

Geometry Favorites (WIP)

People

Last updated December 26, 2022

... special things, I compile...

A Million Dreams

Colloquially, “the problems of all time”; have fun!

(Note: here, ∞_{XY} , $\infty_{\perp XY}$ denote the points at infinity along line XY and along a line perpendicular to \overline{XY} , respectively.)
(also toc moved to p. 2)

🌲 -1 Credits + remarks

Inspired by chapter 11 of EGMO book, and **Geometry At Its Best** by Eric Shen.
Also thanks to collaborators...

Things to do:

- add one more **non-triangular** problem;
- fix toc formatting;

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🌲 O Problems

Remark. Some attempt has been made to deviate from the aforementioned two famous geometry papers.

Problem 1 (SL 1998/G4). Let M and N be two points inside triangle ABC such that $\angle MAB = \angle NAC$ and $\angle MBA = \angle NBC$. Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

Problem 2 (SL 2009/G3). Let ABC be a triangle. The incircle of $\triangle ABC$ touches AB and AC at the points Z and Y , respectively. Let $G = \overline{BY} \cap \overline{CZ}$, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.

Problem 3 (SL 2015/G4). Let ABC be an acute triangle and let M be the midpoint of AC . A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC . Determine all possible values of $\frac{BT}{BM}$.

Problem 4 (SL 2016/G7). Let I be the incentre of a non-equilateral triangle ABC , I_A be the A -excentre, I'_A be the reflection of I_A in BC , and l_A be the reflection of line AI'_A in AI . Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B .

- (a) Prove that P lies on line OI where O is the circumcentre of triangle ABC .
- (b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y . Show that $\angle XIY = 120^\circ$.

Problem 5 (IMO 2008/6). Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C , which is also tangent to the lines AD and CD . Prove that the common external tangents to ω_1 and ω_2 intersect on ω .

Problem 6 (EGMO 2015/6). Let H be the orthocentre and G be the centroid of acute-angled triangle ABC with $AB \neq AC$. The line AG intersects the circumcircle of ABC at A and P . Let P' be the reflection of P in the line BC . Prove that $\angle CAB = 60^\circ$ if and only if $HG = GP'$.

Problem 7 (Iran TST 2018/1/4). Let ABC be a triangle ($\angle A \neq 90^\circ$), with altitudes $\overline{BE}, \overline{CF}$. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N . Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that \overline{AP} bisects \overline{BC} .

Problem 8 (Eric Shen). In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. Let $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{TZ}$, and Q lies on \overline{EF} such that $\overline{PQ} \parallel \overline{BC}$, prove that \overline{AQ} bisects \overline{BC} .

Problem 9 (China TST 2015/2/3). Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G . Let D be the midpoint of \overline{BC} , and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{OD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω .

Problem 10 (SL 2018/G5). Let ABC be a triangle with circumcircle ω and incenter I . A line ℓ meets the lines AI, BI, CI at points D, E, F respectively, all distinct from A, B, C, I . Prove that the circumcircle of the triangle determined by the perpendicular bisectors of $\overline{AD}, \overline{BE}, \overline{CF}$ is tangent to ω .

Problem 11 (MOP 2020/1Z). Let $ABCD$ be a quadrilateral inscribed in circle Ω . Circles ω_A and ω_D are drawn internally tangent to Ω , such that ω_A is tangent to \overline{AB} and \overline{AC} while ω_D is tangent to \overline{DB} and \overline{DC} . Prove that we can draw a line parallel to \overline{AD} which is simultaneously tangent to both ω_A and ω_D .

Problem 12 (Evan Chen Twitch). Let ABC be a triangle and let T be the contact point of the A -mixtilinear incircle with the circumcircle, and let T' be the reflection of T over BC . Prove that the nine-point circle of $T'BC$ is tangent to the incircle.

Problem 13 (IGO 2020/A4). Convex circumscribed quadrilateral $ABCD$ with its incenter I is given such that its incircle is tangent to \overline{AD} , \overline{DC} , \overline{CB} , and \overline{BA} at K , L , M , and N . Let $E = \overline{AD} \cap \overline{BC}$ and $F = \overline{AB} \cap \overline{CD}$. Let $X = \overline{KM} \cap \overline{AB}$ and $Y = \overline{KM} \cap \overline{CD}$. Let $Z = \overline{LN} \cap \overline{AD}$ and $T = \overline{LN} \cap \overline{BC}$.

Prove that the circumcircle of triangle $\triangle XFY$ and the circle with diameter EI are tangent if and only if the circumcircle of triangle $\triangle TEZ$ and the circle with diameter FI are tangent.

Problem 14 (IMO 2019/6). Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC , CA , and AB at D , E , and F , respectively. The line through D perpendicular to EF meets ω at R . Line AR meets ω again at P . The circumcircles of triangle PCE and PBF meet again at Q .

Prove that lines DI and PQ meet on the line through A perpendicular to AI .

Problem 15 (MOP 2019 & USA TST 2019/6). Let ABC be a triangle with incenter I , and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to \overline{BC} at A_1 . Define points B_1 on \overline{CA} and C_1 on \overline{AB} analogously, using the excircles opposite B and C , respectively.

- (a) (MOP 2019) Let E, F be the feet of the altitudes from B, C respectively. Prove that if \overline{EF} touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.
- (b) (USA TST 2019/6) Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then \overline{AD} is tangent to (DB_1C_1) .

Problem 16 (RMM 2012/6 + Fake USAMO 2020/3). In triangle ABC with incenter I and circumcenter O , let the incircle ω touch \overline{BC} , \overline{CA} , \overline{AB} at D, E, F respectively.

- (RMM 2012/6) Let ω_a be the circle through B and C tangent to ω , and define ω_b, ω_c similarly. Finally, let $A' = \omega_b \cap \omega_c$ ($\neq A$), and similarly for points B' and C' . Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO .
- (Fake USAMO 2020/3) Let T be the projection of D to \overline{EF} . The line AT intersects the circumcircle of $\triangle ABC$ again at point $X \neq A$. Circles (AEX) and (AFX) intersect ω again at points $P \neq E$ and $Q \neq F$ respectively. Prove that $\overline{EQ}, \overline{FP}$, and \overline{OI} are also concurrent.

Problem 17 (TSTST 2018/3). Let ABC be an acute triangle with incenter I , circumcenter O , and circumcircle Γ . Let M be the midpoint of \overline{AB} . Ray AI meets \overline{BC} at D . Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y , while line CO meets ω at C and Q . Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D . Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ .

Problem 18 (DeuX MO 2020/II/3). In triangle ABC with circumcenter O and orthocenter H , line OH meets $\overline{AB}, \overline{AC}$ at E, F respectively. Let ω be the circumcircle of triangle AEF with center S , meeting $\overline{(ABC)}$ again at $J \neq A$. Line OH also meets (JSO) again at $D \neq O$. Define $K = (JSO) \cap (ABC)$ ($\neq J$), $M = \overline{JK} \cap \overline{OH}$, and $G = \overline{DK} \cap (ABC)$ ($\neq K$). Prove that (GHM) and (ABC) are tangent to each other.

Problem 19 (USA TST 2021/2). Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ , in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$. Let X be a variable point on the arc V_1V_2 of Γ not containing A or B . Line XA meets line U_1V_1 at C , while line XB meets line U_2V_2 at D .

Prove there exists a fixed point K , independent of X , such that the power of K to the circumcircle of $\triangle XCD$ is constant.

Problem 20 (IMO 2021/3). Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD , respectively. Prove that the lines BC, EF , and O_1O_2 are concurrent.

Problem 21 (USAMO 2021/6). Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}, \overline{BC} \parallel \overline{EF}, \overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y , and Z be the midpoints of $\overline{AD}, \overline{BE}$, and \overline{CF} . Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

Problem 22 (SL 2021/G8). Let ABC be a triangle with circumcircle ω and let Ω_A be the A -excircle. Let X and Y be the intersection points of ω and Ω_A . Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R . Prove that $\overline{AR} \perp \overline{BC}$.

Problem 23 (USEMO 2020/3). Let ABC be an acute triangle with circumcenter O and orthocenter H . Let Γ denote the circumcircle of triangle ABC , and N the midpoint of OH . The tangents to Γ at B and C , and the line through H perpendicular to line AN , determine a triangle whose circumcircle we denote by ω_A . Define ω_B and ω_C similarly.

Prove that the common chords of ω_A, ω_B and ω_C are concurrent on line OH .

Problem 24 (Brazil Olympic Revenge 2021/3). Let I, C, ω and Ω be the incenter, circumcenter, incircle and circumcircle, respectively, of the scalene triangle XYZ with $XZ > YZ > XY$. The incircle ω is tangent to the sides YZ, XZ and XY at the points D, E and F . Let S be the point on Ω such that XS, CI and YZ are concurrent. Let $(XEF) \cap \Omega = R, (RSD) \cap (XEF) = U, SU \cap CI = N, EF \cap YZ = A, EF \cap CI = T$ and $XU \cap YZ = O$.

Prove that $NARUTO$ is cyclic.

1 Solutions

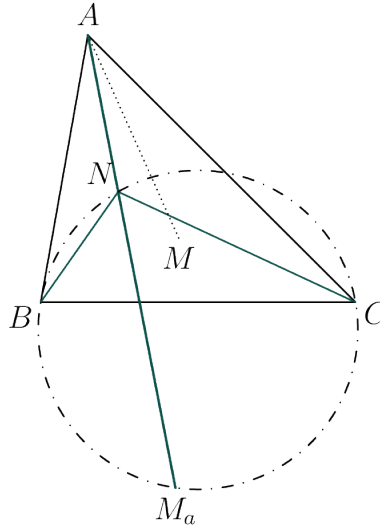
1.1 SL 1998/G4

Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$



Solution 1, by inversion Let i_a denote the inversion at A with power $AB \cdot AC$ composed with reflection in the bisector of $\angle A$. It's well-known that i_a swaps B, C . Let the images of M under i_a be $M_a \in \overline{AN}$, and cyclic variants.

Claim – $M_a \in (BNC)$, and

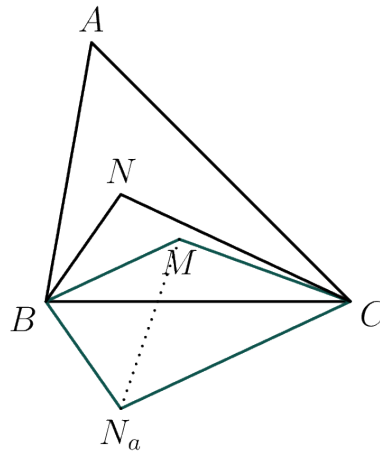
$$\frac{AM \cdot AN}{AB \cdot AC} = \frac{AN}{AM_a}.$$

Proof. The first part by angle chase:

$$\angle BM_aN \stackrel{i_a}{=} \angle MCA = \angle BCN,$$

while the second part is just the inversion distance formula. □

The claim reduces the problem to $\sum_{\text{cyc}} AN/AM_a = 1$, which is just **BAMO 2008/6**.



Solution 2, by area ratios (official / intended)

Claim – For any M, N , we have

$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMC] + [BNC]}{[ABC]}.$$

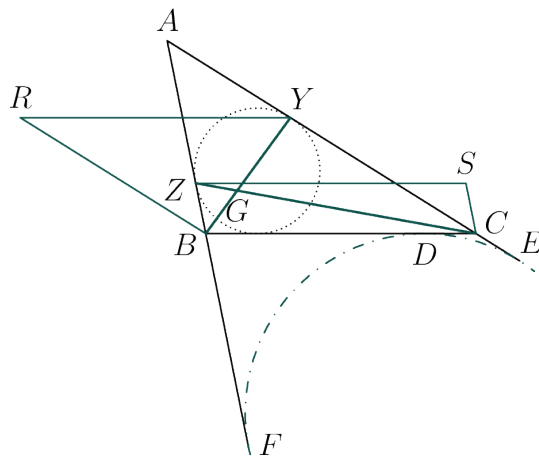
Proof. Reflect N over \overline{BC} to obtain point N_a . Then, because $\angle MBN_a = \angle B$, $(BM \cdot BN)/(BA \cdot BC) = [MBN_a]/[ABC]$. Similarly $(CM \cdot CN)/(CA \cdot CB) = [MCN_a]/[ABC]$, and summing the previous two equations gives

$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMN_aC]}{[ABC]} = \frac{[BMC] + [BNC]}{[ABC]}.$$

Noting that M, N are just isogonal conjugates, we obtain the problem by cyclic summation.

1.2 SL 2009/G3, by Hossein Karke Abadi

Let ABC be a triangle. The incircle of $\triangle ABC$ touches AB and AC at the points Z and Y , respectively. Let $G = \overline{BY} \cap \overline{CZ}$, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.



This is a very “troll” problem. Let (R) , (S) , ω_a denote the point circles at R, S (radius = 0) and the A -excircle respectively. Let ω_a touch \overline{BC} , \overline{CA} , \overline{AB} at D, E, F respectively. Also, for brevity, let $a = BC, b = CA, c = AB, s = (a + b + c)/2$.

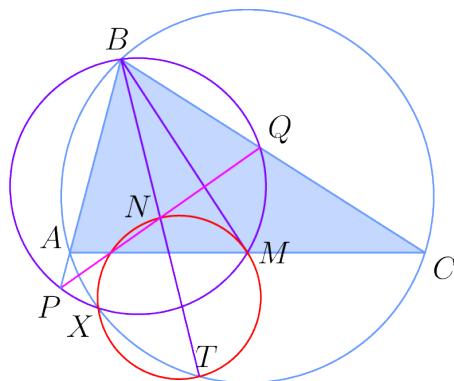
Claim – \overline{BY} is the radical axis of (R) , ω_a .

Proof. $BD = BR = s - c$, while $YE = YR = a$; because \overline{BD} , \overline{YE} touch ω_a , B, Y have powers $(s - c)^2, a^2$ wrt each of (R) , ω_a as promised. \square

By the claim, $G = \overline{BY} \cap \overline{CZ}$ must be the radical center of (R) , (S) , ω_a , implying the desired $GR = GS$.

1.3 SL 2015/G4

Let ABC be an acute triangle and let M be the midpoint of AC . A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC . Determine all possible values of $\frac{BT}{BM}$.



Solution by **CyclicSLscalesTrapezoid**.

