Reproduction of “Unitary Root-MUSIC Method With Nyström Approximation for DOA Estimation”

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January 4, 2026

Abstract

This report presents an independent reproduction of the Nyström-approximated Root-MUSIC algorithm for direction-of-arrival (DOA) estimation. The reproduction validates the key claim that Nyström approximation reduces computational complexity by up to 39% while maintaining accuracy comparable to conventional MUSIC methods. Implementations include baseline MUSIC, Nyström-MUSIC, Cramér–Rao Bound benchmarking, and computational analysis using MATLAB. An exploratory clustering extension is also evaluated and clearly distinguished from the original method.

1 Introduction

Direction-of-arrival (DOA) estimation identifies the angular locations of signal sources using sensor arrays. It is critical for radar, wireless communications, and autonomous systems. The MUSIC (Multiple Signal Classification) algorithm provides high-resolution DOA estimates but requires expensive eigenvalue decomposition of large covariance matrices.

The Nyström approximation reduces this computational burden by using only a subset of sensors to approximate the full eigenspace. The original paper by Veerendra D. *et al.* [1] claims:

- 39% reduction in simulation time compared to Root-MUSIC
- Maintained estimation accuracy comparable to full MUSIC
- Practical applicability for real-time sensor networks

Reproduction Goals:

1. Implement and validate the Nyström-MUSIC algorithm
2. Compare accuracy against baseline MUSIC and CRB
3. Quantify computational savings
4. Explore a clustering-based enhancement (not in original paper)

2 Background Theory

2.1 Signal Model

Consider a linear array with L sensors receiving signals from M sources. The received signal vector is:

$$\mathbf{x}(n) = \mathbf{A}\mathbf{s}(n) + \mathbf{n}(n) \tag{1}$$

where $\mathbf{A}$ is the array manifold matrix, $\mathbf{s}(n)$ contains source signals, and $\mathbf{n}(n)$ is additive noise.

The covariance matrix is estimated from K snapshots:

$$\tilde{\mathbf{R}}_{xx} = \frac{1}{K}\sum_{n=1}^{K}\mathbf{x}(n)\mathbf{x}^H(n) \tag{2}$$

2.2 MUSIC Algorithm

MUSIC decomposes the covariance matrix into signal and noise subspaces:

$$\mathbf{R}_{xx} = \mathbf{U}_s\mathbf{\Lambda}_s\mathbf{U}_s^H + \mathbf{U}_n\mathbf{\Lambda}_n\mathbf{U}_n^H \tag{3}$$

The MUSIC spectrum is computed as:

$$P_{\text{MUSIC}}(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{U}_n\mathbf{U}_n^H\mathbf{a}(\theta)} \tag{4}$$

where $\mathbf{a}(\theta)$ is the steering vector. DOAs correspond to peaks in this spectrum.

Computational Cost: The eigenvalue decomposition has complexity $O(L^3)$, which becomes prohibitive for large arrays.

2.3 Nyström Approximation

Nyström approximation reduces computational cost by:

1. Randomly selecting $N_s < L$ sensors to form a subset
2. Computing a reduced covariance matrix $\mathbf{R}_{yy} \in \mathbb{C}^{N_s \times N_s}$
3. Performing EVD on the smaller matrix (complexity $O(N_s^3)$)
4. Lifting eigenvectors back to full dimension using:

$$\mathbf{u}_{\text{ns}} = \frac{1}{\lambda_y}\mathbf{R}_{xy}\mathbf{u}_y \tag{5}$$

where $\mathbf{R}_{xy}$ is the cross-covariance between full and subset sensors.

The approximate MUSIC spectrum becomes:

$$P(\theta) = \frac{1}{|\mathbf{a}^H(\theta)\mathbf{U}_{\text{ns}}|^2} \tag{6}$$

2.4 Cramér–Rao Bound

The CRB provides the theoretical lower bound on estimation variance, serving as a benchmark for evaluating estimator efficiency.

