Characterize the Entanglement Depth of Indistinguishable Fermionic Degrees of Freedom Specific to Paired Electron Measurement

Junjie Wang
Department of Physics and Astronomy, University of Rochester
(Dated: August 11, 2024)

In this paper, I will start by discussing the recent research work that applies entanglement and quantum bounds in the spin basis and their demand for the development of many complicated degrees of freedom. I start by discussing the entanglement in fermionic systems based on multipartite and multi-mode since these concepts can be applied more conveniently to discuss electron's behave in the crystal lattices. In the periodic lattice, the electrons interacting with phonons can results in Cooper Paired behavior. The last of this paper discussed my idea in the observable characterization of the entanglement results from the paired behavior, which mainly focuses on the free Fermi gas.

Keywords: Quantum Mechanics, Many Body Interaction, Quantum Entanglement, Quantum information, Quantum optics

I. WITNESS ENTANGLEMENTS IN MATTERS

In the paper[1], Yao's Group and Mitrano's Group give perspective to control and modify the quantum entanglement in the material for innovative quantum devices in the future. In their work, they use the light-driven material as the platform for the entanglement and induced the light-driven entanglement based on breaking the equilibrium of the system by the laser, and they prob the entanglement by their laser technique called **time-resolved Resonant Inelastic X-ray Scattering** (trRIXS), and I will introduce more in the following in this section.

One factor that allows them to determine the presence of the entanglement is the term called **Quantum Fisher Information** (QFI)[2] $f_Q(q,t)$ evaluated from various quantifiable factors and one of the important one is **dynamical structure fracture** $\mathcal{S}(q,\omega,t)$, which q is the momentum and ω is the frequency. $\mathcal{S}(q,\omega,t)$ can be measured experimentally by the trRIXS technique in the following expression[1]:

$$\mathcal{S}(q,\omega,t) \approx \frac{\mathcal{I}(q,\omega_i-\omega,\omega_i,t)}{\tau_{core}^2\left|M_{q_i\epsilon_i}^{in}M_{q_s\epsilon_s}^{out}\right|^2}. \quad (1)$$

Note that ω & ω_i denote the energy loss and the photon resonance frequency that can maximize the trRIXS intensity $\mathcal{I}$, the $M_{q_i\epsilon_i}^{in}$ is the matrix element that refers to the electron transition from the core *into* the valence band via ϵ_i polarized photon, and the τ_{core} is the lifetime between core and hole.

Then, the $f_Q(q,t)$ can be evaluated by the following expression from[1]:

$$\int_{-\infty}^{\infty}\mathrm{d}\tau g(\tau;t)^2 f_Q(q,\tau) = 8\sigma_{pr}\sqrt{\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\mathcal{S}(q,\omega,t) \quad (2)$$

$$+\sum_{m=1}^{\infty}\frac{\mathcal{C}_m}{2m!}\frac{\partial^{2m}f_Q}{\partial t^{2m}} \quad (3)$$

$$\mathcal{C}_m = -\frac{\sigma_{pr}^{m-1/2}}{\sqrt{\pi}}\Gamma(m+1/2), \quad (4)$$

and since trRIXS is the ultra-fast, the Equation2 has the factor describe the trRIXS probe envelope $g(\tau;t)$, which changes from its original Gaussian envelope to the delta function.

$$g(\tau;t) = \frac{1}{\sigma_{pr}\sqrt{2\pi}}e^{-(\tau-t)^2/2\sigma_{pr}^2} \quad (5)$$

$$\lim_{\sigma_{pr}\to 0} g(\tau;t) = \delta(\tau-t) \quad (6)$$

Thus, the $f_Q(q,t)$ can be evaluated directly by the Equation2, since

$$\int_{-\infty}^{\infty}\mathrm{d}\tau g(\tau;t)^2 f_Q(q,\tau) = f_Q(q,t). \quad (7)$$

With this value, they compared with the upper bound for the $f_Q(q,t)$ in an entangled system, which infers that exceeding the upper bound signals the exists of entanglement, like two body entangled with each other, k = 2, will make this value greater.

$$f_Q(q,t) \leq 4kn^2 S = 0.25, \quad (8)$$

when k = 1 producible state in an average electron density per site of $n = \mathbb{E}[\sum_{i\sigma} n_{i\sigma}]/N = 0.5$ in quarterly filled system with the total spin $S = 1/2$.

