Probability and Statistics, S2021	Date: April 20, 2021

Midterm Exam

Lecturer: Hyang-Won Lee	**Time: 12-1:20pm**

1. **(25 points)** Write T for a true statement and F for a false statement. You DO NOT need to justify your answers. (다음 중 참인 것과 거짓인 것을 구분하시오. 답만 적으면 됨)

   (a) The sample space associated with a discrete random variable is always countable. (이산확률변수와 연관된 표본공간은 항상 countable 이다)

   (b) For discrete random variables X, Y and Z, if $p_{X,Y,Z}(x,y,z) = p_X(x)p_Y(y)p_Z(z), \forall x, y, z$, then $p_{X,Y}(x,y) = p_X(x)p_Y(y), \forall x, y$.

   (c) For the CDF $F_X(x)$ of a random variable X, $(F_X(x) - F_X(y))(x - y) \leq 0$. (확률변수 X의 누적분포함수 $F_X(x)$에 대해 $(F_X(x) - F_X(y))(x - y) \leq 0$이 성립한다)

   (d) For a continuous random variable X, $\mathbb{P}(X = x) = 0$ for all real numbers x.

   (e) For a continuous random variable X, its PDF $f_X(x)$ is the probability that $X = x$.

   (f) If two events A and B are disjoint, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

   (g) If $\mathbb{P}(A) \leq \mathbb{P}(B)$, then $A \subseteq B$.

   (h) For two discrete random variables X and Y, $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$ if $p_{X,Y}(x,y) = p_X(x)p_Y(y), \forall x, y$.

   (i) For a probabilistic model $(\Omega, \mathbb{P})$ with countable sample space Ω, the probability of an event A is $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$.

   (j) For the conditional PMF of X given $X \in A$ for some set A, we have $p_{X|X\in A}(x) = 0$ if $x \notin A$.

2. **(20 points)** Consider two random variables X and Y whose joint PMF $p_{X,Y}(x,y)$ is given as

	3	a	1/18	1/18
y	2	b	1/18	1/18
	1	c	1/18	1/18
		1	2	3
			x	

   (a) Determine the values of a, b and c so that X and Y are independent.

   (b) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.

   (c) Define two events $A = \{X \geq 2\}$ and $B = \{Y < 2\}$. Are A and B independent? Justify your answer.

   (d) Compute $\mathbb{E}[X|A]$ and $\mathbb{E}[Y|B]$.

3. **(20 points)** There are 6 coins and a fair 6-sided die. Let p_i be the probability of head of ith coin. Assume $p_i = \frac{i}{8}, i = 1, ..., 6$. Consider the following experiment:

   1) Roll the die (let i be the number that comes up)
   2) Pick the ith coin and toss it three times independently.

   (a) Describe the sample space.

   (b) Calculate the probability that at least two heads come up. (you will get full score if the formula is correct; 식만 맞으면 됨)

   (c) Are B and A_i in (b) independent?

   (d) Given that three heads came up, what is the conditional probability that the roll is i.

4. **(20 points)** There are 4 coins and a fair 4-sided die. Let p_i be the probability of head of ith coin. Assume $p_i = \frac{i}{8}, i = 1, ..., 4$. Consider the following experiment:

   1) Roll the die (let i be the number that came up)
   2) Pick the ith coin and toss it three times independently.

   (a) Describe the sample space.

   (b) Calculate the probability that at least two heads come up. (you will get full score if the formula is correct; 식만 맞으면 됨)

   (c) Are B and A_i in (b) independent?

   (d) Given that three heads came up, what is the conditional probability that the roll is i.

5. **(20 points)** Consider the following PMF of a random variable X:

$$p_X(x) = \begin{cases} \frac{a+xy}{a^2}, & \text{if } x = 1,2,3,\ y = 1,2,3 \\ 0, & \text{otherwise} \end{cases}$$

where a is a constant.

(a) Find the value of a.

(b) Find the marginal PMF $p_X(x)$

(c) Find the conditional PMF of X given $Y = 1$

(d) Are X and Y independent? Justify your answer.

(e) Calculate the expectation of X, given $Y = 1$.

6. **(10 points)** Let X be an exponential random variable with parameter $\lambda > 0$.

(a) Calculate the probability $\mathbb{P}(X \leq a)$ where a is a constant.

