

# Select geometry favorites

People

November 18, 2022

This is a collection of problems we've done and found particularly cute/beautiful/hard, inspired by chapter 11 of *Euclidean Geometry in Math Olympiads* and Eric Shen's paper *Geometry At Its Best*. Have fun!

(Note: Here,  $\infty_{XY}$  denotes the point at infinity along line  $XY$ .)

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## 🌲 O Problems

**Remark.** Some attempt has been made to deviate from the aforementioned two famous geometry papers.

:)))

**Problem 1 (SL 1998/G4).** Let  $M$  and  $N$  be two points inside triangle  $ABC$  such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

**Problem 2 (SL 2009/G3).** Let  $ABC$  be a triangle. The incircle of  $ABC$  touches the sides  $AB$  and  $AC$  at the points  $Z$  and  $Y$ , respectively. Let  $G$  be the point where the lines  $BY$  and  $CZ$  meet, and let  $R$  and  $S$  be points such that the two quadrilaterals  $BCYR$  and  $BCSZ$  are parallelogram.

Prove that  $GR = GS$ .

**Problem 3 (SL 2015/G4).** Let  $ABC$  be an acute triangle and let  $M$  be the midpoint of  $AC$ . A circle  $\omega$  passing through  $B$  and  $M$  meets the sides  $AB$  and  $BC$  at points  $P$  and  $Q$  respectively. Let  $T$  be the point such that  $BPTQ$  is a parallelogram. Suppose that  $T$  lies on the circumcircle of  $ABC$ . Determine all possible values of  $\frac{BT}{BM}$ .

**Problem 4 (SL 2016/G7).** Let  $I$  be the incentre of a non-equilateral triangle  $ABC$ ,  $I_A$  be the  $A$ -excentre,  $I'_A$  be the reflection of  $I_A$  in  $BC$ , and  $l_A$  be the reflection of line  $AI'_A$  in  $AI$ . Define points  $I_B, I'_B$  and line  $l_B$  analogously. Let  $P$  be the intersection point of  $l_A$  and  $l_B$ .

- (a) Prove that  $P$  lies on line  $OI$  where  $O$  is the circumcentre of triangle  $ABC$ .
- (b) Let one of the tangents from  $P$  to the incircle of triangle  $ABC$  meet the circumcircle at points  $X$  and  $Y$ . Show that  $\angle XIY = 120^\circ$ .

**Problem 5 (EGMO 2015/6).** Let  $H$  be the orthocentre and  $G$  be the centroid of acute-angled triangle  $ABC$  with  $AB \neq AC$ . The line  $AG$  intersects the circumcircle of  $ABC$  at  $A$  and  $P$ . Let  $P'$  be the reflection of  $P$  in the line  $BC$ . Prove that  $\angle CAB = 60$  if and only if  $HG = GP'$ .

**Problem 6 (Iran TST 2018/1/4).** Let  $ABC$  be a triangle ( $\angle A \neq 90^\circ$ ), with altitudes  $\overline{BE}, \overline{CF}$ . The bisector of  $\angle A$  intersects  $\overline{EF}, \overline{BC}$  at  $M, N$ . Let  $P$  be a point such that  $\overline{MP} \perp \overline{EF}$  and  $\overline{NP} \perp \overline{BC}$ . Prove that  $\overline{AP}$  bisects  $\overline{BC}$ .

**Problem 7 (Eric Shen).** In  $\triangle ABC$ , let  $D, E, F$  be the feet of the altitudes from  $A, B, C$  respectively, and let  $O$  be the circumcenter. Points  $X, Y$  are the projections of  $E, F$  onto  $\overline{AD}$  respectively. Let  $Z = \overline{AO} \cap \overline{EF}$ . There exists a point  $T$  such that  $\angle DTZ = 90^\circ$  and  $AZ = AT$ . If  $P = \overline{AD} \cap \overline{TZ}$ , and  $Q$  lies on  $\overline{EF}$  such that  $\overline{PQ} \parallel \overline{BC}$ , prove that  $\overline{AQ}$  is a median.

**Problem 8 (China TST 2015/2/3).** Let  $\triangle ABC$  be an acute triangle with circumcenter  $O$  and centroid  $G$ . Let  $D$  be the midpoint of  $\overline{BC}$ , and  $E$  be on  $(BC)$  with  $\overline{AE} \perp \overline{BC}$ . Let  $F = \overline{EG} \cap \overline{AD}$ . Define points  $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$  with  $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$ , and  $\overline{MK}, \overline{NL} \perp \overline{BC}$ .

Let  $\omega$  be the circle tangent to  $\overline{OB}, \overline{OC}$  at  $B, C$  respectively. Prove that  $(AMN)$  is tangent to  $\omega$ .

**Problem 9 (IMO 2019/6).** Let  $I$  be the incenter of acute triangle  $ABC$  with  $AB \neq AC$ . The incircle  $\omega$  of  $ABC$

is tangent to sides  $BC$ ,  $CA$ , and  $AB$  at  $D$ ,  $E$ , and  $F$ , respectively. The line through  $D$  perpendicular to  $EF$  meets  $\omega$  at  $R$ . Line  $AR$  meets  $\omega$  again at  $P$ . The circumcircles of triangle  $PCE$  and  $PBF$  meet again at  $Q$ .

Prove that lines  $DI$  and  $PQ$  meet on the line through  $A$  perpendicular to  $AI$ .

**Problem 10 (MOP 2019 & USA TST 2019/6).** Let  $ABC$  be a triangle with incenter  $I$ , and let  $D$  be a point on line  $BC$  satisfying  $\angle AID = 90^\circ$ . Let the excircle of triangle  $ABC$  opposite the vertex  $A$  be tangent to  $\overline{BC}$  at  $A_1$ . Define points  $B_1$  on  $\overline{CA}$  and  $C_1$  on  $\overline{AB}$  analogously, using the excircles opposite  $B$  and  $C$ , respectively.

- (a) (MOP 2019) Let  $E, F$  be the feet of the altitudes from  $B, C$  respectively. Prove that if  $\overline{EF}$  touches the incircle, then quadrilateral  $AB_1A_1C_1$  is cyclic.
- (b) (USA TST 2019/6) Prove that if quadrilateral  $AB_1A_1C_1$  is cyclic, then  $\overline{AD}$  is tangent to the circumcircle of  $\triangle DB_1C_1$ .

**Problem 11 (TSTST 2018/3).** Let  $ABC$  be an acute triangle with incenter  $I$ , circumcenter  $O$ , and circumcircle  $\Gamma$ . Let  $M$  be the midpoint of  $\overline{AB}$ . Ray  $AI$  meets  $\overline{BC}$  at  $D$ . Denote by  $\omega$  and  $\gamma$  the circumcircles of  $\triangle BIC$  and  $\triangle BAD$ , respectively. Line  $MO$  meets  $\omega$  at  $X$  and  $Y$ , while line  $CO$  meets  $\omega$  at  $C$  and  $Q$ . Assume that  $Q$  lies inside  $\triangle ABC$  and  $\angle AQM = \angle ACB$ .

Consider the tangents to  $\omega$  at  $X$  and  $Y$  and the tangents to  $\gamma$  at  $A$  and  $D$ . Given that  $\angle BAC \neq 60^\circ$ , prove that these four lines are concurrent on  $\Gamma$ .

**Problem 12 (RMM 2012/6 & Brazil 2013/6).** In triangle  $ABC$  with incenter  $I$  and circumcenter  $O$ , let the incircle  $\omega$  touch  $\overline{BC}$ ,  $\overline{CA}$ ,  $\overline{AB}$  at  $D, E, F$  respectively.

- (RMM 2012/6) Let  $\omega_a$  be the circle through  $B$  and  $C$  tangent to  $\omega$ , and define  $\omega_b, \omega_c$  similarly. Finally, let  $A' = \omega_b \cap \omega_c$  ( $\neq A$ ), and similarly for points  $B'$  and  $C'$ .
- (Brazil 2013/6) Let  $P$  be the Gergonne point of  $\triangle ABC$ , and its reflections in  $\overline{EF}, \overline{FD}$  and  $\overline{DE}$  be  $P_a, P_b, P_c$ , respectively.

