Geometry Favorites (WIP)

People

Last updated December 24, 2022

> ...special things, I compile...
>
> *A Million Dreams*

Colloquially, "the problems of all time"; have fun!

(Note: here, ∞_{XY}, $\infty_{\perp XY}$ denote the points at infinity along line XY and along a line perpendicular to $\overline{XY}$, respectively.)
(also toc moved to p. 2)

-1 Credits + remarks

Inspired by chapter 11 of EGMO book, and **Geometry At Its Best** by Eric Shen.
Also thanks to collaborators...

Things to do:

- add one more **non-triangular** problem;
- fix toc formatting;

Contents

0 Problems

Remark. Some attempt has been made to deviate from the aformentioned two famous geometry papers.

Problem 1 (SL 1998/G4). Let M and N be two points inside triangle ABC such that $\angle MAB = \angle NAC$ and $\angle MBA = \angle NBC$. Prove that
$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

Problem 2 (SL 2009/G3). Let ABC be a triangle. The incircle of $\triangle ABC$ touches AB and AC at the points Z and Y, respectively. Let $G = \overline{BY} \cap \overline{CZ}$, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.

Problem 3 (SL 2015/G4). Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

Problem 4 (SL 2016/G7). Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120^\circ$.

Problem 5 (IMO 2008/6). Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

Problem 6 (EGMO 2015/6). Let H be the orthocentre and G be the centroid of acute-angled triangle ABC with $AB \neq AC$. The line AG intersects the circumcircle of ABC at A and P. Let P' be the reflection of P in the line BC. Prove that $\angle CAB = 60^\circ$ if and only if $HG = GP'$.

Problem 7 (Iran TST 2018/1/4). Let ABC be a triangle ($\angle A \neq 90^\circ$), with altitudes $\overline{BE}, \overline{CF}$. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N. Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that $\overline{AP}$ bisects $\overline{BC}$.

Problem 8 (Eric Shen). In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. Let $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{TZ}$, and Q lies on $\overline{EF}$ such that $\overline{PQ} \parallel \overline{BC}$, prove that $\overline{AQ}$ bisects $\overline{BC}$.

Problem 9 (China TST 2015/2/3). Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G. Let D be the midpoint of $\overline{BC}$, and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{OD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω.

Problem 10 (SL 2018/G5). Let ABC be a triangle with circumcircle ω and incenter I. A line ℓ meets the lines AI, BI, CI at points D, E, F respectively, all distinct from A, B, C, I. Prove that the circumcircle of the triangle determined by the perpendicular bisectors of $\overline{AD}, \overline{BE}, \overline{CF}$ is tangent to ω.

Problem 11 (MOP 2020/1Z). Let $ABCD$ be a quadrilateral inscribed in circle Ω. Circles ω_A and ω_D are drawn internally tangent to Ω, such that ω_A is tangent to $\overline{AB}$ and $\overline{AC}$ while ω_D is tangent to $\overline{DB}$ and $\overline{DC}$. Prove that we can draw a line parallel to $\overline{AD}$ which is simultaneously tangent to both ω_A and ω_D.

Problem 12 (Evan Chen Twitch). Let ABC be a triangle and let T be the contact point of the A-mixtilinear incircle with the circumcircle, and let T' be the reflection of T over BC. Prove that the nine-point circle of $T'BC$ is tangent to the incircle.

Problem 13 (IGO 2020/A4). Convex circumscribed quadrilateral $ABCD$ with its incenter I is given such that its incircle is tangent to $\overline{AD}$, $\overline{DC}$, $\overline{CB}$, and $\overline{BA}$ at K, L, M, and N. Let $E = \overline{AD} \cap \overline{BC}$ and $F = \overline{AB} \cap \overline{CD}$. Let $X = \overline{KM} \cap \overline{AB}$ and $Y = \overline{KM} \cap \overline{CD}$. Let $Z = \overline{LN} \cap \overline{AD}$ and $T = \overline{LN} \cap \overline{BC}$.

Prove that the circumcircle of triangle $\triangle XFY$ and the circle with diameter EI are tangent if and only if the circumcircle of triangle $\triangle TEZ$ and the circle with diameter FI are tangent.

Problem 14 (IMO 2019/6). Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

Problem 15 (MOP 2019 & USA TST 2019/6). Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

(a) (MOP 2019) Let E, F be the feet of the altitudes from B, C respectively. Prove that if $\overline{EF}$ touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.

(b) (USA TST 2019/6) Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to (DB_1C_1).

Problem 16 (RMM 2012/6 + Fake USAMO 2020/3). In triangle ABC with incenter I and circumcenter O, let the incircle ω touch $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F respectively.

- (RMM 2012/6) Let ω_a be the circle through B and C tangent to ω, and define ω_b, ω_c similarly. Finally, let $A' = \omega_b \cap \omega_c \ (\neq A)$, and similarly for points B' and C'. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.
- (Fake USAMO 2020/3) Let T be the projection of D to $\overline{EF}$. The line AT intersects the circumcircle of $\triangle ABC$ again at point $X \neq A$. Circles (AEX) and (AFX) intersect ω again at points $P \neq E$ and $Q \neq F$ respectively. Prove that $\overline{EQ}$, $\overline{FP}$, and $\overline{OI}$ are also concurrent.

Problem 17 (TSTST 2018/3). Let ABC be an acute triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of $\overline{AB}$. Ray AI meets $\overline{BC}$ at D. Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ.

Problem 18 (DeuX MO 2020/II/3). In triangle ABC with circumcenter O and orthocenter H, line OH meets $\overline{AB}$, $\overline{AC}$ at E, F respectively. Let ω be the circumcircle of triangle AEF with center S, meeting (ABC) again at $J \neq A$. Line OH also meets (JSO) again at $D \neq O$. Define $K = (JSO) \cap (ABC) \ (\neq J)$, $M = \overline{JK} \cap \overline{OH}$, and $G = \overline{DK} \cap (ABC) \ (\neq K)$. Prove that (GHM) and (ABC) are tangent to each other.

Problem 19 (USA TST 2021/2). Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$. Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.

Problem 20 (IMO 2021/3). Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Problem 21 (USAMO 2021/6). Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

Problem 22 (SL 2021/G8). Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.

Problem 23 (USEMO 2020/3). Let ABC be an acute triangle with circumcenter O and orthocenter H. Let Γ denote the circumcircle of triangle ABC, and N the midpoint of OH. The tangents to Γ at B and C, and the line through H perpendicular to line AN, determine a triangle whose circumcircle we denote by ω_A. Define ω_B and ω_C similarly.
Prove that the common chords of ω_A,ω_B and ω_C are concurrent on line OH.

Problem 24 (Brazil Olympic Revenge 2021/3). Let I, C, ω and Ω be the incenter, circumcenter, incircle and circumcircle, respectively, of the scalene triangle XYZ with $XZ > YZ > XY$. The incircle ω is tangent to the sides YZ, XZ and XY at the points D, E and F. Let S be the point on Ω such that XS, CI and YZ are concurrent. Let $(XEF) \cap \Omega = R$, $(RSD) \cap (XEF) = U$, $SU \cap CI = N$, $EF \cap YZ = A$, $EF \cap CI = T$ and $XU \cap YZ = O$.

Prove that $NARUTO$ is cyclic.

1 Solutions

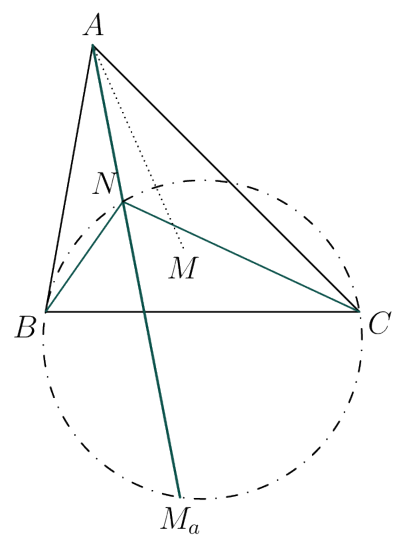
1.1 SL 1998/G4

Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$



Solution 1, by inversion Let i_a denote the inversion at A with power $AB \cdot AC$ composed with reflection in the bisector of $\angle A$. It's well-known that i_a swaps B, C. Let the images of M under i_a be $M_a \in \overline{AN}$, and cyclic variants.

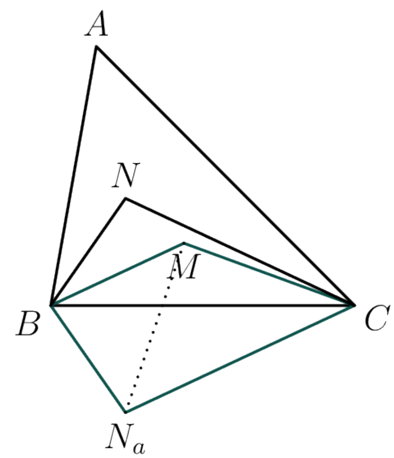
> **Claim –** $M_a \in (BNC)$, and
> $$\frac{AM \cdot AN}{AB \cdot AC} = \frac{AN}{AM_a}.$$

Proof. The first part by angle chase:

$$\measuredangle BM_aN \stackrel{i_a}{=} \measuredangle MCA = \measuredangle BCN,$$

while the second part is just the inversion distance formula. □

The claim reduces the problem to $\sum_{\text{cyc}} AN/AM_a = 1$, which is just **BAMO 2008/6**.



Solution 2, by area ratios (official / intended)

Claim – For any M, N, we have

$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMC] + [BNC]}{[ABC]}.$$

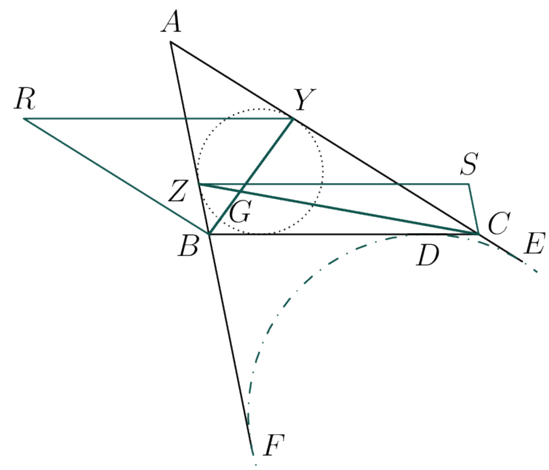
Proof. Reflect N over $\overline{BC}$ to obtain point N_a. Then, because $\angle MBN_a = \angle B$, $(BM \cdot BN)/(BA \cdot BC) = [MBN_a]/[ABC]$. Similarly $(CM \cdot CN)/(CA \cdot CB) = [MCN_a]/[ABC]$, and summing the previous two equations gives

$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMN_aC]}{[ABC]} = \frac{[BMC] + [BNC]}{[ABC]}.$$ □

Noting that M, N are just isogonal conjugates, we obtain the problem by cyclic summation.

1.2 SL 2009/G3, by Hossein Karke Abadi

Let ABC be a triangle. The incircle of $\triangle ABC$ touches AB and AC at the points Z and Y, respectively. Let $G = \overline{BY} \cap \overline{CZ}$, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.



This is a very "troll" problem. Let (R), (S), ω_a denote the point circles at R, S (radius $= 0$) and the A-excircle respectively. Let ω_a touch $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F respectively. Also, for brevity, let $a = BC, b = CA, c = AB, s = (a+b+c)/2$.

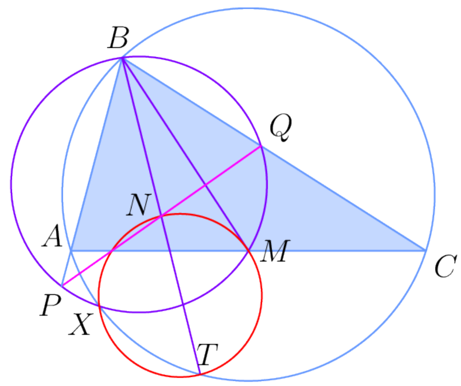
Claim – $\overline{BY}$ is the radical axis of (R), ω_a.

Proof. $BD = BR = s - c$, while $YE = YR = a$; because $\overline{BD}, \overline{YE}$ touch ω_a, B, Y have powers $(s-c)^2, a^2$ wrt each of (R), ω_a as promised. ☐

By the claim, $G = \overline{BY} \cap \overline{CZ}$ must be the radical center of (R), (S), ω_a, implying the desired $GR = GS$.

1.3 SL 2015/G4

Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.



Solution by **CyclicISLscelesTrapezoid**.

The answer is $\sqrt{2}$ only. Let $X = (ABC) \cap (BPMQ) \ (\neq B)$, and let N be the midpoint of $\overline{BT}$.

Claim 1 – $XNMT$ is cyclic.

Proof. Since N is also the midpoint of $\overline{PQ}$, there is a spiral similarity at X sending PNQ to AMC. Thus, we have

$$\measuredangle XMN = \measuredangle XAP = \measuredangle XTB,$$

proving the claim. □

Claim 2 – $\overline{BM}$ is tangent to $(XNMT)$.

Proof. We have

$$\measuredangle XNM = \measuredangle XPA = \measuredangle XPB = \measuredangle XMB.$$ □

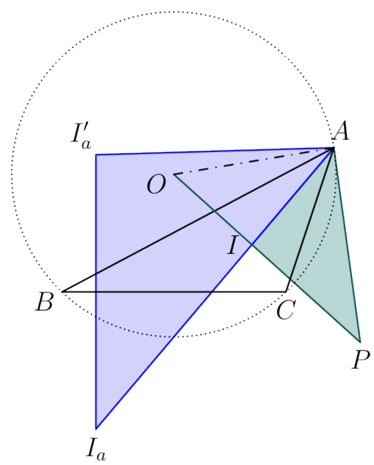
By Power of a Point, $BM^2 = BN \cdot BT = \frac{BT^2}{2}$, so $\frac{BT}{BM} = \sqrt{2}$.

1.4 SL 2016/G7

Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120°$.



Redefine P as the inverse of I wrt (ABC). For the first part we assert more strongly that:

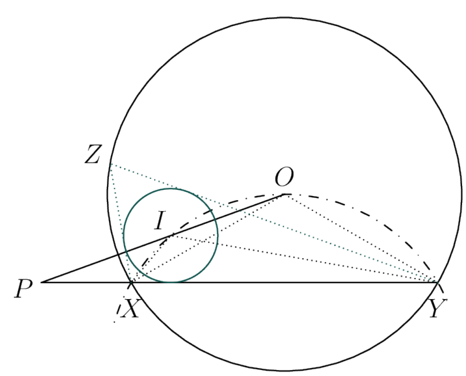
> **Claim –** $\triangle AI_aI'_a \overset{+}{\sim} \triangle API$.

