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Comparative Study of Beam Theories on the Effect of Span-Depth Ratio for Symmetric and Un-symmetric Loadings

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Abstract. In the present work, a comparative study is done between Shear-indeformable (Euler-Bernoulli) model and Shear-deformable (Timoshenko) model for two different types of beams (simply supported and fixed beam) under uniformly distributed loading condition and analysed for different Span-Depth ratio (L/D). For un-symmetric loading, a simply supported beam subjected to external point moment is also analysed for different L/D ratios to check the behaviour of beam. Transverse Displacement is taken as a yardstick. Shape functions for Euler-Bernoulli and Timoshenko beam models are developed in MATLAB software by using the Finite Element Method for calculating deflection at different points on the beam. It was observed that for large L/D ratio, the deflections are same for both the beam theory models but as L/D ratio decreases, the deflections for both the beam theories vary from each other. For small L/D ratio, Timoshenko beam model gives more accurate results, since the Timoshenko beam theory is a higher order beam theory than the Euler-Bernoulli beam theory, it is known to be superior in predicting the response of the deep beam.

1. Introduction

The static and dynamic characteristics of beam element in a structure are evaluated by using the classical or refined beam theories. Euler-Bernoulli beam theory (EBT) is generally used for the analysis of the distribution of stresses and displacements in the beam element with a small thickness. It has been observed that in case of deep beams, the beam section warp under loaded conditions. This is because of shear deformations are significant in the deep sections and cannot be neglected.

As per IS Code – 456 (2000), a beam is considered as a deep beam when the ratio of effective span (L) to overall depth (D) is:-

1. L/D < 2 for simply supported beam
2. L/D < 2.5 for continuous beam

Timoshenko beam theory is the extension of Euler-Bernoulli beam theory. The basic difference in the assumptions of EBT and TBT is that in EBT the plane section remains plane and perpendicular to neutral axis before and after bending, but in TBT plane section before and after bending remains plane but is not perpendicular to the neutral axis. Mosavi et al. [9] investigated beam equation with track motion dynamic stability using fourth order partial differential of the Euler-Bernoulli and Timoshenko model on an Elastic Foundation. Labuschagne et al. [3] considered three different linear theories: Euler-Bernoulli, Timoshenko and two-dimensional elasticity models for a cantilever beam analysis. It is shown that the Timoshenko model gives remarkably accurate results compared to other two models. Edem B. [7] used a new model with two-node Timoshenko beam element for representing beam rotation in a shear deformable beam. Sayyad [5] compared various shear deformation theories for the free vibration analysis of thick isotropic beams. Wang C. [17] presented single span Timoshenko and Euler-Bernoulli beams subjected to transverse loading including the effect of transverse shearing strain for deflection and stresses without f1exural-shear-deformation analysis which is more complicated. Patel R. et al [15] studied the effect of depth of an RCC beam using method of initial functions (MIF). Three different cases were discussed with respect to span-depth ratios which were compared different theories with the results from ANSYS software.

In the present study, MATLAB program is developed to compare various beam theories for the analyses of two different types of beams i.e. simply supported and fixed for different span-depth ratios for symmetric and un-symmetric loadings especially for deep beam section.

2. Mathematical formulation

2.1. Euler Bernoulli beam theory (EBT)

Euler–Bernoulli beam theory (also known as classical beam theory) is a simplified form of the linear theory of elasticity. It covers the case for small deflections of a beam that are subjected to lateral loads only. It is thus a special case of Timoshenko beam theory.

Governing equations (EBT) are as follows

$$\frac{M}{EI} = \frac{d^2y}{dx^2} \tag{1}$$

$$\frac{dM}{dx} = \frac{d^3y}{dx^3} \tag{2}$$

$$\frac{d^4y}{dx^4} = \frac{q}{EI} \tag{3}$$

where, M = Bending Moment; E = Modulus of Elasticity; I = Second Moment Inertia

2.2. Timoshenko beam theory (TBT)

Unlike the Euler-Bernoulli beam, Timoshenko beam theory accounts for shear deformation and rotational inertia effects. Therefore, the Timoshenko beam can be used for the analysis of deep beams. The stiffness of the Timoshenko beam is lower than the Euler-Bernoulli beam, which results in larger deflections under static loading and buckling. As the effect of shear is also considered in TBT, the plane section no longer remains normal to the neutral axis of the beam. The rotation of the plane section is the summation of bending rotation (θ) and rotation due to shear (γ). So the total rotation of the plane can be written as –

$$\frac{dw}{dx} = \theta - \gamma \tag{4}$$

$$\frac{d\left(EI\,\frac{d\theta}{dx}\right)}{dx} = GA\left(\theta - \frac{dw}{dx}\right) \tag{5}$$

where, E= Elastic Modulus; I= Second Moment Inertia; G= Shear Modulus; A=Area of cross-section