Prove that  $P_a \in \overline{AA'}$ , and that  $\overline{AP_aA'}, \overline{BP_bB'}, \overline{CP_cC'}, \overline{IO}$  are concurrent.

**Problem 13 (IMO 2021/3).** Let  $D$  be an interior point of the acute triangle  $ABC$  with  $AB > AC$  so that  $\angle DAB = \angle CAD$ . The point  $E$  on the segment  $AC$  satisfies  $\angle ADE = \angle BCD$ , the point  $F$  on the segment  $AB$  satisfies  $\angle FDA = \angle DBC$ , and the point  $X$  on the line  $AC$  satisfies  $CX = BX$ . Let  $O_1$  and  $O_2$  be the circumcenters of the triangles  $ADC$  and  $EXD$ , respectively. Prove that the lines  $BC, EF$ , and  $O_1O_2$  are concurrent.

**Problem 14 (USAMO 2021/6).** Let  $ABCDEF$  be a convex hexagon satisfying  $\overline{AB} \parallel \overline{DE}, \overline{BC} \parallel \overline{EF}, \overline{CD} \parallel \overline{FA}$ , and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let  $X, Y$ , and  $Z$  be the midpoints of  $\overline{AD}, \overline{BE}$ , and  $\overline{CF}$ . Prove that the circumcenter of  $\triangle ACE$ , the circumcenter of  $\triangle BDF$ , and the orthocenter of  $\triangle XYZ$  are collinear.

**Problem 15 (SL 2021/G8).** Let  $ABC$  be a triangle with circumcircle  $\omega$  and let  $\Omega_A$  be the  $A$ -excircle. Let  $X$  and  $Y$  be the intersection points of  $\omega$  and  $\Omega_A$ . Let  $P$  and  $Q$  be the projections of  $A$  onto the tangent lines to  $\Omega_A$  at  $X$  and  $Y$  respectively. The tangent line at  $P$  to the circumcircle of the triangle  $APX$  intersects the tangent line at  $Q$  to the circumcircle of the triangle  $AQY$  at a point  $R$ . Prove that  $\overline{AR} \perp \overline{BC}$ .

## 1 Solutions

### 1.1 SL 1998/G4

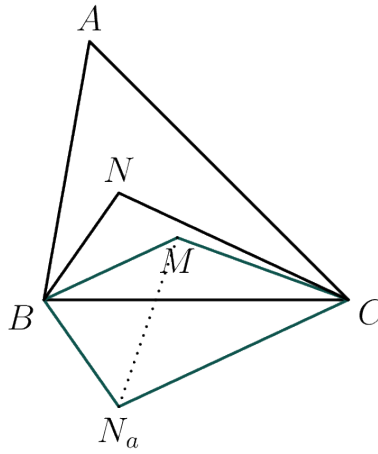
Let  $M$  and  $N$  be two points inside triangle  $ABC$  such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

**Solution 1, by inversion-overlay** Let  $i_a$  denote the inversion at  $A$  with power  $AB \cdot AC$  composed with reflection in the bisector of  $\angle A$ .



**Solution 2, by area ratios (official / intended)**

**Claim –** For any  $M, N$ , we have

$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMC] + [BNC]}{[ABC]}.$$

*Proof.* Reflect  $N$  over  $\overline{BC}$  to obtain point  $N_a$ . Then, because  $\angle MBN_a = \angle B$ ,  $(BM \cdot BN)/(BA \cdot BC) = [MBN_a]/[ABC]$ . Similarly  $(CM \cdot CN)/(CA \cdot CB) = [MCN_a]/[ABC]$ , and summing the previous two equations gives

$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMN_aC]}{[ABC]} = \frac{[BMC] + [BNC]}{[ABC]}.$$

□

Noting that  $M, N$  are just isogonal conjugates, we obtain the problem by cyclic summation.

## 1.2 SL 2015/G4

Let  $ABC$  be an acute triangle and let  $M$  be the midpoint of  $AC$ . A circle  $\omega$  passing through  $B$  and  $M$  meets the sides  $AB$  and  $BC$  at points  $P$  and  $Q$  respectively. Let  $T$  be the point such that  $BPTQ$  is a parallelogram. Suppose that  $T$  lies on the circumcircle of  $ABC$ . Determine all possible values of  $\frac{BT}{BM}$ .

Solution by **CyclicSLscalesTrapezoid**.

The answer is  $\sqrt{2}$  only. Let the  $X \neq B$  be defined as  $(ABC) \cap (BPMQ)$ , and let  $N$  be the midpoint of  $\overline{BT}$ .

**Claim 1** –  $XNMT$  is cyclic.

*Proof.* Since  $N$  is also the midpoint of  $\overline{PQ}$ , there is a spiral similarity at  $X$  sending  $PNQ$  to  $AMC$ . Thus, we have

$$\angle XMN = \angle XAP = \angle XTB,$$

so  $XNMT$  is cyclic. □

**Claim 2** –  $\overline{BM}$  is tangent to the circumcircle of  $XNMT$ .

*Proof.* We have

$$\angle XNM = \angle XPA = \angle XPB = \angle XMB.$$

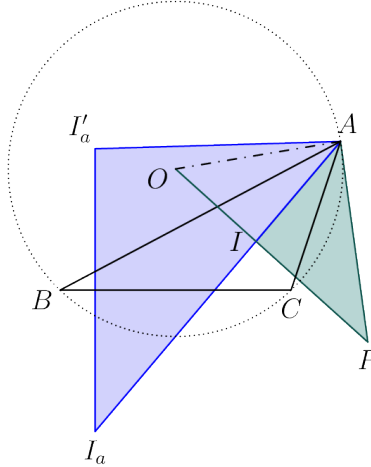
□

By Power of a Point,  $BM^2 = BN \cdot BT = \frac{BT^2}{2}$ , so  $\frac{BT}{BM} = \sqrt{2}$ .

### 1.3 SL 2016/G7

Let  $I$  be the incentre of a non-equilateral triangle  $ABC$ ,  $I_A$  be the  $A$ -excentre,  $I'_A$  be the reflection of  $I_A$  in  $BC$ , and  $l_A$  be the reflection of line  $AI'_A$  in  $AI$ . Define points  $I_B, I'_B$  and line  $l_B$  analogously. Let  $P$  be the intersection point of  $l_A$  and  $l_B$ .

- Prove that  $P$  lies on line  $OI$  where  $O$  is the circumcentre of triangle  $ABC$ .
- Let one of the tangents from  $P$  to the incircle of triangle  $ABC$  meet the circumcircle at points  $X$  and  $Y$ . Show that  $\angle XIY = 120^\circ$ .



Redefine  $P$  as the inverse of  $I$ . For the first part we assert more strongly that:

**Claim –**  $\triangle AI_A I'_A \stackrel{+}{\sim} \triangle API$ .

*Proof.* One of the few uses of SAS similarity? By angle chasing,  $\angle I_A = \angle P$  follows easily. To finish, we show  $I_A I'_A / I_A A = IP / AP$ ; indeed, the first ratio equals  $2 \cos \angle BI_A C = 2 \sin \frac{A}{2}$  because of similar triangles; thus, we're left to length chase  $IP / AP$ ; this becomes

$$\begin{aligned} \frac{OP}{AP} - \frac{OI}{OA} \frac{OA}{AP} &= \frac{OI}{AI} - \frac{OI^2}{OA \cdot AI} = \frac{R}{AI} - \frac{R^2 - 2rR}{R \cdot AI} \\ &= \frac{R - (R - 2r)}{AI} = 2 \sin \frac{A}{2}, \end{aligned}$$

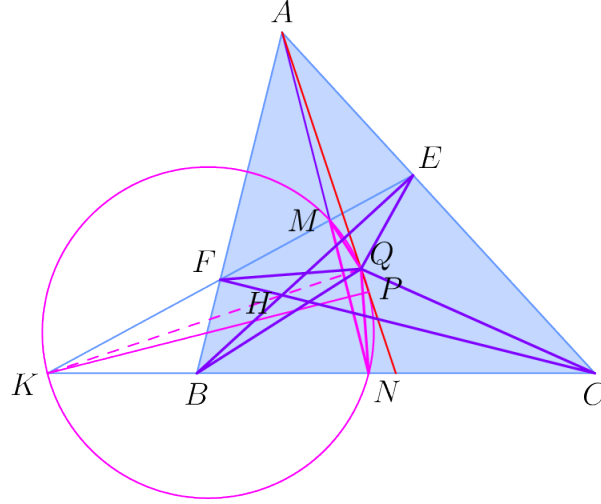
so the ratios are equal, as needed.  $\square$

The claim clearly implies the isogonality.

