An array of car speed is given, and they are positive integers. Assume they are arrayed on a straight line, with the given order (car 0 is front)

If each car's speed is uniformly distributed
what is the expected number of blocks?
Examples:

car	0	1	2	3	4	5	6
→ v	10	9	8	7	6	5	4

7 blocks.

					(4)	(4)	(4)
→ v	10	9	12	4	6	5	8
			(9)				

3 blocks.

→ v	10	11	12	13	14	15	16
		(10)	(10)	(10)	(10)	(10)	(10)

1 block

Scan from left, count rank min, rank 0 $\Rightarrow$ slowest
(front)

if rank < curmin

curmin $\leftarrow$ rank

count++

report count

If speed is uniformly distributed, then the rank permutations are uniformly distributed.

let $X^{(n)}$ be the random var of blocks formed by n cars

then

$$E[X^{(n)}] = \sum_{0 \le i \le n-1} E[X^{(n)} \mid \text{min rank at index } i] \cdot Pr\{\text{min rank at index } i\}$$

Then consider

$$E[X^n \mid \text{min at } i]$$

$$= E[X^n \mid H_i]$$

$$= \frac{1}{|H_i|} \sum_{w \in H_i} X^n(w) = \frac{1}{|H_i|} \sum_{w \in H_i} 1 + X^i(w^i)$$

w^i is w truncated to first i elements

re-order them in rank $\{0, 1, \cdots i-1\}$

then $\dfrac{\# w^i \text{ in sum}}{|H_i|} = Pr\{w^i \text{ in } X^i\}$

switch sample space from X^n to X^i

Thus

$$E[X^n] = \sum_i \Pr\{\text{min at } i\} \cdot \sum_{w \in H_i} \frac{1}{|H_i|} + \frac{X^i(w^i)}{H_i}$$

$$= \sum_i \frac{1}{n} \cdot (1 + E[X^i])$$

incidentally

$$E[X^0] = 0$$

$$E[X^1] = 1$$

$g_n = E[X^n]$ then $g_n = H^{(n)}$ where $H^{(n)}$ is the harmonic series.

since:

$$g_n = \sum_i \frac{1}{n}(1+g^i) = \sum_{0 \le i \le n-1} \frac{1}{n-1}(1+g^i)\frac{n-1}{n}$$

$$= \sum_{0 \le i \le n-2} \frac{1}{n-1}(1+g^i)\frac{n-1}{n} + \frac{1}{n-1}(1+g^{(n-1)})\frac{n-1}{n}$$

$$= \frac{n-1}{n} g_{n-1} + \frac{1}{n} + \frac{1}{n} g_{n-1}$$

$$= \frac{1}{n} + g_{n-1} \checkmark$$