They successfully witness the $f_Q(q,t)$ exceeds 0.25 in the Extended Hubbard Model[1], which is one of the innovations in their work that counts the additional nearest-neighbor interacting potential than the original Hubbard Model[3], when the light pump that applies to the light-driven material reaches a certain value, i.e. pump amplitudes at fixed energy and pump energies at fixed amplitude.

However, what makes this topic interesting is that Equation (1 2 8) is only discussed under the picture that described a pure spin system. Since Fermi has additional spatial degrees of freedom than just spin degrees of freedom, then encounter that may be worthy. There is work that demonstrates the importance of multipartite entanglement of interacting fermions[4], this paper is aiming to provide ways to characterize this kind of entanglement.

II. FERMI ENTANGLEMENT

Entanglement in Fermi degrees of freedom often refers to the fact of the Pauli Exclusive Principle, so that in the mode picture, there is only one or fewer Fermion occupied in each mode. Thus, the N-body wavefunction in the 2nd quantization picture can be expressed as the Fock state that includes the mode occupation status:

$$|\Psi\rangle_N = |n_1, n_2, \cdots, n_M\rangle = (a_1^\dagger)^{n_1} \cdots (a_M^\dagger)^{n_M} |0\rangle . \quad (9)$$

Fermi statistic tells that these creation operators follow Fermion's anti-commutation relation and the occupation number can only be 0 or 1 for each mode.

Then, the entanglement within fermions can either be considered between particles or between different modes, which refers to **mode entanglement** and **particle entanglement** respectively. Additionally, the particle entanglement can also be considered as the minimum mode entanglement[5]. To distinguish them by showing their mathematical form which indicates the entangled in one but unentangled in the other definition:

$$|\psi\rangle_p = a_1^\dagger a_2^\dagger |0\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle) \quad (10)$$

$$|\psi\rangle_m = a_1^\dagger |0\rangle + a_2^\dagger |0\rangle \quad (11)$$

Equation10 indicates the expressions for an entangled particle with separable modes since there are two particles and each corresponds to one mode; Equation11 indicates the expressions for the inseparable modes on only one body, which indicates the dynamics of one mode will affect the other mode.

Nowadays, the entanglement between two body is well studied for both Bosons[2] and Fermions[5], and the two separable body is defined by:

$$\begin{aligned} |12\rangle &= |1\rangle \otimes |2\rangle \\ \mathbf{E}(O_{12}) &= \mathbf{E}(O_1)\mathbf{E}(O_2), \end{aligned}$$

since the observable also follows the separable condition $O_{12} = O_1 \otimes O_2$. For the many body condition, the form and observation for separable states can just be extended to be the tensor product of more states and multiplication of more expectation values for single state.

Moreover, Schrodinger's Equation only describes the particle behavior only in zero absolute temperature, and density matrices in quantum dynamics describe the system in finite temperature. Accordingly, to continue the discussion in Section I, which is in a finite temperature condition, we have to focus on the density matrix of the system, which is referred to **Mixed State**, but the **Pure State** should always by the easier starting point.

III. ELECTRON PAIR

To characterize the entanglement, we should first pick the specific systems as a platform for our discussion, and I would like to start with the number of conserved free Fermi particles, which infers that there is no external background potential and the system can be statistically characterized by the Canonical ensemble. Based on the N body Fermi wavefunction:

$$\begin{aligned} \Psi(\mathbf{r}_1 \dots \mathbf{r}_N) = \mathcal{A}\{&\psi(|\mathbf{r}_1 - \mathbf{r}_2|)\chi_{1,2} \cdots \\ &\psi(|\mathbf{r}_{N-1} - \mathbf{r}_N|)\chi_{N-1,N}\} \\ \chi_{ij} = \frac{1}{\sqrt{2}}&(|\uparrow\rangle_i |\downarrow\rangle_j - |\downarrow\rangle_i |\uparrow\rangle_j), \end{aligned}$$

in this case the $\mathcal{A}\{\dots\}$ represents the normalized combination of Fermi's anti-symmetric relation, sometimes refers to the same function as the Slater Determinant.