The answer is $\sqrt{2}$ only. Let $X = (ABC) \cap (BPMQ)$ ($\neq B$), and let N be the midpoint of \overline{BT} .

Claim 1 – $XNMT$ is cyclic, and \overline{BM} is tangent to this circle..

Proof. Since N is also the midpoint of \overline{PQ} , there is a spiral similarity at X sending PNQ to AMC . Thus, we have

$$\angle XMN = \angle XAP = \angle XTB,$$

proving the concyclicity. For the tangency, check that

$$\angle XNM = \angle XPA = \angle XPB = \angle XMB.$$

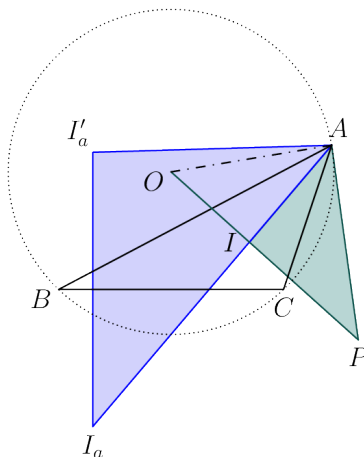
□

By power of a point, $BM^2 = BN \cdot BT = \frac{BT^2}{2}$, so $\frac{BT}{BM} = \sqrt{2}$.

1.4 SL 2016/G7

Let I be the incentre of a non-equilateral triangle ABC , I_A be the A -excentre, I'_A be the reflection of I_A in BC , and l_A be the reflection of line AI'_A in AI . Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B .

- Prove that P lies on line OI where O is the circumcentre of triangle ABC .
- Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y . Show that $\angle XIY = 120^\circ$.



Redefine P as the inverse of I wrt (ABC) . For the first part we assert more strongly that:

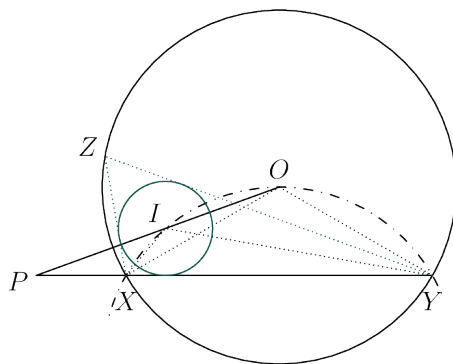
Claim – $\triangle AI_A I'_A \sim \triangle API$.

Proof. By angle chasing, $\angle I_A = \angle P$ follows easily. We contend that $I_A I'_A / I_A A = IP / AP$; indeed, the first ratio equals $2 \cos \angle BI_A C = 2 \sin \frac{A}{2}$ because of similar triangles $I_A BC \sim \triangle I_A I_B I_C$, while

$$\frac{IP}{AP} = \frac{OP}{AP} - \frac{OI}{OA} \frac{OA}{AP} = \frac{OA}{AI} - \frac{OI^2}{OA \cdot AI} = \frac{R}{AI} - \frac{R^2 - 2rR}{R \cdot AI} = \frac{R - (R - 2r)}{AI} = 2 \sin \frac{A}{2},$$

so the ratios are equal. The similarity follows by SAS. \square

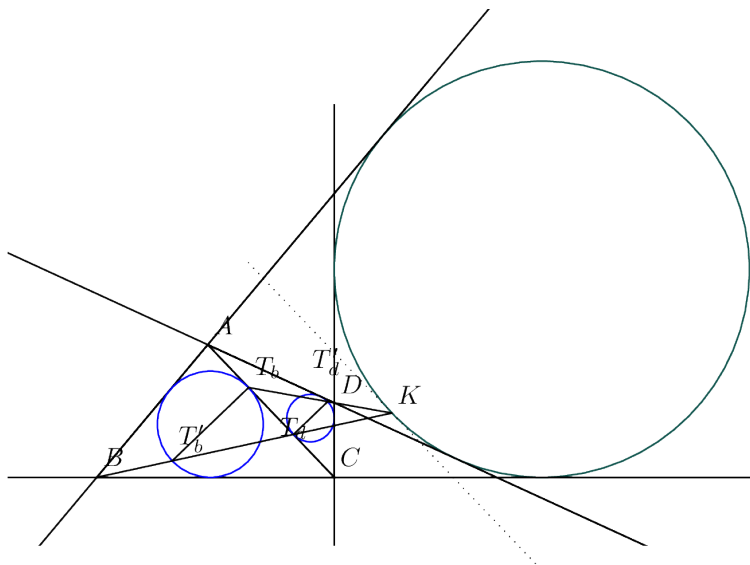
The claim clearly implies the isogonality.



For the second part, using Poncelet, let $Z \in (ABC)$ be the unique point so that $\triangle XYZ, ABC$ share a incircle and circumcircle. Inverting " P, X, Y collinear" wrt the circumcircle gives O, I, X, Y concyclic, or $\angle XOY = \angle XIY$. As it's well-known that $\angle XOY = 2\angle Z$ and $\angle XIY = (\pi + \angle Z)/2$, we must have $\angle Z = \pi/3 \Rightarrow \angle XIY = 2\pi/3$ as needed.

1.5 IMO 2008/6, by Vladimir Shmarov

Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C , which is also tangent to the lines AD and CD . Prove that the common external tangents to ω_1 and ω_2 intersect on ω .



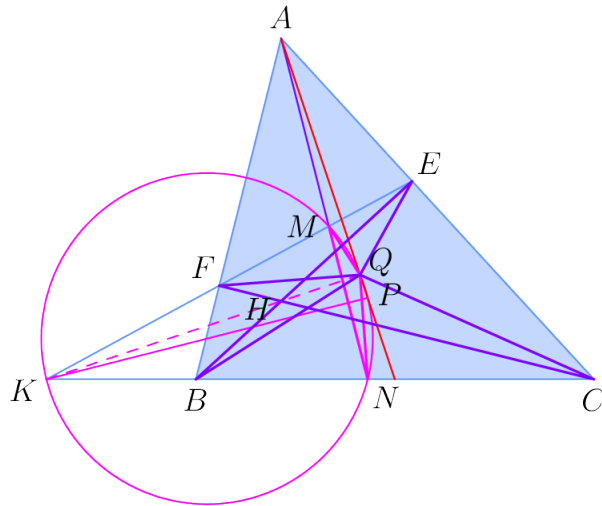
Rename ω_1, ω_2 to ω_b, ω_d ; by Pitot-like reasoning we have $AB + AD = CB + CD$; let T_b, T_d be the intouch points on \overline{AC} ; then T_b, T_d are isotomic by the obtained length condition.

If we let T'_b, T'_d be the antipodes of T_b, T_d on their respective circles, then an EGMO lemma (ch4) implies that B, T_d, T'_b and sym variant are collinear.

Construct the point K' on the "closer" side to the rest of the figure so that the tangent to ω at K is parallel to \overline{AC} . Then by homothety $K' \in \overline{BT_d}, \overline{DT_b}$, so this is the desired exsimilicenter.

1.6 Iran TST 2018/1/4, by Iman Maghsoudi & Hooman Fattahi

Let ABC be a triangle ($\angle A \neq 90^\circ$), with altitudes $\overline{BE}, \overline{CF}$. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N . Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that \overline{AP} bisects \overline{BC} .



Construct $K = \overline{EF} \cap \overline{BC}$, Q as the A -Humpty point, H as the orthocenter of $\triangle ABC$, and $\omega = (KMN)$, so that the P given is the antipode of K on it. Let spiral similarity s at Q take $(E, F) \rightarrow (B, C)$. The main point of the problem is then:

Claim – $MKQN$ cyclic. In other words, $Q \in \omega$.

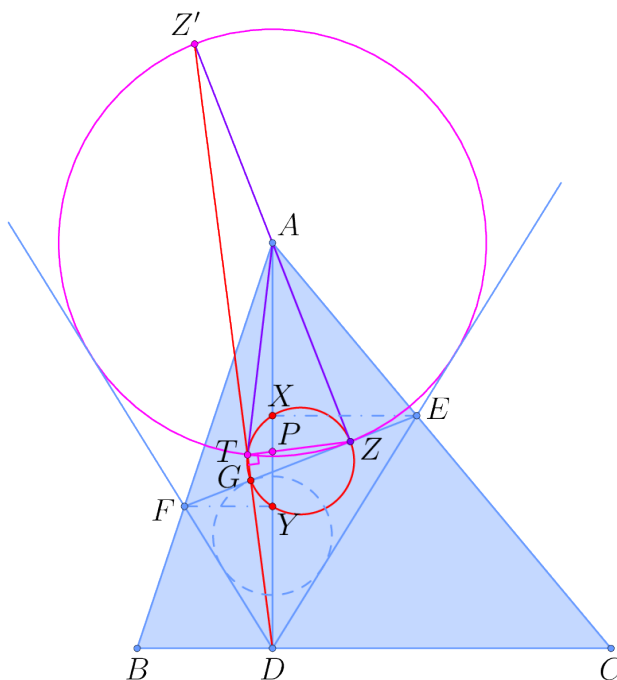
Proof. From angle bisector theorem,

$$\frac{EM}{MF} = \frac{EA}{AF} = \frac{BA}{AC} = \frac{BN}{NC} \Rightarrow (M \xrightarrow{s} N) \Rightarrow \angle MQN = -\arg(s) = \angle(\overline{EF}, \overline{BC}) = \angle MKN. \quad \square$$

Since P is the antipode of K on ω , $\angle KQP = 90^\circ = \angle KQA$, implying that $P \in \overline{AQ}$, the A -median.

1.7 Mock AIME 2019/15', by Eric Shen & Raymond Feng

In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. Let $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{TZ}$, and Q lies on \overline{EF} such that $\overline{PQ} \parallel \overline{BC}$, prove that \overline{AQ} bisects \overline{BC} .



Construct points X, Y as the projections of E, F onto \overline{AD} respectively. *

After drawing a diagram on Geogebra, we obtain:

Characterization of T

T is the harmonic conjugate of Z wrt XY – i.e. it lies on $\omega = (XYZ)$ so that the resulting quadrilateral is harmonic.

In American style, we show that this choice of T lies on ω_a (defined as the circle at A thru Z) and (DZ) ,

Verification (inspired by USA TST 2015/1)

For $AZ = AT$, we use power of a point / length chase to get $AZ^2 = AX \cdot AY$ whence \overline{AZ} touches ω . Hence, by harmonics \overline{AT} is also tangent to ω , so this property follows.

$\angle DTZ = 90^\circ$ is much less straightforward. We define $Z' = 2A - Z$ and $G = E + F - Z$ as the antipodes of Z on the circle at A through Z . By a well-known lemma, D, Z', G collinear (along the cevian through the intouch point in $\triangle DEF$).

But also at the same time, T is on $\omega, \omega_a \Rightarrow \angle ZTG = \angle ZTZ' = \pi/2$ due to antipodes. Hence, $\angle DTZ = \pi/2$, completing the verification.

*Eric Shen originally included these points in the problem statement (as seen in the 2019 version of “Geometry At Its Best”), but I guess the problem’s made harder by deleting them. For me, thinking about their relevance/use was important in making nonzero progress on the problem.

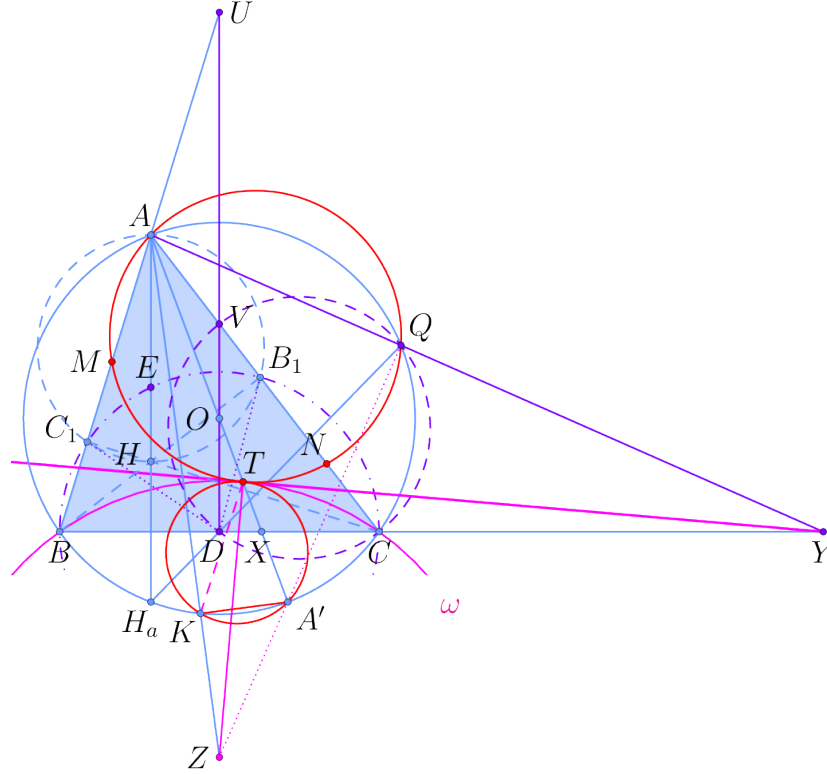
By this definition, we clearly have $(AP; XY) = -1$. From here (the chase is best discovered backwards), harmonic chasing suffices. Define $K = \overline{EF} \cap \overline{A\infty_{BC}}$. Then the bisection is established by

$$(\overline{AQ} \cap \overline{BC}, \infty_{BC}; B, C) \stackrel{A}{=} (QK; FE) \stackrel{\infty_{BC}}{=} (PA; YX) = -1.$$

1.8 China TST 2015/2/3

Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G . Let D be the midpoint of \overline{BC} , and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{OD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω .



