Select geometry favorites

People

November 19, 2022

This is a collection of problems we've done and found particularly cute/beautiful/hard, inspired by chapter 11 of *Euclidean Geometry in Math Olympiads* and Eric Shen's paper *Geometry At Its Best*. Have fun!

(Note: Here, ∞_{XY} denotes the point at infinity along line XY.)

Contents

0 Problems

Remark. Some attempt has been made to deviate from the aformentioned two famous geometry papers.

:)))

Problem 1 (SL 1998/G4). Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

Problem 2 (SL 2009/G3). Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram.
Prove that $GR = GS$.

Problem 3 (SL 2015/G4). Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

Problem 4 (SL 2016/G7). Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120°$.

Problem 5 (EGMO 2015/6). Let H be the orthocentre and G be the centroid of acute-angled triangle ABC with $AB \neq AC$. The line AG intersects the circumcircle of ABC at A and P. Let P' be the reflection of P in the line BC. Prove that $\angle CAB = 60$ if and only if $HG = GP'$.

Problem 6 (Iran TST 2018/1/4). Let ABC be a triangle ($\angle A \neq 90°$), with altitudes $\overline{BE}, \overline{CF}$. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N. Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that $\overline{AP}$ bisects $\overline{BC}$.

Problem 7 (Eric Shen). In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. Points X, Y are the projections of E, F onto $\overline{AD}$ respectively. Let $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90°$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{TZ}$, and Q lies on $\overline{EF}$ such that $\overline{PQ} \parallel \overline{BC}$, prove that $\overline{AQ}$ is a median.

Problem 8 (China TST 2015/2/3). Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G. Let D be the midpoint of $\overline{BC}$, and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{AD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω.

Problem 9 (IMO 2019/6). Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC

is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

Problem 10 (MOP 2019 & USA TST 2019/6). Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

(a) (MOP 2019) Let E, F be the feet of the altitudes from B, C respectively. Prove that if $\overline{EF}$ touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.

(b) (USA TST 2019/6) Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.

Problem 11 (TSTST 2018/3). Let ABC be an acute triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of $\overline{AB}$. Ray AI meets $\overline{BC}$ at D. Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ.

Problem 12 (RMM 2012/6 & Brazil 2013/6). In triangle ABC with incenter I and circumcenter O, let the incircle ω touch $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F respectively.

- (RMM 2012/6) Let ω_a be the circle through B and C tangent to ω, and define ω_b, ω_c similarly. Finally, let $A' = \omega_b \cap \omega_c \;\; (\neq A)$, and similarly for points B' and C'.
- (Brazil 2013/6) Let P be the Gergonne point of $\triangle ABC$, and its reflections in $\overline{EF}$, $\overline{FD}$ and $\overline{DE}$ be P_a, P_b, P_c, respectively.

Prove that $P_a \in \overline{AA'}$, and that $\overline{AP_aA'}$, $\overline{BP_bB'}$, $\overline{CP_cC'}$, $\overline{IO}$ are concurrent.

Problem 13 (IMO 2021/3). Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Problem 14 (USAMO 2021/6). Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

Problem 15 (SL 2021/G8). Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.

1 Solutions

1.1 SL 1998/G4

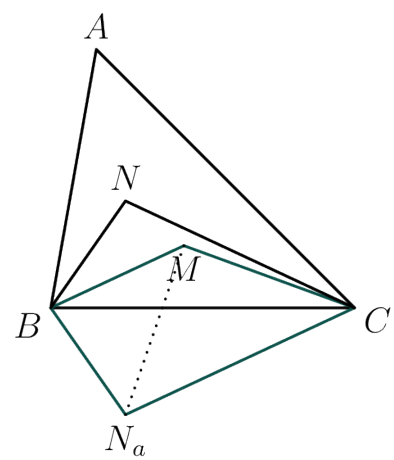
Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

Solution 1, by inversion-overlay Let i_a denote the inversion at A with power $AB \cdot AC$ composed with reflection in the bisector of $\angle A$.



Solution 2, by area ratios (official / intended)

Claim – For any M, N, we have

$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMC] + [BNC]}{[ABC]}.$$

Proof. Reflect N over $\overline{BC}$ to obtain point N_a. Then, because $\angle MBN_a = \angle B$, $(BM \cdot BN)/(BA \cdot BC) = [MBN_a]/[ABC]$. Similarly $(CM \cdot CN)/(CA \cdot CB) = [MCN_a]/[ABC]$, and summing the previous two equations gives

$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMN_aC]}{[ABC]} = \frac{[BMC] + [BNC]}{[ABC]}. \qquad \square$$

Noting that M, N are just isogonal conjugates, we obtain the problem by cyclic summation.

1.2 SL 2015/G4

Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

Solution by **CyclicISLscelesTrapezoid**.

The answer is $\sqrt{2}$ only. Let the $X \neq B$ be defined as $(ABC) \cap (BPMQ)$, and let N be the midpoint of $\overline{BT}$.

> **Claim 1 –** $XNMT$ is cyclic.

Proof. Since N is also the midpoint of $\overline{PQ}$, there is a spiral similarity at X sending PNQ to AMC. Thus, we have

$$\measuredangle XMN = \measuredangle XAP = \measuredangle XTB,$$

so $XNMT$ is cyclic. □

> **Claim 2 –** $\overline{BM}$ is tangent to the circumcircle of $XNMT$.

Proof. We have

$$\measuredangle XNM = \measuredangle XPA = \measuredangle XPB = \measuredangle XMB.$$ □

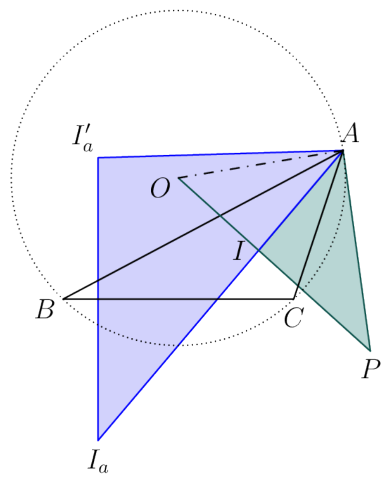
By Power of a Point, $BM^2 = BN \cdot BT = \frac{BT^2}{2}$, so $\frac{BT}{BM} = \sqrt{2}$.

1.3 SL 2016/G7

Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120°$.



Redefine P as the inverse of I. For the first part we assert more strongly that:

Claim – $\triangle AI_aI'_a \overset{+}{\sim} \triangle API$.

