

Select geometry favorites

People

November 22, 2022

This is a collection of problems we've done and found particularly cute/beautiful/hard, inspired by chapter 11 of *Euclidean Geometry in Math Olympiads* and Eric Shen's paper *Geometry At Its Best*. Have fun!

(Note: Here, ∞_{XY} denotes the point at infinity along line XY .)

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🌲 O Problems

Remark. Some attempt has been made to deviate from the aforementioned two famous geometry papers.

Problem 1 (SL 1998/G4). Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

Problem 2 (SL 2009/G3). Let ABC be a triangle. The incircle of $\triangle ABC$ touches AB and AC at the points Z and Y , respectively. Let $G = \overline{BY} \cap \overline{CZ}$, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.

Problem 3 (SL 2015/G4). Let ABC be an acute triangle and let M be the midpoint of AC . A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC . Determine all possible values of $\frac{BT}{BM}$.

Problem 4 (SL 2016/G7). Let I be the incentre of a non-equilateral triangle ABC , I_A be the A -excentre, I'_A be the reflection of I_A in BC , and l_A be the reflection of line AI'_A in AI . Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B .

- (a) Prove that P lies on line OI where O is the circumcentre of triangle ABC .
- (b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y . Show that $\angle XIY = 120^\circ$.

Problem 5 (EGMO 2015/6). Let H be the orthocentre and G be the centroid of acute-angled triangle ABC with $AB \neq AC$. The line AG intersects the circumcircle of ABC at A and P . Let P' be the reflection of P in the line BC . Prove that $\angle CAB = 60^\circ$ if and only if $HG = GP'$.

Problem 6 (Iran TST 2018/1/4). Let ABC be a triangle ($\angle A \neq 90^\circ$), with altitudes $\overline{BE}, \overline{CF}$. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N . Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that \overline{AP} bisects \overline{BC} .

Problem 7 (Eric Shen). In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. Points X, Y are the projections of E, F onto \overline{AD} respectively. Let $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{TZ}$, and Q lies on \overline{EF} such that $\overline{PQ} \parallel \overline{BC}$, prove that \overline{AQ} is a median.

Problem 8 (China TST 2015/2/3). Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G . Let D be the midpoint of \overline{BC} , and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{AD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω .

Problem 9 (IMO 2019/6). Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA , and AB at D, E , and F , respectively. The line through D perpendicular to EF meets ω at R . Line AR meets ω again at P . The circumcircles of triangle PCE and PBF meet again at Q .

Prove that lines DI and PQ meet on the line through A perpendicular to AI .

Problem 10 (MOP 2019 & USA TST 2019/6). Let ABC be a triangle with incenter I , and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to \overline{BC} at A_1 . Define points B_1 on \overline{CA} and C_1 on \overline{AB} analogously, using the excircles opposite B and C , respectively.

- (a) **(MOP 2019)** Let E, F be the feet of the altitudes from B, C respectively. Prove that if \overline{EF} touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.
- (b) **(USA TST 2019/6)** Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then \overline{AD} is tangent to the circumcircle of $\triangle DB_1C_1$.

Problem 11 (TSTST 2018/3). Let ABC be an acute triangle with incenter I , circumcenter O , and circumcircle Γ . Let M be the midpoint of \overline{AB} . Ray AI meets \overline{BC} at D . Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y , while line CO meets γ at C and Q . Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D . Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ .

Problem 12 (RMM 2012/6 & Brazil 2013/6). In triangle ABC with incenter I and circumcenter O , let the incircle ω touch $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F respectively.

- **(RMM 2012/6)** Let ω_a be the circle through B and C tangent to ω , and define ω_b, ω_c similarly. Finally, let $A' = \omega_b \cap \omega_c$ ($\neq A$), and similarly for points B' and C' .
- **(Brazil 2013/6)** Let P be the Gergonne point of $\triangle ABC$, and its reflections in $\overline{EF}, \overline{FD}$ and \overline{DE} be P_a, P_b, P_c , respectively.

Prove that $P_a \in \overline{AA'}$, and that $\overline{AP_aA'}, \overline{BP_bB'}, \overline{CP_cC'}, \overline{IO}$ are concurrent.

Problem 13 (IMO 2021/3). Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD , respectively. Prove that the lines BC, EF , and O_1O_2 are concurrent.

Problem 14 (USAMO 2021/6). Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}, \overline{BC} \parallel \overline{EF}, \overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y , and Z be the midpoints of $\overline{AD}, \overline{BE}$, and \overline{CF} . Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

Problem 15 (SL 2021/G8). Let ABC be a triangle with circumcircle ω and let Ω_A be the A -excircle. Let X and Y be the intersection points of ω and Ω_A . Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R . Prove that $\overline{AR} \perp \overline{BC}$.

1 Solutions

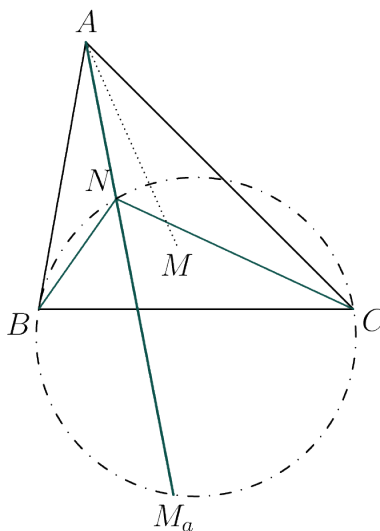
🌲 1.1 SL 1998/G4

Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$



Solution 1, by inversion Let i_a denote the inversion at A with power $AB \cdot AC$ composed with reflection in the bisector of $\angle A$. It's well-known that i_a swaps B, C . Let the images of M under i_a be $M_a \in \overline{AN}$, and cyclic variants.

Claim - $M_a \in (BNC)$, and

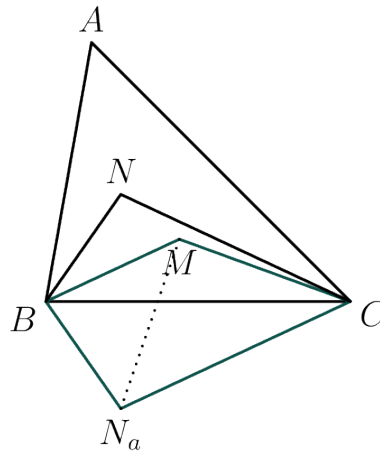
$$\frac{AM \cdot AN}{AB \cdot AC} = \frac{AN}{AM_a}.$$

Proof. The first part by angle chase:

$$\angle BM_aN \stackrel{i_a}{=} \angle MCA = \angle BCN,$$

while the second part is just the inversion distance formula.

The claim reduces the problem to $\sum_{\text{cyc}} AN/AM_a = 1$, which is just **BAMO 2008/6**.



Solution 2, by area ratios (official / intended)

Claim – For any M, N , we have

$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMC] + [BNC]}{[ABC]}.$$

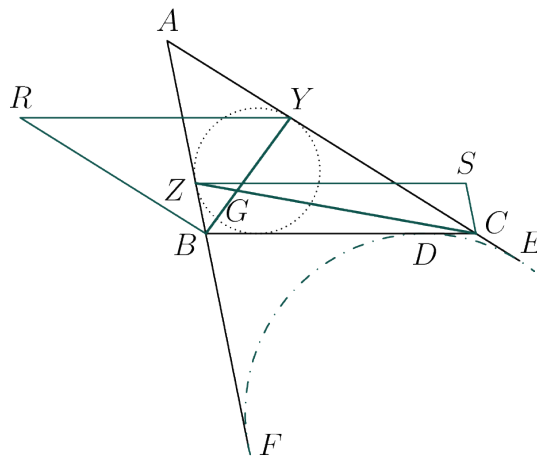
Proof. Reflect N over \overline{BC} to obtain point N_a . Then, because $\angle MBN_a = \angle B$, $(BM \cdot BN)/(BA \cdot BC) = [MBN_a]/[ABC]$. Similarly $(CM \cdot CN)/(CA \cdot CB) = [MCN_a]/[ABC]$, and summing the previous two equations gives

$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMN_aC]}{[ABC]} = \frac{[BMC] + [BNC]}{[ABC]}.$$

Noting that M, N are just isogonal conjugates, we obtain the problem by cyclic summation.

1.2 SL 2009/G3

Let ABC be a triangle. The incircle of $\triangle ABC$ touches AB and AC at the points Z and Y , respectively. Let $G = \overline{BY} \cap \overline{CZ}$, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.