As much of the parallel/perpendicular lines aren't even relevant (just to give us some equal ratios), we simplify as follows:

Problem reworded

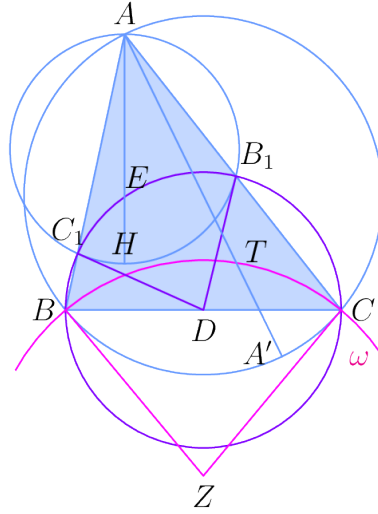
In acute $\triangle ABC$ with circumcenter O and orthocenter H , D is the midpoint of \overline{BC} , and the altitude from A meets (BC) at E (either one works). Let $U, V = \overline{OD} \cap \overline{AB}, \overline{AC}$, respectively; define $M, N \in \overline{AB}, \overline{AC}$ with (lengths directed)

$$UM/MB = VN/NC = AE/EH.$$

Let ω be the circle tangent to segments OB, OC at B, C respectively. Prove that (AMN) is tangent to ω .

We define a load of new points as follows:

- $A' = 2O - A$ as the antipode of A on (ABC) ;
- $T = \overline{AO} \cap \omega$, which we stipulate to be on segment AA' iff E is on segment AH ; WLOG, assume this is the case;
- Q as the harmonic conjugate of A' wrt BC , aka the reflection of the A -orthocenter Miquel point Q_a in the perpendicular bisector \overline{DUV} of \overline{BC} .



First, we get rid of E :

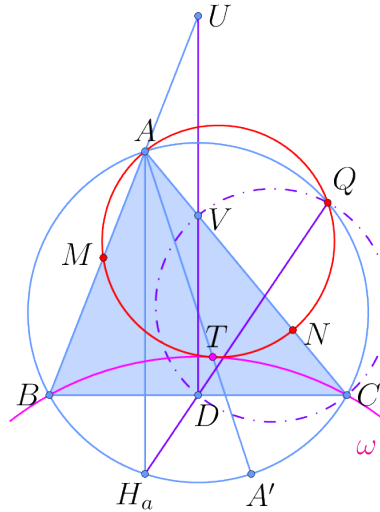
Claim 1 – $AE/EH = AT/TA'$. (lengths still directed)

Proof. (by [v4913](#)) Let B_1, C_1 denote the feet of the respective altitudes from B, C , and r a reflection in the bisector of $\angle A$ composed with a homothety at A with scale factor $AH/AA' = AB_1/AB = AC_1/AC$.

Because $\overline{DB_1}, \overline{DC_1}$ are well-known to touch (AH) , D is the pole of $\overline{B_1C_1}$;

$$\Rightarrow (Z \xrightarrow{r} D) \Rightarrow (\omega \xrightarrow{r} (BC)) \Rightarrow (T_1 \xrightarrow{r} E_1)$$

proving the claim. □



Claim 2 – Q is the Miquel point of $ABCDUV$.

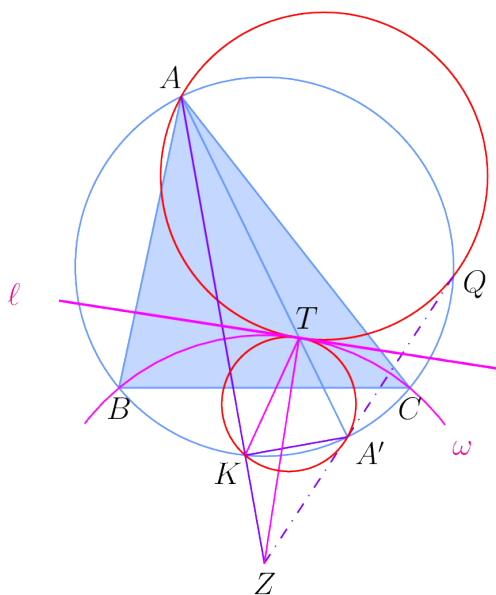
Proof. As we already have $Q \in (ABC)$, sufficient to prove $QDVC$ cyclic. Observe that $Q \in \overline{H_aD}$, which follows by $Q_a \in \overline{A'PH}$ reflected in \overline{DUV} . The result holds by Reim because AH_aQC cyclic and $\overline{DV} \parallel \overline{AH_a}$. □

Consider the spiral similarity s at Q mapping $B, C \rightarrow U, V$. Since $\triangle BA'C \stackrel{+}{\sim} \triangle UAV$, $(A' \xrightarrow{s} A)$. By the length condition $(M \xrightarrow{s} N)$ as well, so $M, N \in (AQT)$.

Finally, we turn to the problem statement:

Claim 3 – AQT_1 touches ω at T_1 .

We present two finishes.

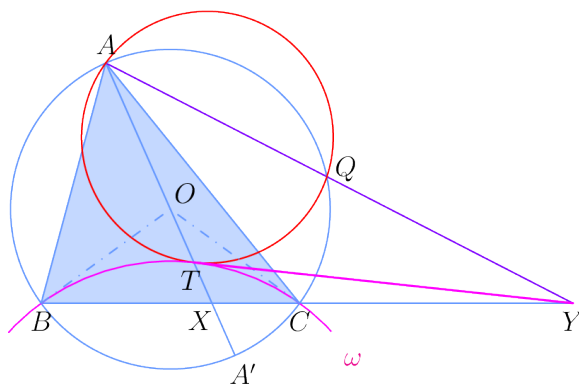


Proof 1, by inversion (v4913) Let $Z \in \overline{QA'}$ be the center of ω aka the polar of \overline{BC} wrt (ABC) , and $*$ denote inversion in ω . Define $K = \overline{AZ} \cap (ABC)$ ($\neq A$) = A^* . Clearly, $(A'Q; BC) = -1 \Rightarrow A' = Q^*$. Finally, let $\ell \perp \overline{ZT}$ denote the tangent to ω at T .

It remains to prove that $(A'KT) = (AQT)^*$ touches ℓ at T (and thus ω as well). We do so by angle chase:

$$\angle(\overline{KT}, \ell) = 90^\circ + \angle KTZ \stackrel{\text{inversion}}{=} 90^\circ + \angle ZAA' = \angle KA'T;$$

inverting back completes the problem.



Proof 2, by polars (crazyeyemoody907) Let $X = \overline{AO} \cap \overline{BC}$, and Y be the pole of \overline{AO} wrt ω , so that \overline{YT} touches ω . Since \overline{AO} contains the pole O of \overline{BC} wrt ω , we also $Y \in \overline{BC}$ by La Hire.

Finally, we contend that A, Q, Y collinear. Indeed, this follows from

$$(\overline{AY} \cap (ABC), A'; B, C) \stackrel{A}{=} (YX; BC) = -1$$

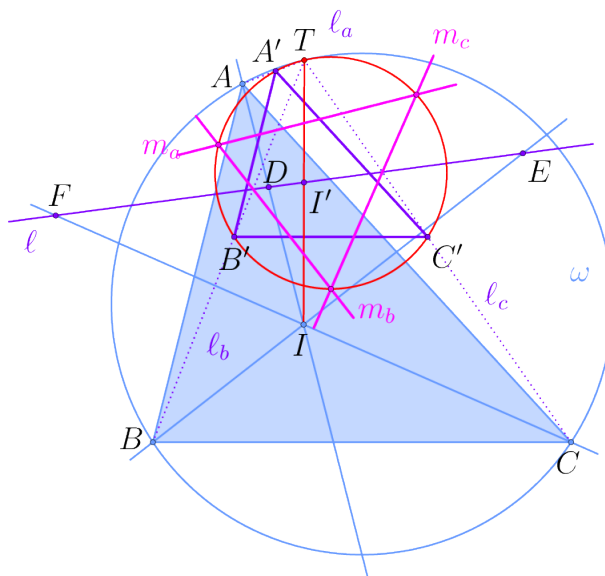
where the last harmonic bundle holds by definition of polar.

We finish by power of a point at Y : $YT^2 = YB \cdot YC = YA \cdot YQ$ means that $(AQT), \omega, \overline{YT}$ are tangent at T . \square

Remark. Should definitely use the first diagram for intimidation purposes.

1.9 SL 2018/G5, by Denmark

Let ABC be a triangle with circumcircle ω and incenter I . A line ℓ meets the lines AI , BI , CI at points D , E , F respectively, all distinct from A , B , C , I . Prove that the circumcircle of the triangle determined by the perpendicular bisectors of \overline{AD} , \overline{BE} , \overline{CF} is tangent to ω .



Solution by **TheUltimate123**.

Let ℓ_a and cyclic variants be the reflections of ℓ in the perpendicular bisectors x_a of \overline{AD} , etc.

Claim – $\ell_a, \ell_b, \ell_c, \omega$ concur at a point T .

Proof. Because

$$\angle(\ell_b, \ell_c) = 2\angle(x_b, x_c) = 2\angle BIC = \angle BAC,$$

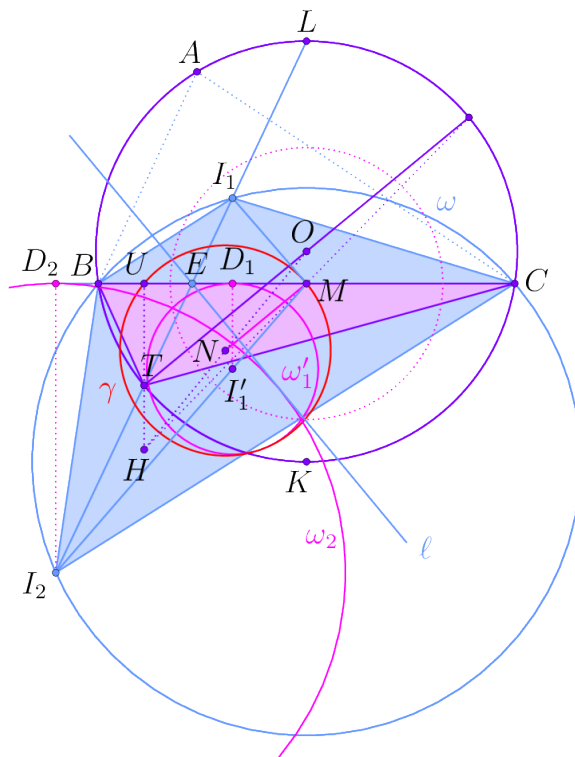
$\ell_b \cap \ell_c \in \omega$; the result follows by symmetry. \square

Let $I' = \overline{TI} \cap \ell$, and consider the homothety h at T mapping $I \rightarrow I'$. Let P' denote the image of point P under h , so I' is the incenter of $\triangle A'B'C'$. Since $\overline{A'I'} \parallel \overline{ADI}$ while $A' \in \ell_a$ and $I' \in \ell$, m_a is also the perpendicular bisector of $\overline{AI'}$.

From here it follows that the pairwise intersections of m_a, m_b, m_c are just the arc midpoints in $(A'B'C')$. By h , $(A'B'C')$, (ABC) tangent at T , hence done.

1.10 Twitch Solves ISL 006.1

Let ABC be a triangle and let T be the contact point of the A -mixtilinear incircle with the circumcircle, and let T' be the reflection of T over BC . Prove that the nine-point circle of $T'BC$ is tangent to the incircle.



Let K, L, M be the midpoints of \widehat{BTC} , \widehat{BAC} , and I_1 be the incenter, so that $\omega = (BI_1C)$; then, let $I_2 = \overline{NT} \cap \omega$ ($\neq I_1$). Clearly, since $\overline{LB}, \overline{LC}$ touch ω , $(BC; I_1I_2) = -1$. Additionally, since $\angle KTL = 90^\circ$, T is the midpoint of $\overline{I_1I_2}$, a Dumpty point...

Recall the following lemma:

Lemma - In $\triangle ABC$ with A -Dumpty point X , \overline{AX} bisects $\angle BXC$.

Reflect the nine-point circle given to obtain the nine-point circle of $\triangle TBC$. We may now safely get rid of A :

Problem simplified

In harmonic quadrilateral BI_1CI_2 , T is the midpoint of $\overline{I_1I_2}$, and I_1' is the reflection of I_1 in \overline{BC} (aka the I_2 -Dumpty point in $\triangle I_2BC$). For $k \in \{1, 2\}$ let D_k be the foot from I_k to \overline{BC} , and ω_k the circle at I_k through D_k . Then the circle ω_1' at I_1' through D_1 touches the nine-point circle γ of $\triangle TBC$.

Recalling the proof of Feuerbach by inverting about the midpoint of a side, we do likewise here. Define...

- U as foot from T to \overline{BC} , and $E = \overline{I_1I_2} \cap \overline{BC}$. By midpoints of harmonic bundles lemma applied to $(D_1D_2; ME) \stackrel{\infty_{BC}}{\equiv} (I_1I_2; NE) = -1$, we have $ME \cdot MU = MD_1 \cdot MD_2$.
- N, O as respective centers of $\gamma, (BTC)$, and H as orthocenter of $\triangle BTC$;

M is the exsimilicenter of ω'_1, ω_2 because it lies on the line of centers $\overline{I'_1 I_2}$ as well as the common external tangent \overline{BC} . Let $*$ denote inversion at M with power $ME \cdot MU = MD_1 \cdot MD_2$, so that $\omega'_1 = \omega_2^*$.

Claim – The reflection ℓ of \overline{BC} in $\overline{I_1 I_2}$ is γ^* .

Proof. Since $E = U^*$, it suffices to prove that $\overline{MN} \perp \ell$. Indeed, $MN \parallel \overline{TO}$ by homothety at H , while reflecting $\overline{TU} \perp \overline{BC}$ in the T -angle bisector $\overline{I_1 T X_2}$ gives $\overline{TO} \perp \ell$. \square

Observe by symmetry about $\overline{I_1 I_2}$ that ℓ also touches ω_2 . Inverting back, we have γ tangent to ω'_1 as required.

