ZGY-ConfigGeo

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> *Remark.* *'cause every night I lie in bed, synthetic geo fills my head…geometry is keeping me awake…*

Contents

1 [50♣] External references

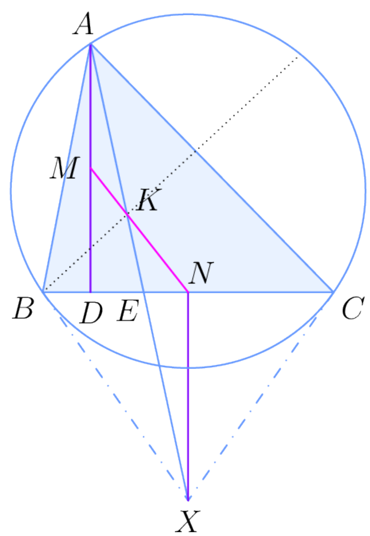
As I really enjoyed quite a few problems on this unit, some of them may be found on **a geo paper** I wrote, namely: [3♣] **TwCh0061**, [5♣] **18SLG5**, [5♣] **12RMM6**, [5♣] **20FakeUSMO3**, [5♣] **20MOP1Z**, [9♣] **20IGOA4**, [9♣] **20DeuXMOII3**, and [9♣] **20USEMO3**.

> *Remark.* Oops, I 'stole' almost all the [9♣] problemsd…also, I should really set up von someday.

2 [13♣] Configs

2.1 [3♣] Schwatt

Let ABC be a triangle with altitude $\overline{AD}$. Let M and N denote the midpoints of $\overline{AD}$ and $\overline{BC}$. Show that line MN passes through the symmedian point K of $\triangle ABC$ (this line is called the A-Schwatt line).



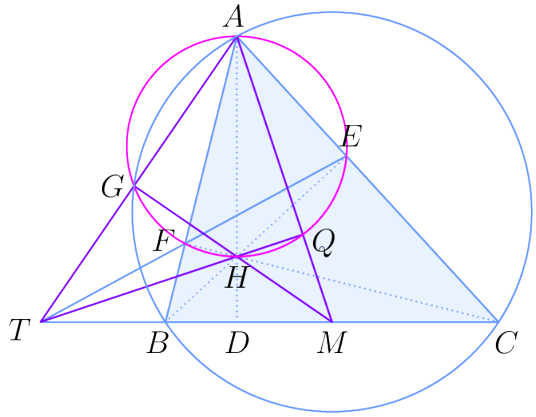
Let the A-symmedian meet $\overline{BC}$ at E, ω be the circumcircle of $\triangle ABC$, and $X \in \overline{AE}$ be the pole of $\overline{BC}$ wrt ω. Clearly, $\overline{AD}, \overline{NX} \perp \overline{BC} \Rightarrow \overline{AD} \parallel \overline{NX}$. Then

$$-1 = (A, C; B, \overline{BK}) \stackrel{B}{=} (AE; KX) \stackrel{N}{=} (A, D; \overline{NK} \cap \overline{AD}, \infty_{AD})$$

so $\overline{NK}$ bisects $\overline{AD}$.

> *Remark.* Instead of using this config to solve TSTST 2016/6, I actually discovered it from that problem…

2.2 [2♣] + [2♣] Humpty Dumpty



Although defined in the next problem, we invoke the notation early: G as A-orthocenter Miquel point, and $T = \overline{AG} \cap \overline{EF} \cap \overline{BC}$ (which exists by radical axis).

By isogonal / antiparallel lines, $\overline{AM}$ is a symmedian in $\triangle AEF$; since $Q \in (AH)$, this is $(AQ; EF) = -1$. First,

> **Lemma** (3b) – T, Q, H collinear.

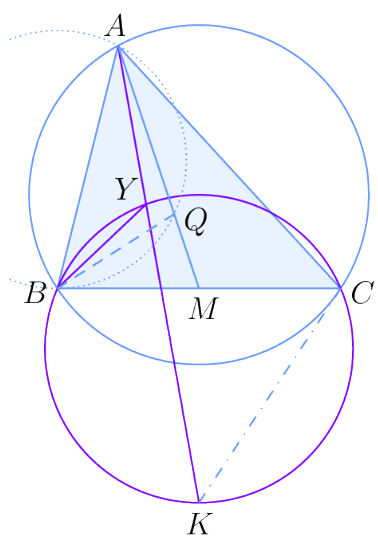
Proof. Harmonics: $(D, \overline{HQ} \cap \overline{BC}; B, C) \stackrel{H}{=} (AQ; EF) = -1$ implies $T = \overline{HQ} \cap \overline{BC}$. □

Second, $\angle MDH = \angle MQH = 90^\circ$ means $MDQH$ cyclic, or in PoP terms, $TQ \cdot TH = TM \cdot TD$. By midpoints of harmonics bundles lemma on $(TD; BC)$, the last product equals $TB \cdot TC$, so **2a: BCHQ cyclic** by PoP converse.

> *Remark.* Iran TST 2018/1/4 should be in the pset… or is it only in the D version?

Third, from the above lemma and $\overline{AHD} \perp \overline{TM}$, **3c: H is orthocenter of triangle ATM**. If A' is the antipode of A on (ABC), then $\angle HGA = \angle A'GA = 90^\circ$ implies **3a: A', M, H, G collinear**.

Fourth, using midpoints of harmonics on $(TD; BC)$ again, we can obtain $MQ \cdot MA = MD \cdot MT = MB^2 = MC^2$, implying **2b: line BC touches (ABQ), (ACQ)**.



Let O be the circumcenter of $\triangle ABC$, K be the pole of $\overline{BC}$ wrt (ABC), $X \neq A$ be the intersection of the A-symmedian with (ABC) and $Y = (A+X)/2$ the A-Dumpty point, so that $Y \in (OBCK)$.

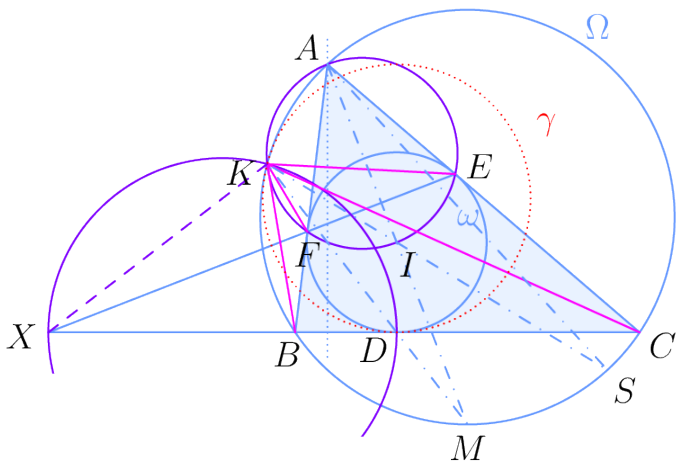
> **Lemma** (2c) – Q, Y isogonal.

Proof. By symmetry, sufficient to prove $\measuredangle ABY = \measuredangle QBC$. Indeed, observing that $\overline{AY}, \overline{AM}$ isogonal, we have

$$\measuredangle KYB = \measuredangle KCB = \measuredangle CAB \Rightarrow \measuredangle ABY = \measuredangle BAY + \measuredangle KYB = \measuredangle CAY = \measuredangle MAB \overset{\text{tangency}}{=} -\measuredangle MAY. \qquad \square$$

2.3 [3♣] Sharky-Devil

> A scary fish and a fiend
>
> 20TSTST2 AoPS thread title



Let M be the midpoint of arc BC excluding A, K be the A-SD point, S be the antipode of A on (ABC). Let Ω, ω denote the circumcircle and incircle respectively.

(a) Angle bisector theorem on $KB/KC \overset{\text{spiral}}{=} BF/EC = BD/DC$ implies that $\overline{KD}$ bisects $\angle BKC \implies K, D, M$ collinear.