This is a very “troll” problem. Let (R) , (S) , ω_a denote the point circles at R, S (radius = 0) and the A -excircle respectively. Let ω_a touch \overline{BC} , \overline{CA} , \overline{AB} at D, E, F respectively. Also, for brevity, let $a = BC$, $b = CA$, $c = AB$, $s = (a + b + c)/2$.

Claim – \overline{BY} is the radical axis of (R) , ω_a .

Proof. $BD = BR = s - c$, while $YE = YR = a$; because \overline{BD} , \overline{YE} touch ω_a , B, Y have powers $(s - c)^2, a^2$ wrt each of (R) , ω_a as promised. \square

By the claim, $G = \overline{BY} \cap \overline{CZ}$ must be the radical center of (R) , (S) , ω_a , implying the desired $GR = GS$.

1.3 SL 2015/G4

Let ABC be an acute triangle and let M be the midpoint of AC . A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC . Determine all possible values of $\frac{BT}{BM}$.

Solution by **CyclicSLscalesTrapezoid**.

The answer is $\sqrt{2}$ only. Let $X = (ABC) \cap (BPMQ)$ ($\neq B$), and let N be the midpoint of \overline{BT} .

Claim 1 – $XNMT$ is cyclic.

Proof. Since N is also the midpoint of \overline{PQ} , there is a spiral similarity at X sending PNQ to AMC . Thus, we have

$$\angle XMN = \angle XAP = \angle XTB,$$

so $XNMT$ is cyclic. □

Claim 2 – \overline{BM} is tangent to the circumcircle of $XNMT$.

Proof. We have

$$\angle XNM = \angle XPA = \angle XPB = \angle XMB.$$

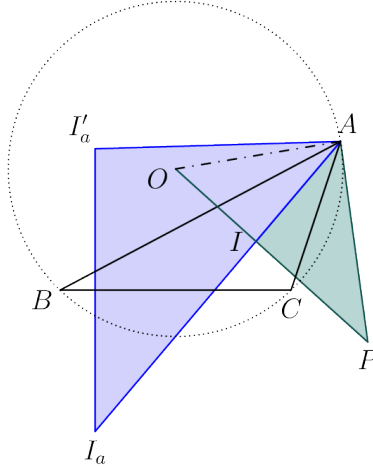
□

By Power of a Point, $BM^2 = BN \cdot BT = \frac{BT^2}{2}$, so $\frac{BT}{BM} = \sqrt{2}$.

1.4 SL 2016/G7

Let I be the incentre of a non-equilateral triangle ABC , I_A be the A -excentre, I'_A be the reflection of I_A in BC , and l_A be the reflection of line AI'_A in AI . Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B .

- Prove that P lies on line OI where O is the circumcentre of triangle ABC .
- Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y . Show that $\angle XIY = 120^\circ$.



Redefine P as the inverse of I wrt (ABC) . For the first part we assert more strongly that:

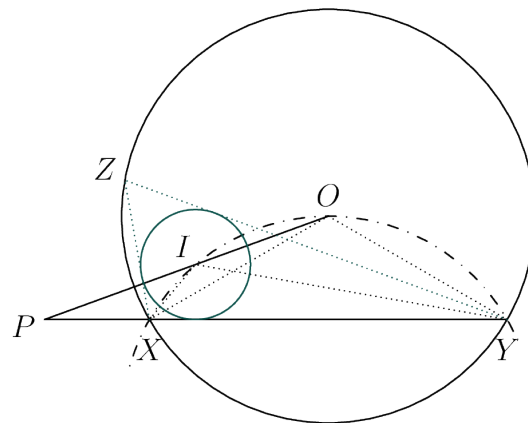
Claim – $\triangle AI_A I'_A \stackrel{+}{\sim} \triangle API$.

Proof. By angle chasing, $\angle I_A = \angle P$ follows easily. To finish, we show $I_A I'_A / I_A A = IP / AP$; indeed, the first ratio equals $2 \cos \angle BI_A C = 2 \sin \frac{A}{2}$ because of similar triangles; thus, we're left to length chase IP / AP ; this becomes

$$\frac{OP}{AP} - \frac{OI}{OA} \frac{OA}{AP} = \frac{OA}{AI} - \frac{OI^2}{OA \cdot AI} = \frac{R}{AI} - \frac{R^2 - 2rR}{R \cdot AI} = \frac{R - (R - 2r)}{AI} = 2 \sin \frac{A}{2},$$

so the ratios are equal, as needed. □

The claim clearly implies the isogonality.



For the second part, using Poncelet, let $Z \in (ABC)$ be the unique point so that $\triangle XYZ, ABC$ share a incircle and circumcircle. Inverting " P, X, Y collinear" wrt the circumcircle gives O, I, X, Y concyclic, or $\angle XOY = \angle XIY$. As it's well-known that $\angle XOY = 2\angle Z$ and $\angle XIY = (\pi + \angle Z)/2$, we must have $\angle Z = \pi/3 \Rightarrow \angle Z = 2\pi/3$ as needed.

🌲 1.5 EGMO 2015/6

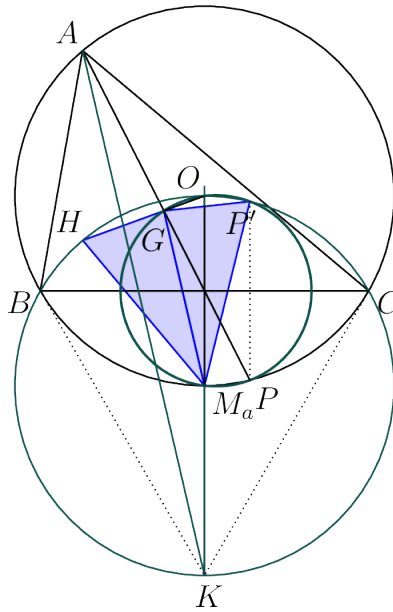
Let H be the orthocentre and G be the centroid of acute-angled triangle ABC with $AB \neq AC$. The line AG intersects the circumcircle of ABC at A and P . Let P' be the reflection of P in the line BC . Prove that $\angle CAB = 60^\circ$ if and only if $HG = GP'$.

I'm just gonna do the 'only if' and not the 'if'.

CyclicISLscalesTrapezoid

🌲 1.5.1 If

🌲 1.5.2 Only if



Let $\omega = (ABC)$, $\omega' = (BCOHP')$, and define M_a as the midpoint of minor arc BC . Then sufficient to prove $\angle HM_aG = \angle PM_aG$. Let $M = (B + C)/2$.

By power of a point at M , we obtain $OP'M_aG$ cyclic. Further, if K is the polar of \overline{BC} wrt (ABC) then $M_a = (2M + K)/3$, $G = (2M + A)/3$ whence $\overline{M_aG}$ parallel to the A -symmedian.

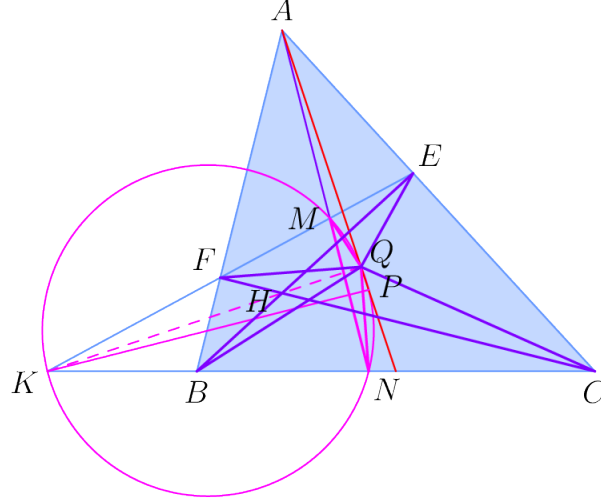
Now we anglechase, using the fact that AOM_aH is a rhombus $\Rightarrow \overline{AO} \parallel \overline{HM_a}$:

$$\angle HM_aG = \angle(\overline{AO}, \overline{M_aG}) = \angle OAK \stackrel{\text{isogonal}}{=} \angle(\overline{AG}, \overline{AH}) \stackrel{(1)}{=} \angle APP' = \angle M_aGP,$$

the end. (Here, (1) follows because $\overline{AH}, \overline{PP'} \perp \overline{BC}$.)

