JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY ANANTAPUR

II B.Tech II-Sem (E.C.E)	**T**	**Tu**	**C**
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(15A02303) CONTROL SYSTEMS ENGINEERING

http://nptel.ac.in/courses/108101037/15

For all the 5 Units

OBJECTIVES:

To make the students learn about:

- Merits and demerits of open loop and closed loop systems; the effects of feedback
- The use of block diagram algebra and Mason's gain formula to find the effective transfer function between two nodes
- Transient and steady state responses , time domain specifications
- The concept of Root loci
- Frequency domain specifications, Bode diagrams and Nyquist plots
- The fundamental aspects of modern control

UNIT – I INTRODUCTION

Open Loop and closed loop control systems and their differences- Examples of control systems- Classification of control systems, Feedback Characteristics, Effects of positive and negative feedback. Mathematical models – Differential equations of Translational and Rotational mechanical systems, and Electrical Systems, Block diagram reduction methods – Signal flow graph - Reduction using Mason's gain formula. Transfer Function of DC Servo motor - AC Servo motor - Synchro transmitter and Receiver

UNIT-II TIME RESPONSE ANALYSIS

Step Response - Impulse Response - Time response of first order systems – Characteristic Equation of Feedback control systems, Transient response of second order systems - Time domain specifications – Steady state response - Steady state errors and error constants

UNIT – III STABILITY

The concept of stability – Routh's stability criterion – Stability and conditional stability – limitations of Routh's stability. The root locus concept - construction of root loci-effects of adding poles and zeros to G(s)H(s) on the root loci.

UNIT – IV FREQUENCY RESPONSE ANALYSIS

Introduction, Frequency domain specifications-Bode diagrams-Determination of Frequency domain specifications and transfer function from the Bode Diagram-Stability Analysis from Bode Plots. Polar Plots-Nyquist Plots- Phase margin and Gain margin-Stability Analysis.

Compensation techniques – Lag, Lead, Lag-Lead Compensator design in frequency Domain.

UNIT – V STATE SPACE ANALYSIS

Concepts of state, state variables and state model, derivation of state models from differential equations. Transfer function models. Block diagrams. Diagonalization. Solving the Time invariant state Equations- State Transition Matrix and it's Properties. System response through State Space models. The concepts of controllability and observability.

OUTCOMES:

After completing the course, the student should be able to do the following:

- Evaluate the effective transfer function of a system from input to output using (i) block diagram reduction techniques (ii) Mason's gain formula
- Compute the steady state errors and transient response characteristics for a given system and excitation
- Determine the absolute stability and relative stability of a system
- Draw root loci
- Design a compensator to accomplish desired performance
- Derive state space model of a given physical system and solve the state equation

TEXT BOOKS:

1. Modern Control Engineering, Katsuhiko Ogata, PEARSON, 1st Impression 2015.
2. Control Systems Engineering, I. J. Nagrath and M. Gopal, New Age International Publishers, 5th edition, 2007, Reprint 2012.

REFERENCE BOOKS:

1. Automatic Control Systems, Farid Golnaraghi and Benjamin. C. Kuo, WILEY, 9th Edition, 2010.
2. Control Systems, Dhanesh N. Manik, CENGAGE Learning, 2012.
3. John J D'Azzo and C. H. Houpis , "Linear Control System Analysis and Design: Conventional and Modern", McGraw - Hill Book Company, 1988.

UNIT-I

INTRODUCTION

The control system is that means by which any quantity of interest in a machine, mechanism or other equipment is maintained or altered in accordance with a desired manner.

When a number of elements or components are connected in a sequence to perform a specific function, that group of elements is called a system. In a system, when the output quantity is controlled by varying the input quantity, the system is called controlled system. The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

Basically, there are two types of control systems, namely open loop and closed loop control systems.

Open-loop System: Any physical system which does not automatically correct the variation in its output, is called open loop system. This means that the output is not fedback to the input for correction.

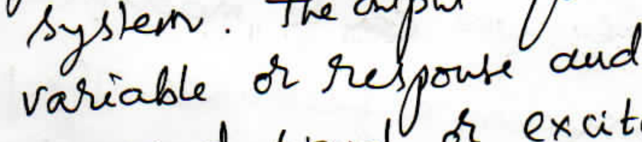


Figure: open-loop systems

In open-loop systems the output is varied by varying the input, but due to external disturbances the system output may change. When the output changes due to disturbances, it is not followed by changes in input to correct

the output. In open loop systems, the changes in output are corrected by changing the input manually.
Ex: Traffic light controller, Combinational circuit etc.

<u>closed-loop System</u>: Control systems in which the output has an effect upon the input quantity in order to maintain the desired output are called closed loop systems.

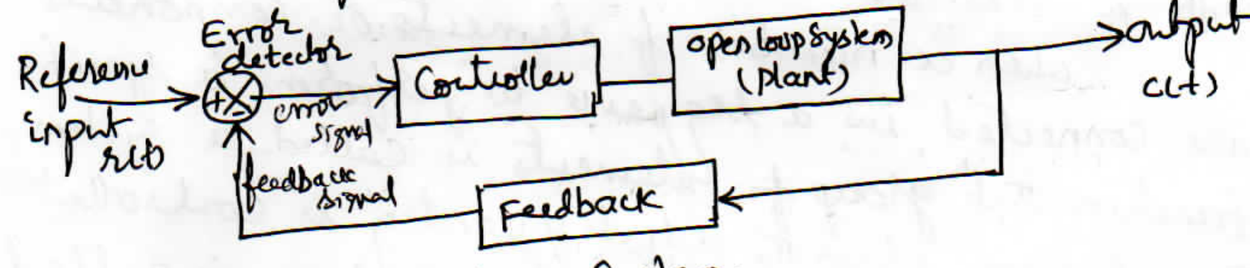


Figure: <u>closed loop System</u>

The open loop system can be modified as closed loop system by providing a feedback. The provision of feedback automatically corrects the changes in output due to disturbances. Hence the closed loop system is also called automatic Control system.

The reference signal corresponds to desired output. The feedback path elements samples the output and converts it to a signal of same type as that of reference signal. The feedback signal proportional to output signal and it is fed to the error detector. The error signal generated by the error detector is the difference between reference signal and feedback signal. The controller modifies and amplifies the error signal to produce better control action. The modified error signal is fed to the plant to correct its output. Ex: Sequential circuits, Driving of automobile

Advantages of open loop systems:

(1) The open loop systems are simple and economical
(2) The open loop systems are easier to correct
(3) Generally the open loop systems are stable.

Disadvantages of open loop systems:

(1) The open loop systems are inaccurate and unreliable
(2) The changes in the output due to external disturb-
-ances are not corrected automatically.

Advantages of closed loop systems:

(1) The closed loop systems are accurate
(2) The closed loop systems are accurate even in the presence of non-linearties.
(3) The sensitivity of the system may be made small to make the systems more stable.
(4) The closed loop systems are less affected by noise

Disadvantages of closed loop systems:

(1) The closed loop systems are complex and costly.
(2) The feedback in closed loop system may lead to oscillatory response.
(3) The feedback reduces the over all gain of the system
(4) Stability is a major problem in closed loop system and more care is needed to design a stable closed loop system.

Examples of Control systems :

(1) Driving of Automobile :

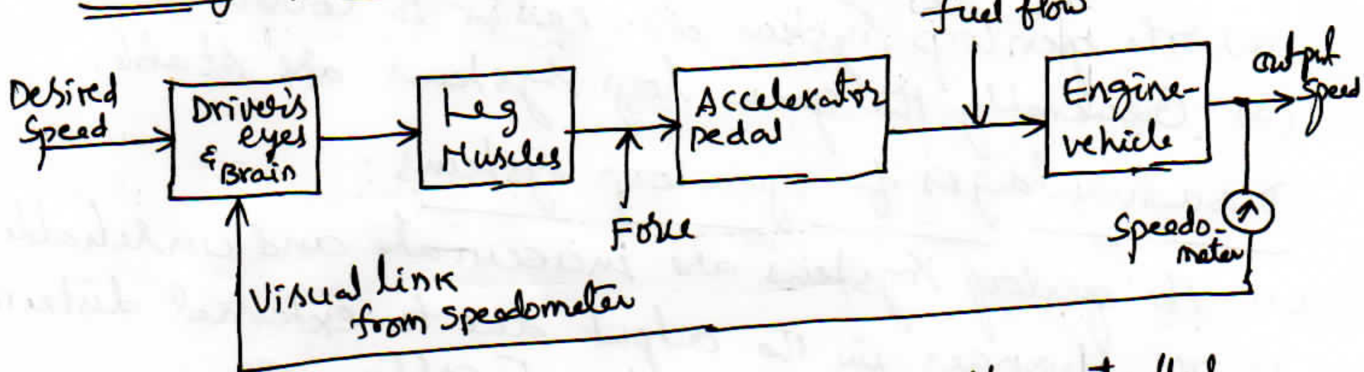


Figure : schematic diagram of a manually controlled closed-loop system.

The automobile driving System (accelerator, Carburator, and engine-vehicle) Constitutes a control system. The speed of the automobile is a function of the position of its accelerator. The desired speed can be maintained by Controlling pressure on the accelerator pedal.

The route, speed and acceleration of the automobile are determined and Controlled by the driver by observing traffic and road conditions and by properly manipulating the accelerator, clutch, gear-lever, brakes and steering wheel etc. Suppose the driver wants to maintain a speed of 50 km, the actual speed of the automobile is measured by the speedometer and indicated on its dial. The driver reads the speed dial visually and compares the actual speed with the desired speed mentally. If there is a deviation of speed from the desired speed, the driver takes the decision to increase or decrease the speed. The decision is executed by change in pressure of foot on the accelerator pedal.

(2) Temperature Control System:

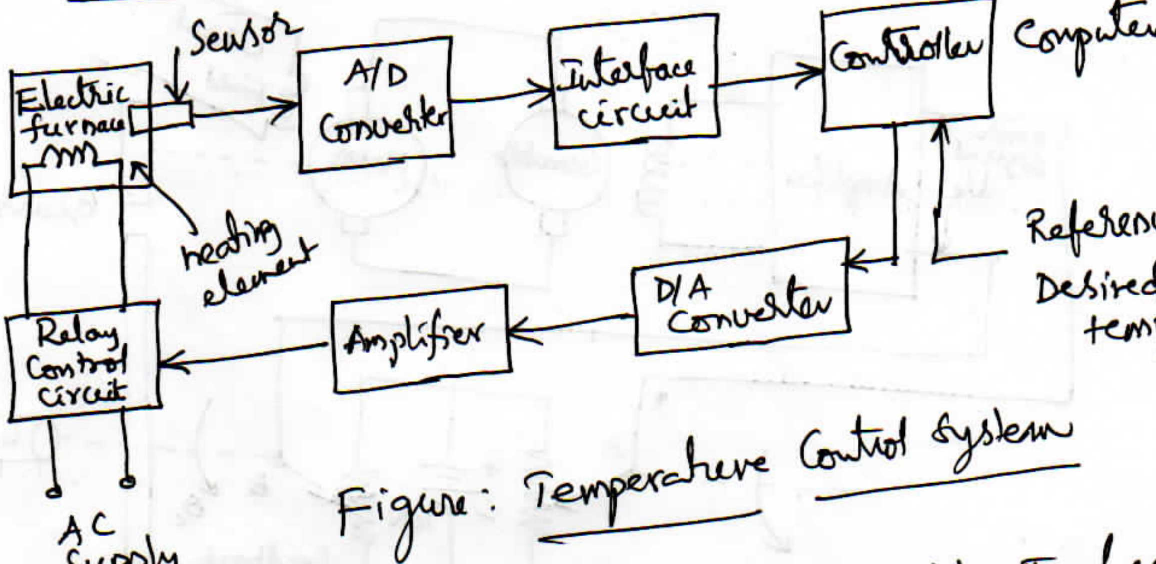


Figure: Temperature Control System

The temperature of the system is controlled by the heat generated by the heating element. The furnace output temperature depends on the time during which the supply to heater remains ON.

The ON and OFF of supply is governed by the time setting of the relay. The temperature of the furnace is measured by sensor and is converted to digital signal by A/D converter.

The switching ON and OFF of the relay is controlled by a controller which is a digital system or computer. The computer reads the actual temperature and compares with desired temperature. If it finds any difference then it sends signal to switch ON or OFF the relay through D/A converter and amplifier. Thus the system automatically corrects any changes in output.

(3) position control System using Servomotor:

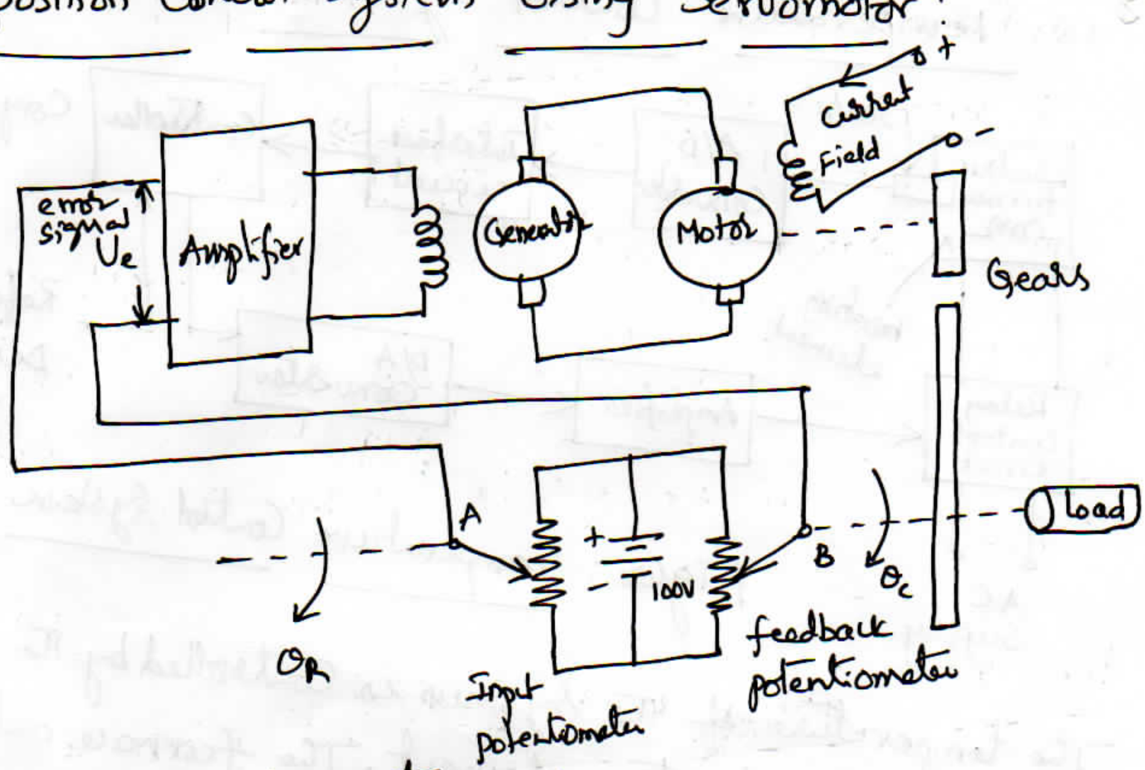


Figure: position Control System

The position Control systems is a closed loop system. The systems consists of a servo motor powered by a generator. The load whose position has to be controlled is connected to motor shaft through gear wheels. potentiometers are used to convert the mechanical motion to electrical signals. The desired position θ_R is set on the input potentiometer and the actual load position θ_c is fed to feedback potentiometer. The difference between two angular positions generates an error signal V_e, which is amplified and fed to generator field circuit. The induced emf in the generator drives the motor in such a way that to get $\theta_c = \theta_R$. If $\theta_c = \theta_R$, then $V_e = 0$ and the motion of the motor is stopped.

The feedback Control systems in which the controlled variable is position or time derivatives of position (velocity and acceleration) are called Servo mechanisms. (Servo mechanism)

Classification of Control Systems: Basically, feedback control systems are classified as
(1) Linear or Non-linear systems
(2) Time-varying or Time-invariant systems

(1) Linear versus Non-linear systems: If the system satisfies the homogeneous and super position principles, then the system is linear otherwise non-linear. Most real-life control systems have non-linear characteristics to some extent.

(2) Time-Invariant versus Time-varying systems: If the parameters of the control system do not change with time, the system is called time-invariant, otherwise time-varying systems. In practice, most of the physical systems contain elements that drift or vary with time. These systems are further classified as continuous-data and Discrete-data control systems.

(i) Continuous-data Control Systems: The signals at various parts of the system are all functions of time 't', the system is said to be continuous-data control system.

These continuous-data control systems are further classified as ac or dc control systems. If the signals in the system are modulated by some form of modulation scheme, then the systems are said to be ac or modulated control systems. On the other hand, if the ac signals are unmodulated, the system is said to be dc or un-modunmodulated Control system.

(ii) Discrete-data control systems: If the signals at one or more points of the systems are in the form of either a pulse-train or a digital code. These systems are further classified into sampled data and digital control systems.

In sampled data control systems, the signals are in the form of pulse train.

In digital control systems, the signals are digitally coded such as binary code to use digital computer.

