

# Select geometry favorites (WIP)

People

December 11, 2022

This is a collection of problems we've done and found particularly cute/beautiful/hard, inspired by chapter 11 of *Euclidean Geometry in Math Olympiads* and Eric Shen's paper *Geometry At Its Best*. Have fun!

(Note: Here,  $\infty_{XY}$  denotes the point at infinity along line  $XY$ .)

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## 🌲 O Problems

**Remark.** Some attempt has been made to deviate from the aforementioned two famous geometry papers.

**Problem 1 (SL 1998/G4).** Let  $M$  and  $N$  be two points inside triangle  $ABC$  such that  $\angle MAB = \angle NAC$  and  $\angle MBA = \angle NBC$ . Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

**Problem 2 (SL 2009/G3).** Let  $ABC$  be a triangle. The incircle of  $\triangle ABC$  touches  $AB$  and  $AC$  at the points  $Z$  and  $Y$ , respectively. Let  $G = \overline{BY} \cap \overline{CZ}$ , and let  $R$  and  $S$  be points such that the two quadrilaterals  $BCYR$  and  $BCSZ$  are parallelograms. Prove that  $GR = GS$ .

**Problem 3 (SL 2015/G4).** Let  $ABC$  be an acute triangle and let  $M$  be the midpoint of  $AC$ . A circle  $\omega$  passing through  $B$  and  $M$  meets the sides  $AB$  and  $BC$  at points  $P$  and  $Q$  respectively. Let  $T$  be the point such that  $BPTQ$  is a parallelogram. Suppose that  $T$  lies on the circumcircle of  $ABC$ . Determine all possible values of  $\frac{BT}{BM}$ .

**Problem 4 (SL 2016/G7).** Let  $I$  be the incentre of a non-equilateral triangle  $ABC$ ,  $I_A$  be the  $A$ -excentre,  $I'_A$  be the reflection of  $I_A$  in  $BC$ , and  $l_A$  be the reflection of line  $AI'_A$  in  $AI$ . Define points  $I_B, I'_B$  and line  $l_B$  analogously. Let  $P$  be the intersection point of  $l_A$  and  $l_B$ .

- (a) Prove that  $P$  lies on line  $OI$  where  $O$  is the circumcentre of triangle  $ABC$ .
- (b) Let one of the tangents from  $P$  to the incircle of triangle  $ABC$  meet the circumcircle at points  $X$  and  $Y$ . Show that  $\angle XIY = 120^\circ$ .

**Problem 5 (IMO 2008/6).** Let  $ABCD$  be a convex quadrilateral with  $BA \neq BC$ . Denote the incircles of triangles  $ABC$  and  $ADC$  by  $\omega_1$  and  $\omega_2$  respectively. Suppose that there exists a circle  $\omega$  tangent to ray  $BA$  beyond  $A$  and to the ray  $BC$  beyond  $C$ , which is also tangent to the lines  $AD$  and  $CD$ . Prove that the common external tangents to  $\omega_1$  and  $\omega_2$  intersect on  $\omega$ .

**Problem 6 (EGMO 2015/6).** Let  $H$  be the orthocentre and  $G$  be the centroid of acute-angled triangle  $ABC$  with  $AB \neq AC$ . The line  $AG$  intersects the circumcircle of  $ABC$  at  $A$  and  $P$ . Let  $P'$  be the reflection of  $P$  in the line  $BC$ . Prove that  $\angle CAB = 60^\circ$  if and only if  $HG = GP'$ .

**Problem 7 (Iran TST 2018/1/4).** Let  $ABC$  be a triangle ( $\angle A \neq 90^\circ$ ), with altitudes  $\overline{BE}, \overline{CF}$ . The bisector of  $\angle A$  intersects  $\overline{EF}, \overline{BC}$  at  $M, N$ . Let  $P$  be a point such that  $\overline{MP} \perp \overline{EF}$  and  $\overline{NP} \perp \overline{BC}$ . Prove that  $\overline{AP}$  bisects  $\overline{BC}$ .

**Problem 8 (Eric Shen).** In  $\triangle ABC$ , let  $D, E, F$  be the feet of the altitudes from  $A, B, C$  respectively, and let  $O$  be the circumcenter. Let  $Z = \overline{AO} \cap \overline{EF}$ . There exists a point  $T$  such that  $\angle DTZ = 90^\circ$  and  $AZ = AT$ . If  $P = \overline{AD} \cap \overline{TZ}$ , and  $Q$  lies on  $\overline{EF}$  such that  $\overline{PQ} \parallel \overline{BC}$ , prove that  $\overline{AQ}$  bisects  $\overline{BC}$ .

**Problem 9 (China TST 2015/2/3).** Let  $\triangle ABC$  be an acute triangle with circumcenter  $O$  and centroid  $G$ . Let  $D$  be the midpoint of  $\overline{BC}$ , and  $E$  be on  $(BC)$  with  $\overline{AE} \perp \overline{BC}$ . Let  $F = \overline{EG} \cap \overline{OD}$ . Define points  $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$  with  $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$ , and  $\overline{MK}, \overline{NL} \perp \overline{BC}$ .

Let  $\omega$  be the circle tangent to  $\overline{OB}, \overline{OC}$  at  $B, C$  respectively. Prove that  $(AMN)$  is tangent to  $\omega$ .

**Problem 10 (SL 2018/G5).** Let  $ABC$  be a triangle with circumcircle  $\omega$  and incenter  $I$ . A line  $\ell$  meets the lines  $AI, BI, CI$  at points  $D, E, F$  respectively, all distinct from  $A, B, C, I$ . Prove that the circumcircle of the triangle determined by the perpendicular bisectors of  $\overline{AD}, \overline{BE}, \overline{CF}$  is tangent to  $\omega$ .

**Problem 11 (MOP 2020/1Z).** Let  $ABCD$  be a quadrilateral inscribed in circle  $\Omega$ . Circles  $\omega_A$  and  $\omega_D$  are drawn internally tangent to  $\Omega$ , such that  $\omega_A$  is tangent to  $\overline{AB}$  and  $\overline{AC}$  while  $\omega_D$  is tangent to  $\overline{DB}$  and  $\overline{DC}$ . Prove that we can draw a line parallel to  $\overline{AD}$  which is simultaneously tangent to both  $\omega_A$  and  $\omega_D$ .

**Problem 12.** Let  $ABC$  be a triangle and let  $T$  be the contact point of the  $A$ -mixtilinear incircle with the circumcircle, and let  $T'$  be the reflection of  $T$  over  $BC$ . Prove that the nine-point circle of  $T'BC$  is tangent to the incircle.

**Problem 13 (IGO 2020/A4).** Convex circumscribed quadrilateral  $ABCD$  with its incenter  $I$  is given such that its incircle is tangent to  $\overline{AD}$ ,  $\overline{DC}$ ,  $\overline{CB}$ , and  $\overline{BA}$  at  $K$ ,  $L$ ,  $M$ , and  $N$ . Let  $E = \overline{AD} \cap \overline{BC}$  and  $F = \overline{AB} \cap \overline{CD}$ . Let  $X = \overline{KM} \cap \overline{AB}$  and  $Y = \overline{KM} \cap \overline{CD}$ . Let  $Z = \overline{LN} \cap \overline{AD}$  and  $T = \overline{LN} \cap \overline{BC}$ .

Prove that the circumcircle of triangle  $\triangle XFY$  and the circle with diameter  $EI$  are tangent if and only if the circumcircle of triangle  $\triangle TEZ$  and the circle with diameter  $FI$  are tangent.

**Problem 14 (IMO 2019/6).** Let  $I$  be the incenter of acute triangle  $ABC$  with  $AB \neq AC$ . The incircle  $\omega$  of  $ABC$  is tangent to sides  $BC$ ,  $CA$ , and  $AB$  at  $D$ ,  $E$ , and  $F$ , respectively. The line through  $D$  perpendicular to  $EF$  meets  $\omega$  at  $R$ . Line  $AR$  meets  $\omega$  again at  $P$ . The circumcircles of triangle  $PCE$  and  $PBF$  meet again at  $Q$ .

Prove that lines  $DI$  and  $PQ$  meet on the line through  $A$  perpendicular to  $AI$ .

**Problem 15 (MOP 2019 & USA TST 2019/6).** Let  $ABC$  be a triangle with incenter  $I$ , and let  $D$  be a point on line  $BC$  satisfying  $\angle AID = 90^\circ$ . Let the excircle of triangle  $ABC$  opposite the vertex  $A$  be tangent to  $\overline{BC}$  at  $A_1$ . Define points  $B_1$  on  $\overline{CA}$  and  $C_1$  on  $\overline{AB}$  analogously, using the excircles opposite  $B$  and  $C$ , respectively.

(a) (MOP 2019) Let  $E, F$  be the feet of the altitudes from  $B, C$  respectively. Prove that if  $\overline{EF}$  touches the incircle, then quadrilateral  $AB_1A_1C_1$  is cyclic.

(b) (USA TST 2019/6) Prove that if quadrilateral  $AB_1A_1C_1$  is cyclic, then  $\overline{AD}$  is tangent to  $(DB_1C_1)$ .

**Problem 16 (RMM 2012/6 + Fake USAMO 2020/3).** In triangle  $ABC$  with incenter  $I$  and circumcenter  $O$ , let the incircle  $\omega$  touch  $\overline{BC}$ ,  $\overline{CA}$ ,  $\overline{AB}$  at  $D, E, F$  respectively.

- (RMM 2012/6) Let  $\omega_a$  be the circle through  $B$  and  $C$  tangent to  $\omega$ , and define  $\omega_b, \omega_c$  similarly. Finally, let  $A' = \omega_b \cap \omega_c$  ( $\neq A$ ), and similarly for points  $B'$  and  $C'$ . Prove that the lines  $AA', BB'$  and  $CC'$  are concurrent at a point on the line  $IO$ .
- (Fake USAMO 2020/3) Let  $T$  be the projection of  $D$  to  $\overline{EF}$ . The line  $AT$  intersects the circumcircle of  $\triangle ABC$  again at point  $X \neq A$ . Circles  $(AEX)$  and  $(AFX)$  intersect  $\omega$  again at points  $P \neq E$  and  $Q \neq F$  respectively. Prove that  $\overline{EQ}, \overline{FP}$ , and  $\overline{OI}$  are also concurrent.

**Problem 17 (TSTST 2018/3).** Let  $ABC$  be an acute triangle with incenter  $I$ , circumcenter  $O$ , and circumcircle  $\Gamma$ . Let  $M$  be the midpoint of  $\overline{AB}$ . Ray  $AI$  meets  $\overline{BC}$  at  $D$ . Denote by  $\omega$  and  $\gamma$  the circumcircles of  $\triangle BIC$  and  $\triangle BAD$ , respectively. Line  $MO$  meets  $\omega$  at  $X$  and  $Y$ , while line  $CO$  meets  $\omega$  at  $C$  and  $Q$ . Assume that  $Q$  lies inside  $\triangle ABC$  and  $\angle AQM = \angle ACB$ .

Consider the tangents to  $\omega$  at  $X$  and  $Y$  and the tangents to  $\gamma$  at  $A$  and  $D$ . Given that  $\angle BAC \neq 60^\circ$ , prove that these four lines are concurrent on  $\Gamma$ .

**Problem 18 (DeuX MO 2020/II/3).** In triangle  $ABC$  with circumcenter  $O$  and orthocenter  $H$ , line  $OH$  meets  $\overline{AB}, \overline{AC}$  at  $E, F$  respectively. Let  $\omega$  be the circumcircle of triangle  $AEF$  with center  $S$ , meeting  $(ABC)$  again at  $J \neq A$ . Line  $OH$  also meets  $(JSO)$  again at  $D \neq O$ . Define  $K = (JSO) \cap (ABC)$  ( $\neq J$ ),  $M = \overline{JK} \cap \overline{OH}$ , and  $G = \overline{DK} \cap (ABC)$  ( $\neq K$ ). Prove that  $(GHM)$  and  $(ABC)$  are tangent to each other.

**Problem 19 (IMO 2021/3).** Let  $D$  be an interior point of the acute triangle  $ABC$  with  $AB > AC$  so that  $\angle DAB = \angle CAD$ . The point  $E$  on the segment  $AC$  satisfies  $\angle ADE = \angle BCD$ , the point  $F$  on the segment  $AB$

satisfies  $\angle FDA = \angle DBC$ , and the point  $X$  on the line  $AC$  satisfies  $CX = BX$ . Let  $O_1$  and  $O_2$  be the circumcenters of the triangles  $ADC$  and  $EXD$ , respectively. Prove that the lines  $BC$ ,  $EF$ , and  $O_1O_2$  are concurrent.

**Problem 20 (USAMO 2021/6).** Let  $ABCDEF$  be a convex hexagon satisfying  $\overline{AB} \parallel \overline{DE}$ ,  $\overline{BC} \parallel \overline{EF}$ ,  $\overline{CD} \parallel \overline{FA}$ , and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let  $X$ ,  $Y$ , and  $Z$  be the midpoints of  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$ . Prove that the circumcenter of  $\triangle ACE$ , the circumcenter of  $\triangle BDF$ , and the orthocenter of  $\triangle XYZ$  are collinear.

**Problem 21 (SL 2021/G8).** Let  $ABC$  be a triangle with circumcircle  $\omega$  and let  $\Omega_A$  be the  $A$ -excircle. Let  $X$  and  $Y$  be the intersection points of  $\omega$  and  $\Omega_A$ . Let  $P$  and  $Q$  be the projections of  $A$  onto the tangent lines to  $\Omega_A$  at  $X$  and  $Y$  respectively. The tangent line at  $P$  to the circumcircle of the triangle  $APX$  intersects the tangent line at  $Q$  to the circumcircle of the triangle  $AQY$  at a point  $R$ . Prove that  $\overline{AR} \perp \overline{BC}$ .

**Problem 22 (USEMO 2020/3).** Let  $ABC$  be an acute triangle with circumcenter  $O$  and orthocenter  $H$ . Let  $\Gamma$  denote the circumcircle of triangle  $ABC$ , and  $N$  the midpoint of  $OH$ . The tangents to  $\Gamma$  at  $B$  and  $C$ , and the line through  $H$  perpendicular to line  $AN$ , determine a triangle whose circumcircle we denote by  $\omega_A$ . Define  $\omega_B$  and  $\omega_C$  similarly.

Prove that the common chords of  $\omega_A, \omega_B$  and  $\omega_C$  are concurrent on line  $OH$ .

# 1 Solutions

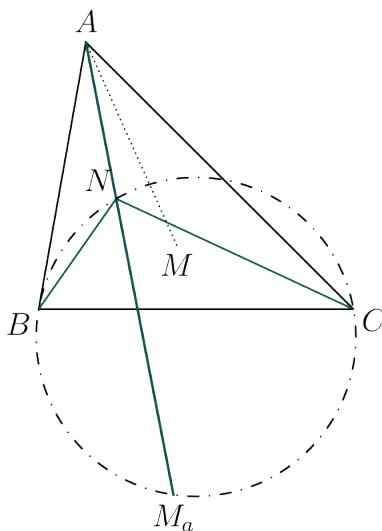
## 1.1 SL 1998/G4

Let  $M$  and  $N$  be two points inside triangle  $ABC$  such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$



**Solution 1, by inversion** Let  $i_a$  denote the inversion at  $A$  with power  $AB \cdot AC$  composed with reflection in the bisector of  $\angle A$ . It's well-known that  $i_a$  swaps  $B, C$ . Let the images of  $M$  under  $i_a$  be  $M_a \in \overline{AN}$ , and cyclic variants.

**Claim** –  $M_a \in (BNC)$ , and

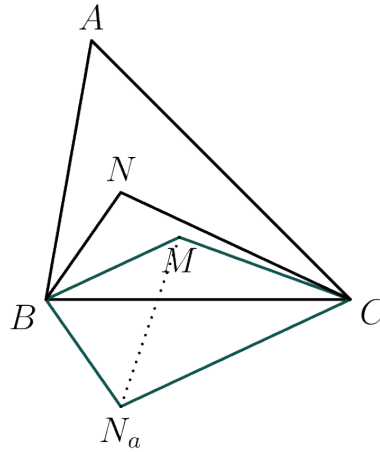
$$\frac{AM \cdot AN}{AB \cdot AC} = \frac{AN}{AM_a}.$$

*Proof.* The first part by angle chase:

$$\angle BM_aN \stackrel{i_a}{=} \angle MCA = \angle BCN,$$

while the second part is just the inversion distance formula. □

The claim reduces the problem to  $\sum_{\text{cyc}} AN/AM_a = 1$ , which is just **BAMO 2008/6**.



**Claim –** For any  $M, N$ , we have

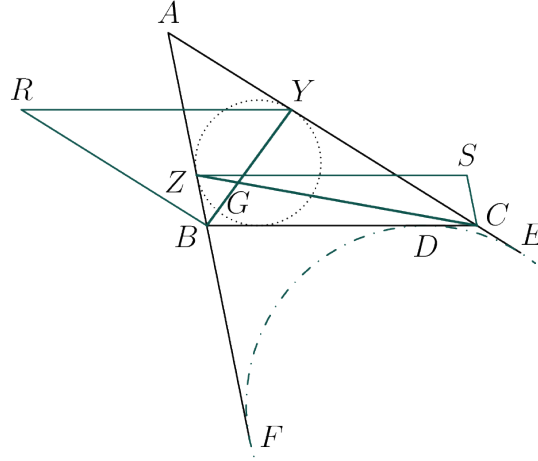
$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMC] + [BNC]}{[ABC]}.$$

$$\frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CB \cdot CA} = \frac{[BMN_aC]}{[ABC]} = \frac{[BMC] + [BNC]}{[ABC]}.$$

Noting that  $M, N$  are just isogonal conjugates, we obtain the problem by cyclic summation.