1.6 Iran TST 2018/1/4, by Iman Maghsoudi & Hooman Fattahi

Let ABC be a triangle ($\angle A \neq 90^\circ$), with altitudes $\overline{BE}, \overline{CF}$. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N . Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that \overline{AP} bisects \overline{BC} .



Construct $K = \overline{EF} \cap \overline{BC}$, Q as the A -Humpty point, H as the orthocenter of $\triangle ABC$, and $\omega = (KMN)$, so that the P given is the antipode of K on it. Let spiral similarity s at Q take $(E, F) \rightarrow (B, C)$. The main point of the problem is then:

Claim – $MKQN$ cyclic. In other words, $Q \in \omega$.

Proof. First, by angle bisector theorem,

$$\frac{EM}{MF} = \frac{EA}{AF} = \frac{BA}{AC} = \frac{BN}{NC}$$

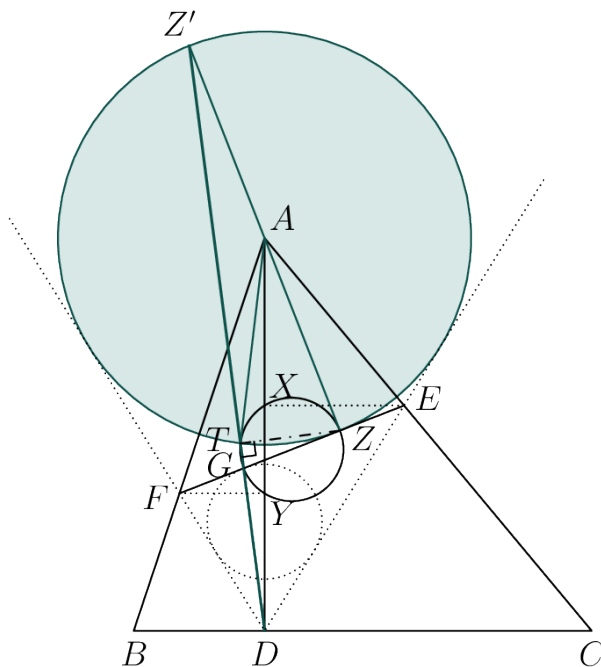
$$\Rightarrow (M \xrightarrow{s} N) \Rightarrow \angle MQN = -\arg(s) = \angle(\overline{EF}, \overline{BC}) = \angle MKN.$$

□

Since P is the antipode of K on ω , $\angle KQP = 90^\circ = \angle KQA$, implying that $P \in \overline{AQ}$, the A -median.

1.7 Mock AIME 2019/15', by Eric Shen & Raymond Feng

In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AZ} \cap \overline{QT}$, and Q lies on \overline{EF} such that $\overline{PQ} \parallel \overline{BC}$, prove that \overline{AQ} bisects \overline{BC} .



Construct points X, Y as the projections of E, F onto \overline{AD} respectively. *

After drawing a diagram on Geogebra, we obtain:

Characterization of T

T is the harmonic conjugate of Z wrt XY – i.e. it lies on $\omega = (XYZ)$ so that the resulting quadrilateral is harmonic.

In American style, we show that this choice of T lies on ω_a (defined as the circle at A thru Z) and (DZ) ,

Verification (inspired by USA TST 2015/1)

For $AZ = AT$, we use power of a point / length chase to get $AZ^2 = AX \cdot AY$ whence \overline{AZ} touches ω . Hence, by harmonics \overline{AT} is also tangent to ω , so this property follows.

$\angle DTZ = 90^\circ$ is much less straightforward. We define $Z' = 2A - Z$ and $G = E + F - Z$ as the antipodes of Z on the circle at A through Z . By a well-known lemma, D, Z', G collinear (along the cevian through the intouch point in $\triangle DEF$).

But also at the same time, T is on $\omega, \omega_a \Rightarrow \angle ZTG = \angle ZTZ' = \pi/2$ due to antipodes. Hence, $\angle DTZ = \pi/2$, completing the verification.

*Eric Shen originally included these points in the problem statement (as seen in the 2019 version of “Geometry At Its Best”), but I guess the problem’s made harder by deleting them. For me, thinking about their relevance/use was important in making nonzero progress on the problem.

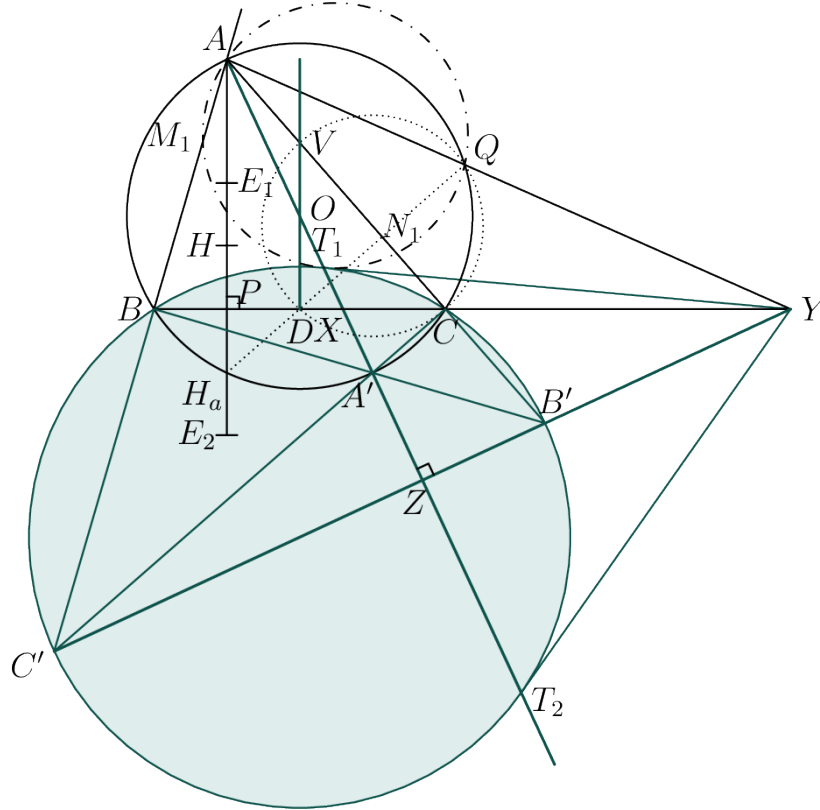
By this definition, we clearly have $(AP; XY) = -1$. From here (the chase is best discovered backwards), harmonic chasing suffices. Define $K = \overline{EF} \cap \overline{A\infty_{BC}}$. Then the bisection is established by

$$(\overline{AQ} \cap \overline{BC}, \infty_{BC}; B, C) \stackrel{A}{=} (QK; FE) \stackrel{\infty_{BC}}{=} (PA; YX) = -1.$$

1.8 China TST 2015/2/3

Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G . Let D be the midpoint of \overline{BC} , and E be on (BC) with $\overline{AE} \perp \overline{BC}$. Let $F = \overline{EG} \cap \overline{AD}$. Define points $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$ with $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$, and $\overline{MK}, \overline{NL} \perp \overline{BC}$.

Let ω be the circle tangent to $\overline{OB}, \overline{OC}$ at B, C respectively. Prove that (AMN) is tangent to ω .



As much of the parallel/perpendicular lines aren't even relevant (just to give us some equal ratios), we simplify as follows:

Problem reworded

In acute $\triangle ABC$ with circumcenter O and orthocenter H , D is the midpoint of \overline{BC} , and the altitude from A meets (BC) at E (either one works). Let $U, V = \overline{OD} \cap \overline{AB}, \overline{AC}$, respectively; define $M, N \in \overline{AB}, \overline{AC}$ with (lengths directed)

$$UM/MB = VN/NC = AE/EH.$$

Let ω be the circle tangent to segments OB, OC at B, C respectively. Prove that (AMN) is tangent to ω .

We define a load of new points as follows.

- $A' = 2O - A$;
- E_1, E_2 be the choices of E with $AE/EH > 0$ and $AE/EH < 0$ respectively. We will only consider M_1, N_1 , because the negative case is identically handled;
- $T_1, T_2 = \overline{AO} \cap \omega, X = \overline{AO} \cap \overline{BC}$, corresponding to E_1, E_2 from earlier;

- $Y = \overline{T_1 T_1} \cap \overline{T_2 T_2} \cap \overline{BC}$ (which exists since $(BC; T_1 T_2) = -1$);
- Q as the harmonic conjugate of A' wrt BC , or equivalently, the reflection of the A -orthocenter Miquel point Q_a in the perpendicular bisector of \overline{BC} , \overline{DUV} .