Proof. By angle chasing, $\angle I_a = \angle P$ follows easily. We contend that $I_aI'_a/I_aA = IP/AP$; indeed, the first ratio equals $2\cos\angle BI_aC = 2\sin\frac{A}{2}$ because of similar triangles $I_aBC \overset{-}{\sim} \triangle I_aI_bI_c$, while

$$\frac{IP}{AP} = \frac{OP}{AP} - \frac{OI}{OA}\frac{OA}{AP} = \frac{OA}{AI} - \frac{OI^2}{OA \cdot AI} = \frac{R}{AI} - \frac{R^2 - 2rR}{R \cdot AI} = \frac{R - (R - 2r)}{AI} = 2\sin\frac{A}{2},$$

so the ratios are equal. The similarity follows by SAS. $\square$

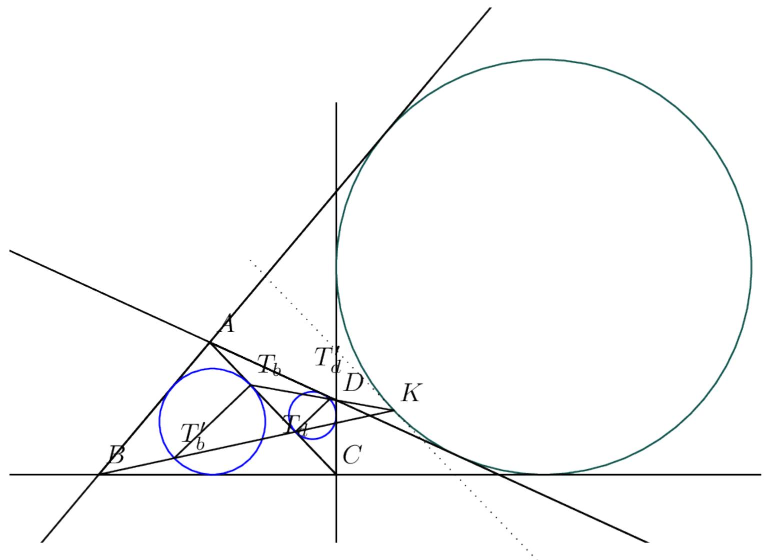
The claim clearly implies the isogonality.



For the second part, using Poncelet, let $Z \in (ABC)$ be the unique point so that $\triangle XYZ, ABC$ share a incircle and circumcircle. Inverting "P, X, Y collinear" wrt the circumcircle gives O, I, X, Y concyclic, or $\angle XOY = \angle XIY$.
As it's well-known that $\angle XOY = 2\angle Z$ and $\angle XIY = (\pi + \angle Z)/2$, we must have $\angle Z = \pi/3 \implies \angle XIY = 2\pi/3$ as needed.

1.5 IMO 2008/6, by Vladimir Shmarov

Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.



Rename ω_1, ω_2 to ω_b, ω_d; by Pitot-like reasoning we have $AB + AD = CB + CD$; let T_b, T_d be the intouch points on $\overline{AC}$; then T_b, T_d are isotomic by the obtained length condition.

If we let T'_b, T'_d be the antipodes of T_b, T_d on their respective circles, then an EGMO lemma (ch4) implies that B, T_d, T'_b and sym variant are collinear.

Construct the point K' on the "closer" side to the rest of the figure so that the tangent to ω at K is parallel to $\overline{AC}$. Then by homothety $K' \in \overline{BT_d}, \overline{DT_b}$, so this is the desired exsimilicenter.

1.6 EGMO 2015/6

Let H be the orthocentre and G be the centroid of acute-angled triangle ABC with $AB \neq AC$. The line AG intersects the circumcircle of ABC at A and P. Let P' be the reflection of P in the line BC. Prove that $\angle CAB = 60^\circ$ if and only if $HG = GP'$.

> I'm just gonna do the 'only if' and not the 'if'.
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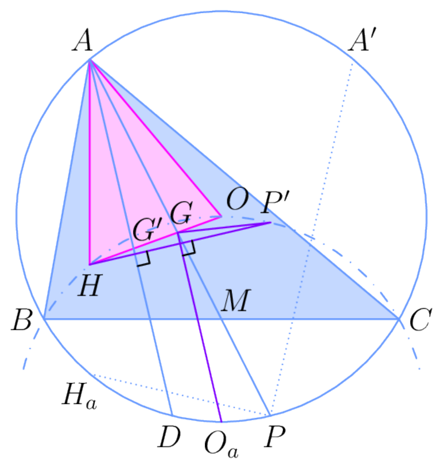
> CyclicISLscelesTrapezoid



Let ℓ be the perpendicular bisector of $\overline{BC}$. Then we unconditionally have:

Claim – $\overline{P'H}$ is perpendicular to the A-symmedian.

Proof. Reflect! Reflect! Reflect! Let D be the intersection of the A-symmedian with (ABC) aka the reflection of P in ℓ, $H_a \in (ABC)$ be the reflection of H in $\overline{BC}$, A' be the reflection of A in ℓ aka the antipode of H_a.

$$\measuredangle(\overline{AD}, \overline{P'H}) = \measuredangle(\overline{AD}, \overline{BC}) + \measuredangle(\overline{BC}, \overline{P'H}) \overset{\text{reflects}}{=} -\measuredangle(\overline{A'P}, \overline{BC}) - \measuredangle(\overline{BC}, \overline{PH_a})$$

$$= -\measuredangle A'PH_a = 90^\circ.$$ □

It's easy to see that O_a – the reflection of the circumcenter O in $\overline{BC}$ – is the center of $(BHP'C)$; $\Longrightarrow O_aH = O_aP = R$ unconditionally. The given length condition is thus equivalent to $\boxed{\overline{O_aG} \perp \overline{HP'}}$, which (by the claim) is in turn equivalent to $\boxed{\overline{O_aG} \parallel \overline{AD}}$.

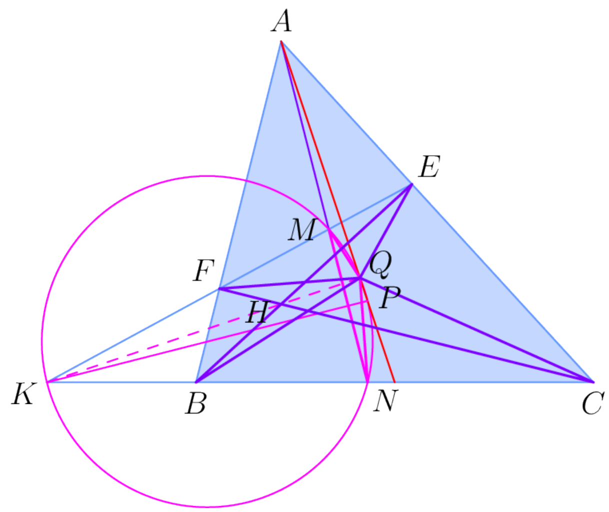
Reflecting yet again, this time in the nine-point center, $(\cdots) \iff \boxed{A, G', D \text{ collinear}}$, where $G' = 2N - G = O + H - G$.

$$\iff \boxed{\overline{AG}, \overline{AG'} \text{ both isogonal and isotomic in } \triangle AHO;}$$

$$\iff \boxed{AH = AO} \iff \boxed{\angle BAC = 60^\circ}.$$

1.7 Iran TST 2018/1/4, by Iman Maghsoudi & Hooman Fattahi

Let ABC be a triangle ($\angle A \neq 90^\circ$), with altitudes $\overline{BE}, \overline{CF}$. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N. Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that $\overline{AP}$ bisects $\overline{BC}$.



Construct $K = \overline{EF} \cap \overline{BC}$, Q as the A-Humpty point, H as the orthocenter of $\triangle ABC$, and $\omega = (KMN)$, so that the P given is the antipode of K on it. Let spiral similarity s at Q take $(E, F) \to (B, C)$. The main point of the problem is then:

Claim – $MKQN$ cyclic. In other words, $Q \in \omega$.

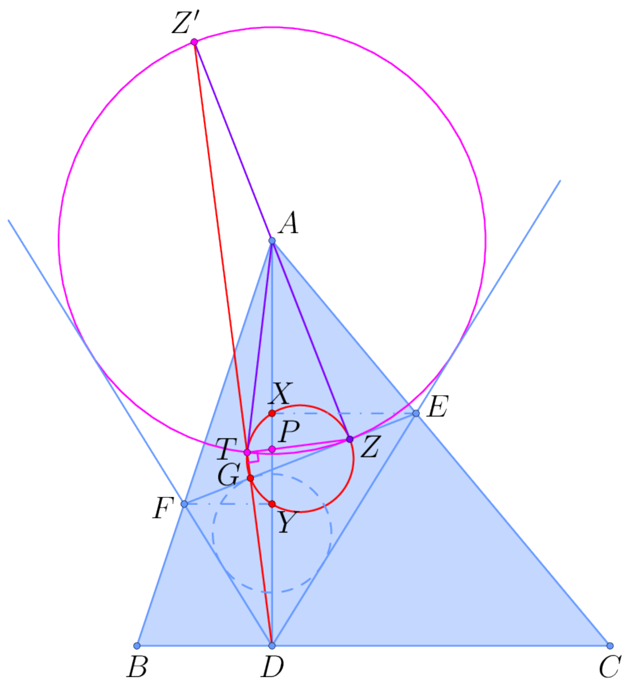
Proof. From angle bisector theorem,

$$\frac{EM}{MF} = \frac{EA}{AF} = \frac{BA}{AC} = \frac{BN}{NC} \Rightarrow (M \xrightarrow{s} N) \Rightarrow \measuredangle MQN = -\arg(s) = \measuredangle(\overline{EF}, \overline{BC}) = \measuredangle MKN. \qquad \square$$

Since P is the antipode of K on ω, $\angle KQP = 90^\circ = \angle KQA$, implying that $P \in \overline{AQ}$, the A-median.

1.8 Mock AIME 2019/15', by Eric Shen & Raymond Feng

In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. Let $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{TZ}$, and Q lies on $\overline{EF}$ such that $\overline{PQ} \parallel \overline{BC}$, prove that $\overline{AQ}$ bisects $\overline{BC}$.



Construct points X, Y as the projections of E, F onto $\overline{AD}$ respectively. *

After drawing a diagram on Geogebra, we obtain:

Characterization of T

T is the harmonic conjugate of Z wrt XY– i.e. it lies on $\omega = (XYZ)$ so that the resulting quadrilateral is harmonic.

In American style, we show that this choice of T lies on ω_a (defined as the circle at A thru Z) and (DZ),

Verification (inspired by USA TST 2015/1)

For $AZ = AT$, we use power of a point / length chase to get $AZ^2 = AX \cdot AY$ whence $\overline{AZ}$ touches ω. Hence, by harmonics $\overline{AT}$ is also tangent to ω, so this property follows.

$\angle DTZ = 90^\circ$ is much less straightforward. We define $Z' = 2A - Z$ and $G = E + F - Z$ as the antipodes of Z on the circle at A through Z. By a well-known lemma, D, Z', G collinear (along the cevian through the intouch point in $\triangle DEF$).

But also at the same time, T is on $\omega, \omega_a \implies \angle ZTG = \angle ZTZ' = \pi/2$ due to antipodes. Hence, $\angle DTZ = \pi/2$, completing the verification.

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*Eric Shen originally included these points in the problem statement (as seen in the 2019 version of "Geometry At Its Best"), but I guess the problem's made harder by deleting them. For me, thinking about their relevance/use was important in making nonzero progress on the problem.

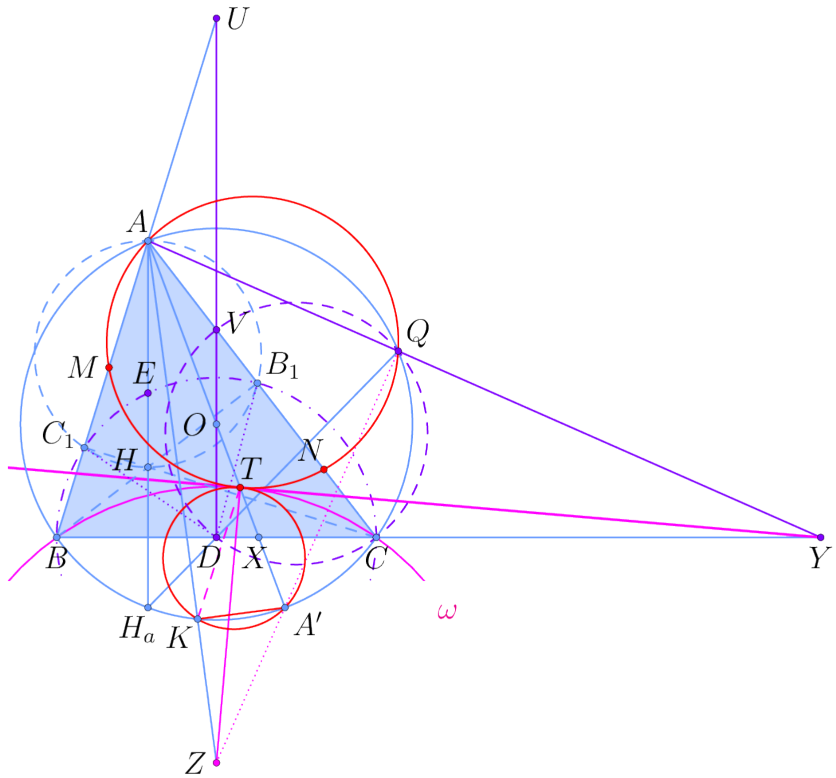
By this definition, we clearly have $(AP; XY) = -1$. From here (the chase is best discovered backwards), harmonic chasing suffices. Define $K = \overline{EF} \cap \overline{A\infty_{BC}}$. Then the bisection is established by

$$(\overline{AQ} \cap \overline{BC}, \infty_{BC}; B, C) \overset{A}{=} (QK; FE) \overset{\infty_{BC}}{=} (PA; YX) = -1.$$

1.9 China TST 2015/2/3

Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G. Let D be the midpoint of $\overline{BC}$, and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{OD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω.



As much of the parallel/perpendicular lines aren't even relevant (just to give us some equal ratios), we simplify as follows:

Problem reworded

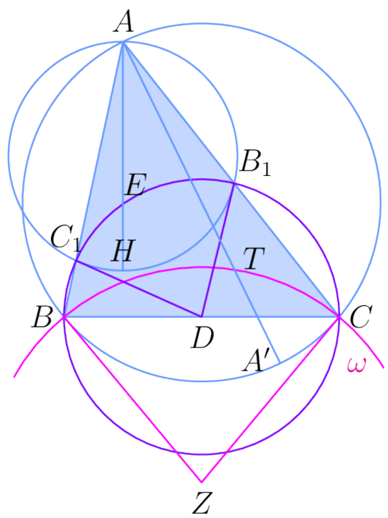
In acute $\triangle ABC$ with circumcenter O and orthocenter H, D is the midpoint of $\overline{BC}$, and the altitude from A meets (BC) at E (either one works). Let $U, V = \overline{OD} \cap \overline{AB}, \overline{AC}$, respectively; define $M, N \in \overline{AB}, \overline{AC}$ with (lengths directed)

$$UM/MB = VN/NC = AE/EH.$$

Let ω be the circle tangent to segments OB, OC at B, C respectively. Prove that (AMN) is tangent to ω.