### 1.2 SL 2009/G3, by Hossein Karke Abadi

Let  $ABC$  be a triangle. The incircle of  $\triangle ABC$  touches  $AB$  and  $AC$  at the points  $Z$  and  $Y$ , respectively. Let  $G = \overline{BY} \cap \overline{CZ}$ , and let  $R$  and  $S$  be points such that the two quadrilaterals  $BCYR$  and  $BCSZ$  are parallelograms. Prove that  $GR = GS$ .



This is a very “troll” problem. Let  $(R)$ ,  $(S)$ ,  $\omega_a$  denote the point circles at  $R, S$  (radius = 0) and the  $A$ -excircle respectively. Let  $\omega_a$  touch  $\overline{BC}$ ,  $\overline{CA}$ ,  $\overline{AB}$  at  $D, E, F$  respectively. Also, for brevity, let  $a = BC$ ,  $b = CA$ ,  $c = AB$ ,  $s = (a + b + c)/2$ .

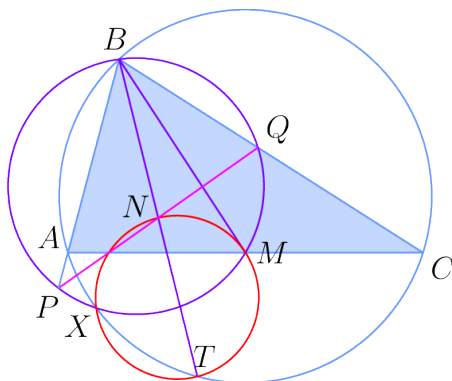
**Claim** –  $\overline{BY}$  is the radical axis of  $(R)$ ,  $\omega_a$ .

*Proof.*  $BD = BR = s - c$ , while  $YE = YR = a$ ; because  $\overline{BD}$ ,  $\overline{YE}$  touch  $\omega_a$ ,  $B, Y$  have powers  $(s - c)^2, a^2$  wrt each of  $(R)$ ,  $\omega_a$  as promised.  $\square$

By the claim,  $G = \overline{BY} \cap \overline{CZ}$  must be the radical center of  $(R)$ ,  $(S)$ ,  $\omega_a$ , implying the desired  $GR = GS$ .

### 1.3 SL 2015/G4

Let  $ABC$  be an acute triangle and let  $M$  be the midpoint of  $AC$ . A circle  $\omega$  passing through  $B$  and  $M$  meets the sides  $AB$  and  $BC$  at points  $P$  and  $Q$  respectively. Let  $T$  be the point such that  $BPTQ$  is a parallelogram. Suppose that  $T$  lies on the circumcircle of  $ABC$ . Determine all possible values of  $\frac{BT}{BM}$ .



Solution by **CyclicSLscalesTrapezoid**.

The answer is  $\sqrt{2}$  only. Let  $X = (ABC) \cap (BPMQ)$  ( $\neq B$ ), and let  $N$  be the midpoint of  $\overline{BT}$ .

**Claim 1** –  $XNMT$  is cyclic.

*Proof.* Since  $N$  is also the midpoint of  $\overline{PQ}$ , there is a spiral similarity at  $X$  sending  $PNQ$  to  $AMC$ . Thus, we have

$$\angle XMN = \angle XAP = \angle XTB,$$

proving the claim. □

**Claim 2** –  $\overline{BM}$  is tangent to  $(XNMT)$ .

*Proof.* We have

$$\angle XNM = \angle XPA = \angle XPB = \angle XMB.$$

□

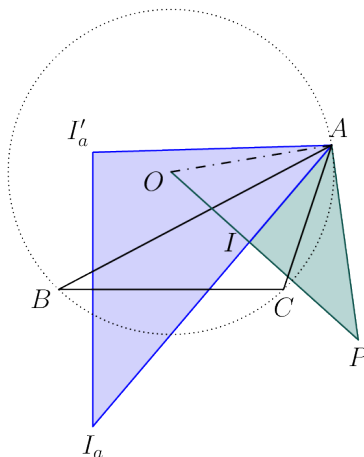
By Power of a Point,  $BM^2 = BN \cdot BT = \frac{BT^2}{2}$ , so  $\frac{BT}{BM} = \sqrt{2}$ .



# 1.4 SL 2016/G7

Let  $I$  be the incentre of a non-equilateral triangle  $ABC$ ,  $I_A$  be the  $A$ -excentre,  $I'_A$  be the reflection of  $I_A$  in  $BC$ , and  $l_A$  be the reflection of line  $AI'_A$  in  $AI$ . Define points  $I_B, I'_B$  and line  $l_B$  analogously. Let  $P$  be the intersection point of  $l_A$  and  $l_B$ .

- Prove that  $P$  lies on line  $OI$  where  $O$  is the circumcentre of triangle  $ABC$ .
- Let one of the tangents from  $P$  to the incircle of triangle  $ABC$  meet the circumcircle at points  $X$  and  $Y$ . Show that  $\angle XIY = 120^\circ$ .



Redefine  $P$  as the inverse of  $I$  wrt  $(ABC)$ . For the first part we assert more strongly that:

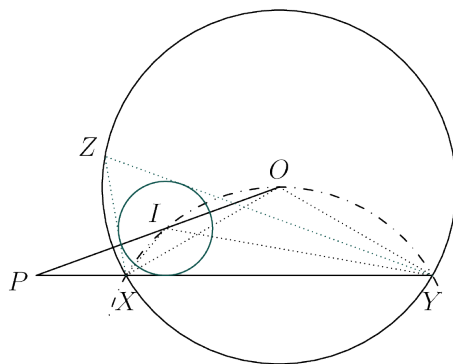
**Claim** –  $\triangle AI_A I'_A \sim \triangle API$ .

*Proof.* By angle chasing,  $\angle I_A = \angle P$  follows easily. We contend that  $I_A I'_A / I_A A = IP / AP$ ; indeed, the first ratio equals  $2 \cos \angle BI_A C = 2 \sin \frac{A}{2}$  because of similar triangles  $I_A BC \sim \triangle I_A I_B I_C$ , while

$$\frac{IP}{AP} = \frac{OP}{AP} - \frac{OI}{OA} \frac{OA}{AP} = \frac{OA}{AI} - \frac{OI^2}{OA \cdot AI} = \frac{R}{AI} - \frac{R^2 - 2rR}{R \cdot AI} = \frac{R - (R - 2r)}{AI} = 2 \sin \frac{A}{2},$$

so the ratios are equal. The similarity follows by SAS.  $\square$

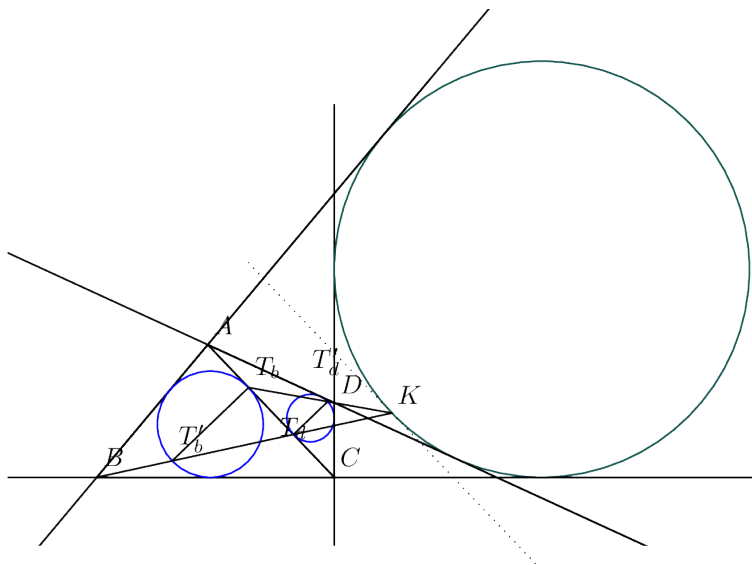
The claim clearly implies the isogonality.



For the second part, using Poncelet, let  $Z \in (ABC)$  be the unique point so that  $\triangle XYZ, ABC$  share a incircle and circumcircle. Inverting “ $P, X, Y$  collinear” wrt the circumcircle gives  $O, I, X, Y$  concyclic, or  $\angle XOY = \angle XIY$ . As it’s well-known that  $\angle XOY = 2\angle Z$  and  $\angle XIY = (\pi + \angle Z)/2$ , we must have  $\angle Z = \pi/3 \Rightarrow \angle XIY = 2\pi/3$  as needed.

### 1.5 IMO 2008/6, by Vladimir Shmarov

Let  $ABCD$  be a convex quadrilateral with  $BA \neq BC$ . Denote the incircles of triangles  $ABC$  and  $ADC$  by  $\omega_1$  and  $\omega_2$  respectively. Suppose that there exists a circle  $\omega$  tangent to ray  $BA$  beyond  $A$  and to the ray  $BC$  beyond  $C$ , which is also tangent to the lines  $AD$  and  $CD$ . Prove that the common external tangents to  $\omega_1$  and  $\omega_2$  intersect on  $\omega$ .



Rename  $\omega_1, \omega_2$  to  $\omega_b, \omega_d$ ; by Pitot-like reasoning we have  $AB + AD = CB + CD$ ; let  $T_b, T_d$  be the intouch points on  $\overline{AC}$ ; then  $T_b, T_d$  are isotomic by the obtained length condition.

If we let  $T'_b, T'_d$  be the antipodes of  $T_b, T_d$  on their respective circles, then an EGMO lemma (ch4) implies that  $B, T_b, T'_b$  and sym variant are collinear.

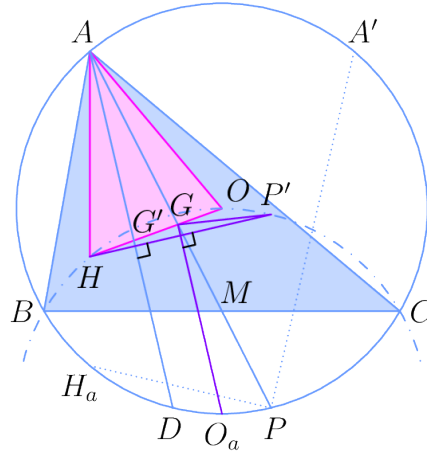
Construct the point  $K'$  on the "closer" side to the rest of the figure so that the tangent to  $\omega$  at  $K$  is parallel to  $\overline{AC}$ . Then by homothety  $K' \in \overline{BT_b}, \overline{DT_b}$ , so this is the desired exsimilicenter.

# 1.6 EGMO 2015/6

Let  $H$  be the orthocentre and  $G$  be the centroid of acute-angled triangle  $ABC$  with  $AB \neq AC$ . The line  $AG$  intersects the circumcircle of  $ABC$  at  $A$  and  $P$ . Let  $P'$  be the reflection of  $P$  in the line  $BC$ . Prove that  $\angle CAB = 60^\circ$  if and only if  $HG = GP'$ .

I'm just gonna do the 'only if' and not the 'if'.

CyclicISLscelesTrapezoid



Let  $\ell$  be the perpendicular bisector of  $\overline{BC}$ . Then we unconditionally have:

**Claim** –  $\overline{P'H}$  is perpendicular to the  $A$ -symmedian.

*Proof.* Reflect! Reflect! Reflect! Let  $D$  be the intersection of the  $A$ -symmedian with  $(ABC)$  aka the reflection of  $P$  in  $\ell$ ,  $H_a \in (ABC)$  be the reflection of  $H$  in  $\overline{BC}$ ,  $A'$  be the reflection of  $A$  in  $\ell$  aka the antipode of  $H_a$ .

$$\begin{aligned} \angle(\overline{AD}, \overline{P'H}) &= \angle(\overline{AD}, \overline{BC}) + \angle(\overline{BC}, \overline{P'H}) \stackrel{\text{reflects}}{=} -\angle(\overline{A'P}, \overline{BC}) - \angle(\overline{BC}, \overline{PH_a}) \\ &= -\angle A'PH_a = 90^\circ. \end{aligned} \quad \square$$

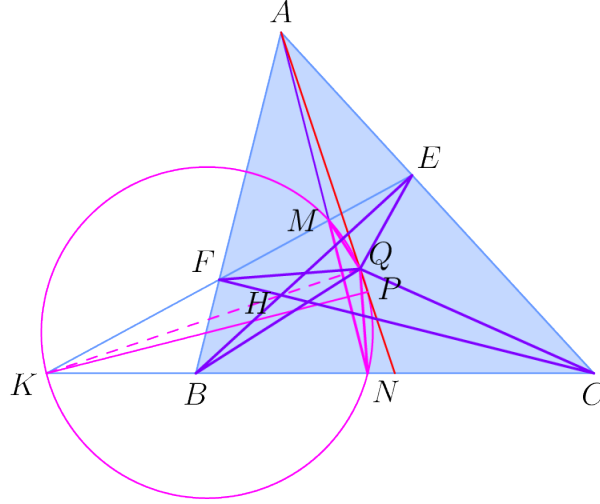
It's easy to see that  $O_a$  – the reflection of the circumcenter  $O$  in  $\overline{BC}$  – is the center of  $(BHP'C)$ ;  $\Rightarrow O_aH = O_aP = R$  unconditionally. The given length condition is thus equivalent to  $\boxed{\overline{O_aG} \perp \overline{HP'}}$ , which (by the claim) is in turn equivalent to  $\boxed{\overline{O_aG} \parallel \overline{AD}}$ .

Reflecting yet again, this time in the nine-point center,  $(\dots) \iff \boxed{A, G', D \text{ collinear}}$ , where  $G' = 2N - G = O + H - G$ .

$$\begin{aligned} &\iff \boxed{\overline{AG}, \overline{AG'} \text{ both isogonal and isotomic in } \triangle AHO}; \\ &\iff \boxed{AH = AO} \iff \boxed{\angle BAC = 60^\circ}. \end{aligned}$$

### 1.7 Iran TST 2018/1/4, by Iman Maghsoudi & Hooman Fattahi

Let  $ABC$  be a triangle ( $\angle A \neq 90^\circ$ ), with altitudes  $\overline{BE}$ ,  $\overline{CF}$ . The bisector of  $\angle A$  intersects  $\overline{EF}$ ,  $\overline{BC}$  at  $M$ ,  $N$ . Let  $P$  be a point such that  $\overline{MP} \perp \overline{EF}$  and  $\overline{NP} \perp \overline{BC}$ . Prove that  $\overline{AP}$  bisects  $\overline{BC}$ .



Construct  $K = \overline{EF} \cap \overline{BC}$ ,  $Q$  as the  $A$ -Humpty point,  $H$  as the orthocenter of  $\triangle ABC$ , and  $\omega = (KMN)$ , so that the  $P$  given is the antipode of  $K$  on it. Let spiral similarity  $s$  at  $Q$  take  $(E, F) \rightarrow (B, C)$ . The main point of the problem is then:

**Claim –**  $MKQN$  cyclic. In other words,  $Q \in \omega$ .

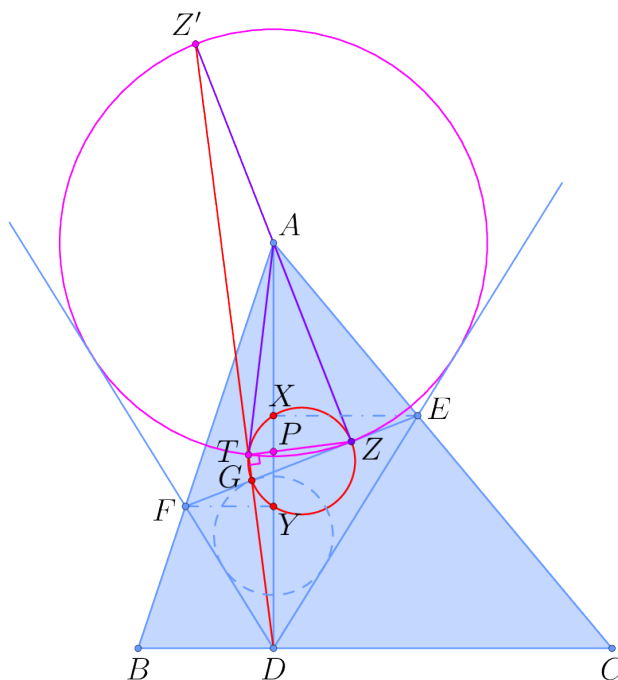
*Proof.* From angle bisector theorem,

$$\frac{EM}{MF} = \frac{EA}{AF} = \frac{BA}{AC} = \frac{BN}{NC} \Rightarrow (M \xrightarrow{s} N) \Rightarrow \angle MQN = -\arg(s) = \angle(\overline{EF}, \overline{BC}) = \angle MKN. \quad \square$$

Since  $P$  is the antipode of  $K$  on  $\omega$ ,  $\angle KQP = 90^\circ = \angle KQA$ , implying that  $P \in \overline{AQ}$ , the  $A$ -median.

