Noah Snyder suggested the idea that it might be possible to prove that $S^2 = \mathrm{id}$ for semisimple, cosemisimple Hopf algebras, using an approach involving framed TQFTs with corners and defects. The idea is to use the fiber functor to vec (or equivalently a rank 1 module category), and "pull theories across defects".

1 Tensor categories

There are various notions for duality in the 3-category TC of tensor categories.

1. The dual of a tensor category $\mathcal{C}$ as an object of the symmetric monoidal 3-category TC
2. Duals of objects within $\mathcal{C}$
3. The adjoint of the $\mathcal{C}$-$\mathcal{D}$-bimodule category seen as a 1-morphism of TC
4. adjoints of functors on bimodule categories

Definition 1.1. An adjunction between functors F and G, $F \dashv G : \mathcal{A} \to \mathcal{B}$ is a couple of natural transformations, the unit $\eta : id_{\mathcal{B}} \to G \circ F$ and the counit $\epsilon : F \circ G \to id_{\mathcal{A}}$, such that for an object $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ the following diagrams commute,

$$\begin{tikzcd} GX \arrow[r, "G\eta_X"] \arrow[dr, "\mathrm{id}_{GX}"'] & GFGX \arrow[d, "\epsilon_{GX}"] & FY \arrow[r, "\eta_{FY}"] \arrow[dr, "\mathrm{id}_{FY}"'] & FGFY \arrow[d, "F\epsilon_Y"] \\ & GX & & FY \end{tikzcd} \tag{1.1}$$

This may be expressed as the following equations,

$$\mathrm{id}_G = (\epsilon \odot id_G) \circ (id_G \odot \eta), \tag{1.2}$$

$$\mathrm{id}_F = (\mathrm{id}_F \odot \epsilon) \circ (\eta \odot \mathrm{id}_F). \tag{1.3}$$

We have the canonical natural isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(FY, X) \cong \mathrm{Hom}_{\mathcal{B}}(X, GY), \tag{1.4}$$

from which we interpret F as the left adjoint and G as the right adjoint. Considering functors as 1-morphisms and natural transformations as 2-morphisms in a 2-category, we may make a similar definition, where $\odot$ will denote horizontal composition of 2-morphisms. If we replace $\odot$ by the tensor product

$\otimes$ in equations (1.2) and (1.3), then the above definition is the definition of rigidity for a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ if we consider F and G as objects in the category $\mathcal{C}$.

Let $\mathcal{C}$ be a monoidal category, considered as a 2-category with 1 object, $*$. Then there is a 2-functor $F : \mathcal{C} \to \mathrm{Cat}$, the 2-category of categories, which is given by

$$F(*) = \mathcal{C} \tag{1.5}$$
$$F(A : * \to *) = - \otimes A \in \mathrm{End}(\mathcal{C}) \tag{1.6}$$
$$F(f : A \Rightarrow B) = - \otimes f : - \otimes A \to - \otimes B \tag{1.7}$$

Note that

$$F(* \xrightarrow{B \circ A} *) = \mathcal{C} \xrightarrow{- \otimes A \otimes B} \mathcal{C}, \tag{1.8}$$

but by convention of [dsps] $A \otimes B := B \circ A$ for 1-morphisms $* \to *$. (A should probably be an algebra object?)

Let Alg be the 2-category with objects algebras, 1-morphisms bimodules and 2-morphisms bimodule maps. Then there is a 2-functor sending an algebra A to mod-A in Cat.

(How is any linear category a Vec bimodule category?)

to be continued...

2 Misc

An object in a symmetric monoidal $(\infty, 3)$- category is called 1-dualizable if it is dual in the usual monoidal categorical sense. It is 2-dualizable if there are series of adjunctions

$$\ldots \mathrm{ev}^{LL} \dashv \mathrm{ev}^{L} \dashv \mathrm{ev} \dashv \mathrm{ev}^{R} \dashv \mathrm{ev}^{RR} \ldots \tag{2.1}$$

and

$$\ldots \mathrm{coev}^{LL} \dashv \mathrm{coev}^{L} \dashv \mathrm{coev} \dashv \mathrm{coev}^{R} \dashv \mathrm{coev}^{RR} \ldots . \tag{2.2}$$

An object is 3-dualizable if for every adjunction (F, G, u, v) in the above chains, the unit u and the counit v are part of an infinite chain of adjunctions.

A symmetric monoidal 3-category $\mathcal{C}$ is 1-dualizable if every object has a dual in the symmetric monoidal 1-category sense. It is 2-dualizable if it is 1-dualizable and if every 1 morphism has a left and right adjoint.
It is 3-dualizable if it is 1- and 2-dualizable and every 2-morphism has a left and right adjoint.

Definition 2.1. Cobordism hypothesis: n-dimensional local framed topological field theories with target a symmetric monoidal (∞, n)-category $\mathcal{C}$ are in one to one correspondence with the n-dualizable objects of $\mathcal{C}$; in fact, the space of such field theories homotopy equivalent to the space of n-dualizable objects of $\mathcal{C}$.

An n-dualizable (object in an)(∞, n)-category is also called a fully dualizable (object in an) (∞, n)-category.

2.1 n-framings

An n-framed k-manifold is a k-manifold M with a trivialization τ of $TM + \mathbb{R}^{n-k}$, the $(n-k)$-fold stabilization of the tangent bundle of M. One can realize this by immersing M into $\mathbb{R}^n$, and taking a trivialization of the tangent bundle along with a trivialization of the normal bundle of the immersion.