Feedback characteristics, Effects of positive and negative

Feedback:

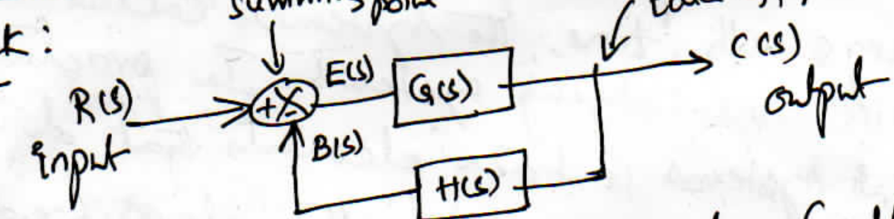


Figure: Negative or Degenerative feedback system

where $G(s)$ = Forward path gain
$H(s)$ = Feedback path gain
$E(s)$ = Error signal
$B(s)$ = Feedback signal

where the output
$$C(s) = E(s)\,G(s)$$
$$= [R(s) - B(s)]\,G(s)$$
$$= [R(s) - C(s)H(s)]\,G(s)$$
$$\therefore C(s)[1+G(s)H(s)] = R(s)\,G(s)$$

$\therefore$ The system Transfer function $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$

Figure: Positive Feedback System

where $\frac{C(s)}{R(s)} = \frac{G(s)}{1-G(s)H(s)}$

The feedback has effects on stability, bandwidth, overall gain, impedance and sensitivity

(i) Effect of feedback on over all gain: Let us assume that the system function $M = \frac{G}{1+GH}$, for convenience. In practical control systems, both G and H are functions of frequency, so the magnitude of $1+GH$ may be greater than 1 in one frequency range but less than 1 in other. Therefore, feedback could increase the gain of the system in one frequency range but decrease it in another.

(ii) Effect of feedback on stability: A system is said to be unstable, if its output is out of control.

we have system gain $M = \frac{G}{GH+1}$; if $GH = -1$ the output of the system is infinite for any finite output, and the system is said to be unstable. Therefore, we may state that feedback can cause a system that is originally stable to become unstable.

Now, let us consider a system with two feedbacks shown in figure, where the output is c(s) and

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH+GF}$$

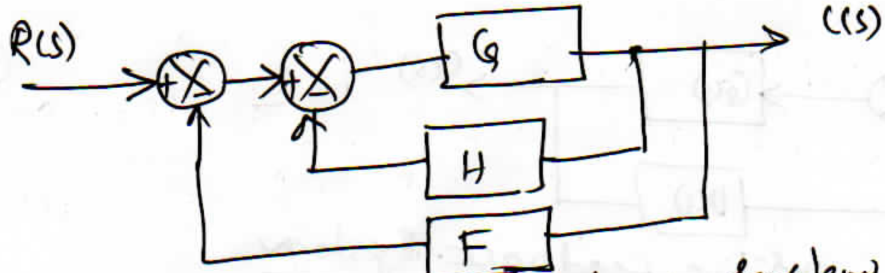


Figure: Negative feedback system

$$\frac{C(s)}{R(s)} = \frac{G}{1+G(H+F)} = \frac{G}{1+GH+GF}$$

If the above system is unstable, because of feed back G, then if $GH = -1$, we will get

$$\frac{C(s)}{R(s)} = \frac{G}{1-1+GF} = \frac{G}{GF} = \frac{1}{F}$$

Now, the over all system can be made stable by properly selecting 'F'.

In practice, GH is a function of frequency, and the stability condition of the closed-loop system depends on the magnitude and phase of GH. Thus, the feedback can improve stability or harmful to stability if is not properly applied.

(iii) Effect of feedback on Sensitivity: In general, a good control systems should be very insensitive to parameter variations but sensitive to the input.

The sensitivity of the gain of the over all system M to the variation in 'G' is defined as

$$S_G^M = \frac{\partial M/M}{\partial G/G} = \frac{\text{percentage change in } M}{\text{percentage change in } G}$$

$$= \frac{\partial M}{\partial G}\cdot\frac{G}{M} = \frac{\partial}{\partial G}\left[\frac{G}{1+GH}\right]\cdot\frac{G}{\left(\frac{G}{1+GH}\right)}$$

$$= \frac{1}{1+GH}$$

Thus, the sensitivity of a closed loop system with respect to variation in G is reduced by a factor 1+GH as compared to that of an open-loop system. (6)

The sensitivity of output M w.r.t feedback 'H' is given by

$$S_H^M = \frac{\partial M/M}{\partial H/H} = \frac{\partial M}{\partial H}\cdot\frac{H}{M} = \frac{\partial}{\partial H}\left(\frac{G}{1+GH}\right)\cdot\frac{H}{\left(\frac{G}{1+GH}\right)}$$

$$= \frac{-GH}{1+GH}$$

In practice, GH is a function of frequency, the magnitude of 1+GH may be less than unity in one frequency range and greater than unity in another. Hence the feedback may increase or decrease sensitivity of the system.

Differential Equations of Translational and Rotational Systems & Electrical Systems:

Mathematical Models of physical Systems:

A physical system is a collection of physical objects connected together to serve an objective. Mathematical representation of the physical model through use of appropriate physical laws is known as mathematical model.

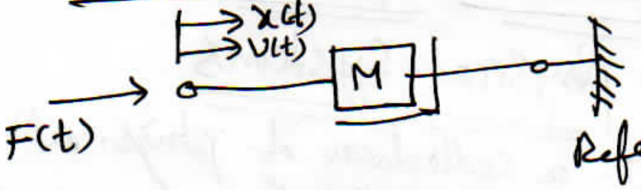
Mathematical models of most physical systems are characterised by differential equations. If the mathematical model obeys superposition and homogeneity principles, then the model is said to be linear. If the coefficients of differential equations are independent of time 't', then the physical model is said to be linear-time invariant.

Mechanical Systems: Mechanical systems are analysed by three idealised elements namely the mass, the spring and the damper, using Newton's law of motion. The motion of mechanical elements can be translatory, rotational or combination of both.

Translational Systems: The motion along a straight line is called the translatory motion. The variables which describe the translatory motion of mechanical systems are velocity, acceleration and displacement. The elements involved in the translatory motion are

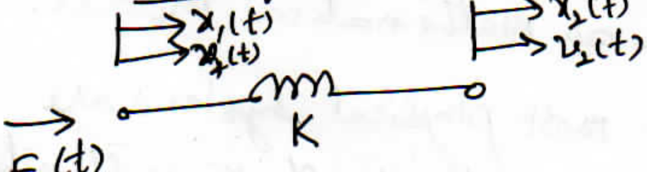
(1) The Mass element:

$$F = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}$$



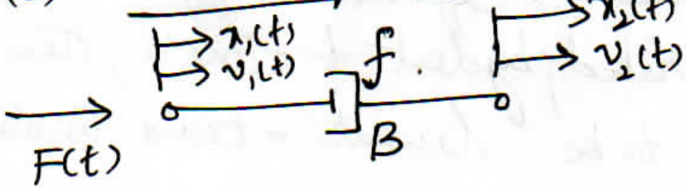
(2) The Spring element

$$F = K(x_1 - x_2) = Kx$$
$$= K\int_{-\infty}^{t}(v_1 - v_2)\,dt$$
$$= K\int_{-\infty}^{t} v\,dt$$



(3) The damper element:

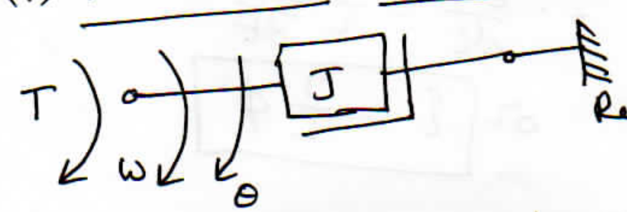
$$F = f(v_1 - v_2)$$
$$= f\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)$$
$$= f\frac{dx}{dt}$$



where x(m), v(m/sec), M(kg), F(Newton), K(N/m), f(N/m/sec), B(N/m/sec)

Rotational Systems: The movement of a body arround its fixed axis is called the rotational motion. The basic elements of rotational motion are moment of inertia (J), spring stiffness (K) and viscous friction coefficient (f or B).

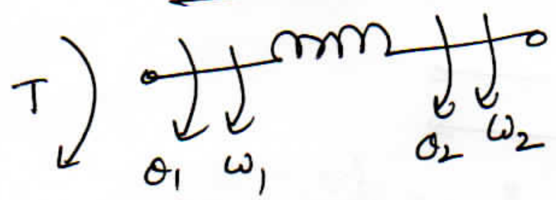
(1) The Moment of Inertia (J)



$$T = J\frac{d\omega}{dt} = J\frac{d^2\theta}{dt^2}$$

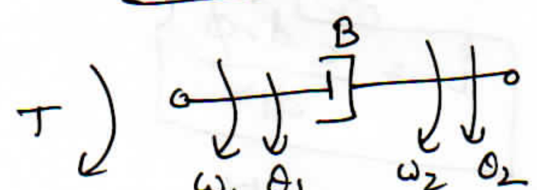
where T is torque in Nm
J is inertia in Kgm^2
ω is angular velocity in rad/sec
θ is angular displacement in rad.

(2) The Torsional Spring element (K)



$$T = K(\theta_1-\theta_2) = K\theta$$
$$= K\int_{-\infty}^{t}(\omega_1-\omega_2)\,dt$$
$$= \int_{\infty}^{t}\omega\,dt$$

(3) The damper element (f or B):

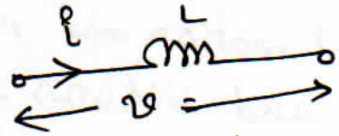


$$T = B(\omega_1-\omega_2) = f(\omega_1-\omega_2)$$
$$= B\left(\frac{d\theta_1}{dt}-\frac{d\theta_2}{dt}\right)$$
$$= B\dot{\theta}$$

where K is in Nm/rad, viscous friction coefficient f or B is (Nm/rad/sec).

Electrical Systems: The passive electric elements are inductor, resistor and Capacitor.

(1) Inductor:



$v = L\frac{di}{dt}$; $i = \frac{dq}{dt}$

$\therefore \boxed{v = L\frac{d^2q}{dt^2}}$

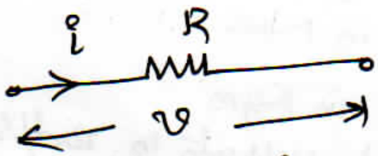
Also $\frac{di}{dt} = \frac{1}{L}v$,

where $v = \frac{d\phi}{dt}$

$\therefore \frac{di}{dt} = \frac{1}{L}\frac{d\phi}{dt}$

or $\boxed{i = \frac{1}{L}\phi}$

(2) Resistor:



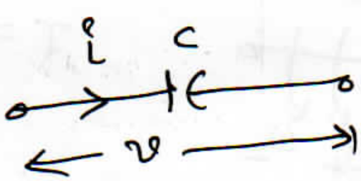
where $v = iR$

or $\boxed{v = R\frac{dq}{dt}}$

Also $i = \frac{1}{R}v = \frac{1}{R}\frac{d\phi}{dt}$

$\therefore \boxed{i = \frac{1}{R}\frac{d\phi}{dt}}$

(3) Capacitor:



where $v = \frac{1}{C}\int i\,dt$

$= \frac{1}{C}\int \frac{dq}{dt}dt$

$\boxed{v = \frac{1}{C}(q)}$

$v = \frac{1}{C}q$

Also $i = C\frac{dv}{dt}$

$\frac{dq}{dt} = C\frac{d^2\phi}{dt^2}$

or $\boxed{i = \frac{C\,d^2\phi}{dt^2}}$

$i = C\frac{d^2\phi}{dt^2}$

Analogous Systems : Systems with identical differential equations are called analogous systems. There are two types of analogy namely

(1) Force (Torque) – voltage analogy.

(2) Force (Torque) – current analogy :

(1) Force (Torque) – voltage Analogy :

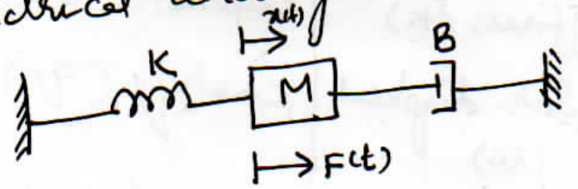
Mechanical system		Electrical system
Translational system	Rotational system	
Force (F)	Torque (T)	voltage (V)
Mass (M)	Inertia (J)	Inductance (L)
Viscous friction coefficient (B)	Viscous friction Coefficient (B)	Resistance (R)
Spring stiffness (K)	Torsional spring stiffness (K)	Reciprocal of Capacitance ($1/c$)
Displacement (x)	Angular displacement (ω)	charge (q)
velocity (v)	Angular velocity (ω)	current (i)

Table : Analogous quantities in Force (Torque) – voltage Analogy :

(2) Force (Torque) – Current Analogy:

Mechanical System: Translational	Mechanical System: Rotational	Electrical System
Force (F)	Torque (T)	Current (i)
Mass (M)	Moment of Inertia (J)	Capacitance (C)
Viscous friction Coefficient (B)	Viscous friction Coefficient (B)	Reciprocal of resistance (1/R)
Spring stiffness (K)	Torsional spring stiffness (K)	Reciprocal of Inductance (1/L)
Displacement (x)	Angular displacement (θ)	flux linkages (ϕ or λ)
velocity 'v'	Angular velocity (ω)	voltage (v)

① Draw the mechanical network, node equations and electrical analogous circuits of the system shown in figure.



$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2+Bs+K}$$

(Sol)

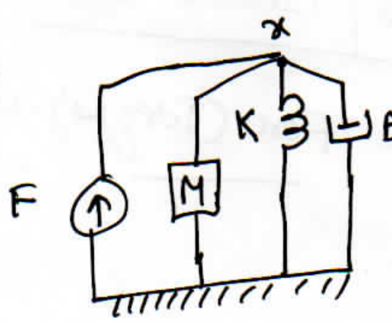


Figure: Mechanical Network

At the node 'x'

$$F = F_M + F_K + F_B$$

$$= M\frac{d^2x}{dt^2} + Kx + B\frac{dx}{dt}$$

$$v = L\frac{d^2q}{dt^2} + \frac{1}{C}q + R\frac{dq}{dt}$$

$$= L\frac{dI}{dt} + \frac{1}{C}\int i\,dt + Ri$$

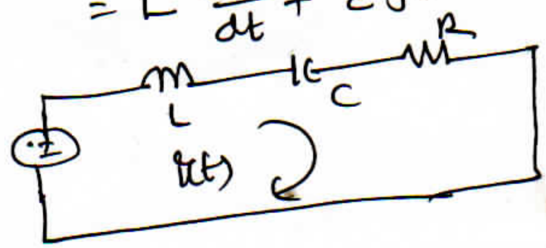
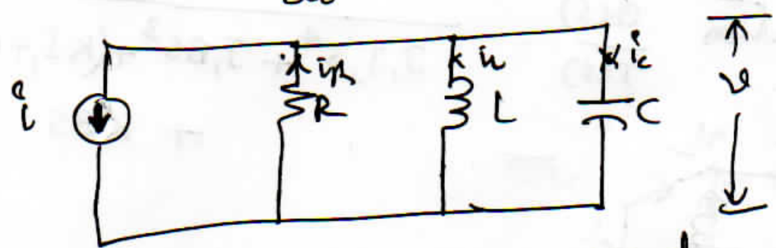


Figure: Force-voltage analogous circuit

In force-current analogy

$$i = C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{C}\phi$$

$$= C\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{C}\int v\,dt$$

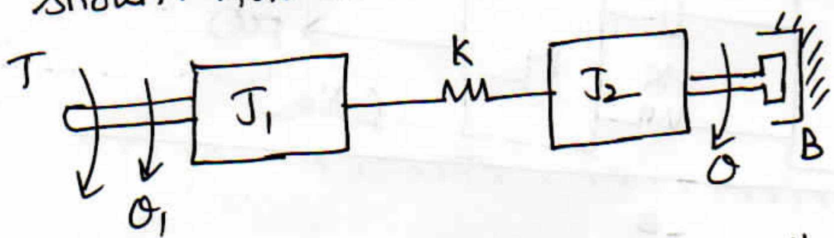


Force - Current analogous circuit

Note (1) The force-current analogous circuit has same structure as that of mechanical network

(2) In force-voltage analogous circuit, the parallel elements may appear in series and vice-versa.

(2) obtain the transfer function of the mechanical system shown. Also draw the electrical analogous circuit.