### 1.8 Mock AIME 2019/15', by Eric Shen & Raymond Feng

In  $\triangle ABC$ , let  $D, E, F$  be the feet of the altitudes from  $A, B, C$  respectively, and let  $O$  be the circumcenter. Let  $Z = \overline{AO} \cap \overline{EF}$ . There exists a point  $T$  such that  $\angle DTZ = 90^\circ$  and  $AZ = AT$ . If  $P = \overline{AD} \cap \overline{TZ}$ , and  $Q$  lies on  $\overline{EF}$  such that  $\overline{PQ} \parallel \overline{BC}$ , prove that  $\overline{AQ}$  bisects  $\overline{BC}$ .



Construct points  $X, Y$  as the projections of  $E, F$  onto  $\overline{AD}$  respectively. \*

After drawing a diagram on Geogebra, we obtain:

#### Characterization of T

$T$  is the harmonic conjugate of  $Z$  wrt  $XY$  – i.e. it lies on  $\omega = (XYZ)$  so that the resulting quadrilateral is harmonic.

In American style, we show that this choice of  $T$  lies on  $\omega_a$  (defined as the circle at  $A$  thru  $Z$ ) and  $(DZ)$ ,

#### Verification (inspired by USA TST 2015/1)

For  $AZ = AT$ , we use power of a point / length chase to get  $AZ^2 = AX \cdot AY$  whence  $\overline{AZ}$  touches  $\omega$ . Hence, by harmonics  $\overline{AT}$  is also tangent to  $\omega$ , so this property follows.

$\angle DTZ = 90^\circ$  is much less straightforward. We define  $Z' = 2A - Z$  and  $G = E + F - Z$  as the antipodes of  $Z$  on the circle at  $A$  through  $Z$ . By a well-known lemma,  $D, Z', G$  collinear (along the cevian through the intouch point in  $\triangle DEF$ ).

But also at the same time,  $T$  is on  $\omega, \omega_a \Rightarrow \angle ZTG = \angle ZTZ' = \pi/2$  due to antipodes. Hence,  $\angle DTZ = \pi/2$ , completing the verification.

\*Eric Shen originally included these points in the problem statement (as seen in the 2019 version of “Geometry At Its Best”), but I guess the problem’s made harder by deleting them. For me, thinking about their relevance/use was important in making nonzero progress on the problem.

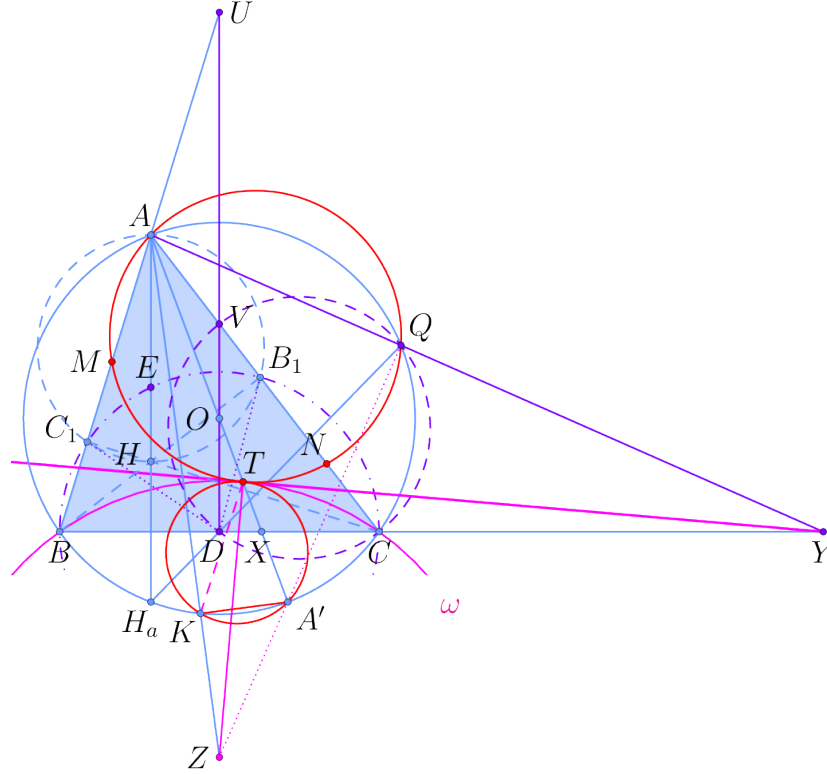
By this definition, we clearly have  $(AP; XY) = -1$ . From here (the chase is best discovered backwards), harmonic chasing suffices. Define  $K = \overline{EF} \cap \overline{A\infty_{BC}}$ . Then the bisection is established by

$$(\overline{AQ} \cap \overline{BC}, \infty_{BC}; B, C) \stackrel{A}{=} (QK; FE) \stackrel{\infty_{BC}}{=} (PA; YX) = -1.$$

### 1.9 China TST 2015/2/3

Let  $\triangle ABC$  be an acute triangle with circumcenter  $O$  and centroid  $G$ . Let  $D$  be the midpoint of  $\overline{BC}$ , and  $E$  be on  $(BC)$  with  $\overline{AE} \perp \overline{BC}$ . Let  $F = \overline{EG} \cap \overline{OD}$ . Define points  $K, L \in \overline{BC}, M \in \overline{AB}, N \in \overline{AC}$  with  $\overline{FK} \parallel \overline{OB}, \overline{FL} \parallel \overline{OC}$ , and  $\overline{MK}, \overline{NL} \perp \overline{BC}$ .

Let  $\omega$  be the circle tangent to  $\overline{OB}, \overline{OC}$  at  $B, C$  respectively. Prove that  $(AMN)$  is tangent to  $\omega$ .



As much of the parallel/perpendicular lines aren't even relevant (just to give us some equal ratios), we simplify as follows:

#### Problem reworded

In acute  $\triangle ABC$  with circumcenter  $O$  and orthocenter  $H$ ,  $D$  is the midpoint of  $\overline{BC}$ , and the altitude from  $A$  meets  $(BC)$  at  $E$  (either one works). Let  $U, V = \overline{OD} \cap \overline{AB}, \overline{AC}$ , respectively; define  $M, N \in \overline{AB}, \overline{AC}$  with (lengths directed)

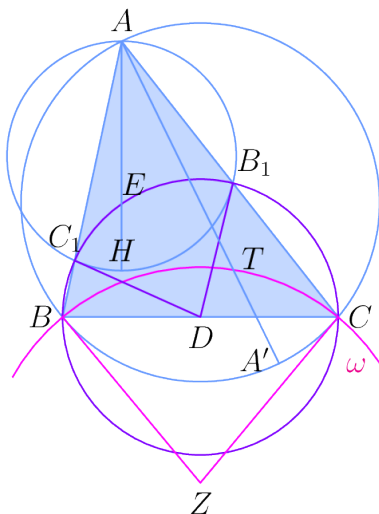
$$UM/MB = VN/NC = AE/EH.$$

Let  $\omega$  be the circle tangent to segments  $OB, OC$  at  $B, C$  respectively. Prove that  $(AMN)$  is tangent to  $\omega$ .

We define a load of new points as follows:

- $A' = 2O - A$  as the antipode of  $A$  on  $(ABC)$ ;
- $T = \overline{AO} \cap \omega$ , which we stipulate to be on segment  $AA'$  iff  $E$  is on segment  $AH$ ; WLOG, assume this is the case;
- $Q$  as the harmonic conjugate of  $A'$  wrt  $BC$ , aka the reflection of the  $A$ -orthocenter Miquel point  $Q_a$  in the perpendicular bisector  $\overline{DUV}$  of  $\overline{BC}$ .





First, we get rid of  $E$ :

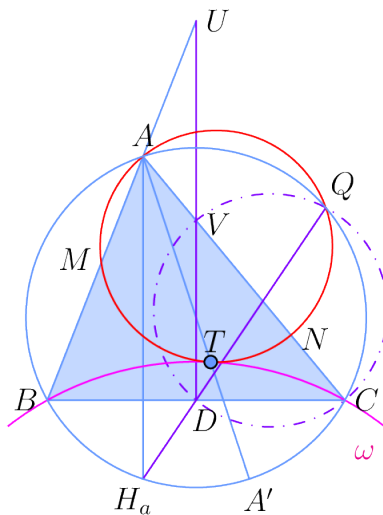
**Claim 1** –  $AE/EH = AT/TA'$ . (lengths still directed)

*Proof.* (by **v4913**) Let  $B_1, C_1$  denote the feet of the respective altitudes from  $B, C$ , and  $r$  a reflection in the bisector of  $\angle A$  composed with a homothety at  $A$  with scale factor  $AH/AA' = AB_1/AB = AC_1/AC$ .

Because  $\overline{DB_1}, \overline{DC_1}$  are well-known to touch  $(AH)$ ,  $D$  is the pole of  $\overline{B_1C_1}$ ;

$$\Rightarrow (Z \xrightarrow{r} D) \Rightarrow (\omega \xrightarrow{r} (BC)) \Rightarrow (T_1 \xrightarrow{r} E_1)$$

proving the claim. □



**Claim 2** –  $Q$  is the Miquel point of  $ABCDUV$ .

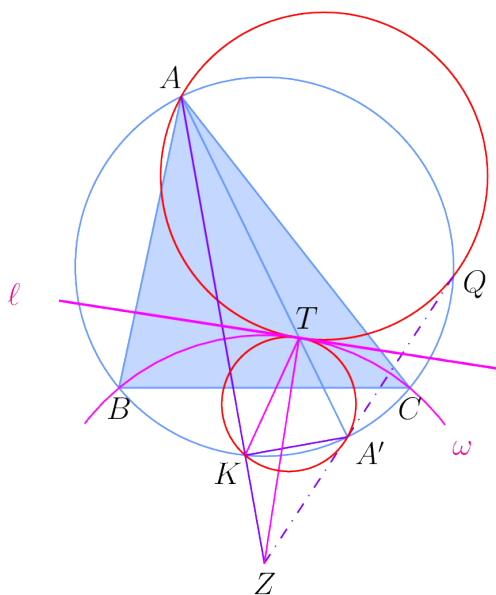
*Proof.* As we already have  $Q \in (ABC)$ , sufficient to prove  $QDVC$  cyclic. Observe that  $Q \in \overline{H_aD}$ , which follows by  $Q_a \in \overline{A'PH}$  reflected in  $\overline{DUV}$ . The result holds by Reim because  $AH_aQC$  cyclic and  $\overline{DV} \parallel \overline{AH_a}$ . □

Consider the spiral similarity  $s$  at  $Q$  mapping  $B, C \rightarrow U, V$ . Since  $\triangle BA'C \stackrel{+}{\sim} \triangle UAV$ ,  $(A' \xrightarrow{s} A)$ . By the length condition  $(M \xrightarrow{s} N)$  as well, so  $M, N \in (AQT)$ .

Finally, we turn to the problem statement:

**Claim 3** –  $AQT_1$  touches  $\omega$  at  $T_1$ .

We present two finishes.

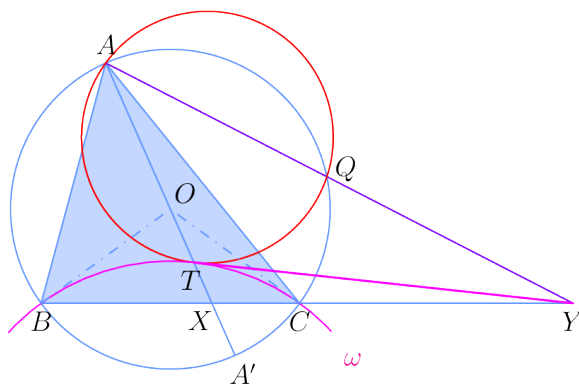


**Proof 1, by inversion (v4913)** Let  $Z \in \overline{QA'}$  be the center of  $\omega$  aka the polar of  $\overline{BC}$  wrt  $(ABC)$ , and  $*$  denote inversion in  $\omega$ . Define  $K = \overline{AZ} \cap (ABC)$  ( $\neq A$ ) =  $A^*$ . Clearly,  $(A'Q; BC) = -1 \Rightarrow A' = Q^*$ . Finally, let  $\ell \perp \overline{ZT}$  denote the tangent to  $\omega$  at  $T$ .

It remains to prove that  $(A'KT) = (AQT)^*$  touches  $\ell$  at  $T$  (and thus  $\omega$  as well). We do so by angle chase:

$$\angle(\overline{KT}, \ell) = 90^\circ + \angle KTZ \stackrel{\text{inversion}}{=} 90^\circ + \angle ZAA' = \angle KA'T;$$

inverting back completes the problem.



**Proof 2, by polars (crazyeyemoody907)** Let  $X = \overline{AO} \cap \overline{BC}$ , and  $Y$  be the pole of  $\overline{AO}$  wrt  $\omega$ , so that  $\overline{YT}$  touches  $\omega$ . Since  $\overline{AO}$  contains the pole  $O$  of  $\overline{BC}$  wrt  $\omega$ , we also  $Y \in \overline{BC}$  by La Hire.

Finally, we contend that  $A, Q, Y$  collinear. Indeed, this follows from

$$(\overline{AY} \cap (ABC), A'; B, C) \stackrel{A}{=} (YX; BC) = -1$$

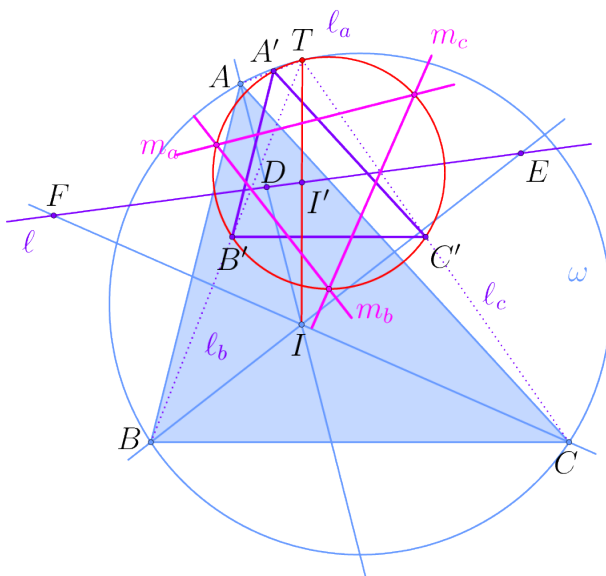
where the last harmonic bundle holds by definition of polar.

We finish by power of a point at  $Y$ :  $YT^2 = YB \cdot YC = YA \cdot YQ$  means that  $(AQT), \omega, \overline{YT}$  are tangent at  $T$ .  $\square$

*Remark.* Should definitely use the first diagram for intimidation purposes.

### 1.10 SL 2018/G5, by Denmark

Let  $ABC$  be a triangle with circumcircle  $\omega$  and incenter  $I$ . A line  $\ell$  meets the lines  $AI$ ,  $BI$ ,  $CI$  at points  $D$ ,  $E$ ,  $F$  respectively, all distinct from  $A$ ,  $B$ ,  $C$ ,  $I$ . Prove that the circumcircle of the triangle determined by the perpendicular bisectors of  $\overline{AD}$ ,  $\overline{BE}$ ,  $\overline{CF}$  is tangent to  $\omega$ .



Solution discovered by **TheUltimate123**.

Let  $\ell_a$  and cyclic variants be the reflections of  $\ell$  in the perpendicular bisectors  $x_a$  of  $\overline{AD}$ , etc.

**Claim** –  $\ell_a, \ell_b, \ell_c, \omega$  concur at a point  $T$ .

*Proof.* Because

$$\angle(\ell_b, \ell_c) = 2\angle(x_b, x_c) = 2\angle BIC = \angle BAC,$$

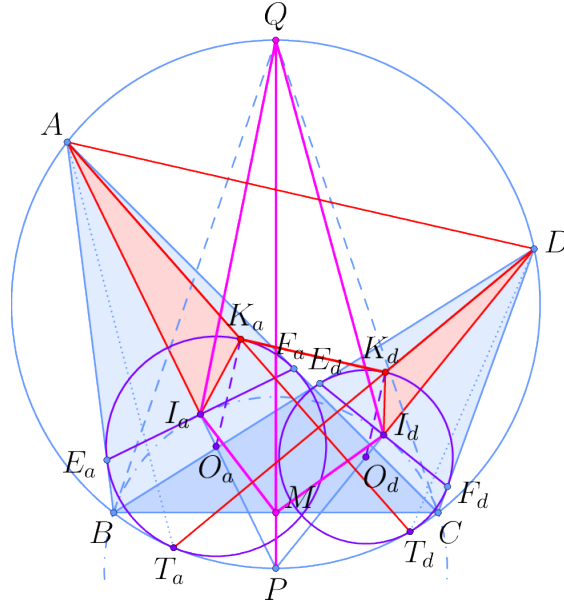
$\ell_b \cap \ell_c \in \omega$ ; the result follows by symmetry.  $\square$

Let  $I' = \overline{TI} \cap \ell$ , and consider the homothety  $h$  at  $T$  mapping  $I \rightarrow I'$ . Let  $P'$  denote the image of point  $P$  under  $h$ , so  $I'$  is the incenter of  $\triangle A'B'C'$ . Since  $\overline{A'T'} \parallel \overline{ADI}$  while  $A' \in \ell_a$  and  $I' \in \ell$ ,  $m_a$  is also the perpendicular bisector of  $\overline{AI'}$ .