Proof. One of the few uses of SAS similarity? By angle chasing, $\angle I_a = \angle P$ follows easily. To finish, we show $I_aI'_a/I_aA = IP/AP$; indeed, the first ratio equals $2\cos\angle BI_aC = 2\sin\frac{A}{2}$ because of similar triangles; thus, we're left to length chase IP/AP; this becomes

$$\frac{OP}{AP} - \frac{OI}{OA}\frac{OA}{AP} = \frac{OI}{AI} - \frac{OI^2}{OA\cdot AI} = \frac{R}{AI} - \frac{R^2-2rR}{R\cdot AI}$$
$$= \frac{R-(R-2r)}{AI} = 2\sin\frac{A}{2},$$

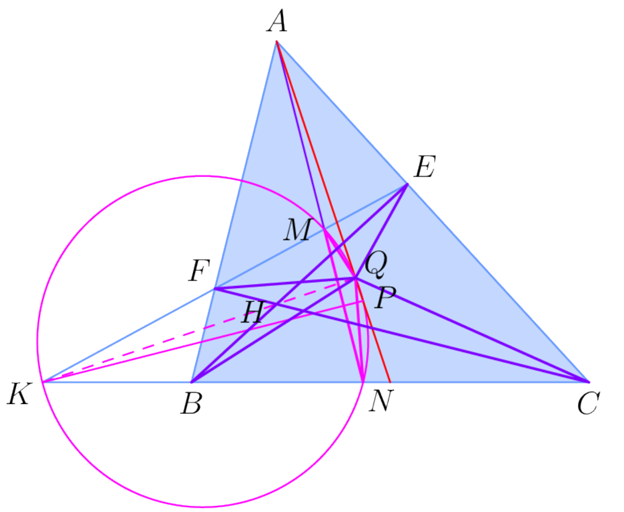
so the ratios are equal, as needed. □

The claim clearly implies the isogonality.

For the second part, using Poncelet, let $Z \in (ABC)$ be the unique point so that $\triangle XYZ, ABC$ share a incircle and circumcircle. By power of a point at P, and the fact that P is the inverse of I wrt (ABC), $PX \cdot PY = PO \cdot PI$ so $XYOI$ cyclic. As $\angle XOY = 2\angle Z$ and $\angle XIY = (\pi + \angle Z)/2$, $Z = \pi/3 \implies \angle XOY = \angle XIY = 2\pi/3$.

1.4 Iran TST 2018/1/4, by Iman Maghsoudi & Hooman Fattahi

Let ABC be a triangle ($\angle A \neq 90^\circ$), with altitudes $\overline{BE}, \overline{CF}$. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N. Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that $\overline{AP}$ bisects $\overline{BC}$.



Construct $K = \overline{EF} \cap \overline{BC}$, Q as the A-Humpty point, H as the orthocenter of $\triangle ABC$, and $\omega = (KMN)$, so that the P given is the antipode of K on it. Let spiral similarity s at Q take $(E, F) \to (B, C)$. The main point of the problem is then:

> **Claim –** $Q \in \omega$.

Proof. First, by angle bisector theorem,

$$\frac{EM}{MF} = \frac{EA}{AF} = \frac{BA}{AC} = \frac{BN}{NC}$$

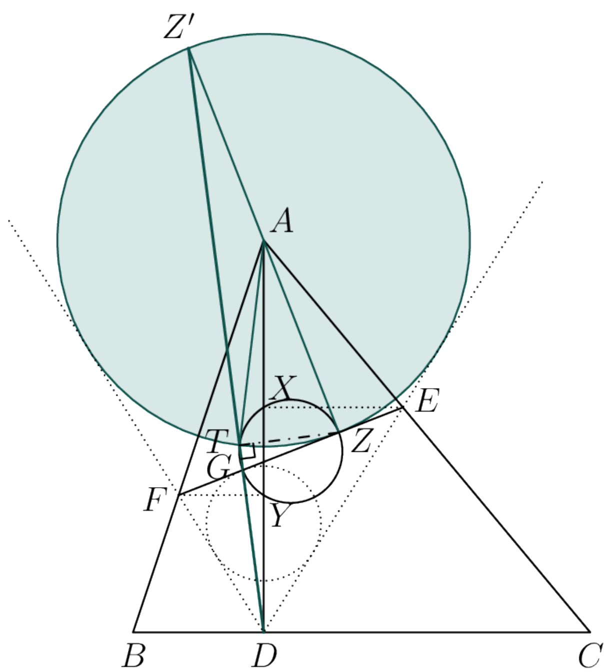
$$\Rightarrow (M \xrightarrow{s} N) \Rightarrow \measuredangle MQN = -\arg(s) = \measuredangle(\overline{EF}, \overline{BC}) = \measuredangle MKN,$$

as desired. □

Since P is the antipode of K on ω, $\angle KQP = 90^\circ = \angle KQA$, implying that $P \in \overline{AQ}$, the A-median.

1.5 Mock AIME 2019/15', by Eric Shen & Raymond Feng

In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AZ} \cap \overline{QT}$, and Q lies on $\overline{EF}$ such that $\overline{PQ} \parallel \overline{BC}$, prove that $\overline{AQ}$ bisects $\overline{BC}$.



Construct points X, Y as the projections of E, F onto $\overline{AD}$ respectively. [*]

After drawing a diagram on Geogebra, we obtain:

Characterization of T

T is the harmonic conjugate of Z wrt XY – i.e. it lies on $\omega = (XYZ)$ so that the resulting quadrilateral is harmonic.

In American style, we show that this choice of T lies on ω_a (defined as the circle at A thru Z) and (DZ),

Verification (inspired by USA TST 2015/1)

For $AZ = AT$, we use power of a point / length chase to get $AZ^2 = AX \cdot AY$ whence $\overline{AZ}$ touches ω. Hence, by harmonics $\overline{AT}$ is also tangent to ω, so this property follows.

$\angle DTZ = 90^\circ$ is much less straightforward. We define $Z' = 2A - Z$ and $G = E + F - Z$ as the antipodes of Z on the circle at A through Z. By a well-known lemma, D, Z', G collinear (along the cevian through the intouch point in $\triangle DEF$).

But also at the same time, T is on $\omega, \omega_a \implies \angle ZTG = \angle ZTZ' = \pi/2$ due to antipodes. Hence, $\angle DTZ = \pi/2$, completing the verification.

[*] Eric Shen originally included these points in the problem statement (as seen in the 2019 version of "Geometry At Its Best"), but I guess the problem's made harder by deleting them. For me, thinking about their relevance/use was important in making nonzero progress on the problem.

