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Apr 2, 2024

Lecture 11 a – Ranking & Importance in Networks.

Note : Dan Larremore guest lecture on Apr 4.
Plans looking forward :
- This week
- Next week : Network sampling / growth
- Next next week : Higher-order networks.
- Week 14–15 : "Ethics & Networks" (Group picture !)
  Followed by three presentation sessions !

"Importance" = node-level statistic that quantify "importance"
$\}$ implies a "model"

$f(i, G) \to \theta_i$

*₁ Structural importance
→ evokes the core-periphery metaphor
↓
centrality scores (from social network analysis) ⇒ closeness, PageRank, harmonic, EVC, betweenness, Bradley-Terry-Luce, k-core

*₂ Dynamical importance : θ_i characterizes i's role in dynamics ran on top of G.
→ influence

Often, $f_d(i, G) \approx f_s(i, G)$, i.e., structural importance is a cheap approximation of dynamical importance.

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ZKC running example

» connectivity : degree centrality

$$\theta_i = k_i$$ influence is prop. to (directed, local) connections.

» centrality from eigenvectors

$$\theta_i = f(\,)$$ influence is related to connections. (indirect, nonlocal)

— eigenvector centrality

EVC :

$$x_i^{(t+1)} = \sum_{j=1}^{n} A_{ij}\, x_i^{(t)}$$

do one update and then normalize $\{x_i\}$

w/ initial condition $x_i^{t=0} = 1$

iteration is slow

Perron-Frobenius thm. $Ax = \lambda_1 x$

Takes $O(n^{2.373})$

⇒ when $\{A_{ij}\}$ is undirected, connected

⇓

sol'n converges

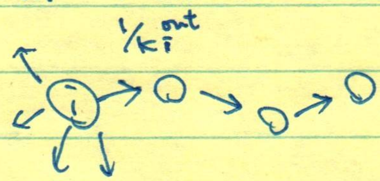
— PageRank

$$x_i = \alpha\left(\sum_{j=1}^{n} A_{ij}\left(\frac{x_j}{k_j^{out}}\right)\right) + \beta$$

"proportional voting"

"universal" basic income

random walk models



1^{st}-order Markov process on G.

$$A + \beta \times K_n$$

complete graph

$$\underline{x} = \alpha A \mathbb{D} \underline{x} + \beta \mathbb{1}$$

$$= \mathbb{D} (D - \alpha A)^{-1} \mathbb{1}.$$

See Notes for.

1) How EVC iterates over the nodes.

2) How degree centrality / EVC / PageRank compare.

Geometric centralities.

Importance ~ positions in G, measured by shortest path.

1) clossness centrality $l_i = \frac{1}{n} \sum_{j} d_{ij}$

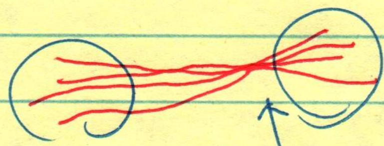
d_{ij} } length of shortest path $i \rightsquigarrow j$

And, $c_i = l_i^{-1}$.

2) harmonic centrality : $c_i = \frac{1}{n-1} \sum_{j \neq i} 1/d_{ij}$

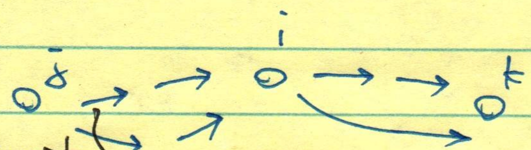
importance ≡ being "close" to everyone else.
model ≡ influence "flows" by shortest path.

3) betweenness centrality



importance ≡ being "in" many information flows
model ≡ influence "flows" by shortest path.

"information broker."



$$b_i = \sum_{jk} \frac{\#\{\text{geodesics } j \to \cdots \to i \to \cdots \to k\}}{\#\{\text{geodesics } j \to \cdots \to k\}} \left(\frac{1}{n^2}\right)$$

optimal definitions.

Advanced material : inferring linear hierarchies.

Ranking → appears in all competitive systems., and rank is related to skill

⟹ Bradley - Terry model (1952) but see also Zermelo (1920's)

⇒ assumes a 1D latent space embedding $\{\pi_i\}$.

