Revenge ELMO 2023: wateRELMOn

Rookie MOPpers

June 2023

No comment, just see below :P

– ELMO Revenge Committee 2023

0 Problems

0.1 Warm-up

For originality purposes we introduce the concept of an official warm-up:

Problem 0. In triangle ABC, let X be the symmedian point. Find the range of all possible values of $\angle BXC$ given

(a) ABC is an acute triangle.

(b) ABC is an obtuse triangle.

0.2 Actual pset

Problem 1. In cyclic quadrilateral $ABCD$ with circumcenter O and circumradius R, define $X = \overline{AB} \cap \overline{CD}$, $Y = \overline{AC} \cap \overline{BD}$, and $Z = \overline{AD} \cap \overline{BC}$. Prove that

$$OX^2 + OY^2 + OZ^2 \geq 2R^2 + 2[ABCD].$$

Problem 2. On an infinite square grid, Gru and his 2022 minions play a game, making moves in a cyclic order with Gru first. On any move, the current player selects 2 adjacent cells of their choice, and paints their shared border. A border cannot be painted over more than once. Gru wins if after any move there is a 2×1 or 1×2 subgrid with its border (comprising of 6 segments) completely colored, but the 1 segment inside it uncolored. Can he guarantee a win?

Problem 3. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\left(\sum_{\text{cyc}} f(x)\right)\left(\sum_{\text{cyc}} xf(y)\right) > \prod_{\text{cyc}} (f(x) + y)$$

for all $x, y, z \in \mathbb{R}^+$. (Here, $\sum_{\text{cyc}} g(x, y, z)$ is shorthand for $g(x, y, z) + g(y, z, x) + g(z, x, y)$.)

Problem 4. On a 5×5 grid $\mathcal{A}$ of integers, each with absolute value $< 10^9$, define a **flip** to be the operation of negating each element in a row / column with negative sum. For example, $(-1, -4, 3, -4, 1) \to (1, 4, -3, 4, -1)$. Determine whether there exists an $\mathcal{A}$ so that it's possible to perform 90 flips on it.

Problem 5. Complex numbers a, b, w, x, y, z, p satisfy

$$\frac{(x-w)\,|a-w|}{(a-w)\,|x-w|} = \text{(cyclic variants)};$$
$$\frac{(z-w)\,|b-w|}{(b-w)\,|z-w|} = \text{(cyclic variants)};$$
$$p = \frac{\sum_{\text{cyc}} \frac{w}{|p-w|}}{\sum_{\text{cyc}} \frac{1}{|p-w|}};$$

where cyclic sums, equations, etc. are wrt w, x, y, z. Prove that there exists a real k such that

$$\sum_{\text{cyc}} \frac{|x-w|\,|a-w|}{|p-w|\,|x-w|} = k \sum_{\text{cyc}} \frac{|z-w|\,|b-w|}{|p-w|\,|z-w|}.$$

Note. In place of this box, a picture of Elmo on fire was supposed to go here instead, but to save ink, we omit it in the printed copies.