The previous section has a pretty different system than the systems of the electron in solid matter. The reason for picking up the free Fermi gas is that most solid state models, which apply the assumptions that only count free electron gas like Drude Model and Sommerfeld Model, give a sufficient approximation on the materials constant[6].

The reason is that original electron-electron interaction $V \propto \frac{1}{\mathbf{r}_i - \mathbf{r}_j}$ is diminished by the screen charge potential noted as Yukawa potential[7] by:

$$V \propto \frac{e^{-\alpha m \mathbf{r}}}{\mathbf{r}},$$

which makes the interaction between electron small enough that each electron can be assumed to be independent between each other. However, following this assumption, each electron is an independent system, which is also known as separable, the entanglement among the system is 0.

Then, to characterize the entanglement under this picture, we have to consider the interaction between

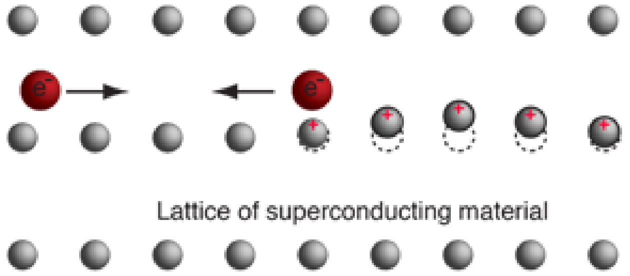

Figure 1. The figure for cooper pair, that two electron is passing through the periodical lattice and the second electron will be affected by the ion vibration caused by the first electron.[9]

the pair of electrons due to phonon interaction. The paired electron is called **Cooper Pair**[8]. The electron is paired, because the results from the small phonon vibration of ions due to the first electron passing through the ion, which its ion displacement can impact the upcoming electron that moves in the opposite direction.

Furthermore, according to the Cooper Pair Theory, this concept is important due to the paired electron has an integer number of spin, which has the similar behavior of bosons, which can make the paired electron occupies more states than the unpaired electrons. One of the examples is the helium super-fluid. In the second quantization picture, people create the electron from the (quasi-) vacuum state. Then, the wavefunction is constructed by creation operators, and the above N-body wavefunction can be written as:

$$|\Psi\rangle_N = \int \prod_i \mathrm{d}^3 r_i \phi(|\mathbf{r}_1 - \mathbf{r}_2|)\psi_\uparrow^\dagger(\mathbf{r}_1)\psi_\downarrow^\dagger(\mathbf{r}_2) \cdots \phi(|\mathbf{r}_{N-1} - \mathbf{r}_N|)\psi_\uparrow^\dagger(\mathbf{r}_{N-1})\psi_\downarrow^\dagger(\mathbf{r}_N)\,|0\rangle\,,$$

which means these electrons are created with specific spins and spatial component. The creation operator is defined in the following:

$$\psi_\sigma^\dagger(\mathbf{r}) = \sum_k \mathbf{c}_{\sigma k}^\dagger \frac{e^{i\mathbf{r}k}}{\sqrt{C_{norm}}} \tag{12}$$

$$\phi(|\mathbf{r}|) = \sum_k \phi(k) \frac{e^{i\mathbf{r}k}}{\sqrt{C_{norm}}}. \tag{13}$$

With the above expression, the paired wavefunction can be expressed as:

$$\mathbf{b}^\dagger = \sum_{k=1}^{M} \phi_k \mathbf{c}_{\uparrow k}^\dagger \mathbf{c}_{\downarrow -k}^\dagger \tag{14}$$

$$|\Psi\rangle_N = \mathbf{b}^{\dagger N/2}\,|0\rangle = |\Psi_{BCS}\rangle\,, \tag{15}$$

for N particle BCS state, which both $u_k, u_k \in \mathbf{C}$, i.e. the complex set. The last line skipped the derivation of the equality, Cooper gives more detailed discussion on deriving Equation15 in his work[8].