(b) Due to the previous part (and the circles-inscribed-in-segments lemma from EGMO book) we may construct a circle γ tangent to $\Omega, \overline{BC}$ at K, D respectively. (This circle is tangent internally to Ω, ω at K, D respectively: $\omega \subseteq \gamma \subseteq \Omega$.) Now, by Monge on (ω, γ, Ω), $\overline{KD}$ passes through the exsimilicenter of (Ω, ω).

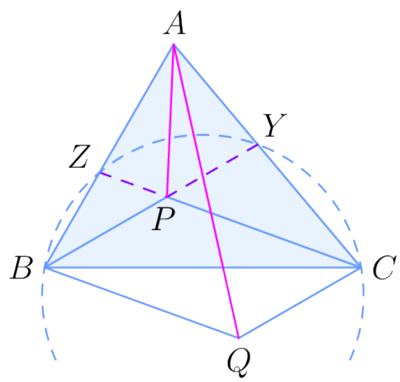
(c) By angle bisector theorem, equivalent to prove $EP/PF = EK/KF$; by spiral, $EK/KF = EC/FB = CD/DB$. Construct $X = \overline{EF} \cap \overline{BC}$ so that $(XD; BC) = -1$. Since $\overline{KD}$ bisects $\angle BKC$, we have $\angle DKX = 90^\circ = DPX$, or $XDKP$ cyclic. In other words, $(XBF), (XCE), (XPD)$ concur at $K \neq X$, so $CD/DB = EP/PF$ follows from spiral.

(d) By radical axis on $((AI), \Omega, (BIC))$, $\overline{AK}, \overline{BC}$, and the line through I perpendicular to $\overline{AI}$ concur.

(e) Let $U = \overline{KD} \cap (AI) \ (\neq K)$. Then, $\overline{UI} \parallel \overline{BC}$ by spiral at A, so $\overline{AU} \perp \overline{UI} \parallel \overline{BC}$.

🌲 2.4 [3♣] First isogonality lemma

In a triangle ABC, let P be an interior point with $\angle ABP = \angle PCA$, and $Q = B + C - P$. Then $\overline{AP}, \overline{AQ}$ isogonal wrt $\angle A$.



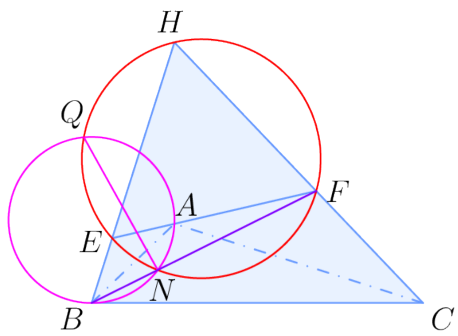
Let $B' = \overline{PB'} \cap \overline{AC}$, $C' = \overline{PC'} \cap \overline{AB}$. Then, consider the composition h of a homothety at A with scale factor $AB/AB' = AC/AC'$ with a reflection in the bisector of $\angle A$. As $(B', C', P) \xrightarrow{h} (B, C, Q)$ by linearity / etc, the result follows.

🌲 3 [65♣] Contest probs

Note. Split into two sections for accessibility reasons.

🌲 3.1 [5♣] 21EGMO3

Let ABC be a triangle with an obtuse angle at A. Let E and F be the intersections of the external bisector of angle A with the altitudes of ABC through B and C respectively. Let M and N be the points on the segments EC and FB respectively such that $\angle EMA = \angle BCA$ and $\angle ANF = \angle ABC$. Prove that the points E, F, N, M lie on a circle.



The problem becomes a lot simpler if we consider problem wrt $\triangle HBC$ where H is the orthocenter (of $\triangle ABC$.) Define:

- Q as the H-orthocenter Miquel point in $\triangle HBC$, aka the A-Humpty point in $\triangle ABC$;
- ω_b as the circle through A touching $\overline{BC}$ at B; it's well-known that $Q \in \omega_b$, while the given angle condition implies $N \in \omega_b$ as well.

Lemma (source?) **–** $HQEF$ cyclic.

Proof. By angle chasing, $\triangle BEA \overset{-}{\sim} \triangle CFA$; thus, if $A' \in (HBC)$ is the reflection of A in $\overline{BC}$ (so that $(QA'; BC) = -1$),

$$\Rightarrow \frac{BE}{CF} = \frac{BA}{CA} = \frac{BA'}{CA'} \overset{\text{harmonics}}{=} \frac{BQ}{CQ},$$

proving the lemma via spiral. □

To finish, all we need is:

Claim – $NQHF$ cyclic.

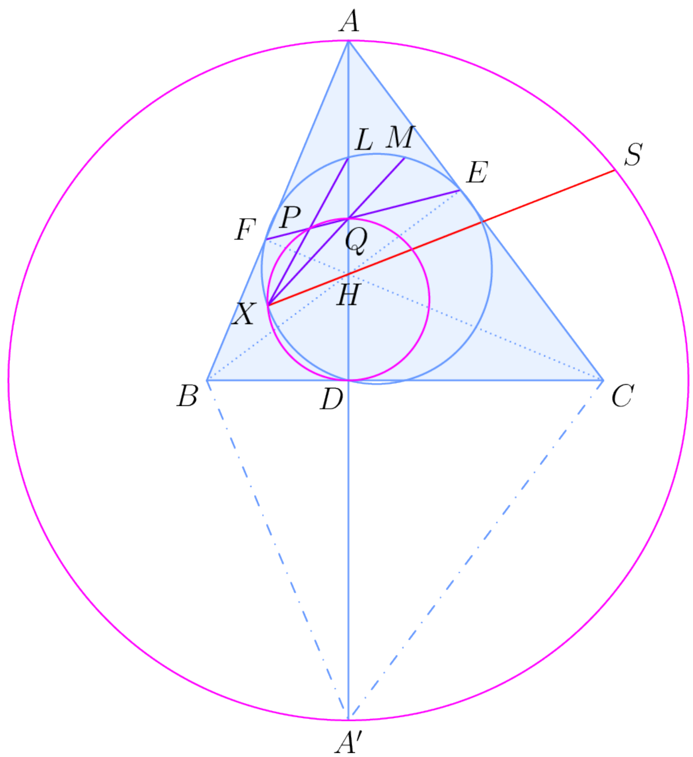
Proof. $\measuredangle QNF = \measuredangle QNB = \measuredangle QBC = \measuredangle QHC = \measuredangle QHF$. □

From above claim, $M, N \in (AQEF)$ completing the proof.

Remark. Everyone's sol is the same…

3.2 [3♣] 19IndTST8

Let ABC be an acute triangle with circumcircle Γ and altitudes $\overline{AD}, \overline{BE}, \overline{CF}$ meeting at H. Let ω be the circumcircle of $\triangle DEF$. Point $S \neq A$ lies on Γ such that $DS = DA$. Line $\overline{AD}$ meets $\overline{EF}$ at Q, and meets ω at $L \neq D$. Point M is chosen such that $\overline{DM}$ is a diameter of ω. Point P lies on $\overline{EF}$ with $\overline{DP} \perp \overline{EF}$. Prove that lines SH, MQ, PL are concurrent.



Obviously, L is the midpoint of minor arc BC, M is the antipode of D (on ω).

Construct $X = (DQ) \cap \omega \quad (\neq D)$. I claim this is the desired concurrency point. Two of the three desired lines are easy to deal with, in American fashion:

> **Claim 1 –** $X \in \overline{LP}, \overline{MQ}$.

Proof. $\angle DPQ = 90^\circ = \angle DXQ$ implies $DQPX$ cyclic. If $X' = \overline{LP} \cap \omega$, then $LP \cdot LX' = LQ \cdot LA = LE^2 = LF^2$, or $DQPX'$ cyclic. Hence, $X' = X$.
To see that $X \in \overline{MQ}$, simply observe that $\angle DXQ = \angle DXM = 90^\circ$ by construction. □

Define $A' = 2D - A$ as the reflection of A in $\overline{BC}$, allowing us to define S more naturally as $(AA') \cap (ABC) \quad (\neq A)$.

Since $\measuredangle BAQ = -\measuredangle BAD = -\measuredangle QEB$, $BA'EQ$ cyclic. For the last line, we can actually show:

> **Claim 2 –** If i denotes the inversion at H with (negative) power $p = HA \cdot HD = HB \cdot HE = HC \cdot HF$, then $S \overset{i}{\leftrightarrow} X$.