(Sol) The mechanical network is as shown in figure

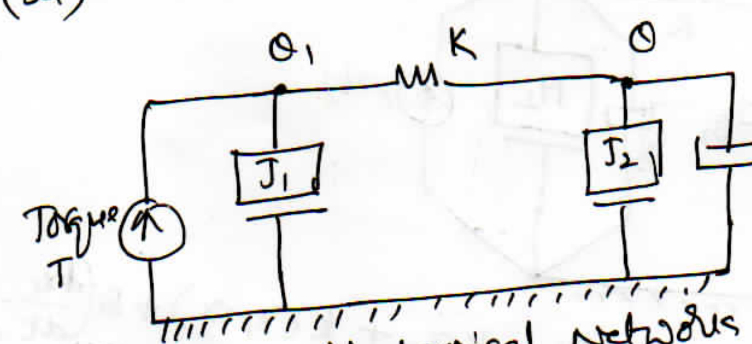


Figure : Mechanical Networks

At node θ_1, $J_1\frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T \longrightarrow (1)$

At node θ, $J_2\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0 \longrightarrow (2)$

Applying Laplace transform

$$(J_1 s^2 + K)\,\theta_1(s) + K\theta(s) = T(s) \longrightarrow ③$$

and $$(J_2 s^2 + Bs + K)\,\theta(s) = K\theta_1(s) \longrightarrow ④$$

∴ The Transfer function $$\frac{\theta(s)}{T(s)} = \frac{K}{J_1 J_2 s^4 + J_1 B s^3 + (KJ_1 + KJ_2)s^2 + KBs}$$

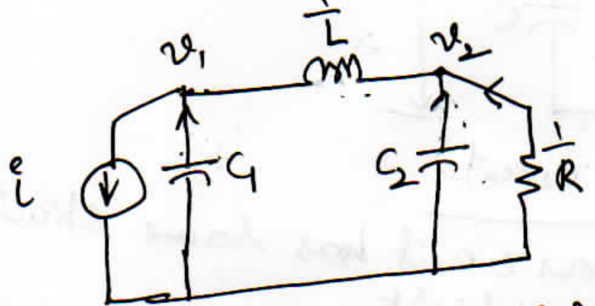
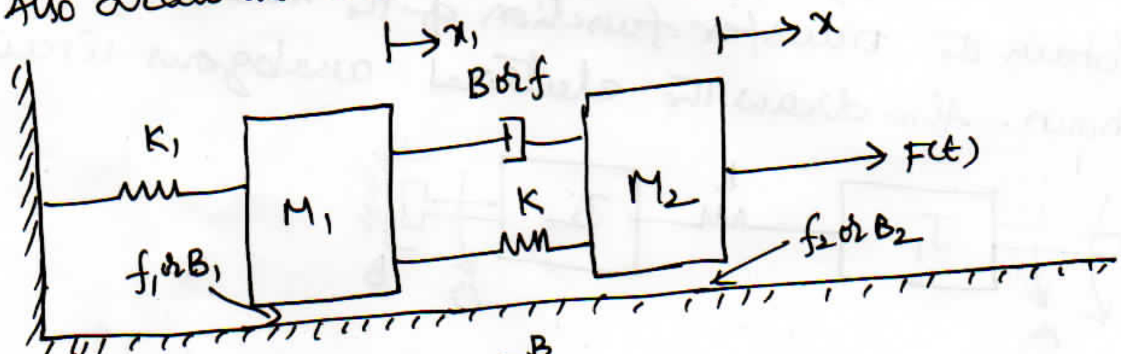
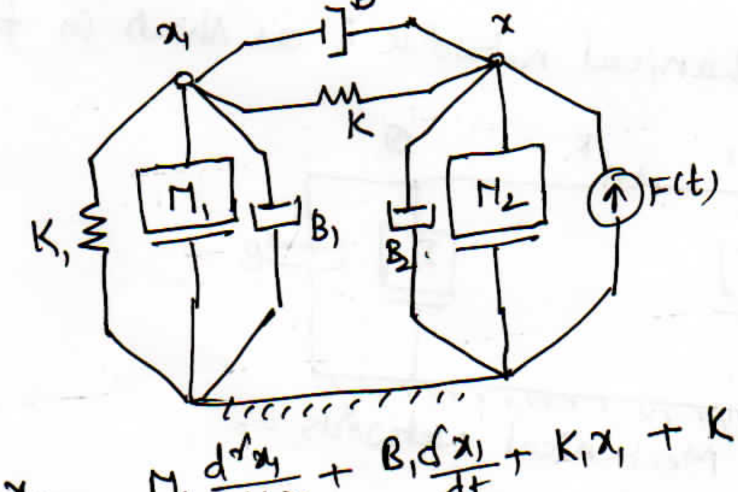


Figure: Force (Torque) – current analogous circuit

(3) Draw the mechanical network and write the node equations. Also draw the electrical analog circuit.



(Sol)



At node x_1, $$M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + K_1x_1 + K(x_1 - x) + B\left(\frac{dx_1}{dt} - \frac{dx}{dt}\right) = 0$$

At node x, $$\left.\begin{aligned} M_2\frac{d^2x}{dt^2} + K(x - x_1) + B\left(\frac{dx}{dt} - \frac{dx_1}{dt}\right) \\ + B_2\frac{dx}{dt} \end{aligned}\right\} = F(t)$$

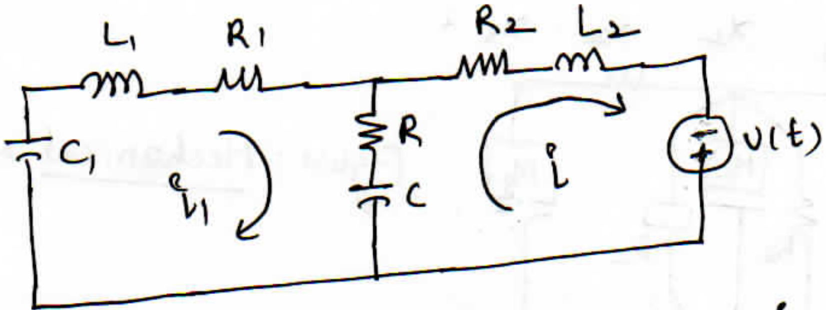


Figure: Force-voltage analogous circuit.

for 1st mesh

$$\frac{1}{C}\int(i_1-i)\,dt + L_1\frac{di}{dt^2} + R_1 i_1 + R(i_1-i) + \frac{1}{C_1}\int i_1\,dt = 0$$

for 2nd Mesh

$$\frac{1}{C}\int(i-i_1)\,dt + R(i-i_1) + R_2 i + L_2\frac{di}{dt} - v(t) = 0$$

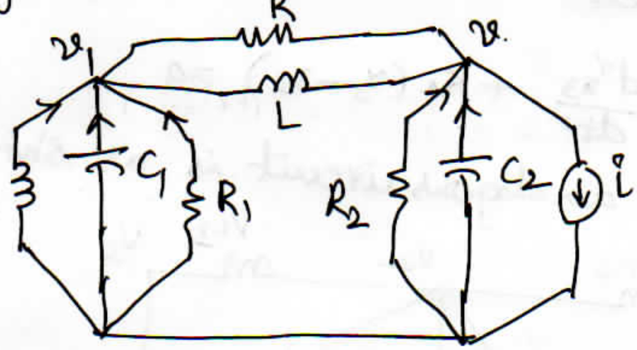


Figure: Force-Current analogous circuit

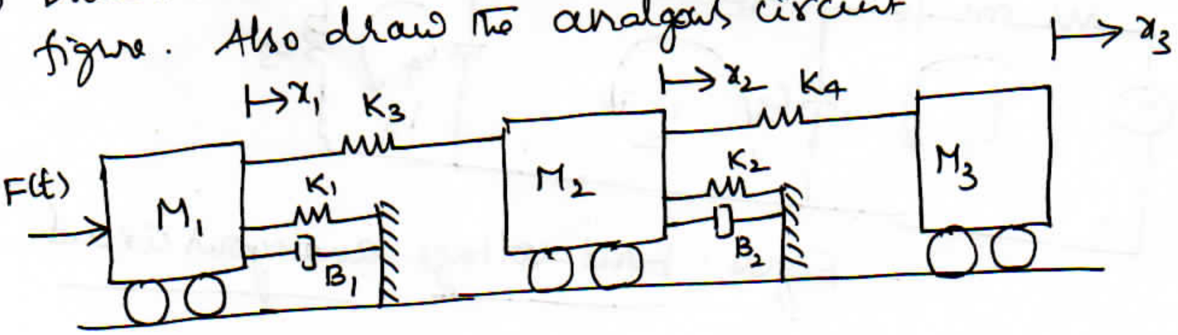
at node v_1,

$$\frac{1}{L_1}\int v_1\,dt + C_1\frac{dv_1}{dt} + \frac{v_1}{R_1} + \frac{v_1-v}{R} + \frac{1}{L}\int(v_1-v)\,dt = 0$$

at node v;

$$C_2\frac{dv}{dt} + \frac{v}{R_2} + \frac{v-v_1}{R} + \frac{1}{L}\int(v-v_1)\,dt = i$$

(3) Draw the mechanical network of the system shown in figure. Also draw the analogous circuit



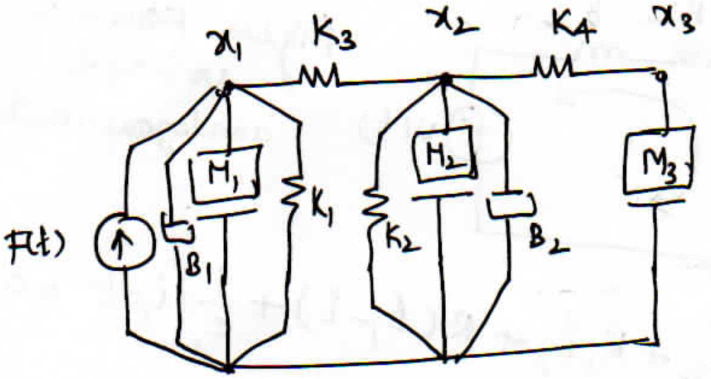


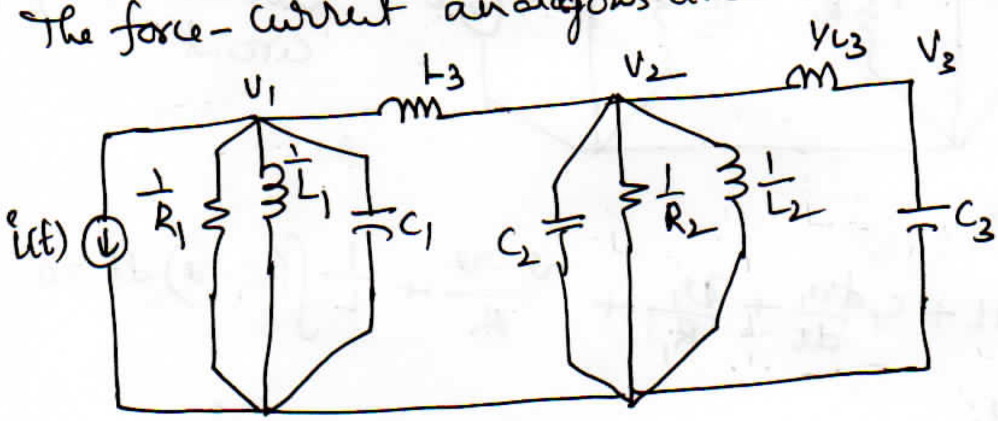
Figure : Mechanical Network

at node x_1, $M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+K_1x_1+K_3(x_1-x_2) = F(t)$

at node x_2, $M_2\frac{d^2x_2}{dt^2}+B_2\frac{dx_2}{dt}+K_2x_2+K_3(x_2-x_1)+K_4(x_2-x_3)=0$

at node x_3, $M_3\frac{d^2x_3}{dt^2}+K_4(x_3-x_2)=0$

The force-current analogous circuit is as shown in figure



Force-current Analogous circuit

The force-voltage analogous circuit is as follows

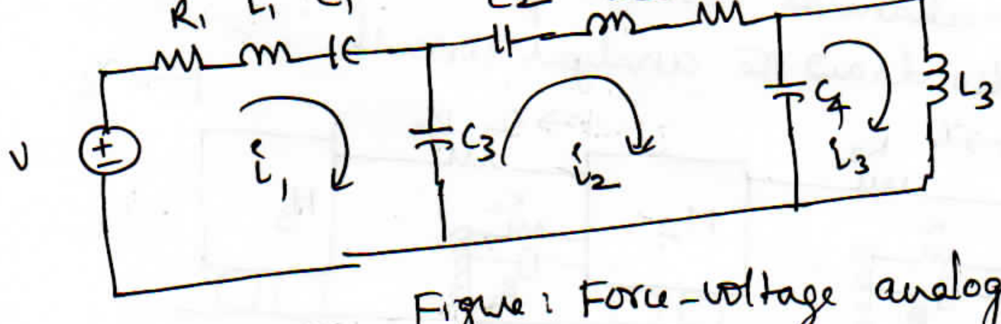
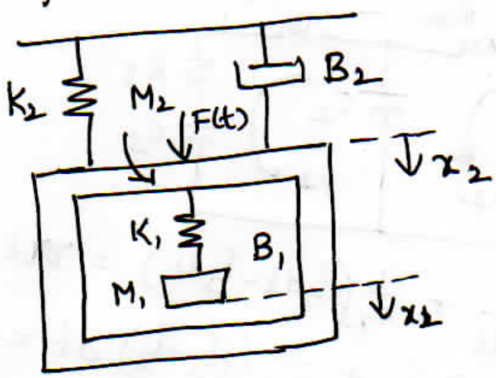
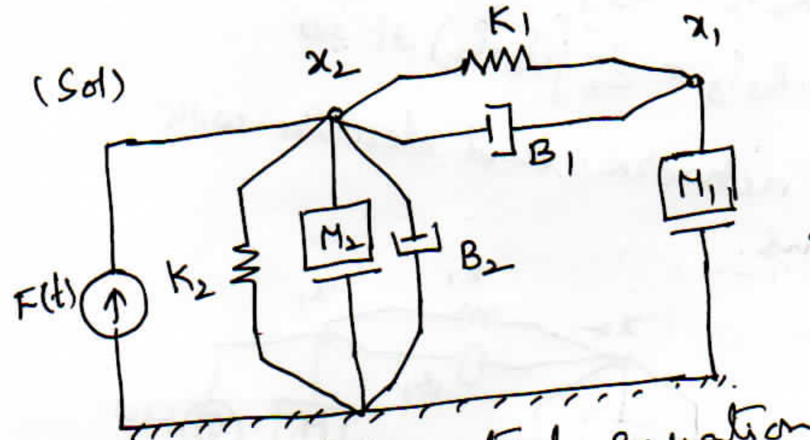


Figure : Force-voltage analogous circuit

(1) Draw the mechanical network and write the differential equations for the system shown. (II)



(Sol)



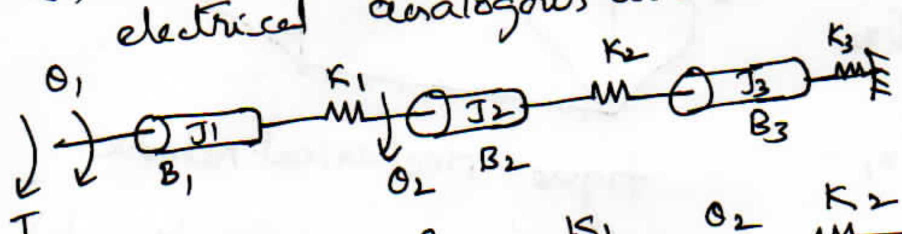
at 'x_2'

$$F(t) = M_2 \frac{d^2 x_2}{dt^2} + K_2 x_2 + B_2 x_2 + K_1(x_2 - x_1) + B_1\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)$$

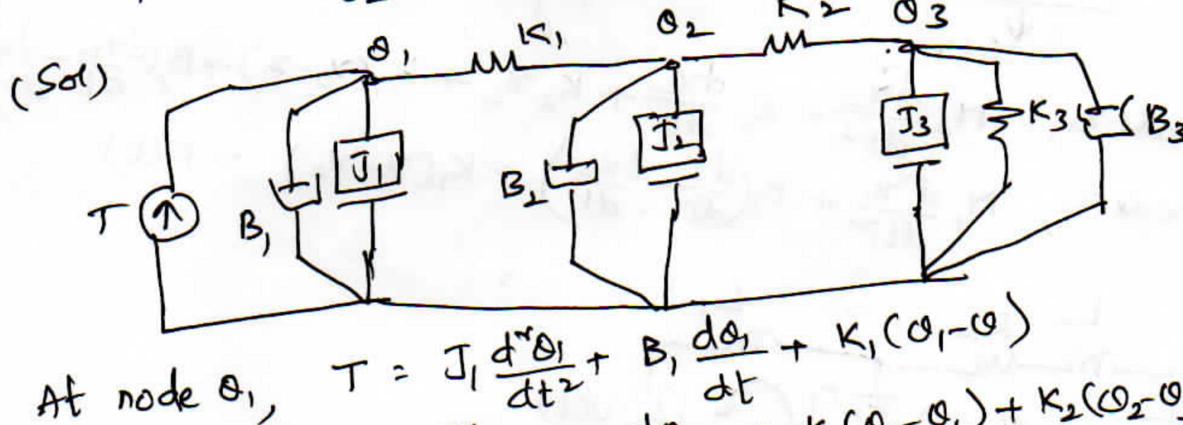
at 'x_1',

$$M_1 \frac{d^2 x_1}{dt^2} + K_1(x_1 - x_2) + B\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) = 0$$

(2) obtain differential equations and also draw the electrical analogous circuit



(Sol)

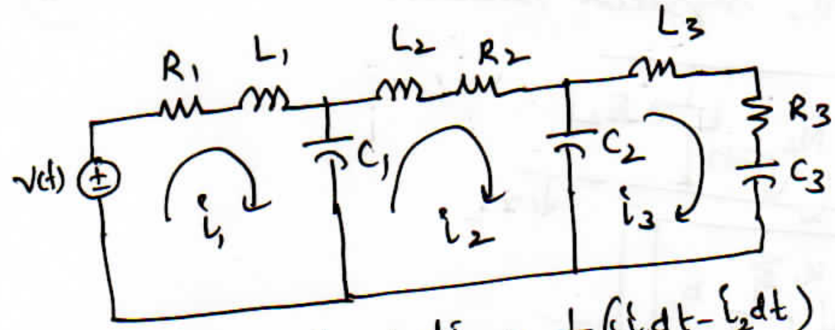


At node θ_1, $T = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K_1(\theta_1 - \theta)$

At node θ_2, $J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_1(\theta_2 - \theta_1) + K_2(\theta_2 - \theta_3) = 0$

At node θ_3, $J_3 \frac{d^2\theta_3}{dt^2} + K_3\theta_3 + B_3 \frac{d\theta_3}{dt} + K_2(\theta_3 - \theta_2) = 0$

(Torque)
The force-voltage analogous circuit is as follows

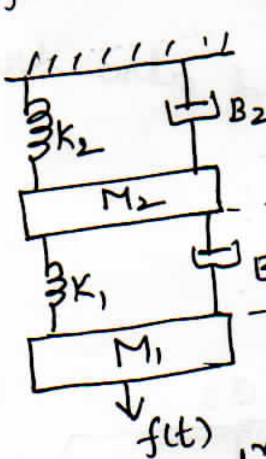


for Mesh ① $R_1 i_1 + L_1 \frac{di}{dt} + \frac{1}{C_1}\int (i_1 dt - i_2 dt) = v(t)$

for mesh ② $L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int (i_2 - i_3)\, dt = 0$

for mesh ③ $L_3 \frac{di_3}{dt} + R_3 i_3 + \frac{1}{C_3}\int (i_3 - i_2)\, dt = 0$

② Draw the mechanical network and describe with differential equations.