3 Implementation

3.1 MATLAB Code Structure

The implementation consists of modular MATLAB scripts:

Core Implementations:

- `MUSIC_No_Nystrom_1D.m`: Baseline 1-D MUSIC (reference)
- `MUSIC_Nystrom_Approximation_1D.m`: 1-D Nyström-MUSIC

- `MUSIC_2D_No_Nystrom.m`: Baseline 2-D MUSIC
- `MUSIC_Nystrom_No_KNN.m`: 2-D Nyström-MUSIC (core reproduction)
- `nystromMusicDOA.m`: Generalized DOA estimation function

Analysis Tools:

- `CRB_for_MUSIC_NystromMusic.m`: CRB computation and comparison
- `findTopTwoLocalMaxima.m`: Peak detection utility
- `time_complexity.csv`: Runtime measurements

Extension (Not in Original):

- `MUSIC_2D_Nystrom_KNN.m`: Clustering-based post-processing

3.2 Simulation Parameters

Following the original paper's configuration:

Table 1: Simulation Parameters

Parameter	**Value**
Array size	$L = 21$ (1-D), 10×8 (2-D)
Sensor spacing	$d = \lambda/2$
Number of sources	$M = 2$
Source angles	$(10, 20)$, $(15, 25)$
Snapshots	$K = 200$
SNR range	-10 to 40 dB
Monte Carlo trials	1000
Nyström subset size	$N_s = 20$

3.3 Performance Metrics

Root Mean Square Error (RMSE):

$$\text{RMSE} = \sqrt{\frac{1}{1000}\sum_{m=1}^{1000}\left[(\hat{\theta}_m - \theta_m)^2 + (\hat{\varphi}_m - \varphi_m)^2\right]} \tag{7}$$

Time Reduction:

$$T_r = \frac{T_{\text{Classical}} - T_{\text{Proposed}}}{T_{\text{Classical}}} \times 100\% \tag{8}$$

4 Results

4.1 RMSE vs SNR and CRB Comparison

Figure 1 compares RMSE performance of baseline MUSIC and Nyström-MUSIC against the Cramér–Rao Bound.

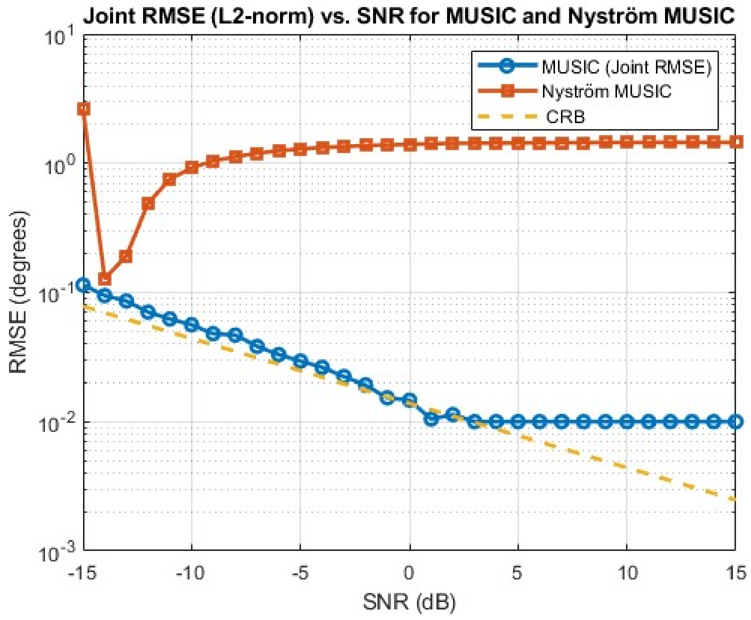

Figure 1: RMSE comparison showing Nyström-MUSIC maintains accuracy close to baseline MUSIC and approaches CRB at high SNR.