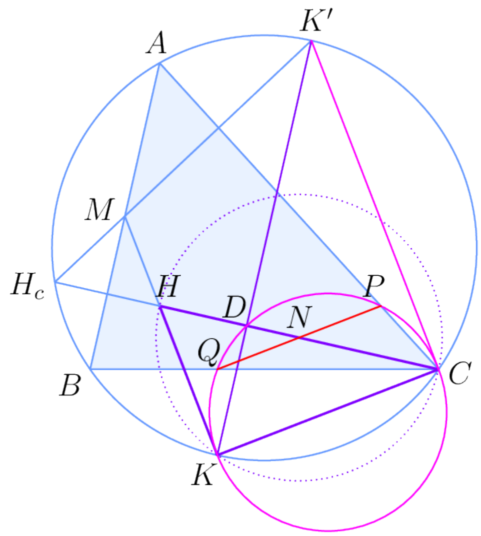
Proof. It's well-known that i swaps $(ABC), \omega$.

I claim that i swaps (Q, A'). Indeed applying power of a point converse to claim 2 implies $HA' \cdot HQ = HB \cdot HE = p$.

As H, A, D, Q, A' all collinear, it follows that i swaps $((AA'), (DQ))$ and thus (S, X) as well. □

3.3 [3♣] 19Shrg20

Let O be the circumcenter of triangle ABC, H be its orthocenter, and M be the midpoint of AB. The line MH meets the line passing through O and parallel to AB at point K lying on the circumcircle of ABC. Let P be the projection of K onto AC. Prove that $PH \parallel BC$.



Let H_c be the reflection of H in $\overline{AB}$, $D = \overline{CH_c} \cap (CH)$, $Q = \overline{BC} \cap (CH)$, and $N = \overline{PQ} \cap \overline{CH}$; s denote the spiral at K mapping $P, Q, D \to A, B, H_c$. Finally let K' be the reflection of K in the perpendicular bisector of $\overline{AB}$. By the given condition this is also the antipode of K in the $\Omega = (ABC)$, so that $\angle K'CK = 90^\circ$ whence $\overline{CK'}$ touches (CK).

Claim 1 – N is the midpoint of $\overline{PQ}$.

Proof. It's well-known that $\overline{H_cK'}$ passes through M, i.e. it bisects $\overline{AB}$; applying s^{-1} means $\overline{CD}$ bisects $\overline{PQ}$ as deisred. □

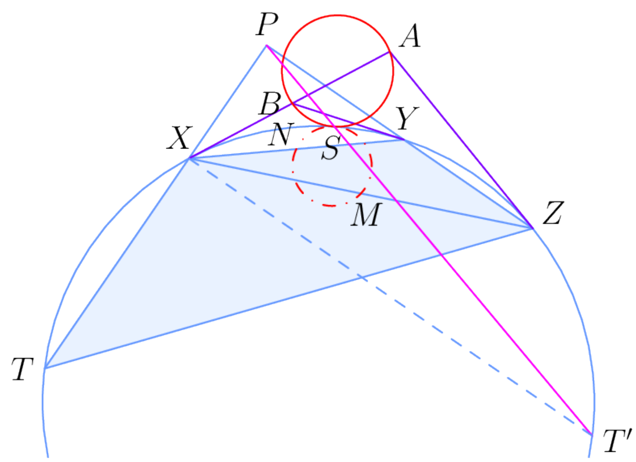
Claim 2 – N is also the midpoint of $\overline{CH}$.

Proof. Applying s^{-1} to "$\overline{KK'} \parallel \overline{AB}$" means that $\overline{KC} \parallel \overline{PQ}$, so $CPQK$ is a cyclic isosceles trapezoid. Thus $N = \frac{P+Q}{2}$ is on the common perpendicular bisector of $\overline{CK}$, $\overline{PQ}$. But in right $\triangle HKC$, since it's on a perpendicular bisector of a side and the hypotenuse, it must be the circumcenter, hence $NC = NH$ as required. □

It follows that $CPHQ$ is a parallelogram, completing the proof.

3.4 [5♣] 16ChnTST26

The diagonals of a cyclic quadrilateral $ABCD$ intersect at P, and there exists a circle Γ tangent to the extensions of $\overline{AB}$, $\overline{BC}$, $\overline{AD}$, $\overline{DC}$ at X, Y, Z, T respectively. Circle Ω passes through points A, B, and is externally tangent to circle Γ at S. Prove that $\overline{SP} \perp \overline{ST}$.

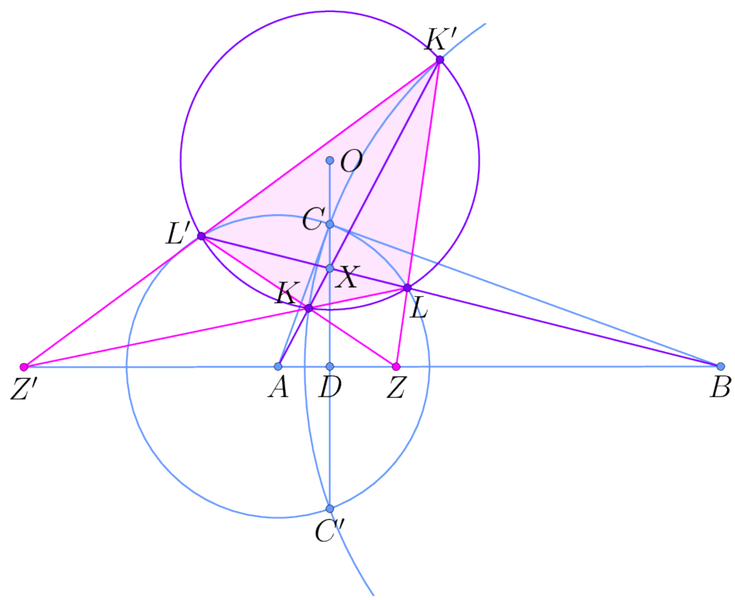


Solution outline with **CyclicISLscelesTrapezoid** and **v4913**. (*AoPS post link*)

By Brianchon's on $AXBCTD$ and $AZDCYB$, $\overline{XY}$ and $\overline{YZ}$ intersect at P. By angle chasing, $\overline{XT}$ and $\overline{YZ}$ are perpendicular. Let M and N be the midpoints of $\overline{XZ}$ and $\overline{XY}$, respectively, let T' be the antipode of T with respect to Γ, and redefine S as the second intersection of $\overline{PT'}$ with Γ. By inversion about Γ, it suffices to show that Γ is tangent to the circumcircle of SMN at S. By angle chasing, $XYZT'$ is a cyclic isosceles trapezoid, so we are done by **SL 2011/G4**.

3.5 [3♣] 13DecTST3

Let ABC be a scalene triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C.Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. The circumcircle of triangle DKL intersects segment AB at a second point T (other than D). Prove that $\angle ACT = \angle BCT$.



Let ω_a, ω_b be the circles A, B through C, $K' = \overline{AX} \cap \omega_b$ $(\neq K)$, and similarly for L'. Let $C' \in \omega_a \cap \omega_b$ be the reflection of C in $\overline{AB}$.

Claim 1 – $KLK'L'$ is harmonic.

Proof. The quadrilateral is cyclic by power of a point at X: $XK \cdot XK' = XC \cdot XC' = XL \cdot XL'$.

Call its circumcircle Ω. Meanwhile, power of a point at A means it's harmonic too:

$$AK \cdot AK' = AC^2 = AL^2 = AL'^2 \Rightarrow \overline{AL}, \overline{AL'} \text{ touch } \Omega.$$ □

Let O be the center of Ω, and $Z = \overline{KL'} \cap \overline{LK'}$, $Z' = \overline{KL} \cap \overline{K'L'}$ which both lie on $\overline{AB}$ by Brokard. As $\overline{AB}$ is the polar of $X = \overline{KK'} \cap \overline{LL'}$ wrt Ω, D is the Miquel point of $KLK'L'$, whence $Z \in (DKL)$ and $Z = T$.