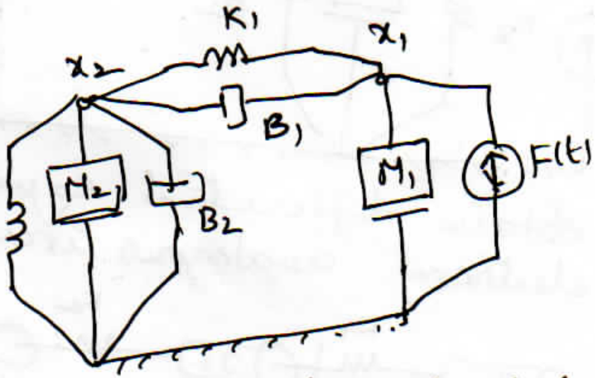


figure: Mechanical network

At Node x_2, $M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + K_1(x_2 - x_1) + B_1\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) = 0$

at node x_1, $M_1 \frac{d^2 x_1}{dt^2} + B_1\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) + K_1(x_1 - x_2) = F(t)$

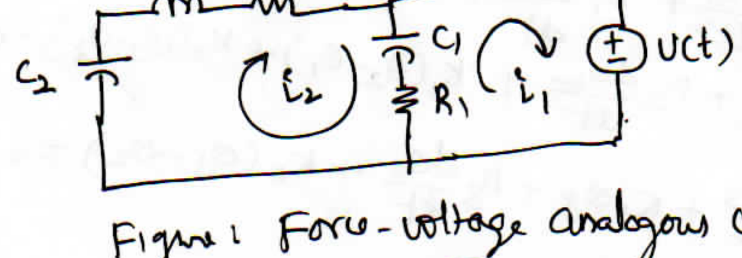
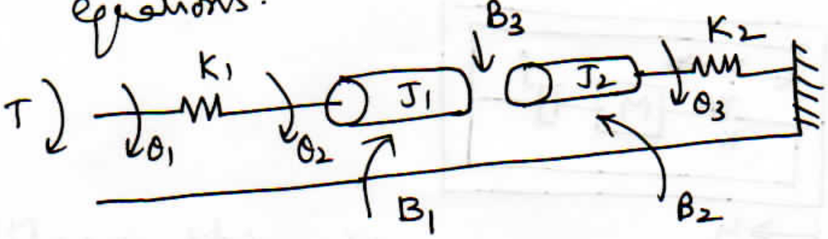
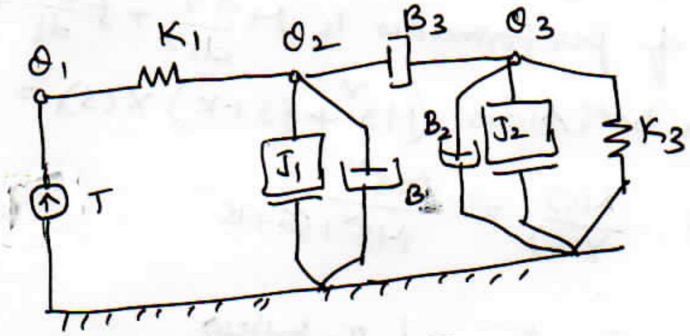


Figure: Force-voltage analogous circuit

① Draw the mechanical network and write the differential equations. (12)



(Sol)



At node θ_1, $K_1(\theta_1-\theta_2) = T$

node θ_2, $J_1 \frac{d^2\theta_2}{dt^2} + B_1 \frac{d\theta_2}{dt} + K_1(\theta_2-\theta_1) + B_3\left(\frac{d\theta_2}{dt} - \frac{d\theta_3}{dt}\right) = 0$

node θ_3, $J_2 \frac{d^2\theta_3}{dt^2} + B_2 \frac{d\theta_3}{dt} + K_3\theta_3 + B_3\left(\frac{d\theta_3}{dt} - \frac{d\theta_2}{dt}\right) = 0$

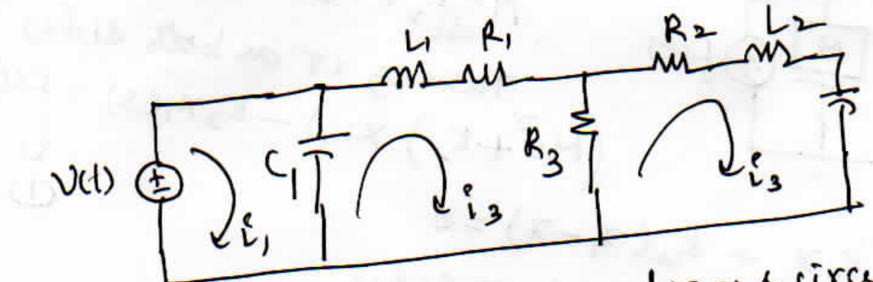
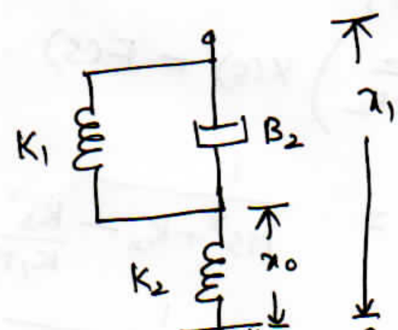


Figure: voltage-Torque analogous circuit

② Find the transfer function of the System.



(Sol) The equation of performance is

$$B_2\left(\frac{dx_1}{dt} - \frac{dx_0}{dt}\right) + K_1(x_1 - x_0) = K_2 x_0$$

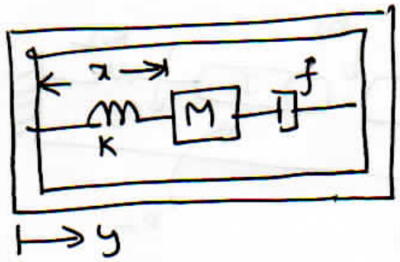
Taking LT on both sides

$$X_1(s)\left[B_2 S + K_1\right] = X_0(s)\left[B_2 S + K_1 + K_2\right]$$

∴ Transfer function

$$\frac{X_0(s)}{X_1(s)} = \frac{B_2 S + K_1}{B_2 S + K_2 + K_1}$$

② Findout the transfer function of the mechanical accelerator.

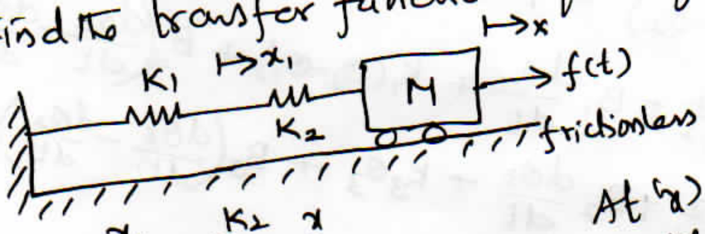


(Sol) The equation of performance is $M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx = M\frac{d^2y}{dt^2}$

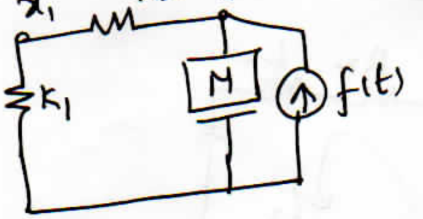
Taking LT on both sides $(Ms^2 + fs + K)\,X(s) = Ms^2 Y(s)$

$\therefore$ TF $\quad \frac{X(s)}{Y(s)} = \frac{Ms^2}{Ms^2 + fs + K}$

(3) Find the transfer function of the system



(Sol)



At 'x'

$$M\frac{d^2x}{dt^2} + K_2(x - x_1) = f(t)$$

Taking LT on both sides

$$(Ms^2 + K_2)\,X(s) - K_2X_1(s) = F(s) \quad \text{①}$$

At node 'x_1', $\quad K_1x_1 + K_2(x_1 - x) = 0$

Taking LT on both sides

$$(K_1 + K_2)\,X_1(s) = K_2X(s) \longrightarrow \text{②}$$

Substituting eq ② in eq ①,

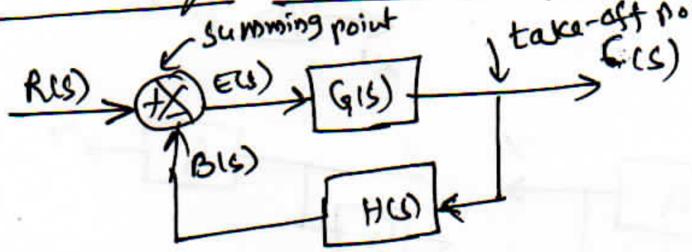
$$(Ms^2 + K_2)\,X(s) - K_2\left(\frac{K_2}{K_1 + K_2}\right)X(s) = F(s)$$

$\therefore$ Transfer function $\quad \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + K_2 - \frac{K_2^2}{K_1 + K_2}}$

$$= \frac{1}{Ms^2 + K_2\left[1 - \frac{K_2}{K_1 + K_2}\right]}$$

$$= \frac{1}{Ms^2 + \frac{K_1K_2}{K_1 + K_2}}$$

Block Diagram Reduction Techniques:



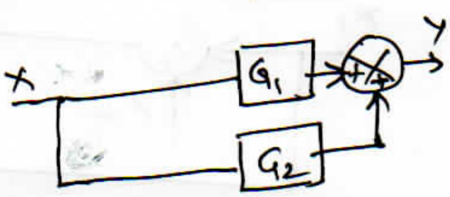
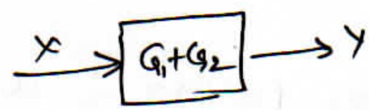
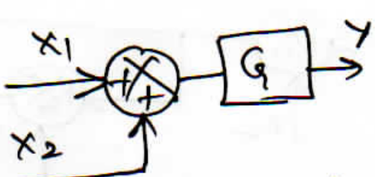
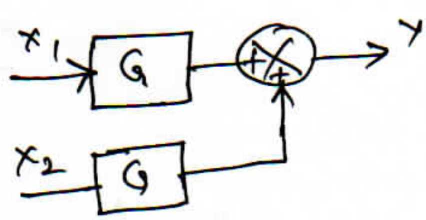
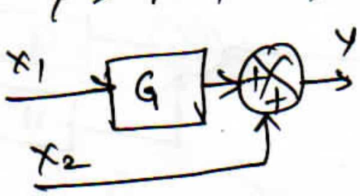
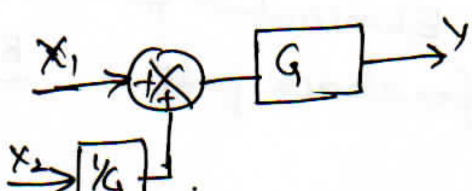
$R(s)$ = Reference input
$C(s)$ = output or Controlled Variable
$E(s)$ = Actuating signal or Error signal
$B(s)$ = Feedback Signal

$C(s) = E(s)G(s)$

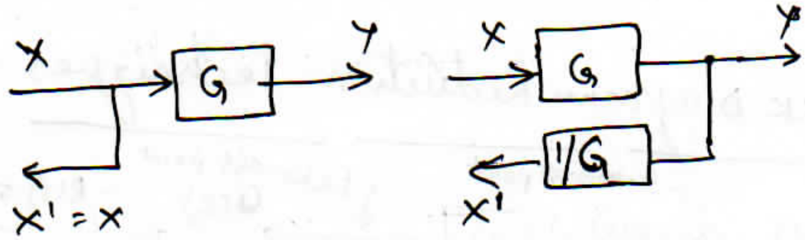
$\therefore \frac{C(s)}{E(s)} = G(s)$ is Forward path Transfer function

$B(s) = C(s)H(s) \Rightarrow \frac{B(s)}{C(s)} = H(s)$ = Feedback Transfer function

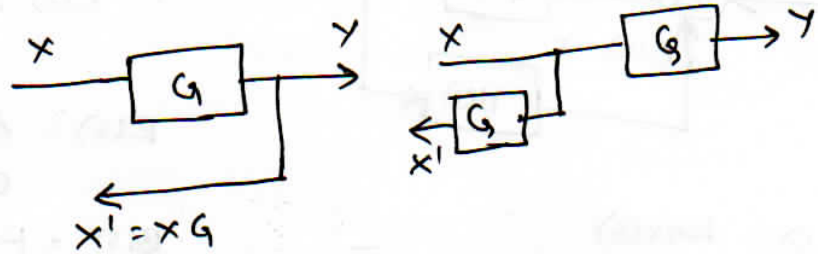
Block Diagram Reduction Albera:

Rule	Original Diagram	Equivalent diagram
(1) Combing blocks in Cascade	 $y = G_1G_2x$	
(2) Combing blocks in parallel	 $y = (G_1 + G_2)x$	
(3) Moving a summing point after a block	 $y = G(x_1 + x_2)$	
(4) Moving a summing point ahead of a block	 $y = x_1G + x_2$	

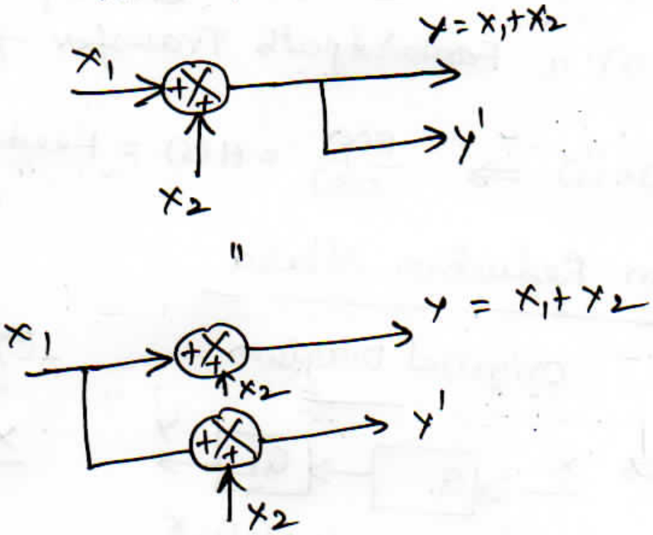
(5) Moving a take-off point after a block



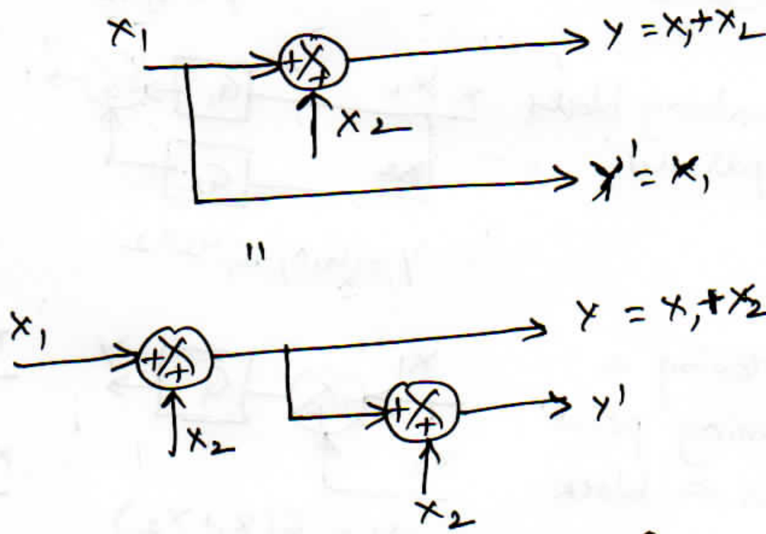
(6) Moving a take-off point ahead of a block



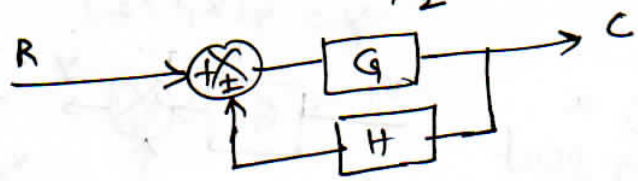
(7) Moving of a take-off point ahead of a summing point



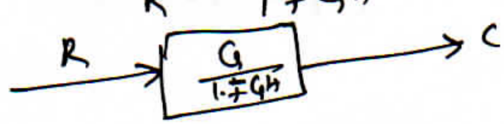
(8) Moving of a take-off point after a summing point



(9) Elimination of feedback path

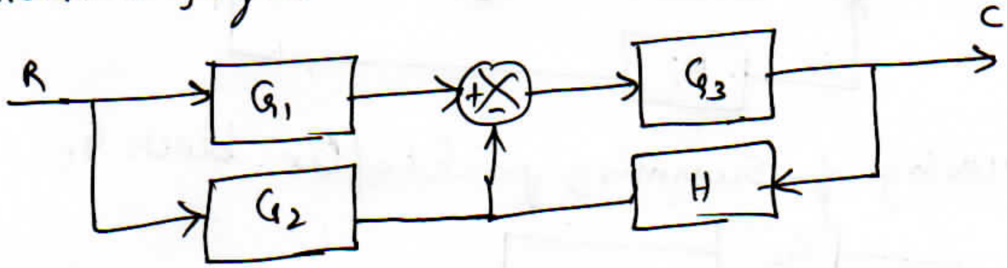


$$\frac{C}{R} = \frac{G}{1 \mp GH}$$



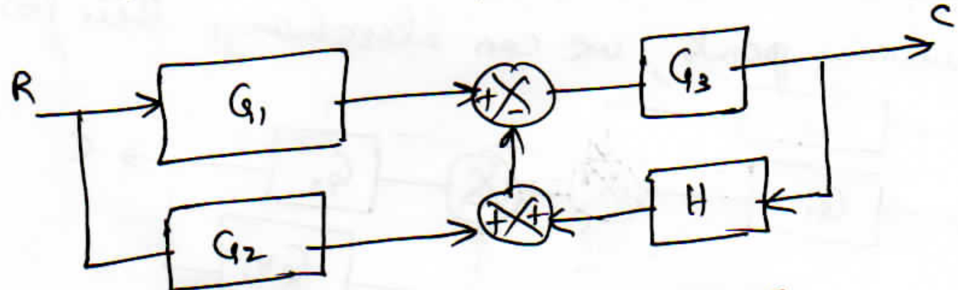
① Determine the transfer function of the block diagrams shown in figure (14)