⇒ generate a network (directed) via pairwise comparisons.

$$\mathbb{P}(A_{ij}=1) = \pi_i / \pi_i + \pi_j$$

probability that i "beats" j

(alternative, logit-type, formulation

$$\mathbb{P}(A_{ij}=1) = \exp(\pi_i) / \exp(\pi_i) + \exp(\pi_j),$$

nice consequence ⇒ $\mathbb{P}(A_{ij}=1) \propto \pi_i - \pi_j$)

Then, given digraph G of pairwise comparisons

A_{ij} = # wins by i over j

we can infer $\{\pi_i\}$.

$$\text{MLE} : \log \mathcal{L} = \sum_{ij} \left(A_{ij} \ln \pi_i - A_{ij} \ln(\pi_i + \pi_j) \right)$$

we solve this by : ① choose random initial π

② iteratively update ....

$$\pi_i' = k_i^{out} \left(\sum_{j \neq i} \frac{A_{ij} + A_{ji}}{\pi_i + \pi_j} \right)$$

convex optimization
MLE is unique.

$$\pi_i = \frac{\pi_i'}{\sum_j \pi_j'} \quad , \text{ where } k_i^{out} \equiv \sum_j A_{ij}$$

(see Peel & Clauset 2015).

$\Longrightarrow$ Minimum violation rankings

$$\mathbb{P}(A_{ij} = 1) = \begin{cases} 1 & \text{if } \pi_i > \pi_j \\ 0 & \text{otherwise} \end{cases}$$

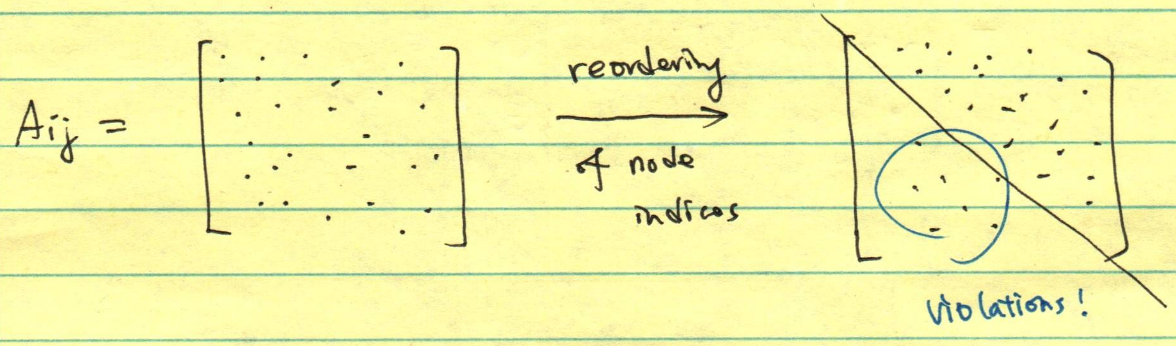
"strict hierarchy" ("hard" version of the BT model)

≫ used in inferring dominance behavior in animal behaviors.

Inference

$$\hat{\pi} = \inf_{\pi_i \in [n]} \sum_{ij} A_{ij} \times \text{sign}(\pi_i - \pi_j)$$

all permutations of n nodes



→ Spring Rank (2018)

» models edges as springs, a la Hook's law.

» solve

$$\underset{\pi \in \mathbb{R}^N}{\text{minimize}} \quad H(\pi) = \frac{1}{2} \sum_{ij} A_{ij} \left(\pi_i - \pi_j - 1 \right)^2 .$$

inference thru convex optimization (find $\nabla(H) = 0$)

see De Bacco, Larremore, (2018)

and Ten (2023)

↗ w/ Stephen Becker, CU APPM

Idea : when nodes have annotations (timestamp or category),

we may leverage them to inform a better ranking

Solve :

$$\text{minimize} \quad H + \left(\lambda \, \Omega (L \pi) \right)$$

examples :

to smooth rankings over time

$$\Omega_1 = \sum_t \| \pi^t - \pi^{t-1} \|_1$$

$$\Omega_2 = \sum_t \| \pi^t - \pi^{t-1} \|_2^2$$

examples (cont'd):

OR, to enforce modeler's belief that
"Same node category, same mean ranking"

$$\Omega_{group} = \sum_{z_i \neq z_j} \left| \underset{\substack{\uparrow \\ \text{over the same category}}}{\langle \pi_i \rangle} - \langle \pi_j \rangle \right|$$

key to solution

Algm: Gradient-based, krylov methods (sum-of-squares form, sparse iterative solver)

↓

Gradient-based, first-order methods
(dual-based proximal gradient methods ...)

Thurs (Apr 4, 2024)

Prof. Dan Larremore Guest Lecture!