For the second part, using Poncelet, let  $Z \in (ABC)$  be the unique point so that  $\triangle XYZ, ABC$  share a incircle and circumcircle. By power of a point at  $P$ , and the fact that  $P$  is the inverse of  $I$  wrt  $(ABC)$ ,  $PX \cdot PY = PO \cdot PI$  so  $XYOI$  cyclic. As  $\angle XOY = 2\angle Z$  and  $\angle XIY = (\pi + \angle Z)/2$ ,  $Z = \pi/3 \Rightarrow \angle XOY = \angle XIY = 2\pi/3$ .

### 1.4 Iran TST 2018/1/4, by Iman Maghsoudi & Hooman Fattahi

Let  $ABC$  be a triangle ( $\angle A \neq 90^\circ$ ), with altitudes  $\overline{BE}$ ,  $\overline{CF}$ . The bisector of  $\angle A$  intersects  $\overline{EF}$ ,  $\overline{BC}$  at  $M$ ,  $N$ . Let  $P$  be a point such that  $\overline{MP} \perp \overline{EF}$  and  $\overline{NP} \perp \overline{BC}$ . Prove that  $\overline{AP}$  bisects  $\overline{BC}$ .



Construct  $K = \overline{EF} \cap \overline{BC}$ ,  $Q$  as the  $A$ -Humpty point,  $H$  as the orthocenter of  $\triangle ABC$ , and  $\omega = (KMN)$ , so that the  $P$  given is the antipode of  $K$  on it. Let spiral similarity  $s$  at  $Q$  take  $(E, F) \rightarrow (B, C)$ . The main point of the problem is then:

**Claim –**  $Q \in \omega$ .

*Proof.* First, by angle bisector theorem,

$$\frac{EM}{MF} = \frac{EA}{AF} = \frac{BA}{AC} = \frac{BN}{NC}$$

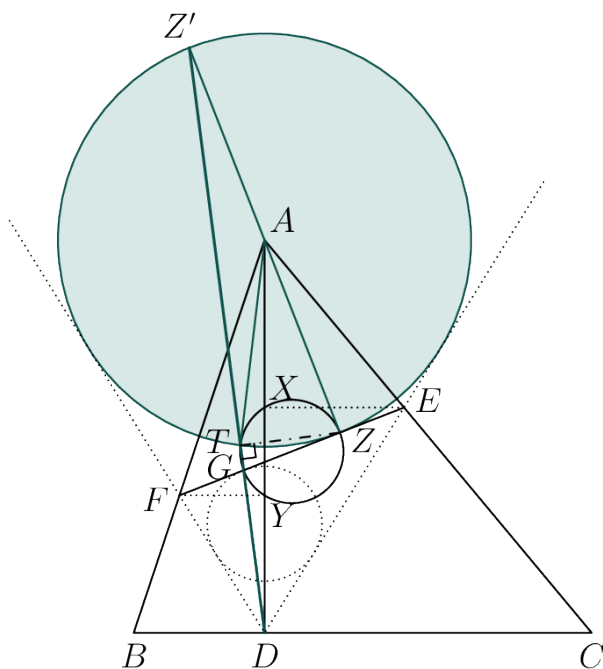
$$\Rightarrow (M \xrightarrow{s} N) \Rightarrow \angle MQN = -\arg(s) = \angle(\overline{EF}, \overline{BC}) = \angle MKN,$$

as desired. □

Since  $P$  is the antipode of  $K$  on  $\omega$ ,  $\angle KQP = 90^\circ = \angle KQA$ , implying that  $P \in \overline{AQ}$ , the  $A$ -median.

### 1.5 Mock AIME 2019/15', by Eric Shen & Raymond Feng

In  $\triangle ABC$ , let  $D, E, F$  be the feet of the altitudes from  $A, B, C$  respectively, and let  $O$  be the circumcenter. There exists a point  $T$  such that  $\angle DTZ = 90^\circ$  and  $AZ = AT$ . If  $P = \overline{AZ} \cap \overline{QT}$ , and  $Q$  lies on  $\overline{EF}$  such that  $\overline{PQ} \parallel \overline{BC}$ , prove that  $\overline{AQ}$  bisects  $\overline{BC}$ .



Construct points  $X, Y$  as the projections of  $E, F$  onto  $\overline{AD}$  respectively. \*

After drawing a diagram on Geogebra, we obtain:

#### Characterization of T

$T$  is the harmonic conjugate of  $Z$  wrt  $XY$  – i.e. it lies on  $\omega = (XYZ)$  so that the resulting quadrilateral is harmonic.

In American style, we show that this choice of  $T$  lies on  $\omega_a$  (defined as the circle at  $A$  thru  $Z$ ) and  $(DZ)$ ,

#### Verification (inspired by USA TST 2015/1)

For  $AZ = AT$ , we use power of a point / length chase to get  $AZ^2 = AX \cdot AY$  whence  $\overline{AZ}$  touches  $\omega$ . Hence, by harmonics  $\overline{AT}$  is also tangent to  $\omega$ , so this property follows.

$\angle DTZ = 90^\circ$  is much less straightforward. We define  $Z' = 2A - Z$  and  $G = E + F - Z$  as the antipodes of  $Z$  on the circle at  $A$  through  $Z$ . By a well-known lemma,  $D, Z', G$  collinear (along the cevian through the intouch point in  $\triangle DEF$ ).

But also at the same time,  $T$  is on  $\omega$ ,  $\omega_a \Rightarrow \angle ZTG = \angle ZTZ' = \pi/2$  due to antipodes. Hence,  $\angle DTZ = \pi/2$ , completing the verification.

\*Eric Shen originally included these points in the problem statement (as seen in the 2019 version of “Geometry At Its Best”), but I guess the problem’s made harder by deleting them. For me, thinking about their relevance/use was important in making nonzero progress on the problem.



By this definition, we clearly have  $(AP; XY) = -1$ . From here (the chase is best discovered backwards), harmonic chasing suffices. Define  $K = \overline{EF} \cap \overline{A\infty_{BC}}$ . Then

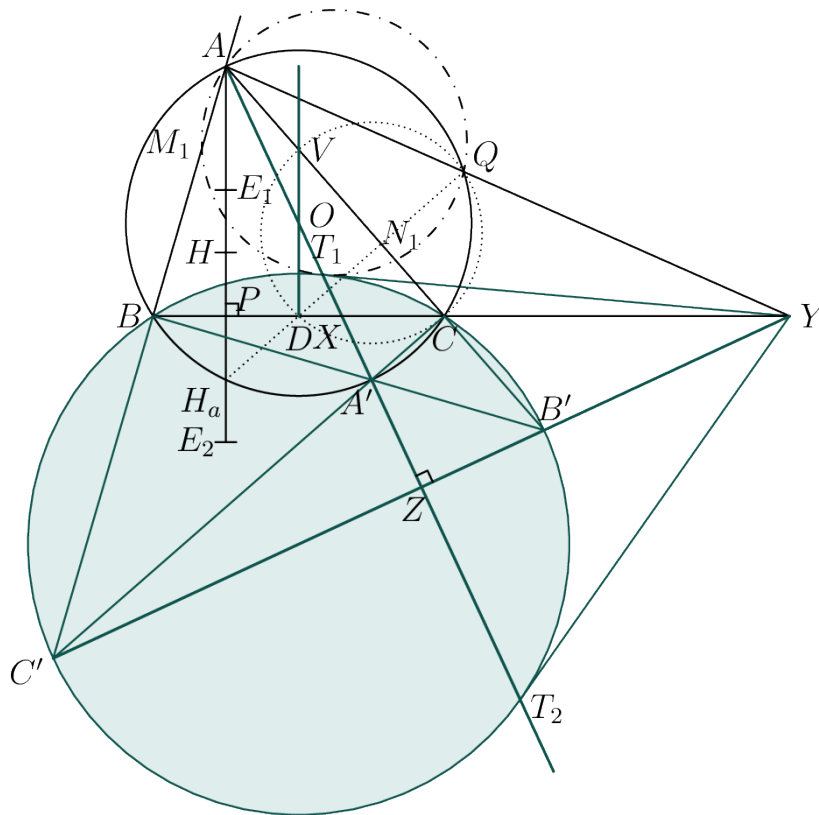
$$(\overline{AQ} \cap \overline{BC}, \infty_{BC}; B, C) \stackrel{A}{=} (QK; FE) \stackrel{\infty_{BC}}{=} (PA; YX) = -1,$$

the end!