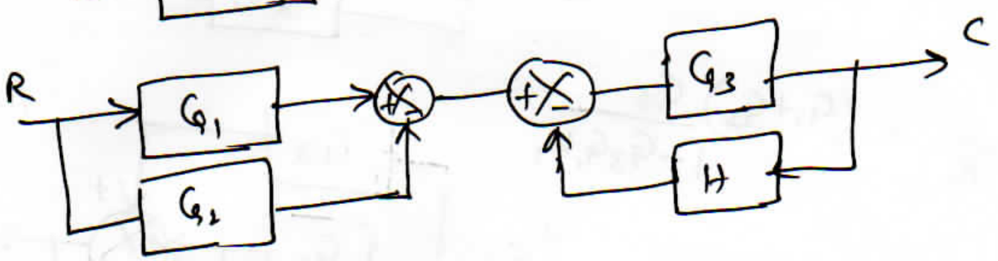
(i)



$\therefore$ Transfer function $\frac{C}{R} = \frac{(G_1 - G_2)\, G_3}{1 + G_3 H}$

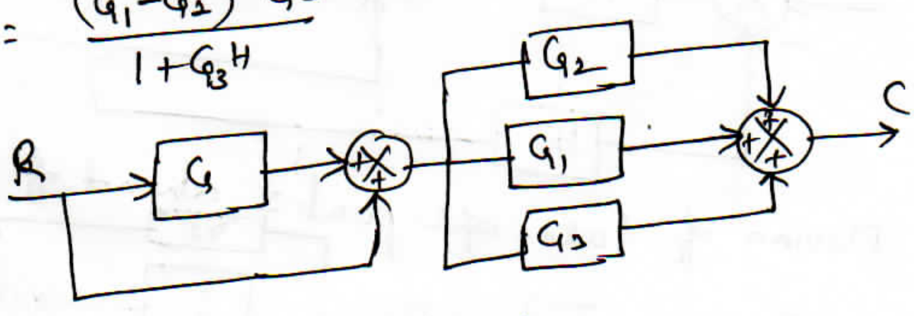
(ii)





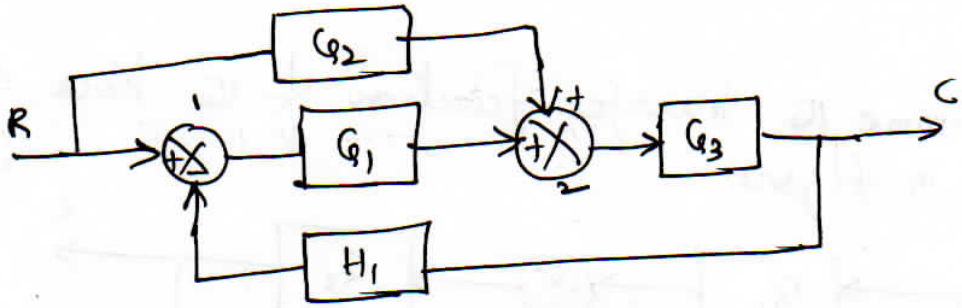
$$\frac{C}{R} = \frac{(G_1 - G_2)\ G_3}{1 + G_3 H}$$

(iii)

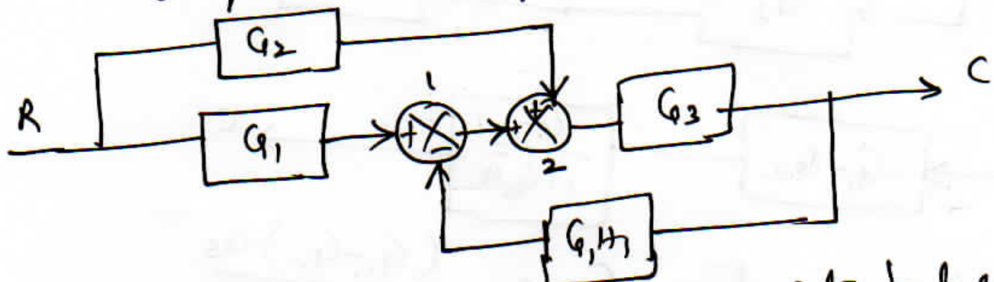


$$\frac{C}{R} = (1+G)\,(G_1 + G_2 + G_3)$$

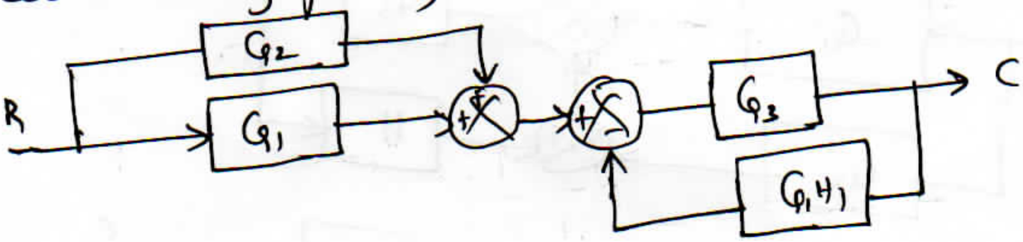
(iv)



(Sol) Moving of summing point 1 after block G_1

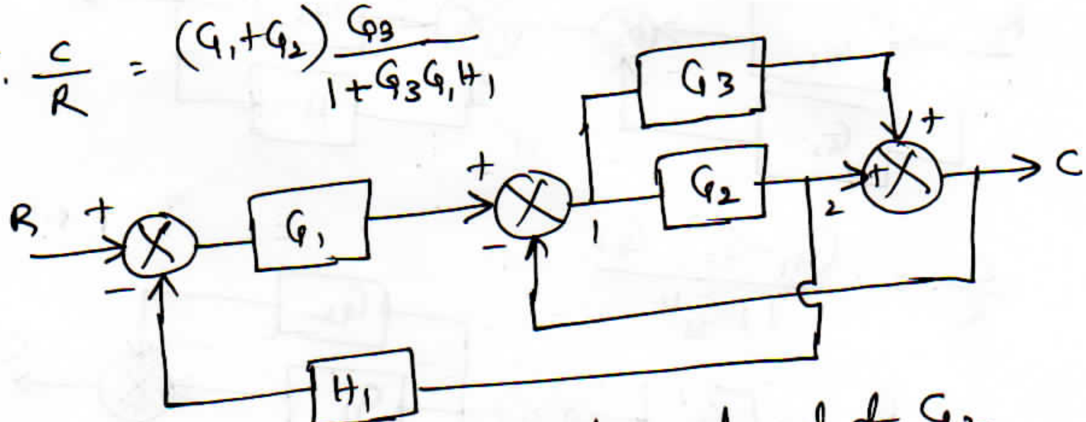


If there are no blocks & take-off points between two summing points, we can interchange their positions

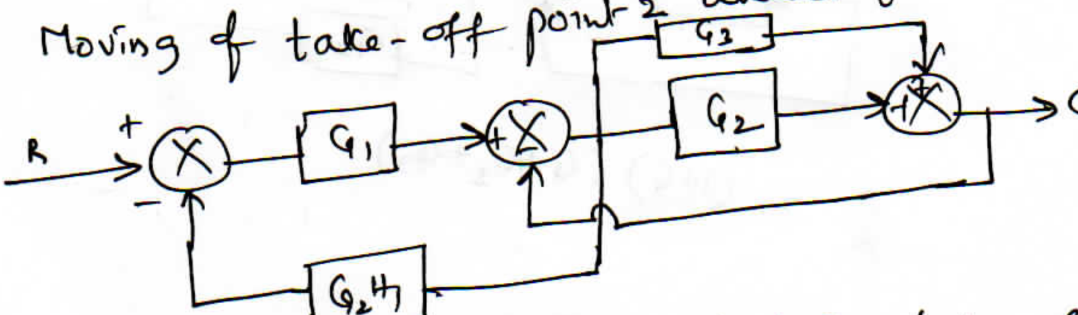


$$\therefore \frac{C}{R} = (G_1+G_2)\frac{G_3}{1+G_3G_1H_1}$$

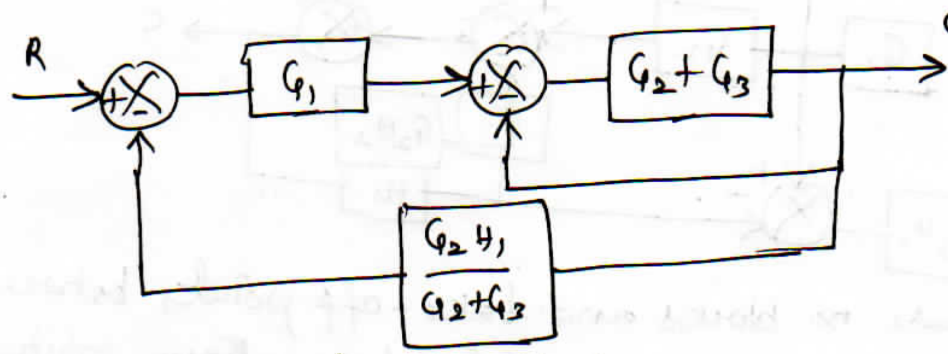
(v)



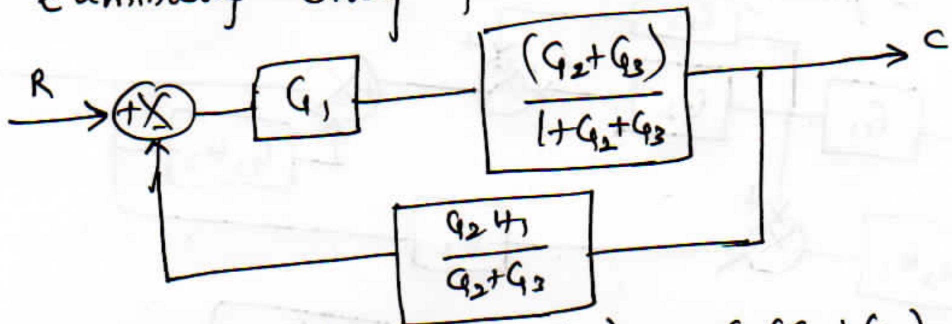
(Sol) Moving of take-off point 2 ahead of G_2



Combining blocks G_2 & G_3 and moving take-off point after the combination.

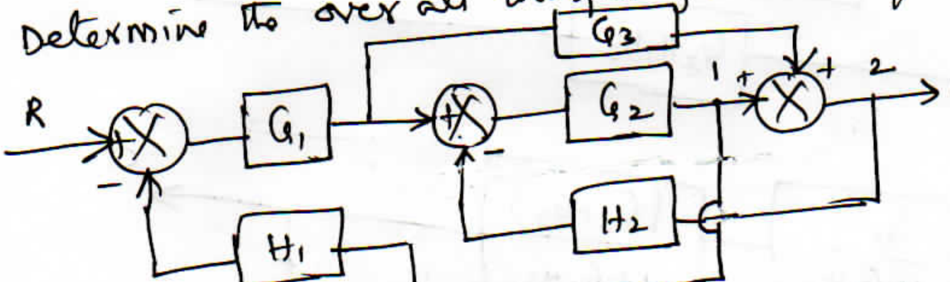


Eliminating unity feedback

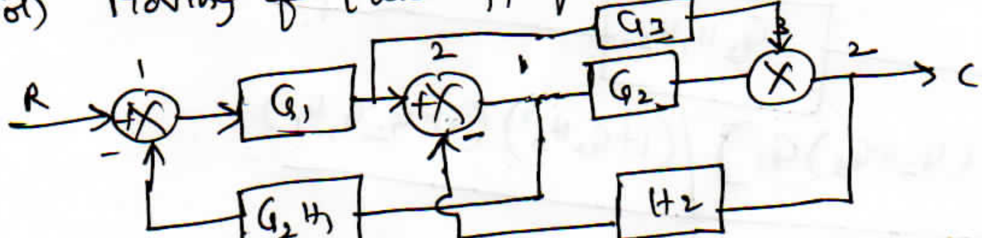


$$\frac{C}{R} = \frac{(G_2+G_3)G_1/(1+G_2+G_3)}{1+\frac{(G_2+G_3)}{(1+G_2+G_3)}G_1\cdot\frac{G_2H_1}{(G_2+G_3)}} = \frac{G_1(G_2+G_3)}{1+G_2+G_3+G_1G_2H_1}$$

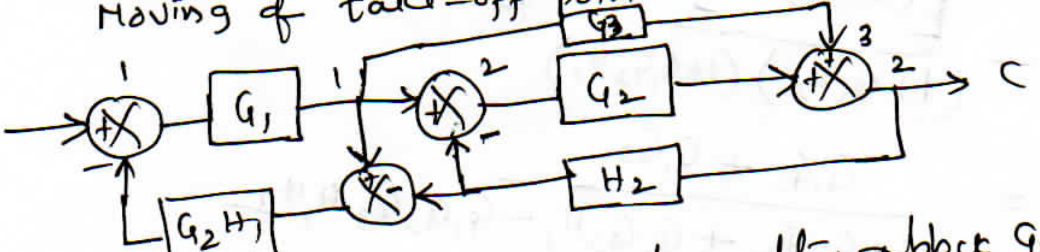
(vi) Determine the over all transfer function of the system



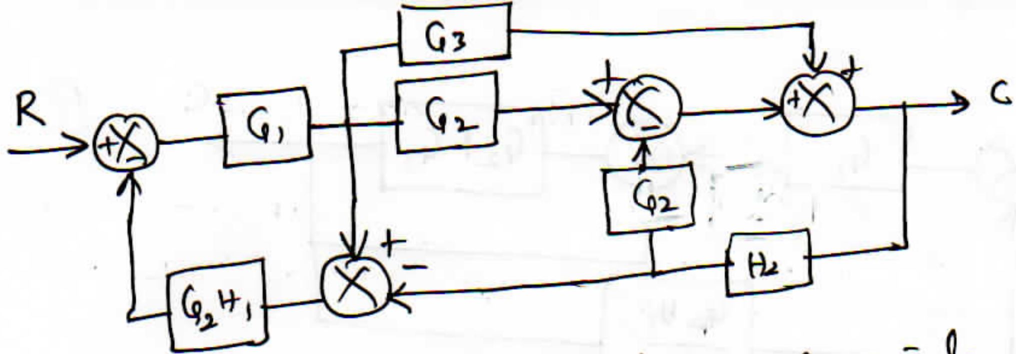
(Sol) Moving of take-off point 1 ahead of block G_2



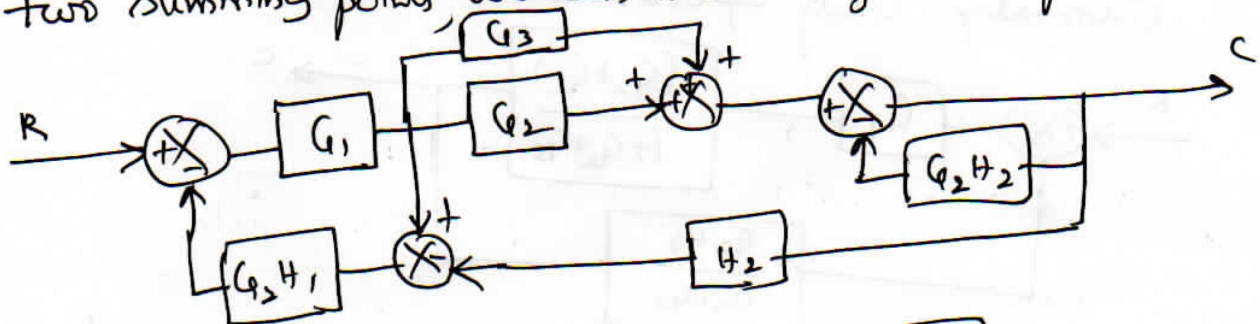
Moving of take-off point 1 ahead of summing point 2

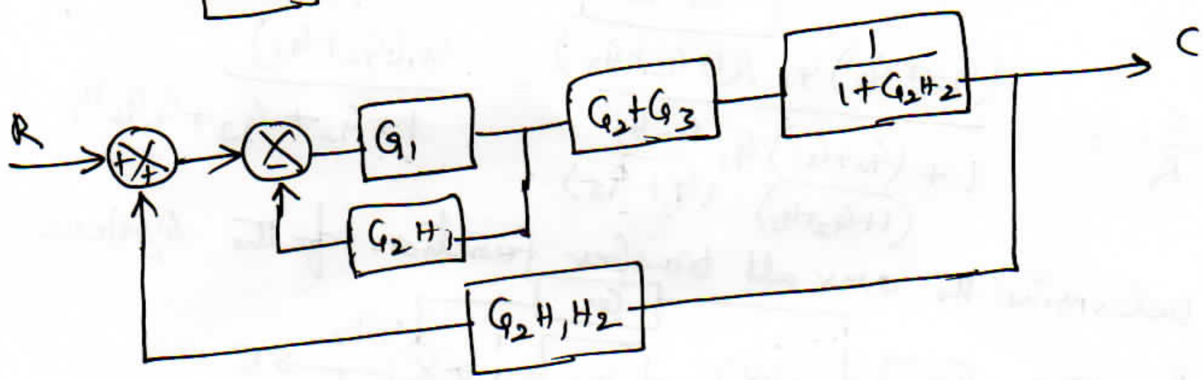


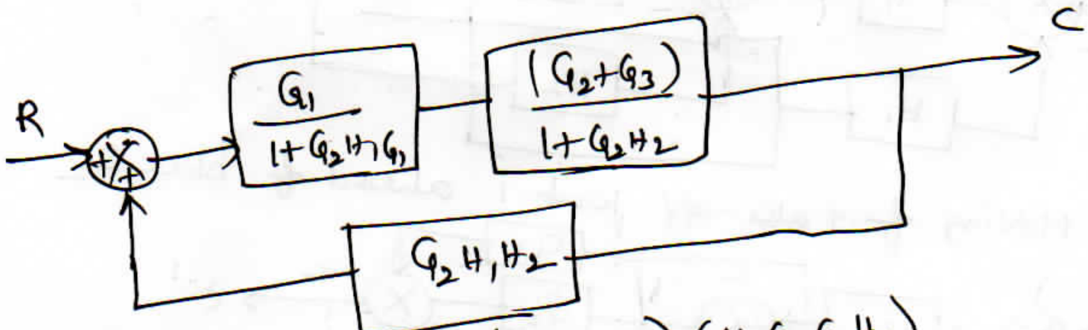
moving of summing point 2 after a block G_2



If there are no blocks and take-off points between two summing points, we can interchange their position.