We define a load of new points as follows:

- $A' = 2O - A$ as the antipode of A on (ABC);
- $T = \overline{AO} \cap \omega$, which we stipulate to be on segment AA' iff E is on segment AH; WLOG, assume this is the case;
- Q as the harmonic conjugate of A' wrt BC, aka the reflection of the A-orthocenter Miquel point Q_a in the perpendicular bisector $\overline{DUV}$ of $\overline{BC}$.



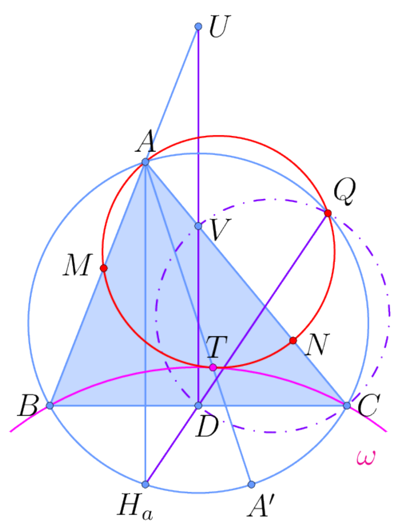
First, we get rid of E:

Claim 1 – $AE/EH = AT/TA'$. (lengths still directed)

Proof. (by **v4913**) Let B_1, C_1 denote the feet of the respective altitudes from B, C, and r a reflection in the bisector of $\angle A$ composed with a homothety at A with scale factor $AH/AA' = AB_1/AB = AC_1/AC$.
Because $\overline{DB_1}$, $\overline{DC_1}$ are well-known to touch (AH), D is the pole of $\overline{B_1C_1}$;

$$\implies (Z \xrightarrow{r} D) \implies (\omega \xrightarrow{r} (BC)) \implies (T_1 \xrightarrow{r} E_1)$$

proving the claim. □



Claim 2 – Q is the Miquel point of $ABCDUV$.

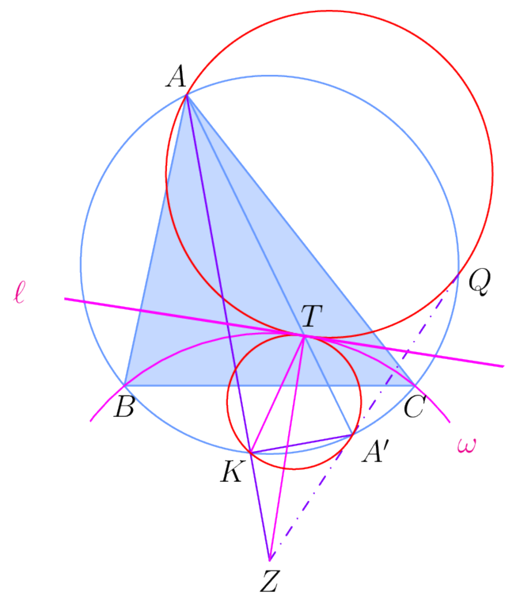
Proof. As we already have $Q \in (ABC)$, sufficient to prove $QDVC$ cyclic. Observe that $Q \in \overline{H_aD}$, which follows by $Q_a \in \overline{A'PH}$ reflected in $\overline{DUV}$. The result holds by Reim because AH_aQC cyclic and $\overline{DV} \parallel \overline{AH_a}$. □

Consider the spiral similarity s at Q mapping $B, C \to U, V$. Since $\triangle BA'C \overset{+}{\sim} \triangle UAV$, $(A' \overset{s}{\to} A)$. By the length condition $(M \overset{s}{\to} N)$ as well, so $M, N \in (AQT)$.

Finally, we turn to the problem statement:

Claim 3 – AQT_1 touches ω at T_1.

We present two finishes.

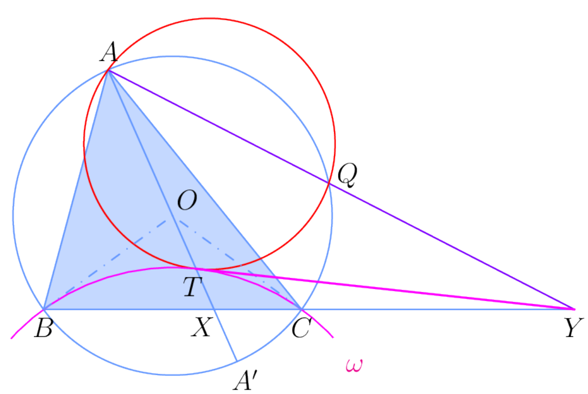


Proof 1, by inversion (v4913) Let $Z \in \overline{QA'}$ be the center of ω aka the polar of $\overline{BC}$ wrt (ABC), and * denote inversion in ω. Define $K = \overline{AZ} \cap (ABC) \quad (\neq A) = A^*$. Clearly, $(A'Q; BC) = -1 \Rightarrow A' = Q^*$. Finally, let $\ell \perp \overline{ZT}$ denote the tangent to ω at T.

It remains to prove that $(A'KT) = (AQT)^*$ touches ℓ at T (and thus ω as well). We do so by angle chase:

$$\measuredangle(\overline{KT}, \ell) = 90^\circ + \measuredangle KTZ \overset{\text{inversion}}{=} 90^\circ + \measuredangle ZAA' = \measuredangle KA'T;$$

inverting back completes the problem. □



Proof 2, by polars (crazyeyemoody907) Let $X = \overline{AO} \cap \overline{BC}$, and Y be the pole of $\overline{AO}$ wrt ω, so that $\overline{YT}$ touches ω. Since $\overline{AO}$ contains the pole O of $\overline{BC}$ wrt ω, we also $Y \in \overline{BC}$ by La Hire.

Finally, we contend that A, Q, Y collinear. Indeed, this follows from

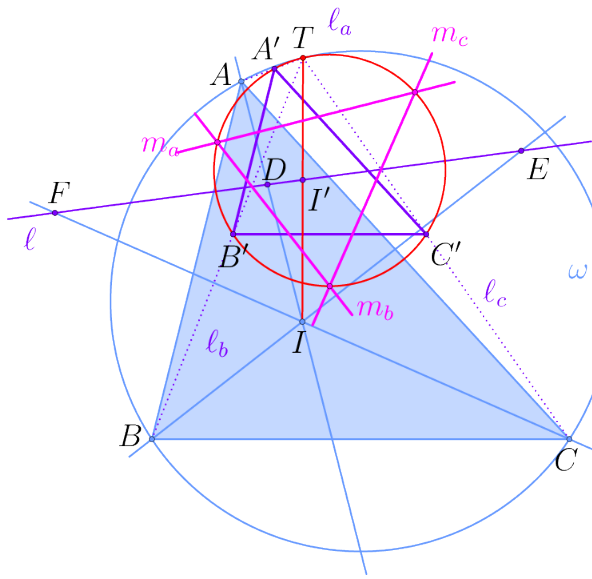
$$(\overline{AY} \cap (ABC), A'; B, C) \stackrel{A}{=} (YX; BC) = -1$$

where the last harmonic bundle holds by definition of polar.
We finish by power of a point at Y: $YT^2 = YB \cdot YC = YA \cdot YQ$ means that (AQT), ω, $\overline{YT}$ are tangent at T. $\square$

Remark. Should definitely use the first diagram for intimidation purposes.

1.10 SL 2018/G5, by Denmark

Let ABC be a triangle with circumcircle ω and incenter I. A line ℓ meets the lines AI, BI, CI at points D, E, F respectively, all distinct from A, B, C, I. Prove that the circumcircle of the triangle determined by the perpendicular bisectors of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ is tangent to ω.



Solution by **TheUltimate123**.

Let ℓ_a and cyclic variants be the reflections of ℓ in the perpendicular bisectors x_a of $\overline{AD}$, etc.

> **Claim –** ℓ_a, ℓ_b, ℓ_c, ω concur at a point T.

Proof. Because

$$\measuredangle(\ell_b, \ell_c) = 2\measuredangle(x_b, x_c) = 2\measuredangle BIC = \measuredangle BAC,$$

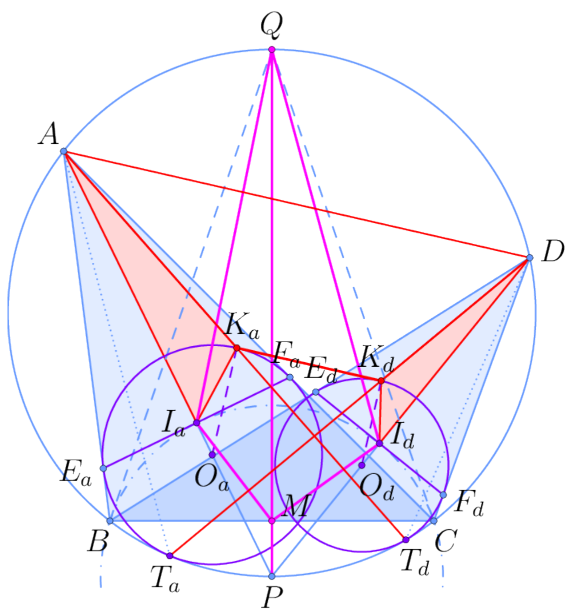
$\ell_b \cap \ell_c \in \omega$; the result follows by symmetry. □

Let $I' = \overline{TI} \cap \ell$, and consider the homothety h at T mapping $I \longrightarrow I'$. Let P' denote the image of point P under h, so I' is the incenter of $\triangle A'B'C'$. Since $\overline{A'I'} \parallel \overline{ADI}$ while $A' \in \ell_a$ and $I' \in \ell$, m_a is also the perpendicular bisector of $\overline{AI}$.

From here it follows that the pairwise intersections of m_a, m_b, m_c are just the arc midpoints in $(A'B'C')$. By h, $(A'B'C')$, (ABC) tangent at T, hence done.

1.11 MOP 2020/1Z, by Evan Chen

Let $ABCD$ be a quadrilateral inscribed in circle Ω. Circles ω_A and ω_D are drawn internally tangent to Ω, such that ω_A is tangent to $\overline{AB}$ and $\overline{AC}$ while ω_D is tangent to $\overline{DB}$ and $\overline{DC}$. Prove that we can draw a line parallel to $\overline{AD}$ which is simultaneously tangent to both ω_A and ω_D.



Solution by **v4913**. Define…

- P, Q as the respective midpoints of $\widehat{BC}, \widehat{BAC}$, I_a, I_d as the respective incenters of ω_a, ω_d, and M as the midpoint of $\overline{BC}$;
- O_a, O_d as respective centers of ω_a, ω_d, and $\gamma = (BI_aI_dC)$ (with center P), so that $\overline{QB}, \overline{QC}$ touch γ;
- $E_a, F_a, T_a = \omega_a \cap \overline{AB}, \overline{AC}, \Omega$; K_a as the intersection of $\overline{AT_d}$ with ω_a closer to A, and their symmetric variants. It's well-known that Q, I_a, T_a collinear, and that I_a is the midpoint of $\overline{E_aF_a}$;
- s_a as the spiral similarity mapping $\gamma \longrightarrow \omega_a$ and thus $Q, B, C, M \longrightarrow A, E_a, F_a, I_a$. Since $\measuredangle K_aAF_a = \frac{1}{2}\widehat{T_dC} = \measuredangle I_dQC$ by design, we also have $(K_a \xrightarrow{s_a} I_d)$.

We contend that $\overline{K_aK_d}$ is the desired tangent, using the following two parts:

Claim 1 – $\overline{O_aK_a}, \overline{O_dK_d} \perp \overline{AD}$.

Proof. We angle chase:

$$\measuredangle(\overline{O_aK_a}, \overline{AD}) = \measuredangle O_aK_aA + \measuredangle K_aAD \stackrel{s_a}{=} \measuredangle PI_dQ + \measuredangle T_dQD = \measuredangle(\overline{PI_d}, \overline{QD}) = \frac{1}{2}\widehat{PQ} = 90^\circ.$$

The claim follows by symmetry. □

Claim 2 – $\overline{K_aK_d} \parallel \overline{AD}$.

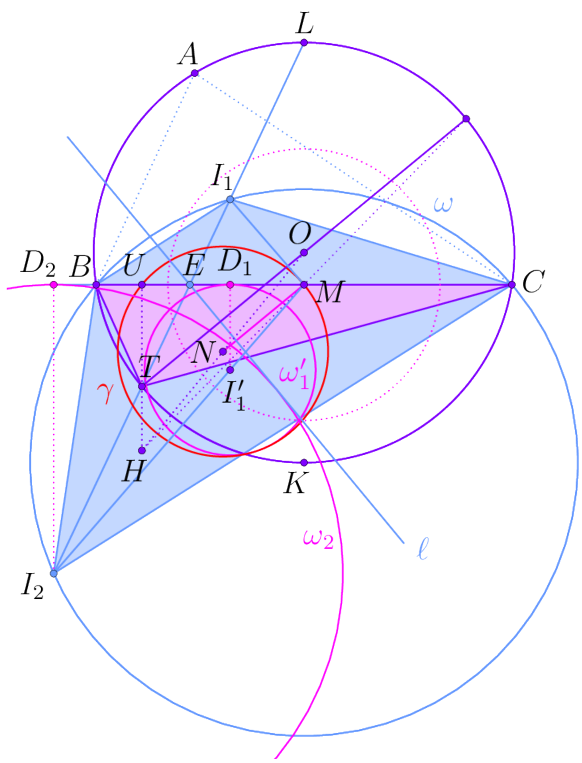
Proof. Let $X = \overline{AT_d} \cap \overline{DT_a}$, so that it'll suffice to prove $AK_a/AX = DK_d/DX$. Indeed, using s_a, $AK_a = QI_d \cdot \frac{AI_a}{QM}$ and similarly $DK_d = QI_a \cdot \frac{DI_d}{QM}$. We thus have:

$$\frac{AK_a}{DK_d} = \frac{AI_a/QI_a}{DI_d/QI_d} = \frac{AT_a/QP}{DT_d/QP} = \frac{AX}{DX}. \qquad \square$$

From the previous two claims, $\overline{O_aK_a}, \overline{O_dK_d} \perp \overline{K_aK_d}, \overline{AD}$ so $\overline{K_aK_d}$ touches both ω_a, ω_d while also parallel to $\overline{AD}$, as required.

1.12 Twitch Solves ISL 006.1

Let ABC be a triangle and let T be the contact point of the A-mixtilinear incircle with the circumcircle, and let T' be the reflection of T over BC. Prove that the nine-point circle of $T'BC$ is tangent to the incircle.