From here it follows that the pairwise intersections of  $m_a, m_b, m_c$  are just the arc midpoints in  $(A'B'C')$ . By  $h$ ,  $(A'B'C')$ ,  $(ABC)$  tangent at  $T$ , hence done.

### 1.11 MOP 2020/1Z, by Evan Chen

Let  $ABCD$  be a quadrilateral inscribed in circle  $\Omega$ . Circles  $\omega_A$  and  $\omega_D$  are drawn internally tangent to  $\Omega$ , such that  $\omega_A$  is tangent to  $\overline{AB}$  and  $\overline{AC}$  while  $\omega_D$  is tangent to  $\overline{DB}$  and  $\overline{DC}$ . Prove that we can draw a line parallel to  $\overline{AD}$  which is simultaneously tangent to both  $\omega_A$  and  $\omega_D$ .



Solution by **v4913**. Define...

- $P, Q$  as the respective midpoints of  $\widehat{BC}, \widehat{BAC}$ ,  $I_a, I_d$  as the respective incenters of  $\omega_a, \omega_d$ , and  $M$  as the midpoint of  $\overline{BC}$ ;
- $O_a, O_d$  as respective centers of  $\omega_a, \omega_d$ , and  $\gamma = (BI_a I_d C)$  (with center  $P$ ), so that  $\overline{QB}, \overline{QC}$  touch  $\gamma$ ;
- $E_a, F_a, T_a = \omega_a \cap \overline{AB}, \overline{AC}, \Omega$ ;  $K_a$  as the intersection of  $\overline{AT_d}$  with  $\omega_a$  closer to  $A$ , and their symmetric variants. It's well-known that  $Q, I_a, T_a$  collinear, and that  $I_a$  is the midpoint of  $\overline{E_a F_a}$ ;
- $s_a$  as the spiral similarity mapping  $\gamma \rightarrow \omega_a$  and thus  $Q, B, C, M \rightarrow A, E_a, F_a, I_a$ . Since  $\angle K_a A F_a = \frac{1}{2} \widehat{T_d C} = \angle I_d Q C$  by design, we also have  $(K_a \xrightarrow{s_a} I_d)$ .

We contend that  $\overline{K_a K_d}$  is the desired tangent, using the following two parts:

**Claim 1** –  $\overline{O_a K_a}, \overline{O_d K_d} \perp \overline{AD}$ .

*Proof.* We angle chase:

$$\angle(\overline{O_a K_a}, \overline{AD}) = \angle O_a K_a A + \angle K_a A D \stackrel{s_a}{=} \angle P I_d Q + \angle T_d Q D = \angle(\overline{P I_d}, \overline{Q D}) = \frac{1}{2} \widehat{P Q} = 90^\circ.$$

The claim follows by symmetry. □

**Claim 2** –  $\overline{K_a K_d} \parallel \overline{AD}$ .

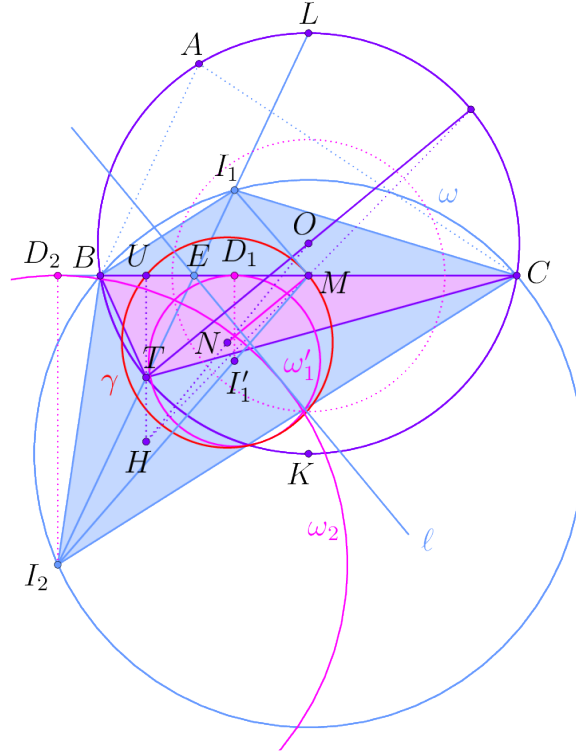
*Proof.* Let  $X = \overline{AT_d} \cap \overline{DT_a}$ , so that it'll suffice to prove  $AK_a/AX = DK_d/DX$ . Indeed, using  $s_a$ ,  $AK_a = QI_d \cdot \frac{AI_a}{QM}$  and similarly  $DK_d = QI_a \cdot \frac{DI_d}{QM}$ . We thus have:

$$\frac{AK_a}{DK_d} = \frac{AI_a/QI_a}{DI_d/QI_d} = \frac{AT_a/QP}{DT_d/QP} = \frac{AX}{DX}. \quad \square$$

From the previous two claims,  $\overline{O_a K_a}, \overline{O_d K_d} \perp \overline{K_a K_d}, \overline{AD}$  so  $\overline{K_a K_d}$  touches both  $\omega_a, \omega_d$  while also parallel to  $\overline{AD}$ , as required.

### 1.12 (Source?)

Let  $ABC$  be a triangle and let  $T$  be the contact point of the  $A$ -mixtilinear incircle with the circumcircle, and let  $T'$  be the reflection of  $T$  over  $BC$ . Prove that the nine-point circle of  $T'BC$  is tangent to the incircle.



Let  $K, L, M$  be the midpoints of  $\widehat{BTC}$ ,  $\widehat{BAC}$ , and  $I_1$  be the incenter, so that  $\omega = (BI_1C)$ ; then, let  $I_2 = \overline{NT} \cap \omega$  ( $\neq I_1$ ). Clearly, since  $\overline{LB}, \overline{LC}$  touch  $\omega$ ,  $(BC; I_1I_2) = -1$ . Additionally, since  $\angle KTL = 90^\circ$ ,  $T$  is the midpoint of  $\overline{I_1I_2}$ , a Dumpty point...

Recall the following lemma:

**Lemma -** In  $\triangle ABC$  with  $A$ -Dumpty point  $X$ ,  $\overline{AX}$  bisects  $\angle BXC$ .

Reflect the nine-point circle given to obtain the nine-point circle of  $\triangle TBC$ . We may now safely get rid of  $A$ :

#### Problem simplified

In harmonic quadrilateral  $BI_1CI_2$ ,  $T$  is the midpoint of  $\overline{I_1I_2}$ , and  $I_1'$  is the reflection of  $I_1$  in  $\overline{BC}$  (aka the  $I_2$ -Dumpty point in  $\triangle I_2BC$ ). For  $k \in \{1, 2\}$  let  $D_k$  be the foot from  $I_k$  to  $\overline{BC}$ , and  $\omega_k$  the circle at  $I_k$  through  $D_k$ . Then the circle  $\omega_1'$  at  $I_1'$  through  $D_1$  touches the nine-point circle  $\gamma$  of  $\triangle TBC$ .

Recalling the proof of Feuerbach by inverting about the midpoint of a side, we do likewise here. Define...

- $U$  as foot from  $T$  to  $\overline{BC}$ , and  $E = \overline{I_1I_2} \cap \overline{BC}$ . By midpoints of harmonic bundles lemma applied to  $(D_1D_2; ME) \stackrel{\infty_{BC}}{\equiv} (I_1I_2; NE) = -1$ , we have  $ME \cdot MU = MD_1 \cdot MD_2$ .
- $N, O$  as respective centers of  $\gamma, (BTC)$ , and  $H$  as orthocenter of  $\triangle BTC$ ;

$M$  is the exsimilicenter of  $\omega'_1, \omega_2$  because it lies on the line of centers  $\overline{I'_1 I_2}$  as well as the common external tangent  $\overline{BC}$ . Let  $*$  denote inversion at  $M$  with power  $ME \cdot MU = MD_1 \cdot MD_2$ , so that  $\omega'_1 = \omega_2^*$ .

**Claim –** The reflection  $\ell$  of  $\overline{BC}$  in  $\overline{I_1 I_2}$  is  $\gamma^*$ .

*Proof.* Since  $E = U^*$ , it suffices to prove that  $\overline{MN} \perp \ell$ . Indeed,  $MN \parallel \overline{TO}$  by homothety at  $H$ , while reflecting  $\overline{TU} \perp \overline{BC}$  in the  $T$ -angle bisector  $\overline{I_1 T X_2}$  gives  $\overline{TO} \perp \ell$ .  $\square$

Observe by symmetry about  $\overline{I_1 I_2}$  that  $\ell$  also touches  $\omega_2$ . Inverting back, we have  $\gamma$  tangent to  $\omega'_1$  as required.



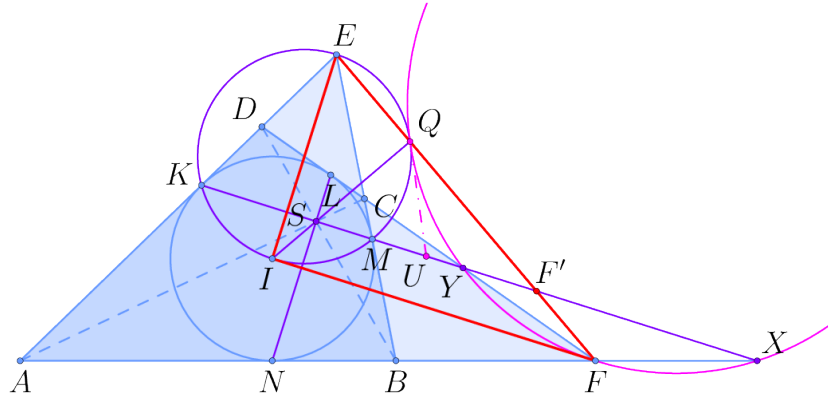
### 1.13 IGO 2020/A4

Convex circumscribed quadrilateral  $ABCD$  with its incenter  $I$  is given such that its incircle is tangent to  $\overline{AD}$ ,  $\overline{DC}$ ,  $\overline{CB}$ , and  $\overline{BA}$  at  $K$ ,  $L$ ,  $M$ , and  $N$ . Let  $E = \overline{AD} \cap \overline{BC}$  and  $F = \overline{AB} \cap \overline{CD}$ . Let  $X = \overline{KM} \cap \overline{AB}$  and  $Y = \overline{KM} \cap \overline{CD}$ . Let  $Z = \overline{LN} \cap \overline{AD}$  and  $T = \overline{LN} \cap \overline{BC}$ .

Prove that the circumcircle of triangle  $\triangle XFY$  and the circle with diameter  $EI$  are tangent if and only if the circumcircle of triangle  $\triangle TEZ$  and the circle with diameter  $FI$  are tangent.

imagine doing both directions

CyclicISLscalesTrapezoid



We contend that  $(XFY), (EI)$  tangent  $\iff \overline{KM} \perp \overline{LN}$ , which is just another expression of ‘ $ABCD$  bicentric’.

Call the incircle  $\omega$ . Define  $S = \overline{KM} \cap \overline{LN}$  and  $Q$  as the Miquel point of  $KLMN$  aka the inverse of  $S$  wrt  $\omega$ , which obviously lies on the polar  $\overline{EF}$  of  $S$  wrt  $\omega$ . It follows that  $\angle SQE = \angle SQF = 90^\circ$ .

Let  $F' = \overline{QF} \cap \overline{KM}$   $U$  be the midpoint of  $\overline{SF'}$ .

**Claim** –  $\overline{QF}$  always bisects  $\angle XQY$ , and  $UM \cdot UK = US^2 = UX \cdot UY$ , so  $U$  lies on the radical axis of  $(EIMK), (XFY)$ .

*Proof.* By Brianchon on  $ABMCDK$ ,  $S = \overline{AB} \cap \overline{CD}$  as well. Apply DIT to  $\overline{KM}$  and quadrilateral  $ABCD$  and project to  $Q$ , to obtain an involutive pairing  $i : Q(XY; SS; KM)$ . The last two pairs imply that  $i$  is just reflection in  $\overline{QS}$ , so  $\overline{QS}$  bisects  $\angle XQY$ . As  $\overline{QF} \perp \overline{QS}$ , it must also bisect the same angle:  $i : Q(F'F')$ .

By these right angles and angle bisections,  $(SF'; MK) = (SF'; XY) = -1$ , so the last result follows by midpoints of harmonic bundles lemma.  $\square$

Now because  $\angle SQF' = 90^\circ$ , we have  $UQ = US = UF' \implies (EI), (XQY)$  tangent at  $Q$ ; this means that the desired is equivalent to  $FQXY$  cyclic.

**Note.** Actually, there are two circles through  $X, Y$ , but one of them is extraneous by configuration issues.

Next, we show that this happens iff  $FX = FY$ .

**“If” direction** Since  $\angle FQX = \angle YQF$ , triangles  $FQX, FQY$  have equal circumradii, so their circumcircles either coincide or are reflections of each other in  $\overline{QF}$ . If they were to be reflections, we’d obtain two possibilities, each

absurd in the context of the problem: (i)  $Q, X, Y$  collinear  $\Rightarrow \overline{XY}, \overline{EF}$  coincide; and (ii)  $X, Y$  reflections in  $\overline{QF} \Rightarrow \overline{XY}, \overline{SQ} \perp \overline{EF}, \overline{XY} \parallel \overline{SQ}$ . Thus  $(FQX) = (FQY)$  as required.

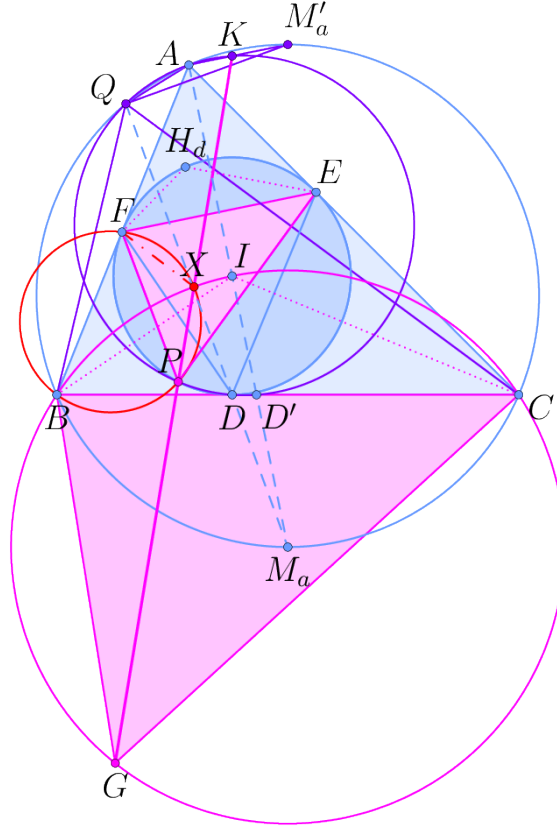
**“Only if” direction**  $\angle YQF = \angle FQX \Rightarrow \widehat{YF} = \widehat{FX}$ , trivial;

To finish the problem, observe that  $FX = FY$  is equivalent to  $\overline{MK}$  making equal angles with  $\overline{AB}, \overline{CD}$ . As  $FN = FL$  as well,  $\overline{LN}$  (always) makes equal angles with the same two lines. Since  $K, L, M, N$  form a convex polygon, this is in turn equivalent to  $\overline{KM} \perp \overline{LN}$ , completing the proof.

### 1.14 IMO 2019/6, by Anant Mudgal

Let  $I$  be the incenter of acute triangle  $ABC$  with  $AB \neq AC$ . The incircle  $\omega$  of  $ABC$  is tangent to sides  $BC$ ,  $CA$ , and  $AB$  at  $D$ ,  $E$ , and  $F$ , respectively. The line through  $D$  perpendicular to  $\overline{EF}$  meets  $\omega$  at  $R$ . Line  $AR$  meets  $\omega$  again at  $P$ . The circumcircles of triangle  $PCE$  and  $PBF$  meet again at  $Q$ .

Prove that lines  $DI$  and  $PQ$  meet on the line through  $A$  perpendicular to  $\overline{AI}$ .

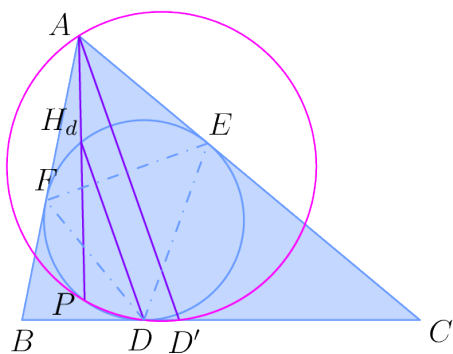


Observe that  $P$  is the  $D$ -orthocenter Miquel in  $\triangle DEF$ . Define  $K$  as the intersection of the  $A$ -external bisector with  $\overline{AD}$ . We make the following definitions...