### 1.6 China TST 2015/2/3

Let  $\triangle ABC$  be an acute triangle with circumcenter  $O$  and centroid  $G$ . Let  $D$  be the midpoint of  $\overline{BC}$ , and  $E$  be on  $(BC)$  with  $\overline{AE} \perp \overline{BC}$ . Let  $F = \overline{EG} \cap \overline{AD}$ . Define points  $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$  with  $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$ , and  $\overline{MK}, \overline{NL} \perp \overline{BC}$ .

Let  $\omega$  be the circle tangent to  $\overline{OB}, \overline{OC}$  at  $B, C$  respectively. Prove that  $(AMN)$  is tangent to  $\omega$ .



As much of the parallel/perpendicular lines aren't even relevant (just to give us some equal ratios), we simplify as follows:

#### Problem reworded

In acute  $\triangle ABC$  with circumcenter  $O$  and orthocenter  $H$ ,  $D$  is the midpoint of  $\overline{BC}$ , and the altitude from  $A$  meets  $(BC)$  at  $E$  (either one works). Let  $U, V = \overline{OD} \cap \overline{AB}, \overline{AC}$ , respectively; define  $M, N \in \overline{AB}, \overline{AC}$  with (lengths directed)

$$UM/MB = VN/NC = AE/EH.$$

Let  $\omega$  be the circle tangent to segments  $OB, OC$  at  $B, C$  respectively. Prove that  $(AMN)$  is tangent to  $\omega$ .

We define a load of new points as follows.

- $A' = 2O - A$ ;
- $E_1, E_2$  be the choices of  $E$  with  $AE/EH > 0$  and  $AE/EH < 0$  respectively. We will only consider  $M_1, N_1$ , because the negative case is identically handled;
- $T_1, T_2 = \overline{AO} \cap \omega, X = \overline{AO} \cap \overline{BC}$ , corresponding to  $E_1, E_2$  from earlier;

- $Y = \overline{T_1 T_1} \cap \overline{T_2 T_2} \cap \overline{BC}$  (which exists since  $(BC; T_1 T_2) = -1$ );
- $Q$  as the harmonic conjugate of  $A'$  wrt  $BC$ , or equivalently, the reflection of the  $A$ -orthocenter Miquel point  $Q_a$  in the perpendicular bisector of  $\overline{BC}$ ,  $\overline{DUV}$ .

**Claim 1** –  $Q$  is the Miquel point of  $ABCDUV$ .

*Proof.* As we already have  $Q \in (ABC)$ , sufficient to prove  $QDVC$  cyclic. Observe that  $Q \in \overline{H_a D}$ , which follows by  $Q_a \in \overline{A'PH}$  reflected in  $\overline{DUV}$ . The result holds by Reim because  $AH_a QC$  cyclic and  $\overline{DV} \parallel \overline{AH_a}$ .  $\square$

**Claim 2** –  $(AQT_1)$  touches  $\omega$ ,  $\overline{YT_1}$  at  $T_1$ .

*Proof.* Sufficient to show  $Q \in \overline{AY}$ , so that the claim will follow by power of a point at  $Y$ . Indeed,

$$-1 = (BC; XY) \stackrel{A}{=} (B, C; A', \overline{AY} \cap (ABC))$$

proving the claim.  $\square$

**Claim 3** –  $AE_1/E_1H = AT_1/T_1A'$ .

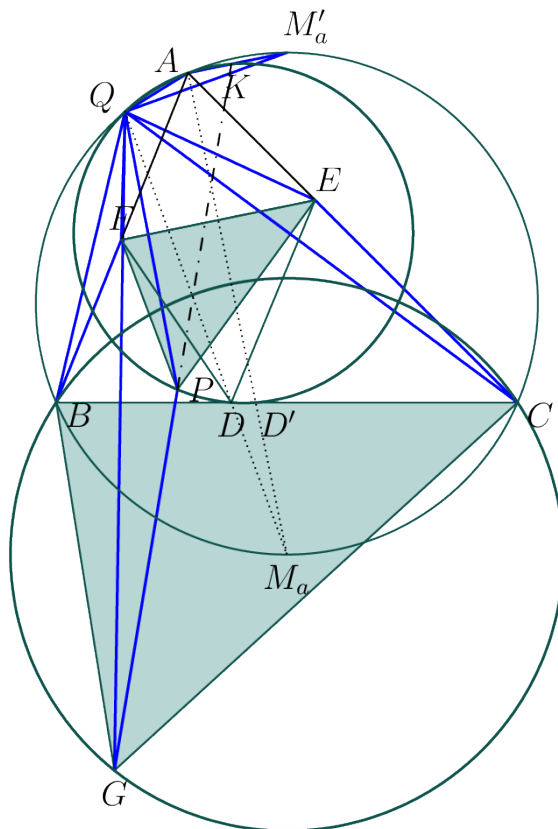
*Proof.* Define  $B' = \overline{A'B} \cap \overline{AC}$ ,  $C' = \overline{A'C} \cap \overline{AB}$ . Using the logic of **USA TST 2007/5**, we know that  $\triangle ABC \sim \triangle AB'C'$ , and that  $Q$  is the  $A$ -orthocenter Miquel point in  $\triangle AB'C'$ . Next, let  $P, Z$  be the foot from  $A$  to  $\overline{BC}, \overline{B'C'}$  respectively. If  $r$  denotes the reflection + homothety at  $A$  that maps  $B, C \Rightarrow B', C'$ , then observing that  $(E_1, P, E_2) \xrightarrow{r} (T_1, Z, T_2)$  wins.  $\square$

To finish the problem, observe  $M_1, N_1 \in (AQT_1)$  follows by spiral similarity at  $Q$ , completing the proof.

### 1.7 IMO 2019/6, by Anant Mudgal

Let  $I$  be the incenter of acute triangle  $ABC$  with  $AB \neq AC$ . The incircle  $\omega$  of  $ABC$  is tangent to sides  $BC$ ,  $CA$ , and  $AB$  at  $D$ ,  $E$ , and  $F$ , respectively. The line through  $D$  perpendicular to  $\overline{EF}$  meets  $\omega$  at  $R$ . Line  $AR$  meets  $\omega$  again at  $P$ . The circumcircles of triangle  $PCE$  and  $PBF$  meet again at  $Q$ .

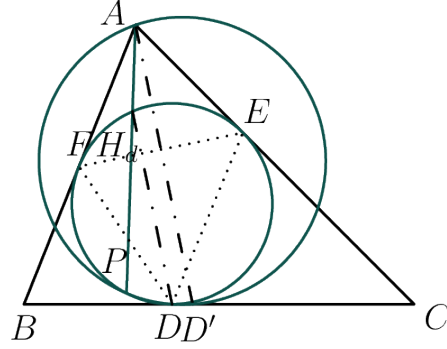
Prove that lines  $DI$  and  $PQ$  meet on the line through  $A$  perpendicular to  $\overline{AI}$ .