Claim 1 – Q is the Miquel point of $ABCDUV$.

Proof. As we already have $Q \in (ABC)$, sufficient to prove $QDVC$ cyclic. Observe that $Q \in \overline{H_a D}$, which follows by $Q_a \in \overline{A'PH}$ reflected in \overline{DUV} . The result holds by Reim because $AH_a QC$ cyclic and $\overline{DV} \parallel \overline{AH_a}$. \square

Claim 2 – (AQT_1) touches ω , $\overline{YT_1}$ at T_1 .

Proof. Sufficient to show $Q \in \overline{AY}$, so that the claim will follow by power of a point at Y . Indeed,

$$-1 = (BC; XY) \stackrel{A}{=} (B, C; A', \overline{AY} \cap (ABC)) \Rightarrow Q = \overline{AY} \cap (ABC). \quad \square$$

Claim 3 – $AE_1/E_1H = AT_1/T_1A'$.

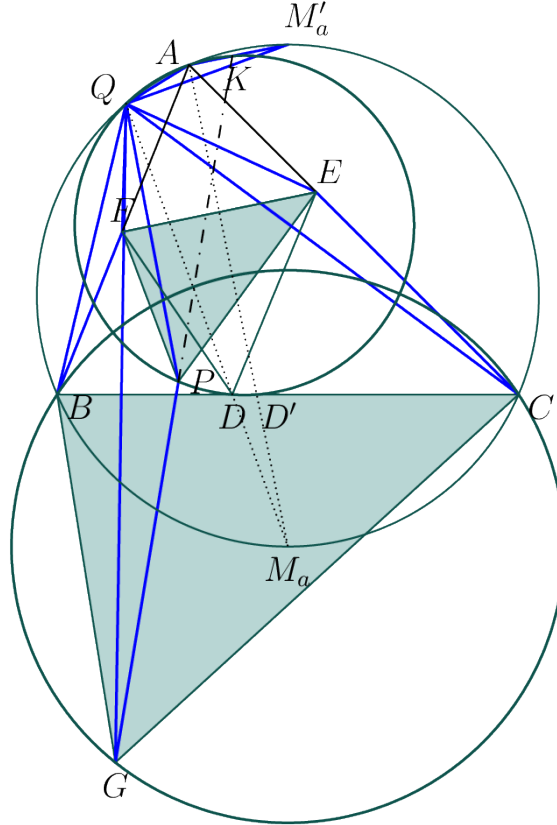
Proof. Define $B' = \overline{A'B} \cap \overline{AC}$, $C' = \overline{A'C} \cap \overline{AB}$. Using the logic of **USA TST 2007/5**, we know that $\triangle ABC \sim \triangle AB'C'$, and that Q is the A -orthocenter Miquel point in $\triangle AB'C'$. Next, let P, Z be the foot from A to $\overline{BC}, \overline{B'C'}$ respectively. If r denotes the reflection + homothety at A that maps $B, C \Rightarrow B', C'$, then observing that $(E_1, P, E_2) \xrightarrow{r} (T_1, Z, T_2)$ wins. \square

To finish the problem, observe $M_1, N_1 \in (AQT_1)$ follows by spiral similarity at Q , completing the proof.

1.9 IMO 2019/6, by Anant Mudgal

Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC , CA , and AB at D , E , and F , respectively. The line through D perpendicular to \overline{EF} meets ω at R . Line AR meets ω again at P . The circumcircles of triangle PCE and PBF meet again at Q .

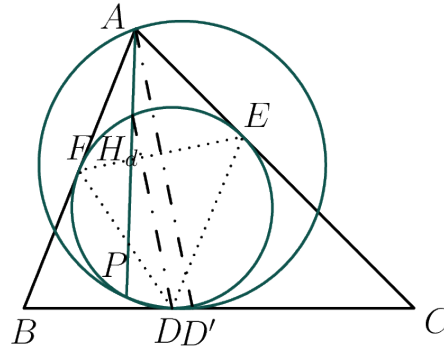
Prove that lines DI and PQ meet on the line through A perpendicular to \overline{AI} .



Observe that P is the D -orthocenter Miquel in $\triangle DEF$. Define K as the intersection of the A -external bisector with \overline{AD} . We make the following definitions...

- Let ω , ω_a denote the incircle and (BIC) respectively;
- Define X as intersection of *segment* PK with ω . Let Q instead denote the A -SD point;
- G be the harmonic conjugate of I wrt BC , D' as the foot of the A -angle bisector; M_a as the midpoint of arc BC exc. A ; M'_a as the antipode of M_a on (ABC) ;
- H as orthocenter of $\triangle DEF$, and H_d its reflection over \overline{EF} .

\Rightarrow because $MB^2 = MD \cdot MQ = MD' \cdot MA$, $Q \in (ADD'K)$.



Claim 1 - $P \in (ADD'KQ)$.

Proof. Observe that $(PH_d; EF) = -1$ whence A, P, H_d collinear. Then because $\overline{DH_d} \parallel \overline{AI}$ because both perpendicular to \overline{EF} . Hence result by degenerate Reim. \square

Claim 2 – $\Delta PFE \overset{+}{\sim} \Delta GBC$.

Proof. Proceed by spiral at Q . Observe that $\triangle H_d E F \stackrel{+}{\sim} \triangle ICB$ by angle chase. Because $(H_d P; EF) = (IG; BC) = -1$, the needed similarity follows. \square

Claim 3 – K, G, P collinear.

Proof. An angle chase, using the previous two claims:

$$\angle QPK \stackrel{\text{claim 1}}{=} \angle QAK = \angle QAM_a \stackrel{\text{spiral}}{=} \angle QPG. \quad \square$$

Using last two claims, we may chase:

$$\angle PXB = \angle GXB = \angle GCB^{\text{spiral}} = \angle PEF = \angle PFB$$

whence $PXFB$ (and also $PXEC$ by symmetry) cyclic.

This completes the proof. \square

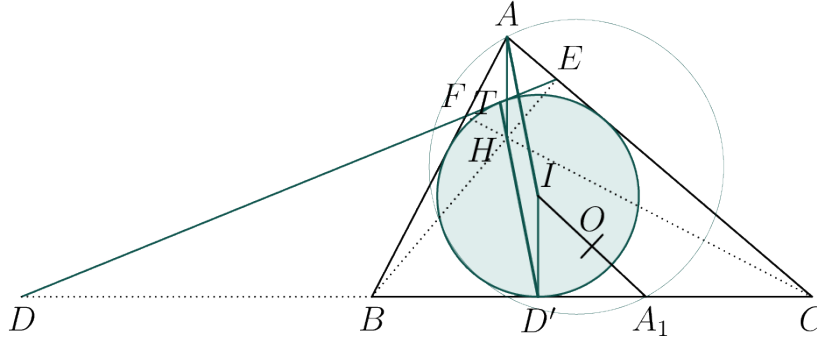
Remark. ggb way too op

1.10 MOP + USA TST, by Ankan Bhattacharya

Let ABC be a triangle with incenter I , and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to \overline{BC} at A_1 . Define points B_1 on \overline{CA} and C_1 on \overline{AB} analogously, using the excircles opposite B and C , respectively.

1.10.1 MOP

Let E, F be the feet of the altitudes from B, C respectively. Prove that if \overline{EF} touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.



Call the incircle ω .

Claim 1 – D, E, F are collinear.

Proof. We will prove that the tangent line from D is antiparallel to \overline{BC} wrt $\angle A$. Indeed, this line is found by reflecting \overline{DBC} over \overline{DI} , a line perpendicular to \overline{AI} , so we win. \square

Let ω touch \overline{DEF} at a point T , and let D' denote the A -intouch point.

Claim 2 – $\overline{AI} \parallel \overline{HD'}$; hence $AID'H$ is a parallelogram and $AH = r$, the inradius of $\triangle ABC$.