- Let  $\omega$ ,  $\omega_a$  denote the incircle and  $(BIC)$  respectively;
- Define  $X$  as intersection of *segment*  $PK$  with  $\omega$ . Let  $Q$  instead denote the  $A$ -SD point;
- $G$  be the harmonic conjugate of  $I$  wrt  $BC$ ,  $D'$  as the foot of the  $A$ -angle bisector;  $M_a$  as the midpoint of arc  $BC$  exc.  $A$ ;  $M'_a$  as the antipode of  $M_a$  on  $(ABC)$ ;
- $H$  as orthocenter of  $\triangle DEF$ , and  $H_d$  its reflection over  $\overline{EF}$ .

$\Rightarrow$  because  $MB^2 = MD \cdot MQ = MD' \cdot MA$ ,  $Q \in (ADD'K)$ .

Thus we want to show that  $PXFB$  cyclic. ( $PXEC$  cyclic would follow from symmetry, proving that  $X$  was indeed the point constructed in the problem.)



**Claim 1 –  $P \in (ADD'KQ)$ .**

*Proof.* Observe that  $(PH_d; EF) = -1$  whence  $A, P, H_d$  collinear. Then because  $\overline{DH_d} \parallel \overline{AI}$  because both perpendicular to  $\overline{EF}$ . Hence result by degenerate Reim.  $\square$

**Claim 2 -  $\Delta PFE \stackrel{+}{\sim} \Delta GBC$ .**

*Proof.* Proceed by spiral at  $Q$ . Observe that  $\triangle H_d EF \stackrel{+}{\sim} \triangle ICB$  by angle chase. Because  $(H_d P; EF) = (IG; BC) = -1$ , the needed similarity follows.  $\square$

**Claim 3** –  $K, G, P$  collinear.

*Proof.* An angle chase, using the previous two claims:

$$\angle QPK \stackrel{\text{claim 1}}{=} \angle QAK = \angle QAM_a \stackrel{\text{spiral}}{=} \angle QPG. \quad \square$$

Using last two claims, we may angle chase:

$$\angle PXB = \angle GXB = \angle GCB \stackrel{\text{spiral}}{=} \angle PEF = \angle PFB,$$

or  $PXFB$  cyclic.

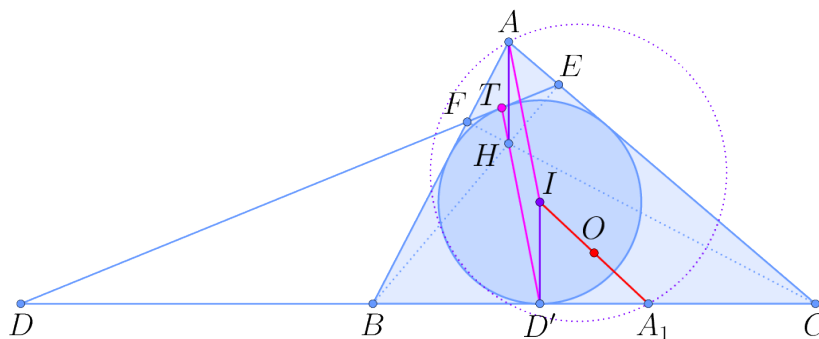
*Remark.* ggb way too op

### 1.15 MOP + USA TST, by Ankan Bhattacharya

Let  $ABC$  be a triangle with incenter  $I$ , and let  $D$  be a point on line  $BC$  satisfying  $\angle AID = 90^\circ$ . Let the excircle of triangle  $ABC$  opposite the vertex  $A$  be tangent to  $\overline{BC}$  at  $A_1$ . Define points  $B_1$  on  $\overline{CA}$  and  $C_1$  on  $\overline{AB}$  analogously, using the excircles opposite  $B$  and  $C$ , respectively.

🌲 1.15.1 MOP 2019/(?)

Let  $E, F$  be the feet of the altitudes from  $B, C$  respectively. Prove that if  $\overline{EF}$  touches the incircle, then quadrilateral  $AB_1A_1C_1$  is cyclic.



Call the incircle  $\omega$ .

**Claim 1 -**  $D, E, F$  are collinear.

*Proof.* We will prove that the tangent line from  $D$  is antiparallel to  $\overline{BC}$  wrt  $\angle A$ . Indeed, this line is found by reflecting  $\overline{DBC}$  over  $\overline{DI}$ , a line perpendicular to  $\overline{AI}$ , so we win.  $\square$

Let  $\omega$  touch  $\overline{DEF}$  at a point  $T$ , and let  $D'$  denote the  $\mathcal{A}$ -intouch point.

**Claim 2 -  $\overline{AI} \parallel \overline{HD'}$ ; hence  $AID'H$  is a parallelogram and  $AH = r$ , the inradius of  $\triangle ABC$ .**

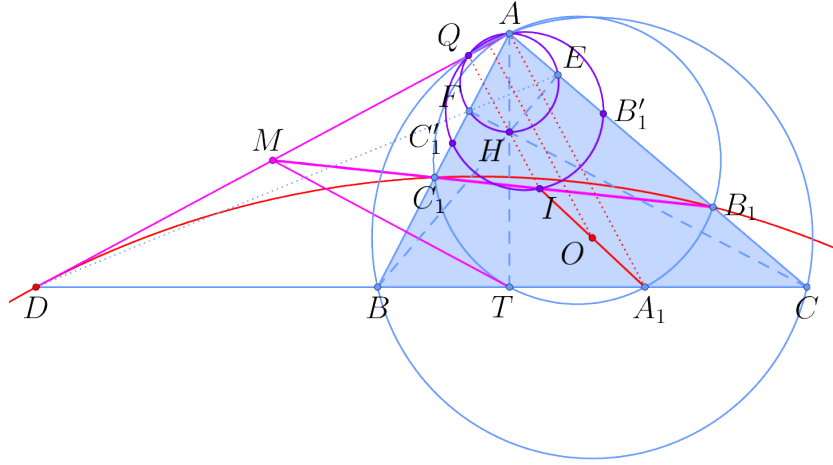
*Proof.* Because  $BCEF$  is tangential, it follows by degenerate Brianchon that lines  $BE, CF, DT'$  concur, i.e.  $H \in \overline{TD'}$ . Observe that  $DT = DD'$ ; then  $\overline{THD'} \perp \overline{DI}$  by symmetry, while  $\overline{AI} \perp \overline{DI}$  is given; the lines are thus parallel as claimed.  $\square$

Now, let  $\omega_a$ , etc denote  $(AB_1C_1)$ , etc, respectively. We observe that because the perpendicular from  $A_1$  to  $\overline{BC}$  and its cyclic variants all concur at the point  $2O - I$ , it follows that all three circles must concur at this point by Miquel spam.

But because  $r/2 = AH/2$  is the distance from  $O$  to  $\overline{BC}$ , we actually have  $2O - I = A_1$  (also because of their feet onto  $\overline{BC}$ ). Hence  $A_1 \in \omega_a$  as desired.

### 1.15.2 USA TST 2019/6

Prove that if quadrilateral  $AB_1A_1C_1$  is cyclic, then  $\overline{AD}$  is tangent to the circumcircle of  $\triangle DB_1C_1$ .



From MOP 2019, we make the following observations:

- By its converse,  $D, E, F$  collinear; then, if  $T$  is the foot from  $A$  to  $\overline{BC}$ , we have  $(TD; BC) = -1$ .
- As  $A_1$  is the Bevan point  $2O - I$ , its projections onto  $\overline{AC}, \overline{AB}$  are  $B_1, C_1$  respectively. It follows that  $A, A_1$  are antipodes on  $\omega_a$ .
- Since  $BCEF$  is bicentric, if the incircle touches  $\overline{AC}, \overline{AB}$  at  $B'_1, C'_1$ , then  $BC'_1/FC'_1 = CB'_1/EB'_1$ , so the  $A$ -incenter and orthocenter Miquel points coincide, say at  $Q \in (ABC)$ .

From the last item,  $\angle AQI = \angle AQH = 90^\circ$ .

**Claim** –  $\overline{AD}$  touches  $\omega_a$ .

*Proof.* Since  $(ABC) \cap (AH) = \{A, Q\}$ , the projection of  $O$  onto  $\overline{AQD}$  is  $\frac{A+Q}{2}$ . At the same time, the above implies  $Q$  is the projection of  $I$  onto  $\overline{AQD}$ . By linearity the projection of  $A_1 = 2O - I$  onto  $\overline{AD}$  is  $2\frac{A+Q}{2} - Q = A$  – in other words,  $\angle A_1AD = 90^\circ$ . This proves the tangency as  $\overline{AA_1}$  is a diameter of  $\omega_a$ .  $\square$

Let  $M = \frac{A+D}{2}$ , so  $\overline{MT}$  touches  $\omega_a$  as well by symmetry in the perpendicular bisector  $M \infty_{BC}$  of  $\overline{AT}$ . Now,  $(AT; B_1C_1) \stackrel{A}{=} (DT; CB) = -1$  means  $M \in \overline{B_1C_1}$ .

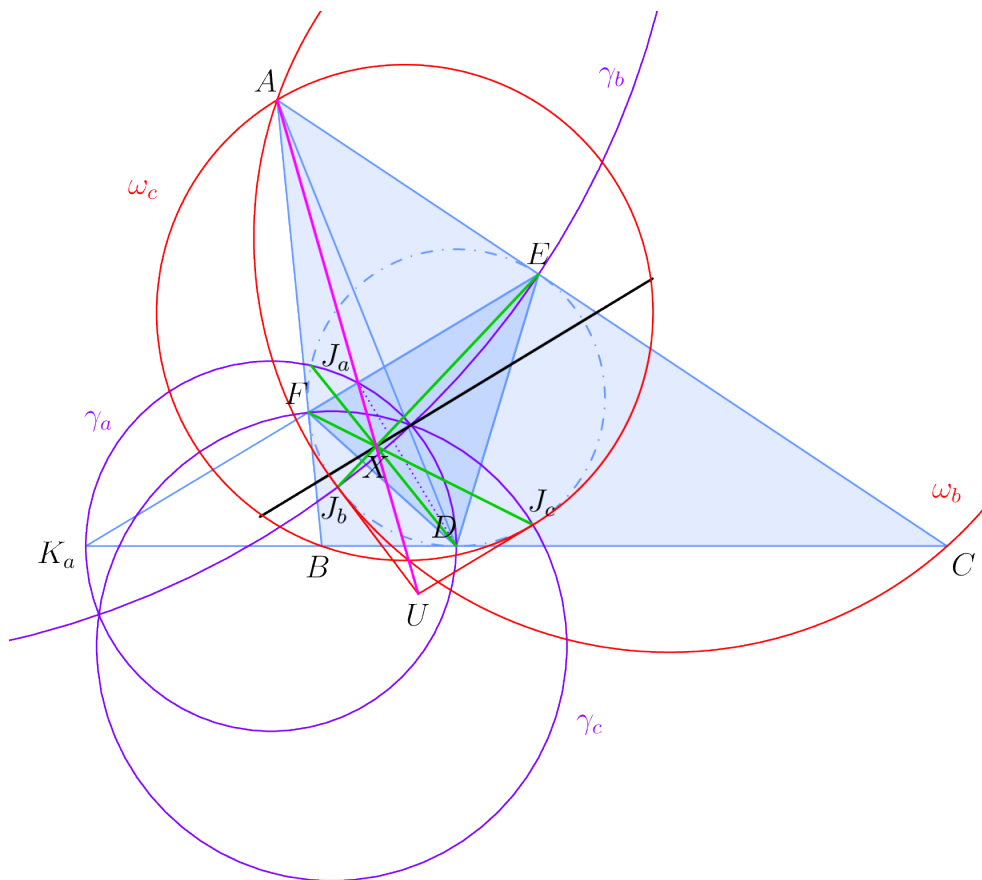
Finish by power of a point converse:  $MD^2 = MA^2 = MB_1 \cdot MC_1$  gives the needed tangency.

## 1.16 RMM + Fake USAMO

### 1.16.1 RMM 2012/6, by Fedor Ivlev

Let  $ABC$  be a triangle and let  $I$  and  $O$  denote its incentre and circumcentre respectively. Let  $\omega_A$  be the circle through  $B$  and  $C$  which is tangent to the incircle of the triangle  $ABC$ ; the circles  $\omega_B$  and  $\omega_C$  are defined similarly. The circles  $\omega_B$  and  $\omega_C$  meet at a point  $A'$  distinct from  $A$ ; the points  $B'$  and  $C'$  are defined similarly. Prove that the lines  $AA'$ ,  $BB'$  and  $CC'$  are concurrent at a point on the line  $IO$ .

Let  $K_a = \overline{EF} \cap \overline{BC}$ ,  $\gamma_a = (K_a D)$ ,  $J_a = \omega_a \cap \gamma_a \cap \omega$  (and cyclic variants), and  $H$  and  $\ell$  denote the orthocenter and Euler line of  $\triangle DEF$ , respectively. Also, let  $I_a, I_b, I_c$  be the excenters of  $\triangle ABC$ .

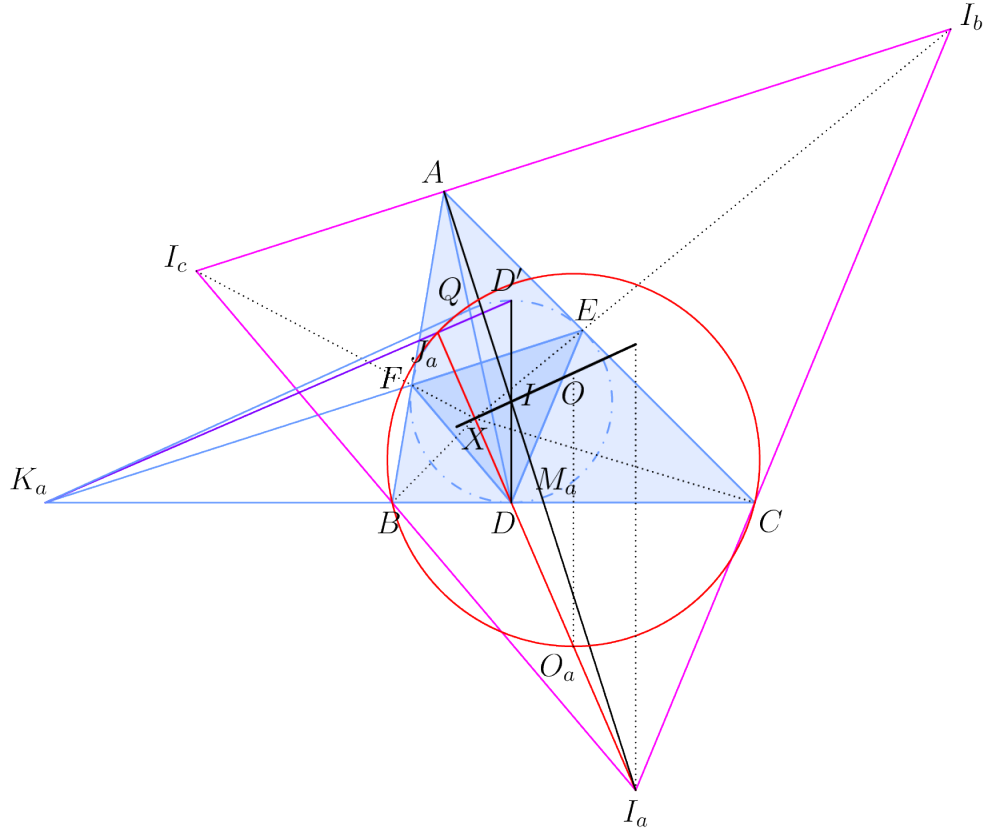


**Solution 1, by radical axes** Observe that  $\ell$  is just  $\overline{OI}$ , and that  $\gamma_a$ , etc are coaxial Apollonian circles. Define  $X$  as the radical center of  $\gamma_a, \gamma_b, \gamma_c, \omega$  (which exists since the former 3 circles are coaxial). We'll show this is the desired concurrency point in both problems. Clearly,  $\overline{DJ_a}$  is the raxis of  $(\gamma_a, \omega)$ , i.e.  $X \in \overline{DJ_a}$ .

**Lemma 1** –  $\ell$  is the raxis of  $\gamma_a$  and variants.

*Proof.* Let  $T_a$  denote the foot of  $D$  onto  $\overline{EF}$ , which is obviously on  $\gamma_a$ . Then  $H$  has power  $HD \cdot HT_a$  (= variants) wrt the  $\gamma$ 's, hence on raxis; Meanwhile  $I$  has power  $r^2$  wrt all circles by orthogonality, hence also on raxis, done.  $\square$

Let tangents to  $\omega$  at  $J_b, J_c$  meet at  $U$ ; then,  $\overline{AU}$  is the raxis of  $\omega_b, \omega_c$ . Clearly this is the polar of  $\overline{J_b J_c} \cap \overline{EF}$ . Recalling that  $X = \overline{EJ_b} \cap \overline{FJ_c}$ , follows by Brokard that  $X \in \overline{AU}$ , the end.



**Solution 2, by homothety (v4913)** Let  $D'$  be the antipode of  $D$  on  $\omega$ ,  $Q = \overline{AD} \cap \omega$  ( $\neq D$ ); then, because  $(EF; DQ) = -1$ ,  $\overline{K_aQ}$  touches  $\omega$  as well. Also, because  $\angle DJ_aD' = \angle DJ_aK_a = 90^\circ$ ,  $K_a, J_a, D'$  are collinear, whence  $(DQ; J_aD') = -1$ .