2.3. Stiffness matrix

The finite element stiffness matrix for Euler Bernoulli and Timoshenko beam models are-

For Euler Bernoulli beam –

$$K_E = \frac{\boldsymbol{EI}}{\boldsymbol{L^3}}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{6}$$

For Timoshenko beam –

$$K_E = \frac{EI}{(1+\emptyset)L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4+\emptyset)L^2 & -6L & (2-\emptyset)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2-\emptyset)L^2 & -6L & (4+\emptyset)L^2 \end{bmatrix} \tag{7}$$

$$\text{where, } \emptyset = \frac{12EI}{L^2GA} \tag{8}$$

Shape Function for EBT

$$[N] = [\, N_1 \;\; N_2 \;\; N_3 \;\; N_4 \,] \tag{9}$$

where,

$$\left.\begin{aligned} N_1 &= 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}; \\ N_2 &= x - \frac{2x^2}{L} + \frac{x^3}{L^2}; \\ N_3 &= \frac{3x^2}{L^2} - \frac{2x^2}{L^3}; \\ N_4 &= \frac{x^3}{L^2} - \frac{x^2}{L} \end{aligned}\right\} \tag{11}$$

Shape Function for TBT

$$[N] = [\, N_1 \;\; N_2 \;\; N_3 \;\; N_4 \,] \tag{10}$$

where,

$$\left.\begin{aligned} N_1 &= 1 - \frac{x}{L}\;; \\ N_2 &= \frac{-x^2}{2L} + \frac{x}{2}\;; \\ N_3 &= \frac{x}{L}; \\ N_4 &= \frac{x^2}{2L} - \frac{x}{2} \end{aligned}\right\} \tag{12}$$

Accuracy of the formulated/generated shape function of TBT is examined with the previously available investigations. The geometric, loading and the boundary conditions are same as stated in the literature, Patel R. et al. [8].Table 1 illustrates the values of transverse deflection of the simply supported beam with L/D ratio of 7.5. The present model is in good agreement with the transverse deflection obtained by Patel R. et al [14].

Table 1. Validation of programming

L/D ratio	Deflection Patel et al. [14]	Deflection Present model	Percentage difference $\frac{\delta 1 - \delta 2}{\delta 1} \times 100$
7.5	177.58 (mm)	182.42 (mm)	2.73 %

3. Beam under consideration

The analysis is carried out for a beam (Figure 1)made up of isotropic material subjected to uniformly distributed load (UDL), q = 20kN/m with dimensions as length, L = 3 m, width, B = unity and depth, D = 0.375, 1, 1.2, and 1.5m. The following material properties are taken, Elastic Modulus, E = 22,360×10^{3}kN/m^2, Shear Modulus, G = 10,164×10^3 kN/m^2, δ_E= maximum deflection of Euler Bernoulli beam element, δ_T = maximum deflection of Timoshenko beam element.

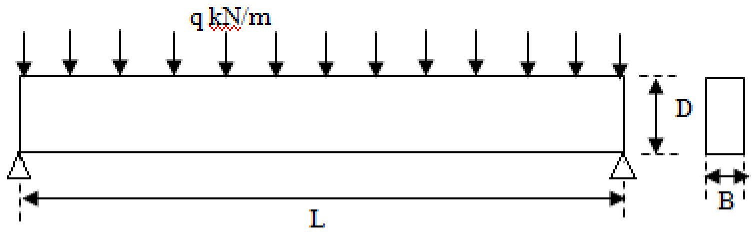

Figure 1. Schematic diagram of beam under consideration

4. Results and discussions

In present work, the analyses carried out were as follows-

Case 1. Beam under consideration was examined for symmetrical loading subjected to uniformly distributed load for simply supported and fixed beam.