Claim 2 – $\angle ZCZ' = 90°$.

Proof. Equivalent to prove $DC^2 = DZ \cdot DZ'$. O is the orthocenter of $\triangle XZZ'$ by Brokard, while it's also the orthocenter of $\triangle XAB$ because $\overline{AO} \perp \overline{LL'}$, $\overline{BO} \perp \overline{KK'}$. Recall that in a triangle ABC with orthocenter H and D the foot of the A-altitude, $DB \cdot DC = DH \cdot DA$. Thus, applying the result to $\triangle XZZ'$, XAB, we obtain

$$DZ \cdot DZ' = DO \cdot DX = DA \cdot DB = DC^2$$

as needed. □

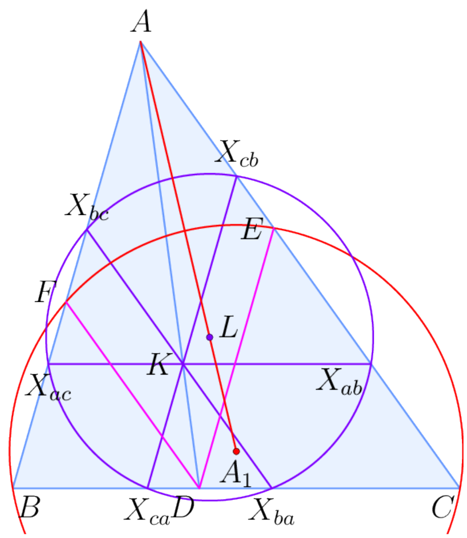
Finally, since $(AB; ZZ') = -1$ (Ceva-Menelaus) and $\angle ZCZ' = \angle ACB = 90°$, $\overline{CZ}$ bisects $\angle ACB$ by a well-known result.

3.6 [3♣] 05ChnTST

Let ω be the circumcircle of acute triangle ABC. The tangents to ω at B and C intersect at P, and $D = \overline{AP} \cap \overline{BC}$. Points E, F are on $\overline{AC}$ and $\overline{AB}$, respectively, such that $\overline{DE} \parallel \overline{BA}$ and $\overline{DF} \parallel \overline{CA}$.

(a) Prove that points F, B, C, and E are concyclic.

(b) Let A_1 denote the circumcenter of cyclic quadrilateral $FBCE$. Points B_1 and C_1 are defined similarly. Prove that $\overline{AA_1}$, $\overline{BB_1}$,and $\overline{CC_1}$ are concurrent.



For part **(a)**, since $\overline{AD}$ is a symmedian in $\triangle ABC$ and a median in $\triangle AEF$, $\overline{BC}$, $\overline{EF}$ are antiparallel wrt $\angle A$.

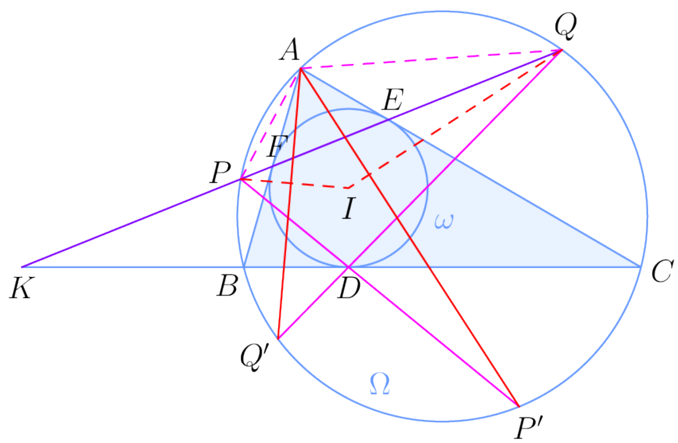
For part **(b)**, we'll show that $\overline{AA_1}$ passes through the center L of the Lemoine circle ω. Define K as the symmedian point, and $X_{bc} = \overline{K\infty AC} \cap \overline{AB}$ and its five other variants. Consider homothety h at A mapping $D \to K$. By parallel lines, this homothety also maps $BCEF \to X_{ac}X_{ab}X_{cb}X_{bc}$ and thus their circumcenters $A_1 \to L$ as well. Hence $L \in \overline{AA_1}$ as required.

Remark. wth is the lemoine circle

3.7 [5♣] 19SLG6

Let ABC be a triangle with incenter I whose incircle touches sides BC, CA, AB at D, E, F. Line EF meets the circumcircle of $\triangle ABC$ at two points P and Q. Prove that

$$\angle DPA + \angle AQD = \angle QIP.$$



If Ω, ω are the circumcircle and incircle respectively, define $P' = \overline{PD} \cap \omega \ (\neq P)$, and Q' similarly.

> **Claim –** $\overline{AQ'}$ is the polar of P wrt ω. Thus, $\overline{AQ'} \perp \overline{PI}$.

Proof. Since A is obviously on that polar, it suffices to prove $(P, \overline{AQ} \cap \overline{EF}; E, F) = -1$. Indeed,

$$(P, \overline{AQ} \cap \overline{EF}; E, F) \overset{A}{=} (PQ'; BC) \overset{Q}{=} (KD; BC) = -1.$$

□

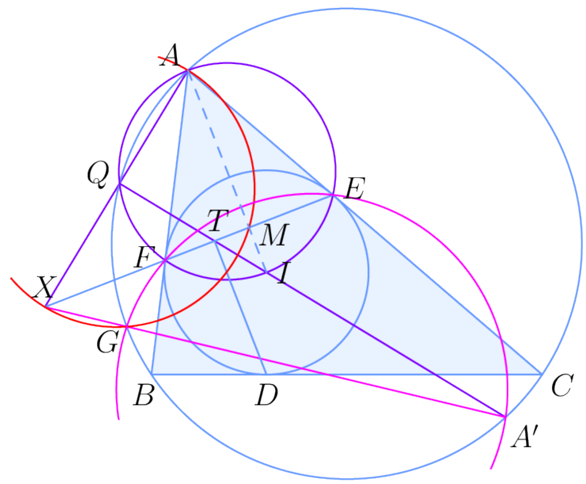
Now the problem is a simple angle chase: the claim implies $\measuredangle QIP = \measuredangle P'AQ'$, while (arcs directed mod 360°)

$$\measuredangle DPA + \measuredangle AQD = \frac{1}{2}(\widehat{P'A} + \widehat{AQ'}) = \measuredangle P'AQ'$$

as well, as required.

3.8 [3♣] 19ESLG3

Let $\triangle ABC$ be an acute triangle with incenter I and circumcenter O. The incircle touches sides BC, CA, and AB at D, E, and F respectively, and A' is the reflection of A over O. The circumcircles of ABC and $A'EF$ meet at G, and the circumcircles of AMG and $A'EF$ meet at a point $H \neq G$, where M is the midpoint of EF. Prove that if GH and EF meet at T, then $DT \perp EF$.



Redefine T as the foot from D to $\overline{EF}$, so that we want T on the radical axis of (AMG), $(A'EF)$. Construct Q as the A-SD point.

By radical axis on (AI), $(A'EFG)$, (ABC), there exists a point $X = \overline{AQ} \cap \overline{EF} \cap \overline{AG}$.