We start with  $X$  as the similicenter of homothetic triangles  $DEF, I_aI_bI_c$ . Let homothety  $h$  at  $X$  with scale factor  $r$  map  $(D, E, F) \rightarrow (I_a, I_b, I_c)$ . This must also map their circumcenters to each other, i.e.  $I \xrightarrow{H} 2O - I$ , whence  $X \in \overline{OI}$ .

Also, let  $M_a$  be the midpoint of  $\overline{BC}$ ,  $O_a \in \overline{DJ_a}$  be the midpoint of arc  $BC$  on  $\omega_a$  not containing  $J_a$  (and variants).

**Lemma 2 (SL 2002/G7)** –  $J_a, D, I_a$  collinear.

*Proof.* Harmonics:

$$-1 = (DQ; D'J_a) \stackrel{D}{=} (\overline{BC} \cap \overline{AI}, \overline{J_aD} \cap \overline{AI}; I, A),$$

implying that  $\overline{J_aD} \cap \overline{AI}$  is the  $A$ -excenter. □

Hence,  $X = \overline{DJ_a} \cap \overline{EJ_b} \cap \overline{FJ_c}$ .

**Claim** –  $O_a$  is the midpoint of  $\overline{DI_a}$ .

*Proof.* By symmetry,  $M_a$  is the foot of  $O_a$  onto  $\overline{BC}$ , while it's well-known that  $2M - D$  is the foot of  $I_a$  onto  $\overline{BC}$ .  $M$  obviously being the midpoint of the segment with endpoints  $D, 2M - D$  implies the claim by parallel lines. □



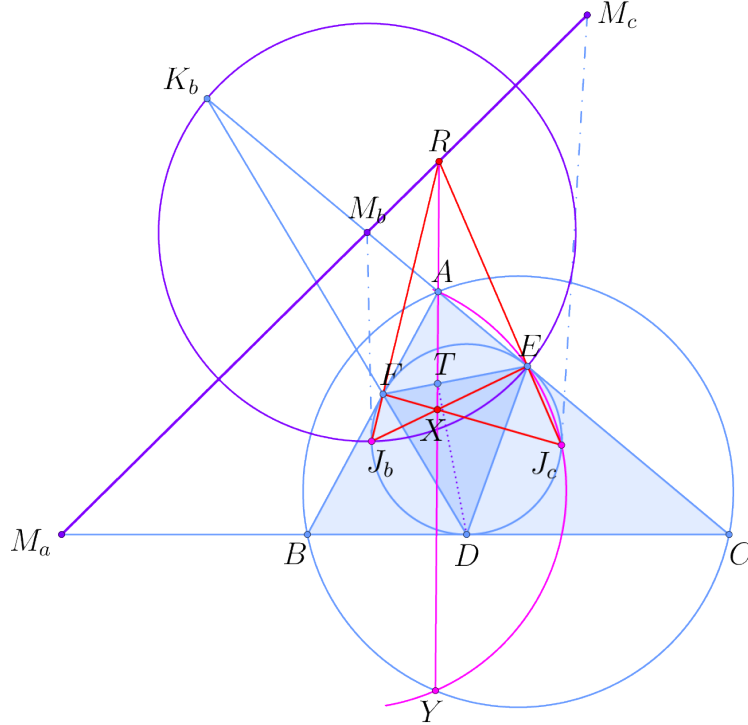
Therefore there must exist a homothety  $h'$  at  $X$  with scale factor  $(1+r)/2$ , mapping  $(D, E, F) \rightarrow (O_a, O_b, O_c)$ .  
To show that our  $X$  is indeed the radical center of  $\omega_a, \omega_b, \omega_c$ , compute

$$\text{Pow}(X, \omega_a) = XJ_a \cdot XO_a \stackrel{h'}{=} \frac{1+r}{2} XJ_a \cdot XD = \frac{\text{Pow}(X, \omega)}{(r+1)/2},$$

a symmetric quantity wrt  $a, b, c$ .

### 1.16.2 Fake USAMO 2020/3 (author?)

Let  $\triangle ABC$  be a scalene triangle with circumcenter  $O$ , incenter  $I$ , and incircle  $\omega$ . Let  $\omega$  touch the sides  $\overline{BC}$ ,  $\overline{CA}$ , and  $\overline{AB}$  at points  $D$ ,  $E$ , and  $F$  respectively. Let  $T$  be the projection of  $D$  to  $\overline{EF}$ . The line  $AT$  intersects the circumcircle of  $\triangle ABC$  again at point  $X \neq A$ . The circumcircles of  $\triangle AEX$  and  $\triangle AFX$  intersect  $\omega$  again at points  $P \neq E$  and  $Q \neq F$  respectively. Prove that the lines  $EQ$ ,  $FP$ , and  $OI$  are concurrent.



Retain the point definitions from the previous two problems, renaming the  $X$  given in the problem to  $Y$ . For consistency of notation we let  $X$  denote the simlicenter of triangles  $I_a I_b I_c$ ,  $DEF$  (as before). We'll show that  $AEYJ_c$  cyclic –  $P, Q$  are just the  $J_b, J_c$  from earlier.

**Claim 1** –  $\overline{EJ_c}, \overline{FJ_b}, \overline{AT}$  are concurrent at some point  $R$  on the polar of  $X$  wrt  $\omega$ .

*Proof.* I claim that  $\overline{AT}$  is the polar of  $\overline{EF} \cap \overline{J_b J_c}$  wrt  $\omega$ . Indeed, this is just Brokard. By Brokard again,  $\overline{EJ_b} \cap \overline{FJ_c}$  is on  $\overline{AT}$  as well as the polar of  $X$  wrt  $\omega$ .  $\square$

Let  $M_a$  be midpoint of  $\overline{K_a D}$  (and cyclic variants).

**Lemma** – The polar of  $X$  wrt  $\omega$  is the radical axis of  $\Omega, \omega$ .

*Proof.* As  $\gamma_b \perp \omega$ , and  $\overline{M_b E}$  touches  $\omega$ ,  $\overline{M_b J_b}$  must also touch it; in other words  $M_b$  is the pole of  $\overline{EJ_b}$  wrt  $\omega$ . As  $X \in \overline{EJ_b}$ ,  $\overline{M_a M_b M_c}$  is the polar of  $X$  wrt  $\omega$  by la Hire.

It remains to prove that  $M_a$  (and thus cyclic variants by symmetry) is on the radical axis of  $\Omega, \omega$ . Indeed, by the midpoints of harmonic bundles lemma on  $(K_a D; BC)$ ,

$$\text{Pow}(M, \omega) = M_a D^2 = MB \cdot MC = \text{Pow}(M, \Omega)$$

$\square$

From the previous two claims,

$$RA \cdot RY = \text{Pow}(R, \Omega) = \text{Pow}(R, \omega) = RE \cdot RJ_c \Rightarrow AEJ_c Y \text{ cyclic,}$$

completing the proof.

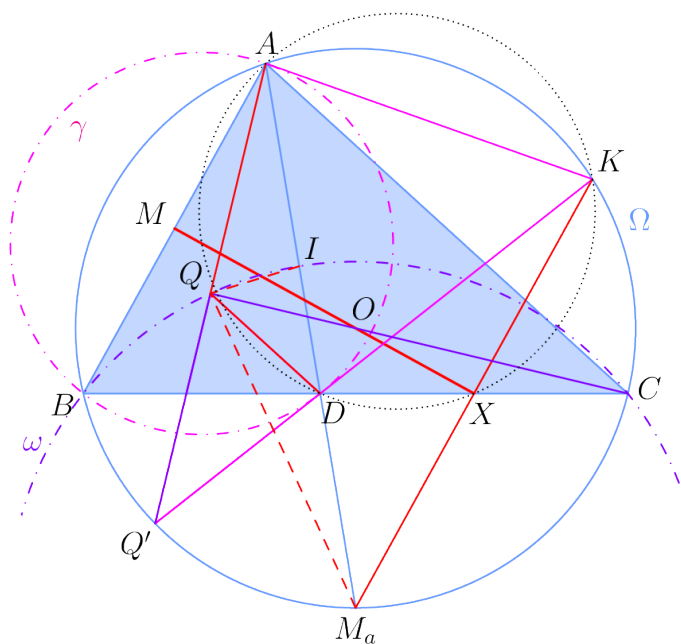
### 1.17 TSTST 2018/3, by Evan Chen & Yannick Yao

Let  $ABC$  be an acute triangle with incenter  $I$ , circumcenter  $O$ , and circumcircle  $\Gamma$ . Let  $M$  be the midpoint of  $\overline{AB}$ . Ray  $AI$  meets  $\overline{BC}$  at  $D$ . Denote by  $\omega$  and  $\gamma$  the circumcircles of  $\triangle BIC$  and  $\triangle BAD$ , respectively. Line  $MO$  meets  $\omega$  at  $X$  and  $Y$ , while line  $CO$  meets  $\omega$  at  $C$  and  $Q$ . Assume that  $Q$  lies inside  $\triangle ABC$  and  $\angle AQM = \angle ACB$ .

Consider the tangents to  $\omega$  at  $X$  and  $Y$  and the tangents to  $\gamma$  at  $A$  and  $D$ . Given that  $\angle BAC \neq 60^\circ$ , prove that these four lines are concurrent on  $\Gamma$ .

Geometry is the science of correct reasoning on incorrect figures.

George Pólya / Henri Poincaré



The given angle condition implies  $AMQO$  cyclic, or  $\angle AQC = \angle AMO = \pi/2$ . We make the following definitions:

- $\Omega = (ABC)$ ,  $M_a$  as the center of  $\omega$  and midpoint of  $\overline{BC}$ ;
- $Q' = 2Q - A$  as the reflection of  $A$  in  $\overline{QOC}$  – this lies on  $\Omega$  by symmetry about  $\overline{CO}$ ;
- $K \in \Omega$  as the reflection of  $M_a$  in  $\overline{MO}$ , the perpendicular bisector of  $\overline{AB}$ .

In the meat of the problem we'll use the following easily findable arcs/angles frequently:

$$\widehat{BM_a} = \widehat{CM_a} = \widehat{AK} = A, \text{ and } \widehat{AC} = \widehat{CQ'} = \widehat{M_aK} \Rightarrow \angle AQ'C = B.$$

#### Observation

$\overline{QI}$  bisects  $\angle AQD$ . (Holds because  $Q \in \gamma$ , the Apollonian circle wrt  $A, D$  through  $I$ .)

**Claim 1** -  $\triangle QQ'D \stackrel{+}{\sim} \triangle M_a AC$ .

*Proof.* First, we'll show  $\angle QQ'D = \angle B$ , a massive angle chase:

$$\begin{aligned} \angle M_a AQ &= \angle CAQ' - \angle CAM_a = B - \frac{A}{2}, \text{ and } \angle M_a IQ = \frac{\pi - \angle IM_a Q}{2} = \frac{\pi}{2} - \angle ICO = B + \frac{C}{2}; \\ \Rightarrow \angle AQI &= \angle M_a IQ - \angle M_a AQ = \frac{\pi - B}{2}. \end{aligned}$$

Applying the observation gives  $\angle Q'QD = B$ .

To conclude, note that

$$\frac{Q'Q}{QD} = \frac{AQ}{QD} = \frac{AI}{ID} = \frac{AC}{CD} = \frac{AM_a}{M_a C},$$

establishing the similarity by SAS.

(The similarity is negative due to lack of configuration issues, and inspection.) □

**Claim 2** -  $Q', D, K$  collinear.

*Proof.* Angle chase again:  $\angle AQ'D \stackrel{\text{claim 1}}{=} -\angle M_a AC = \widehat{AC}/2 = \widehat{AK}/2 = \angle AQ'K$ . □

### Part 1: $\overline{KA}$ and $\overline{KD}$ touch $\gamma$

Even more angle chasing, taking advantage of claim 2...

$$\angle KAD = \frac{\widehat{KM_a}}{2} = \frac{\widehat{AC}}{2} = \angle ABD, \text{ while } \angle KDA = \frac{\widehat{KA} + \widehat{QM_a}}{2} = \frac{\widehat{Q'C}}{2} = B = \angle DBA,$$

proving the tangencies.

The other, more elegant part of the problem...

**Claim 3** -  $\overline{MO}, \overline{BC}, \overline{KM_a}$  ( $ADK$ ) all concur at a point  $X$ .

*Proof.* Let  $X_1 = \overline{MO} \cap \overline{BC}$ ,  $X_2 = \overline{KM_a} \cap \overline{BC}$ .

- $X_1 \in (ADK)$  by similarity: observe by (omitted) angle chase that  $\triangle AXB \stackrel{+}{\sim} \triangle AKD$ , whence  $\angle AXD = \angle AKD$ ;
- $X_2 \in (ADK)$  (by contrast) is by power of a point at  $M_a$ :

$$M_a B^2 = M_a C^2 = M_a X_2 \cdot M_a K = M_a A \cdot M_a D.$$

As  $X_1 = X_2 = (ADK) \cap \overline{BC}$  ( $\neq D$ ), the claim is proven. □

Because  $\overline{M_a K} \parallel \overline{AB} \perp \overline{MO}$ , and  $X = \overline{MO} \cap \overline{M_a K}$  is the inverse of  $K$  wrt  $\omega$  (by the second equation in previous claim's proof),  $\overline{MO}$  is the polar of  $K$  wrt  $\omega$ , completing the problem.

**Remark. (crazyeyemoody907)** For me, it took forever to discover the proof of claim 1. Here are some useless formulations of its statement I harped upon:

- $(AC; KM_a) = -1$  which directly implies the first part:

$$(B, \overline{BK} \cap \gamma; A, D) \stackrel{B}{=} (M_a K; AC) = -1;$$

Since “ $\overline{KA}$  touches  $\gamma$ ” is very easily provable,  $K$  would be polar of  $\overline{AD}$  wrt  $\gamma$  as promised...

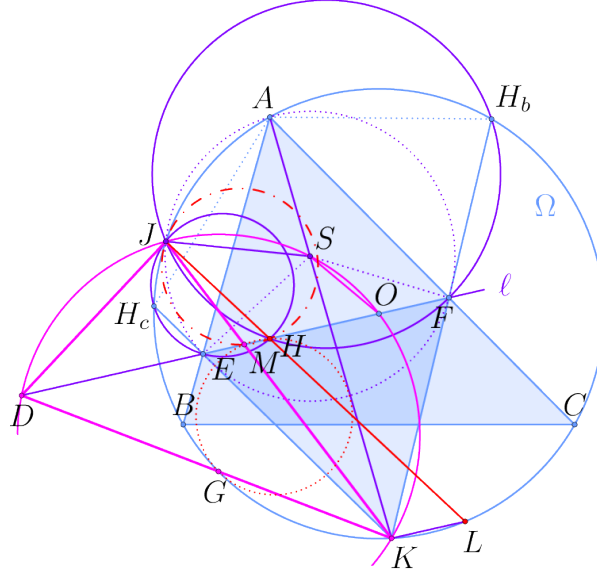
- $BDQQ'$  cyclic ( $\iff \overline{QD} \parallel \overline{AC}$  by Reim)

In fact, this means post-solve that  $\overline{BQ} \parallel \overline{Q'DK}$ ...in hindsight, equally useless...

**Remark.** (Note to self?) When angle chasing in a synthetic problem fails, try length ratios! Examples: SL 2016/G7, GGG 4.5;

♣ 1.18 DeuX MO 2020/II/3, by Hao Minyan (China)

In triangle  $ABC$  with circumcenter  $O$  and orthocenter  $H$ , line  $OH$  meets  $\overline{AB}, \overline{AC}$  at  $E, F$  respectively. Let  $\omega$  be the circumcircle of triangle  $AEF$  with center  $S$ , meeting  $(ABC)$  again at  $J \neq A$ . Line  $OH$  also meets  $(JSO)$  again at  $D \neq O$ . Define  $K = (JSO) \cap (ABC) \ (\neq J), M = \overline{JK} \cap \overline{OH}$ , and  $G = \overline{DK} \cap (ABC) \ (\neq K)$ . Prove that  $(GHM)$  and  $(ABC)$  are tangent to each other.



Solution by [crazyeyemoody907, v4913](#).

Let  $\Omega = (ABC)$ ,  $H_b, H_c$  be the respective reflections of  $H$  in  $\overline{AC}, \overline{AB}$ , and  $\ell = \overline{EFOH}$ . Redefine  $K = \overline{H_cE} \cap \overline{H_bF}$  (we'll see this is an equivalent definition). As  $\overline{EA}, \overline{FA}$  are external angle bisectors wrt  $\triangle KEF$ , we have  $\angle EKF = \pi - 2A$ .

**Claim 1** –  $J \in (HEH_c), (HFH_b)$ .

*Proof.* Let  $J' = (HEH_c) \cap (HFH_b) \ (\neq H)$ . Then:

$$\angle H_c J' H_b = \angle H_c J' H + \angle H J' H_b = \angle H_c E H + \angle H F H_b = \angle (\overline{H_c E}, \overline{H_b F}) = \angle H_b K H_c = \angle H_b A H_c \Rightarrow J' \in \Omega.$$

The construction of  $J'$  implies that  $\overline{J'E}, \overline{J'F}$  respectively bisect  $\angle H_c J' H, \angle H_b J' H$ , and thus

$$\angle E J' F = \frac{1}{2} \angle H_b J' H_c = \angle BAC = \angle EAF \Rightarrow J' \in (AEF),$$

finishing the claim. □

Let  $L = \overline{JH} \cap \Omega \ (\neq J)$ ; then, as  $JH_c KL, JH_c EH$  cyclic,  $\ell \parallel \overline{KL}$  by Reim. By homothety,  $(JHM)$  touches  $(JKL) = \Omega$ .