By this definition, we clearly have $(AP;XY) = -1$. From here (the chase is best discovered backwards), harmonic chasing suffices. Define $K = \overline{EF} \cap \overline{A\infty_{BC}}$. Then

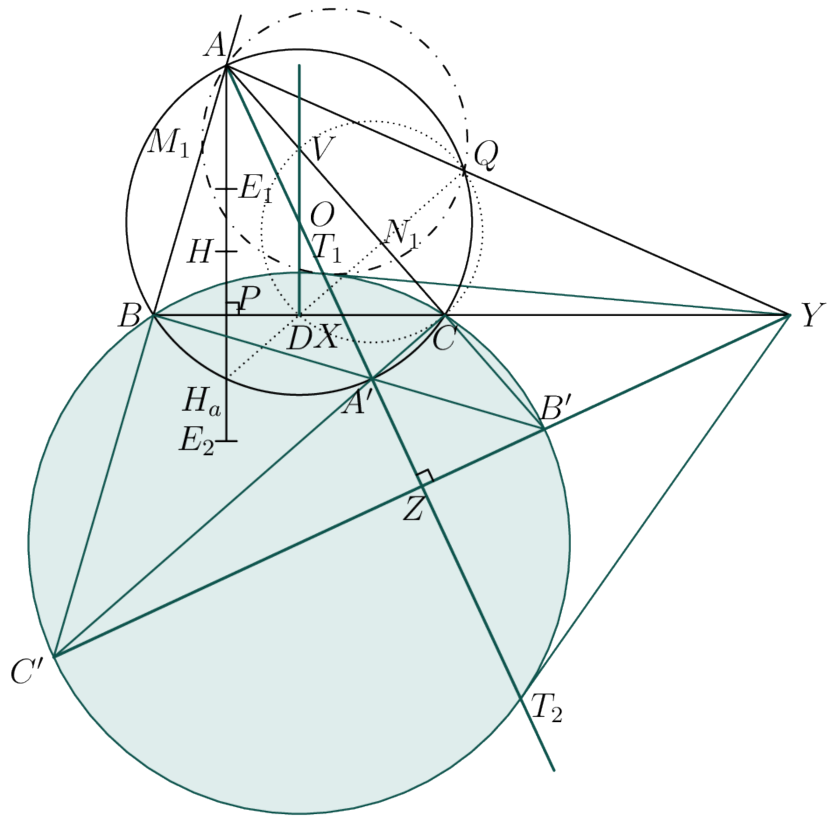
$$(\overline{AQ} \cap \overline{BC}, \infty_{BC}; B, C) \stackrel{A}{=} (QK;FE) \stackrel{\infty_{BC}}{=} (PA;YX) = -1,$$

the end!

1.6 China TST 2015/2/3

Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G. Let D be the midpoint of $\overline{BC}$, and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{AD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω.



As much of the parallel/perpendicular lines aren't even relevant (just to give us some equal ratios), we simplify as follows:

Problem reworded

In acute $\triangle ABC$ with circumcenter O and orthocenter H, D is the midpoint of $\overline{BC}$, and the altitude from A meets (BC) at E (either one works). Let $U, V = \overline{OD} \cap \overline{AB}, \overline{AC}$, respectively; define $M, N \in \overline{AB}, \overline{AC}$ with (lengths directed)

$$UM/MB = VN/NC = AE/EH.$$

Let ω be the circle tangent to segments OB, OC at B, C respectively. Prove that (AMN) is tangent to ω.

We define a load of new points as follows.

- $A' = 2O - A$;
- E_1, E_2 be the choices of E with $AE/EH > 0$ and $AE/EH < 0$ respectively. We will only consider M_1, N_1, because the negative case is identically handled;
- $T_1, T_2 = \overline{AO} \cap \omega, X = \overline{AO} \cap \overline{BC}$, corresponding to E_1, E_2 from earlier;

- $Y = \overline{T_1T_1} \cap \overline{T_2T_2} \cap \overline{BC}$ (which exists since $(BC; T_1T_2) = -1$);
- Q as the harmonic conjugate of A' wrt BC, or equivalently, the reflection of the A-orthocenter Miquel point Q_a in the perpendicular bisector of $\overline{BC}$, $\overline{DUV}$.

Claim 1 – Q is the Miquel point of $ABCDUV$.

Proof. As we already have $Q \in (ABC)$, sufficient to prove $QDVC$ cyclic. Observe that $Q \in \overline{H_aD}$, which follows by $Q_a \in \overline{A'PH}$ reflected in $\overline{DUV}$. The result holds by Reim because AH_aQC cyclic and $\overline{DV} \parallel \overline{AH_a}$. □

Claim 2 – (AQT_1) touches ω, $\overline{YT_1}$ at T_1.

Proof. Sufficient to show $Q \in \overline{AY}$, so that the claim will follow by power of a point at Y. Indeed,

$$-1 = (BC;XY) \stackrel{A}{=} (B, C; A', \overline{AY} \cap (ABC))$$

proving the claim. □

Claim 3 – $AE_1/E_1H = AT_1/T_1A'$.

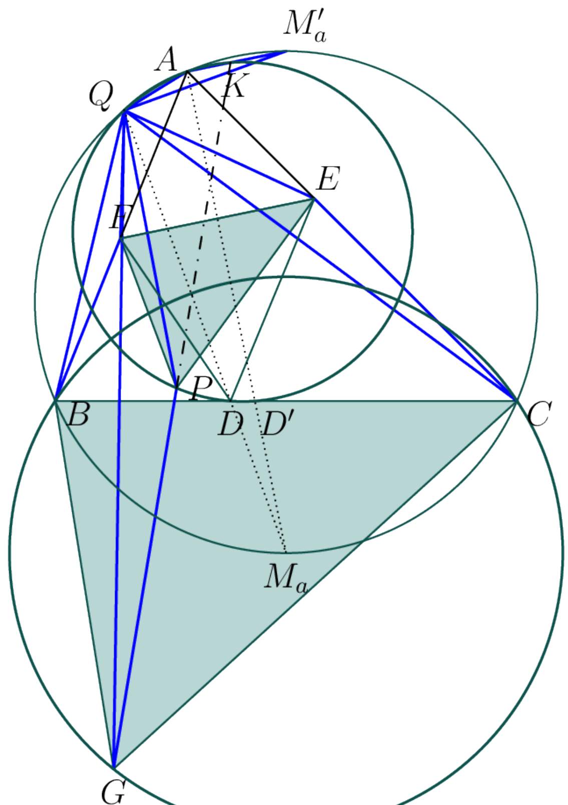
Proof. Define $B' = \overline{A'B} \cap \overline{AC}$, $C' = \overline{A'C} \cap \overline{AB}$. Using the logic of **USA TST 2007/5**, we know that $\triangle ABC \overset{-}{\sim} \triangle AB'C'$, and that Q is the A-orthocenter Miquel point in $\triangle AB'C'$. Next, let P, Z be the foot from A to $\overline{BC}, \overline{B'C'}$ respectively. If r denotes the reflection + homothety at A that maps $B, C \Rightarrow B', C'$, then observing that $(E_1, P, E_2) \overset{r}{\Rightarrow} (T_1, Z, T_2)$ wins. □

To finish the problem, observe $M_1, N_1 \in (AQT_1)$ follows by spiral similarity at Q, completing the proof.