Claim – *$AGMX$, $IMQX$ cyclic.*

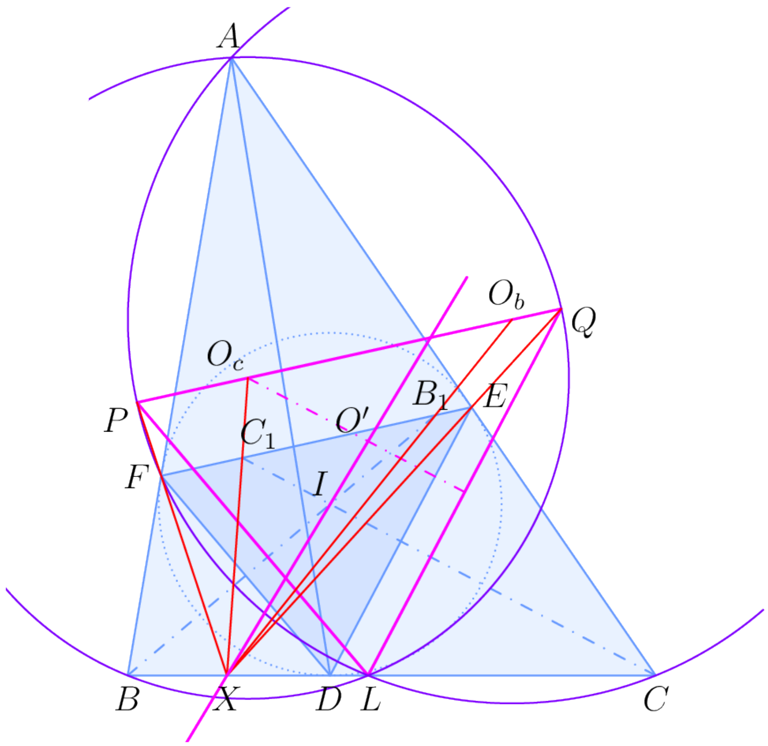
Proof. The first follows from $\angle AGX = \angle AGA' = 90^\circ = \angle AMX$, while the second from $\angle IQX = 90^\circ = \angle IMX$. □

Finish by power of a point at T: $\operatorname{Pow}(T, (A'EF)) = TE \cdot TF = TQ \cdot TI = TM \cdot TX = \operatorname{Pow}(T, (AMGX))$, as required.

🌲 3.9 [5♣] 13Shrg19

Let ABC be a triangle with circumcenter O and incenter I. The incircle is tangent to sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at A_0, B_0, C_0. Point L lies on $\overline{BC}$ so that $\angle BAL = \angle CAL$. The perpendicular bisector of $\overline{AL}$ meets BI and CI at Q and P, respectively. Let C_1 and B_1 denote the projections of B and C onto lines CI and BI. Let O_1 and O_2 denote the circumcenters of triangles ABL and ACL.

Prove that the six lines BC, PC_0, QB_0, C_1O_1, B_1O_2, and OI are concurrent.



Rename A_0, B_0, C_0, O_1, O_2 to D, E, F, O_c, O_b respectively.

> **Claim 1 –** $\triangle LQP$,$\triangle DEF$ are homothetic.

Proof. Observe that P, Q are midpoints of $\widehat{AL}$ on (ACL), (ABL) respectively, so that $\angle ALQ = \frac{B}{2}$; thus

$$\angle QLC = \frac{B}{2} + \angle ABL + \angle LAB = \frac{A+B}{2} = \frac{\pi - C}{2} = \angle ELC \Rightarrow \overline{LQ} \parallel \overline{DE}$$

and its cyclic variant, $\overline{LP} \parallel \overline{DF}$. Additionally $\overline{PQ}, \overline{EF} \perp \overline{AI}$ (by design) implies $\overline{PQ} \parallel \overline{EF}$; as the triangles have parallel sides, they're indeed homothetic. □

Let $X = \overline{BC} \cap \overline{QE} \cap \overline{PF}$ be the similicenter of these two triangles. I is the orthocenter of $\triangle LPQ$ because $\overline{CP} \perp \overline{DE} \parallel \overline{LQ}$ and $\overline{BQ} \perp \overline{LP}$ analogously.

> **Claim 2 –** $\overline{OI}$ is the common Euler line of $\triangle DEF$, $\triangle LPQ$, and passes through X.

Proof. It's well-known that $\overline{OI}$ is the Euler line of $\triangle DEF$. By homothety, the Euler line of $\triangle DEF$ is parallel to that of $\triangle LPQ$.
However, since these parallel lines share a point I (not at infinity), they must coincide. In order for a line to map to itself under a homothety, it must pass through the center– in other words, $X \in \overline{OI}$. □

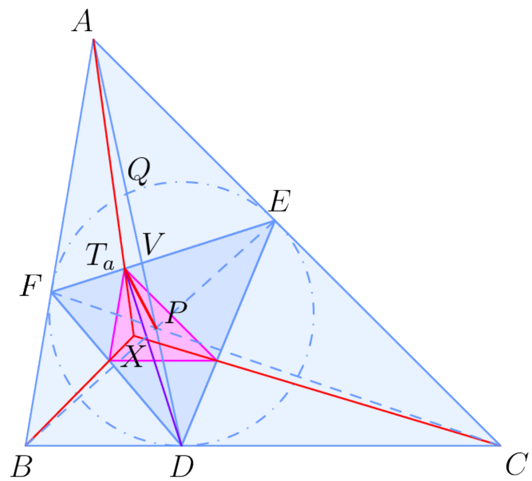
Let O' be the circumcenter of $\triangle LPQ$. It remains to prove that:

Claim 3 – O_c, C_1 correspond under the homothety.

Proof. Recall that $\overline{CI}$ is the perpendicular bisector of $\overline{DE}$ while $O_cL = O_cQ$ and $O'L = O'Q$ by design means $\overline{O_cO'}$ is the that of $\overline{LQ}$. By Iran lemma, $C_1 = \overline{CI} \cap \overline{EF}$, so it corresponds with $O_c = \overline{O_cO'} \cap \overline{PQ}$. □

🌲 3.10 [3♣] 13Bra6

The incircle of triangle ABC touches sides BC, CA and AB at points D, E and F, respectively. Let P be the intersection of lines AD and BE. The reflections of P with respect to EF, FD and DE are X, Y and Z, respectively. Prove that lines AX, BY and CZ are concurrent at a point on line IO, where I and O are the incenter and circumcenter of triangle ABC.

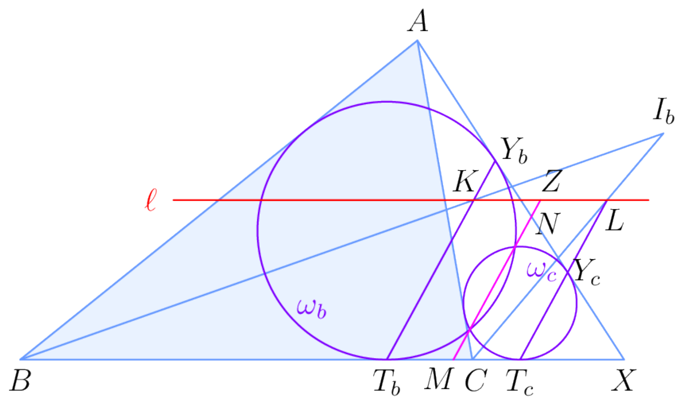


Let T_a be the foot from D to $\overline{EF}$, and X be the similicenter of homothetic triangles DEF, $I_aI_bI_c$. Clearly, it must also be the similicenter of their respective orthic triangles, so A, T_a, X collinear.

Next, let $V = \overline{AD} \cap \overline{EF}$, so that $(DV;AP) = -1$. Because $\angle DT_aV = 90^\circ$, $\overline{EF}$ must bisect $\angle AT_aP$, whence $P_a \in \overline{AT_aA'}$. Considering triangles ABC, DEF, and the orthic triangle of $\triangle DEF$, the concurrency holds by cevian nest.

3.11 [5♣] 04SLG7

For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.



Let I_b be the B-excenter, and ℓ be the A-midline in $\triangle ABX, ACX$. Also, let ω_b, ω_c be the incircles of $\triangle ABX, ACX$ respectively, respectively touching $\overline{BC}$ at T_b, T_c and $\overline{AX}$ at Y_b, Y_c. Finally, let M, N be respective midpoints of $\overline{T_bT_c}$, $\overline{Y_bY_c}$, so that $\overline{MNPQ}$ is the radical axis of ω_b, ω_c.