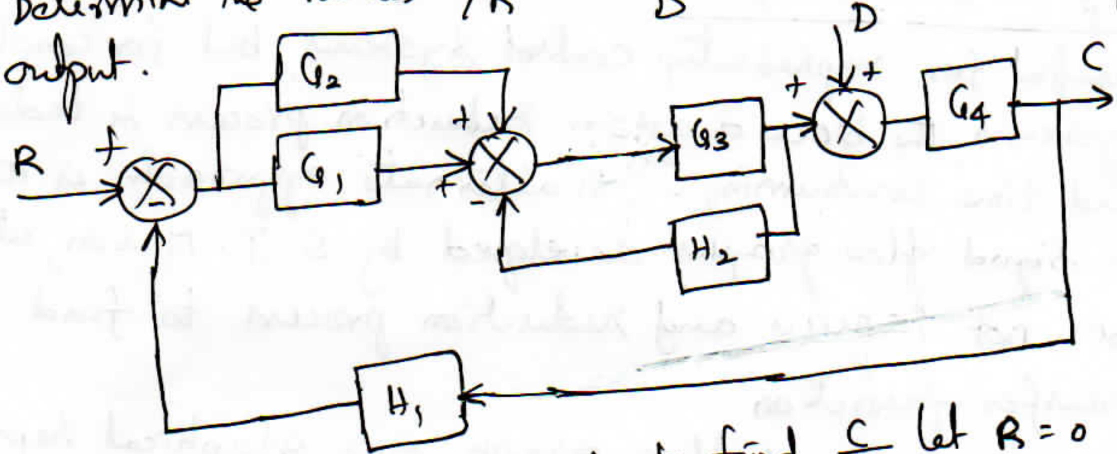




$$\frac{C}{R} = \frac{\left[(G_2+G_3)G_1 \right] \Big/ (1+G_2H_2)(1+G_2G_1H_1)}{1 - \dfrac{\left[(G_2+G_3)G_1 \right]}{(1+G_2H_2)(1+G_1G_2H_1)} - G_2H_1H_2}$$

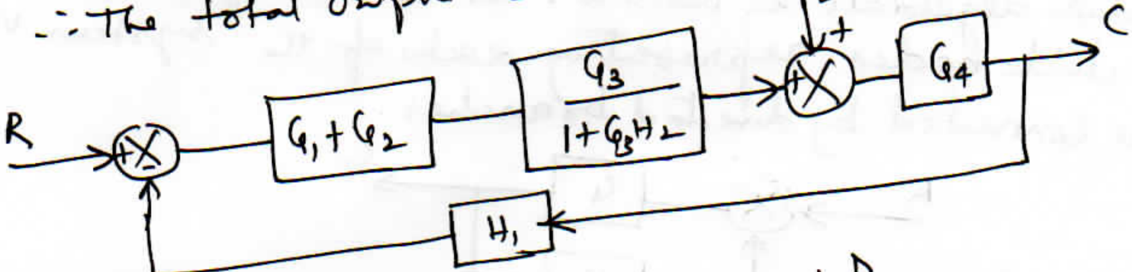
$$= \frac{G_1G_2 + G_1G_3}{1+G_2H_2 + G_1G_2H_1 - G_1G_2G_3H_1H_2}$$

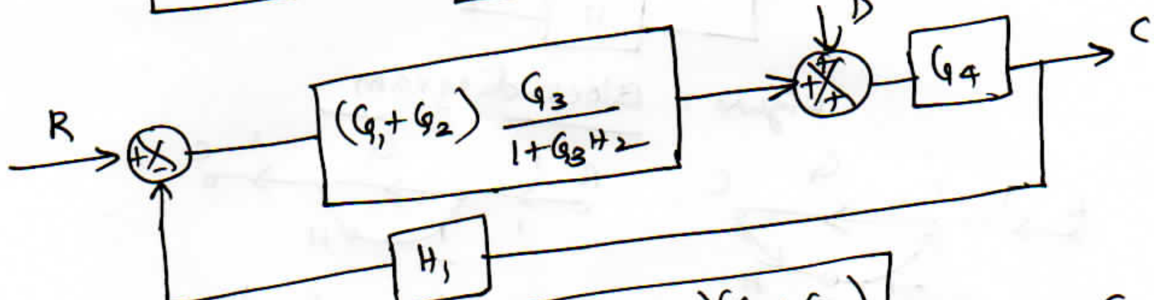
(4) Determine the ratios C/R and $\frac{C}{D}$ also find the total output. (16)



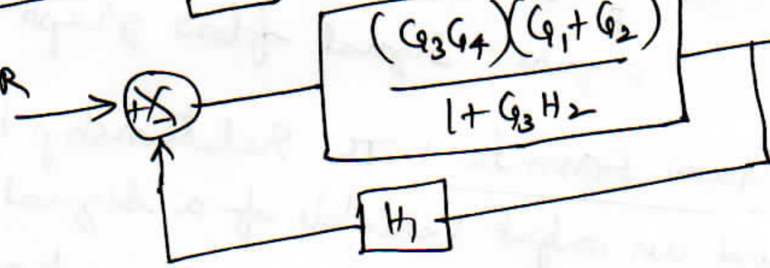
(Sol) To find $\frac{C}{R}$, let $D=0$ and to find $\frac{C}{D}$ let $R=0$

$\therefore$ The total output $C = \left(\frac{C}{R}\right) R + \left(\frac{C}{D}\right) D$



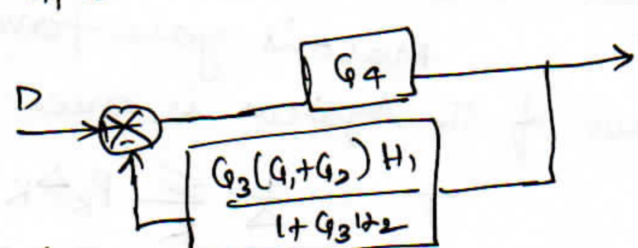


(i) $\frac{C}{R} = ?$



$$\frac{C}{R} = \frac{G_1G_3G_4 + G_2G_3G_4}{1+G_3H_2 + G_1G_3G_4H_1 + G_2G_3G_4H_1}$$

(ii) To find $\frac{C}{D}$ let $R=0$



$$\frac{C}{D} = \frac{G_4(1+G_3H_2)}{1+G_3H_2+G_1G_3G_4H_1+G_2G_3G_4H_1};$$

Total o/p $C = \frac{(G_1G_3G_4+G_2G_3G_4)R}{(\;''\;)} + \frac{G_4(1+G_3H_2)D}{(\;''\;)}$

Signal Flow Graphs: Block diagrams are very useful for representing control systems, but for complicated systems, the block diagram reduction process is tedious and time consuming. An alternate approach is that of signal flow graphs developed by S. J. Mason, which does not require any reduction process to find the transfer function.

A signal flow graph is a graphical representation of the relationships between the variables of a set of linear algebraic equations. It consists of a network in which nodes representing each of the system variables are connected by directed branches.

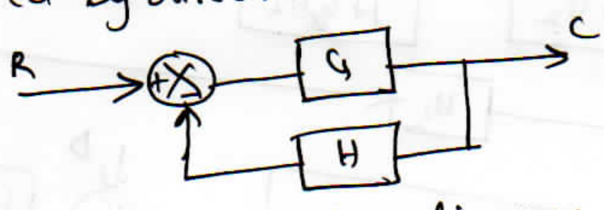


Figure : Block diagram

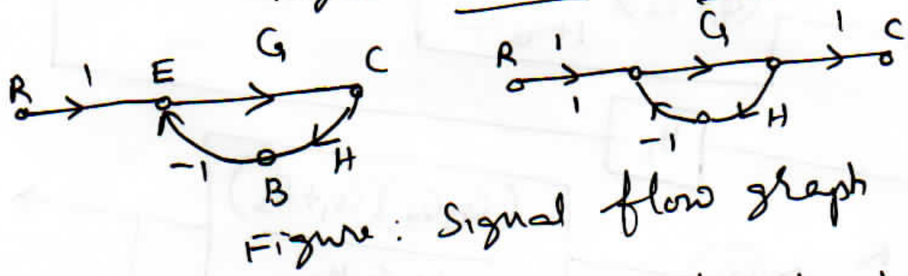


Figure : Signal flow graph

Mason's Gain Formula: The relationship between an input variable and an output variable of a signal flow graph is given by the net gain between the input and output nodes and is known as the over all gain of the system.

Mason's gain formula to determine the overall gain of the system is given by

$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

where P_K = path gain of K^{th} forward path

Δ = Determinant of the graph

= 1 − (Sum of loop gains of all individual loops)

+ (Sum of gain products of all possible Combinations of two non-touching loops)

− (Sum of gain products of all possible Combinations of three non-touching loops)

+ - - - -

$$\therefore \Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \cdots$$

where P_{mr} = gain product of m-th possible Combinations of 'r' non-touching loops

Δ_K = The value of Δ for the part of the graph not-touching the K^{th} forward path.

T = over all gain of the system.

① Draw the signal flow graph and find the over all gain of the system equations given by

$$x_2 = a_{12}x_1 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5$$

$$x_3 = a_{23}x_2$$

$$x_4 = a_{34}x_3 + a_{44}x_4$$

$$x_5 = a_{35}x_3 + a_{45}x_4$$

where x_1 is the input variable and x_5 is the output variable

(1) Node: It represents a system variable which is equal to the sum of all incoming signals at the node

(2) Branch: A signal travels along a branch from one node to another in the direction indicated by the branch arrow and in the process gets multiplied by the gain or transmittance of the branch.

(3) Notation: a_{ij} is the transmittance of the branch directed from node x_i to node x_j.

(4) Input node or Source: It is a node with only outgoing branches.

(5) output node or Sink: It is a node only with incoming branches. This does not meet always. In that case,

where 'C' is output node

an additional branch with unit gain may be introduced in order to meet the specified condition

(6) path: It is the traversal of connected branches in the direction of the branch arrows such that no node is traversed more than once.

(7) Forward path: It is a path from the input node to the output node.

(8) Loop: Loop is a path which originates and terminates at the same node.

(9) Non-touching loops: Loops are said to be non-touching if they do not possess any common node.

(10) Forward path gain: It is the product of the branch gains encountered in traversing a forward path.

(11) Loop gain: It is the product of branch gains encountered in traversing a loop.

Construction of Signal flow graph:

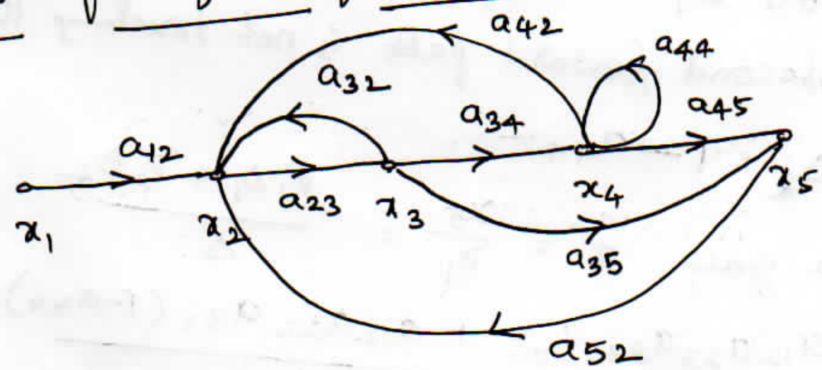


Figure: Signal flow graph

(1) There are two forward paths with path gains

$P_1 = a_{12}a_{23}a_{34}a_{45}$; $P_2 = a_{12}a_{23}a_{35}$

(2) There are five individual loops with loop gains

$P_{11} = a_{23}a_{32}$; $P_{21} = a_{23}a_{34}a_{42}$

$P_{31} = a_{44}$; $P_{41} = a_{23}a_{34}a_{45}a_{52}$

$P_{51} = a_{23}a_{35}a_{52}$

(3) There are two possible combinations of two non-touching loops with loop gain products

$P_{12} = a_{23}a_{32}a_{44}$

$P_{22} = a_{23}a_{35}a_{52}a_{44}$

4) There are no combinations of three-non-touching loops, four non-touching loops etc.

Therefore $P_{m3} = P_{m4} = \cdots = 0$

Hence $\Delta = 1 - (a_{23}a_{32} + a_{23}a_{34}a_{42} + a_{44} + a_{23}a_{34}a_{45}a_{52} + a_{23}a_{35}a_{52}) + (a_{23}a_{32}a_{44} + a_{23}a_{35}a_{52}a_{44})$

(5) The first forward path is in touch with all the loops

Therefore $\Delta_1 = 1$

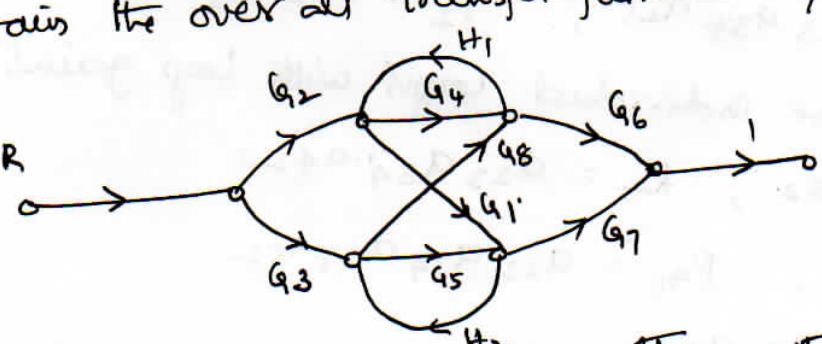
The second forward path is not touching the loop a_{44}

$\therefore \Delta_2 = 1 - a_{44}$

$\therefore$ The gain $T = \dfrac{x_5}{x_1} = \dfrac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$

$$= \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{23}a_{35}(1-a_{44})}{1 - a_{23}a_{32} - a_{23}a_{34}a_{42} - a_{44} - a_{23}a_{34}a_{45}a_{52} + a_{23}a_{32}a_{44} + a_{23}a_{35}a_{52}a_{44}}$$

(2) obtain the over all transfer function C/R.



(Sol) There are six forward paths with gains

$P_1 = G_2G_4G_6$; $P_2 = G_2G_1G_7$; $P_3 = G_2G_1H_2G_8G_6$

$P_4 = G_3G_5G_7$ $P_5 = G_3G_8G_6$; $P_6 = G_3G_8H_1G_1G_7$

(2) There are 3 individual loops.

$P_{11} = G_4 H_1$; $P_{21} = G_5 H_2$

$P_{31} = G_1 H_2 G_8 H_1$

(3) There is only one combination of non-touching loops

$P_{12} = P_{11} P_{21} = G_4 G_5 H_1 H_2$

(4) There are no combinations of three non-touching loops

$\therefore P_{m3} = P_{m4} = 0$

$\therefore \Delta = 1 - P_{m1} + P_{m2} - P_{m3} + \cdots$

$= 1 - (G_4 H_1 + G_5 H_2 + G_1 H_2 G_8 H_1) + G_4 G_5 H_1 H_2$

(5) Forward path P_1 is not touching $G_5 H_2$

$\therefore \Delta_1 = 1 - G_5 H_2$

Forward path P_4 is not touching the loop $G_4 H_1$

$\therefore \Delta_4 = 1 - G_4 H_1$

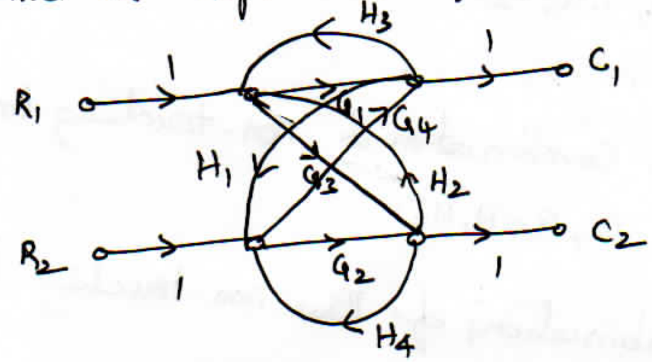
Remaining forward paths touching all the loops

$\therefore \Delta_2 = \Delta_3 = \Delta_5 = \Delta_6 = 1$

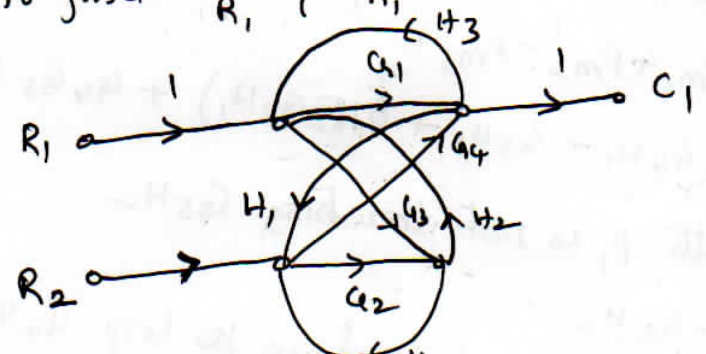
$\therefore$ The transfer function $\frac{C}{R} = \dfrac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6}{1 - \sum_m P_{m1} + \sum_m P_{m2}}$

$$\frac{C}{R} = \frac{G_2 G_4 G_6 (1 - G_5 H_2) + G_2 G_1 G_7 + G_2 G_1 H_2 G_8 G_6 + G_3 G_5 G_7 (1 - G_4 H_1) + G_3 G_8 G_6 + G_3 G_8 H_1 G_1 G_7}{1 - (G_4 H_1 + G_5 H_2 + G_1 H_2 G_8 H_1) + G_4 G_5 H_1 H_2}$$

(2) Find the expressions for the outputs C_1 and C_2.