Let K, L, M be the midpoints of $\widehat{BTC}, \widehat{BAC}$,, and I_1 be the incenter, so that $\omega = (BI_1C)$; then, let $I_2 = \overline{NT} \cap \omega \ (\neq I_1)$. Clearly, since $\overline{LB}, \overline{LC}$ touch ω, $(BC; I_1I_2) = -1$. Additionally, since $\angle KTL = 90^\circ$, T is the midpoint of $\overline{I_1I_2}$, a Dumpty point…

Recall the following lemma:

> **Lemma –** In $\triangle ABC$ with A-Dumpty point X, $\overline{AX}$ bisects $\angle BXC$.

Reflect the nine-point circle given to obtain the nine-point circle of $\triangle TBC$. We may now safely get rid of A:

Problem simplified

In harmonic quadrilateral BI_1CI_2, T is the midpoint of $\overline{I_1I_2}$, and I_1' is the reflection of I_1 in $\overline{BC}$ (aka the I_2-Humpty point in $\triangle I_2BC$). For $k \in \{1, 2\}$ let D_k be the foot from I_k to $\overline{BC}$, and ω_k the circle at I_k through D_k. Then the circle ω_1' at I_1' through D_1 touches the nine-point circle γ of $\triangle TBC$.

Recalling the proof of Feuerbach by inverting about the midpoint of a side, we do likewise here. Define…

- U as foot from T to $\overline{BC}$, and $E = \overline{I_1I_2} \cap \overline{BC}$. By midpoints of harmonic bundles lemma applied to $(D_1D_2; ME) \overset{\infty_{\perp BC}}{=} (I_1I_2; NE) = -1$, we have $ME \cdot MU = MD_1 \cdot MD_2$.
- N, O as respective centers of γ, (BTC), and H as orthocenter of $\triangle BTC$;

M is the exsimilicenter of ω_1', ω_2 because it lies on the line of centers $\overline{I_1' I_2}$ as well as the common external tangent $\overline{BC}$. Let * denote inversion at M with power $ME \cdot MU = MD_1 \cdot MD_2$, so that $\omega_1' = \omega_2^*$.

Claim – The reflection ℓ of $\overline{BC}$ in $\overline{I_1 I_2}$ is γ^*.

Proof. Since $E = U^*$, it suffices to prove that $\overline{MN} \perp \ell$. Indeed, $MN \parallel \overline{TO}$ by homothety at H, while reflecting $\overline{TU} \perp \overline{BC}$ in the T-angle bisector $\overline{I_1 T X_2}$ gives $\overline{TO} \perp \ell$. □

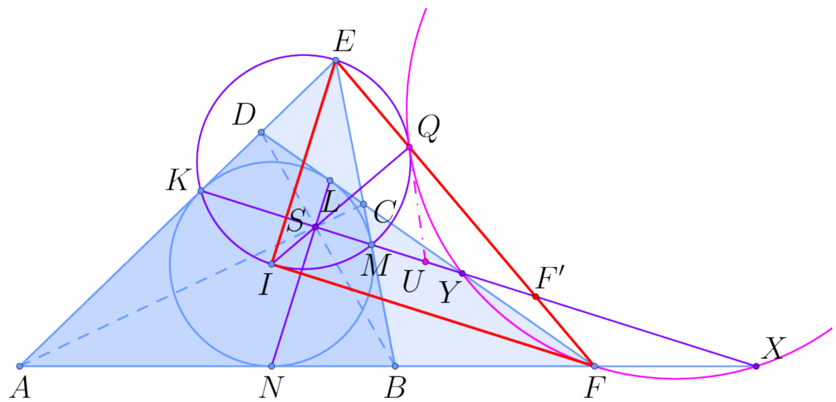
Observe by symmetry about $\overline{I_1 I_2}$ that ℓ also touches ω_2. Inverting back, we have γ tangent to ω_1' as required.

1.13 IGO 2020/A4

Convex circumscribed quadrilateral $ABCD$ with its incenter I is given such that its incircle is tangent to $\overline{AD}$, $\overline{DC}$, $\overline{CB}$, and $\overline{BA}$ at K, L, M, and N. Let $E = \overline{AD} \cap \overline{BC}$ and $F = \overline{AB} \cap \overline{CD}$. Let $X = \overline{KM} \cap \overline{AB}$ and $Y = \overline{KM} \cap \overline{CD}$. Let $Z = \overline{LN} \cap \overline{AD}$ and $T = \overline{LN} \cap \overline{BC}$.

Prove that the circumcircle of triangle $\triangle XFY$ and the circle with diameter EI are tangent if and only if the circumcircle of triangle $\triangle TEZ$ and the circle with diameter FI are tangent.

> imagine doing both directions
>
> CyclicISLscelesTrapezoid



We contend that (XFY), (EI) tangent $\iff$ $\overline{KM} \perp \overline{LN}$, which is just another expression of '$ABCD$ bicentric'.

Call the incircle ω. Define $S = \overline{KM} \cap \overline{LN}$ and Q as the Miquel point of $KLMN$ aka the inverse of S wrt ω, which obviously lies on the polar $\overline{EF}$ of S wrt ω. It follows that $\angle SQE = \angle SQF = 90°$.

Let $F' = \overline{QF} \cap \overline{KM}$ U be the midpoint of $\overline{SF'}$.

> **Claim –** $\overline{QF}$ always bisects $\angle XQY$, and $UM \cdot UK = US^2 = UX \cdot UY$, so U lies on the radical axis of $(EIMK)$, (XFY).

Proof. By Brianchon on $ABMCDK$, $S = \overline{AB} \cap \overline{CD}$ as well. Apply DIT to $\overline{KM}$ and quadrilateral $ABCD$ and project to Q, to obtain an involutive pairing $i : Q(XY; SS; KM)$. The last two pairs imply that i is just reflection in $\overline{QS}$, so $\overline{QS}$ bisects $\angle XQY$. As $\overline{QF} \perp \overline{QS}$, it must also bisect the same angle: $i : Q(F'F')$.
By these right angles and angle bisections, $(SF'; MK) = (SF'; XY) = -1$, so the last result follows by midpoints of harmonic bundles lemma. □

Now because $\angle SQF' = 90°$, we have $UQ = US = UF' \implies (EI), (XQY)$ tangent at Q; this means that the desired is equivalent to $FQXY$ cyclic.

> *Note.* Actually, there are two circles through X, Y, but one of them is extraneous by configuration issues.

Next, we show that this happens iff $FX = FY$.

"If" direction Since $\angle FQX = \angle YQF$, triangles FQX, FQY have equal circumradii, so their circumcircles either coincide or are reflections of each other in $\overline{QF}$. If they were to be reflections, we'd obtain two possibilities, each absurd in the context of the problem: (i) Q, X, Y collinear $\Rightarrow \overline{XY}, \overline{EF}$ coincide; and (ii) X, Y reflections in $\overline{QF} \Rightarrow \overline{XY}, \overline{SQ} \perp \overline{EF}, \overline{XY} \parallel \overline{SQ}$. Thus $(FQX) = (FQY)$ as required.

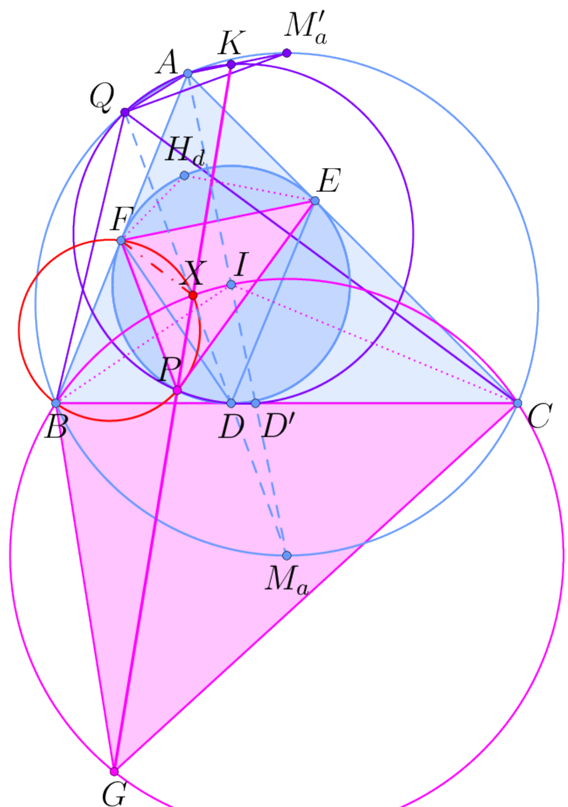
"Only if" direction $\angle YQF = \angle FQX \Rightarrow \widehat{YF} = \widehat{FX}$, trivial;

To finish the problem, observe that $FX = FY$ is equivalent to $\overline{MK}$ making equal angles with $\overline{AB}, \overline{CD}$. As $FN = FL$ as well, $\overline{LN}$ (always) makes equal angles with the same two lines. Since K, L, M, N form a convex polygon, this is in turn equivalent to $\overline{KM} \perp \overline{LN}$, completing the proof.

1.14 IMO 2019/6, by Anant Mudgal

Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to $\overline{EF}$ meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

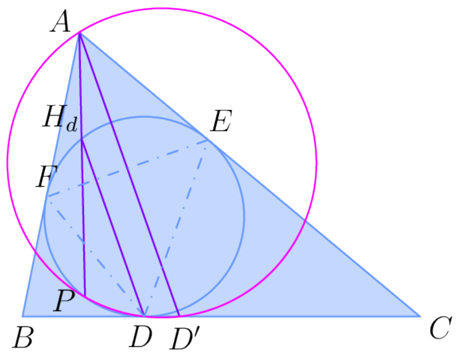
Prove that lines DI and PQ meet on the line through A perpendicular to $\overline{AI}$.



Observe that P is the D-orthocenter Miquel in $\triangle DEF$. Define K as the intersection of the A-external bisector with $\overline{AD}$. We make the following definitions...

- Let ω, ω_a denote the incircle and (BIC) respectively;
- Define X as intersection of *segment* PK with ω. Let Q instead denote the A-SD point;
- G be the harmonic conjugate of I wrt BC, D' as the foot of the A-angle bisector; M_a as the midpoint of arc BC exc. A; M_a' as the antipode of M_a on (ABC);
- H as orthocenter of $\triangle DEF$, and H_d its reflection over $\overline{EF}$.

$\Rightarrow$ because $MB^2 = MD \cdot MQ = MD' \cdot MA$, $Q \in (ADD'K)$.
Thus we want to show that $PXFB$ cyclic. ($PXEC$ cyclic would follow from symmetry, proving that X was indeed the point constructed in the problem.)



Claim 1 – $P \in (ADD'KQ)$.

Proof. Observe that $(PH_d; EF) = -1$ whence A, P, H_d collinear. Then because $\overline{DH_d} \parallel \overline{AI}$ because both perpendicular to $\overline{EF}$. Hence result by degenerate Reim. □

Claim 2 – $\triangle PFE \overset{+}{\sim} \triangle GBC$.

Proof. Proceed by spiral at Q. Observe that $\triangle H_dEF \overset{+}{\sim} \triangle ICB$ by angle chase. Because $(H_dP; EF) = (IG; BC) = -1$, the needed similarity follows. □

Claim 3 – K, G, P collinear.

Proof. An angle chase, using the previous two claims:

$$\measuredangle QPK \overset{\text{claim 1}}{=} \measuredangle QAK = \measuredangle QAM_a \overset{\text{spiral}}{=} \measuredangle QPG.$$ □

Using last two claims, we may angle chase:

$$\angle PXB = \angle GXB = \angle GCB \overset{\text{spiral}}{=} \angle PEF = \angle PFB,$$

or $PXFB$ cyclic.

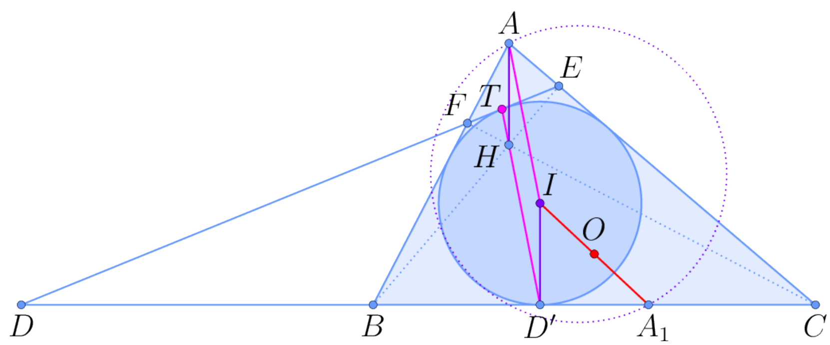
Remark. ggb way too op

1.15 MOP + USA TST, by Ankan Bhattacharya

Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90°$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

1.15.1 MOP 2019/(?)

Let E, F be the feet of the altitudes from B, C respectively. Prove that if $\overline{EF}$ touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.



Call the incircle ω.

Claim 1 – D, E, F are collinear.

Proof. We will prove that the tangent line from D is antiparallel to $\overline{BC}$ wrt $\angle A$. Indeed, this line is found by reflecting $\overline{DBC}$ over $\overline{DI}$, a line perpendicular to $\overline{AI}$, so we win. □

Let ω touch $\overline{DEF}$ at a point T, and let D' denote the A-intouch point.

Claim 2 – $\overline{AI} \parallel \overline{HD'}$; hence $AID'H$ is a parallelogram and $AH = r$, the inradius of $\triangle ABC$.

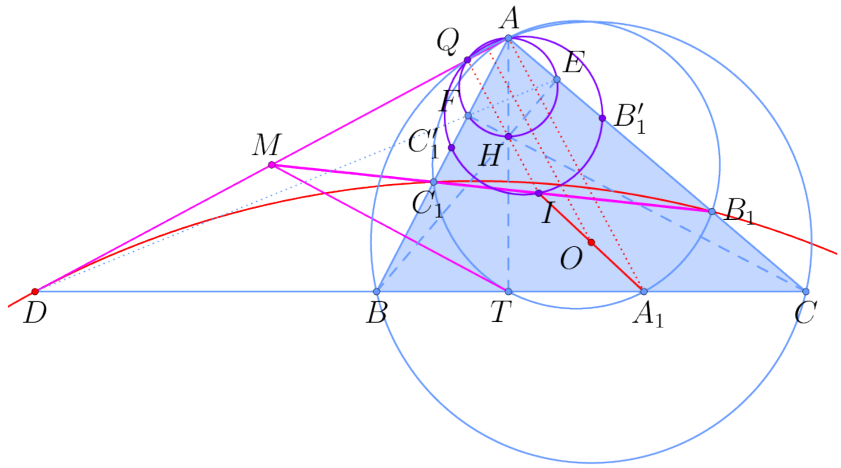
Proof. Because $BCEF$ is tangential, it follows by degenerate Brianchon that lines BE, CF, DT' concur, i.e. $H \in \overline{TD'}$. Observe that $DT = DD'$; then $\overline{THD'} \perp \overline{DI}$ by symmetry, while $\overline{AI} \perp \overline{DI}$ is given; the lines are thus parallel as claimed. □

Now, let ω_a, etc denote (AB_1C_1), etc, respectively. We observe that because the perpendicular from A_1 to $\overline{BC}$ and its cyclic variants all concur at the point $2O - I$, it follows that all three circles must concur at this point by Miquel spam.