Case 2. For unsymmetrical loading the beam was subjected to moment at the left end and examined for deflection using the formulated EBT and TBT for simply supported beam.

The variation of deflection (mm) with distance along the beam is discussed in figure 2 to figure 4 and in table 2 to table 4.

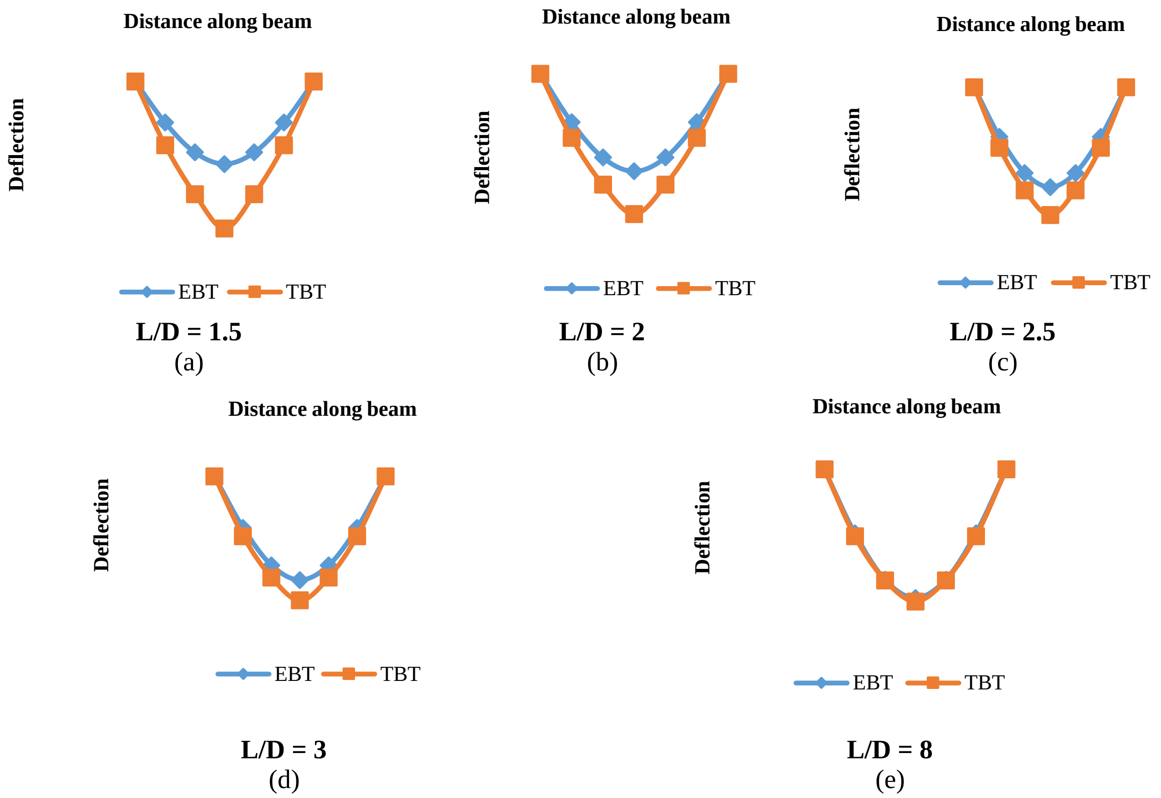

Figure 2. Variation in deflection (mm) of simply supported beams with different L/D ratio calculated using Euler and Timoshenko theories.