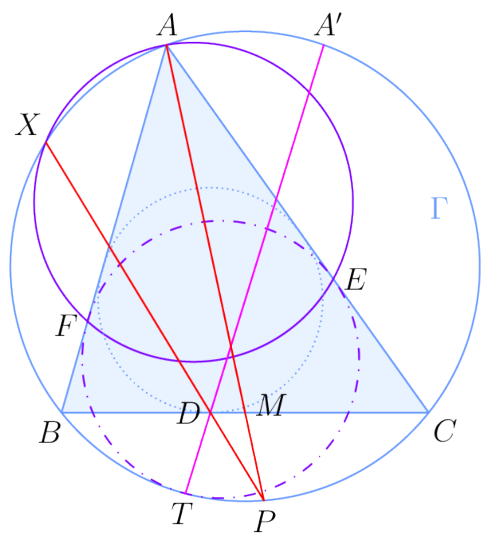
Recall the following result:

> **Lemma –** In a triangle ABC, if the incircle touches $\overline{AB}, \overline{AC}$ at X, Y, then the B-midline, the C-angle bisector, and $\overline{XY}$ are concurrent.

Consider the fixed points $K = \overline{BI_b} \cap \ell \ (\in \overline{T_bY_b})$ and $L = \overline{CI_b} \cap \ell \ (\in \overline{T_cY_c})$. It's routine to show that $\overline{MN}$ is midway between the parallel lines $\overline{T_bY_b}$, $\overline{T_cY_c}$ and thus passes through $Z = \frac{K+L}{2}$, also a fixed point.

♣ 3.12 [5♣] 16SLG2

Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. Denote by D the foot of the perpendicular from I to $\overline{BC}$. The line through I perpendicular to $\overline{AI}$ meets sides AB and AC at F and E respectively. Suppose the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.



Let $A' \in \Gamma$ denote the reflection of A in the perpendicular bisector of $\overline{BC}$, and T denote the contact point of the A-mixtilinear incircle with Γ. Since $\overline{TD}, \overline{TA}$ isogonal wrt $\angle BTC$, $A' \in \overline{TD}$. It's well-known that E, F lie on said mixtilinear incircle.

Claim – $(XT;BC) = -1$.

Proof. By a well-known lemma, $\overline{TE}, \overline{TF}$ respectively bisect $\angle ATC, \angle ATB$, so

$$\frac{FB}{FT}\frac{FA}{TA} = \frac{EA}{TA} = \frac{EC}{TC} \Rightarrow \frac{XB}{XC} \overset{\text{spiral}}{=} \frac{FB}{EC} = \frac{TB}{TC}.$$ □

Now, let $P = \overline{XD} \cap \Gamma \ (\neq X)$. Then

$$(\overline{AP} \cap \overline{BC}, \infty_{BC}; B, C) \overset{A}{=} (PA'; BC) \overset{D}{=} (XT; BC) \overset{\text{claim}}{=} -1$$

means $\overline{AP}$ bisects $\overline{BC}$. In other words, $P = \overline{XD} \cap \overline{AM} \cap \Gamma$ as required.

Remark. Should be [3♣] instead of [5♣]? No intention of casting aspersion, oops...

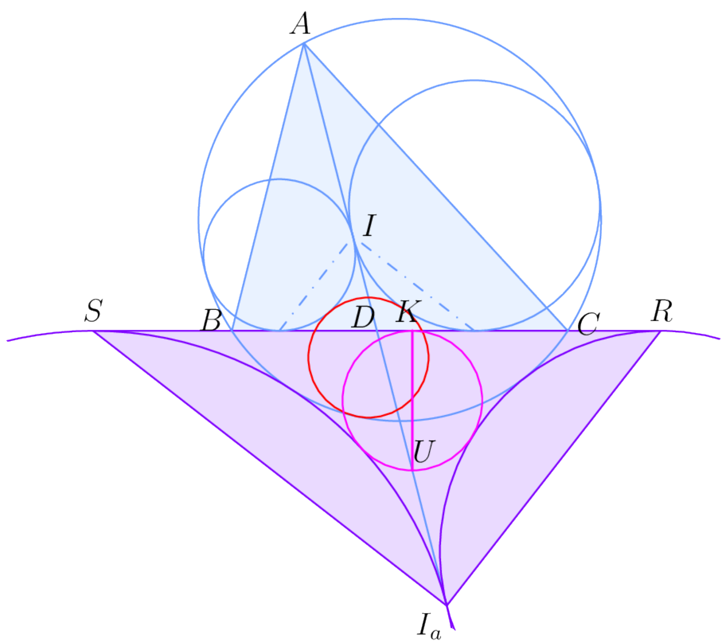
3.13 [9♣] Grant's Amerigeo

(in ARCH, `H1746385`.)

Convex pentagon $ABMCN$ is inscribed in circle Γ with diameter $\overline{MN}$, with $BM = CM$. Two distinct circles ω_B and ω_C are drawn, each tangent to segments AM and BC, and internally tangent to Γ. Finally, we draw a circle γ externally tangent to ω_B and ω_C, and internally tangent to Γ at a point W on arc $\widehat{BMC}$ of Γ.

(a) Prove that $\overline{AM}$ and $\overline{WN}$ meet on γ.

(b) Prove that $\overline{AM}$ passes through one of the intersections of γ and the A-mixtilinear incircle.



(asy'd without Geogebra conversion, despite lack of productivity in doing so...)

Let I, I_a, α be the incenter, A-excenter, and A-excircle respectively. Define $D = \overline{AI} \cap \overline{BC}$ and $T = \alpha \cap \overline{BC}$.

Claim 1 – ω_b, ω_c touch $\overline{AD}$ at I.

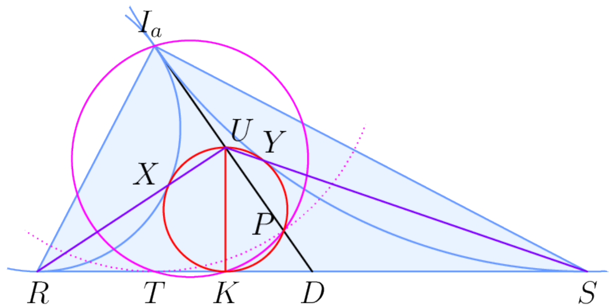
Proof. Recalling the properties of curvilinear incircles, the line through the tangency points of ω_b with $\overline{AD}$, $\overline{BD}$ passes through I. It follows that the former tangency point is simply I. □

Now, let i denote the inversion about A with power $AB \cdot AC = AM \cdot AD = AI \cdot AI_a$ composed with a reflection in $\overline{AM}$. Let the images of ω_b, ω_c under i (which we call ω_b', ω_c') touch $\overline{BC}$ at R, S, so that $DI_a = DR = DS$.

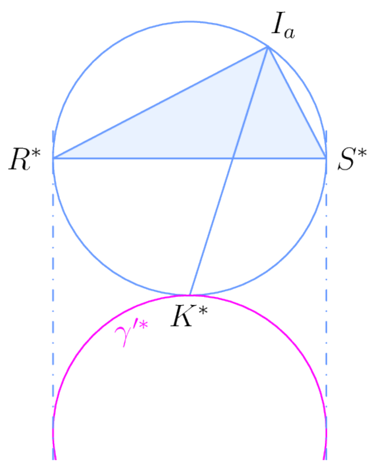
Lemma – If two circles α, *beta* touch a segment AB at A, B respectively, and each other at P, then $\angle APB = 90°$.

Proof. If M is the midpoint of $\overline{AB}$, then since it's on the radical axis of α, β, we have $MA = MB = MP$, which implies the result. □

$\Longrightarrow \triangle RI_aS$ right. Also, let γ map to a circle γ' tangent to ω_b', ω_c', and $\overline{BC}$ at some point K.



Claim 2 – $\overline{I_aK}$ bisects $\angle RI_aS$.



Proof. Invert at I_a with arbitrary power; then, if R, S map to R', S', γ' maps to a circle γ'' tangent to (the tangents to $I_aR'S'$ at R', S') and $(I_aR'S')$ itself. By symmetry (about the perpendicular bisector of $\overline{R'S'}$), γ'' touches $\widehat{BC}$ at its midpoint. The angle bisection directly follows. □

Let U be the antipode of K on γ'. To get rid of $\triangle ABC$, we'll need:

Claim 3 – $U \in \overline{AI_aD}$.