But because $r/2 = AH/2$ is the distance from O to $\overline{BC}$, we actually have $2O - I = A_1$ (also because of their feet onto $\overline{BC}$). Hence $A_1 \in \omega_a$ as desired.

1.15.2 USA TST 2019/6

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.



From MOP 2019, we make the following observations:

- By its converse, D, E, F collinear; then, if T is the foot from A to $\overline{BC}$, we have $(TD; BC) = -1$.
- As A_1 is the Bevan point $2O - I$, its projections onto $\overline{AC}, \overline{AB}$ are B_1, C_1 respectively. It follows that A, A_1 are antipodes on ω_a.
- Since $BCEF$ is bicentric, if the incircle touches $\overline{AC}, \overline{AB}$ at B_1', C_1', then $BC_1'/FC_1' = CB_1'/EB_1'$, so the A-incenter and orthocenter Miquel points coincide, say at $Q \in (ABC)$.

From the last item, $\angle AQI = \angle AQH = 90^\circ$.

Claim – $\overline{AD}$ touches ω_a.

Proof. Since $(ABC) \cap (AH) = \{A, Q\}$, the projection of O onto $\overline{AQD}$ is $\frac{A+Q}{2}$. At the same time, the above implies Q is the projection of I onto $\overline{AQD}$. By linearity the projection of $A_1 = 2O - I$ onto $\overline{AD}$ is $2\frac{A+Q}{2} - Q = A$– in other words, $\angle A_1AD = 90^\circ$. This proves the tangency as $\overline{AA_1}$ is a diameter of ω_a. $\square$

Let $M = \frac{A+D}{2}$, so $\overline{MT}$ touches ω_a as well by symmetry in the perpendicular bisector $M\infty_{BC}$ of $\overline{AT}$. Now, $(AT; B_1C_1) \overset{A}{=} (DT; CB) = -1$ means $M \in \overline{B_1C_1}$.

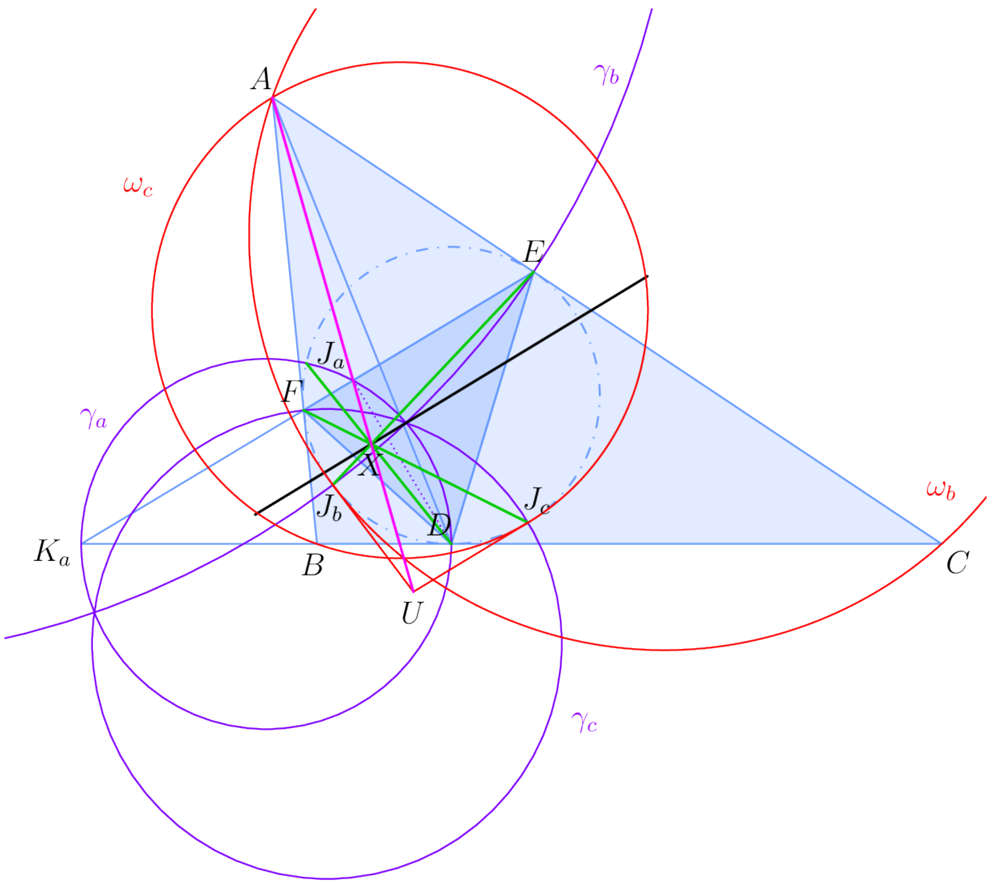
Finish by power of a point converse: $MD^2 = MA^2 = MB_1 \cdot MC_1$ gives the needed tangency.

1.16 RMM + Fake USAMO

1.16.1 RMM 2012/6, by Fedor Ivlev

Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.

Let $K_a = \overline{EF} \cap \overline{BC}$, $\gamma_a = (K_aD)$, $J_a = \omega_a \cap \gamma_a \cap \omega$ (and cyclic variants), and H and ℓ denote the orthocenter and Euler line of $\triangle DEF$, respectively. Also, let I_a, I_b, I_c be the excenters of $\triangle ABC$.

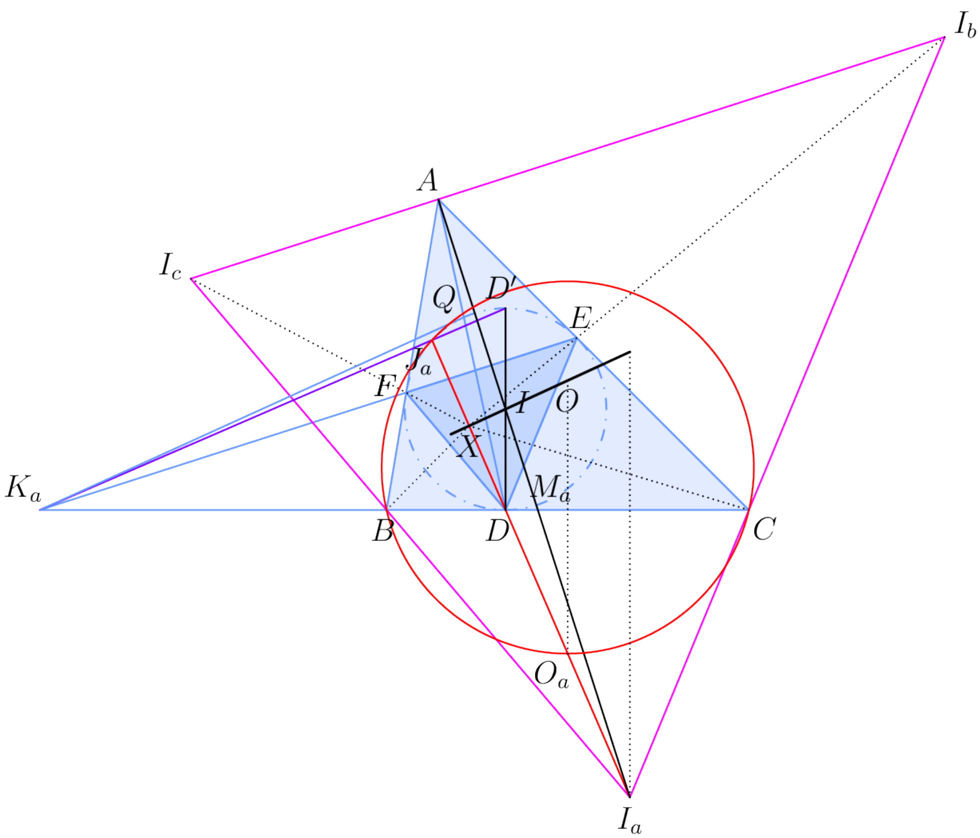


Solution 1, by radical axes Observe that ℓ is just $\overline{OI}$, and that γ_a, etc are coaxial Apollonian circles. Define X as the radical center of $\gamma_a, \gamma_b, \gamma_c, \omega$ (which exists since the former 3 circles are coaxial). We'll show this is the desired concurrency point in both problems. Clearly, $\overline{DJ_a}$ is the raxis of (γ_a, ω), i.e. $X \in \overline{DJ_a}$.

Lemma 1 – ℓ is the raxis of γ_a and variants.

Proof. Let T_a denote the foot of D onto $\overline{EF}$, which is obviously on γ_a. Then H has power $HD \cdot HT_a$ (= variants) wrt the γ's, hence on raxis; Meanwhile I has power r^2 wrt all circles by orthogonality, hence also on raxis, done. □

Let tangents to ω at J_b, J_c meet at U; then, $\overline{AU}$ is the raxis of ω_b, ω_c. Clearly this is the polar of $\overline{J_bJ_c} \cap \overline{EF}$. Recalling that $X = \overline{EJ_b} \cap \overline{FJ_c}$, follows by Brokard that $X \in \overline{AU}$, the end.



Solution 2, by homothety (v4913) Let D' be the antipode of D on ω, $Q = \overline{AD} \cap \omega$ $(\neq D)$; then, because $(EF; DQ) = -1$, $\overline{K_aQ}$ touches ω as well. Also, because $\angle DJ_aD' = \angle DJ_aK_a = 90^\circ$, K_a, J_a, D' are collinear, whence $(DQ; J_aD') = -1$.

We start with X as the similicenter of homothetic triangles $DEF, I_aI_bI_c$. Let homothety h at X with scale factor r map $(D, E, F) \to (I_a, I_b, I_c)$, This must also map their circumcenters to each other, i.e. $I \overset{H}{\Rightarrow} 2O - I$, whence $X \in \overline{OI}$.

Also, let M_a be the midpoint of $\overline{BC}$, $O_a \in \overline{DJ_a}$ be the midpoint of arc BC on ω_a not containing J_a (and variants).

Lemma 2 (SL 2002/G7) – J_a, D, I_a collinear.

Proof. Harmonics:

$$-1 = (DQ; D'J_a) \overset{D}{=} (\overline{BC} \cap \overline{AI}, \overline{J_aD} \cap \overline{AI}; I, A),$$

implying that $\overline{J_aD} \cap \overline{AI}$ is the A-excenter. □

Hence, $X = \overline{DJ_a} \cap \overline{EJ_b} \cap \overline{FJ_c}$.

Claim – O_a is the midpoint of $\overline{DI_a}$.

Proof. By symmetry, M_a is the foot of O_a onto $\overline{BC}$, while it's well-known that $2M - D$ is the foot of I_a onto $\overline{BC}$. M obviously being the midpoint of the segment with endpoints $D, 2M - D$ implies the claim by parallel lines. □

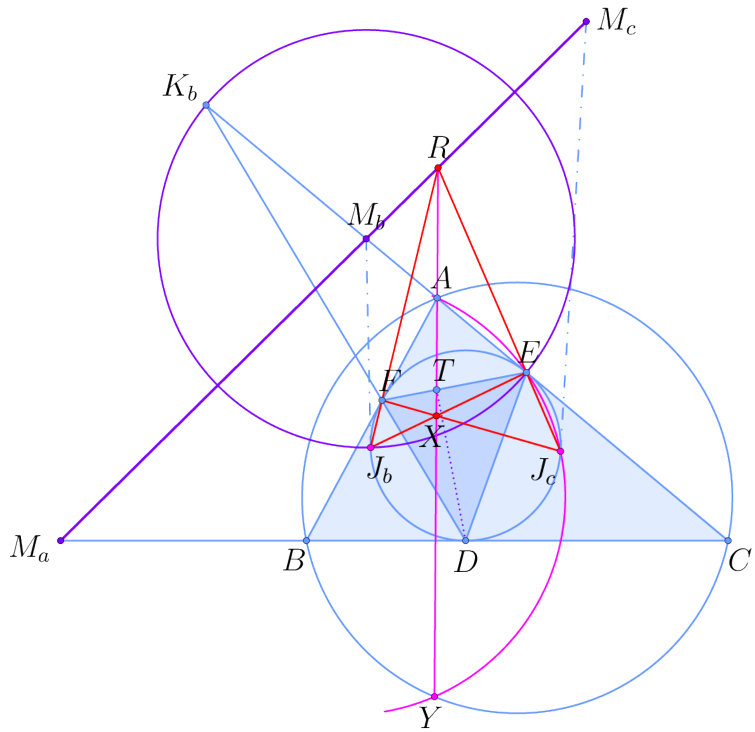
Therefore there must exist a homothety h' at X with scale factor $(1 + r)/2$, mapping $(D, E, F) \to (O_a, O_b, O_c)$. To show that our X is indeed the radical center of $\omega_a, \omega_b, \omega_c$, compute

$$\operatorname{Pow}(X, \omega_a) = XJ_a \cdot XO_a \overset{h'}{=} \frac{1+r}{2} XJ_a \cdot XD = \frac{\operatorname{Pow}(X, \omega)}{(r+1)/2},$$

a symmetric quantity wrt a, b, c.

1.16.2 Fake USAMO 2020/3 (author?)

Let $\triangle ABC$ be a scalene triangle with circumcenter O, incenter I, and incircle ω. Let ω touch the sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F respectively. Let T be the projection of D to $\overline{EF}$. The line AT intersects the circumcircle of $\triangle ABC$ again at point $X \neq A$. The circumcircles of $\triangle AEX$ and $\triangle AFX$ intersect ω again at points $P \neq E$ and $Q \neq F$ respectively. Prove that the lines EQ, FP, and OI are concurrent.



Retain the point definitions from the previous two problems, renaming the X given in the problem to Y. For consistency of notation we let X denote the similicenter of triangles $I_aI_bI_c$, DEF (as before). We'll show that $AEYJ_c$ cyclic– P, Q are just the J_b, J_c from earlier.

Claim 1 – $\overline{EJ_c}$, $\overline{FJ_b}$, $\overline{AT}$ are concurrent at some point R on the polar of X wrt ω.

Proof. I claim that $\overline{AT}$ is the polar of $\overline{EF} \cap \overline{J_bJ_c}$ wrt ω. Indeed, this is just Brokard. By Brokard again, $\overline{EJ_b} \cap \overline{FJ_c}$ is on $\overline{AT}$ as well as the polar of X wrt ω. □

Let M_a be midpoint of $\overline{K_aD}$ (and cyclic variants).

Lemma – The polar of X wrt ω is the radical axis of Ω, ω.