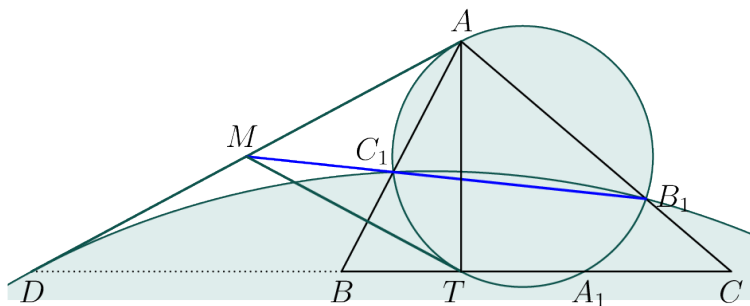
Proof. Because $BCEF$ is tangential, it follows by degenerate Brianchon that lines BE, CF, DT' concur, i.e. $H \in \overline{TD'}$. Observe that $DT = DD'$; then $\overline{THD'} \perp \overline{DI}$ by symmetry, while $\overline{AI} \perp \overline{DI}$ is given; the lines are thus parallel as claimed. \square

Now, let ω_a , etc denote (AB_1C_1) , etc, respectively. We observe that because the perpendicular from A_1 to \overline{BC} and its cyclic variants all concur at the point $2O - I$, it follows that all three circles must concur at this point by Miquel spam.

But because $r/2 = AH/2$ is the distance from O to \overline{BC} , we actually have $2O - I = A_1$ (also because of their feet onto \overline{BC}). Hence $A_1 \in \omega_a$ as desired.

1.10.2 USA TST 2019/6

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then \overline{AD} is tangent to the circumcircle of $\triangle DB_1C_1$.



We first make some observations from working backwards on the previous part.

First, $\overline{I_a A_1}$ is unconditionally the raxis of ω_b, ω_c , which is because $2O - I, A_1, I_a$ lie on the same line $\perp \overline{BC}$. Thus, if A_1 is to lie on ω_a , then by anglechase, ω_b, ω_c touch at A_1 whence $I_a A_1 \perp \overline{BC}$.

Also, by MOP 2019 converse (which follows by uniqueness wrt $\angle A$) we have D, E, F collinear. If T is the foot of A onto \overline{BC} , it follows that $(DT; BC) = -1$.

Claim 1 – The A -SD point coincides with the A - orthocenter Miquel.

Proof. Since $BF/CE = \cos B/\cos C = (s-c)/(s-b)$ from 19MOP, result follows by spiral.

Next, we have A, A_1 antipodes on ω_a , which follows by angle chasing, observing that ω_b, ω_c touch at A_1 / etc.

Claim 2 - \overline{AD} is tangent to ω_a .

Proof. Recall that \overline{ADQ} is perpendicular to $\overline{HIQ'}$; thus, equivalent to show $\overline{HQ} \parallel \overline{AA'}$ which is another angle chase. \square

By radical axis/etc, it suffices to show that the midpoint M of \overline{AD} lies on $\overline{B_1C_1}$. By symmetry about the perpendicular bisector of \overline{AT} , \overline{MA} , \overline{MT} touch ω_a , so this is equivalent to $(AT; B_1C_1) = -1$. Indeed, $(AT; B_1C_1) \stackrel{A}{=} (DT; BC) = -1$ as needed. From here the problem follows by power of a point converse on $MD^2 = MA^2 = MB_1 \cdot MC_1$.

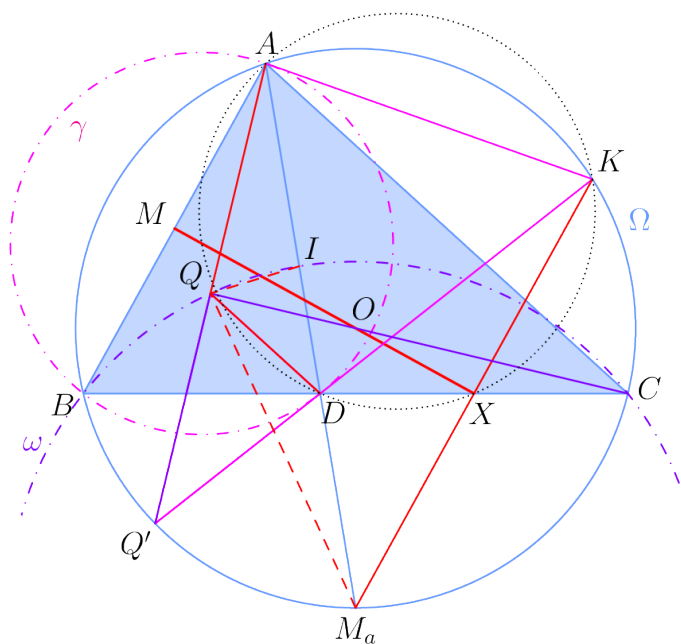
1.11 TSTST 2018/3, by Evan Chen & Yannick Yao

Let ABC be an acute triangle with incenter I , circumcenter O , and circumcircle Γ . Let M be the midpoint of \overline{AB} . Ray AI meets \overline{BC} at D . Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y , while line CO meets ω at C and Q . Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D . Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ .

Geometry is the science of correct reasoning on incorrect figures.

George Pólya / Henri Poincaré



The given angle condition implies $AMQO$ cyclic, or $\angle AQC = \angle AMO = \pi/2$. We make the following definitions:

- $\Omega = (ABC)$, M_a as the center of ω and midpoint of \overline{BC} ;
- $Q' = 2Q - A$ as the reflection of A in \overline{QOC} – this lies on Ω by symmetry about \overline{CO} ;
- $K \in \Omega$ as the reflection of M_a in \overline{MO} , the perpendicular bisector of \overline{AB} .

In the meat of the problem we'll use the following easily findable arcs/angles frequently:

$$\widehat{BM_a} = \widehat{CM_a} = \widehat{AK} = A, \text{ and } \widehat{AC} = \widehat{CQ'} = \widehat{M_aK} \Rightarrow \angle AQ'C = B.$$

Observation

\overline{QI} bisects $\angle AQD$. (Holds because $Q \in \gamma$, the Apollonian circle wrt A, D through I .)

Claim 1 - $\triangle QQ'D \stackrel{+}{\sim} \triangle M_a AC$.

Proof. First, we'll show $\angle QQ'D = \angle B$, a massive angle chase:

$$\begin{aligned} \angle M_a AQ &= \angle CAQ' - \angle CAM_a = B - \frac{A}{2}, \text{ and } \angle M_a IQ = \frac{\pi - \angle IM_a Q}{2} = \frac{\pi}{2} - \angle ICO = B + \frac{C}{2}; \\ \Rightarrow \angle AQI &= \angle M_a IQ - \angle M_a AQ = \frac{\pi - B}{2}. \end{aligned}$$

Applying the observation gives $\angle Q'QD = B$.

To conclude, note that

$$\frac{Q'Q}{QD} = \frac{AQ}{QD} = \frac{AI}{ID} = \frac{AC}{CD} = \frac{AM_a}{M_a C},$$

establishing the similarity by SAS.

(The similarity is negative due to lack of configuration issues, and inspection.) □

Claim 2 - Q', D, K collinear.

Proof. Angle chase again: $\angle AQ'D \stackrel{\text{claim 1}}{=} -\angle M_a AC = \widehat{AC}/2 = \widehat{AK}/2 = \angle AQ'K$. □

Part 1: \overline{KA} and \overline{KD} touch γ

Even more angle chasing, taking advantage of claim 2...

$$\angle KAD = \frac{\widehat{KM_a}}{2} = \frac{\widehat{AC}}{2} = \angle ABD, \text{ while } \angle KDA = \frac{\widehat{KA} + \widehat{QM_a}}{2} = \frac{\widehat{Q'C}}{2} = B = \angle DBA,$$

proving the tangencies.

The other, more elegant part of the problem...

Claim 3 - $\overline{MO}, \overline{BC}, \overline{KM_a}$ (ADK) all concur at a point X .

Proof. Let $X_1 = \overline{MO} \cap \overline{BC}$, $X_2 = \overline{KM_a} \cap \overline{BC}$.

- $X_1 \in (ADK)$ by similarity: observe by (omitted) angle chase that $\triangle AXB \stackrel{+}{\sim} \triangle AKD$, whence $\angle AXD = \angle AKD$;
- $X_2 \in (ADK)$ (by contrast) is by power of a point at M_a :

$$M_a B^2 = M_a C^2 = M_a X_2 \cdot M_a K = M_a A \cdot M_a D.$$

As $X_1 = X_2 = (ADK) \cap \overline{BC}$ ($\neq D$), the claim is proven. □

Because $\overline{M_a K} \parallel \overline{AB} \perp \overline{MO}$, and $X = \overline{MO} \cap \overline{M_a K}$ is the inverse of K wrt ω (by the second equation in previous claim's proof), \overline{MO} is the polar of K wrt ω , completing the problem.