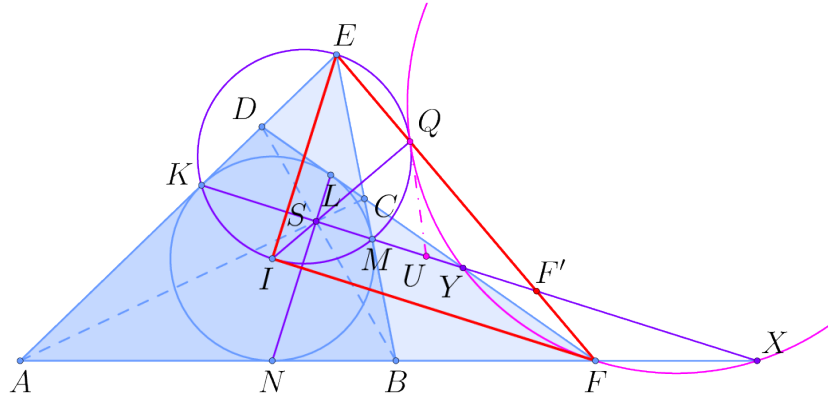
1.11 IGO 2020/A4

Convex circumscribed quadrilateral $ABCD$ with its incenter I is given such that its incircle is tangent to \overline{AD} , \overline{DC} , \overline{CB} , and \overline{BA} at K , L , M , and N . Let $E = \overline{AD} \cap \overline{BC}$ and $F = \overline{AB} \cap \overline{CD}$. Let $X = \overline{KM} \cap \overline{AB}$ and $Y = \overline{KM} \cap \overline{CD}$. Let $Z = \overline{LN} \cap \overline{AD}$ and $T = \overline{LN} \cap \overline{BC}$.

Prove that the circumcircle of triangle $\triangle XFY$ and the circle with diameter EI are tangent if and only if the circumcircle of triangle $\triangle TEZ$ and the circle with diameter FI are tangent.

imagine doing both directions

CyclicISLscalesTrapezoid



We contend that $(XFY), (EI)$ tangent $\iff \overline{KM} \perp \overline{LN}$, which is just another expression of ‘ $ABCD$ bicentric’.

Call the incircle ω . Define $S = \overline{KM} \cap \overline{LN}$ and Q as the Miquel point of $KLMN$ aka the inverse of S wrt ω , which obviously lies on the polar \overline{EF} of S wrt ω . It follows that $\angle SQE = \angle SQF = 90^\circ$.

Let $F' = \overline{QF} \cap \overline{KM}$ U be the midpoint of $\overline{SF'}$.

Claim – \overline{QF} always bisects $\angle XQY$, and $UM \cdot UK = US^2 = UX \cdot UY$, so U lies on the radical axis of $(EIMK), (XFY)$.

Proof. By Brianchon on $ABMCDK$, $S = \overline{AB} \cap \overline{CD}$ as well. Apply DIT to \overline{KM} and quadrilateral $ABCD$ and project to Q , to obtain an involutive pairing $i : Q(XY; SS; KM)$. The last two pairs imply that i is just reflection in \overline{QS} , so \overline{QS} bisects $\angle XQY$. As $\overline{QF} \perp \overline{QS}$, it must also bisect the same angle: $i : Q(F'F')$.

By these right angles and angle bisections, $(SF'; MK) = (SF'; XY) = -1$, so the last result follows by midpoints of harmonic bundles lemma. \square

Now because $\angle SQF' = 90^\circ$, we have $UQ = US = UF' \implies (EI), (XQY)$ tangent at Q ; this means that the desired is equivalent to $FQXY$ cyclic.

Note. Actually, there are two circles through X, Y , but one of them is extraneous by configuration issues.

Next, we show that this happens iff $FX = FY$.

“If” direction Since $\angle FQX = \angle YQF$, triangles FQX, FQY have equal circumradii, so their circumcircles either coincide or are reflections of each other in \overline{QF} . If they were to be reflections, we’d obtain two possibilities, each

absurd in the context of the problem: (i) Q, X, Y collinear $\Rightarrow \overline{XY}, \overline{EF}$ coincide; and (ii) X, Y reflections in $\overline{QF} \Rightarrow \overline{XY}, \overline{SQ} \perp \overline{EF}, \overline{XY} \parallel \overline{SQ}$. Thus $(FQX) = (FQY)$ as required.

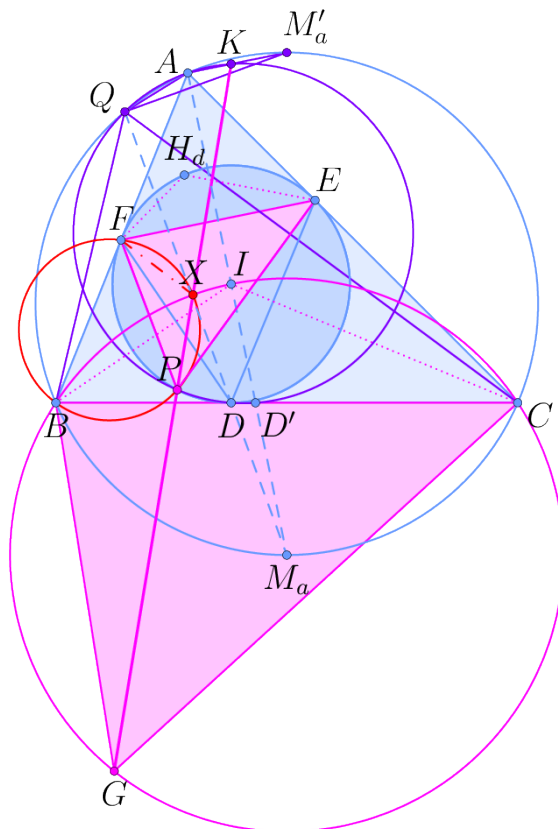
“Only if” direction $\angle YQF = \angle FQX \Rightarrow \widehat{YF} = \widehat{FX}$, trivial;

To finish the problem, observe that $FX = FY$ is equivalent to \overline{MK} making equal angles with $\overline{AB}, \overline{CD}$. As $FN = FL$ as well, \overline{LN} (always) makes equal angles with the same two lines. Since K, L, M, N form a convex polygon, this is in turn equivalent to $\overline{KM} \perp \overline{LN}$, completing the proof.

1.12 IMO 2019/6, by Anant Mudgal

Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC , CA , and AB at D , E , and F , respectively. The line through D perpendicular to \overline{EF} meets ω at R . Line AR meets ω again at P . The circumcircles of triangle PCE and PBF meet again at Q .

Prove that lines DI and PQ meet on the line through A perpendicular to \overline{AI} .

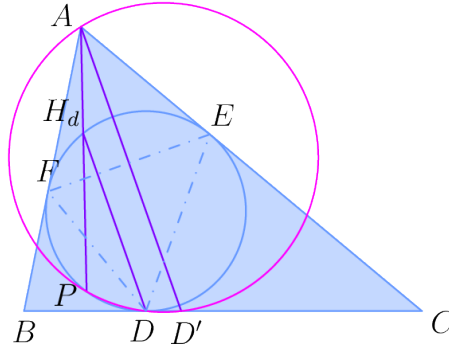


Observe that P is the D -orthocenter Miquel in $\triangle DEF$. Define K as the intersection of the A -external bisector with \overline{AD} . We make the following definitions...

- Let ω , ω_a denote the incircle and (BIC) respectively;
- Define X as intersection of *segment* PK with ω . Let Q instead denote the A -SD point;
- G be the harmonic conjugate of I wrt BC , D' as the foot of the A -angle bisector; M_a as the midpoint of arc BC exc. A ; M'_a as the antipode of M_a on (ABC) ;
- H as orthocenter of $\triangle DEF$, and H_d its reflection over \overline{EF} .

\Rightarrow because $MB^2 = MD \cdot MQ = MD' \cdot MA$, $Q \in (ADD'K)$.

Thus we want to show that $PXFB$ cyclic. ($PXEC$ cyclic would follow from symmetry, proving that X was indeed the point constructed in the problem.)



Claim 1 – $P \in (ADD'KQ)$.

Proof. Observe that $(PH_d; EF) = -1$ whence A, P, H_d collinear. Then because $\overline{DH_d} \parallel \overline{AI}$ because both perpendicular to \overline{EF} . Hence result by degenerate Reim. \square

Claim 2 – $\triangle PFE \stackrel{+}{\sim} \triangle GBC$.

Proof. Proceed by spiral at Q . Observe that $\triangle H_dEF \stackrel{+}{\sim} \triangle ICB$ by angle chase. Because $(H_dP; EF) = (IG; BC) = -1$, the needed similarity follows. \square

Claim 3 – K, G, P collinear.

Proof. An angle chase, using the previous two claims:

$$\angle QPK \stackrel{\text{claim 1}}{=} \angle QAK = \angle QAM_a \stackrel{\text{spiral}}{=} \angle QPG.$$

\square

Using last two claims, we may angle chase:

$$\angle PXB = \angle GXB = \angle GCB \stackrel{\text{spiral}}{=} \angle PEF = \angle PFB,$$

or $PXFB$ cyclic.

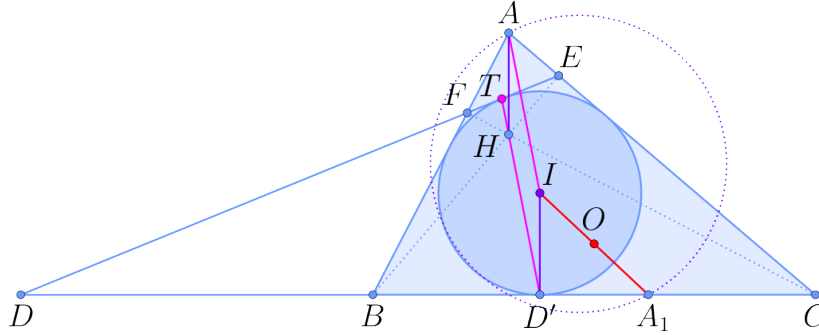
Remark. ggb way too op

1.13 MOP + USA TST, by Ankan Bhattacharya

Let ABC be a triangle with incenter I , and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to \overline{BC} at A_1 . Define points B_1 on \overline{CA} and C_1 on \overline{AB} analogously, using the excircles opposite B and C , respectively.

1.13.1 MOP 2019/(?)

Let E, F be the feet of the altitudes from B, C respectively. Prove that if \overline{EF} touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.



Call the incircle ω .

Claim 1 – D, E, F are collinear.

Proof. We will prove that the tangent line from D is antiparallel to \overline{BC} wrt $\angle A$. Indeed, this line is found by reflecting \overline{DBC} over \overline{DI} , a line perpendicular to \overline{AI} , so we win. \square

Let ω touch \overline{DEF} at a point T , and let D' denote the A -intouch point.

Claim 2 – $\overline{AI} \parallel \overline{HD'}$; hence $AID'H$ is a parallelogram and $AH = r$, the inradius of $\triangle ABC$.

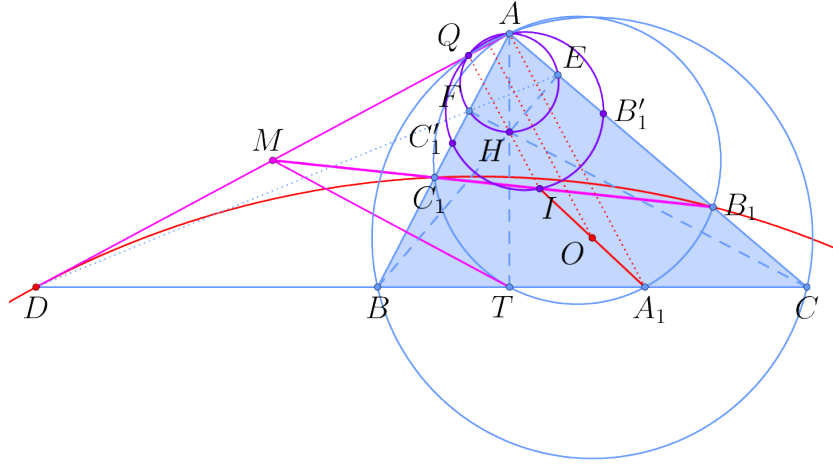
Proof. Because $BCEF$ is tangential, it follows by degenerate Brianchon that lines BE, CF, DT' concur, i.e. $H \in \overline{TD'}$. Observe that $DT = DD'$; then $\overline{THD'} \perp \overline{DI}$ by symmetry, while $\overline{AI} \perp \overline{DI}$ is given; the lines are thus parallel as claimed. \square

Now, let ω_a , etc denote (AB_1C_1) , etc, respectively. We observe that because the perpendicular from A_1 to \overline{BC} and its cyclic variants all concur at the point $2O - I$, it follows that all three circles must concur at this point by Miquel spam.

But because $r/2 = AH/2$ is the distance from O to \overline{BC} , we actually have $2O - I = A_1$ (also because of their feet onto \overline{BC}). Hence $A_1 \in \omega_a$ as desired.

1.13.2 USA TST 2019/6

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then \overline{AD} is tangent to the circumcircle of $\triangle DB_1C_1$.



From MOP 2019, we make the following observations:

- By its converse, D, E, F collinear; then, if T is the foot from A to \overline{BC} , we have $(TD; BC) = -1$.
- As A_1 is the Bevan point $2O - I$, its projections onto $\overline{AC}, \overline{AB}$ are B_1, C_1 respectively. It follows that A, A_1 are antipodes on ω_a .
- Since $BCEF$ is bicentric, if the incircle touches $\overline{AC}, \overline{AB}$ at B'_1, C'_1 , then $BC'_1/FC'_1 = CB'_1/EB'_1$, so the A -incenter and orthocenter Miquel points coincide, say at $Q \in (ABC)$.

From the last item, $\angle AQI = \angle AQH = 90^\circ$.

Claim – \overline{AD} touches ω_a .

Proof. Since $(ABC) \cap (AH) = \{A, Q\}$, the projection of O onto \overline{AQD} is $\frac{A+Q}{2}$. At the same time, the above implies Q is the projection of I onto \overline{AQD} . By linearity the projection of $A_1 = 2O - I$ onto \overline{AD} is $2\frac{A+Q}{2} - Q = A$ – in other words, $\angle A_1AD = 90^\circ$. This proves the tangency as $\overline{AA_1}$ is a diameter of ω_a . \square

Let $M = \frac{A+D}{2}$, so \overline{MT} touches ω_a as well by symmetry in the perpendicular bisector $M \infty_{BC}$ of \overline{AT} . Now, $(AT; B_1C_1) \stackrel{A}{=} (DT; CB) = -1$ means $M \in \overline{B_1C_1}$.