Proof. As $\gamma_b \perp \omega$, and $\overline{M_bE}$ touches ω, $\overline{M_bJ_b}$ must also touch it; in other words M_b is the pole of $\overline{EJ_b}$ wrt ω. As $X \in \overline{EJ_b}$, $\overline{M_aM_bM_c}$ is the polar of X wrt ω by la Hire.
It remains to prove that M_a (and thus cyclic variants by symmetry) is on the radical axis of Ω, ω. Indeed, by the midpoints of harmonic bundles lemma on $(K_aD; BC)$,

$$\operatorname{Pow}(M, \omega) = M_aD^2 = MB \cdot MC = \operatorname{Pow}(M, \Omega)$$ □

From the previous two claims,

$$RA \cdot RY = \mathrm{Pow}(R, \Omega) = \mathrm{Pow}(R, \omega) = RE \cdot RJ_c \Rightarrow AEJ_cY \text{ cyclic},$$

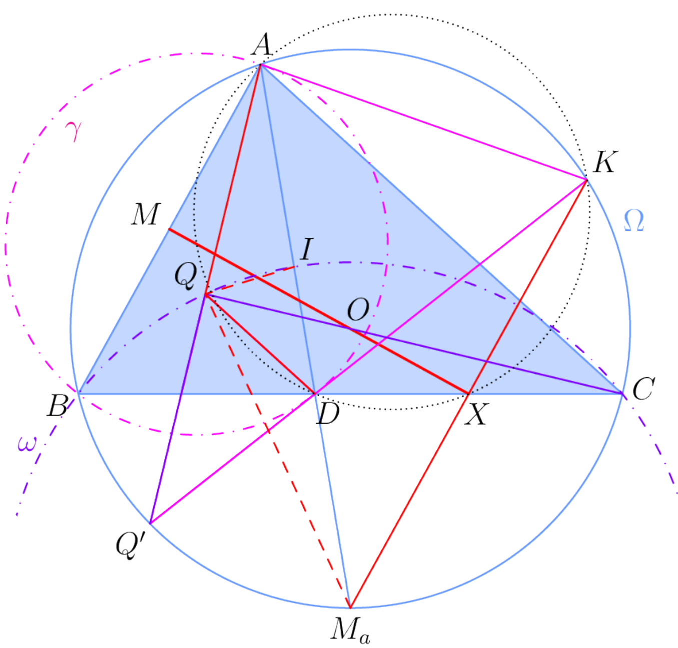
completing the proof.

1.17 TSTST 2018/3, by Evan Chen & Yannick Yao

Let ABC be an acute triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of $\overline{AB}$. Ray AI meets $\overline{BC}$ at D. Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ.

> Geometry is the science of correct reasoning on incorrect figures.
>
> George Pólya / Henri Poincaré



The given angle condition implies $AMQO$ cyclic, or $\angle AQC = \angle AMO = \pi/2$. We make the following definitions:

- $\Omega = (ABC)$, M_a as the center of ω and midpoint of $\widehat{BC}$;
- $Q' = 2Q - A$ as the reflection of A in $\overline{QOC}$– this lies on Ω by symmetry about $\overline{CO}$;
- $K \in \Omega$ as the reflection of M_a in $\overline{MO}$, the perpendicular bisector of $\overline{AB}$.

In the meat of the problem we'll use the following easily findable arcs/angles frequently:

$$\widehat{BM_a} = \widehat{CM_a} = \widehat{AK} = A, \text{ and } \widehat{AC} = \widehat{CQ'} = \widehat{M_aK} \Rightarrow \angle AQ'C = B.$$

Observation

$\overline{QI}$ bisects $\angle AQD$. (Holds because $Q \in \gamma$, the Apollonian circle wrt A, D through I.)

Claim 1 – $\triangle QQ'D \overset{+}{\sim} \triangle M_aAC$.

Proof. First, we'll show $\angle QQ'D = \angle B$, a massive angle chase:

$$\angle M_aAQ = \angle CAQ' - \angle CAM_a = B - \frac{A}{2}, \text{ and } \angle M_aIQ = \frac{\pi - \angle IM_aQ}{2} = \frac{\pi}{2} - \angle ICO = B + \frac{C}{2};$$

$$\Rightarrow \angle AQI = \angle M_aIQ - \angle M_aAQ = \frac{\pi - B}{2}.$$

Applying the observation gives $\angle Q'QD = B$.

To conclude, note that

$$\frac{Q'Q}{QD} = \frac{AQ}{QD} = \frac{AI}{ID} = \frac{AC}{CD} = \frac{AM_a}{M_aC},$$

establishing the similarity by SAS.

(The similarity is negative due to lack of configuration issues, and inspection.) ◻

Claim 2 – Q', D, K collinear.

Proof. Angle chase again: $\measuredangle AQ'D \overset{\text{claim 1}}{=} -\measuredangle M_aAC = \widehat{AC}/2 = \widehat{AK}/2 = \measuredangle AQ'K$. ◻

Part 1: $\overline{KA}$ and $\overline{KD}$ touch γ

Even more angle chasing, taking advantage of claim 2...

$$\measuredangle KAD = \frac{\widehat{KM_a}}{2} = \frac{\widehat{AC}}{2} = \measuredangle ABD, \text{ while } \angle KDA = \frac{\widehat{KA} + \widehat{QM_a}}{2} = \frac{\widehat{Q'C}}{2} = B = \angle DBA,$$

proving the tangencies.

The other, more elegant part of the problem...

Claim 3 – $\overline{MO}, \overline{BC}, \overline{KM_a}, (ADK)$ all concur at a point X.

Proof. Let $X_1 = \overline{MO} \cap \overline{BC}, X_2 = \overline{KM_a} \cap \overline{BC}$.

- $X_1 \in (ADK)$ by similarity: observe by (omitted) angle chase that $\triangle AXB \overset{+}{\sim} \triangle AKD$, whence $\measuredangle AXD = \measuredangle AKD$;
- $X_2 \in (ADK)$ (by contrast) is by power of a point at M_a:

$$M_aB^2 = M_aC^2 = M_aX_2 \cdot M_aK = M_aA \cdot M_aD.$$

As $X_1 = X_2 = (ADK) \cap \overline{BC}$ $(\neq D)$, the claim is proven. ◻

Because $\overline{M_aK} \parallel \overline{AB} \perp \overline{MO}$, and $X = \overline{MO} \cap \overline{M_aK}$ is the inverse of K wrt ω (by the second equation in previous claim's proof), $\overline{MO}$ is the polar of K wrt ω, completing the problem.

Remark. **(crazyeyemoody907)** For me, it took forever to discover the proof of claim 1. Here are some useless formulations of its statement I harped upon:

- $(AC; KM_a) = -1$ which directly implies the first part:

$$(B, \overline{BK} \cap \gamma; A, D) \stackrel{B}{=} (M_aK; AC) = -1;$$

  Since "$\overline{KA}$ touches γ" is very easily provable, K would be polar of $\overline{AD}$ wrt γ as promised...

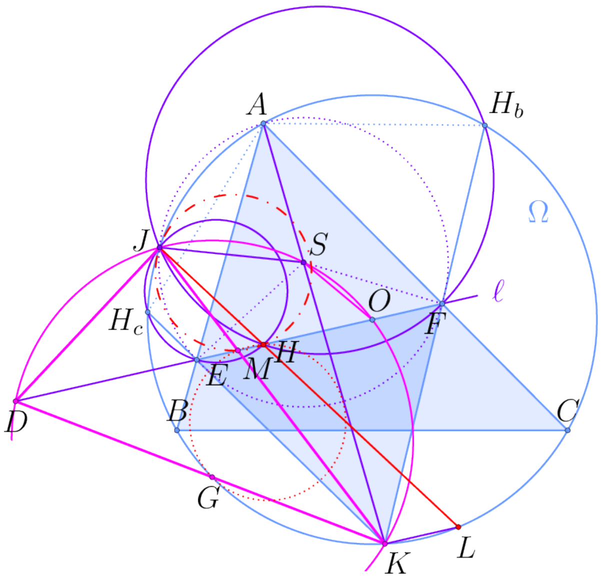
- $BDQQ'$ cyclic ($\iff$ $\overline{QD} \parallel \overline{AC}$ by Reim)

In fact, this means post-solve that $\overline{BQ} \parallel \overline{Q'DK}$... in hindsight, equally useless...

Remark. (Note to self?) When angle chasing in a synthetic problem fails, try length ratios! Examples: SL 2016/G7, GGG 4.5;

1.18 DeuX MO 2020/II/3, by Hao Minyan (China)

In triangle ABC with circumcenter O and orthocenter H, line OH meets $\overline{AB}, \overline{AC}$ at E, F respectively. Let ω be the circumcircle of triangle AEF with center S, meeting (ABC) again at $J \neq A$. Line OH also meets (JSO) again at $D \neq O$. Define $K = (JSO) \cap (ABC) \quad (\neq J)$, $M = \overline{JK} \cap \overline{OH}$, and $G = \overline{DK} \cap (ABC) \quad (\neq K)$. Prove that (GHM) and (ABC) are tangent to each other.



Solution by **crazyeyemoody907**, **v4913**.

Let $\Omega = (ABC)$, H_b, H_c be the respective reflections of H in $\overline{AC}, \overline{AB}$, and $\ell = \overline{EFOH}$. Redefine $K = \overline{H_cE} \cap \overline{H_bF}$ (we'll see this is an equivalent definition). As $\overline{EA}, \overline{FA}$ are external angle bisectors wrt $\triangle KEF$, we have $\angle EKF = \pi - 2A$.

Claim 1 – $J \in (HEH_c), (HFH_b)$.

Proof. Let $J' = (HEH_c) \cap (HFH_b) \quad (\neq H)$. Then:

$$\measuredangle H_cJ'H_b = \measuredangle H_cJ'H + \measuredangle HJ'H_b = \measuredangle H_cEH + \measuredangle HFH_b = \measuredangle(\overline{H_cE}, \overline{H_bF}) = \measuredangle H_bKH_c = \measuredangle H_bAH_c \Rightarrow J' \in \Omega.$$

The construction of J' implies that $\overline{J'E}, \overline{J'F}$ respectively bisect $\angle H_cJ'H$, $\angle H_bJ'H$, and thus

$$\angle EJ'F = \frac{1}{2}\angle H_bJ'H_c = \angle BAC = \angle EAF \Rightarrow J' \in (AEF),$$

finishing the claim. □

Let $L = \overline{JH} \cap \Omega \quad (\neq J)$; then, as JH_cKL, JH_cEH cyclic, $\ell \parallel \overline{KL}$ by Reim. By homothety, (JHM) touches $(JKL) = \Omega$.

Claim 2 – For the K defined in solution, $K \in \overline{AS}, (JSO)$.

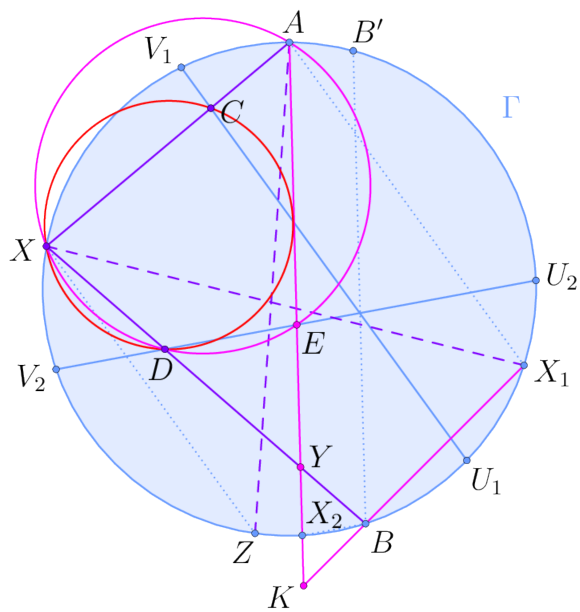
Proof. Since $\measuredangle ESF = 2\measuredangle BAC = \measuredangle EKF$, we have $KESF$ cyclic; as $SE = SF, AH_b = AH_c, A, S$ both lie on bisector of $\angle EKF$.
Next, we prove that O is the midpoint of $\widehat{JSK}$ on (JSK). Because $\overline{OS}$ is the perpendicular bisector of $\overline{AJ}$ by symmetry, it externally bisects $\angle JSK$ as $K \in \overline{AS}$. At the same time, $OJ = OK$ means O is on the perpendicular bisector of $\overline{JK}$. These two properties imply that O is the claimed arc midpoint. $\square$

From here, as $DJKO$ cyclic and $OJ = OK$, $\overline{DO}$ bisects $\angle JDK$, and $G = \overline{DK} \cap \Omega$ is the reflection of J in ℓ by symmetry. Reflecting "(JHM) touches Ω" over ℓ completes the proof.

1.19 USA TST 2021/2, by Andrew Gu & Frank Han

Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$. Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.



Clearly, the problem statement should hold for any $X \in \Gamma$; here, all lengths are directed.
Let X_1, X_2 be the respective reflections of A, B in the perpendicular bisectors of $\overline{U_1V_1}$, $\overline{U_2V_2}$. We assert that $K = \overline{AX_2} \cap \overline{BX_1}$ fits the bill. For brevity, let '$\leftrightarrow$' denote 'is a constant multiple of', so '$x \leftrightarrow 1$' is a shorthand for 'x is constant'.

By Reim, $E = \overline{BX} \cap \overline{AX_2}$ lies on (ADX), so $\operatorname{Pow}(K, (ADX)) = KE \cdot KA \leftrightarrow 1$. Now, in the spirit of linpop, let $f(P) = \operatorname{Pow}(P, (ADX)) - \operatorname{Pow}(P, (XCD))$, so that because $f(Y) = 0$, we have

$$f(K) = -\frac{KY}{YA} f(A) = AC \cdot AX \frac{KY}{AY}.$$

The rest is a wild length chase; let B', Z be the respective reflections of B, X in the perpendicular bisector of $\overline{U_1V_1}$, so that $XX_1 = AZ$ and $\overline{AZ}, \overline{ACX}$ isogonal wrt $\angle U_1AV_1$. Then, observing that all lengths not involving X, C, D, Y are fixed,

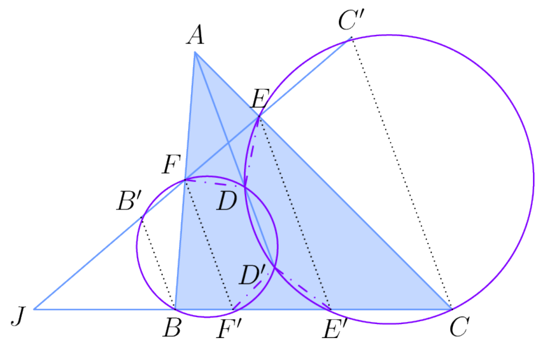
$$\frac{KY}{AY} = (KA; Y\infty_{AK}) \stackrel{B}{=} (X_1A; XB') \leftrightarrow \frac{X_1X}{AX} = \frac{AZ}{AX};$$
$$\Rightarrow f(K) \leftrightarrow AC \cdot AZ = AU_1 \cdot AV_1 \leftrightarrow 1,$$

where the last equality follows because Z, C swapped by inversion at A with power $AU_1 \cdot AV_1$ composed with reflection in the angle bisector of $\angle U_1AV_1$, so we win.