Observe that  $P$  is the  $D$ -orthocenter Miquel in  $\triangle DEF$ . Define  $K$  as the intersection of the  $A$ -external bisector with  $\overline{AD}$ . We make the following definitions...

- Let  $\omega, \omega_a$  denote the incircle and  $(BIC)$  respectively;
- Define  $X$  as intersection of *segment*  $PK$  with  $\omega$ . Let  $Q$  instead denote the  $A$ -SD point;
- $G$  be the harmonic conjugate of  $I$  wrt  $BC$ ,  $D'$  as the foot of the  $A$ -angle bisector;  $M_a$  as the midpoint of arc  $BC$  exc.  $A$ ;  $M'_a$  as the antipode of  $M_a$  on  $(ABC)$ ;
- $H$  as orthocenter of  $\triangle DEF$ , and  $H_d$  its reflection over  $\overline{EF}$ .

$\Rightarrow$  because  $MB^2 = MD \cdot MQ = MD' \cdot MA$ ,  $Q \in (ADD'K)$ .



**Claim 1** -  $P \in (ADD'KQ)$ .

*Proof.* Observe that  $(PH_d; EF) = -1$  whence  $A, P, H_d$  collinear. Then because  $\overline{DH_d} \parallel \overline{AI}$  because both perpendicular to  $\overline{EF}$ . Hence result by degenerate Reim.  $\square$

**Claim 2** -  $\triangle PFE \stackrel{+}{\sim} \triangle GBC$ .

*Proof.* Proceed by spiral at  $Q$ . Observe that  $\triangle H_d EF \stackrel{+}{\sim} \triangle ICB$  by angle chase. Because  $(H_d P; EF) = (IG; BC) = -1$ , the needed similarity follows.  $\square$

**Claim 3** -  $K, G, P$  collinear.

*Proof.* An angle chase, using the previous two claims:

$$\angle QPK \stackrel{\text{claim 1}}{=} \angle QAK = \angle QAM_a \stackrel{\text{spiral}}{=} \angle QPG.$$

$\square$

Using last two claims, we may chase:

$$\angle PXB = \angle GXB = \angle GCB \stackrel{\text{spiral}}{=} \angle PEF = \angle PFB$$

whence  $PXFB$  (and also  $PXEC$  by symmetry) cyclic.

This completes the proof.

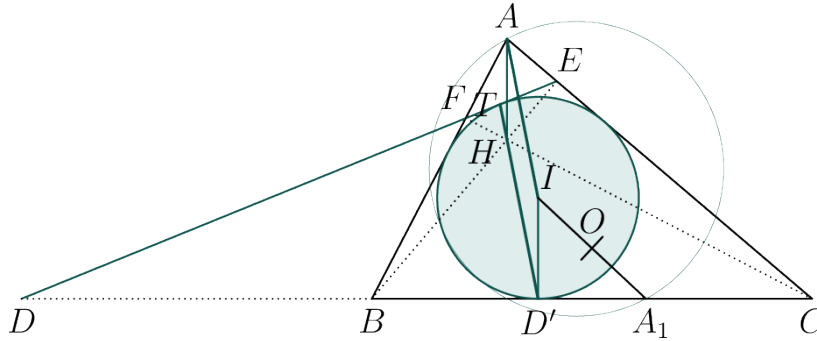
*Remark.* ggb way too op

## 1.8 MOP + USA TST, by Ankan Bhattacharya

Let  $ABC$  be a triangle with incenter  $I$ , and let  $D$  be a point on line  $BC$  satisfying  $\angle AID = 90^\circ$ . Let the excircle of triangle  $ABC$  opposite the vertex  $A$  be tangent to  $\overline{BC}$  at  $A_1$ . Define points  $B_1$  on  $\overline{CA}$  and  $C_1$  on  $\overline{AB}$  analogously, using the excircles opposite  $B$  and  $C$ , respectively.

### 1.8.1 MOP

Let  $E, F$  be the feet of the altitudes from  $B, C$  respectively. Prove that if  $\overline{EF}$  touches the incircle, then quadrilateral  $AB_1A_1C_1$  is cyclic.



Call the incircle  $\omega$ .

**Claim 1** –  $D, E, F$  are collinear.

*Proof.* We will prove that the tangent line from  $D$  is antiparallel to  $\overline{BC}$  wrt  $\angle A$ . Indeed, this line is found by reflecting  $\overline{DBC}$  over  $\overline{DI}$ , a line perpendicular to  $\overline{AI}$ , so we win.  $\square$

Let  $\omega$  touch  $\overline{DEF}$  at a point  $T$ , and let  $D'$  denote the  $A$ -intouch point.

**Claim 2** –  $\overline{AI} \parallel \overline{HD'}$ ; hence  $AID'H$  is a parallelogram and  $AH = r$ , the inradius of  $\triangle ABC$ .

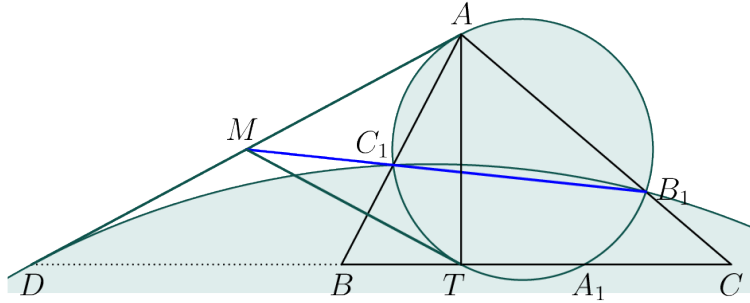
*Proof.* Because  $BCEF$  is tangential, it follows by degenerate Brianchon that lines  $BE, CF, DT'$  concur, i.e.  $H \in \overline{TD'}$ . Observe that  $DT = DD'$ ; then  $\overline{THD'} \perp \overline{DI}$  by symmetry, while  $\overline{AI} \perp \overline{DI}$  is given; the lines are thus parallel as claimed.  $\square$

Now, let  $\omega_a$ , etc denote  $(AB_1C_1)$ , etc, respectively. We observe that because the perpendicular from  $A_1$  to  $\overline{BC}$  and its cyclic variants all concur at the point  $2O - I$ , it follows that all three circles must concur at this point by Miquel spam.

But because  $r/2 = AH/2$  is the distance from  $O$  to  $\overline{BC}$ , we actually have  $2O - I = A_1$  (also because of their feet onto  $\overline{BC}$ ). Hence  $A_1 \in \omega_a$  as desired.

### 1.8.2 USA TST 2019/6

Prove that if quadrilateral  $AB_1A_1C_1$  is cyclic, then  $\overline{AD}$  is tangent to the circumcircle of  $\triangle DB_1C_1$ .



We first make some observations from working backwards on the previous part.

First,  $\overline{I_aA_1}$  is unconditionally the axis of  $\omega_b, \omega_c$ , which is because  $2O - I, A_1, I_a$  lie on the same line  $\perp \overline{BC}$ . Thus, if  $A_1$  is to lie on  $\omega_a$ , then by anglechase,  $\omega_b, \omega_c$  touch at  $A_1$  whence  $I_aA_1 \perp \overline{BC}$ .

Also, by MOP 2019 converse (which follows by uniqueness wrt  $\angle A$ ) we have  $D, E, F$  collinear. If  $T$  is the foot of  $A$  onto  $\overline{BC}$ , it follows that  $(DT; BC) = -1$ .

**Claim 1** – The  $A$ -SD point coincides with the  $A$ - orthocenter Miquel.

*Proof.* Since  $BF/CE = \cos B/\cos C = (s-c)/(s-b)$  from 19MOP, result follows by spiral.  $\square$

Next, we have  $A, A_1$  antipodes on  $\omega_a$ , which follows by angle chasing, observing that  $\omega_b, \omega_c$  touch at  $A_1$  / etc.