1.7 IMO 2019/6, by Anant Mudgal

Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to $\overline{EF}$ meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

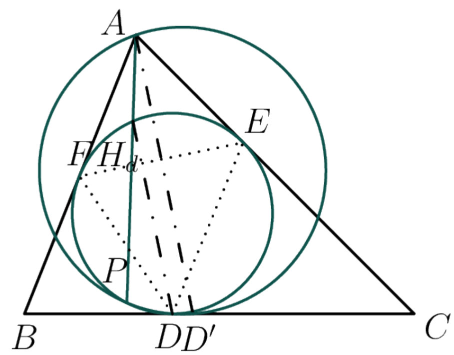
Prove that lines DI and PQ meet on the line through A perpendicular to $\overline{AI}$.



Observe that P is the D-orthocenter Miquel in $\triangle DEF$. Define K as the intersection of the A-external bisector with $\overline{AD}$. We make the following definitions…

- Let ω, ω_a denote the incircle and (BIC) respectively;
- Define X as intersection of *segment* PK with ω. Let Q instead denote the A-SD point;
- G be the harmonic conjugate of I wrt BC, D' as the foot of the A-angle bisector; M_a as the midpoint of arc BC exc. A; M_a' as the antipode of M_a on (ABC);
- H as orthocenter of $\triangle DEF$, and H_d its reflection over $\overline{EF}$.

$\Rightarrow$ because $MB^2 = MD \cdot MQ = MD' \cdot MA$, $Q \in (ADD'K)$.



Claim 1 – $P \in (ADD'KQ)$.

Proof. Observe that $(PH_d;EF) = -1$ whence A, P, H_d collinear. Then because $\overline{DH_d} \parallel \overline{AI}$ because both perpendicular to $\overline{EF}$. Hence result by degenerate Reim. □

Claim 2 – $\triangle PFE \overset{+}{\sim} \triangle GBC$.

Proof. Proceed by spiral at Q. Observe that $\triangle H_dEF \overset{+}{\sim} \triangle ICB$ by angle chase. Because $(H_dP;EF) = (IG;BC) = -1$, the needed similarity follows. □

Claim 3 – K, G, P collinear.

Proof. An angle chase, using the previous two claims:

$$\measuredangle QPK \overset{\text{claim 1}}{=} \measuredangle QAK = \measuredangle QAM_a \overset{\text{spiral}}{=} \measuredangle QPG. \qquad \square$$

Using last two claims, we may chase:

$$\angle PXB = \angle GXB = \angle GCB \overset{\text{spiral}}{=} \angle PEF = \angle PFB$$

whence $PXFB$ (and also $PXEC$ by symmetry) cyclic.

This completes the proof.

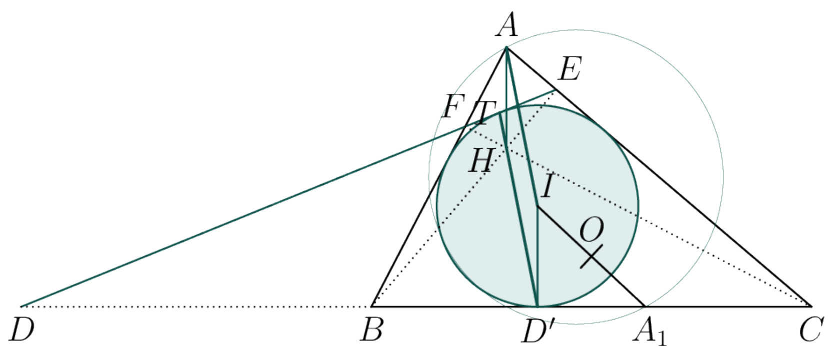
Remark. ggb way too op

1.8 MOP + USA TST, by Ankan Bhattacharya

Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90°$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

1.8.1 MOP

Let E, F be the feet of the altitudes from B, C respectively. Prove that if $\overline{EF}$ touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.



Call the incircle ω.

Claim 1 – D, E, F are collinear.

Proof. We will prove that the tangent line from D is antiparallel to $\overline{BC}$ wrt $\angle A$. Indeed, this line is found by reflecting $\overline{DBC}$ over $\overline{DI}$, a line perpendicular to $\overline{AI}$, so we win. □

Let ω touch $\overline{DEF}$ at a point T, and let D' denote the A-intouch point.

Claim 2 – $\overline{AI} \parallel \overline{HD'}$; hence $AID'H$ is a parallelogram and $AH = r$, the inradius of $\triangle ABC$.

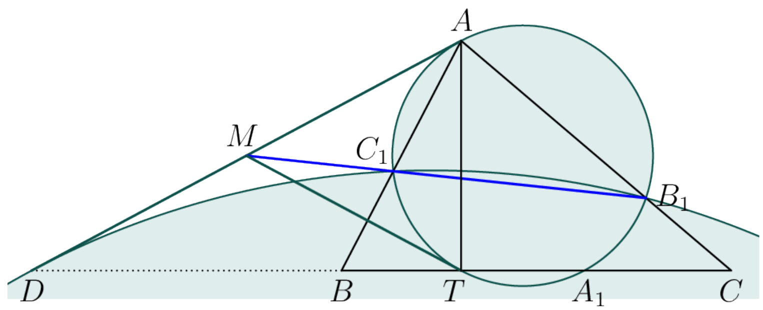
Proof. Because $BCEF$ is tangential, it follows by degenerate Brianchon that lines BE, CF, DT' concur, i.e. $H \in \overline{TD'}$. Observe that $DT = DD'$; then $\overline{THD'} \perp \overline{DI}$ by symmetry, while $\overline{AI} \perp \overline{DI}$ is given; the lines are thus parallel as claimed. □

Now, let ω_a, etc denote (AB_1C_1), etc, respectively. We observe that because the perpendicular from A_1 to $\overline{BC}$ and its cyclic variants all concur at the point $2O - I$, it follows that all three circles must concur at this point by Miquel spam.

But because $r/2 = AH/2$ is the distance from O to $\overline{BC}$, we actually have $2O - I = A_1$ (also because of their feet onto $\overline{BC}$). Hence $A_1 \in \omega_a$ as desired.

1.8.2 USA TST 2019/6

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.



We first make some observations from working backwards on the previous part.