Remark. How on earth would someone find K? I considered the degenerate cases when (XCD) is a straight line (which occur when $X = X_1, X_2$, hence their names).

1.20 IMO 2021/3

Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.



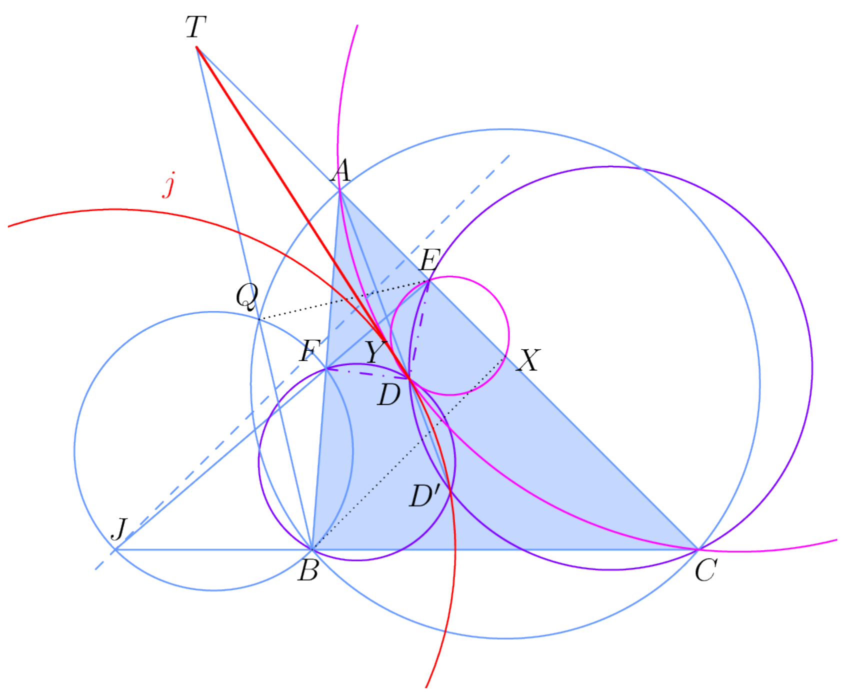
Solution by **v4913**.

Let $J = \overline{EF} \cap \overline{BC}$, and $D' \in \overline{AD}$ be the isogonal conjugate of D wrt $\triangle ABC$. The given angle conditions imply that $BDD'F$, $CDD'E$ are cyclic, while power of a point at A implies $BCEF$ cyclic as well.

> **Claim 1 –** J is the exsimilicenter of (EDC), (FDB); hence, $JD = JD'$ by symmetry.

Proof. Construct $E_1 = (CDD'E) \cap \overline{BC}$ $(\neq C)$, $F_1 = (BDD'F) \cap \overline{BC}$ $(\neq B)$. By isogonality, $DF = D'F'$ and $DE = D'E'$ whence $DD'E'E$, $DD'F'F$ are both cyclic isosceles trapezoids. $\overline{DD'}, \overline{EE'}, \overline{FF'}$ share a perpendicular bisector b, and in fact, this is the bisector of $\angle J$, i.e. $JE = JE'$, $JF = JF'$.

Reflect B, C over b to obtain B', C'; then, because $JB/JF' = JB/JF = JE/JC = JE'/JC$, there is a homothety at J mapping $(B, B', F, F') \to (E', E, C', C)$ and thus their circumcircles $(BB'DD') \to (CC'DD')$ as well. □



Let $Y = (ADC) \cap (EXD) \quad (\neq D)$, Q be the Miquel point of $ABCJEF$, and j the circle at J through D, D'. Observing that $\overline{O_1 O_2}$ is the perpendicular bisector of $\overline{DY}$, it remains to prove $Y \in j$.

> **Claim 2 –** $XQEB$ is cyclic.

Proof. This is a simple angle chase: using cyclic quadrilaterals $(ABCQ)$, $(JFBQ)$, $(ECJQ)$, and $(AEFQ)$, we obtain

$$\measuredangle EQB = \measuredangle EQA + \measuredangle AQB = \measuredangle ACB + \measuredangle EFA = 2\measuredangle ACB = \measuredangle EXB$$ □

Next, we characterize the radical axis of j, (JBF) – it's perpendicular to the line of centers and through A:

> **Claim 3 –** The line through B and the center of (JBF) is perpendicular to $\overline{AC}$.

Proof. This is equivalent to "t_b, the tangent to (JBF) at J, is parallel to $\overline{AC}$". Because $\measuredangle(\overline{BJC}, t_b) = \measuredangle BFJ = \measuredangle JCE$, the result follows. □

Because $\operatorname{Pow}(A, j) = AD \cdot AD' = AQ \cdot AJ = \operatorname{Pow}(A, (JBQF))$, A is on the radical axis of j, (JBF). By the previous claim, it follows that $\overline{AC}$ is the radical axis of j, (JBF).

To finish, define $T = \overline{DY} \cap \overline{AC} \cap \overline{BQ}$ as the radical center of (JBF), (ABC), (EXD), (ADC), and the phantom point $Y' = \overline{TD} \cap j \quad (\neq D)$. Because T is on $\overline{AC}$, the radical axis of j, (JBF), we have (lengths directed)

$$TY' \cdot TD = \operatorname{Pow}(T, j) = \operatorname{Pow}(T, (JBF)) = \operatorname{Pow}(T, (ABCQ)) = TA \cdot TC = TY \cdot TD \Rightarrow Y = Y',$$

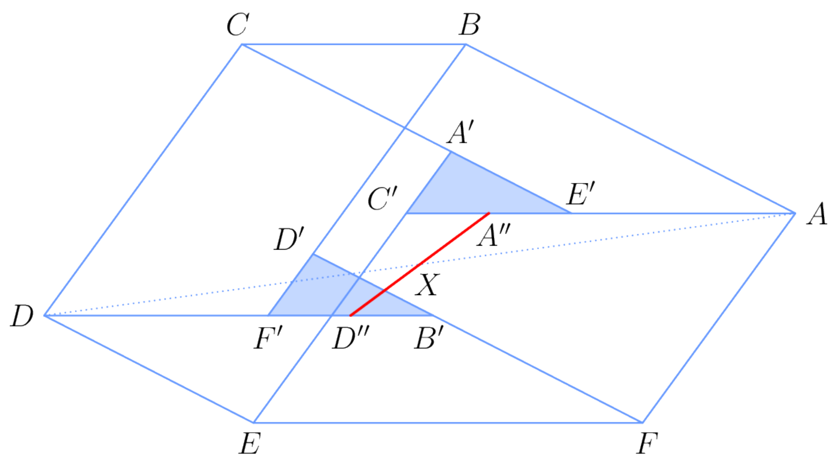
the end!

1.21 USAMO 2021/6, by Ankan Bhattacharya

Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.



Construct parallelogram $CDEA'$ and cyclic variants: $A' = C + E - D$, etc. We may compute using vectors that $\triangle B'D'F'$ is a translation of $\triangle A'C'E'$ by the vector $(B+D+F)-(A+C+E)$. In particular, they're congruent.

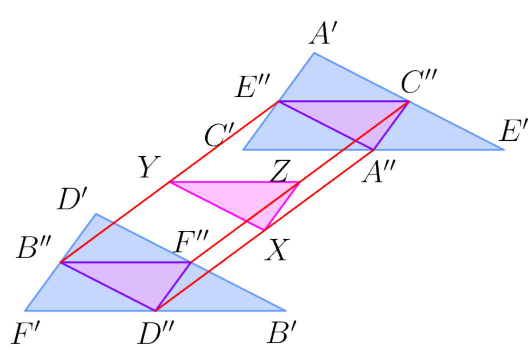
> **Claim 1 –** A, C, E have same power wrt $(A'C'E')$; in other words, $\triangle ACE, A'C'E'$ share a circumcenter.

Proof. Observing that $\operatorname{Pow}(A, (A'C'E')) = AC' \cdot AE' = BC \cdot EF$ by parallelograms, this claim follows by the given length condition. □

Next, construct $A'' = \frac{C'+E'}{2}$ and cyclic variants. The circumcenter of $\triangle A'C'E'$ is then the orthocenter of $\triangle A''C''E''$.

> **Claim 2 –** $X = \frac{A''+D''}{2}$.

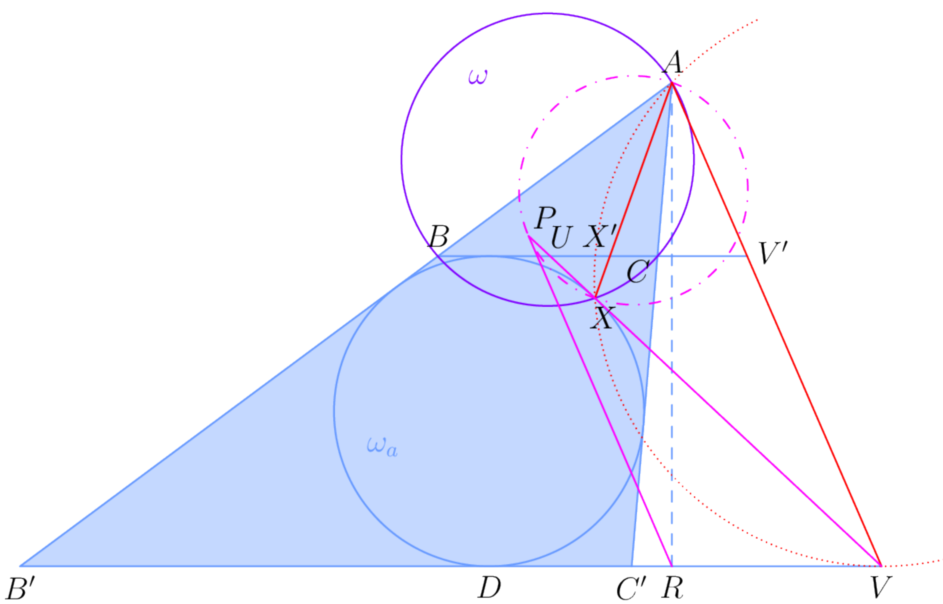
Proof. Using vectors, $B' + C' = E' + F' = A + D \implies \frac{A+D}{2} = \frac{B'+C'+E'+F'}{4} = \frac{A''+D''}{2}$. □



By claim 2 + symmetry, $\triangle XYZ$ is the vector average of (congruent) triangles $A''C''E''$, $B''D''F''$, so their orthocenters are collinear.

1.22 SL 2021/G8

Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.



Solution by **crazyeyemoody907**.

Let the antipode of the A-extouch point be D, and the tangent to ω_a at D intersect $\overline{AB}, \overline{AC}$ at B', C' respectively. Also, construct the tangent line to ω_a at X, meeting $\overline{BC}, \overline{B'C'}$ at U, V respectively. Finally, let $X' = \overline{AX} \cap \overline{BC}$, $V' = \overline{AV} \cap \overline{BC}$.

Claim 1 – $AXUV'$ cyclic.

Proof. Apply DDIT to A, $UXV\infty_{BC}$ (with inconic ω_a), and project onto $\overline{BC}$, to obtain an involutive pairing (B, C), (U, V'), (∞_{BC}, X') – or equivalently, $X'B \cdot X'C = X'U \cdot X'V'$. By power of a point, $X'B \cdot X'C = X'A \cdot X'X$, so the claim follows from power of a point converse on $X'U \cdot X'V = X'A \cdot X'X$. □

Claim 2 – $\overline{DV}$ is tangent to (AXV).

Proof. Angle chase using previous claim, and the fact that $\overline{BC} \parallel \overline{B'C'}$:

$$\measuredangle XAV \overset{\text{claim 1}}{=} \measuredangle XUV' = \measuredangle XVD. \quad \square$$

Redefine R as the foot from A to $\overline{B'C'}$. It remains to show,

Claim 3 – $\overline{PR}$ touches (APX').

Proof. Since $\angle VPA = \angle VRA = 90^\circ$, $APRV$ cyclic, so we may anglechase as follows:

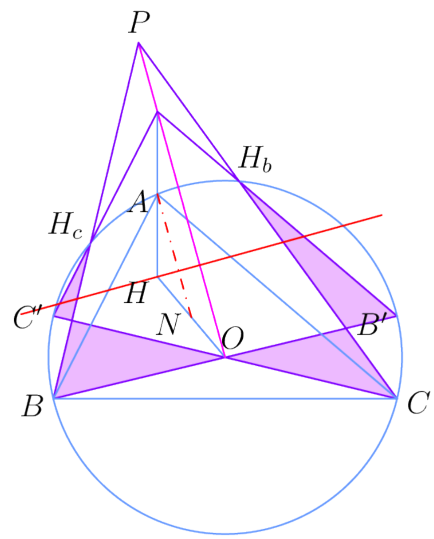
$$\measuredangle APR = \measuredangle AVR \overset{\text{claim 2}}{=} \measuredangle AXV = \measuredangle AXP. \quad \square$$

1.23 USEMO 2020/3, by Anant Mudgal

Let ABC be an acute triangle with circumcenter O and orthocenter H. Let Γ denote the circumcircle of triangle ABC, and N the midpoint of $\overline{OH}$. The tangents to Γ at B and C, and the line through H perpendicular to line AN, determine a triangle whose circumcircle we denote by ω_A. Define ω_B and ω_C similarly.

Prove that the common chords of ω_A, ω_B, and ω_C are concurrent on line OH.

Let H_a, A' denote the respective reflections of H in $\overline{BC}$, A in O, and their symmetric variants.



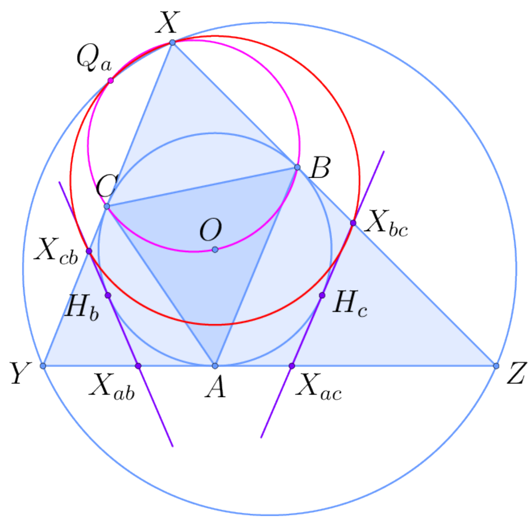
> **Claim 1 –** The polar ℓ_a of $\overline{BH_c} \cap \overline{CH_b}$ passes through H and is perpendicular to $\overline{AN}$.