**Claim 2** – For the  $K$  defined in solution,  $K \in \overline{AS}, (JSO)$ .

*Proof.* Since  $\angle ESF = 2\angle BAC = \angle EKF$ , we have  $KESF$  cyclic; as  $SE = SF$ ,  $AH_b = AH_c$ ,  $A, S$  both lie on bisector of  $\angle EKF$ .

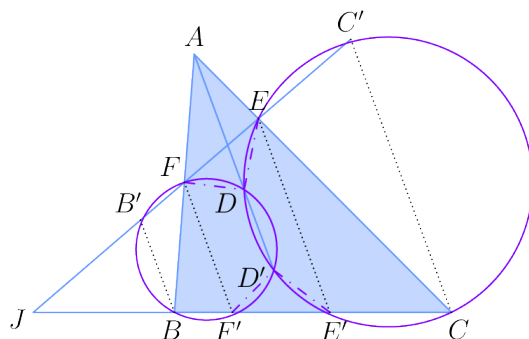
Next, we prove that  $O$  is the midpoint of  $\widehat{JSK}$  on  $(JSK)$ . Because  $\overline{OS}$  is the perpendicular bisector of  $\overline{AJ}$  by symmetry, it externally bisects  $\angle JSK$  as  $K \in \overline{AS}$ . At the same time,  $OJ = OK$  means  $O$  is on the perpendicular bisector of  $\overline{JK}$ . These two properties imply that  $O$  is the claimed arc midpoint.  $\square$

From here, as  $DJKO$  cyclic and  $OJ = OK$ ,  $\overline{DO}$  bisects  $\angle JDK$ , and  $G = \overline{DK} \cap \Omega$  is the reflection of  $J$  in  $\ell$  by symmetry. Reflecting “ $(JHM)$  touches  $\Omega$ ” over  $\ell$  completes the proof.



# 1.19 IMO 2021/3

Let  $D$  be an interior point of the acute triangle  $ABC$  with  $AB > AC$  so that  $\angle DAB = \angle CAD$ . The point  $E$  on the segment  $AC$  satisfies  $\angle ADE = \angle BCD$ , the point  $F$  on the segment  $AB$  satisfies  $\angle FDA = \angle DBC$ , and the point  $X$  on the line  $AC$  satisfies  $CX = BX$ . Let  $O_1$  and  $O_2$  be the circumcenters of the triangles  $ADC$  and  $EXD$ , respectively. Prove that the lines  $BC$ ,  $EF$ , and  $O_1O_2$  are concurrent.



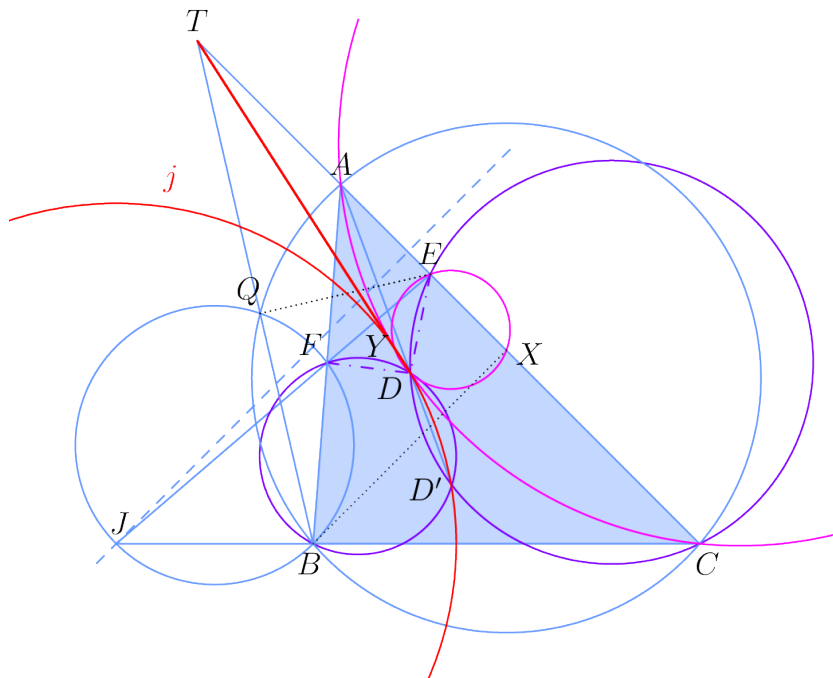
Solution by **v4913**.

Let  $J = \overline{EF} \cap \overline{BC}$ , and  $D' \in \overline{AD}$  be the isogonal conjugate of  $D$  wrt  $\triangle ABC$ . The given angle conditions imply that  $BDD'F$ ,  $CDD'E$  are cyclic, while power of a point at  $A$  implies  $BCEF$  cyclic as well.

**Claim 1** –  $J$  is the exsimilicenter of  $(EDC)$ ,  $(FDB)$ ; hence,  $JD = JD'$  by symmetry.

*Proof.* Construct  $E_1 = (CDD'E) \cap \overline{BC}$  ( $\neq C$ ),  $F_1 = (BDD'F) \cap \overline{BC}$  ( $\neq B$ ). By isogonality,  $DF = D'F'$  and  $DE = D'E'$  whence  $DD'E'E$ ,  $DD'F'F$  are both cyclic isosceles trapezoids.  $\overline{DD'}$ ,  $\overline{EE'}$ ,  $\overline{FF'}$  share a perpendicular bisector  $b$ , and in fact, this is the bisector of  $\angle J$ , i.e.  $JE = JE'$ ,  $JF = JF'$ .

Reflect  $B, C$  over  $b$  to obtain  $B', C'$ ; then, because  $JB/JF' = JB/JF = JE/JC = JE'/JC$ , there is a homothety at  $J$  mapping  $(B, B', F, F') \rightarrow (E', E, C', C)$  and thus their circumcircles  $(BB'DD') \rightarrow (CC'DD')$  as well.  $\square$



Let  $Y = (ADC) \cap (EXD)$  ( $\neq D$ ),  $Q$  be the Miquel point of  $ABCJEF$ , and  $j$  the circle at  $J$  through  $D, D'$ . Observing that  $\overline{O_1 O_2}$  is the perpendicular bisector of  $\overline{DY}$ , it remains to prove  $Y \in j$ .

**Claim 2** –  $XQEB$  is cyclic.

*Proof.* This is a simple angle chase: using cyclic quadrilaterals  $(ABCQ)$ ,  $(JFBQ)$ ,  $(ECJQ)$ , and  $(AEFQ)$ , we obtain

$$\angle EQB = \angle EQA + \angle AQB = \angle ACB + \angle EFA = 2\angle ACB = \angle EXB \quad \square$$

Next, we characterize the radical axis of  $j$ ,  $(JBF)$  – it's perpendicular to the line of centers and through  $A$ :

**Claim 3** – The line through  $B$  and the center of  $(JBF)$  is perpendicular to  $\overline{AC}$ .

*Proof.* This is equivalent to “ $t_b$ , the tangent to  $(JBF)$  at  $J$ , is parallel to  $\overline{AC}$ ”. Because  $\angle(\overline{BJC}, t_b) = \angle BFJ = \angle JCE$ , the result follows.  $\square$

Because  $\text{Pow}(A, j) = AD \cdot AD' = AQ \cdot AJ = \text{Pow}(A, (JBQF))$ ,  $A$  is on the radical axis of  $j$ ,  $(JBF)$ . By the previous claim, it follows that  $\overline{AC}$  is the radical axis of  $j$ ,  $(JBF)$ .

To finish, define  $T = \overline{DY} \cap \overline{AC} \cap \overline{BQ}$  as the radical center of  $(JBF)$ ,  $(ABC)$ ,  $(EXD)$ ,  $(ADC)$ , and the phantom point  $Y' = \overline{TD} \cap j$  ( $\neq D$ ). Because  $T$  is on  $\overline{AC}$ , the radical axis of  $j$ ,  $(JBF)$ , we have (lengths directed)

$$TY' \cdot TD = \text{Pow}(T, j) = \text{Pow}(T, (JBF)) = \text{Pow}(T, (ABCQ)) = TA \cdot TC = TY \cdot TD \Rightarrow Y = Y',$$

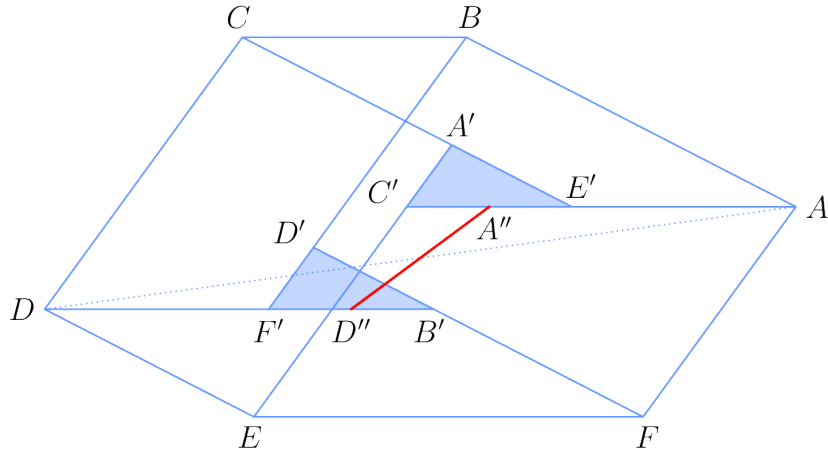
the end!

### 1.20 USAMO 2021/6, by Ankan Bhattacharya

Let  $ABCDEF$  be a convex hexagon satisfying  $\overline{AB} \parallel \overline{DE}$ ,  $\overline{BC} \parallel \overline{EF}$ ,  $\overline{CD} \parallel \overline{FA}$ , and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let  $X$ ,  $Y$ , and  $Z$  be the midpoints of  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$ . Prove that the circumcenter of  $\triangle ACE$ , the circumcenter of  $\triangle BDF$ , and the orthocenter of  $\triangle XYZ$  are collinear.



Construct parallelogram  $CDEA'$  and cyclic variants:  $A' = C + E - D$ , etc. We may compute using vectors that  $\triangle B'D'F'$  is a translation of  $\triangle A'C'E'$  by the vector  $(B+D+F) - (A+C+E)$ . In particular, they're congruent.

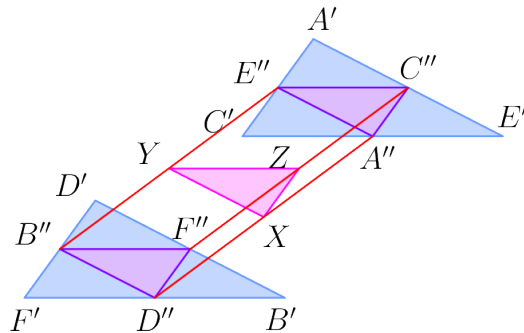
**Claim 1** –  $A, C, E$  have same power wrt  $(A'C'E')$ ; in other words,  $\triangle ACE, A'C'E'$  share a circumcenter.

*Proof.* Observing that  $\text{Pow}(A, (A'C'E')) = AC' \cdot AE' = BC \cdot EF$  by parallelograms, this claim follows by the given length condition.  $\square$

Next, construct  $A'' = \frac{C'+E'}{2}$  and cyclic variants. The circumcenter of  $\triangle A'C'E'$  is then the orthocenter of  $\triangle A''C''E''$ .

**Claim 2** –  $X = \frac{A''+D''}{2}$ .

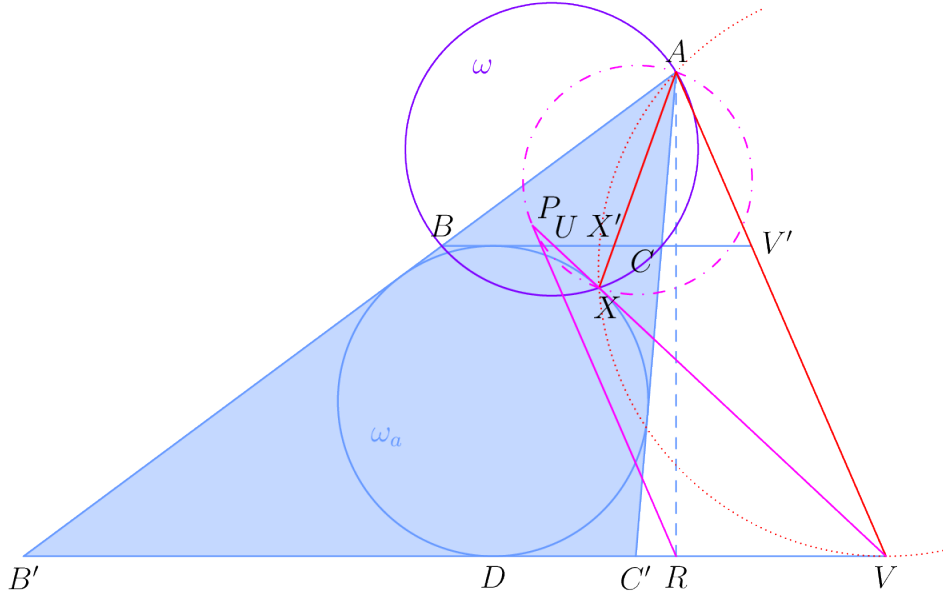
*Proof.* Using vectors,  $B' + C' = E' + F' = A + D \Rightarrow \frac{A+D}{2} = \frac{B'+C'+E'+F'}{4} = \frac{A''+D''}{2}$ .  $\square$



By claim 2 + symmetry,  $\triangle XYZ$  is the vector average of (congruent) triangles  $A''C''E''$ ,  $B''D''F''$ , so their orthocenters are collinear.

### 1.21 SL 2021/G8

Let  $ABC$  be a triangle with circumcircle  $\omega$  and let  $\Omega_A$  be the  $A$ -excircle. Let  $X$  and  $Y$  be the intersection points of  $\omega$  and  $\Omega_A$ . Let  $P$  and  $Q$  be the projections of  $A$  onto the tangent lines to  $\Omega_A$  at  $X$  and  $Y$  respectively. The tangent line at  $P$  to the circumcircle of the triangle  $APX$  intersects the tangent line at  $Q$  to the circumcircle of the triangle  $AQY$  at a point  $R$ . Prove that  $\overline{AR} \perp \overline{BC}$ .



Solution by [crazyeyemoody907](#).

Let the antipode of the  $A$ -extouch point be  $D$ , and the tangent to  $\omega_a$  at  $D$  intersect  $\overline{AB}, \overline{AC}$  at  $B', C'$  respectively. Also, construct the tangent line to  $\omega_a$  at  $X$ , meeting  $\overline{BC}, \overline{B'C'}$  at  $U, V$  respectively. Finally, let  $X' = \overline{AX} \cap \overline{BC}$ ,  $V' = \overline{AV} \cap \overline{BC}$ .

**Claim 1** -  $AXUV'$  cyclic.

*Proof.* Apply DDIT to  $A, UXV \infty_{BC}$  (with inconic  $\omega_a$ ), and project onto  $\overline{BC}$ , to obtain an involutive pairing  $(B, C), (U, V'), (\infty_{BC}, X')$  – or equivalently,  $X'B \cdot X'C = X'U \cdot X'V'$ . By power of a point,  $X'B \cdot X'C = X'A \cdot X'X$ , so the claim follows from power of a point converse on  $X'U \cdot X'V' = X'A \cdot X'X$ .  $\square$

**Claim 2** -  $\overline{DV}$  is tangent to  $(AXV)$ .

*Proof.* Angle chase using previous claim, and the fact that  $\overline{BC} \parallel \overline{B'C'}$ :

$$\angle XAV \stackrel{\text{claim 1}}{=} \angle XUV' = \angle XVD.$$

$\square$

Redefine  $R$  as the foot from  $A$  to  $\overline{B'C'}$ . It remains to show,

**Claim 3** -  $\overline{PR}$  touches  $(APX')$ .

*Proof.* Since  $\angle VPA = \angle VRA = 90^\circ$ ,  $APRV$  cyclic, so we may anglechase as follows:

$$\angle APR = \angle AVR \stackrel{\text{claim 2}}{=} \angle AXV = \angle AXP.$$

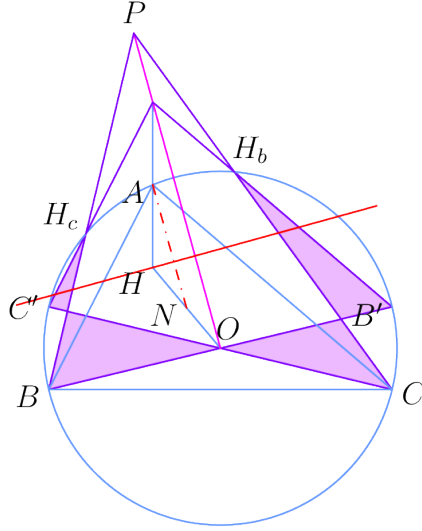
$\square$

### 1.22 USEMO 2020/3, by Anant Mudgal

Let  $ABC$  be an acute triangle with circumcenter  $O$  and orthocenter  $H$ . Let  $\Gamma$  denote the circumcircle of triangle  $ABC$ , and  $N$  the midpoint of  $\overline{OH}$ . The tangents to  $\Gamma$  at  $B$  and  $C$ , and the line through  $H$  perpendicular to line  $AN$ , determine a triangle whose circumcircle we denote by  $\omega_A$ . Define  $\omega_B$  and  $\omega_C$  similarly.