Proof. Let $X = \omega_b' \cap \gamma'$, $Y = \omega_c' \cap \gamma'$. By homothety/etc, we obtain $U \in \overline{RX}, \overline{SY}$, while by similar triangles,

$$UR \cdot UX = UK^2 = US \cdot UY$$

means U is on the radical axis of ω_b', ω_c', which is $\overline{I_aD}$. (In fact, $UK = UI_a \dots$) □

Now we can obtain **(a)**: $\overline{WN} \overset{i}{\Leftrightarrow} (AEK)$, so it's equivalent to show:

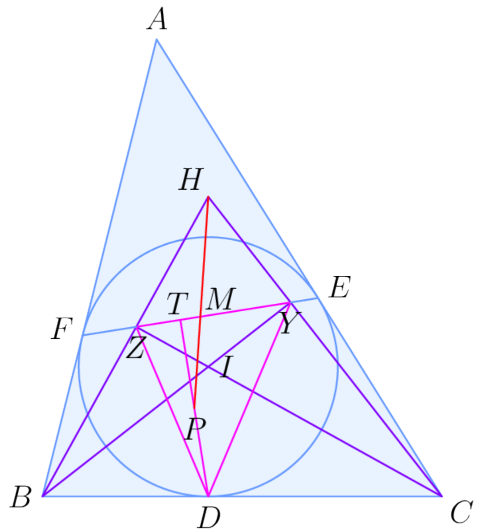
Claim 4 – $UKAE$ cyclic.

Proof. $\angle UKE = 90^\circ = \angle UAE$ using last claim. □

Finally, let $P \in \overline{I_aD}$ be the reflection of T in $\overline{AK}$, so that $AT = AP \Rightarrow T \in \alpha$. Performing i on the statement of **(b)** changes it to "α, $\overline{I_aD}$ meet at a point (namely, P). Indeed, $P \in \gamma'$ because $\angle KPU = \angle KPA = 90^\circ$, the end!

3.14 [3♣] 09IrnTST9

In triangle ABC, D, E and F are the points of tangency of incircle with the center of I to BC, CA and AB respectively. Let M be the foot of the perpendicular from D to EF. P is on DM such that $DP = MP$. If H is the orthocenter of BIC, prove that PH bisects EF.



Recalling the lemma famously associated with this problem, let $Y, Z \in \overline{EF}$ be the feet from B, C to $\overline{CI}$, $\overline{BI}$ respectively. Then, we can get rid of triangle ABC as follows:

Iran TST 2009/9 reduced

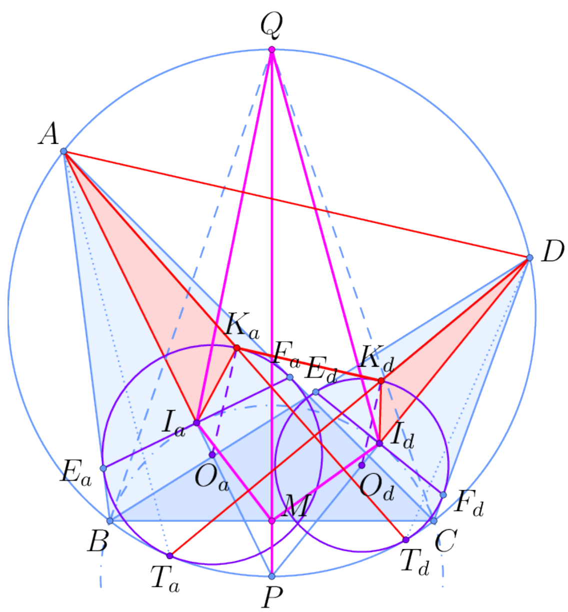
In triangle HBC, D, Y, Z are the respective feet of altitudes, and I is the orthocenter. If M is the foot from I to $\overline{YZ}$ and P is the midpoint of the altitude from D to $\overline{YZ}$, then P, M, H are collinear

But this is just the below lemma from EGMO chapter 4, applied to $\triangle DYZ$, so we're done…

Lemma (EGMO Ch. 4) **–** In a triangle ABC, the midpoint of the A-altitude, the intouch point on $\overline{BC}$, and the A-excenter are collinear.

3.15 [5♣] 20MOP1Z

Let $ABCD$ be a quadrilateral inscribed in circle Ω. Circles ω_A and ω_D are drawn internally tangent to Ω, such that ω_A is tangent to $\overline{AB}$ and $\overline{AC}$ while ω_D is tangent to $\overline{DB}$ and $\overline{DC}$. Prove that we can draw a line parallel to $\overline{AD}$ which is simultaneously tangent to both ω_A and ω_D.



Solution by **v4913**. Define…

- P, Q as the respective midpoints of $\widehat{BC}, \widehat{BAC}$, I_a, I_d as the respective incenters of ω_a, ω_d, and M as the midpoint of $\overline{BC}$;
- O_a, O_d as respective centers of ω_a, ω_d, and $\gamma = (BI_aI_dC)$ (with center P), so that $\overline{QB}, \overline{QC}$ touch γ;
- $E_a, F_a, T_a = \omega_a \cap \overline{AB}, \overline{AC}, \Omega$; K_a as the intersection of $\overline{AT_d}$ with ω_a closer to A, and their symmetric variants. It's well-known that Q, I_a, T_a collinear, and that I_a is the midpoint of $\overline{E_aF_a}$;
- s_a as the spiral similarity mapping $\gamma \to \omega_a$ and thus $Q, B, C, M \to A, E_a, F_a, I_a$. Since $\measuredangle K_aAF_a = \frac{1}{2}\widehat{T_dC} = \measuredangle I_dQC$ by design, we also have $(K_a \xrightarrow{s_a} I_d)$.

We contend that $\overline{K_aK_d}$ is the desired tangent, using the following two parts:

Claim 1 – $\overline{O_aK_a}, \overline{O_dK_d} \perp \overline{AD}$.

Proof. We angle chase:

$$\measuredangle(\overline{O_aK_a}, \overline{AD}) = \measuredangle O_aK_aA + \measuredangle K_aAD \overset{s_a}{=} \measuredangle PI_dQ + \measuredangle T_dQD = \measuredangle(\overline{PI_d}, \overline{QD}) = \frac{1}{2}\widehat{PQ} = 90^\circ.$$

The claim follows by symmetry. □

Claim 2 – $\overline{K_aK_d} \parallel \overline{AD}$.

Proof. Let $X = \overline{AT_d} \cap \overline{DT_a}$, so that it'll suffice to prove $AK_a/AX = DK_d/DX$. Indeed, using s_a, $AK_a = QI_d \cdot \frac{AI_a}{QM}$ and similarly $DK_d = QI_a \cdot \frac{DI_d}{QM}$. We thus have:

$$\frac{AK_a}{DK_d} = \frac{AI_a/QI_a}{DI_d/QI_d} = \frac{AT_a/QP}{DT_d/QP} = \frac{AX}{DX}. \qquad \square$$

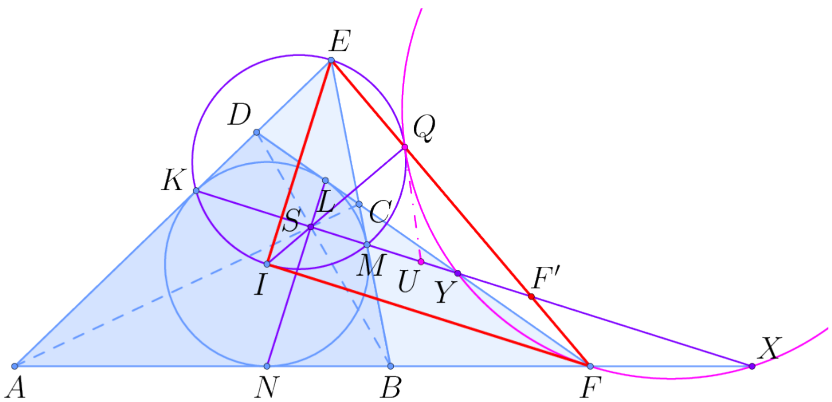
From the previous two claims, $\overline{O_aK_a}, \overline{O_dK_d} \perp \overline{K_aK_d}, \overline{AD}$ so $\overline{K_aK_d}$ touches both ω_a, ω_d while also parallel to $\overline{AD}$, as required.