Remark. (crazyeyemoody907) For me, it took forever to discover the proof of claim 1. Here are some useless formulations of its statement I harped upon:

- $(AC; KM_a) = -1$ which directly implies the first part:

$$(B, \overline{BK} \cap \gamma; A, D) \stackrel{B}{=} (M_a K; AC) = -1;$$

Since “ \overline{KA} touches γ ” is very easily provable, K would be polar of \overline{AD} wrt γ as promised...

- $BDQQ'$ cyclic ($\iff \overline{QD} \parallel \overline{AC}$ by Reim)

In fact, this means post-solve that $\overline{BQ} \parallel \overline{Q'DK}$...in hindsight, equally useless...

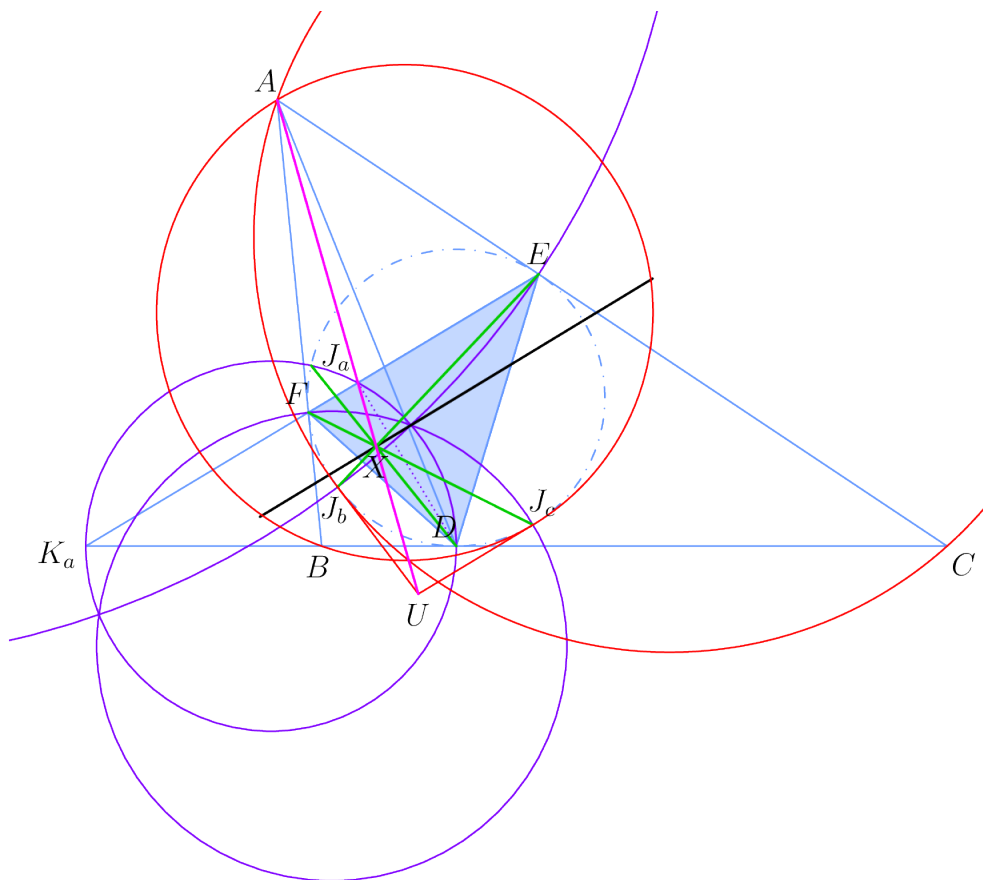
Remark. (Note to self?) When angle chasing in a synthetic problem fails, try length ratios! Examples: SL 2016/G7, GGG 4.5;

1.12 RMM + Brazil

1.12.1 RMM 2012/6

Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC ; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A ; the points B' and C' are defined similarly. Prove that the lines AA' , BB' and CC' are concurrent at a point on the line IO .

Let $K_a = \overline{EF} \cap \overline{BC}$, $\gamma_a = (K_a D)$, $J_a = \omega_a \cap \gamma_a \cap \omega$ (and cyclic variants), and H and ℓ denote the orthocenter and Euler line of $\triangle DEF$, respectively. Also, let I_a, I_b, I_c be the excenters of $\triangle ABC$.

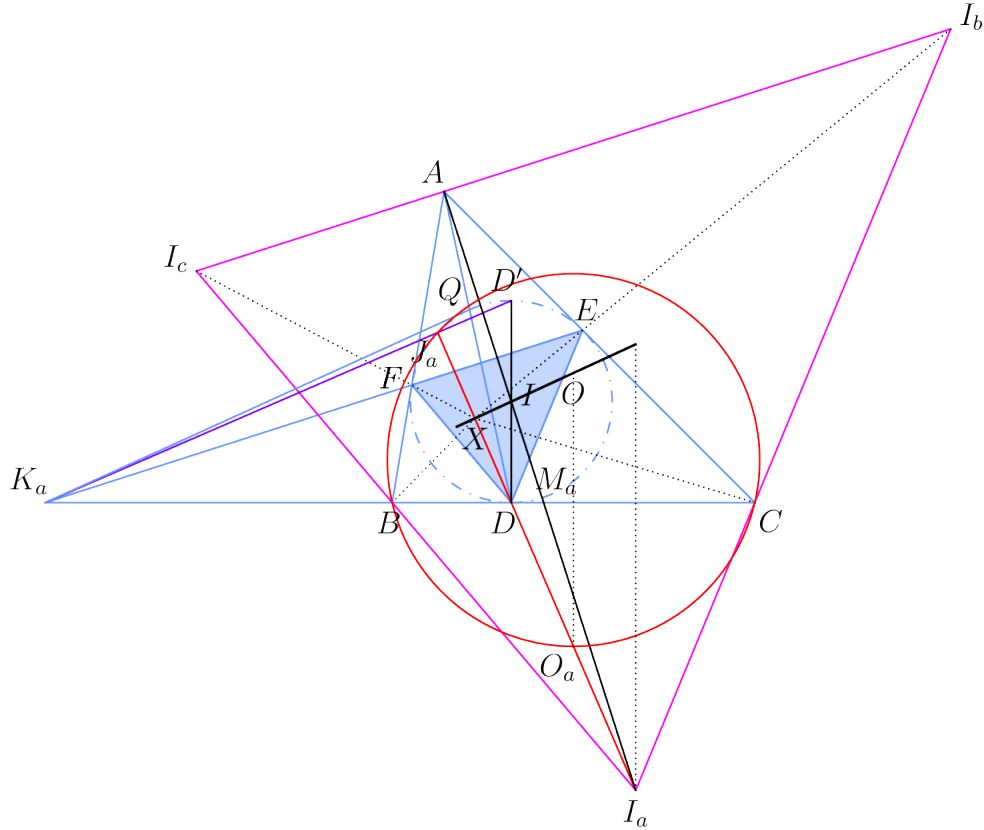


Solution 1, by radical axes Observe that ℓ is just \overline{OI} , and that γ_a , etc are coaxial Apollonian circles. Define X as the radical center of $\gamma_a, \gamma_b, \gamma_c, \omega$ (which exists since the former 3 circles are coaxial). We'll show this is the desired concurrency point in both problems. Clearly, $\overline{DJ_a}$ is the raxis of (γ_a, ω) , i.e. $X \in \overline{DJ_a}$.

Lemma 1 – ℓ is the raxis of γ_a and variants.

Proof. Let T_a denote the foot of D onto \overline{EF} , which is obviously on γ_a . Then H has power $HD \cdot HT_a$ (= variants) wrt the γ 's, hence on raxis; Meanwhile I has power r^2 wrt all circles by orthogonality, hence also on raxis, done. \square

Let tangents to ω at J_b, J_c meet at U ; then, \overline{AU} is the raxis of ω_b, ω_c . Clearly this is the polar of $\overline{J_b J_c} \cap \overline{EF}$. Recalling that $X = \overline{EJ_b} \cap \overline{FJ_c}$, follows by Brokard that $X \in \overline{AU}$, the end.



Solution 2, by homothety (v4913) Let D' be the antipode of D on ω , $Q = \overline{AD} \cap \omega$ ($\neq D$); then, because $(EF; DQ) = -1$, $\overline{K_aQ}$ touches ω as well. Also, because $\angle DJ_aD' = \angle DJ_aK_a = 90^\circ$, K_a, J_a, D' are collinear, whence $(DQ; J_aD') = -1$.