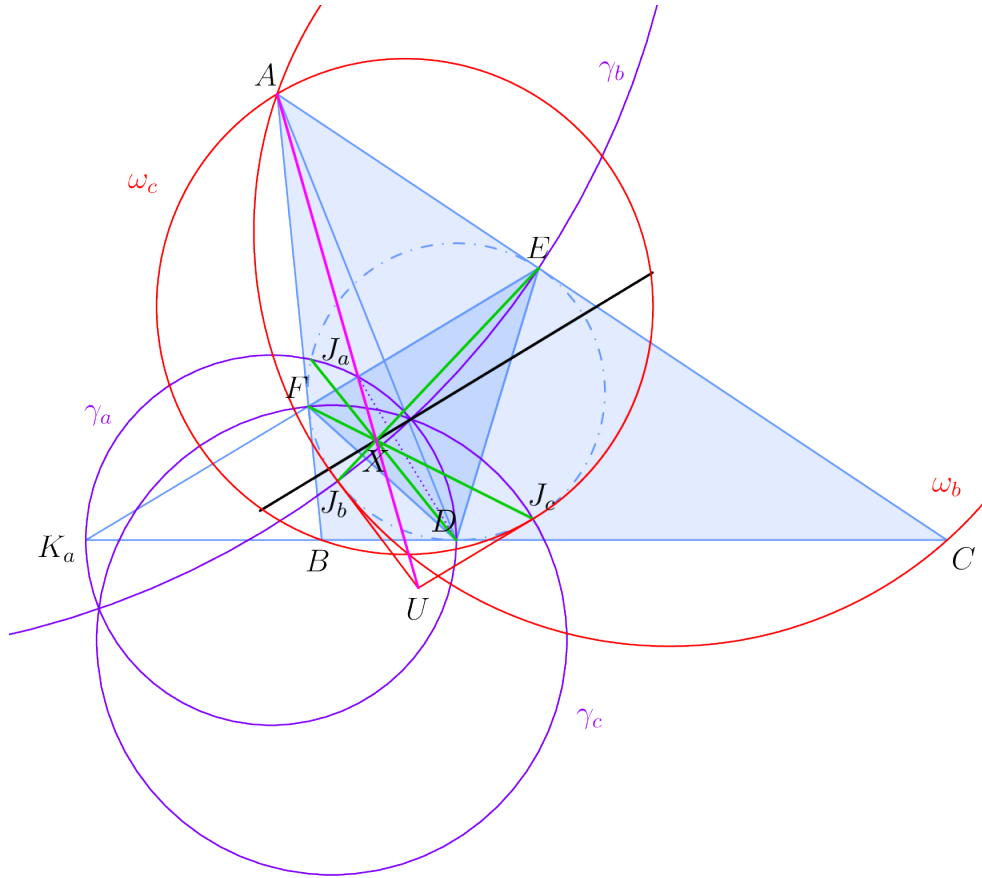
Finish by power of a point converse: $MD^2 = MA^2 = MB_1 \cdot MC_1$ gives the needed tangency.

1.14 RMM + Fake USAMO

1.14.1 RMM 2012/6, by Fedor Ivlev

Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC ; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A ; the points B' and C' are defined similarly. Prove that the lines AA' , BB' and CC' are concurrent at a point on the line IO .

Let $K_a = \overline{EF} \cap \overline{BC}$, $\gamma_a = (K_a D)$, $J_a = \omega_a \cap \gamma_a \cap \omega$ (and cyclic variants), and H and ℓ denote the orthocenter and Euler line of $\triangle DEF$, respectively. Also, let I_a, I_b, I_c be the excenters of $\triangle ABC$.

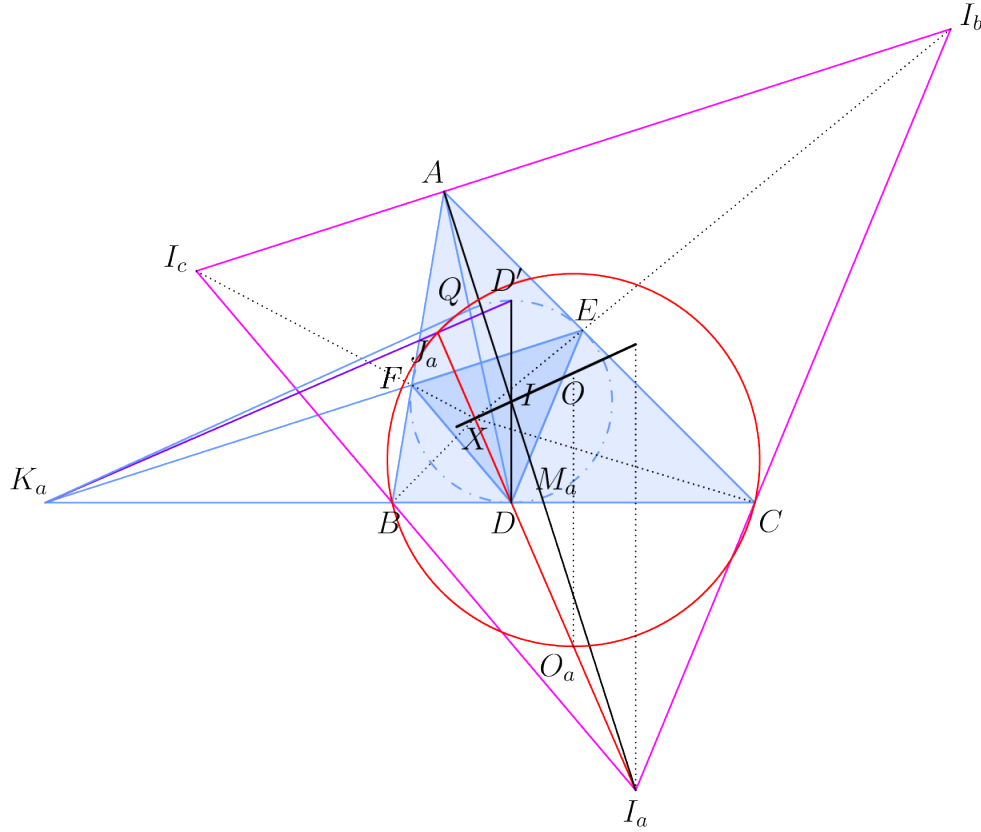


Solution 1, by radical axes Observe that ℓ is just \overline{OI} , and that γ_a , etc are coaxial Apollonian circles. Define X as the radical center of $\gamma_a, \gamma_b, \gamma_c, \omega$ (which exists since the former 3 circles are coaxial). We'll show this is the desired concurrency point in both problems. Clearly, $\overline{DJ_a}$ is the raxis of (γ_a, ω) , i.e. $X \in \overline{DJ_a}$.

Lemma 1 – ℓ is the raxis of γ_a and variants.

Proof. Let T_a denote the foot of D onto \overline{EF} , which is obviously on γ_a . Then H has power $HD \cdot HT_a$ (= variants) wrt the γ 's, hence on raxis; Meanwhile I has power r^2 wrt all circles by orthogonality, hence also on raxis, done. \square

Let tangents to ω at J_b, J_c meet at U ; then, \overline{AU} is the raxis of ω_b, ω_c . Clearly this is the polar of $\overline{J_b J_c} \cap \overline{EF}$. Recalling that $X = \overline{EJ_b} \cap \overline{FJ_c}$, follows by Brokard that $X \in \overline{AU}$, the end.



Solution 2, by homothety (v4913) Let D' be the antipode of D on ω , $Q = \overline{AD} \cap \omega$ ($\neq D$); then, because $(EF; DQ) = -1$, $\overline{K_aQ}$ touches ω as well. Also, because $\angle DJ_aD' = \angle DJ_aK_a = 90^\circ$, K_a, J_a, D' are collinear, whence $(DQ; J_aD') = -1$.

We start with X as the similicenter of homothetic triangles $DEF, I_aI_bI_c$. Let homothety h at X with scale factor r map $(D, E, F) \rightarrow (I_a, I_b, I_c)$. This must also map their circumcenters to each other, i.e. $I \xrightarrow{h} 2O - I$, whence $X \in \overline{OI}$.

Also, let M_a be the midpoint of \overline{BC} , $O_a \in \overline{DJ_a}$ be the midpoint of arc BC on ω_a not containing J_a (and variants).

Lemma 2 (SL 2002/G7) – J_a, D, I_a collinear.

Proof. Harmonics:

$$-1 = (DQ; D'J_a) \stackrel{D}{=} (\overline{BC} \cap \overline{AI}, \overline{J_aD} \cap \overline{AI}; I, A),$$

implying that $\overline{J_aD} \cap \overline{AI}$ is the A -excenter. □

Hence, $X = \overline{DJ_a} \cap \overline{EJ_b} \cap \overline{FJ_c}$.

Claim – O_a is the midpoint of $\overline{DI_a}$.

Proof. By symmetry, M_a is the foot of O_a onto \overline{BC} , while it's well-known that $2M - D$ is the foot of I_a onto \overline{BC} . M obviously being the midpoint of the segment with endpoints $D, 2M - D$ implies the claim by parallel lines. □

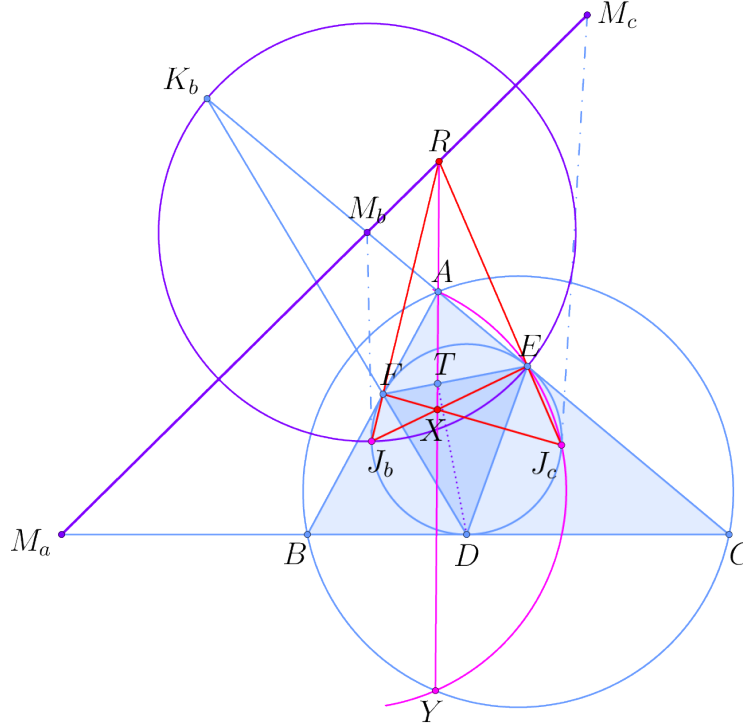
Therefore there must exist a homothety h' at X with scale factor $(1+r)/2$, mapping $(D, E, F) \rightarrow (O_a, O_b, O_c)$.
To show that our X is indeed the radical center of $\omega_a, \omega_b, \omega_c$, compute

$$\text{Pow}(X, \omega_a) = XJ_a \cdot XO_a \stackrel{h'}{=} \frac{1+r}{2} XJ_a \cdot XD = \frac{\text{Pow}(X, \omega)}{(r+1)/2},$$

a symmetric quantity wrt a, b, c .

1.14.2 Fake USAMO 2020/3 (author?)

Let $\triangle ABC$ be a scalene triangle with circumcenter O , incenter I , and incircle ω . Let ω touch the sides \overline{BC} , \overline{CA} , and \overline{AB} at points D , E , and F respectively. Let T be the projection of D to \overline{EF} . The line AT intersects the circumcircle of $\triangle ABC$ again at point $X \neq A$. The circumcircles of $\triangle AEX$ and $\triangle AFX$ intersect ω again at points $P \neq E$ and $Q \neq F$ respectively. Prove that the lines EQ , FP , and OI are concurrent.



Retain the point definitions from the previous two problems, renaming the X given in the problem to Y . For consistency of notation we let X denote the simlicenter of triangles $I_a I_b I_c$, DEF (as before). We'll show that $AEYJ_c$ cyclic – P, Q are just the J_b, J_c from earlier.

Claim 1 – $\overline{EJ_c}, \overline{FJ_b}, \overline{AT}$ are concurrent at some point R on the polar of X wrt ω .

Proof. I claim that \overline{AT} is the polar of $\overline{EF} \cap \overline{J_b J_c}$ wrt ω . Indeed, this is just Brokard. By Brokard again, $\overline{EJ_b} \cap \overline{FJ_c}$ is on \overline{AT} as well as the polar of X wrt ω . \square

Let M_a be midpoint of $\overline{K_a D}$ (and cyclic variants).

Lemma – The polar of X wrt ω is the radical axis of Ω, ω .

Proof. As $\gamma_b \perp \omega$, and $\overline{M_b E}$ touches ω , $\overline{M_b J_b}$ must also touch it; in other words M_b is the pole of $\overline{EJ_b}$ wrt ω . As $X \in \overline{EJ_b}$, $\overline{M_a M_b M_c}$ is the polar of X wrt ω by la Hire.

It remains to prove that M_a (and thus cyclic variants by symmetry) is on the radical axis of Ω, ω . Indeed, by the midpoints of harmonic bundles lemma on $(K_a D; BC)$,

$$\text{Pow}(M, \omega) = M_a D^2 = MB \cdot MC = \text{Pow}(M, \Omega)$$

\square

From the previous two claims,

$$RA \cdot RY = \text{Pow}(R, \Omega) = \text{Pow}(R, \omega) = RE \cdot RJ_c \Rightarrow AEJ_c Y \text{ cyclic},$$

completing the proof.

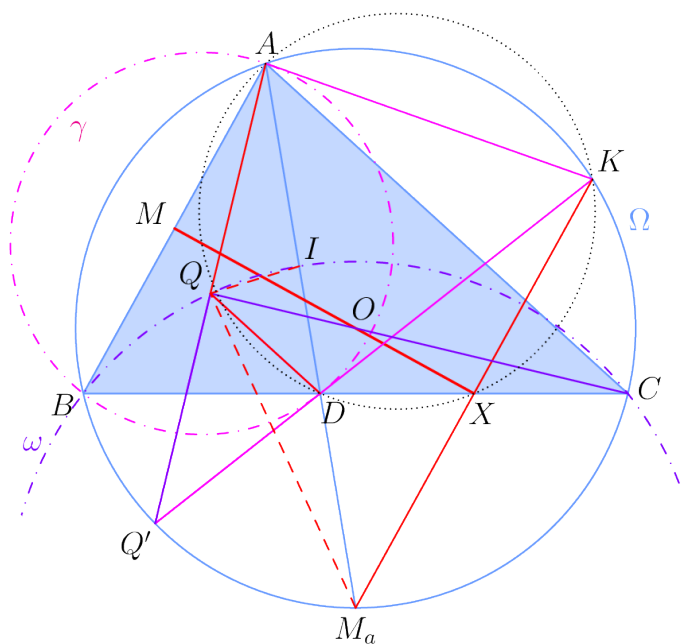
1.15 TSTST 2018/3, by Evan Chen & Yannick Yao

Let ABC be an acute triangle with incenter I , circumcenter O , and circumcircle Γ . Let M be the midpoint of \overline{AB} . Ray AI meets \overline{BC} at D . Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y , while line CO meets ω at C and Q . Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D . Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ .

Geometry is the science of correct reasoning on incorrect figures.

George Pólya / Henri Poincaré



The given angle condition implies $AMQO$ cyclic, or $\angle AQC = \angle AMO = \pi/2$. We make the following definitions:

- $\Omega = (ABC)$, M_a as the center of ω and midpoint of \overline{BC} ;
- $Q' = 2Q - A$ as the reflection of A in \overline{QOC} – this lies on Ω by symmetry about \overline{CO} ;
- $K \in \Omega$ as the reflection of M_a in \overline{MO} , the perpendicular bisector of \overline{AB} .

In the meat of the problem we'll use the following easily findable arcs/angles frequently:

$$\widehat{BM_a} = \widehat{CM_a} = \widehat{AK} = A, \text{ and } \widehat{AC} = \widehat{CQ'} = \widehat{M_aK} \Rightarrow \angle AQ'C = B.$$

Observation

\overline{QI} bisects $\angle AQD$. (Holds because $Q \in \gamma$, the Apollonian circle wrt A, D through I .)

Claim 1 - $\triangle QQ'D \stackrel{+}{\sim} \triangle M_a AC$.

Proof. First, we'll show $\angle QQ'D = \angle B$, a massive angle chase:

$$\begin{aligned} \angle M_a AQ &= \angle CAQ' - \angle CAM_a = B - \frac{A}{2}, \text{ and } \angle M_a IQ = \frac{\pi - \angle IM_a Q}{2} = \frac{\pi}{2} - \angle ICO = B + \frac{C}{2}; \\ \Rightarrow \angle AQI &= \angle M_a IQ - \angle M_a AQ = \frac{\pi - B}{2}. \end{aligned}$$

Applying the observation gives $\angle Q'QD = B$.

To conclude, note that

$$\frac{Q'Q}{QD} = \frac{AQ}{QD} = \frac{AI}{ID} = \frac{AC}{CD} = \frac{AM_a}{M_a C},$$

establishing the similarity by SAS.

(The similarity is negative due to lack of configuration issues, and inspection.) □

Claim 2 - Q', D, K collinear.

Proof. Angle chase again: $\angle AQ'D \stackrel{\text{claim 1}}{=} -\angle M_a AC = \widehat{AC}/2 = \widehat{AK}/2 = \angle AQ'K$. □

Part 1: \overline{KA} and \overline{KD} touch γ

Even more angle chasing, taking advantage of claim 2...

$$\angle KAD = \frac{\widehat{KM_a}}{2} = \frac{\widehat{AC}}{2} = \angle ABD, \text{ while } \angle KDA = \frac{\widehat{KA} + \widehat{QM_a}}{2} = \frac{\widehat{Q'C}}{2} = B = \angle DBA,$$

proving the tangencies.

The other, more elegant part of the problem...

Claim 3 - $\overline{MO}, \overline{BC}, \overline{KM_a}$ (ADK) all concur at a point X .

Proof. Let $X_1 = \overline{MO} \cap \overline{BC}$, $X_2 = \overline{KM_a} \cap \overline{BC}$.

- $X_1 \in (ADK)$ by similarity: observe by (omitted) angle chase that $\triangle AXB \stackrel{+}{\sim} \triangle AKD$, whence $\angle AXD = \angle AKD$;
- $X_2 \in (ADK)$ (by contrast) is by power of a point at M_a :

$$M_a B^2 = M_a C^2 = M_a X_2 \cdot M_a K = M_a A \cdot M_a D.$$

As $X_1 = X_2 = (ADK) \cap \overline{BC}$ ($\neq D$), the claim is proven. □

Because $\overline{M_a K} \parallel \overline{AB} \perp \overline{MO}$, and $X = \overline{MO} \cap \overline{M_a K}$ is the inverse of K wrt ω (by the second equation in previous claim's proof), \overline{MO} is the polar of K wrt ω , completing the problem.

Remark. (crazyeyemoody907) For me, it took forever to discover the proof of claim 1. Here are some useless formulations of its statement I harped upon:

- $(AC; KM_a) = -1$ which directly implies the first part:

$$(B, \overline{BK} \cap \gamma; A, D) \stackrel{B}{=} (M_a K; AC) = -1;$$

Since “ \overline{KA} touches γ ” is very easily provable, K would be polar of \overline{AD} wrt γ as promised...

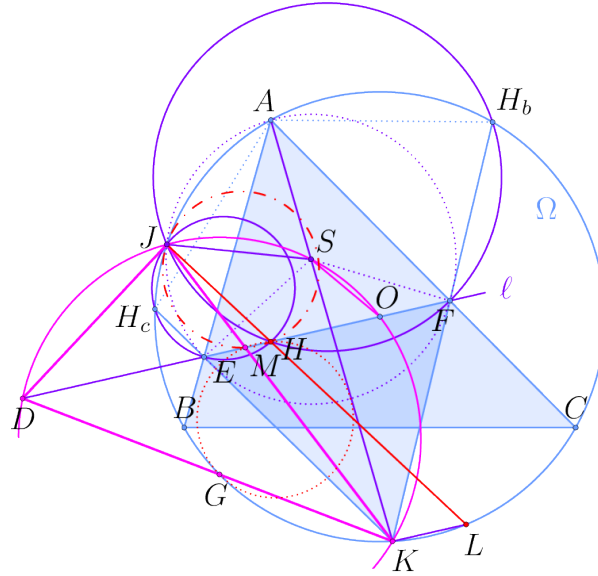
- $BDQQ'$ cyclic ($\iff \overline{QD} \parallel \overline{AC}$ by Reim)

In fact, this means post-solve that $\overline{BQ} \parallel \overline{Q'DK}$...in hindsight, equally useless...

Remark. (Note to self?) When angle chasing in a synthetic problem fails, try length ratios! Examples: SL 2016/G7, GGG 4.5;

1.16 DeuX MO 2020/II/3, by Hao Minyan (China)

In triangle ABC with circumcenter O and orthocenter H , line OH meets $\overline{AB}, \overline{AC}$ at E, F respectively. Let ω be the circumcircle of triangle AEF with center S , meeting (ABC) again at $J \neq A$. Line OH also meets (JSO) again at $D \neq O$. Define $K = (JSO) \cap (ABC) \ (\neq J)$, $M = \overline{JK} \cap \overline{OH}$, and $G = \overline{DK} \cap (ABC) \ (\neq K)$. Prove that (GHM) and (ABC) are tangent to each other.



Solution by [crazyeyemoody907, v4913](#).

Let $\Omega = (ABC)$, H_b, H_c be the respective reflections of H in $\overline{AC}, \overline{AB}$, and $\ell = \overline{EF}$. Redefine $K = \overline{H_cE} \cap \overline{H_bF}$ (we'll see this is an equivalent definition). As $\overline{EA}, \overline{FA}$ are external angle bisectors wrt $\triangle KEF$, we have $\angle EKF = \pi - 2A$.

Claim 1 – $J \in (HEH_c), (HFH_b)$.

Proof. Let $J' = (HEH_c) \cap (HFH_b) \ (\neq H)$. Then:

$$\angle H_c J' H_b = \angle H_c J' H + \angle H J' H_b = \angle H_c E H + \angle H F H_b = \angle (\overline{H_c E}, \overline{H_b F}) = \angle H_b K H_c = \angle H_b A H_c \Rightarrow J' \in \Omega.$$

The construction of J' implies that $\overline{J'E}, \overline{J'F}$ respectively bisect $\angle H_c J' H, \angle H_b J' H$, and thus

$$\angle E J' F = \frac{1}{2} \angle H_b J' H_c = \angle BAC = \angle EAF \Rightarrow J' \in (AEF),$$

finishing the claim. □

Let $L = \overline{JH} \cap \Omega \ (\neq J)$; then, as $JH_c KL, JH_c EH$ cyclic, $\ell \parallel \overline{KL}$ by Reim. By homothety, (JHM) touches $(JKL) = \Omega$.

Claim 2 – For the K defined in solution, $K \in \overline{AS}, (JSO)$.

Proof. Since $\angle ESF = 2\angle BAC = \angle EKF$, we have $KESF$ cyclic; as $SE = SF$, $AH_b = AH_c$, A, S both lie on bisector of $\angle EKF$.

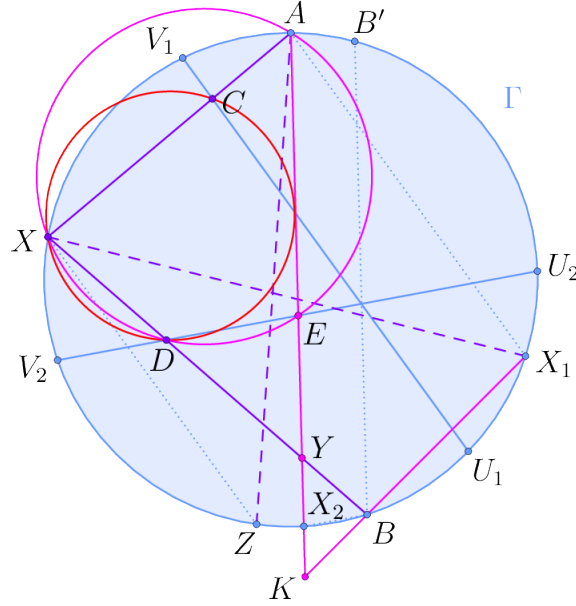
Next, we prove that O is the midpoint of \widehat{JSK} on (JSK) . Because \overline{OS} is the perpendicular bisector of \overline{AJ} by symmetry, it externally bisects $\angle JSK$ as $K \in \overline{AS}$. At the same time, $OJ = OK$ means O is on the perpendicular bisector of \overline{JK} . These two properties imply that O is the claimed arc midpoint. \square

From here, as $DJKO$ cyclic and $OJ = OK$, \overline{DO} bisects $\angle JDK$, and $G = \overline{DK} \cap \Omega$ is the reflection of J in ℓ by symmetry. Reflecting “ (JHM) touches Ω ” over ℓ completes the proof.

1.17 USA TST 2021/2, by Andrew Gu & Frank Han

Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ , in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$. Let X be a variable point on the arc V_1V_2 of Γ not containing A or B . Line XA meets line U_1V_1 at C , while line XB meets line U_2V_2 at D .

Prove there exists a fixed point K , independent of X , such that the power of K to the circumcircle of $\triangle XCD$ is constant.



Clearly, the problem statement should hold for any $X \in \Gamma$; here, all lengths are directed.

Let X_1, X_2 be the respective reflections of A, B in the perpendicular bisectors of $\overline{U_1V_1}, \overline{U_2V_2}$. We assert that $K = \overline{AX_2} \cap \overline{BX_1}$ fits the bill. For brevity, let ' \leftrightarrow ' denote 'is a constant multiple of', so ' $x \leftrightarrow 1$ ' is a shorthand for ' x is constant'.

By Reim, $E = \overline{BX} \cap \overline{AX_2}$ lies on (ADX) , so $\text{Pow}(K, (ADX)) = KE \cdot KA \leftrightarrow 1$. Now, in the spirit of linpop, let $f(P) = \text{Pow}(P, (ADX)) - \text{Pow}(P, (XCD))$, so that because $f(Y) = 0$, we have

$$f(K) = -\frac{KY}{YA}f(A) = AC \cdot AX \frac{KY}{AY}.$$

The rest is a wild length chase; let B', Z be the respective reflections of B, X in the perpendicular bisector of $\overline{U_1V_1}$, so that $XX_1 = AZ$ and $\overline{AZ}, \overline{ACX}$ isogonal wrt $\angle U_1AV_1$. Then, observing that all lengths not involving X, C, D, Y are fixed,

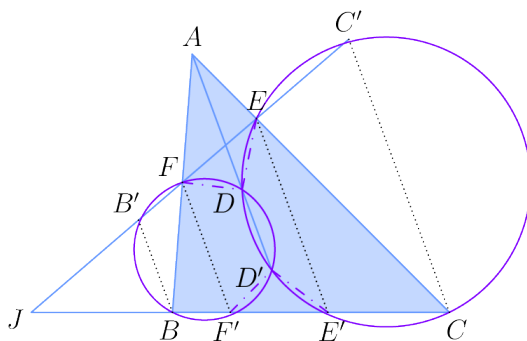
$$\begin{aligned} \frac{KY}{AY} &= (KA; Y \infty_{AK}) \stackrel{B}{=} (X_1A; XB') \leftrightarrow \frac{X_1X}{AX} = \frac{AZ}{AX}; \\ &\Rightarrow f(K) \leftrightarrow AC \cdot AZ = AU_1 \cdot AV_1 \leftrightarrow 1, \end{aligned}$$

where the last equality follows because Z, C swapped by inversion at A with power $AU_1 \cdot AV_1$ composed with reflection in the angle bisector of $\angle U_1AV_1$, so we win.

Remark. How on earth would someone find K ? I considered the degenerate cases when (XCD) is a straight line (which occur when $X = X_1, X_2$, hence their names).

1.18 IMO 2021/3

Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD , respectively. Prove that the lines BC , EF , and O_1O_2 are concurrent.