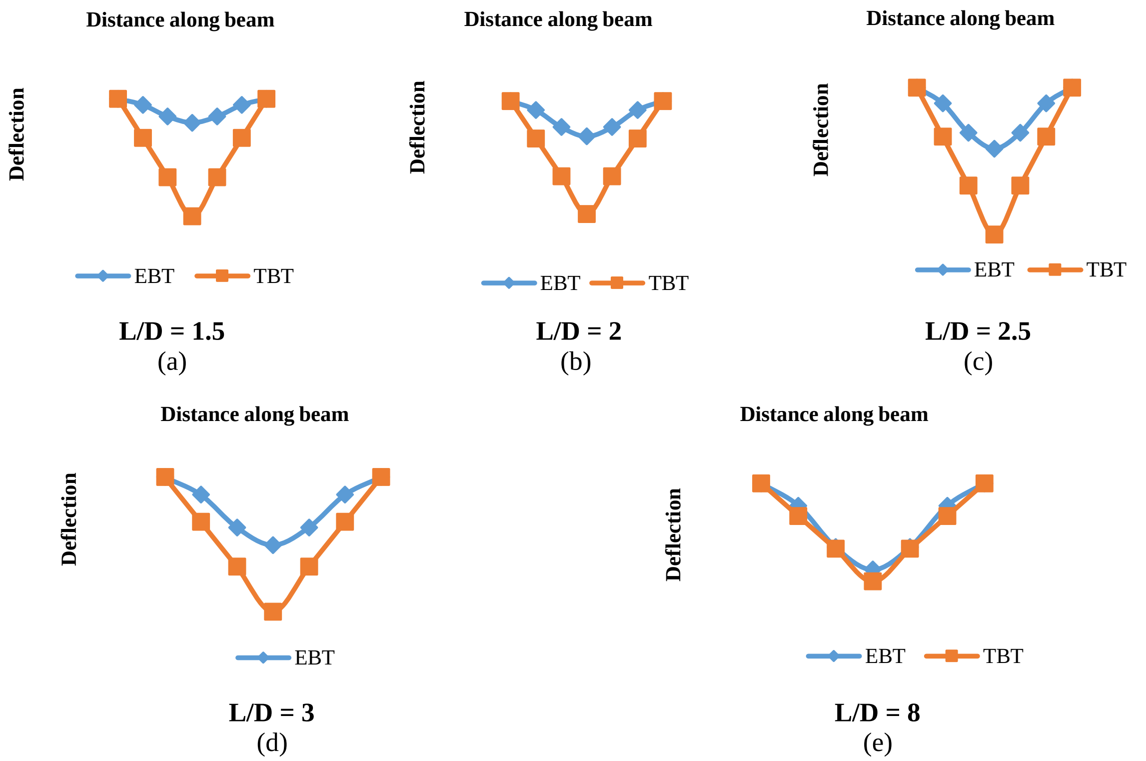

Figure 3. Variation in deflection (mm) of fixed beams with different L/D ratio calculated using Euler and Timoshenko theories.

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As observed in figure 2 to figure 3 and table 2 and table 3 for different beams i.e. simply supported and fixed with the increase of L/D ratio for symmetric loading, the deflections calculated using the Timoshenko theory and Euler theory are so close and beams with lower L/D ratio variation in deflection was large because of the shear effect in Timoshenko theory which is not included in Euler theory.

Table 2. Variation in deflection of simply supported beams calculated using Euler and Timoshenko theories.

Sl. No.	L/D ratio	δ_E(mm)	δ_T(mm)	$\frac{\delta_T - \delta_E}{\delta_T} \times 100\ \%$
1	1.5	-1.42×10^{-3}	-2.52×10^{-3}	36.89
2	2	-3.35×10^{-3}	-4.83×10^{-3}	30.64
3	2.5	-6.55×10^{-3}	-8.40×10^{-3}	22.02
4	3	-1.13×10^{-2}	-1.35×10^{-2}	16.29
5	8	-2.15×10^{-1}	-2.21×10^{-1}	2.71

Table 3. Variation in Deflection of fixed beam calculated using Euler and Timoshenko theories.

Sl. No.	L/D ratio	δ_E(mm)	δ_T(mm)	$\frac{\delta_T - \delta_E}{\delta_T} \times 100\ \%$
1	1.5	-0.00028301	-0.0013899	79.64
2	2	-0.00067084	-0.0021466	68.75
3	2.5	-0.0013102	-0.003155	58.47
4	3	-0.0022641	-0.0044778	49.44
5	8	-0.0429338	-0.048837	12.09

