Youth EUCLID MO 2022

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January 18, 2023

The test was held in mid-October 2022.

Contents

0 Acknowledgements

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Sponsors

None yet :p

1 Problems

Have fun :)

Problem 1. Determine all pairs of rational numbers $x, y > 0$ satisfying

$$3x^2 + 2xy + 3y^2 = \frac{1}{x^2} + \frac{1}{y^2}.$$

Problem 2. Let a, b, c, d, e, f be positive integers. Evan is building with a large supply of three types of blocks:

1. Blocks with width a, length 1, height 1
2. Blocks with width 1, length b, height 1
3. Blocks with width 1, length 1, height c

If Evan can place blocks to form a rectangular prism with width d, length e, height f, show he could build a prism with identical dimensions and orientation with blocks of just one type.

(Evan cannot change a block's orientation, so he cannot rotate a block or flip it on a side during building.)

Problem 3. Variable triangles ABC and DEF share a fixed incircle ω and circumcircle Ω. Let ω_a be the A-mixtilinear incircle in $\triangle ABC$, and similarly for ω_d. Determine (as the triangles vary) the locus of the intersection of the common external tangents to these two circles.

(The X-mixtilinear incircle of a triangle XYZ is the circle tangent to segments XY, XZ as well as the circumcircle internally.)

2 Solutions

2.1 YEMO 1, by Neal Yan

Determine all pairs of rational numbers $x, y > 0$ satisfying

$$3x^2 + 2xy + 3y^2 = \frac{1}{x^2} + \frac{1}{y^2}.$$

Rearrange the equation as

$$(x^3 - y^3)^2 = (x^2 + y^2)^3 - (x^2 + y^2);$$

now, recalling that the elliptic curve $y^2 = x^3 - x$ has no nontrivial rational points, it follows that $x^3 - y^3 = 0$. Plugging in $x = y$ into the given equation yields no solutions, so we are done.

Disclaimer– *why is elliptic curves on a high school math contest?*

This problem takes deep inspiration from Revenge ELMO 2022/2, in which the associated Diophantine equation rearranges as

$$(2xy - 3x - 3y)^2 = (xy - 1)^3 - (xy - 1),$$

the exact same elliptic curve equation as used in this problem.

(**squareman** notes that the $n = 4$ case of Fermat's last theorem was intended as the punchline instead, but due to $\mathbb{Q}$-birational equivalence (as noted in Niven's *Introduction to the theory of numbers*), I got the elliptic curve equation via a much more direct rearrangement.)

Remark. Haha, the disclaimer was longer than the solution…

2.2 YEMO 2, by Evan Chang

Let a, b, c, d, e, f be positive integers. Evan is building with a large supply of three types of blocks:

1. Blocks with width a, length 1, height 1
2. Blocks with width 1, length b, height 1
3. Blocks with width 1, length 1, height c

If Evan can place blocks to form a rectangular prism with width d, length e, height f, show he could build a prism with identical dimensions and orientation with blocks of just one type.

(Evan cannot change a block's orientation, so he cannot rotate a block or flip it on a side during building.)

Define $\omega_a, \omega_b, \omega_c$ as arbitrary primitive a, b, cth roots of unity respectively, and refer to the given blocks collectively as **beams**.

The pith of the problem: *assign to each unit cube (x, y, z) the complex number $\omega_a^x \omega_b^y \omega_c^z$.*

Claim – The total sum of all labels is zero.

Proof. Observe that in each beam, the sum is

$$\text{something} \cdot (\omega_a^1 + \cdots + \omega_a^a) \text{ (or cyclic variants)} = 0.$$

Sum over all beams, hence done. □

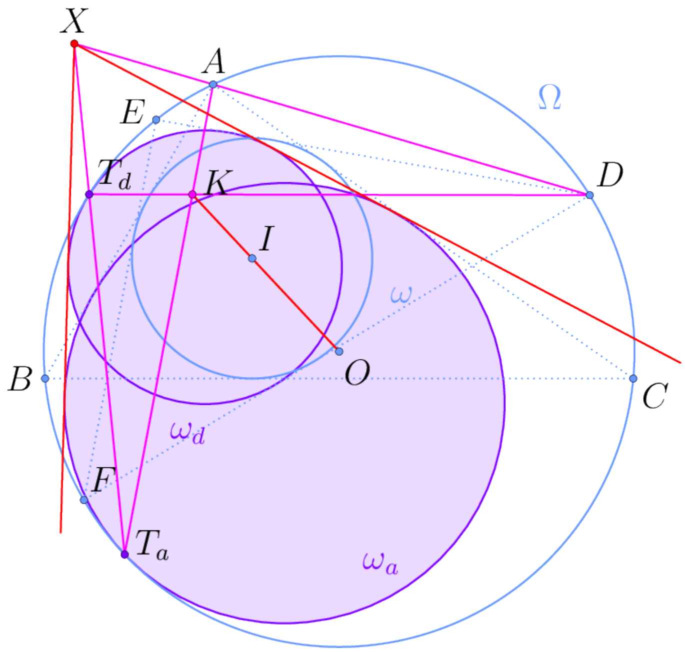
Meanwhile, the sum of the labels can be computed globally as

$$\prod_{\text{cyc}} (\omega_a^0 + \omega_a^1 + \cdots + \omega_a^{d-1}).$$

$a \mid d$, $b \mid e$, or $c \mid f$ follows directly.

2.3 YEMO 3, by Neal Yan

Variable triangles ABC and DEF share a fixed incircle ω and circumcircle Ω. Let ω_a be the A-mixtilinear incircle in $\triangle ABC$, and similarly for ω_d. Determine (as the triangles vary) the locus of the intersection of the common external tangents to these two circles.



Let the mixtilinears touch Ω at T_a, T_d, and let K, X denotes the exsimilicenters of (Ω, ω) (fixed) and (ω_a, ω_d), the desired. Applying Monge to all possible triplets out of the four circles implies that $K = \overline{AT_a} \cap \overline{DT_D}$ while $X = \overline{AD} \cap \overline{T_aT_d}$. By Brokard, it follows that X lies on the polar of K wrt Ω, a **fixed line**.