(Sol) To find $\frac{C_1}{R_1}$ & $\frac{C_2}{R_1}$ assume $R_2 = 0$



$\frac{C_1}{R_1} = ?$ (1) $P_1 = G_1$; $P_2 = G_3 H_4 G_4$

(2) $\sum P_{m1} = ?$ $P_{11} = G_2 H_4$; $P_{21} = G_1 H_3$

$P_{31} = G_3 H_2$ $P_{41} = G_4 H_1$

$P_{51} = G_1 H_1 G_2 H_2$

$P_{61} = G_3 H_4 G_4 H_3$

(3) There is only one Combination of non-touching Loops. $\sum P_{m2} = ?$

$P_{12} = P_{11} P_{21} = G_1 G_2 H_3 H_4$

$\sum_m P_m = \sum_m P_{m4} = 0$

$\therefore \Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2}$

$= 1 - (G_2 H_4 + G_1 H_3 + G_3 H_2 + G_4 H_1 + G_1 H_1 G_2 H_2 + G_3 H_4 G_4 H_3) + G_1 G_2 H_3 H_4$

(4) $\Delta_1 = 1 - G_2H_4$; $\Delta_2 = 1$

$$\therefore \frac{C_1}{R_1} = \frac{G_1(1-G_2H_4) + G_3G_4H_4}{\Delta} \longrightarrow (1)$$

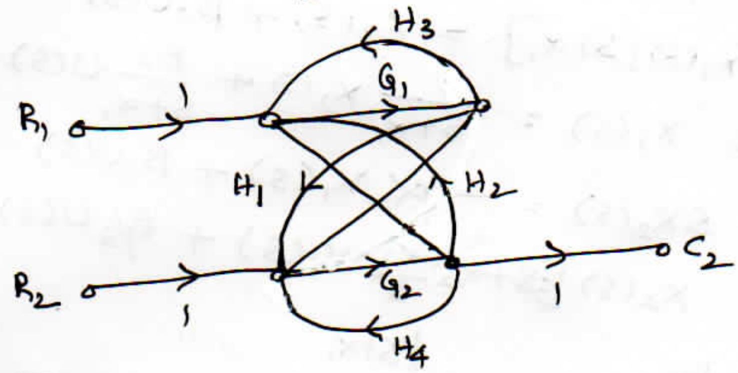
$\frac{C_1}{R_2} = ?$ $P_1 = G_4$; $P_2 = H_2G_1G_2$

$\Delta_1 = 1 - G_3H_2$; $\Delta_2 = 1$

$$\therefore \frac{C_1}{R_2} = \frac{G_4 + G_1H_2}{\Delta}$$

$$\therefore \text{output } C_1 = \frac{[G_1(1-G_2H_4) + G_3G_4H_4]R_1 + [G_4(1-G_3H_2) + G_1G_2H_2]R_2}{\Delta}$$

To find C_2:



$\frac{C_2}{R_1} = ?$ $P_1 = G_3$; $P_2 = G_1H_1G_2$

$\Delta_1 = 1 - G_4H_1$; $\Delta_2 = 1$

$$\therefore \frac{C_2}{R_1} = \frac{G_3(1-G_4H_1) + G_1G_2H_1}{\Delta}$$

$\frac{C_2}{R_2} = ?$ $P_1 = G_2$; $P_2 = G_4H_3G_3$

$\Delta_1 = 1 - G_1H_3$ $\Delta_2 = 1$

$$\frac{C_2}{R_2} = \frac{G_2(1-G_1H_3) + G_4G_3H_3}{\Delta}$$

$$\therefore \text{output } C_2 = \frac{[G_3(1-G_4H_1) + G_1G_2H_1]R_1 + [G_2(1-G_1H_3) + G_4G_3H_3]R_2}{\Delta}$$

C_1 is independent of R_2 if $G_4(1-G_3H_2) + G_1G_2H_2 = 0$

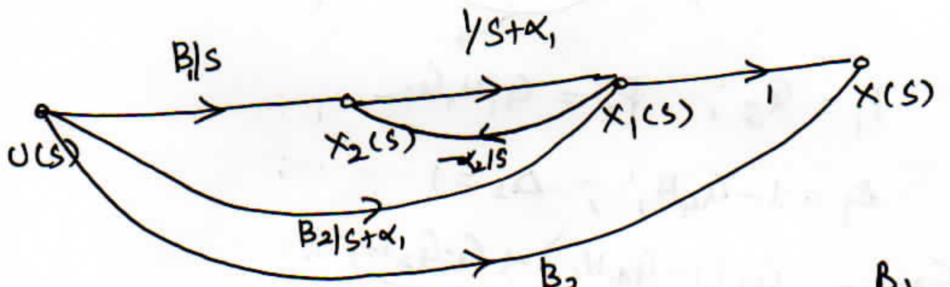
C_2 is independent of R_1 if $G_3(1-G_4H_1) + G_1G_2H_1 = 0$

(4) For the system represented by the following equations, find the transfer function $X(s)/U(s)$ by using signal flow graph technique.

$$x = x_1 + \beta_3 u$$
$$\dot{x}_1 = -\alpha_1 x_1 + x_2 + \beta_2 u$$
$$\dot{x}_2 = -\alpha_2 x_1 + \beta_1 u$$

Sol) Taking LT of the equations

$$X(s) = X_1(s) + \beta_3 U(s) \longrightarrow ①$$
$$sX_1(s) = -\alpha_1 X_1(s) + X_2(s) + \beta_2 U(s)$$
$$X_1(s)[s+\alpha_1] = X_2(s) + \beta_2 U(s)$$
$$\text{or } X_1(s) = \frac{1}{s+\alpha_1} X_2(s) + \frac{\beta_2}{s+\alpha_1} U(s) \longrightarrow ②$$
$$sX_2(s) = -\alpha_2 X_1(s) + \beta_1 U(s)$$
$$X_2(s) = -\frac{\alpha_2}{s} X_1(s) + \beta_1/s\, U(s) \longrightarrow ③$$



(1) Number of forward paths $P_1 = \frac{\beta_1}{s(s+\alpha_1)}$ $\quad \Delta_1 = 1$

$P_2 = \frac{\beta_2}{(s+\alpha_1)}$ $\quad \Delta_2 = 1$

$\sum_m P_{m1} = ?$

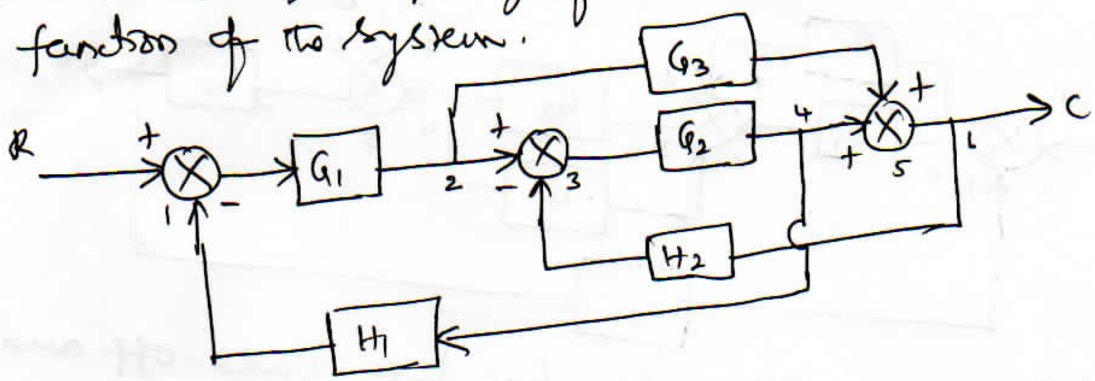
$P_{11} = -\frac{\alpha_2}{s(s+\alpha_1)}$ $\qquad P_3 = \beta_3$ $\quad \Delta_3 = 1$

$$\therefore \text{Transfer function } \frac{X(s)}{U(s)} = \frac{\frac{\beta_1}{s(s+\alpha_1)} + \frac{\beta_2}{(s+\alpha_1)} + \beta_3}{1 - (-\alpha_2/s(\alpha_1+s))}$$

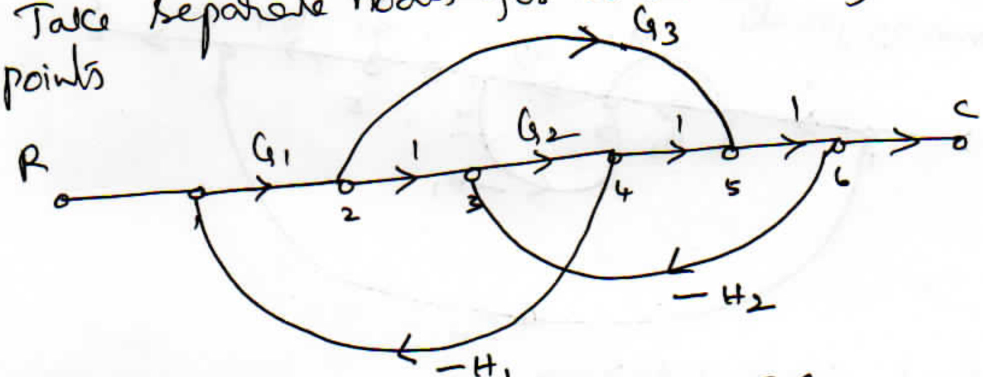
$$= \frac{\beta_3 s(s+\alpha_1) + \beta_2 s + \beta_1}{s(s+\alpha_1) + \alpha_2}$$

$$= \frac{(s^2+\alpha_1 s)\beta_3 + \beta_2 s + \beta_1}{s^2 + \alpha_1 s + \alpha_2}$$

① Draw the Signal flow graph and determine the transfer function of the system.



(Sol) Take separate nodes for both summing and take-off points



(i) Number of forward paths $P_1 = G_1G_2$

$P_2 = G_1G_3$

(ii) Number of individual loops $\sum_m P_{m1} = ?$

$P_{11} = G_2(-H_2) = -G_2H_2$; $P_{21} = G_1G_2(-H_1)$

$P_{31} = G_1G_3(-H_2)G_2(-H_1) = G_1G_2G_3H_1H_2$

Every forward path touching all the loops, hence

$\Delta_1 = \Delta_2 = 1$

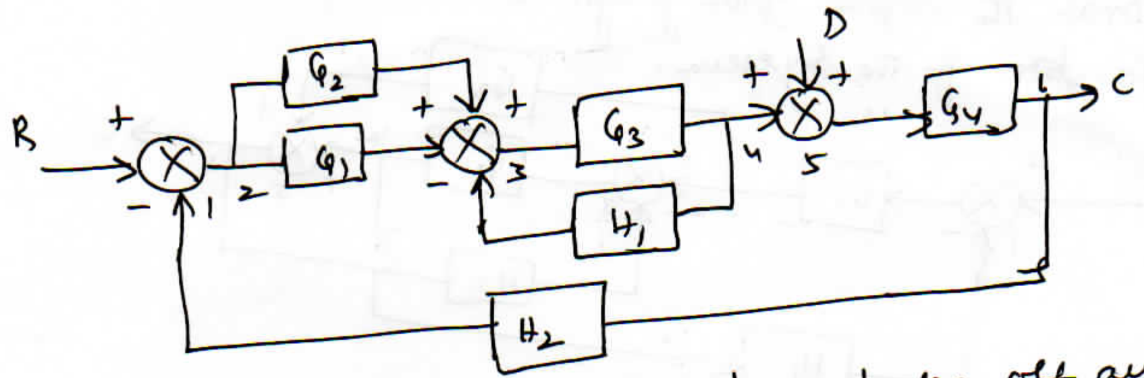
There are no combinations of non-touching loops

hence $\sum_m P_{m2} = \sum_m P_{m3} = 0$

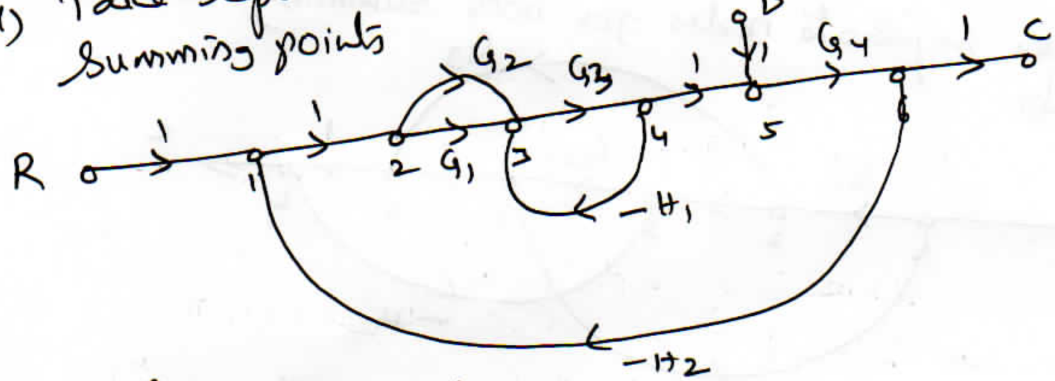
$$\therefore \frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2 + G_1G_3}{1 - \{-G_2H_2 - G_1G_2H_1 + G_1G_2G_3H_1H_2\}}$$

$$= \frac{G_1G_2 + G_1G_3}{1 + G_2H_2 + G_1G_2H_1 - G_1G_2G_3H_1H_2}$$

② Using Mason's gain formula determine $\frac{C}{R}$



(Sol) Take separate nodes for both take-off and summing points



To find C/R; let $D = 0$

$\therefore\ P_1 = G_1 G_3 G_4 \qquad \Delta_1 = 1$

$P_2 = G_2 G_3 G_4 \qquad \Delta_2 = 1$

$P_{11} = -G_3 H_1 \qquad P_{21} = -G_1 G_3 G_4 H_2$

$P_{31} = -G_2 G_3 G_4 H_2$

and $\sum_m P_{m2} = \sum_m P_{m3} = 0$

$\therefore\ \Delta = 1 - \sum_m P_{m1}$

$$\therefore\ \frac{C}{R} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{1 - \sum_m P_{m1}}$$

$$= \frac{G_1 G_3 G_4 (1) + G_2 G_3 G_4 (1)}{1 + G_3 H_1 + G_1 G_3 G_4 H_2 + G_2 G_3 G_4 H_2}$$

Transfer function of DC Servo Motor:

There are two types of DC Motors namely
(1) Field Controlled DC Motor (2) Armature Controlled DC Motor

Field Controlled DC Servo Motor:

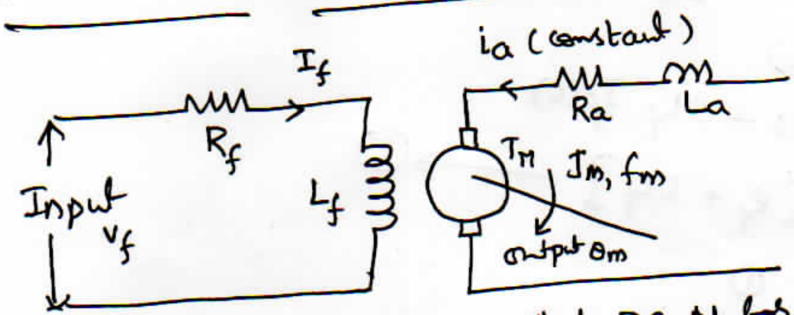


Figure: Field Controlled DC Motor

The input voltage V_f is applied to the field winding which has a resistance R_f and inductance L_f. The armature current i_a supplied to the armature is kept constant and thus the motor shaft is controlled by the input voltage V_f. The field current I_f produces a flux in the machine which inturn produces a torque at the motor shaft. The moment of inertia and the coefficient of viscous friction at the motor shaft are J_m and f_m respectively. The angular shift in the motor shaft is θ_m and the corresponding angular velocity beeing ω_m.

Since the armature current i_a kept constant, the relationship between the develped motor torque T_M and the field current i_f is given by

$$T_M \propto i_f \quad \text{or}$$

$$T_M = K_f \, i_f \longrightarrow (1)$$

where K_f is motor torque constant in Nm/A.

The Relation V_f and i_f is given by

$$V_f = R_f i_f + L_f \frac{di_f}{dt} \longrightarrow (2)$$

The relation between T_M, J_m and f_m is given by

$$T_M = J_m \frac{d^2\theta_m}{dt^2} + f_m \frac{d\theta_m}{dt} \longrightarrow (3)$$

Taking LT of eq (2)

$$V_f(s) = R_f I_f(s) + sL_f I_f(s)$$

$$= I_f(s)[R_f + sL_f] \longrightarrow (I)$$

Taking LT of eq (1)

$$T_M(s) = K_f I_f(s) \longrightarrow (II)$$

Taking LT of eq (3)

$$T_M(s) = [J_m s^2 + f_m s]\, \theta_m(s) \longrightarrow (III)$$

The relation between V_f, I_f and T_M and θ_m is shown in figure

Figure : Block Diagram representation of field Controlled DC Motor.