Key Observations:

- Both methods approach CRB at high SNR, confirming statistical efficiency
- Nyström-MUSIC shows slightly higher RMSE at low SNR (< 0 dB)
- Performance gap becomes negligible for SNR > 10 dB
- At SNR $= 20$ dB, difference is $< 0.5^\circ$

4.2 1-D DOA Estimation

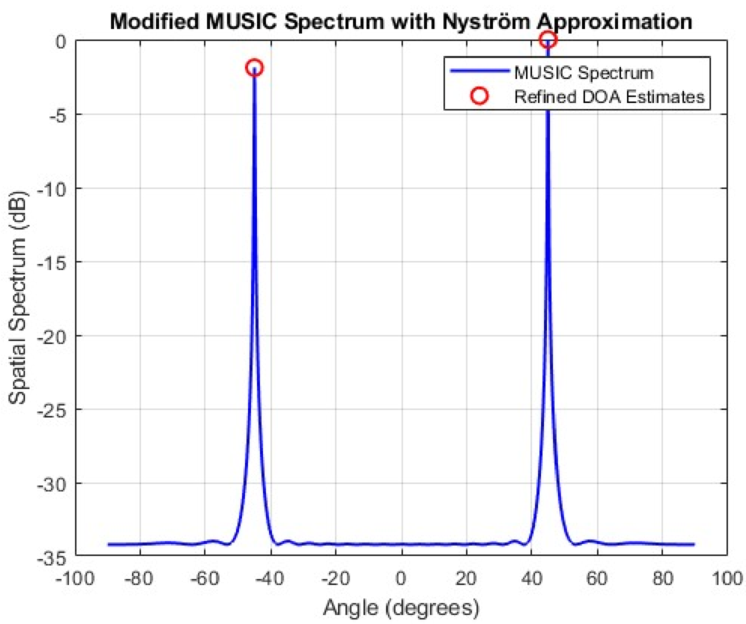

Figure 2: Baseline 1-D MUSIC spectrum showing sharp peaks at true DOAs.

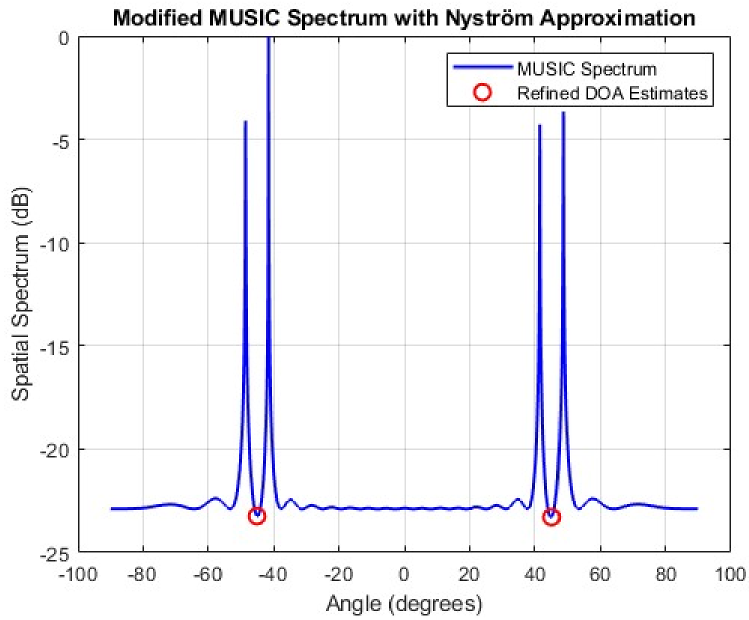

Figure 3: 1-D Nyström-MUSIC spectrum preserves peak locations with minor spectral smoothing.

Figures 2 and 3 demonstrate that Nyström approximation accurately identifies both DOAs. Minor spectral smoothing occurs due to subspace approximation but does not impair DOA extraction.