For the BCS wavefunction, it refers to the condition where the particle number is not conserving, which it is composed by linear combination of probability λ distribution of the average particle number N[10],

$$|\Psi_{BCS}^N\rangle = \sum_{N=0}^{2M} \lambda_N\,|\Psi_{BCS}\rangle = \prod_k (u_k + v_k \mathbf{c}_{\uparrow k}^\dagger \mathbf{c}_{\downarrow -k}^\dagger)\,|0\rangle \tag{16}$$

Their Hamiltonian for the interacting fermions is expressed in the following form:

$$\mathcal{H} = \sum_{k,\sigma} \epsilon_k \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{\sigma k} + \frac{V_0}{C_{norm}} \sum_{k,k'} \mathbf{c}_{k\uparrow}^\dagger \mathbf{c}_{-k\downarrow}^\dagger \mathbf{c}_{k'\downarrow} \mathbf{c}_{-k'\uparrow}, \tag{17}$$

with the assumption that only focus on the center of mass of the pair has zero-momentum[8].

IV. PAIRED ENTANGLEMENT

Continue the discussion of the Cooper pair, since two electrons with spin 1/2 paired together, behaved like particles that have the integer of spin number, which is like Bosons.

For the entanglement involving paired states, the entanglement appears a little different than previously in Equation10 and Equation11. For an example of an unpaired state,

$$|\Psi^*\rangle = C(a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger + a_5^\dagger a_6^\dagger a_7^\dagger a_8^\dagger)\,|0\rangle\,,$$

which is an example of the mode and particle entanglement if we use the preceding definition.

Furthermore, the paired state between two particles is entangled if the Reduced Density Matrix (RDM) constructed From N particle BCS state, i.e. Equation15, has mode number $M > 2N - 3$[10]. The RDM is defined by:

$$\mathcal{O}^\rho_{(ij)(kl)} = Tr[\rho a_i^\dagger a_j^\dagger a_k^\dagger a_l^\dagger] \tag{18}$$

$$Tr[\mathcal{O}^\rho] = \mathbf{E}(N_{op}^2) - \mathbf{E}(N_{op}). \tag{19}$$

Pair Detection

Continue to follow the preceding work on paired state[10], since the above discussion shows that entanglement is not equivalent to paired state, people should check if the state is paired. The definition of that concept is also straightforward, which the expectation value of the paired state cannot be duplicated by the unpaired state, and the mathematical expression should follow:

$$\begin{aligned}\mathcal{C}_{\vec{\mathcal{O}}} =&\{\vec{v} = Tr[\mathcal{O}\rho_u]; \rho_u \in S_u\} && (20)\\ \vec{v}_p =&Tr[\mathcal{O}\rho_p] \notin \mathcal{C}_{\vec{\mathcal{O}}}, && (21)\end{aligned}$$

and all the sets above is a convex set.

There is an example of a pairing detecting for symmetric version of bcs state Equation 15:

$$|\Psi_{sym}\rangle = \left(\sum_{k=1}^{M} \phi_k(\mathbf{c}^\dagger_{\uparrow k}\mathbf{c}^\dagger_{\downarrow -k} + \mathbf{c}^\dagger_{\uparrow k+M}\mathbf{c}^\dagger_{\downarrow -k-M})\right)^{N/2} |0\rangle \tag{22}$$

and its pair detector is:

$$\begin{aligned}\mathcal{H}(v_k) =&\sum_{k=1}^{M} H_k\\ H_k =&2(1-\epsilon-|v_k|^2)N_k\\ &-4(\mathbf{c}^\dagger_{\uparrow k}\mathbf{c}^\dagger_{\downarrow -k}\mathbf{c}_{\uparrow k+M}\mathbf{c}_{\downarrow -k-M} + H.c.)\\ N_k =&n_k + n_{-k} + n_{k+M} + n_{-(k+M)},\end{aligned}$$

which n is the number density $0 \leq 1v_k \leq 1-\epsilon$.

