (AQT_1)$ follows by spiral similarity at Q, hence done.

1.5 MOP & USA TST 2019

Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

1.5.1 MOP

Let E, F be the feet of the altitudes from B, C respectively. Prove that if $\overline{EF}$ touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.



Call the incircle ω.

Claim 1 – D, E, F are collinear.

Proof. We will prove that the tangent line from D is antiparallel to $\overline{BC}$ wrt $\angle A$. Indeed, this line is found by reflecting $\overline{DBC}$ over $\overline{DI}$, a line perpendicular to $\overline{AI}$, so we win. □

Let ω touch $\overline{DEF}$ at a point T, and let D' denote the A-intouch point.

Claim 2 – $\overline{AI} \parallel \overline{HD'}$; hence $AID'H$ is a parallelogram and $AH = r$, the inradius of $\triangle ABC$.

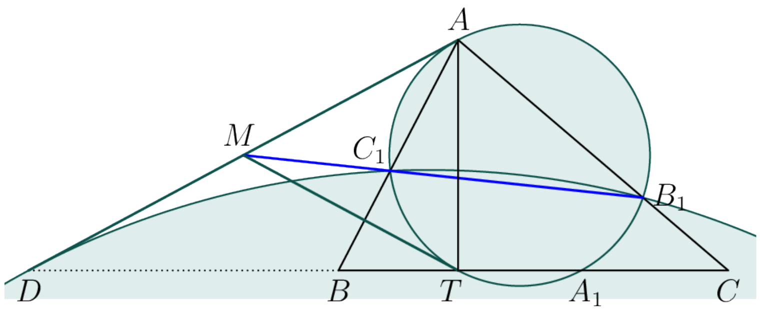
Proof. Because $BCEF$ is tangential, it follows by degenerate Brianchon that lines BE, CF, DT' concur, i.e. $H \in \overline{TD'}$. Observe that $DT = DD'$; then $\overline{THD'} \perp \overline{DI}$ by symmetry, while $\overline{AI} \perp \overline{DI}$ is given; the lines are thus parallel as claimed. □

Now, let ω_a, etc denote (AB_1C_1), etc, respectively. We observe that because the perpendicular from A_1 to $\overline{BC}$ and its cyclic variants all concur at the point $2O - I$, it follows that all three circles must concur at this point by Miquel spam.

But because $r/2 = AH/2$ is the distance from O to $\overline{BC}$, we actually have $2O - I = A_1$ (also because of their feet onto $\overline{BC}$). Hence $A_1 \in \omega_a$ as desired.

1.5.2 USA TST 2019/6

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.



We first make some observations from working backwards on the previous part.

First, $\overline{I_aA_1}$ is unconditionally the raxis of ω_b, ω_c, which is because $2O - I$, A_1, I_a lie on the same line $\perp \overline{BC}$. Thus, if A_1 is to lie on ω_a, then by anglechase, ω_b, ω_c touch at A_1 whence $I_aA_1 \perp \overline{BC}$.

Also, by 19MOP converse we have D, E, F collinear by uniqueness. If T is the foot of A onto $\overline{BC}$, it follows that $(DT; BC) = -1$.

> **Claim 1 –** The A-SD point coincides with the A- orthocenter Miquel.

Proof. Since $BF/CE = \cos B/\cos C = (s-c)/(s-b)$ from 19MOP, result follows by spiral. □

> **Claim 2 –** A_1 is the antipode of A on ω_a.

Proof. Angle chase, observing that ω_b, ω_c touch at A_1 / etc. □

> **Claim 3 –** $\overline{AD}$ is tangent to ω_a.

Proof. Recall that $\overline{ADQ}$ is perpendicular to $\overline{HIQ'}$; thus, equivalent to show $\overline{HQ} \parallel \overline{AA'}$ which is not hard. □

By radical axis/etc, it suffices to show that the midpoint M of $\overline{AD}$ lies on $\overline{B_1C_1}$. By symmetry, $\overline{MA}$, $\overline{TA}$ touch ω_a.

> **Claim 4 –** $(AT; B_1C_1) = -1$.

Proof. $(AT; B_1C_1) \stackrel{A}{=} (DT; BC) = -1$, the end. □

Result follows from PoP converse on $MD^2 = MA^2 = MB_1 \cdot MC_1$.

1.6 IMO 2021/3

Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

(probably gonna make v write this one)

1.7 SL 2021/G8

Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.

Remark. Brianchon exists?