3.16 [9♣] 20IGOA4

Convex circumscribed quadrilateral $ABCD$ with its incenter I is given such that its incircle is tangent to $\overline{AD}$, $\overline{DC}$, $\overline{CB}$, and $\overline{BA}$ at K, L, M, and N. Let $E = \overline{AD} \cap \overline{BC}$ and $F = \overline{AB} \cap \overline{CD}$. Let $X = \overline{KM} \cap \overline{AB}$ and $Y = \overline{KM} \cap \overline{CD}$. Let $Z = \overline{LN} \cap \overline{AD}$ and $T = \overline{LN} \cap \overline{BC}$.

Prove that the circumcircle of triangle $\triangle XFY$ and the circle with diameter EI are tangent if and only if the circumcircle of triangle $\triangle TEZ$ and the circle with diameter FI are tangent.

> imagine doing both directions
>
> CyclicISLscelesTrapezoid



We contend that (XFY), (EI) tangent $\iff$ $\overline{KM} \perp \overline{LN}$, which is just another expression of '$ABCD$ bicentric'.

Call the incircle ω. Define $S = \overline{KM} \cap \overline{LN}$ and Q as the Miquel point of $KLMN$ aka the inverse of S wrt ω, which obviously lies on the polar $\overline{EF}$ of S wrt ω. It follows that $\angle SQE = \angle SQF = 90°$.

Let $F' = \overline{QF} \cap \overline{KM}$ U be the midpoint of $\overline{SF'}$.

> **Claim –** $\overline{QF}$ always bisects $\angle XQY$, and $UM \cdot UK = US^2 = UX \cdot UY$, so U lies on the radical axis of $(EIMK)$, (XFY).

Proof. By Brianchon on $ABMCDK$, $S = \overline{AB} \cap \overline{CD}$ as well. Apply DIT to $\overline{KM}$ and quadrilateral $ABCD$ and project to Q, to obtain an involutive pairing $i : Q(XY; SS; KM)$. The last two pairs imply that i is just reflection in $\overline{QS}$, so $\overline{QS}$ bisects $\angle XQY$. As $\overline{QF} \perp \overline{QS}$, it must also bisect the same angle: $i : Q(F'F')$.
By these right angles and angle bisections, $(SF'; MK) = (SF'; XY) = -1$, so the last result follows by midpoints of harmonic bundles lemma. □

Now because $\angle SQF' = 90°$, we have $UQ = US = UF' \implies (EI), (XQY)$ tangent at Q; this means that the desired is equivalent to $FQXY$ cyclic.

> *Note.* Actually, there are two circles through X, Y, but one of them is extraneous by configuration issues.

Next, we show that this happens iff $FX = FY$.

"If" direction Since $\angle FQX = \angle YQF$, triangles FQX, FQY have equal circumradii, so their circumcircles either coincide or are reflections of each other in $\overline{QF}$. If they were to be reflections, we'd obtain two possibilities, each absurd in the context of the problem: (i) Q, X, Y collinear $\Rightarrow \overline{XY}, \overline{EF}$ coincide; and (ii) X, Y reflections in $\overline{QF} \Rightarrow \overline{XY}, \overline{SQ} \perp \overline{EF}, \overline{XY} \parallel \overline{SQ}$. Thus $(FQX) = (FQY)$ as required.

"Only if" direction $\angle YQF = \angle FQX \Rightarrow \widehat{YF} = \widehat{FX}$, trivial;

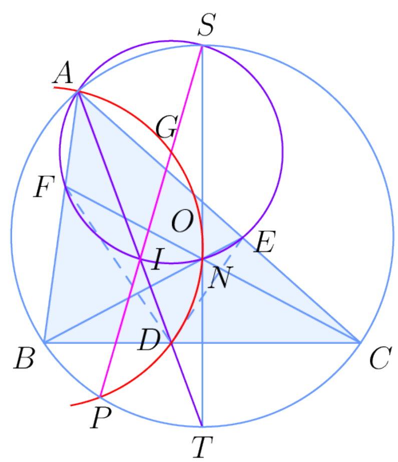
To finish the problem, observe that $FX = FY$ is equivalent to $\overline{MK}$ making equal angles with $\overline{AB}, \overline{CD}$. As $FN = FL$ as well, $\overline{LN}$ (always) makes equal angles with the same two lines. Since K, L, M, N form a convex polygon, this is in turn equivalent to $\overline{KM} \perp \overline{LN}$, completing the proof.

🌲 3.17 [5♣] 21CHMMC6

Let ABC be a triangle with circumcenter O. The interior bisector of $\angle BAC$ intersects BC at D. Circle ω_A is tangent to segments AB and AC and internally tangent to the circumcircle at P. Let E and F be the points at which the B-excircle and C-excircle are tangent to AC and AB.Suppose that lines BE and CF pass through a common point N on the circumcircle of AEF.

(a) Prove that the circumcircle of PDO passes through N.

(b) Suppose that $PD/BC = 2/7$. Find, with proof, the value of $\cos \angle BAC$.



First part Let $\Omega = (ABC)$, $\omega = (AEF)$, the latter with center G. Also let S, T be the respective midpoints of $\widehat{BC}, \widehat{BAC}$. Since $BF = CE$, $S \in \omega$ by spiral. Also, as customary in bash solutions, let $a = BC, b = CA, c = AB, s = \frac{a+b+c}{2}$.

Claim 1 – D is the Miquel point of $AENF$.

Proof. Let D' be the Miquel point of $AENF$ and thus $BFCE$ as well. Again, since $BF = CE$ and there exists a spiral similarity at D' mapping $B, F \to C, E$, that spiral similarity must in fact be a rotation. Thus, $D'B = D'E$, so $\overline{AD'}$ bisects $\angle BAC$. Additionally, by Brokard, D' lies on the line through $B = \overline{EN} \cap \overline{AF}$ and $C = \overline{FN} \cap \overline{AE}$, which pins down the position of D'. □

Remark. This looks like one of **CyclicISLscelesTrapezoid**'s discarded problem ideas…

Claim 2 – $b + c = 2a$ and $I \in \omega$ as well.

Proof. Since $AFCD$ cyclic by definition of Miquel point, power of a point gives

$$a\frac{ac}{b+c} = BD \cdot BC = BF \cdot BA = c(s-a) \Rightarrow (2a-b-c)(a+b+c) = 0 \Rightarrow b+c = 2a.$$

Now if we let E', F' be the feet from I to $\overline{AC}, \overline{AB}$, then $\triangle FF'I \cong \triangle EE'I$ implies $\angle EIF = \angle E'IF' = \pi - A$, and $I \in \omega$ as claimed. □

Claim 3 – *N is the reflection of T in $\overline{BC}$.*

Proof. Using the cyclic quads associated with Miquel points, we've $\angle NBD = \angle CFD = \angle CAD = \frac{A}{2}$; similarly, $\angle NCD = \frac{A}{2} = \angle NDC$. Noting that $T \neq N$ also satisfies these angle conditions, it follows that the two points are indeed reflections in $\overline{BC}$. □

From the last claim, N must be the foot from I to $\overline{ST}$, while it's well-known that P is the foot from T to $\overline{SI}$. From the second claim, D is the midpoint of $\overline{TI}$. Obviously, G, O are the respective midpoints of $\overline{SI}, \overline{ST}$ while A is the foot from S to $\overline{IT}$.

To conclude, A, G, O, N, D, P lie on the nine-point circle of $\triangle IST$.

Second part Let $PD = 2, BC = 7$. Since $\angle IPT = \angle INT$ by the claims from the previous part, D is the center of $(INTP)$; we then obtain $TI = 4$ so $\triangle TBC$ has sides $4, 4, 7$. We obtain

$$\cos A = -\cos \angle BTC = \frac{7^2 - 2 \cdot 4^2}{2 \cdot 4^2} = \boxed{\frac{17}{32}}.$$