Prove that the common chords of  $\omega_A$ ,  $\omega_B$ , and  $\omega_C$  are concurrent on line  $OH$ .

Let  $H_a, A'$  denote the respective reflections of  $H$  in  $\overline{BC}$ ,  $A$  in  $O$ , and their symmetric variants.



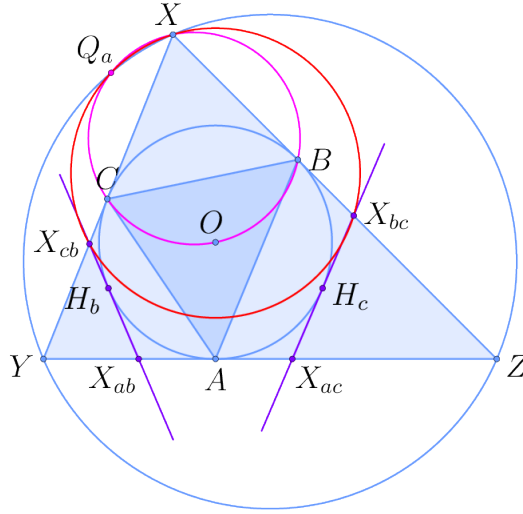
**Claim 1** – The polar  $\ell_a$  of  $\overline{BH_c} \cap \overline{CH_b}$  passes through  $H$  and is perpendicular to  $\overline{AN}$ .

*Proof.* Let  $P = \overline{BH_c} \cap \overline{CH_b}$  and  $S = 2A - H$ .  $H \in \ell_a$  is just Brokard, so it suffices to prove  $\overline{AN} \parallel \overline{OP}$ . By Pascal on  $BB'H_bCC'H_c$ , we have  $P, O, S$  collinear. Taking a homothety at  $H$  with scale factor  $\frac{1}{2}$  maps the latter two points to  $N, A$ , which implies the required parallel lines.  $\square$

In  $\triangle ABC$ , let  $X_{bc}$  be the pole of  $\overline{BH_c}$  wrt  $\Gamma$  (and 5 other variants),  $X, Y, Z$  be the poles of the sides,  $D, E, F$  be the feet of the altitudes. Clearly,  $\ell_a = \overline{X_{bc}X_{cb}}$ .

**Note.** Here, the condition  $\triangle ABC$  acute comes in:  $\Gamma$  is the incircle, not excircle, of  $\triangle XYZ$ .

We'll show that  $\overline{XD}$  is the radical axis of  $\omega_b, \omega_c$ . (By a somewhat-known configuration (say, **Brazil 2013/6**),  $\overline{XD} \cap \overline{YE} \cap \overline{ZF}$  lies on the Euler line.) Also let  $Q_a, Q_b, Q_c$  be the SD points of  $\triangle XYZ$ .



**Claim 2** –  $Q_a$  lies on  $\omega_a$ .

*Proof.* By spiral similarity, it suffices to prove  $YX_{bc}/YC = ZX_{cb}/ZB$ . By antiparallel lines,  $\triangle XYZ \sim \triangle X_{ab}YX_{cb}, X_{ac}X_{bc}Z$ . But since  $\Gamma$  is the  $Y$ -excircle of  $\triangle X_{ab}YX_{cb}$ , we have  $YX_{cb}/YC = a/s$ . Similarly  $ZX_{bc}/ZB = a/s$  as well. (In some awful notation,  $a = YZ, b = ZX, c = XY$  and  $s = \frac{a+b+c}{2}$ .)  $\square$

Let  $L = \overline{YQ_b} \cap \overline{ZQ_c}$ .

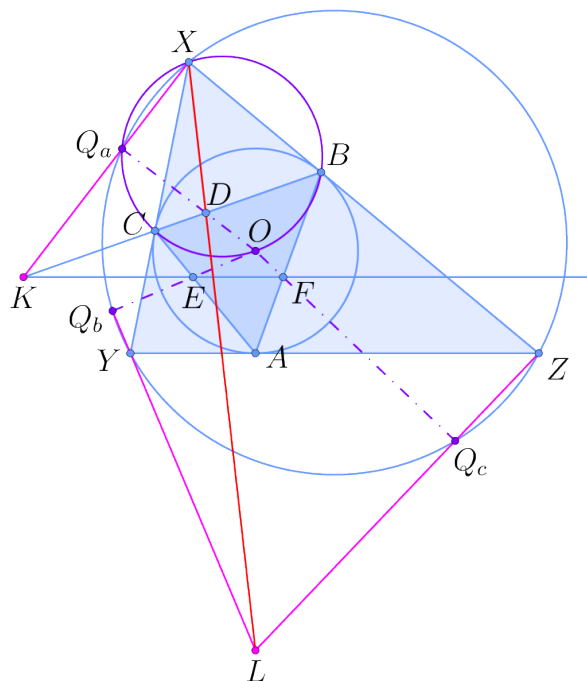
**Claim 3** –  $\overline{XL}$  is the radical axis of  $\omega_b, \omega_c$ .

*Proof.* By antiparallel lines again,  $YX_{ba}X_{ca}$  cyclic, so that

$$\text{Pow}(X, \omega_b) = XX_{ca} \cdot XY = XX_{ba} \cdot XZ = \text{Pow}(X, \omega_c), \text{ while}$$

$$\text{Pow}(L, \omega_b) = LY \cdot LQ_b = LZ \cdot LQ_c = \text{Pow}(L, \omega_c). \quad \square$$

It remains to prove  $X, D, L$  collinear.



**Claim 4** –  $L$  is the pole of  $\overline{EF}$  wrt  $\Gamma$ .

*Proof.* Since  $Q_a$  is the inverse of  $D$  wrt  $\Gamma$  and  $\angle OQ_aX = 90^\circ$ ,  $\overline{XQ_a}$  is the polar of  $D$  wrt  $\Gamma$ . Similarly,  $\overline{YQ_b}$ ,  $\overline{ZQ_c}$  are the respective polars of  $E, F$  wrt  $\Gamma$ . The claim is then established by la Hire.  $\square$

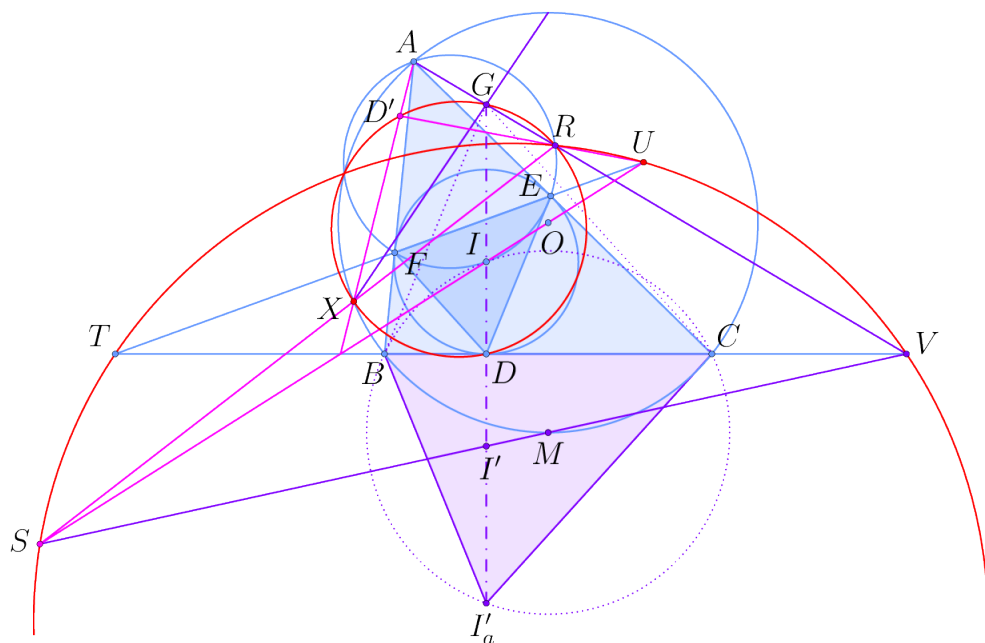
**Claim 5** –  $\overline{BC}$ ,  $\overline{EF}$ ,  $\overline{XQ_a}$  concurrent.

*Proof.* Let  $K = \overline{EF} \cap \overline{BC}$  so that  $(KD; BC) = -1$ . Because  $\overline{Q_aO}$  bisects  $\angle BQ_aC$ ,  $\angle KQ_aO = 90^\circ = \angle AQ_aO \Rightarrow X, Q_a, K$  collinear.  $\square$

Taking poles wrt  $\Gamma$  in the last claim gives the desired collinearity.

**Remark.** The problem can be bary'd wrt  $\triangle XYZ$  after the first claim, but it's monstrous from my experience a long time ago, oops

### 1.23 Brazil OlyRevenge 2021/3 (aka “Naruto”)



Solution by [crazyeyemoody907](#), [CyclidSLscalesTrapezoid](#) with [Eyed](#), v4913.

**Warning.** This problem is not meant for neither the faint-hearted nor freehand geometers like the paper’s author(s). If Geogebra’s to be used any time, it’d be now.

We do a massive refactoring and simplification; consider the following equivalent problem, a breakdown of the given, despite being longer:

#### Naruto simplified

In triangle  $ABC$  with circumcircle  $\Omega$  centered at  $O$ , the incircle  $\omega$  centered at  $I$  touches the sides at  $D, E, F$ . Let  $I', I'_a$  be the respective reflections of  $I$  and the orthocenter of  $\triangle BIC$  in  $\overline{BC}$ , and  $M$  the midpoint of arc  $BC$  on  $\Omega$ . Further define:

- $S$  as the intersection of the Euler lines  $\overline{OI}$  of  $\triangle DEF$ ,  $\overline{MI'}$  of  $\triangle I'_aBC$ ;
- $T = \overline{EF} \cap \overline{BC}$ ,  $U = \overline{EF} \cap \overline{OI}$ ,  $V = \overline{MI'} \cap \overline{BC}$ ,  $R = \overline{AV} \cap (AI)$ ;
- $K = \overline{OI} \cap \overline{BC}$ ;

Prove that (a)  $Q, R, S, T, U, V$  are concyclic, and (b)  $\overline{AK}, \Omega, (QRD), \overline{RS}$  concurrent;

**(a) The concyclicity** Let the spiral similarity  $s$  at  $Q$  with (directed) angle  $\theta$  map  $E, F \rightarrow C, B$  and thus  $D, I$  and the orthocenter of  $\triangle DEF$  to  $I', M, I'_a$  respectively. Clearly,  $S$  is the intersection of the Euler lines of two triangles related by  $s$ :  $DEF, I'_aCB$ .

By design, we have  $U \xrightarrow{s} V$ , so

$$\angle VQU = \theta = \angle(\overline{BC}, \overline{EF}) \stackrel{s}{=} \angle(\overline{MI'}, \overline{OI}) = \angle VSU,$$



whence  $Q, S, T, U, V$  concyclic. To see that the last point is also concyclic with the other five, let  $N$  be the midpoint of  $\widehat{BAC}$ , so that  $\overline{NA}$  touches  $(AI)$ . Indeed, then

$$\angle QRV = \angle QRA = \angle QAN \stackrel{s}{=} \angle QUV$$

as needed.

**Remark.** In fact, by design,  $S$  is the exsimilicenter of the incircle and the circle at  $O$  with radius half that of  $\Omega$ , so it's actually the inverse of  $I$  wrt  $\Omega$ .

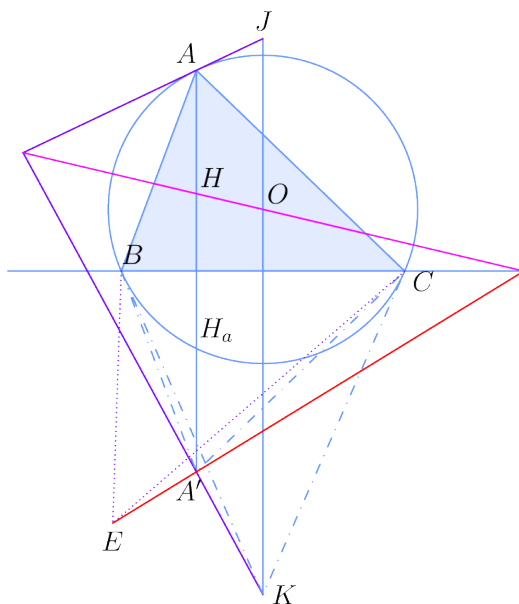
**(b) The concurrence** Let  $D'$  be the reflection of  $D$  in  $\overline{EF}$ , and  $G$  the orthocenter of  $\triangle BIC$ , so that  $D' \xrightarrow{s} G$ . We easily have  $DD'GQ$  cyclic. As  $\angle(\overline{AD'}, \overline{NG}) = \theta$ , the point  $X = \overline{AD'} \cap \overline{NG}$  lies on both  $(DD'GQ)$ ,  $\Omega$ . We require the following result(s):

### Theorem: weird concurrences

In a scalene triangle  $ABC$  with circumcenter  $O$ , circumcircle  $\Omega$ , and orthocenter  $H$ .

- (a) let  $K$  be the polar of  $\overline{BC}$  wrt  $\Omega$ , and  $A'$  be the reflection of  $A$  in  $\overline{BC}$ . Then  $\overline{OH}$ ,  $\overline{A'K}$  and the tangent to  $\Omega$  at  $A$  are concurrent.
- (b) Let  $E$  be the reflection of the point  $E_0$  (such that  $A$  is the incenter or excenter of  $\triangle E_0BC$ ) in the perpendicular bisector of  $\overline{BC}$ . Then  $\overline{OH}$ ,  $\overline{BC}$ ,  $\overline{EA'}$  are also concurrent.

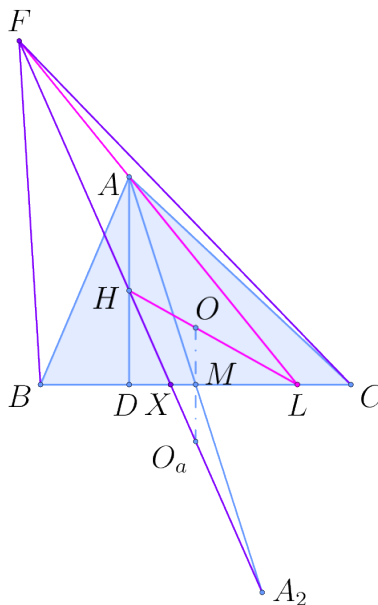
(parentheses used above for easier grammatical parsing)



*Proof.* These two parts actually aren't connected at all...

**Part (a), by CyclicalScelesTrapezoid** Let  $J$  be the intersection of the tangent to  $\Omega$  at  $A$  with the perpendicular bisector of  $\overline{BC}$ , and  $H_a \in \Omega$  be the reflection of  $H$  in  $\overline{BC}$ . We contend that the triples  $(A, H, A')$ ,  $(J, O, K)$  are homothetic. Indeed, they lie on parallel lines. To finish, check that (if  $R$  denotes the radius of  $\Omega$ )

$$JO = \frac{R}{\cos(B-C)}, OK = \frac{R}{\cos A}, AH = 2R \cos A, HA' = AH_a = 2R \cos(B-C) \Rightarrow \frac{JO}{OK} = \frac{AH}{HA'}.$$



**Part (b), by crazyeyemoody907** Let  $F = B + C - E_0$ , and  $A_2 = B + C - A$ , so that  $A_2$  is an incenter or excenter of  $\triangle FBC$ . Since  $H$  is the antipode of  $A_2$  on  $(BA_2C)$ , it is another incenter / excenter. To prove that  $A, L, F$  collinear where  $X = \overline{FHA_2} \cap \overline{BC}$ ,  $L = \overline{OH} \cap \overline{BC}$ , verify that (where  $O_a \in \overline{HA_2}$  is the reflection of  $O$  in  $\overline{BC}$ )

$$(\overline{AF} \cap \overline{BC}, X; D, M) \stackrel{A}{=} (FX; HA_2) = -1 \text{ while } (DM; XL) \stackrel{H}{=} (\infty_{\perp BC} M; O_a O) = -1.$$

□

Returning to the problem, applying respective parts of the theorem to  $\triangle DEF, I'_a BC$ , we obtain  $(A, D', K)$  and  $(A, G, V)$  collinear. Since  $R \in (\overline{UVQ}), \overline{GV}$ , and  $Q$  is the Miquel point of  $D'GVU$ , we must have  $R = \overline{D'U} \cap \overline{GV}$  – an intersection of opposite sides. Hence, by definition of Miquel point,  $R \in (\overline{QD'G})$ .

It remains to prove that  $R, X, S$  collinear. In fact, there is a spiral similarity at  $Q$  mapping  $D', X \rightarrow U, S$  since  $Q \in (\overline{URS}), (D'XR)$ , so we're done!