We start with X as the similicenter of homothetic triangles $DEF, I_aI_bI_c$. Let homothety h at X with scale factor r map $(D, E, F) \rightarrow (I_a, I_b, I_c)$. This must also map their circumcenters to each other, i.e. $I \xrightarrow{h} 2O - I$, whence $X \in \overline{OI}$.

Also, let M_a be the midpoint of \overline{BC} , $O_a \in \overline{DJ_a}$ be the midpoint of arc BC on ω_a not containing J_a (and variants).

Lemma 2 (SL 2002/G7) – J_a, D, I_a collinear.

Proof. Harmonics:

$$-1 = (DQ; D'J_a) \stackrel{D}{=} (\overline{BC} \cap \overline{AI}, \overline{J_aD} \cap \overline{AI}; I, A),$$

implying that $\overline{J_aD} \cap \overline{AI}$ is the A -excenter. □

Hence, $X = \overline{DJ_a} \cap \overline{EJ_b} \cap \overline{FJ_c}$.

Claim – O_a is the midpoint of $\overline{DI_a}$.

Proof. By symmetry, M_a is the foot of O_a onto \overline{BC} , while it's well-known that $2M - D$ is the foot of I_a onto \overline{BC} . M obviously being the midpoint of the segment with endpoints $D, 2M - D$ implies the claim by parallel lines. □

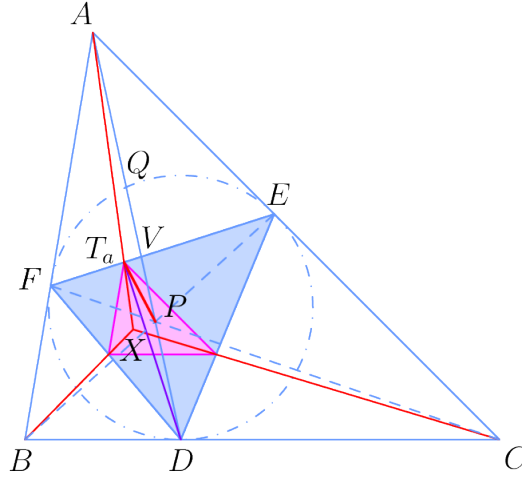
Therefore there must exist a homothety h' at X with scale factor $(1+r)/2$, mapping $(D, E, F) \rightarrow (O_a, O_b, O_c)$.
To show that our X is indeed the radical center of $\omega_a, \omega_b, \omega_c$, compute

$$\text{Pow}(X, \omega_a) = XJ_a \cdot XO_a \stackrel{h'}{=} \frac{1+r}{2} XJ_a \cdot XD = \frac{\text{Pow}(X, \omega)}{(r+1)/2},$$

a symmetric quantity wrt a, b, c .

1.12.2 Brazil 2013/6

The incircle of triangle ABC touches sides BC , CA and AB at points D , E and F , respectively. Let P be the intersection of lines AD and BE . The reflections of P with respect to EF , FD and DE are X , Y and Z , respectively. Prove that lines AX , BY and CZ are concurrent at a point on line IO , where I and O are the incenter and circumcenter of triangle ABC .



(We continue to use terminology from the previous subsection.) Let T_a be the projection of D onto \overline{EF} . As promised in the refactored statement in the problem section,

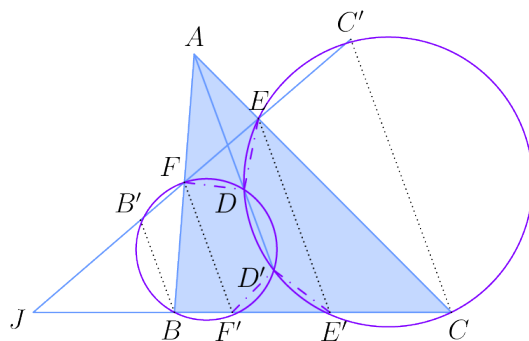
Claim – $T_a \in \overline{AXA'}$.

Proof. Because X is the similicenter of triangles DEF , $I_a I_b I_c$, it must also be similicenter of their orthic triangles. It follows that $T_a \in \overline{AX}$, as needed. \square

Next, let $V = \overline{AD} \cap \overline{EF}$, so that $(DV; AP) = -1$. Because $\angle DT_a V = 90^\circ$, \overline{EF} must bisect $\angle AT_a P$, whence $P_a \in \overline{AT_a A'}$. Considering triangles ABC , DEF , and the orthic triangle of $\triangle DEF$, the concurrency holds by cevian nest.

1.13 IMO 2021/3

Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD , respectively. Prove that the lines BC , EF , and O_1O_2 are concurrent.



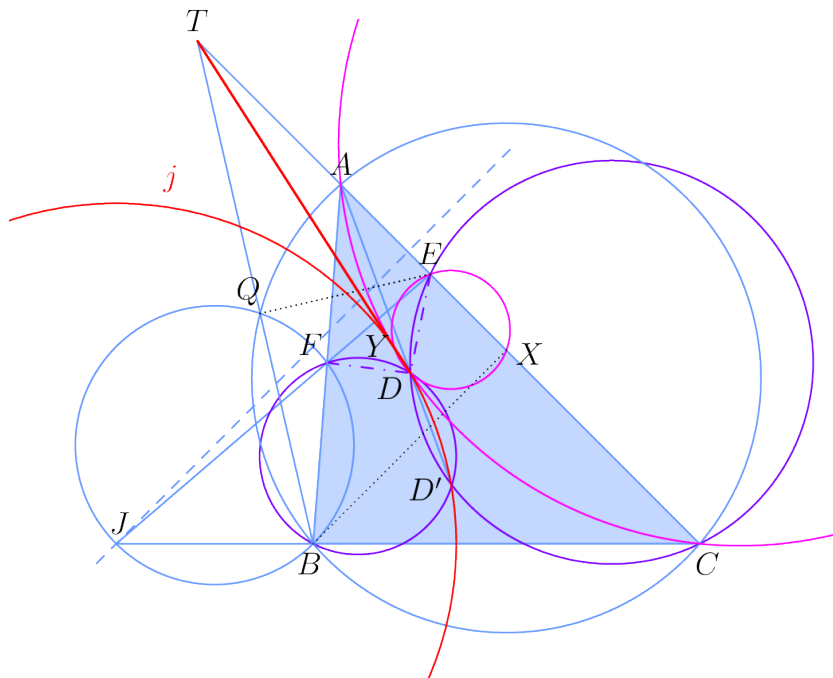
Solution by **v4913**.

Let $J = \overline{EF} \cap \overline{BC}$, and $D' \in \overline{AD}$ be the isogonal conjugate of D wrt $\triangle ABC$. The given angle conditions imply that $BDD'F$, $CDD'E$ are cyclic, while power of a point at A implies $BCEF$ cyclic as well.

Claim 1 – J is the exsimilicenter of (EDC) , (FDB) ; hence, $JD = JD'$ by symmetry.

Proof. Construct $E_1 = (CDD'E) \cap \overline{BC}$ ($\neq C$), $F_1 = (BDD'F) \cap \overline{BC}$ ($\neq B$). By isogonality, $DF = D'F'$ and $DE = D'E'$ whence $DD'E'E$, $DD'F'F$ are both cyclic isosceles trapezoids. $\overline{DD'}$, $\overline{EE'}$, $\overline{FF'}$ share a perpendicular bisector b , and in fact, this is the bisector of $\angle J$, i.e. $JE = JE'$, $JF = JF'$.

Reflect B, C over b to obtain B', C' ; then, because $JB/JF' = JB/JF = JE/JC = JE'/JC$, there is a homothety at J mapping $(B, B', F, F') \rightarrow (E', E, C', C)$ and thus their circumcircles $(BB'DD') \rightarrow (CC'DD')$ as well. \square



Let $Y = (ADC) \cap (EXD)$ ($\neq D$), Q be the Miquel point of $ABCJEF$, and j the circle at J through D, D' . Observing that $\overline{O_1O_2}$ is the perpendicular bisector of \overline{DY} , it remains to prove $Y \in j$.

Claim 2 – $XQEB$ is cyclic.