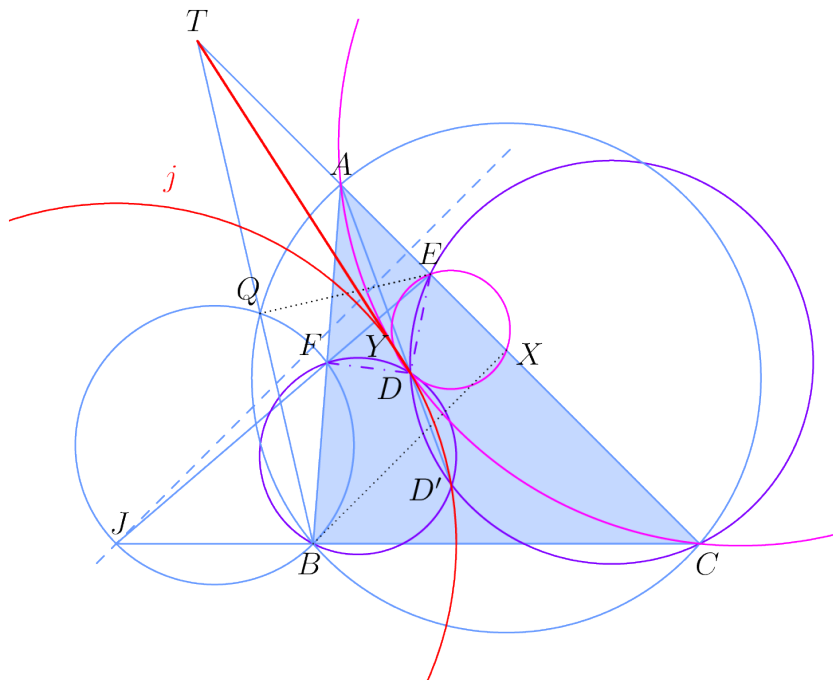
Solution by **v4913**.

Let $J = \overline{EF} \cap \overline{BC}$, and $D' \in \overline{AD}$ be the isogonal conjugate of D wrt $\triangle ABC$. The given angle conditions imply that $BDD'F$, $CDD'E$ are cyclic, while power of a point at A implies $BCEF$ cyclic as well.

Claim 1 – J is the exsimilicenter of (EDC) , (FDB) ; hence, $JD = JD'$ by symmetry.

Proof. Construct $E_1 = (CDD'E) \cap \overline{BC}$ ($\neq C$), $F_1 = (BDD'F) \cap \overline{BC}$ ($\neq B$). By isogonality, $DF = D'F'$ and $DE = D'E'$ whence $DD'E'E$, $DD'F'F$ are both cyclic isosceles trapezoids. $\overline{DD'}$, $\overline{EE'}$, $\overline{FF'}$ share a perpendicular bisector b , and in fact, this is the bisector of $\angle J$, i.e. $JE = JE'$, $JF = JF'$.

Reflect B, C over b to obtain B', C' ; then, because $JB/JF' = JB/JF = JE/JC = JE'/JC$, there is a homothety at J mapping $(B, B', F, F') \rightarrow (E', E, C', C)$ and thus their circumcircles $(BB'DD') \rightarrow (CC'DD')$ as well. \square



Let $Y = (ADC) \cap (EXD)$ ($\neq D$), Q be the Miquel point of $ABCJEF$, and j the circle at J through D, D' . Observing that $\overline{O_1 O_2}$ is the perpendicular bisector of \overline{DY} , it remains to prove $Y \in j$.

Claim 2 – $XQEB$ is cyclic.

Proof. This is a simple angle chase: using cyclic quadrilaterals $(ABCQ)$, $(JFBQ)$, $(ECJQ)$, and $(AEFQ)$, we obtain

$$\angle EQB = \angle EQA + \angle AQB = \angle ACB + \angle EFA = 2\angle ACB = \angle EXB \quad \square$$

Next, we characterize the radical axis of j , (JBF) – it's perpendicular to the line of centers and through A :

Claim 3 – The line through B and the center of (JBF) is perpendicular to \overline{AC} .

Proof. This is equivalent to “ t_b , the tangent to (JBF) at J , is parallel to \overline{AC} ”. Because $\angle(\overline{BJC}, t_b) = \angle BFJ = \angle JCE$, the result follows. \square

Because $\text{Pow}(A, j) = AD \cdot AD' = AQ \cdot AJ = \text{Pow}(A, (JBQF))$, A is on the radical axis of j , (JBF) . By the previous claim, it follows that \overline{AC} is the radical axis of j , (JBF) .

To finish, define $T = \overline{DY} \cap \overline{AC} \cap \overline{BQ}$ as the radical center of (JBF) , (ABC) , (EXD) , (ADC) , and the phantom point $Y' = \overline{TD} \cap j$ ($\neq D$). Because T is on \overline{AC} , the radical axis of j , (JBF) , we have (lengths directed)

$$TY' \cdot TD = \text{Pow}(T, j) = \text{Pow}(T, (JBF)) = \text{Pow}(T, (ABCQ)) = TA \cdot TC = TY \cdot TD \Rightarrow Y = Y',$$

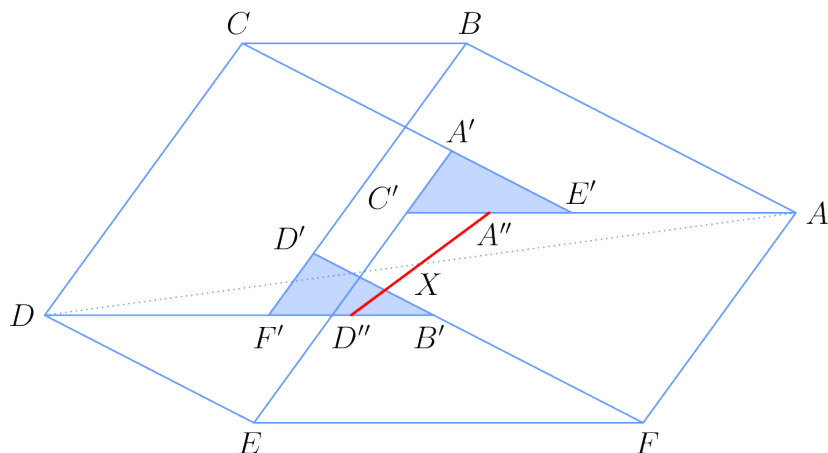
the end!

1.19 USAMO 2021/6, by Ankan Bhattacharya

Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X , Y , and Z be the midpoints of \overline{AD} , \overline{BE} , and \overline{CF} . Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.



Construct parallelogram $CDEA'$ and cyclic variants: $A' = C + E - D$, etc. We may compute using vectors that $\triangle B'D'F'$ is a translation of $\triangle A'C'E'$ by the vector $(B+D+F) - (A+C+E)$. In particular, they're congruent.

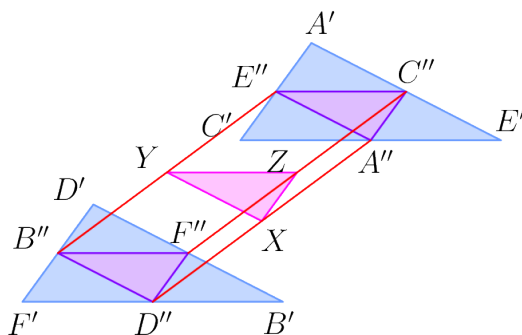
Claim 1 - A, C, E have same power wrt $(A'C'E')$; in other words, $\triangle ACE, A'C'E'$ share a circumcenter.

Proof. Observing that $\text{Pow}(A, (A'C'E')) = AC' \cdot AE' = BC \cdot EF$ by parallelograms, this claim follows by the given length condition. \square

Next, construct $A'' = \frac{C'+E'}{2}$ and cyclic variants. The circumcenter of $\triangle A'C'E'$ is then the orthocenter of $\triangle A''C''E''$.

Claim 2 - $X = \frac{A''+D''}{2}$.

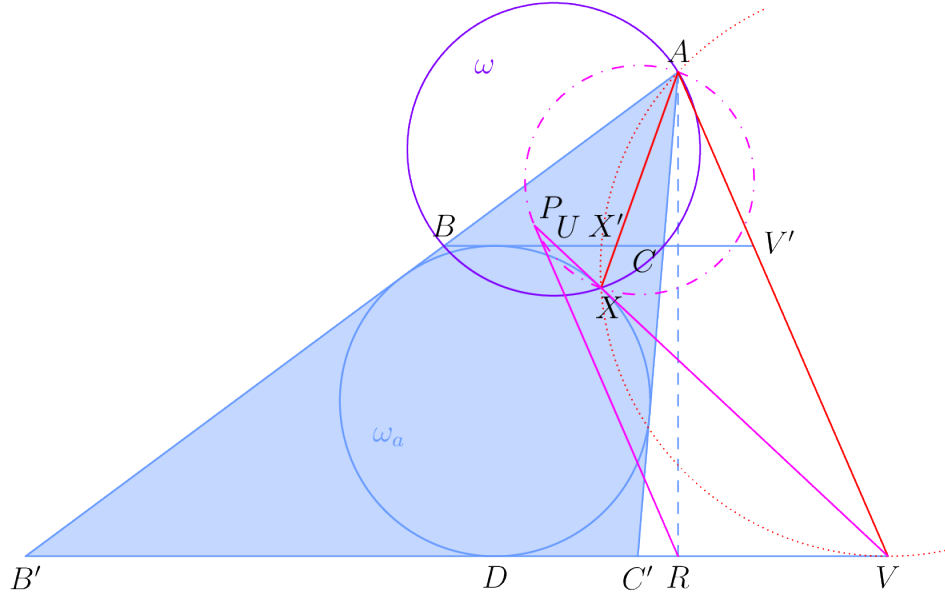
Proof. Using vectors, $B' + C' = E' + F' = A + D \Rightarrow \frac{A+D}{2} = \frac{B'+C'+E'+F'}{4} = \frac{A''+D''}{2}$. \square



By claim 2 + symmetry, $\triangle XYZ$ is the vector average of (congruent) triangles $A''C''E''$, $B''D''F''$, so their orthocenters are collinear.

1.20 SL 2021/G8

Let ABC be a triangle with circumcircle ω and let Ω_A be the A -excircle. Let X and Y be the intersection points of ω and Ω_A . Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R . Prove that $\overline{AR} \perp \overline{BC}$.



Solution by [crazyeyemoody907](#).

Let the antipode of the A -extouch point be D , and the tangent to ω_a at D intersect $\overline{AB}, \overline{AC}$ at B', C' respectively. Also, construct the tangent line to ω_a at X , meeting $\overline{BC}, \overline{B'C'}$ at U, V respectively. Finally, let $X' = \overline{AX} \cap \overline{BC}$, $V' = \overline{AV} \cap \overline{BC}$.

Claim 1 - $AXUV'$ cyclic.

Proof. Apply DDIT to $A, UXV \infty_{BC}$ (with inconic ω_a), and project onto \overline{BC} , to obtain an involutive pairing $(B, C), (U, V'), (\infty_{BC}, X')$ – or equivalently, $X'B \cdot X'C = X'U \cdot X'V'$. By power of a point, $X'B \cdot X'C = X'A \cdot X'X$, so the claim follows from power of a point converse on $X'U \cdot X'V' = X'A \cdot X'X$. \square

Claim 2 - \overline{DV} is tangent to (AXV) .

Proof. Angle chase using previous claim, and the fact that $\overline{BC} \parallel \overline{B'C'}$:

$$\angle XAV \stackrel{\text{claim 1}}{=} \angle XUV' = \angle XVD.$$

\square

Redefine R as the foot from A to $\overline{B'C'}$. It remains to show,

Claim 3 - \overline{PR} touches (APX') .

Proof. Since $\angle VPA = \angle VRA = 90^\circ$, $APRV$ cyclic, so we may anglechase as follows:

$$\angle APR = \angle AVR \stackrel{\text{claim 2}}{=} \angle AXV = \angle AXP.$$

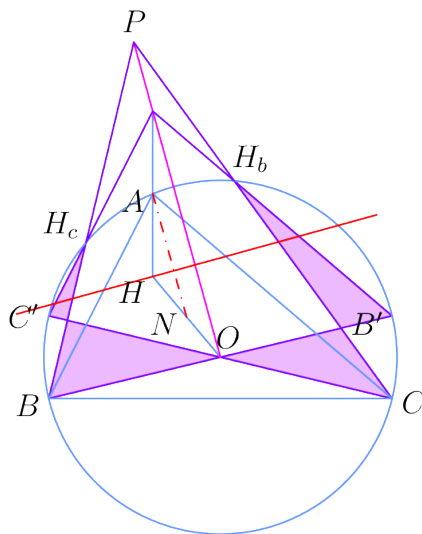
\square

1.21 USEMO 2020/3, by Anant Mudgal

Let ABC be an acute triangle with circumcenter O and orthocenter H . Let Γ denote the circumcircle of triangle ABC , and N the midpoint of \overline{OH} . The tangents to Γ at B and C , and the line through H perpendicular to line AN , determine a triangle whose circumcircle we denote by ω_A . Define ω_B and ω_C similarly.

Prove that the common chords of ω_A , ω_B , and ω_C are concurrent on line OH .

Let H_a, A' denote the respective reflections of H in \overline{BC} , A in O , and their symmetric variants.



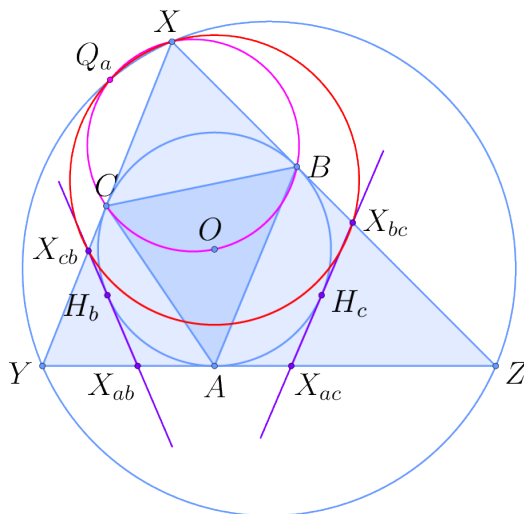
Claim 1 – The polar ℓ_a of $\overline{BH_c} \cap \overline{CH_b}$ passes through H and is perpendicular to \overline{AN} .

Proof. Let $P = \overline{BH_c} \cap \overline{CH_b}$ and $S = 2A - H$. $H \in \ell_a$ is just Brokard, so it suffices to prove $\overline{AN} \parallel \overline{OP}$. By Pascal on $BB'H_bCC'H_c$, we have P, O, S collinear. Taking a homothety at H with scale factor $\frac{1}{2}$ maps the latter two points to N, A , which implies the required parallel lines. \square

In $\triangle ABC$, let X_{bc} be the pole of $\overline{BH_c}$ wrt Γ (and 5 other variants), X, Y, Z be the poles of the sides, D, E, F be the feet of the altitudes. Clearly, $\ell_a = \overline{X_{bc}X_{cb}}$.