V. ENTANGLEMENT CHARACTERIZATION

In this section, we want to introduce operators that can give out puts that can characterize the system entanglement form. Then, there should be outputs that are related to the system's feature that follows a statistical relationship.

If people use $\mathbf{P}(\mu|\theta)$ give the probability of the outcome μ measuring the quantity θ, the Bayesian estimation framework should be:

$$\mathbf{P}(\theta|\mu) = \frac{\mathbf{P}(\mu|\theta)\mathbf{P}(\theta)}{\mathbf{P}(\mu)},$$

which allows people to estimate θ given the series of outcome $\vec{\mu} = [\mu_1, \mu_2, \cdots, \mu_x]$ for x independent measurements.

To characterize the entanglement with paired state, we should focus on two features:

Measure the number of existing pairs

The previous work on pair detection on not number conserving state Equation16 is in the form

$$\mathcal{H} = \sum_{k=1}^{M} 2(1-\epsilon-|v_k|^2)(n_k+n_{-k}) - 2(v_k u_k^* \mathbf{c}^\dagger_{\uparrow k}\mathbf{c}^\dagger_{\downarrow -k} + H.c.)$$

[10], and the pair existence when the $\langle\mathcal{H}\rangle = Tr[\rho\mathcal{H}] = -4\epsilon\sum_k v_k < 0$.

(Start from here, it will start to be ridiculous.) Checking the number of paired electrons should be the next step, and the intuition is to use the Pauli exclusive principle that $b^\dagger b^\dagger |0\rangle = 0$. Then, I think the measurement witness operator should be the linear combination of terms that contains

$$\mathbf{c}_{\downarrow -k}\mathbf{c}_{\uparrow k}\mathbf{c}^\dagger_{\uparrow k}\mathbf{c}^\dagger_{\downarrow -k} \tag{23}$$

$$\mathbf{c}_{\downarrow \pm k}\mathbf{c}^\dagger_{\downarrow \pm k} + \mathbf{c}_{\uparrow \pm k}\mathbf{c}^\dagger_{\uparrow \pm k}. \tag{24}$$

Then, the witness operator should be:

$$\mathcal{W} = \sum_{k,\sigma} \lambda_p \mathbf{c}_{\downarrow -k}\mathbf{c}_{\uparrow k}\mathbf{c}^\dagger_{\uparrow k}\mathbf{c}^\dagger_{\downarrow -k} + \lambda_u(\mathbf{c}_{\downarrow \pm k}\mathbf{c}^\dagger_{\downarrow \pm k} + \mathbf{c}_{\uparrow \pm k}\mathbf{c}^\dagger_{\uparrow \pm k}), \tag{25}$$

which can make the expectation value a result from $\lambda_p \& \lambda_u$. Then, people may use the Bayesian framework to analyse the likelihood of each number of pairs.

Measure the entanglement depth between pairs

The concept of cooper pair emphasizes the coupling between two electrons due to the attractive force caused by the displacement of the photon. However, it is still not sure whether the displacement of ions can effect the third or more electrons; then it is also necessary to find the entanglement between more bodies. (Start from here, it will start to be ridiculous.) Then, the "paired" creation operator for 3 electrons, for example, should be:

$$\mathbf{b}'^\dagger = \sum_{k=1}^{M} \phi_k \mathbf{c}^\dagger_{\uparrow k}\mathbf{c}^\dagger_{\downarrow -k}\mathbf{c}^\dagger_{\downarrow k}, \tag{26}$$

which means the same displacement of ion attract 1 additional electrons to the original BCS theory.