**Claim 2** –  $\overline{AD}$  is tangent to  $\omega_a$ .

*Proof.* Recall that  $\overline{ADQ}$  is perpendicular to  $\overline{HIQ'}$ ; thus, equivalent to show  $\overline{HQ} \parallel \overline{AA'}$  which is another angle chase.  $\square$

By radical axis/etc, it suffices to show that the midpoint  $M$  of  $\overline{AD}$  lies on  $\overline{B_1C_1}$ . By symmetry,  $\overline{MA}, \overline{TA}$  touch  $\omega_a$ .

**Claim 3** –  $(AT; B_1C_1) = -1$ .

*Proof.* Harmonics:  $(AT; B_1C_1) \stackrel{A}{=} (DT; BC) = -1$ , as claimed.  $\square$

From here the problem follows by power of a point converse on  $MD^2 = MA^2 = MB_1 \cdot MC_1$ .

### 1.9 TSTST 2018/3, by Evan Chen & Yannick Yao

Let  $ABC$  be an acute triangle with incenter  $I$ , circumcenter  $O$ , and circumcircle  $\Gamma$ . Let  $M$  be the midpoint of  $\overline{AB}$ . Ray  $AI$  meets  $\overline{BC}$  at  $D$ . Denote by  $\omega$  and  $\gamma$  the circumcircles of  $\triangle BIC$  and  $\triangle BAD$ , respectively. Line  $MO$  meets  $\omega$  at  $X$  and  $Y$ , while line  $CO$  meets  $\omega$  at  $C$  and  $Q$ . Assume that  $Q$  lies inside  $\triangle ABC$  and  $\angle AQM = \angle ACB$ .

Consider the tangents to  $\omega$  at  $X$  and  $Y$  and the tangents to  $\gamma$  at  $A$  and  $D$ . Given that  $\angle BAC \neq 60^\circ$ , prove that these four lines are concurrent on  $\Gamma$ .

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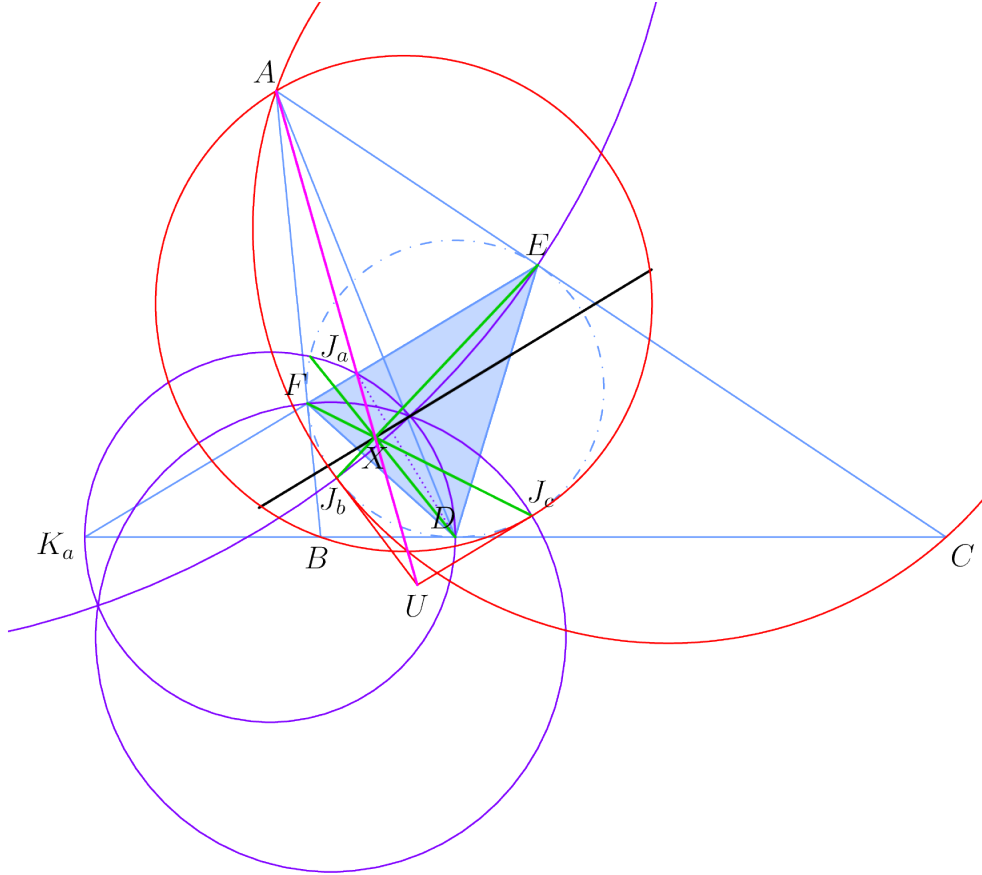


## 🌲 1.10 RMM + Brazil

### 🌲 1.10.1 RMM 2012/6

Let  $ABC$  be a triangle and let  $I$  and  $O$  denote its incentre and circumcentre respectively. Let  $\omega_A$  be the circle through  $B$  and  $C$  which is tangent to the incircle of the triangle  $ABC$ ; the circles  $\omega_B$  and  $\omega_C$  are defined similarly. The circles  $\omega_B$  and  $\omega_C$  meet at a point  $A'$  distinct from  $A$ ; the points  $B'$  and  $C'$  are defined similarly. Prove that the lines  $AA'$ ,  $BB'$  and  $CC'$  are concurrent at a point on the line  $IO$ .

Let  $K_a = \overline{EF} \cap \overline{BC}$ ,  $\gamma_a = (K_a D)$ ,  $J_a = \omega_a \cap \gamma_a \cap \omega$  (and cyclic variants), and  $H$  and  $\ell$  denote the orthocenter and Euler line of  $\triangle DEF$ , respectively. Also, let  $I_a, I_b, I_c$  be the excenters of  $\triangle ABC$

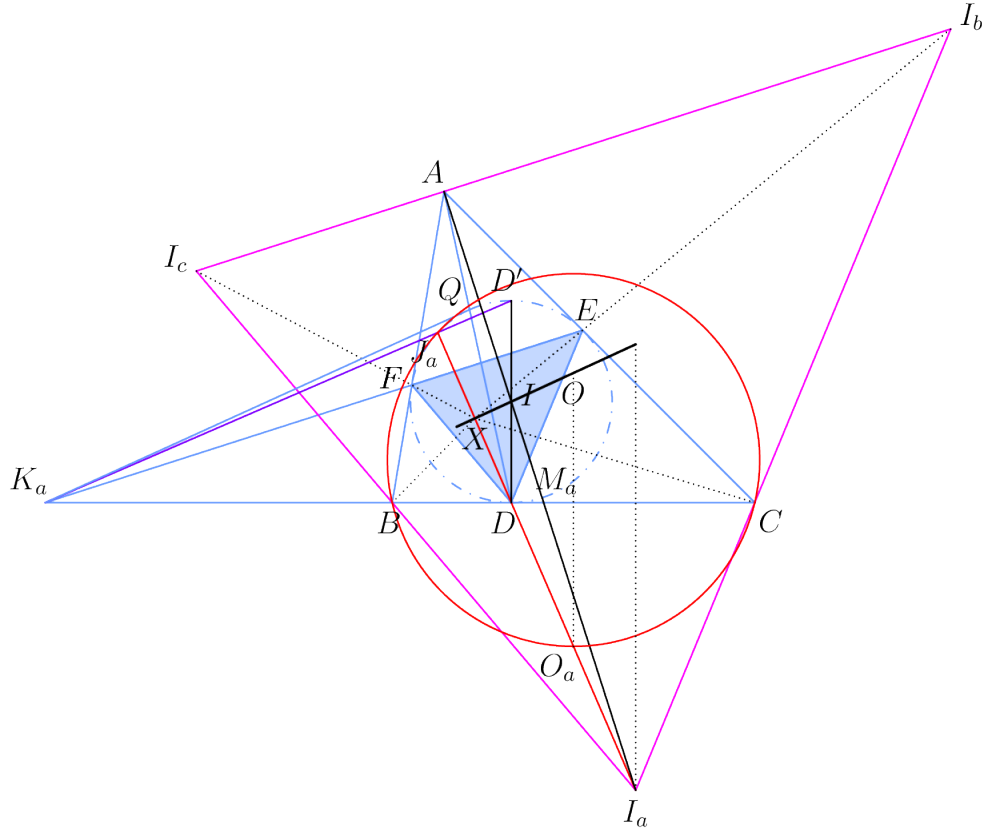


**Solution 1, by radical axes** Observe that  $\ell$  is just  $\overline{OI}$ , and that  $\gamma_a$ , etc are coaxial Apollonian circles. Define  $X$  as the radical center of  $\gamma_a, \gamma_b, \gamma_c, \omega$  (which exists since the former 3 circles are coaxial). We'll show this is the desired concurrency point in both problems. Clearly,  $\overline{DJ_a}$  is the raxis of  $(\gamma_a, \omega)$ , i.e.  $X \in \overline{DJ_a}$ .