First, $\overline{I_aA_1}$ is unconditionally the raxis of ω_b, ω_c, which is because $2O - I, A_1, I_a$ lie on the same line $\perp \overline{BC}$. Thus, if A_1 is to lie on ω_a, then by anglechase, ω_b, ω_c touch at A_1 whence $I_aA_1 \perp \overline{BC}$.

Also, by MOP 2019 converse (which follows by uniqueness wrt $\angle A$) we have D, E, F collinear. If T is the foot of A onto $\overline{BC}$, it follows that $(DT;BC) = -1$.

Claim 1 – The A-SD point coincides with the A- orthocenter Miquel.

Proof. Since $BF/CE = \cos B/\cos C = (s-c)/(s-b)$ from 19MOP, result follows by spiral. □

Next, we have A, A_1 antipodes on ω_a, which follows by angle chasing, observing that ω_b, ω_c touch at A_1 / etc.

Claim 2 – $\overline{AD}$ is tangent to ω_a.

Proof. Recall that $\overline{ADQ}$ is perpendicular to $\overline{HIQ'}$; thus, equivalent to show $\overline{HQ} \parallel \overline{AA'}$ which is another angle chase. □

By radical axis/etc, it suffices to show that the midpoint M of $\overline{AD}$ lies on $\overline{B_1C_1}$. By symmetry, $\overline{MA}$, $\overline{TA}$ touch ω_a.

Claim 3 – $(AT;B_1C_1) = -1$.

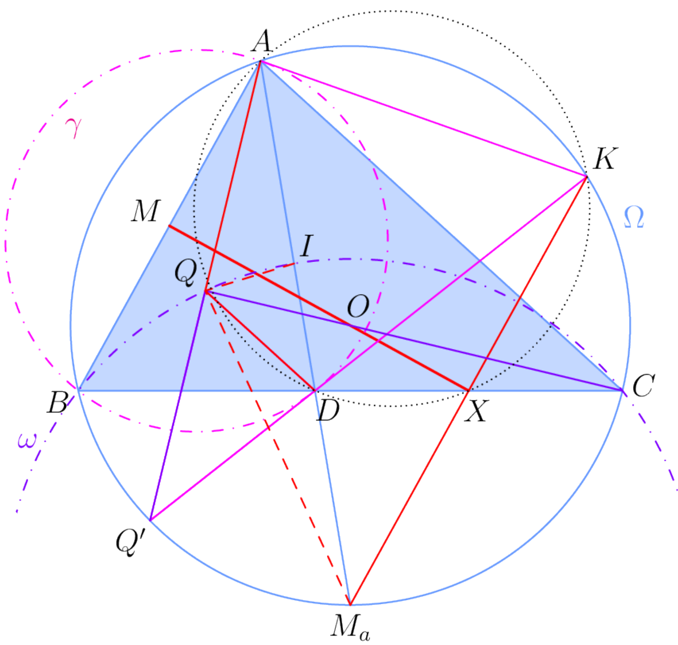
Proof. Harmonics: $(AT;B_1C_1) \overset{A}{=} (DT;BC) = -1$, as claimed. □

From here the problem follows by power of a point converse on $MD^2 = MA^2 = MB_1 \cdot MC_1$.

1.9 TSTST 2018/3, by Evan Chen & Yannick Yao

Let ABC be an acute triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of $\overline{AB}$. Ray AI meets $\overline{BC}$ at D. Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ.



The given angle condition implies $AMQO$ cyclic, or $\angle AQC = \angle AMO = \pi/2$. We make the following definitions:

- $\Omega = (ABC)$, M_a as the center of ω and midpoint of $\widehat{BC}$;
- $Q' = 2Q - A$ as the reflection of A in $\overline{QOC}$ – this lies on Ω by symmetry about $\overline{CO}$;
- $K \in \Omega$ as the reflection of M_a in $\overline{MO}$, the perpendicular bisector of $\overline{AB}$.

In the meat of the problem we'll use the following easily findable arcs/angles frequently:

$$\widehat{BM_a} = \widehat{CM_a} = \widehat{AK} = A, \text{and } \widehat{AC} = \widehat{CQ'} = \widehat{M_aK} \implies \angle AQ'C = B.$$

Observation
$\overline{QI}$ bisects $\angle AQD$. (Holds because $Q \in \gamma$, the Apollonian circle wrt A, D through I.)

Claim 1 – $\triangle QQ'D \overset{+}{\sim} \triangle M_aAC$.

Proof. First, we'll show $\angle QQ'D = \angle B$, a massive angle chase:

$$\angle M_aAQ = \angle CAQ' - \angle CAM_a = B - \frac{A}{2}, \text{ and } \angle M_aIQ = \frac{\pi - \angle IM_aQ}{2} = \frac{\pi}{2} - \angle ICO = B + \frac{C}{2};$$

$$\implies \angle AQI = \angle M_aIQ - \angle M_aAQ = \frac{\pi - B}{2}.$$

Applying the observation gives $\angle Q'QD = B$.

To conclude, note that

$$\frac{Q'Q}{QD} = \frac{AQ}{QD} = \frac{AI}{ID} = \frac{AC}{CD} = \frac{AM_a}{M_aC},$$

establishing the similarity by SAS.

(The similarity is negative due to lack of configuration issues, and inspection.) □

Claim 2 – Q', D, K collinear.

Proof. Angle chase again: $\measuredangle AQ'D \overset{\text{claim 1}}{=} -\measuredangle M_aAC = \widehat{AC}/2 = \widehat{AK}/2 = \measuredangle AQ'K$. □

Part 1: $\overline{KA}$ and $\overline{KD}$ touch γ

Even more angle chasing, taking advantage of claim 2...

$$\measuredangle KAD = \frac{\widehat{KM_a}}{2} = \frac{\widehat{AC}}{2} = \measuredangle ABD, \text{ while } \angle KDA = \frac{\widehat{KA} + \widehat{QM_a}}{2} = \frac{\widehat{Q'C}}{2} = B = \angle DBA,$$

proving the tangencies.

The other, more elegant part of the problem...

Claim 3 – $\overline{MO}, \overline{BC}, \overline{KM_a}, (ADK)$ all concur at a point X.

Proof. Let $X_1 = \overline{MO} \cap \overline{BC}, X_2 = \overline{KM_a} \cap \overline{BC}$.

- $X_1 \in (ADK)$ by similarity: observe by (omitted) angle chase that $\triangle AXB \overset{+}{\sim} \triangle AKD$, whence $\measuredangle AXD = \measuredangle AKD$;
- $X_2 \in (ADK)$ (by contrast) is by power of a point at M_a:

$$M_aB^2 = M_aC^2 = M_aX_2 \cdot M_aK = M_aA \cdot M_aD.$$

As $X_1 = X_2 = (ADK) \cap \overline{BC}$ $(\neq D)$, the claim is proven. □

Because $\overline{M_aK} \parallel \overline{AB} \perp \overline{MO}$, and $X = \overline{MO} \cap \overline{M_aK}$ is the inverse of K wrt ω (by the second equation in previous claim's proof), $\overline{MO}$ is the polar of K wrt ω, completing the problem.