Proof. Let $P = \overline{BH_c} \cap \overline{CH_b}$ and $S = 2A - H$. $H \in \ell_a$ is just Brokard, so it suffices to prove $\overline{AN} \parallel \overline{OP}$. By Pascal on $BB'H_bCC'H_c$, we have P, O, S collinear. Taking a homothety at H with scale factor $\frac{1}{2}$ maps the latter two points to N, A, which implies the required parallel lines. □

In $\triangle ABC$, let X_bc be the pole of $\overline{BH_c}$ wrt Γ (and 5 other variants), X, Y, Z be the poles of the sides, D, E, F be the feet of the altitudes. Clearly, $\ell_a = \overline{X_{bc}X_{cb}}$.

> *Note.* Here, the condition $\triangle ABC$ acute comes in: Γ is the incircle, not excircle, of $\triangle XYZ$.

We'll show that $\overline{XD}$ is the radical axis of ω_b, ω_c. (By a somewhat-known configuration (say, **Brazil 2013/6**), $\overline{XD} \cap \overline{YE} \cap \overline{ZF}$ lies on the Euler line.) Also let Q_a, Q_b, Q_c be the SD points of $\triangle XYZ$.



Claim 2 – Q_a lies on ω_a.

Proof. By spiral similarity, it suffices to prove $YX_{bc}/YC = ZX_{cb}/ZB$. By antiparallel lines, $\triangle XYZ \overset{-}{\sim} \triangle X_{ab}YX_{cb}, X_{ac}X_{bc}Z$. But since Γ is the Y-excircle of $\triangle X_{ab}YX_{cb}$, we have $YX_{cb}/YC = a/s$. Similarly $ZX_{bc}/ZB = a/s$ as well.
(In some awful notation, $a = YZ$, $b = ZX$, $c = XY$ and $s = \frac{a+b+c}{2}$.) □

Let $L = \overline{YQ_b} \cap \overline{ZQ_c}$.

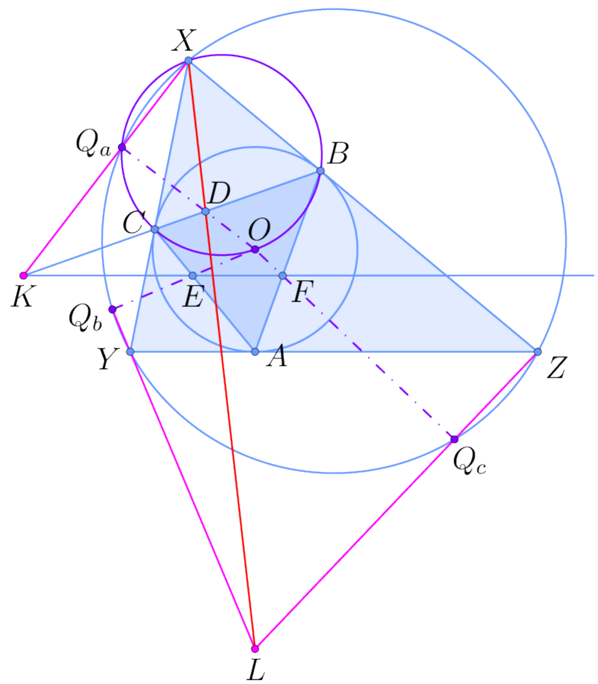
Claim 3 – $\overline{XL}$ is the radical axis of ω_b, ω_c.

Proof. By antiparallel lines again, $YZX_{ba}X_{ca}$ cyclic, so that

$$\text{Pow}(X, \omega_b) = XX_{ca} \cdot XY = XX_{ba} \cdot XZ = \text{Pow}(X, \omega_c), \text{ while}$$

$$\text{Pow}(L, \omega_b) = LY \cdot LQ_b = LZ \cdot LQ_c = \text{Pow}(L, \omega_c).$$ □

It remains to prove X, D, L collinear.



Claim 4 – L is the pole of $\overline{EF}$ wrt Γ.

Proof. Since Q_a is the inverse of D wrt Γ and $\angle OQ_aX = 90°$, $\overline{XQ_a}$ is the polar of D wrt Γ. Similarly, $\overline{YQ_b}$, $\overline{ZQ_c}$ are the respective polars of E, F wrt Γ. The claim is then established by la Hire. □

Claim 5 – $\overline{BC}, \overline{EF}, \overline{XQ_a}$ concurrent.

Proof. Let $K = \overline{EF} \cap \overline{BC}$ so that $(KD; BC) = -1$. Because $\overline{Q_aO}$ bisects $\angle BQ_aC$, $\angle KQ_aO = 90° = \angle AQ_aO \Rightarrow$ X, Q_a, K collinear. □

Taking poles wrt Γ in the last claim gives the desired collinearity.

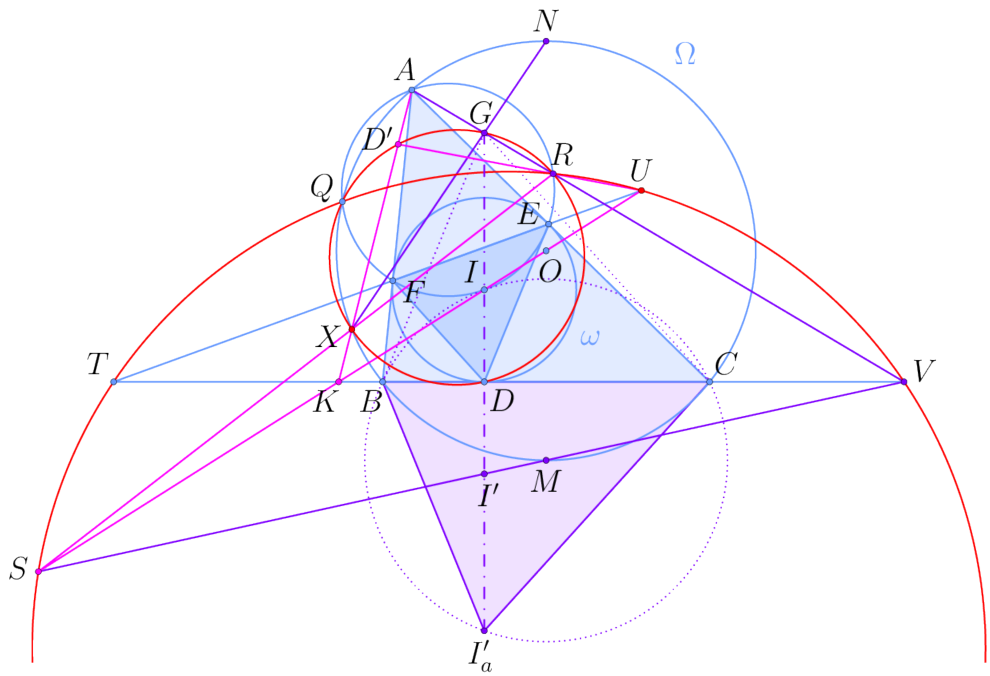
Remark. The problem can be bary'd wrt $\triangle XYZ$ after the first claim, but it's monstrous from my experience a long time ago, oops

1.24 Brazil Olympic Revenge 2021/3, by Joao P.R. Viana Costa

Let I, C, ω and Ω be the incenter, circumcenter, incircle and circumcircle, respectively, of the scalene triangle XYZ with $XZ > YZ > XY$. The incircle ω is tangent to the sides YZ, XZ and XY at the points D, E and F. Let S be the point on Ω such that XS, CI and YZ are concurrent. Let $(XEF) \cap \Omega = R$, $(RSD) \cap (XEF) = U$, $SU \cap CI = N$, $EF \cap YZ = A$, $EF \cap CI = T$ and $XU \cap YZ = O$.

Prove that $NARUTO$ is cyclic.

Colloquially known as "Naruto".



Solution by **crazyeyemoody907**, **CyclicISLscelesTrapezoid** with **Eyed**, **v4913**.

> *Warning.* This problem is not meant for neither the faint-hearted nor freehand geometers like the paper's author(s). If Geogebra's to be used any time, it'd be now.

We do a massive refactoring and simplification; consider the following equivalent problem, a breakdown of the given, despite being longer:

Naruto simplified

In triangle ABC with circumcircle Ω centered at O, the incircle ω centered at I touches the sides at D, E, F. Let I', I'_a be the respective reflections of I and the orthocenter of $\triangle BIC$ in $\overline{BC}$, and M the midpoint of arc BC on Ω. Further define:

- S as the intersection of the Euler lines $\overline{OI}$ of $\triangle DEF$, $\overline{MI'}$ of $\triangle I'_aBC$;
- $T = \overline{EF} \cap \overline{BC}$, $U = \overline{EF} \cap \overline{OI}$, $V = \overline{MI'} \cap \overline{BC}$, $R = \overline{AV} \cap (AI)$;
- $K = \overline{OI} \cap \overline{BC}$;

Prove that (a) Q, R, S, T, U, V are concyclic, and (b) $\overline{AK}$, Ω, (QRD), $\overline{RS}$ concurrent;

(a) The concyclicity Let the spiral similarity s at Q with (directed) angle θ map $E, F \to C, B$ and thus D, I and the orthocenter of $\triangle DEF$ to I', M, I'_a respectively. Clearly, S is the intersection of the Euler lines of two triangles related by s: $DEF, I'_a CB$.
By design, we have $U \xrightarrow{s} V$, so

$$\measuredangle VQU = \theta = \measuredangle(\overline{BC}, \overline{EF}) \stackrel{s}{=} \measuredangle(\overline{MI'}, \overline{OI}) = \measuredangle VSU,$$

whence Q, S, T, U, V concyclic. To see that the last point is also concyclic with the other five, let N be the midpoint of $\widehat{BAC}$, so that $\overline{NA}$ touches (AI). Indeed, then

$$\measuredangle QRV = \measuredangle QRA = \measuredangle QAN \stackrel{s}{=} \measuredangle QUV$$

as needed.

> *Remark.* In fact, by design, S is the exsimilicenter of the incircle and the circle at O with radius half that of Ω, so it's actually the inverse of I wrt Ω.

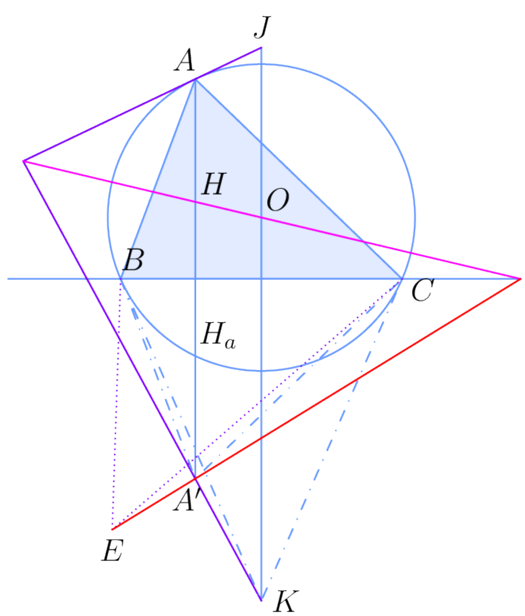
(b) The concurrence Let D' be the reflection of D in $\overline{EF}$, and G the orthocenter of $\triangle BIC$, so that $D' \xrightarrow{s} G$. We easily have $DD'GQ$ cyclic. As $\measuredangle(\overline{AD'}, \overline{NG}) = \theta$, the point $X = \overline{AD'} \cap \overline{NG}$ lies on both $(DD'GQ)$, Ω. We require the following result(s):

Theorem: weird concurrences
In a scalene triangle ABC with circumcenter O, circumcircle Ω, and orthocenter H.

(a) let K be the polar of $\overline{BC}$ wrt Ω, and A' be the reflection of A in $\overline{BC}$. Then $\overline{OH}, \overline{A'K}$ and the tangent to Ω at A are concurrent.

(b) Let E be the reflection of the point E_0 (such that A is the incenter or excenter of $\triangle E_0BC$) in the perpendicular bisector of $\overline{BC}$. Then $\overline{OH}, \overline{BC}, \overline{EA'}$ are also concurrent.

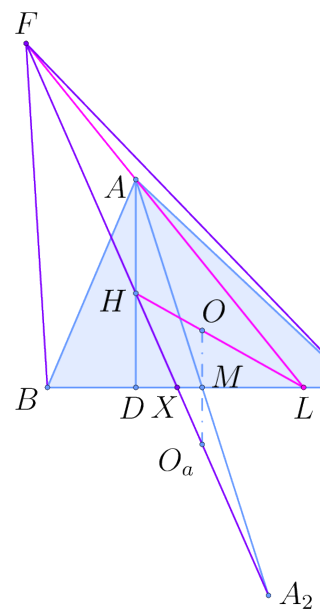
(parentheses used above for easier grammatical parsing)



Proof. These two parts actually aren't connected at all…

Part (a), by CyclicISLscelesTrapezoid Let J be the intersection of the tangent to Ω at A with the perpendicular bisector of $\overline{BC}$, and $H_a \in \Omega$ be the reflection of H in $\overline{BC}$. We contend that the triples (A, H, A'), (J, O, K) are homothetic. Indeed, they lie on parallel lines. To finish, check that (if R denotes the radius of Ω)

$$JO = \frac{R}{\cos(B-C)}, OK = \frac{R}{\cos A}, AH = 2R\cos A, HA' = AH_a = 2R\cos(B-C) \Rightarrow \frac{JO}{OK} = \frac{AH}{HA'}.$$



Part (b), by crazyeyemoody907 Let $F = B + C - E_0$, and $A_2 = B + C - A$, so that A_2 is an incenter or excenter of $\triangle FBC$. Since H is the antipode of A_2 on (BA_2C), it is another incenter / excenter. To prove that A, L, F collinear where $X = \overline{FHA_2} \cap \overline{BC}, L = \overline{OH} \cap \overline{BC}$, verify that (where $O_a \in \overline{H_aA_2}$ is the reflection of O in $\overline{BC}$)

$$(\overline{AF} \cap \overline{BC}, X; D, M) \overset{A}{=} (FX; HA_2) = -1 \text{ while } (DM; XL) \overset{H}{=} (\infty_{\perp BC}M; O_aO) = -1. \qquad \square$$

Returning to the problem, applying respective parts of the theorem to $\triangle DEF, I_a'BC$, we obtain (A, D', K) and (A, G, V) collinear. Since $R \in (UVQ), \overline{GV}$, and Q is the Miquel point of $D'GVU$, we must have $R = \overline{D'U} \cap \overline{GV}$ – an intersection of opposite sides. Hence, by definition of Miquel point, $R \in (QD'G)$.

It remains to prove that R, X, S collinear. In fact, there is a spiral similarity at Q mapping $D', X \to U, S$ since $Q \in (URS), (D'XR)$, so we're done!