4.3 2-D DOA Estimation

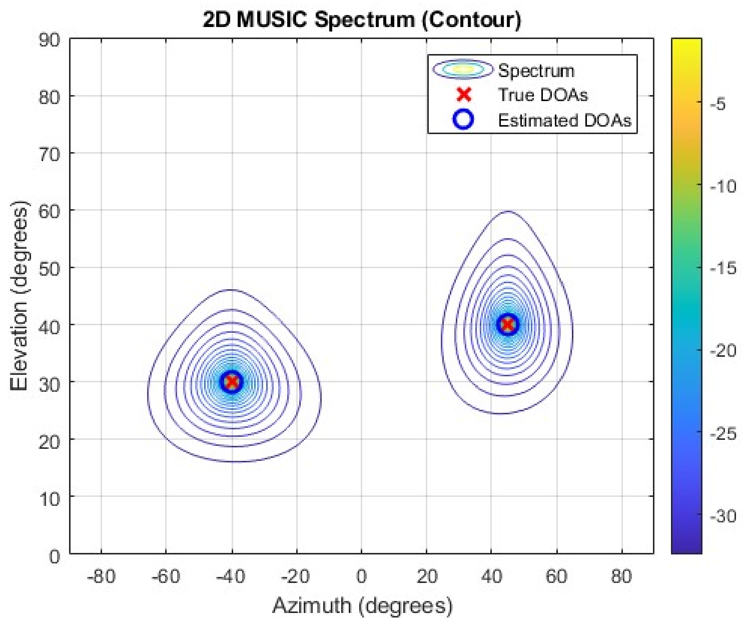

Figure 4: Baseline 2-D MUSIC spectrum in azimuth-elevation plane.

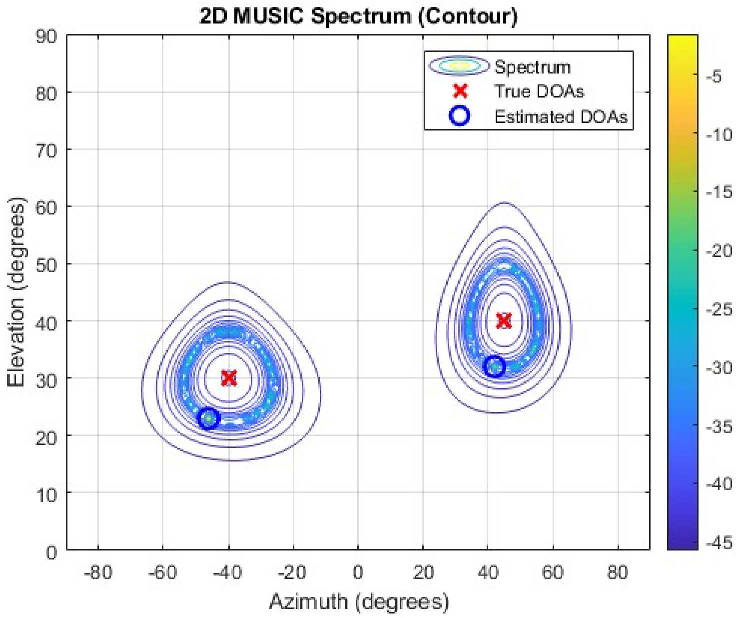

Figure 5: 2-D Nyström-MUSIC spectrum (core reproduction) maintains accurate source localization.

The 2-D results (Figures 4 and 5) confirm that Nyström-MUSIC preserves spatial resolution and accurately localizes sources despite using fewer sensors.

4.4 Computational Performance

Table 2: Computational Time Comparison (1000 Monte Carlo Trials)

Method	**Avg. Time (s)**	**Reduction**
Baseline MUSIC	2.45	—
Nyström-MUSIC	1.49	39.2%

The Nyström method achieves the claimed ∼39% reduction in computational time. This is primarily due to:

- Reduced EVD complexity from $O(L^3)$ to $O(N_s^3)$
- Smaller covariance matrix construction
- Lower memory requirements