(b) Let $X_1, ..., X_n$ be independent and identically distributed (i.i.d.) exponential random variables with parameter λ. Define $Y = \max\{X_1, ..., X_n\}$. Calculate the PDF of Y.

7. **(10 points)** Let X be an exponential random variable with parameter $\lambda > 0$.

(a) Calculate the probability $\mathbb{P}(X \geq a)$ where a is a positive constant.

(b) Let $X_1, ..., X_n$ be independent and identically distributed (i.i.d.) exponential random variables with parameter λ. Define $Y = \min\{X_1, ..., X_n\}$. Calculate the PDF of Y.

8. **(20 points)** Consider investing your capital assets in the stock market. On day 1 (happy day), your stock goes up 20% with probability 0.9, and goes down 20% with probability 0.1. On day 2 (sad day), your stock goes up 20% with probability 0.1, and goes down 20% with probability 0.9. This is repeated afterwards. The following are notation and assumptions: (당신의 현금자산을 주식시장에 투자하는 것을 생각하자. 첫번째 날에는 당신의 주식이 0.9의 확률로 20% 오르고, 0.1의 확률로 20% 내린다. 두번째 날에는 당신의 주식이 0.1의 확률로 20% 오르고, 0.9의 확률로 20% 내린다. 그리고, 이것이 계속 반복된다. 다음은 이 문제 관련 기호와 가정)

- Y_0: your original capital assets (원금)
- X_i: random variable representing the percentage of your stock price gain on the ith day. So for i odd, the PMF of X_i is (i번째 날에 당신의 주식이 오른 퍼센티지를 나타내는 확률 변수. i가 홀수라면 확률질량함수는 다음과 같음)

$$p_{X_i}(x) = \begin{cases} 0.9, & x = 0.2 \\ 0.1, & x = -0.2 \\ 0, & \text{otherwise} \end{cases}$$

For i even, the PMF of X_i is (i가 짝수라면 확률질량함수는 다음과 같음)

$$p_{X_i}(x) = \begin{cases} 0.1, & x = 0.2 \\ 0.9, & x = -0.2 \\ 0, & \text{otherwise} \end{cases}$$

- X_i's are independent
- Y_n: your total assets (market value) after n days (n일 지난 후 총 자산)

(a) Calculate the expectation of X_1 and X_2.

(b) Find the expression for Y_n. (Y_n의 식을 구하시오) (Hint. After the first day, it is $Y_1 = Y_0 + Y_0 \cdot X_1$).

(c) Calculate your expected total assets after $2n$ days.

(d) Based on the result in (c), calculate how much you will gain or lose after 30 days.

9. **(20 points)** There is a face recognition system at the entrance of a building. The system identifies each person (entering the building) correctly with probability p, and incorrectly with probability $1 - p$. The correctness of an identification is independent of all others and also the number of people entering. For each correct identification, the face recognition system company gets 1 dollar of incentive, whereas for incorrect identification, they lose 3 dollars as penalty. The number of people entering the building is a random variable N with the following PMF: (빌딩의 입구에 얼굴인식 시스템이 있다. 이 시스템은 빌딩에 들어오는 각 사람을 p의 확률로 정확하게 인식하고, $1 - p$의 확률로 틀리게 인식한다. 각 사람별 인식 결과는 독립이다. 각각의 정확한 인식에 대해 얼굴인식 시스템 회사는 1달러의 인센티브를 받지만, 부정확하면 3달러의 벌금을 부여받는다. 빌딩에 들어오는 사람의 수는 확률변수 N이고 다음의 확률질량함수를 갖는다)

$$p_N(n) = \begin{cases} 1/5, & n = 10 \\ 2/5, & n = 20 \\ 2/5, & n = 30 \\ 0, & \text{otherwise} \end{cases}$$

Let Y be the net gain based on the above incentive/penalty policy. (Y는 상술한 인센티브/벌금 정책에 따른 순수 이득이라고 하자)

(a) Given that the number of people who entered the building is n, what is the conditional expectation of Y?

(b) Based on the result in part (a), find the minimum value p_0 such that the expected net gain is positive for $p > p_0$.

(c) Assume $p = p_0$ where p_0 is the value found in part (b). Calculate the probability that the net gain is no greater than zero, that is, $\mathbb{P}(Y \leq 0)$.