Note. Here, the condition $\triangle ABC$ acute comes in: Γ is the incircle, not excircle, of $\triangle XYZ$.

We'll show that \overline{XD} is the radical axis of ω_b, ω_c . (By a somewhat-known configuration (say, **Brazil 2013/6**), $\overline{XD} \cap \overline{YE} \cap \overline{ZF}$ lies on the Euler line.) Also let Q_a, Q_b, Q_c be the SD points of $\triangle XYZ$.



Claim 2 – Q_a lies on ω_a .

Proof. By spiral similarity, it suffices to prove $YX_{bc}/YC = ZX_{cb}/ZB$. By antiparallel lines, $\triangle XYZ \sim \triangle X_{ab}YX_{cb}, X_{ac}X_{bc}Z$. But since Γ is the Y -excircle of $\triangle X_{ab}YX_{cb}$, we have $YX_{cb}/YC = a/s$. Similarly $ZX_{bc}/ZB = a/s$ as well.

(In some awful notation, $a = YZ, b = ZX, c = XY$ and $s = \frac{a+b+c}{2}$.) □

Let $L = \overline{YQ_b} \cap \overline{ZQ_c}$.

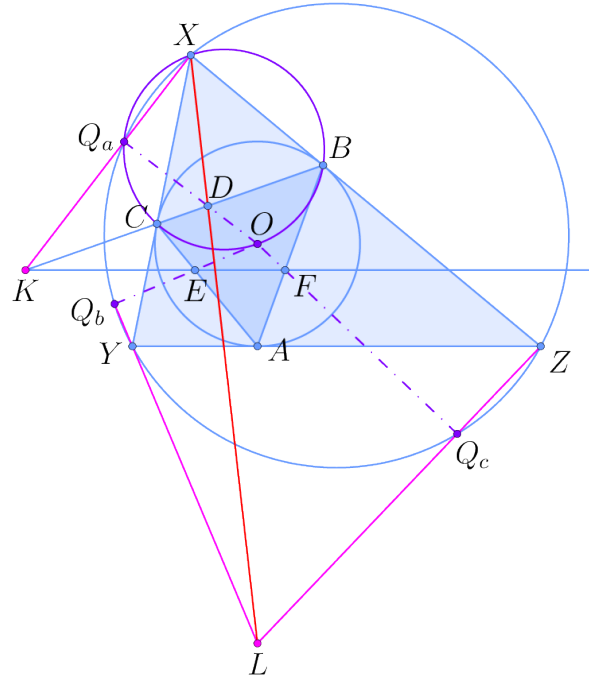
Claim 3 – \overline{XL} is the radical axis of ω_b, ω_c .

Proof. By antiparallel lines again, $YZX_{ba}X_{ca}$ cyclic, so that

$$\text{Pow}(X, \omega_b) = XX_{ca} \cdot XY = XX_{ba} \cdot XZ = \text{Pow}(X, \omega_c), \text{ while}$$

$$\text{Pow}(L, \omega_b) = LY \cdot LQ_b = LZ \cdot LQ_c = \text{Pow}(L, \omega_c). \quad \square$$

It remains to prove X, D, L collinear.



Claim 4 – L is the pole of \overline{EF} wrt Γ .

Proof. Since Q_a is the inverse of D wrt Γ and $\angle OQ_aX = 90^\circ$, $\overline{XQ_a}$ is the polar of D wrt Γ . Similarly, $\overline{YQ_b}$, $\overline{ZQ_c}$ are the respective polars of E, F wrt Γ . The claim is then established by la Hire. \square

Claim 5 – \overline{BC} , \overline{EF} , $\overline{XQ_a}$ concurrent.

Proof. Let $K = \overline{EF} \cap \overline{BC}$ so that $(KD; BC) = -1$. Because $\overline{Q_aO}$ bisects $\angle BQ_aC$, $\angle KQ_aO = 90^\circ = \angle AQ_aO \Rightarrow X, Q_a, K$ collinear. \square

Taking poles wrt Γ in the last claim gives the desired collinearity.

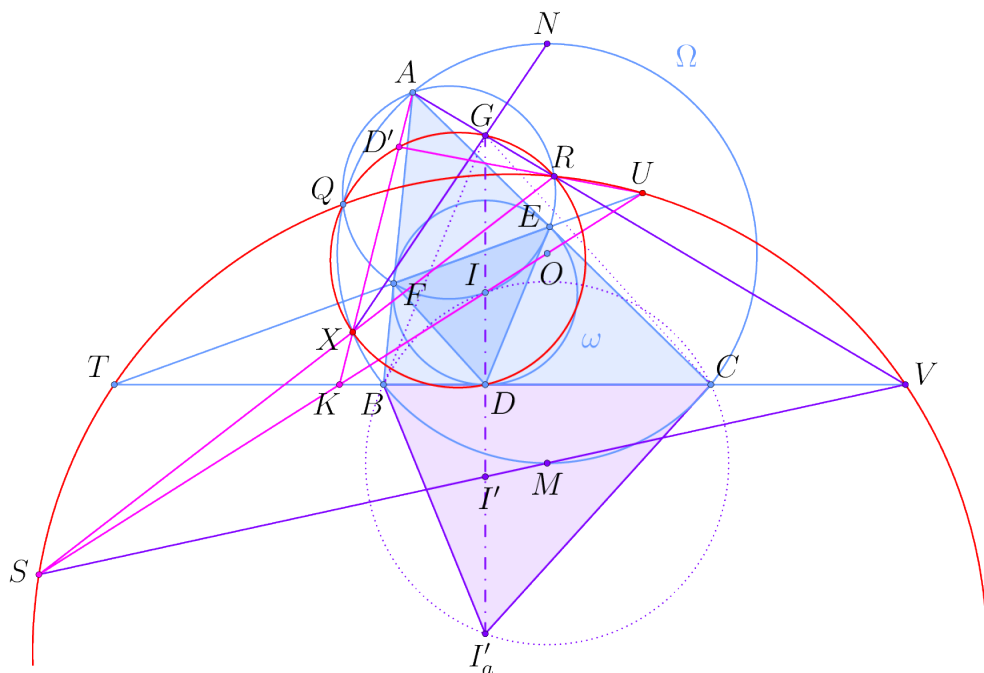
Remark. The problem can be bary'd wrt $\triangle XYZ$ after the first claim, but it's monstrous from my experience a long time ago, oops

1.22 Brazil Olympic Revenge 2021/3, by Joao P.R. Viana Costa

Let I, C, ω and Ω be the incenter, circumcenter, incircle and circumcircle, respectively, of the scalene triangle XYZ with $XZ > YZ > XY$. The incircle ω is tangent to the sides YZ, XZ and XY at the points D, E and F . Let S be the point on Ω such that XS, CI and YZ are concurrent. Let $(XEF) \cap \Omega = R, (RSD) \cap (XEF) = U, SU \cap CI = N, EF \cap YZ = A, EF \cap CI = T$ and $XU \cap YZ = O$.

Prove that $NARUTO$ is cyclic.

Colloquially known as “Naruto”.



Solution by [crazyeyemoody907](#), [CyclicSLscalesTrapezoid](#) with [Eyed](#), v4913.

Warning. This problem is not meant for neither the faint-hearted nor freehand geometers like the paper's author(s). If Geogebra's to be used any time, it'd be now.

We do a massive refactoring and simplification; consider the following equivalent problem, a breakdown of the given, despite being longer:

Naruto simplified

In triangle ABC with circumcircle Ω centered at O , the incircle ω centered at I touches the sides at D, E, F . Let I', I'_a be the respective reflections of I and the orthocenter of $\triangle BIC$ in \overline{BC} , and M the midpoint of arc BC on Ω . Further define:

- S as the intersection of the Euler lines \overline{OI} of $\triangle DEF$, $\overline{MI'}$ of $\triangle I'_aBC$;
- $T = \overline{EF} \cap \overline{BC}$, $U = \overline{EF} \cap \overline{OI}$, $V = \overline{MI'} \cap \overline{BC}$, $R = \overline{AV} \cap (AI)$;
- $K = \overline{OI} \cap \overline{BC}$;

Prove that (a) Q, R, S, T, U, V are concyclic, and (b) $\overline{AK}, \Omega, (QRD), \overline{RS}$ concurrent;

(a) The concyclicity Let the spiral similarity s at Q with (directed) angle θ map $E, F \rightarrow C, B$ and thus D, I and the orthocenter of $\triangle DEF$ to I', M, I'_a respectively. Clearly, S is the intersection of the Euler lines of two triangles related by s : $DEF, I'_a CB$.

By design, we have $U \xrightarrow{s} V$, so

$$\angle VQU = \theta = \angle(\overline{BC}, \overline{EF}) \stackrel{s}{=} \angle(\overline{MI'}, \overline{OI}) = \angle VSU,$$

whence Q, S, T, U, V concyclic. To see that the last point is also concyclic with the other five, let N be the midpoint of \widehat{BAC} , so that \overline{NA} touches (AI) . Indeed, then

$$\angle QRV = \angle QRA = \angle QAN \stackrel{s}{=} \angle QUV$$

as needed.

Remark. In fact, by design, S is the exsimilicenter of the incircle and the circle at O with radius half that of Ω , so it's actually the inverse of I wrt Ω .

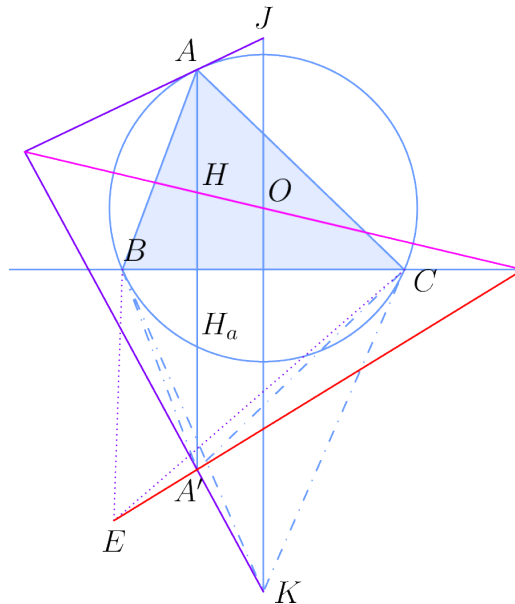
(b) The concurrence Let D' be the reflection of D in \overline{EF} , and G the orthocenter of $\triangle BIC$, so that $D' \xrightarrow{s} G$. We easily have $DD'GQ$ cyclic. As $\angle(\overline{AD'}, \overline{NG}) = \theta$, the point $X = \overline{AD'} \cap \overline{NG}$ lies on both $(DD'GQ), \Omega$. We require the following result(s):

Theorem: weird concurrences

In a scalene triangle ABC with circumcenter O , circumcircle Ω , and orthocenter H .

- (a) let K be the polar of \overline{BC} wrt Ω , and A' be the reflection of A in \overline{BC} . Then $\overline{OH}, \overline{A'K}$ and the tangent to Ω at A are concurrent.
- (b) Let E be the reflection of the point E_0 (such that A is the incenter or excenter of $\triangle E_0BC$) in the perpendicular bisector of \overline{BC} . Then $\overline{OH}, \overline{BC}, \overline{EA'}$ are also concurrent.

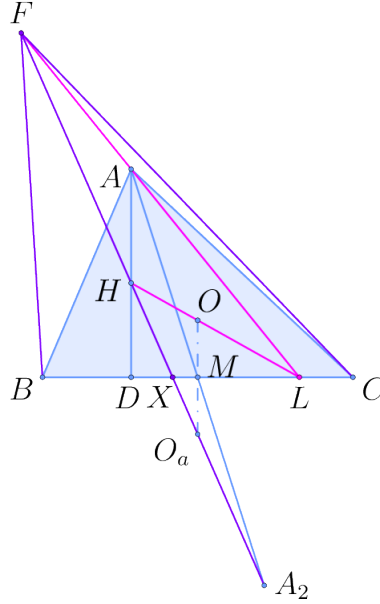
(parentheses used above for easier grammatical parsing)



Proof. These two parts actually aren't connected at all...

Part (a), by CyclicalScelesTrapezoid Let J be the intersection of the tangent to Ω at A with the perpendicular bisector of \overline{BC} , and $H_a \in \Omega$ be the reflection of H in \overline{BC} . We contend that the triples (A, H, A') , (J, O, K) are homothetic. Indeed, they lie on parallel lines. To finish, check that (if R denotes the radius of Ω)

$$JO = \frac{R}{\cos(B-C)}, OK = \frac{R}{\cos A}, AH = 2R \cos A, HA' = AH_a = 2R \cos(B-C) \Rightarrow \frac{JO}{OK} = \frac{AH}{HA'}.$$



Part (b), by crazyeyemoody907 Let $F = B + C - E_0$, and $A_2 = B + C - A$, so that A_2 is an incenter or excenter of $\triangle FBC$. Since H is the antipode of A_2 on (BA_2C) , it is another incenter / excenter. To prove that A, L, F collinear where $X = \overline{FHA_2} \cap \overline{BC}$, $L = \overline{OH} \cap \overline{BC}$, verify that (where $O_a \in \overline{HA_2}$ is the reflection of O in \overline{BC})

$$(\overline{AF} \cap \overline{BC}, X; D, M) \stackrel{A}{=} (FX; HA_2) = -1 \text{ while } (DM; XL) \stackrel{H}{=} (\infty_{\perp BC} M; O_a O) = -1. \quad \square$$

Returning to the problem, applying respective parts of the theorem to $\triangle DEF, I'_a BC$, we obtain (A, D', K) and (A, G, V) collinear. Since $R \in (UVQ), \overline{GV}$, and Q is the Miquel point of $D'GVU$, we must have $R = \overline{D'U} \cap \overline{GV}$ – an intersection of opposite sides. Hence, by definition of Miquel point, $R \in (QD'G)$.

It remains to prove that R, X, S collinear. In fact, there is a spiral similarity at Q mapping $D', X \rightarrow U, S$ since $Q \in (URS), (D'XR)$, so we're done!