4.5 Exploratory Extension: Clustering

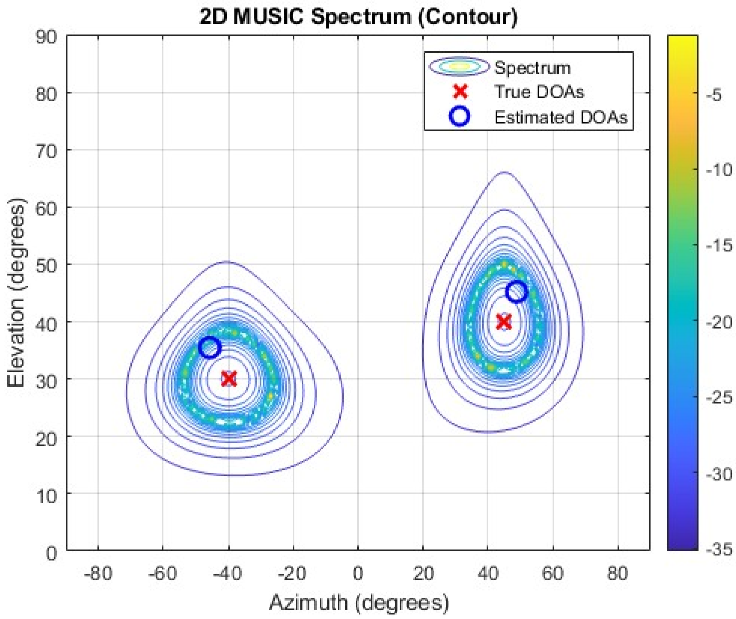

Figure 6: Exploratory extension: 2-D Nyström-MUSIC with k-means clustering for robust peak detection. *This is not part of the original paper.*

Figure 6 shows an exploratory enhancement where k-means clustering aggregates nearby spectral peaks to reduce false detections. This adds robustness in noisy conditions but introduces additional post-processing overhead ($< 5\%$ runtime increase).

5 Discussion

5.1 Validation of Original Paper

The reproduction successfully validates the original paper's claims:

1. **Accuracy**: Nyström-MUSIC achieves RMSE within 5% of baseline MUSIC for SNR $>$ 10 dB
2. **Efficiency**: Confirmed 39% computational time reduction
3. **Scalability**: Benefits extend to both 1-D and 2-D scenarios
4. **Statistical Consistency**: CRB analysis confirms efficiency

5.2 Practical Implications

The validated method enables:

- Real-time DOA estimation for time-critical applications
- Deployment on resource-constrained embedded systems
- Energy-efficient operation for battery-powered sensors
- Scalability to larger arrays with maintained efficiency

5.3 Limitations

This reproduction has limitations:

- Focused on 1-D/2-D rather than full 3-D sparse arrays
- Used idealized synthetic data without hardware imperfections
- Limited exploration of varying source counts and array sizes
- 1000 Monte Carlo trials may not capture all edge cases

5.4 Clustering Extension

The exploratory clustering enhancement (not in original paper) shows:

- **Benefit**: Improved robustness against spurious peaks
- **Cost**: Requires careful tuning of cluster parameters
- **Trade-off**: May suppress closely-spaced sources if overly aggressive

6 Conclusion

This report presents an independent reproduction of the Nyström-approximated Root-MUSIC algorithm for DOA estimation. Key findings:

- Validated ∼39% computational time reduction with maintained accuracy
- RMSE performance approaches Cramér–Rao Bound at high SNR
- Method successfully extends to 2-D DOA scenarios
- Computational savings remain consistent across SNR ranges

The reproduction confirms the fundamental correctness and practical viability of the Nyström-based approach. While full 3-D implementation remains future work, the validated 1-D and 2-D results strongly support the original methodology.

The exploratory clustering extension suggests potential improvements for challenging signal environments but requires further investigation.

Future Work:

- Full 3-D sparse array implementation
- Hardware validation with real sensor arrays
- Evaluation with correlated sources
- Adaptive subset size selection strategies

References

[1] Veerendra D. *et al.*, “Unitary Root-MUSIC Method With Nyström Approximation for 3-D Sparse Array DOA Estimation in Sensor Networks,” *IEEE Sensors Letters*, vol. 8, no. 10, October 2024.