Probability and Statistics, S2023	Posted: March 28, 2023
Problem Set 3	
Lecturer: Hyang-Won Lee	**Due: April 4, 2023**

Random variables, discrete random variables, probability mass functions, expectation, variance

1. (10 points) Determine whether the following random variables are discrete or not.

   (a) X is a random variable such that

   $$X = i, \text{ with probability } 2^{-i}, i = 1, 2, 3, ...$$

   (b) Draw a number ω from the interval $[-1, 1]$, and define

   $$X(\omega) = \begin{cases} \omega^2, & \text{if } \omega \geq 0 \\ -1, & \text{if } \omega < 0 \end{cases}$$

   (c) Draw a number ω from the interval $[-1, 1]$, and define

   $$X(\omega) = \omega^2$$

(d) Draw a number ω from the interval $[-1, 1]$, and define

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \geq 0 \\ -1, & \text{if } \omega < 0 \end{cases}$$

2. (30 points) The random variable X has PMF

$$p_X(x) = \begin{cases} cx^2, & \text{if } x = 1, 2, 3, 4 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate the value of c.

(b) Calculate the probability that X is an even number.

(c) Calculate the probability that X is greater than 2.

(d) Calculate the expectations of X and X^2.

(e) Calculate the variance of X

(f) Compare the variances of X^2, X and $\sqrt{X}$, without actually calculating the variances. (계산하지 않고, X^2, X, $\sqrt{X}$의 분산의 대소를 비교해 보시오)

3. The random variable X has the probability mass function given as

$$p_X(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}, x = 0, ..., k$$

where n, k, N are all integers, and satisfy

$$k < n \quad \text{and} \quad N > n + k$$

(a) Prove that $p_X(x)$ is indeed a probability mass function, i.e., $\sum_{x=0}^{k} p_X(x) = 1$ (Hint. Argue or show that $\binom{N}{n} = \sum_{x=0}^{k} \binom{k}{x}\binom{N-k}{n-x}$.)

(b) Show that $\mathbb{E}[X] = \frac{nk}{N}$.

(c) Show that $\mathrm{var}(X) = \frac{N-n}{N-1} \cdot n \cdot \frac{k}{N}(1 - \frac{k}{N})$.

4. (20 points) Consider a lottery. Each ticket is 1,000 WON and has six numbers out of $\{1, 2, ..., 45\}$. A ticket is a winner if its six numbers all match the six numbers drawn at random at the end of a week (assume partial matchings are not rewarded). For each ticket sold, 500WON is added to the pot for the winners. If there are k winners, the prize money is split equally among the winners. Suppose that you bought a ticket in a week in which the total of $2n$ tickets were sold. (티켓 한 장이 1000원인 로또를 생각하자. 각 티켓에는 6개의 숫자가 있고, 만약 이 숫자가 모두 추첨으로 뽑힌 숫자와 맞으면 상금을 받는다. 부분적으로 맞는 것은 상금이 없다고 가정하자. 티켓이 한 장 팔릴 때마다 500원이 당첨금으로 적립된다. 만약 k개의 티켓이 당첨되면 적립된 당첨금을 동일하게 배분한다. 당신이 티켓을 한 장 샀고, 그 주에 총 $2n$장의 티켓이 팔렸다고 가정하자.)

   (a) What is the probability p that an arbitrary ticket is a winning ticket? (임의의 티켓 하나가 당첨 티켓일 확률을 구하시오)

   (b) Suppose that your ticket is a winning ticket. Let K_n be the number of other winning tickets (besides yours). Assume that whether a ticket becomes a winning ticket or is independent of all other tickets. Find the PMF of K_n. (Hint. There are $2n - 1$ tickets besides yours.)

   (c) Again, your ticket is a winning ticket. Let W_n be the prize money you get for your winning ticket. Calculate the expected value of W_n. Simplify the expression as much as you can. (Hint. When there are k other winners, you get $\frac{1}{k+1}$ of the total prize money.)

   (d) Does the expected prize money increase linearly with the sales of tickets? If the result in part (c) is not simplified enough, it may be hard to see the trend. In this case, you may write a code to numerically plot the expected value as n increases. (판매된 티켓수가 증가함에 따라 상금이 선형적으로 증가합니까? (c)에서의 결과를 간단하게 정리 못했을 경우 경향을 보기 힘들 수 있음. 이 경우 컴퓨터로 수치적으로 그래프를 그리고 확인해도 됨)