Proof. This is a simple angle chase: using cyclic quadrilaterals $(ABCQ)$, $(JFBQ)$, $(ECJQ)$, and $(AEFQ)$, we obtain

$$\angle EQB = \angle EQA + \angle AQB = \angle ACB + \angle EFA = 2\angle ACB = \angle EXB \quad \square$$

Next, we characterize the radical axis of j , (JBF) – it's perpendicular to the line of centers and through A :

Claim 3 – The line through B and the center of (JBF) is perpendicular to \overline{AC} .

Proof. This is equivalent to “ t_b , the tangent to (JBF) at J , is parallel to \overline{AC} ”. Because $\angle(\overline{BJC}, t_b) = \angle BFJ = \angle JCE$, the result follows. \square

Because $\text{Pow}(A, j) = AD \cdot AD' = AQ \cdot AJ = \text{Pow}(A, (JBQF))$, A is on the radical axis of j , (JBF) . By the previous claim, it follows that \overline{AC} is the radical axis of j , (JBF) .

To finish, define $T = \overline{DY} \cap \overline{AC} \cap \overline{BQ}$ as the radical center of (JBF) , (ABC) , (EXD) , (ADC) , and the phantom point $Y' = \overline{TD} \cap j$ ($\neq D$). Because T is on \overline{AC} , the radical axis of j , (JBF) , we have (lengths directed)

$$TY' \cdot TD = \text{Pow}(T, j) = \text{Pow}(T, (JBF)) = \text{Pow}(T, (ABCQ)) = TA \cdot TC = TY \cdot TD \Rightarrow Y = Y',$$

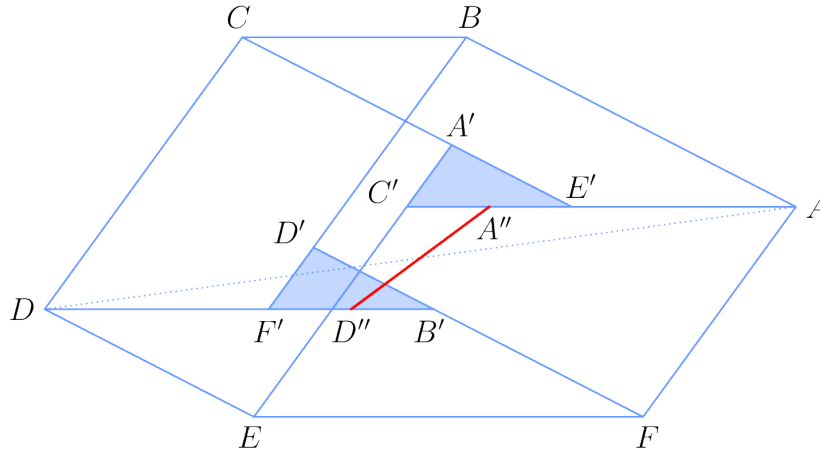
the end!

1.14 USAMO 2021/6, by Ankan Bhattacharya

Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X , Y , and Z be the midpoints of \overline{AD} , \overline{BE} , and \overline{CF} . Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.



Construct parallelogram $CDEA'$ and cyclic variants: $A' = C + E - D$, etc. We may compute using vectors that $\triangle B'D'F'$ is a translation of $\triangle A'C'E'$ by the vector $(B+D+F) - (A+C+E)$. In particular, they're congruent.

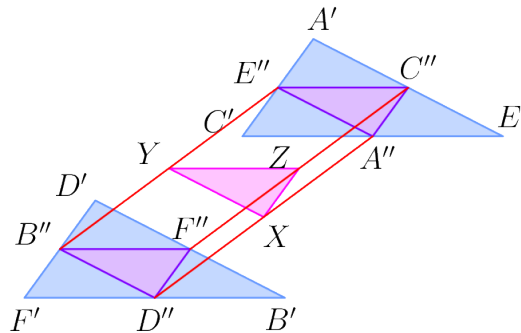
Claim 1 - A, C, E have same power wrt $(A'C'E')$; in other words, $\triangle ACE, A'C'E'$ share a circumcenter.

Proof. Observing that $\text{Pow}(A, (A'C'E')) = AC' \cdot AE' = BC \cdot EF$ by parallelograms, this claim follows by the given length condition. \square

Next, construct $A'' = \frac{C'+E'}{2}$ and cyclic variants. The circumcenter of $\triangle A'C'E'$ is then the orthocenter of $\triangle A''C''E''$.

Claim 2 - $X = \frac{A''+D''}{2}$.

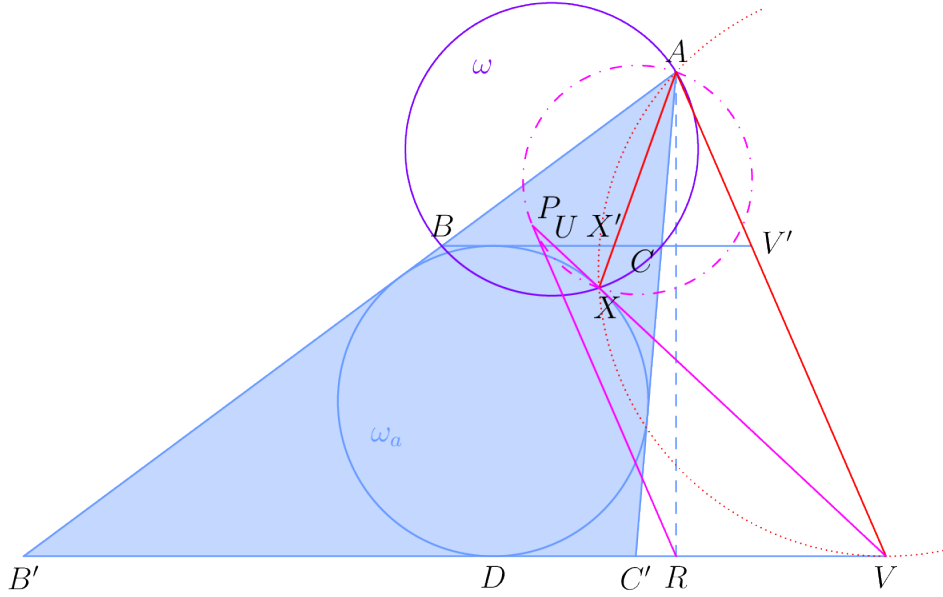
Proof. Using vectors, $B' + C' = E' + F' = A + D \Rightarrow \frac{A+D}{2} = \frac{B'+C'+E'+F'}{4} = \frac{A''+D''}{2}$. \square



By claim 2 + symmetry, $\triangle XYZ$ is the vector average of (congruent) triangles $A''C''E'', B''D''F''$, so their orthocenters are collinear.

1.15 SL 2021/G8

Let ABC be a triangle with circumcircle ω and let Ω_A be the A -excircle. Let X and Y be the intersection points of ω and Ω_A . Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R . Prove that $\overline{AR} \perp \overline{BC}$.



Solution by [crazyeyemoody907](#).

Let the antipode of the A -extouch point be D ; let the tangent to ω_a at D intersect $\overline{AB}, \overline{AC}$ at B', C' respectively. Redefine R as the foot from A to $\overline{B'C'}$; let line x be tangent to ω_a at X , $U = x \cap BC$, and $V = x \cap \overline{B'C'}$. Finally, let $X' = \overline{AX} \cap \overline{BC}$, $V' = \overline{AV} \cap \overline{BC}$.

Claim 1 - $AXUV'$ cyclic.

Proof. Apply DDIT to $A, UXV \infty_{BC}$, and project onto \overline{BC} , to obtain an involutive pairing $(B, C), (U, V'), (\infty_{BC}, X')$ – or equivalently, $X'B \cdot X'C = X'U \cdot X'V'$. By power of a point, $X'B \cdot X'C = X'A \cdot X'X$, so the claim follows from power of a point converse on $X'U \cdot X'V = X'A \cdot X'X$. \square

Claim 2 - \overline{DV} is tangent to (AXV) .

Proof. Angle chase using previous claim, and the fact that $\overline{BC} \parallel \overline{B'C'}$:

$$\angle XAV \stackrel{\text{claim 1}}{=} \angle XUV' = \angle XVD.$$

\square

Redefine R as the foot from A to $\overline{B'C'}$. It remains to show,

Claim 3 - \overline{PR} touches (APX') .

Proof. Since $\angle VPA = \angle VRA = 90^\circ$, $APRV$ cyclic, so we may anglechase as follows:

$$\angle APR = \angle AVR \stackrel{\text{claim 2}}{=} \angle AXV = \angle AXP.$$

\square