The entanglement depth is initially introduced as "the minimum number of particles forming multi-particle entangled states in the sample" [11], and the more straight forward example will be[12]:

$$\begin{aligned}\mathcal{D}(|1234\rangle \otimes |56\rangle) =&4\\ \mathcal{D}(|12\rangle \otimes |34\rangle \otimes |56\rangle) =&2, etc.\end{aligned}$$

Note that the example above describe the entangled depth specifies to particle entanglement, and the particle-mode entanglement should be:

$$\begin{aligned}\mathcal{D}((a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger + a_5^\dagger a_6^\dagger a_7^\dagger a_8^\dagger)|0\rangle) =&4\\ \mathcal{D}((a_1^\dagger a_2^\dagger + a_3^\dagger + a_4^\dagger a_5^\dagger + a_6^\dagger + a_7^\dagger a_8^\dagger)|0\rangle) =&5,\end{aligned}$$

which selects the most entangled body that take into account both the particle and mode perspective.

One of the intuitive ways to tackle this is to use the bipartite[1], to divide the mode-particle system into two subsystems and see the entanglement between the place where the division happens. Benatti et. al. provides the preliminary work and idea on this topic, and I would like to continue working on applying their idea to this scenario[14].

If we consider the Equation9, which is an mode unentangled state, can be divided into:

$$|\Psi\rangle = ((a_1^\dagger)^{n_1}\cdots(a_m^\dagger)^{n_m})\cdot((a_{m+1}^\dagger)^{n_{m+1}}\cdots(a_M^\dagger)^{n_M})\,|0\rangle\,,$$

which is called $(m, M-m)$ separable. However, in the general case the Fock state Equation9 is separable if and only if it be decomposed into[14]:

$$\begin{aligned}
|\Psi\rangle &= \sum_{\{k\},\{\alpha\}} \mathcal{C}_{\{k\},\{\alpha\}}\,|k_1,\cdots,k_m;\alpha_{m+1},\cdots,\alpha_M\rangle\\
1 &= \sum_{\{k\},\{\alpha\}} \left|\mathcal{C}_{\{k\},\{\alpha\}}\right|^2\\
&|k_1,\cdots,k_m;\alpha_{m+1},\cdots,\alpha_M\rangle\\
&=(a_1^\dagger)^{k_1}\cdots(a_m^\dagger)^{k_m}(a_{m+1}^\dagger)^{\alpha_{m+1}}\cdots(a_M^\dagger)^{\alpha_M}\,|0\rangle\,,
\end{aligned}$$

here the n notation is replaced by k and α for distinction after separation, and note that $k,\alpha\in\{0,1\}$, since Fermion can only occupy two states (or mode). Then, for the first $m\le M$ mode, it will have the number of Fermions $\mathbf{k}\le N$, the total number of Fermions, and the rest of them occupy in the rest M-m modes:

$$\mathbf{k} = \sum_m k_m$$
$$N-\mathbf{k} = \sum_{i=m+1}^{M}\alpha_i.$$

Since $\mathbf{k}$ Fermions occupy the m modes, the number ways they can occupy will count as $\binom{m}{\mathbf{k}}$, which is similar for the rest M-m modes. The basis will become:

$$|\mathbf{k},\sigma;N-\mathbf{k},\sigma'\rangle \tag{27}$$
$$\sigma\in\binom{m}{\mathbf{k}},\sigma'\in\binom{M-m}{N-\mathbf{k}}, \tag{28}$$

with the orthonormal property of basis vector.

$$\langle\mathbf{k},\sigma;N-\mathbf{k},\sigma'|\mathbf{l},\tau;N-\mathbf{l},\tau'\rangle = \delta_{k,l}\delta_{\sigma,\tau}\delta_{\sigma',\tau'}$$

[1] Even current work can divide the system into tripartite[13], it is now only developed for the entanglement in particle specified system.

Then, extending to the mixed state, the density matrix can be expressed as[14]:

$$\rho = \sum_{\mathbf{k},\mathbf{l}}\sum_{\sigma\sigma';\tau\tau'}\rho_{\mathbf{k}\sigma\sigma';\mathbf{l}\tau\tau'}\,|\mathbf{k},\sigma;N-\mathbf{k},\sigma'\rangle\,\langle\mathbf{l},\tau;N-\mathbf{l},\tau'| \tag{29}$$
$$1 = \sum_{\mathbf{k},\mathbf{l}}\sum_{\sigma\sigma';\tau\tau'}\rho_{\mathbf{k}\sigma\sigma';\mathbf{l}\tau\tau'} \tag{30}$$

Since $\mathbf{k}$ and $\mathbf{l}$ represent the number of particles occupies the first m modes, it has the upper bound for the summation of $min\{m,N\}$ since each mode can only occupy at most 1 particle. Similarly, it will have a lower bound, which people always assume to be 0 but actually is not, is $max\{0,N-(M-m)\}$, if the N is greater than the rest of M-m modes, it means it at least has N-(M-m) particles in the first m modes, and we should only consider the ways of different occupations.