**Lemma 1** –  $\ell$  is the raxis of  $\gamma_a$  and variants.

*Proof.* Let  $T_a$  denote the foot of  $D$  onto  $\overline{EF}$ , which is obviously on  $\gamma_a$ . Then  $H$  has power  $HD \cdot HT_a$  (= variants) wrt the  $\gamma$ 's, hence on raxis; Meanwhile  $I$  has power  $r^2$  wrt all circles by orthogonality, hence also on raxis, done.  $\square$

Let tangents to  $\omega$  at  $J_b, J_c$  meet at  $U$ ; then,  $\overline{AU}$  is the raxis of  $\omega_b, \omega_c$ . Clearly this is the polar of  $\overline{J_b J_c} \cap \overline{EF}$ . Recalling that  $X = \overline{EJ_b} \cap \overline{FJ_c}$ , follows by Brokard that  $X \in \overline{AU}$ , the end.



**Solution 2, by homothety (v4913)** Let  $D'$  be the antipode of  $D$  on  $\omega$ ,  $Q = \overline{AD} \cap \omega$  ( $\neq D$ ); then, because  $(EF; DQ) = -1$ ,  $\overline{K_aQ}$  touches  $\omega$  as well. Also, because  $\angle DJ_aD' = \angle DJ_aK_a = 90^\circ$ ,  $K_a, J_a, D'$  are collinear, whence  $(DQ; J_aD') = -1$ .

We start with  $X$  as the similicenter of homothetic triangles  $DEF, I_aI_bI_c$ . Let homothety  $h$  at  $X$  with scale factor  $r$  map  $(D, E, F) \rightarrow (I_a, I_b, I_c)$ . This must also map their circumcenters to each other, i.e.  $I \xrightarrow{H} 2O - I$ , whence  $X \in \overline{OI}$ .

Also, let  $M_a$  be the midpoint of  $\overline{BC}$ ,  $O_a \in \overline{DJ_a}$  be the midpoint of arc  $BC$  on  $\omega_a$  not containing  $J_a$  (and variants).

**Lemma 2** –  $J_a, D, I_a$  collinear.

*Proof.* Harmonics:

$$-1 = (DQ; D'J_a) \stackrel{D}{=} (\overline{BC} \cap \overline{AI}, \overline{J_aD} \cap \overline{AI}; I, A),$$

implying that  $\overline{J_aD} \cap \overline{AI}$  is the  $A$ -excenter. □

Hence,  $X = \overline{DJ_a} \cap \overline{EJ_b} \cap \overline{FJ_c}$ .

**Claim** –  $O_a$  is the midpoint of  $\overline{DI_a}$ .

*Proof.* By symmetry,  $M_a$  is the foot of  $O_a$  onto  $\overline{BC}$ , while it's well-known that  $2M - D$  is the foot of  $I_a$  onto  $\overline{BC}$ .  $M$  obviously being the midpoint of the segment with endpoints  $D, 2M - D$  implies the claim by parallel lines. □

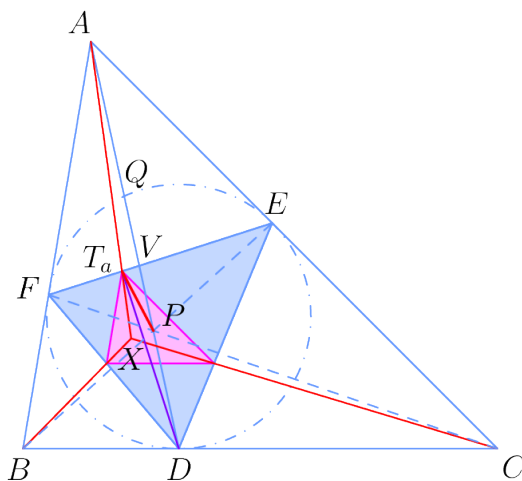
Therefore there must exist a homothety  $h'$  at  $X$  with scale factor  $(1+r)/2$ , mapping  $(D, E, F) \rightarrow (O_a, O_b, O_c)$ .  
To show that our  $X$  is indeed the radical center of  $\omega_a, \omega_b, \omega_c$ , compute

$$\text{Pow}(X, \omega_a) = XJ_a \cdot XO_a \stackrel{h'}{=} \frac{1+r}{2} XJ_a \cdot XD = \frac{\text{Pow}(X, \omega)}{(r+1)/2},$$

a symmetric quantity wrt  $a, b, c$ .

### 1.10.2 Brazil 2013/6

The incircle of triangle  $ABC$  touches sides  $BC$ ,  $CA$  and  $AB$  at points  $D$ ,  $E$  and  $F$ , respectively. Let  $P$  be the intersection of lines  $AD$  and  $BE$ . The reflections of  $P$  with respect to  $EF$ ,  $FD$  and  $DE$  are  $X$ ,  $Y$  and  $Z$ , respectively. Prove that lines  $AX$ ,  $BY$  and  $CZ$  are concurrent at a point on line  $IO$ , where  $I$  and  $O$  are the incenter and circumcenter of triangle  $ABC$ .



(We continue to use terminology from the previous subsection.) Let  $T_a$  be the projection of  $D$  onto  $\overline{EF}$ . As promised in the refactored statement in the problem section,

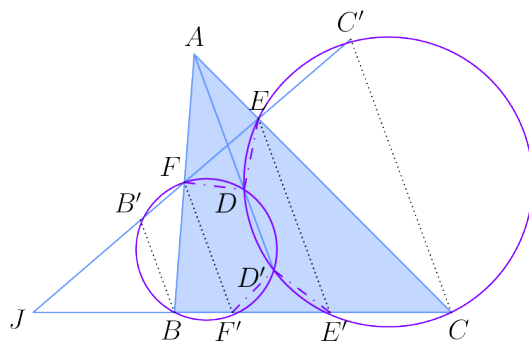
**Claim –**  $T_a \in \overline{AXA'}$ .

*Proof.* Because  $X$  is the similicenter of triangles  $DEF$ ,  $I_a I_b I_c$ , it must also be similicenter of their orthic triangles. It follows that  $T_a \in \overline{AX}$ , as needed.  $\square$

Next, let  $V = \overline{AD} \cap \overline{EF}$ , so that  $(DV; AP) = -1$ . Because  $\angle DT_a V = 90^\circ$ ,  $\overline{EF}$  must bisect  $\angle AT_a P$ , whence  $P_a \in \overline{AT_a A'}$ . Considering triangles  $ABC$ ,  $DEF$ , and the orthic triangle of  $\triangle DEF$ , the concurrency holds by cevian nest.

# 1.11 IMO 2021/3

Let  $D$  be an interior point of the acute triangle  $ABC$  with  $AB > AC$  so that  $\angle DAB = \angle CAD$ . The point  $E$  on the segment  $AC$  satisfies  $\angle ADE = \angle BCD$ , the point  $F$  on the segment  $AB$  satisfies  $\angle FDA = \angle DBC$ , and the point  $X$  on the line  $AC$  satisfies  $CX = BX$ . Let  $O_1$  and  $O_2$  be the circumcenters of the triangles  $ADC$  and  $EXD$ , respectively. Prove that the lines  $BC$ ,  $EF$ , and  $O_1O_2$  are concurrent.