Remark. **(crazyeyemoody907)** For me, it took forever to discover the proof of claim 1. Here are some useless formulations of its statement I harped upon:

- $(AC; KM_a) = -1$ which directly implies the first part:

$$(B, \overline{BK} \cap \gamma; A, D) \overset{B}{=} (M_aK; AC) = -1;$$

  Since "$\overline{KA}$ touches γ" is very easily provable, K would be polar of $\overline{AD}$ wrt γ as promised...
- $BDQQ'$ cyclic ($\iff$ $\overline{QD} \parallel \overline{AC}$ by Reim)

In fact, this means post-solve that $\overline{BQ} \parallel \overline{Q'DK}$ ... in hindsight, equally useless...

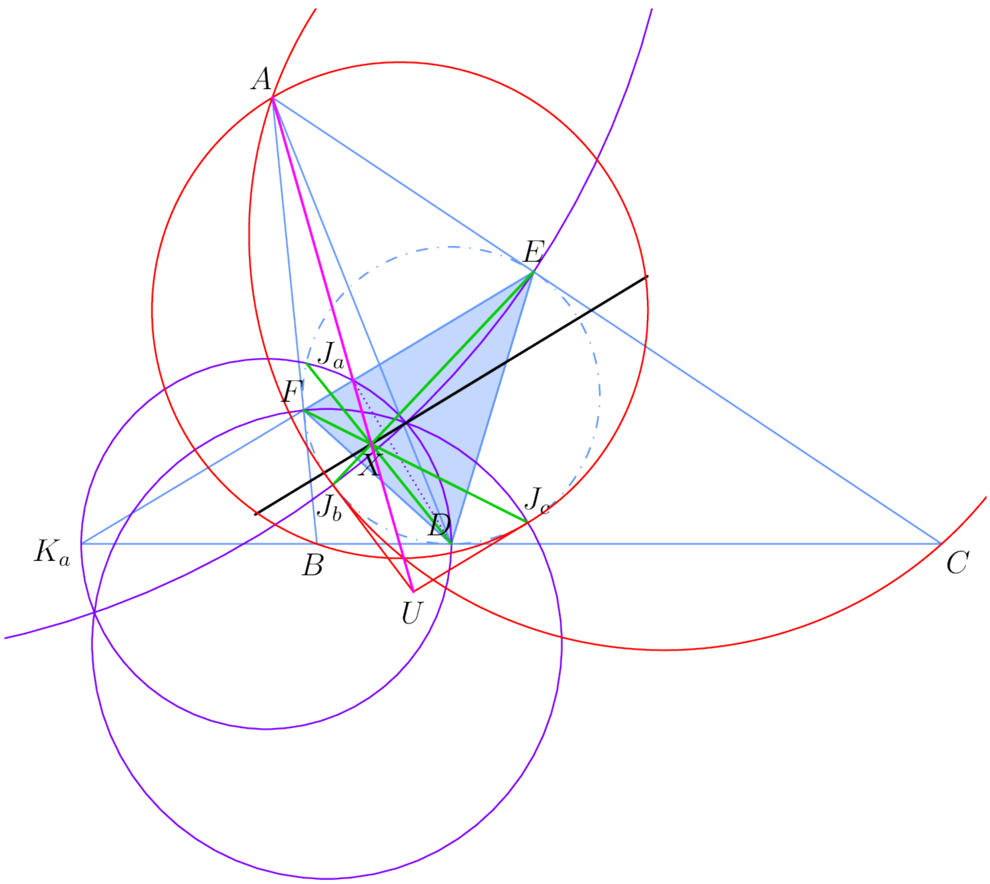
Remark. (Note to self?) When angle chasing in a synthetic problem fails, try length ratios! Examples: SL 2016/G7, GGG 4.5;

1.10 RMM + Brazil

1.10.1 RMM 2012/6

Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.

Let $K_a = \overline{EF} \cap \overline{BC}$, $\gamma_a = (K_a D)$, $J_a = \omega_a \cap \gamma_a \cap \omega$ (and cyclic variants), and H and ℓ denote the orthocenter and Euler line of $\triangle DEF$, respectively. Also, let I_a, I_b, I_c be the excenters of $\triangle ABC$

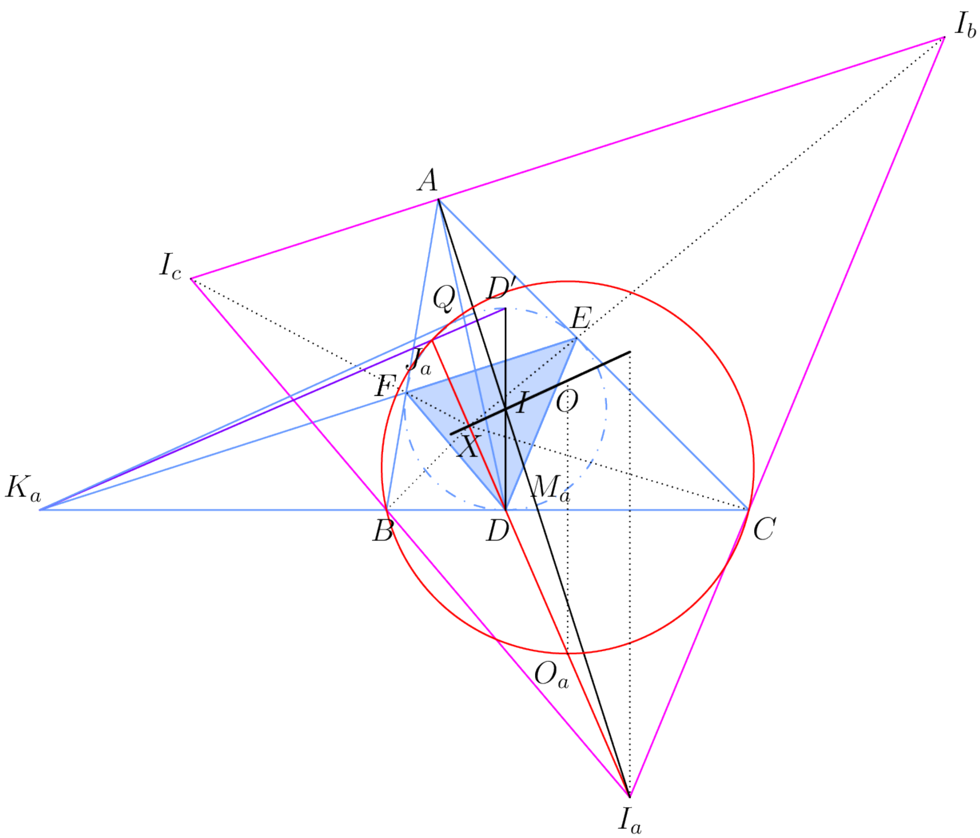


Solution 1, by radical axes Observe that ℓ is just $\overline{OI}$, and that γ_a, etc are coaxial Apollonian circles. Define X as the radical center of $\gamma_a, \gamma_b, \gamma_c, \omega$ (which exists since the former 3 circles are coaxial). We'll show this is the desired concurrency point in both problems. Clearly, $\overline{DJ_a}$ is the raxis of (γ_a, ω), i.e. $X \in \overline{DJ_a}$.