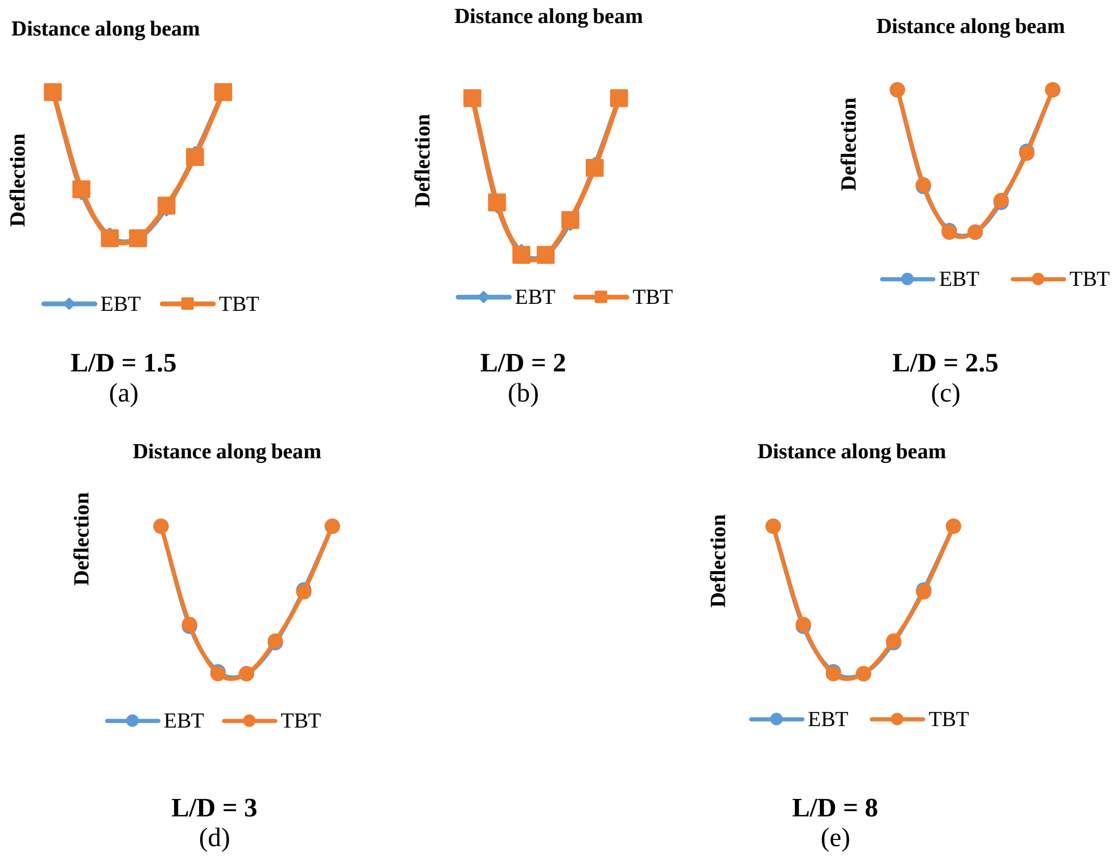

Figure 4. Variation in deflection (mm) of simply supported beams with different L/D ratio calculated using Euler and Timoshenko theories.

Similarly, when simply supported beam was subjected to moment at left end for un-symmetric loading i.e. Case-2, the results show that the Timoshenko beam model is remarkably accurate. As observed from figure 4, the TBT and EBT gave same values of deflection because the shear effect is negligible for this loading condition as only moment is applied in beam section.

Table 4. Variation in deflection of simply supported beams calculated using Euler and Timoshenko theories.

S.No.	L/D ratio	δ_E(mm)	δ_T(mm)	$\frac{\delta_T - \delta_E}{\delta_T} \times 100$ %
1.	1.5	-1.162×10^{-3}	-1.180×10^{-3}	1.525
2.	2	-2.753×10^{-3}	-2.795×10^{-3}	1.502
3.	2.5	-5.378×10^{-3}	-5.458×10^{-3}	1.465
4.	3	-9.295×10^{-3}	-9.432×10^{-3}	1.442
5.	8	-176.27×10^{-3}	-178.84×10^{-3}	1.437

5. Conclusions

In the present work, a MATLAB program is developed for the comparative study of beam theories to study the effect of span-depth ratio for different beam sections subjected to symmetrical UDL and un-symmetrical loading. The main objective was to formulate shape function for TBT and a program which can be used for the analysis of deep beam sections. Based on the obtained results, the following conclusions can be drawn –

1. The developed MATLAB programme is capable of analysing different deep beam sections and yields good agreement with literature.
2. The MATLAB programme can be considered for different design parameters like depth for deep beam, boundary conditions, and loading conditions.

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