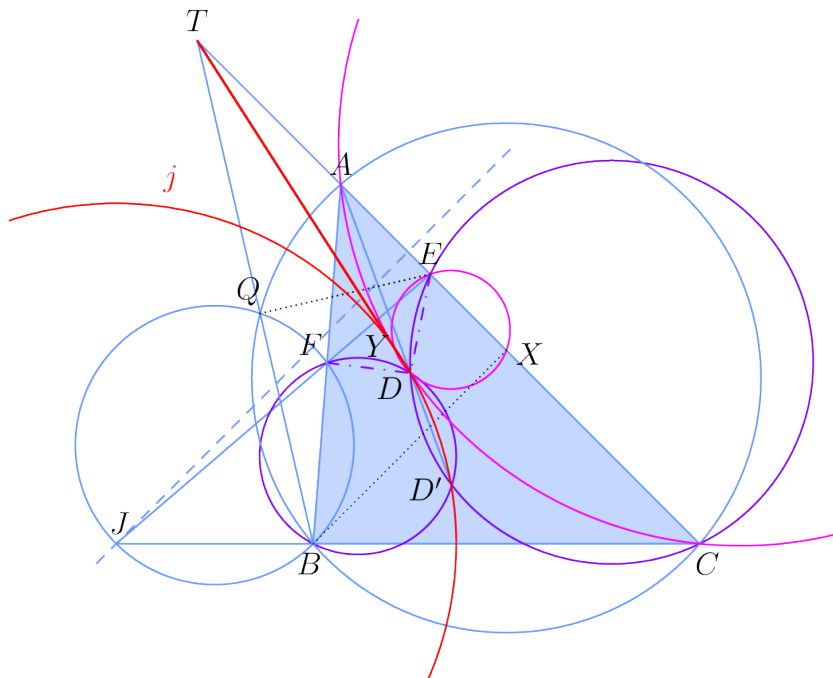
Solution by **v4913**.

Let  $J = \overline{EF} \cap \overline{BC}$ , and  $D' \in \overline{AD}$  be the isogonal conjugate of  $D$  wrt  $\triangle ABC$ . The given angle conditions imply that  $BDD'F$ ,  $CDD'E$  are cyclic, while power of a point at  $A$  implies  $BCEF$  cyclic as well.

**Claim 1** –  $J$  is the exsimilicenter of  $(EDC)$ ,  $(FDB)$ ; hence,  $JD = JD'$  by symmetry.

*Proof.* Construct  $E_1 = (CDD'E) \cap \overline{BC}$  ( $\neq C$ ),  $F_1 = (BDD'F) \cap \overline{BC}$  ( $\neq B$ ). By isogonality,  $DF = D'F'$  and  $DE = D'E'$  whence  $DD'E'E$ ,  $DD'F'F$  are both cyclic isosceles trapezoids.  $\overline{DD'}$ ,  $\overline{EE'}$ ,  $\overline{FF'}$  share a perpendicular bisector  $b$ , and in fact, this is the bisector of  $\angle J$ , i.e.  $JE = JE'$ ,  $JF = JF'$ .

Reflect  $B, C$  over  $b$  to obtain  $B', C'$ ; then, because  $JB/JF' = JB/JF = JE/JC = JE'/JC$ , there is a homothety at  $J$  mapping  $(B, B', F, F') \rightarrow (E', E, C', C)$  and thus their circumcircles  $(BB'DD') \rightarrow (CC'DD')$  as well.  $\square$



Let  $Y = (ADC) \cap (EXD)$  ( $\neq D$ ),  $Q$  be the Miquel point of  $ABCJEF$ , and  $j$  the circle at  $J$  through  $D, D'$ . Observing that  $\overline{O_1O_2}$  is the perpendicular bisector of  $\overline{DY}$ , it remains to prove  $Y \in j$ .

**Claim 2** –  $XQEB$  is cyclic.

*Proof.* This is a simple angle chase: using cyclic quadrilaterals  $(ABCQ)$ ,  $(JFBQ)$ ,  $(ECJQ)$ , and  $(AEFQ)$ , we obtain

$$\angle EQB = \angle EQA + \angle AQB = \angle ACB + \angle EFA = 2\angle ACB = \angle EXB$$

as desired.  $\square$

Next, we characterize the radical axis of  $j$ ,  $(JBF)$  – it's perpendicular to the line of centers and through  $A$ :

**Claim 3** – The line through  $B$  and the center of  $(JBF)$  is perpendicular to  $\overline{AC}$ .

*Proof.* This is equivalent to “ $t_b$ , the tangent to  $(JBF)$  at  $J$ , is parallel to  $\overline{AC}$ ”. Because  $\angle(\overline{BJC}, t_b) = \angle BFJ = \angle JCE$ , the result follows.  $\square$

Because  $\text{Pow}(A, j) = AD \cdot AD' = AQ \cdot AJ = \text{Pow}(A, (JBQF))$ ,  $A$  is on the radical axis of  $j$ ,  $(JBF)$ . By the previous claim, it follows that  $\overline{AC}$  is the radical axis of  $j$ ,  $(JBF)$ .

To finish, define  $T = \overline{DY} \cap \overline{AC} \cap \overline{BQ}$  as the radical center of  $(JBF)$ ,  $(ABC)$ ,  $(EXD)$ ,  $(ADC)$ , and the phantom point  $Y' = \overline{TD} \cap j$  ( $\neq D$ ). Because  $T$  is on  $\overline{AC}$ , the radical axis of  $j$ ,  $(JBF)$ , we have (lengths directed)

$$TY' \cdot TD = \text{Pow}(T, j) = \text{Pow}(T, (JBF)) = \text{Pow}(T, (ABCQ)) = TA \cdot TC = TY \cdot TD \Rightarrow Y = Y',$$

the end!

🌲 1.12 USAMO 2021/6, by Ankan Bhattacharya

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### 1.13 SL 2021/G8

Let  $ABC$  be a triangle with circumcircle  $\omega$  and let  $\Omega_A$  be the  $A$ -excircle. Let  $X$  and  $Y$  be the intersection points of  $\omega$  and  $\Omega_A$ . Let  $P$  and  $Q$  be the projections of  $A$  onto the tangent lines to  $\Omega_A$  at  $X$  and  $Y$  respectively. The tangent line at  $P$  to the circumcircle of the triangle  $APX$  intersects the tangent line at  $Q$  to the circumcircle of the triangle  $AQY$  at a point  $R$ . Prove that  $\overline{AR} \perp \overline{BC}$ .

**Solution 1, by Brianchon (from AoPS)**<sup>†</sup> (WIP) Redefine  $R$  as intersection of tangent at  $D'$  and  $A$ -altitude and prove  $PR$  is tangent to  $\omega_{XPA}$ . Let us denote some points:  $D'D' \cap AB = B', D'D' \cap AC = C', D'D' \cap XX = B', XX \cap AB = X'$  and  $CX \cap AP' = M$ . apply Brianchon to the polar reciprocal  $DD'XXB''C''$ :

1.  $DD \cap D'D' = \infty$
2.  $D'D' \cap XX = P'$
3.  $XX \cap XX = X$
4.  $XX \cap B''B'' = X'$
5.  $B''B'' \cap C''C'' = A$
6.  $C''C'' \cap DD = C'$

and lines 14, 25, 36 must be concurrent. Since  $AP' \cap CX = M$  we can imply that  $MX' \parallel BC$  By angle chase  $\angle MX'A = \angle P'B'A = 180 - \angle AB'C' = 180 - \angle ABC = 180 - \angle AXC = \angle MXA$  so  $MXX'A$  is concyclic. Again by angle chase  $\angle MAX = \angle MX'P' = \angle X'P'B' = \angle PAR$  (since  $P'APR$  is concyclic) thus  $\angle XAP = \angle P'AR = \angle P'PR$  and we are done.

**Solution 2, by DDIT (CyclicSLscalesTrapezoid)**

<sup>†</sup><https://artofproblemsolving.com/community/c6h2882551p25740378>