> **Lemma 1 –** ℓ is the raxis of γ_a and variants.

Proof. Let T_a denote the foot of D onto $\overline{EF}$, which is obviously on γ_a. Then H has power $HD \cdot HT_a$ (= variants) wrt the γ's, hence on raxis; Meanwhile I has power r^2 wrt all circles by orthogonality, hence also on raxis, done. □

Let tangents to ω at J_b, J_c meet at U; then, $\overline{AU}$ is the raxis of ω_b, ω_c. Clearly this is the polar of $\overline{J_bJ_c} \cap \overline{EF}$. Recalling that $X = \overline{EJ_b} \cap \overline{FJ_c}$, follows by Brokard that $X \in \overline{AU}$, the end.



Solution 2, by homothety (v4913) Let D' be the antipode of D on ω, $Q = \overline{AD} \cap \omega \ (\neq D)$; then, because $(EF; DQ) = -1$, $\overline{K_a Q}$ touches ω as well. Also, because $\angle DJ_aD' = \angle DJ_aK_a = 90^\circ$, K_a, J_a, D' are collinear, whence $(DQ; J_aD') = -1$.

We start with X as the similicenter of homothetic triangles $DEF, I_aI_bI_c$. Let homothety h at X with scale factor r map $(D, E, F) \rightarrow (I_a, I_b, I_c)$, This must also map their circumcenters to each other, i.e. $I \overset{H}{\Rightarrow} 2O - I$, whence $X \in \overline{OI}$.

Also, let M_a be the midpoint of $\overline{BC}$, $O_a \in \overline{DJ_a}$ be the midpoint of arc BC on ω_a not containing J_a (and variants).

Lemma 2 – J_a, D, I_a collinear.

Proof. Harmonics:

$$-1 = (DQ; D'J_a) \overset{D}{=} (\overline{BC} \cap \overline{AI}, \overline{JaD} \cap \overline{AI}; I, A),$$

implying that $\overline{J_aD} \cap \overline{AI}$ is the A-excenter. □

Hence, $X = \overline{DJ_a} \cap \overline{EJ_b} \cap \overline{FJ_c}$.

Claim – O_a is the midpoint of $\overline{DI_a}$.

Proof. By symmetry, M_a is the foot of O_a onto $\overline{BC}$, while it's well-known that $2M - D$ is the foot of I_a onto $\overline{BC}$. M obviously being the midpoint of the segment with endpoints $D, 2M - D$ implies the claim by parallel lines. □

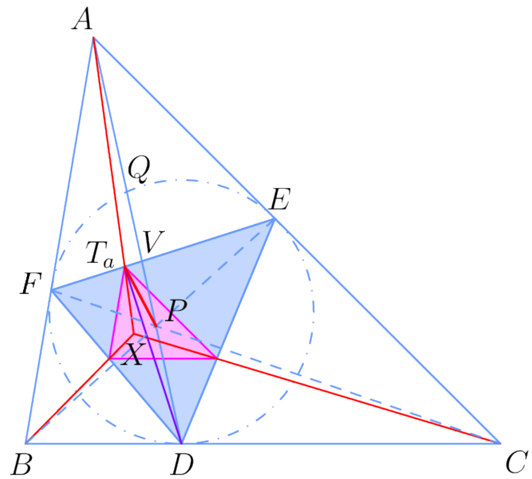
Therefore there must exist a homothety h' at X with scale factor $(1+r)/2$, mapping $(D, E, F) \to (O_a, O_b, O_c)$. To show that our X is indeed the radical center of $\omega_a, \omega_b, \omega_c$, compute

$$\operatorname{Pow}(X, \omega_a) = XJ_a \cdot XO_a \stackrel{h'}{=} \frac{1+r}{2} XJ_a \cdot XD = \frac{\operatorname{Pow}(X, \omega)}{(r+1)/2},$$

a symmetric quantity wrt a, b, c.

1.10.2 Brazil 2013/6

The incircle of triangle ABC touches sides BC, CA and AB at points D, E and F, respectively. Let P be the intersection of lines AD and BE. The reflections of P with respect to EF, FD and DE are X, Y and Z, respectively. Prove that lines AX, BY and CZ are concurrent at a point on line IO, where I and O are the incenter and circumcenter of triangle ABC.



(We continue to use terminology from the previous subsubsection.) Let T_a be the projection of D onto $\overline{EF}$. As promised in the refactored statement in the problem section,

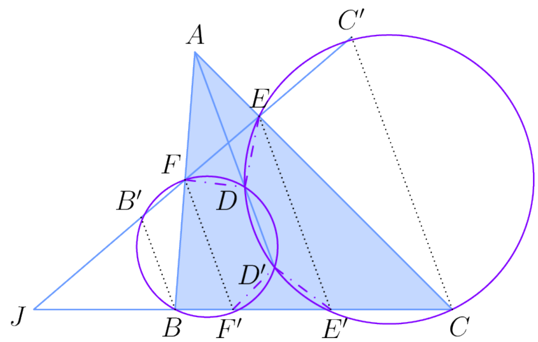
Claim – $T_a \in \overline{AXA'}$.

Proof. Because X is the similicenter of triangles DEF, $I_aI_bI_c$, it must also be similicenter of their orthic triangles. It follows that $T_a \in \overline{AX}$, as needed. □

Next, let $V = \overline{AD} \cap \overline{EF}$, so that $(DV;AP) = -1$. Because $\angle DT_aV = 90°$, $\overline{EF}$ must bisect $\angle AT_aP$, whence $P_a \in \overline{AT_aA'}$. Considering triangles ABC, DEF, and the orthic triangle of $\triangle DEF$, the concurrency holds by cevian nest.

1.11 IMO 2021/3

Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.