Recall the paired state Equation15 for the fixed number of electrons, if the system has a total number of electrons that are not perfectly paired, the number of paired electrons is not number conserving, which people should use Equation16. Additionally, in the discussion of screen potential and the free electron approximation, the paired state is inseparable and the unpaired electron is free.

$$|\Psi\rangle = \left(\sum_{k=1}^{M}\phi_k\mathbf{c}_{\uparrow k}^\dagger\mathbf{c}_{\downarrow -k}^\dagger\right)^{N/2}|0\rangle \tag{31}$$
$$= \left(\sum_{k=1}^{M}\mathbf{b}_k^\dagger\right)^{N/2}|0\rangle \tag{32}$$

(I should find a way to represent the paired state into a divisible form, and prove the off-diagonal element is 0) Expanding the above equation to visualize much straight forward and change N/2 to n, the number of pairs:

$$|\Psi\rangle = \left(\sum_{i=1}^{M}\mathbf{b}_i^\dagger\right)^{n}|0\rangle \tag{33}$$
$$= \sum_{\mathbf{k}_1+\cdots+\mathbf{k}_M}\frac{n!}{\mathbf{k}_1!\cdots\mathbf{k}_M!}(\mathbf{b}_1^\dagger)^{\mathbf{k}_1}\cdots(\mathbf{b}_M^\dagger)^{\mathbf{k}_M}\,|0\rangle\,, \tag{34}$$

the sum is over different combinations of $\mathbf{k}$, which represents the number in each mode, and the $\mathbf{k}$ can exceed 1 since now the pair behaves like a boson. If I want to do the separation, I should do something like:

$$|\Psi\rangle = \sum_{\mathbf{k}_1+\cdots+\mathbf{k}_m+\mathbf{k}_{m+1}+\cdots+\mathbf{k}_M} \tag{35}$$

$$\mathcal{C}(\mathbf{b}_1^\dagger)^{\mathbf{k}_1}\cdots(\mathbf{b}_m^\dagger)^{\mathbf{k}_m}(\mathbf{b}_{m+1}^\dagger)^{\mathbf{k}_{m+1}}\cdots(\mathbf{b}_M^\dagger)^{\mathbf{k}_M}|0\rangle \tag{36}$$

$$\mathcal{C} = \frac{n!}{\mathbf{k}_1!\cdots\mathbf{k}_m\mathbf{k}_{m+1}\cdots\mathbf{k}_M!} \tag{37}$$

The rest discussion is going to be following boson's bipartite analysis[15]. If the off-diagonal of the density matrix like Equation29 exists, it will show that the pairing involves more than one pair or entanglement between pairs.

VI. DISCUSSION

In this paper, I started from the results given by Yao's Group and Mitrano's Group and try to tackle the concern they mentioned in their paper[1], that characteristic of the Fermi system entanglement, and my interested system is specified in a BCS picture. Then, in the BCS picture, I try to work on finding the number of the paired electron by a witness operator, which is a challenge mentioned in previous works in paired state[10], and the entanglement depth involves between different body, which is one of the contemporary challenge[12].

However, I have not finished this work, but I provide the idea for solving those challenges: I provide the reasonable format of the number measurement Witness**??**; I also provide the reasonable ways to calculate the entanglement depth using the bipartite analysis by different ways to separate the system.

ACKNOWLEDGEMENTS

I gratefully acknowledge the stimulating discussion with Prof. Gabriel Landi, Songbo Xie, and the support from Department of Physics in University of Rochester.

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