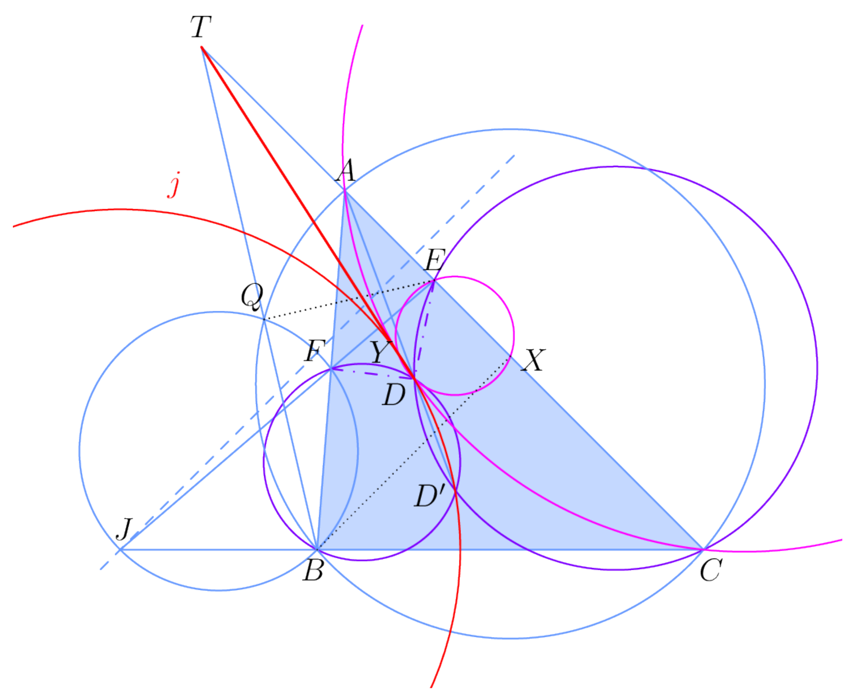
Solution by **v4913**.

Let $J = \overline{EF} \cap \overline{BC}$, and $D' \in \overline{AD}$ be the isogonal conjugate of D wrt $\triangle ABC$. The given angle conditions imply that $BDD'F$, $CDD'E$ are cyclic, while power of a point at A implies $BCEF$ cyclic as well.

Claim 1 – J is the exsimilicenter of (EDC), (FDB); hence, $JD = JD'$ by symmetry.

Proof. Construct $E_1 = (CDD'E) \cap \overline{BC}$ $(\neq C)$, $F_1 = (BDD'F) \cap \overline{BC}$ $(\neq B)$. By isogonality, $DF = D'F'$ and $DE = D'E'$ whence $DD'E'E$, $DD'F'F$ are both cyclic isosceles trapezoids. $\overline{DD'}, \overline{EE'}, \overline{FF'}$ share a perpendicular bisector b, and in fact, this is the bisector of $\angle J$, i.e. $JE = JE'$, $JF = JF'$.

Reflect B, C over b to obtain B', C'; then, because $JB/JF' = JB/JF = JE/JC = JE'/JC$, there is a homothety at J mapping $(B, B', F, F') \to (E', E, C', C)$ and thus their circumcircles $(BB'DD') \to (CC'DD')$ as well. □



Let $Y = (ADC) \cap (EXD) \quad (\neq D)$, Q be the Miquel point of $ABCJEF$, and j the circle at J through D, D'. Observing that $\overline{O_1O_2}$ is the perpendicular bisector of $\overline{DY}$, it remains to prove $Y \in j$.

Claim 2 – $XQEB$ is cyclic.

Proof. This is a simple angle chase: using cyclic quadrilaterals $(ABCQ)$, $(JFBQ)$, $(ECJQ)$, and $(AEFQ)$, we obtain

$$\measuredangle EQB = \measuredangle EQA + \measuredangle AQB = \measuredangle ACB + \measuredangle EFA = 2\measuredangle ACB = \measuredangle EXB$$

as desired. □

Next, we characterize the radical axis of j, (JBF)– it's perpendicular to the line of centers and through A:

Claim 3 – The line through B and the center of (JBF) is perpendicular to $\overline{AC}$.

Proof. This is equivalent to "t_b, the tangent to (JBF) at J, is parallel to $\overline{AC}$". Because $\measuredangle(\overline{BJC}, t_b) = \measuredangle BFJ = \measuredangle JCE$, the result follows. □

Because $\operatorname{Pow}(A, j) = AD \cdot AD' = AQ \cdot AJ = \operatorname{Pow}(A, (JBQF))$, A is on the radical axis of j, (JBF). By the previous claim, it follows that $\overline{AC}$ is the radical axis of j, (JBF).

To finish, define $T = \overline{DY} \cap \overline{AC} \cap \overline{BQ}$ as the radical center of (JBF), (ABC), (EXD), (ADC), and the phantom point $Y' = \overline{TD} \cap j \quad (\neq D)$. Because T is on $\overline{AC}$, the radical axis of j, (JBF), we have (lengths directed)

$$TY' \cdot TD = \operatorname{Pow}(T, j) = \operatorname{Pow}(T, (JBF)) = \operatorname{Pow}(T, (ABCQ)) = TA \cdot TC = TY \cdot TD \Rightarrow Y = Y',$$

the end!

1.12 USAMO 2021/6, by Ankan Bhattacharya

1.13 SL 2021/G8

Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.

Solution 1, by Brianchon (from AoPS) † (WIP) Redefine R as intersection of tangent at D' and A-altitude and prove PR is tangent to ω_{XPA}. Let us denote some points: $D'D' \cap AB = B'$,$D'D' \cap AC = C'$,$D'D' \cap XX = B'$,$XX \cap AB = X'$ and $CX \cap AP' = M$. apply Brianchon to the polar reciprocal $DD'XXB"C"$:

1. $DD \cap D'D' = \infty$
2. $D'D' \cap XX = P'$
3. $XX \cap XX = X$
4. $XX \cap B"B" = X'$
5. $B"B" \cap C"C" = A$
6. $C"C" \cap DD = C'$

and lines 14, 25, 36 must be concurrent. Since $AP' \cap CX = M$ we can imply that $MX'||BC$ By angle chase $\angle MX'A = \angle P'B'A = 180 - \angle AB'C' = 180 - \angle ABC = 180 - \angle AXC = \angle MXA$ so $MXX'A$ is concyclic. Again by angle chase $\angle MAX = \angle MX'P' = \angle X'P'B' = \angle PAR$ (since $P'APR$ is concyclic) thus $\angle XAP = \angle P'AR = \angle P'PR$ and we are done.

Solution 2, by DDIT (CyclicISLscelesTrapezoid)

† https://artofproblemsolving.com/community/c6h2882551p25740378