The transfer function relating the input and output is given by

$$\frac{\theta_M(s)}{V_f(s)} = \frac{K_f}{(R_f + sL_f)(J_m s^2 + f_m s)}$$

The relation between angular velocity ω_m and angular displacement θ_m is given by

$$\omega_m = \frac{d}{dt}\theta_m \quad \text{Taking LT} \quad \omega_m(s) = s\,\theta_m(s)$$

$$\therefore \frac{\omega_m(s)/s}{V_f(s)} = \frac{K_f}{(R_f + sL_f)(J_m s^2 + f_m s)}$$

$$\text{or} \quad \frac{\omega_m(s)}{V_f(s)} = \frac{K_f}{(R_f + sL_f)(J_m s + f_m)}$$

(2) Armature Controlled DC Motor: The relation between applied armature voltage V_a and motor shaft displacement θ_m can be derived as follows.

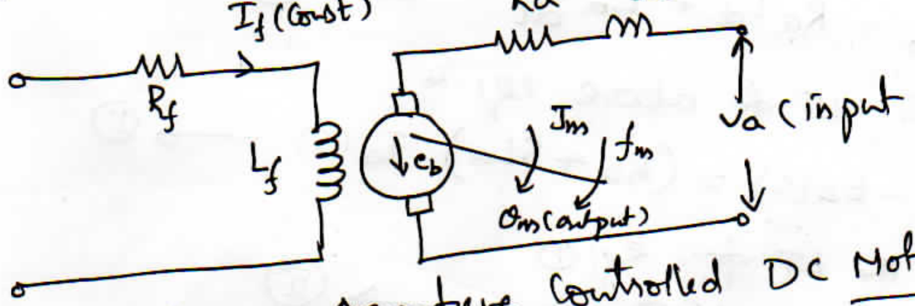


Figure: Armature Controlled DC Motor

The input voltage V_a is applied to the armature which has a resistance of R_a and inductance of L_a. The field current supplied to the field winding is kept constant and thus the armature input voltage V_a controls the motor shaft output θ_m. The moment of inertia and the coefficient of viscous friction at the motor shaft being J_m and f_m respectively. The angular shift in the motor shaft being θ_m and the motor shaft velocity is being ω_m.

As the field current I_f is kept constant, the relation between the torque developed T_M and i_a is

$$T_M \propto i_a$$

$$\text{or } T_M = K_T i_a \longrightarrow (1)$$

where K_T is motor torque constant K_T in Nm/A

The applied input voltage V_a is being opposed by the back emf e_b developed in armature. The relation between e_b and the motor speed ω_m is given by

$$e_b \propto \omega_m \text{, where } \omega_m = \frac{d}{dt}\theta_m$$

$$\therefore e_b = K_b \frac{d}{dt}\theta_m \longrightarrow (2)$$

where K_b is the back emf Constant expressed in V/(rad/sec).

The resultant KVL equation of armature circuit is

$$V_a - e_b = R_a i_a + L_a \frac{d}{dt} i_a \longrightarrow ③$$

Taking the LT of above eq is

$$V_a(s) - E_b(s) = (R_a + sL_a)\, I_a(s) \longrightarrow \text{I}$$

Taking the LT of eq ①

$$T_M(s) = K_T I_a(s) \longrightarrow \text{II}$$

Taking the LT of eq ②

$$T_M(s) = (J_m s^2 + f_m s)\, \theta_m(s) \longrightarrow \text{III}$$

- The relation between all the above eqs is as follows

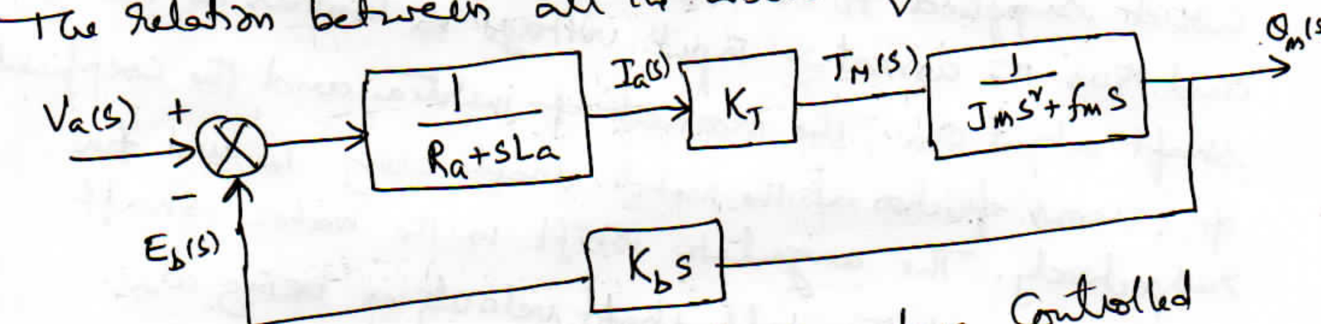
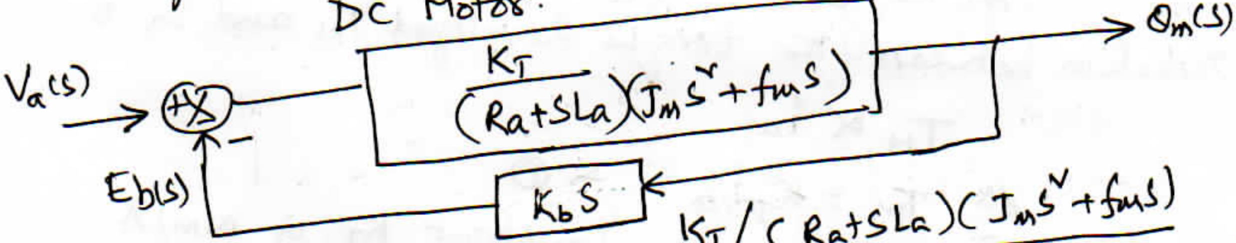


Figure : Block Diagram of Armature Controlled DC Motor.



$$\therefore \frac{\theta_m(s)}{V_a(s)} = \frac{K_T / (R_a + sL_a)(J_m s^2 + f_m s)}{1 + \left[K_T / (sL_a + R_a)(J_m s^2 + f_m s) \right] s K_b}$$

$$= \frac{K_T}{s(R_a + sL_a)(J_m s + f_m) + s K_T K_b}$$

If the armature inductance L_a is neglected

$$\frac{\theta_m(s)}{V_a(s)} = \frac{K_T}{s R_a (J_m s + f_m) + s K_T K_b}$$

$$\therefore \frac{\theta_m(s)}{V_a(s)} = \frac{K_T}{s(sR_aJ_m + R_af_m + K_TK_b)}$$

$$= \frac{K_T/(R_af_m + K_TK_b)}{s\left[\frac{sR_aJ_m}{R_af_m + K_TK_b} + 1\right]}$$

$$\text{or } \frac{\theta_m(s)}{V_a(s)} = \frac{K_m}{s(1+sT_M)} \longrightarrow (i)$$

where $K_m = \frac{K_T}{R_af_m + K_TK_b}$ is motor gain Constant

$T_m = \frac{R_aJ_m}{(R_af_m + K_TK_b)}$ is motor time Constant

Also $\omega_m = \frac{d}{dt}\theta_m$ and $\omega_m(s) = s\theta_m(s)$

$$\therefore \frac{\omega_m(s)/s}{V_a(s)} = \frac{K_m}{s(1+sT_M)}$$

$$\text{or } \frac{\omega_m(s)}{V_a(s)} = \frac{K_m}{(1+sT_m)}$$

<u>The relation between torque Constant</u> K_T and <u>back emf Constant</u> K_b:- The mechanical power output of the motor is $T_M\omega_m$, which is equal to armature input e_bi_a

Therefore $T_M\omega_M = e_bi_a$

where $K_M = K_Ti_a$ and $e_b = K_b\omega_m$

$$\therefore K_Ti_a\omega_m = K_b\omega_mi_a$$

$$\text{Hence } \boxed{K_T = K_b}$$

Transfer function of AC Servo Motor :

The transfer function of AC Servo Motor relates the angular shift θ_m in the shaft to the input control $V_c(t)$.

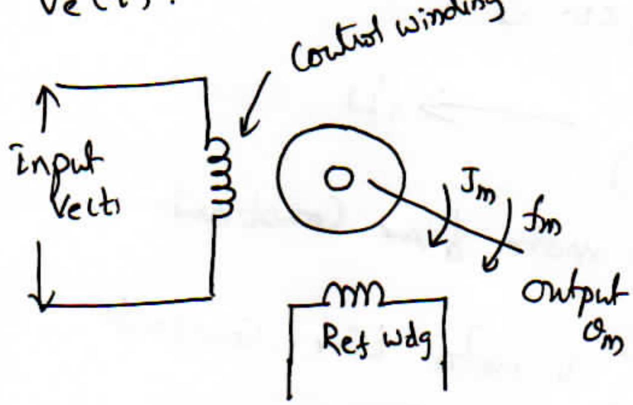


Figure (1) : Two-phase AC Servo Motor.

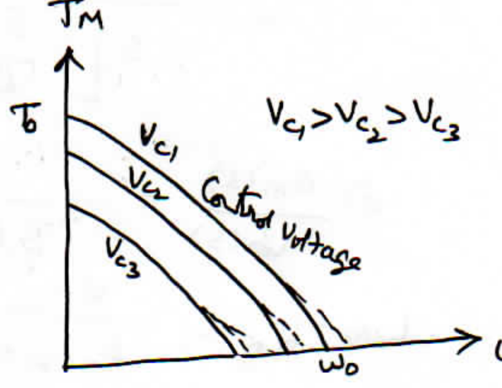


Figure (2) : Torque Speed characteristics of a two phase AC Servo Motor

Two-phase ac servo motor is a two phase induction motor having drag cup type rotor construction. The control voltage $V_c(t)$ is applied to the control winding and a fixed voltage having a phase difference of 90° w.r.t control voltage having a phase difference of 90° w.r.t control winding voltage is applied to the reference winding. The control voltage results in the development of the motor torque T_M. The torque-speed characteristics of motor are shown in figure (2).

The moment of inertia and the viscous friction coefficient of Motor are given by J_m and f_m respectively. The angular shift of motor shaft and velocity are given by θ_m and ω_m respectively.

From the Torque-Speed characteristics, the dynamic relation between the motor torque and the speed is given by

$$T_M = m\omega_m + KV_c \longrightarrow ①$$

where m and K can be derived as follows

(i) when the speed $\omega_m = 0$, the torque is T_0 (stalling torque) and this stalling torque is proportional to the control voltage V_c.

$$\therefore T_0 = K V_c \quad \text{or} \quad K = \frac{T_0}{V_c} \quad \text{in Nm/V}$$

(ii) The slope of the torque - speed characteristics is

$$m = -\frac{T_0}{\omega_0} \quad \text{in Nm/rad/sec}$$

$$\text{also } \omega_m = \frac{d\theta_m}{dt};$$

Now equation ① can be expressed as

$$T_M = m\frac{d\theta_m}{dt} + K V_c \longrightarrow ②$$

$$\text{Also } T_M = J_m \frac{d^2\theta_m}{dt^2} + f_m \frac{d\theta_m}{dt} \longrightarrow ③$$

Taking LT of eq ② $\quad T_M(s) = m s\,\theta_m(s) + K V_c(s) \rightarrow$ (i)

Taking LT of eq ③; $\quad T_M(s) = (s^2 J_m + s f_m)\,\theta_m(s) \longrightarrow$ (ii)

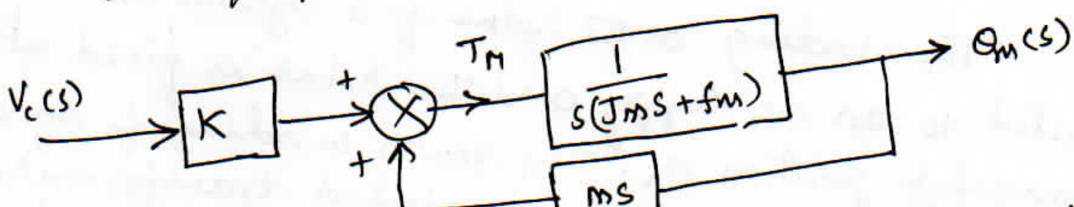
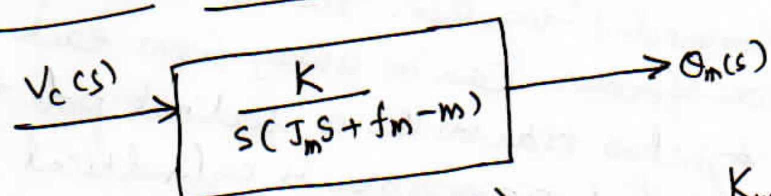


Figure: Block Diagram representation of A.C Servo Motor



$$\therefore \text{TF } \frac{\theta_m(s)}{V_c(s)} = \frac{\left(K/(f_m - f)\right)}{s\left(\frac{J_m s}{f_m - f} + 1\right)} = \frac{K_m}{s(1+sT_M)}$$

where $K_m = \frac{K}{f_m - f}$ is motor gain Constant

$T_M = \frac{J_m}{f_m - f}$ is motor time constant

$$\text{Also } \frac{[\theta_m(s)/s]}{V_c(s)} = \frac{\omega_m(s)}{V_c(s)} = \frac{K_m}{(1+sT_M)}$$

Synchro Error Detector (Selsyn):

The Synchro transmitter and Synchro Control transformer Converts an angular position difference into a proportional ac voltage.

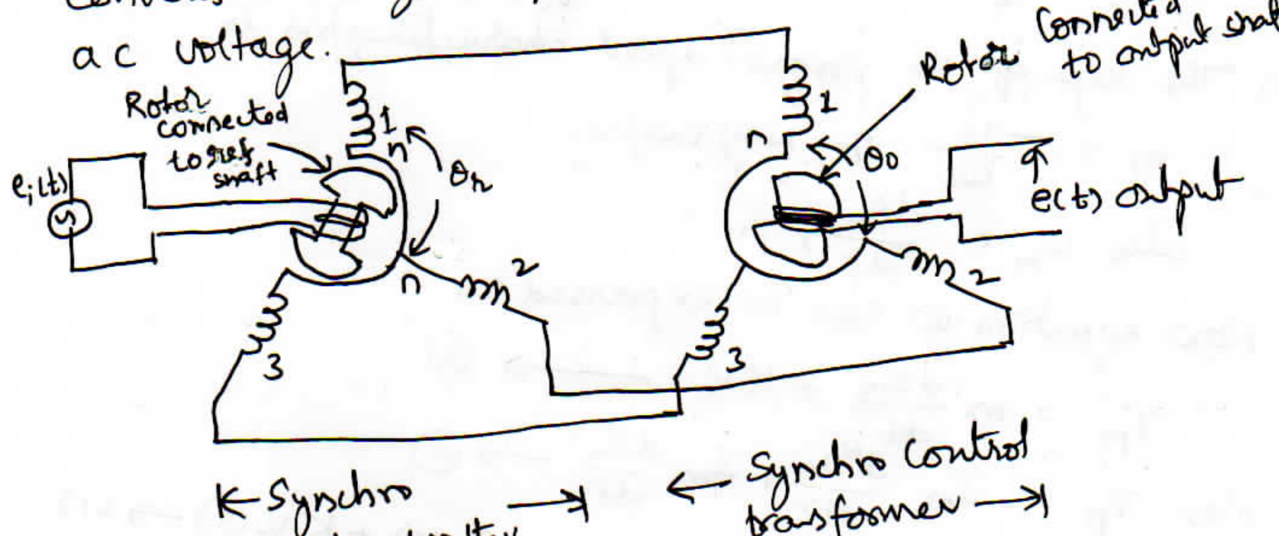


Figure : Synchro Error Detector

The winding on the rotor of a synchro transmitter is Connected to an ac supply and this rotor is fixed at a desired angular position θ_r. The stator winding of synchro transmitter and also that of synchro Control transformer are wound at 120° in space on the stator. The two stator windings are Connected together. The locations of transmitter and control transformer Can be away from each other. The rot of synchro transmitter is salient pole type and that of synchro Control transformer is cylindrical type.

The rotor of the control transformer is Coupled to the output shaft of the control system. If the position of the output shaft is indicated as θ_0, this results in an angular error $\theta_e = (\theta_r - \theta_0)$ between the positions of reference and output shafts.

The process of Conversion of the angular difference into a proportional voltage is explained as follows.

If $e_1(t) = E_m \sin(2\pi f t)$ is applied to the rotor $\rightarrow ①$ winding of the synchro transmitter, then if $\theta_r = 0$, the corresponding voltage induced by transformer section across the stator winding 1n is given by

$$e_{1n} = K E_m \sin(2\pi f t) \longrightarrow ②$$

where K is constant of proportionality

As the stator windings 2n and 3n are 240° and 120° apart in anti-clockwise direction w.r.t the winding 1n, the voltages induced across them are

$$e_{2n} = K E_m \sin(2\pi f t) \cos 240°$$
$$= -0.5 K E_m \sin(2\pi f t) \longrightarrow ③$$

$$e_{3n} = K E_m \sin(2\pi f t) \cos 120°$$
$$= -0.5 K E_m \sin(2\pi f t) \longrightarrow ④$$

Now, if the rotor of the synchro transmitter shifts in anti-clockwise direction through an angle θ, the voltages induced in stator coil are

$$e_{1n} = K E_m \sin 2\pi f t \cos\theta \longrightarrow ⑤$$
$$e_{2n} = K E_m \sin 2\pi f t \cos(240 - \theta) \longrightarrow ⑥$$
$$e_{3n} = K E_m \sin 2\pi f t \cos(120 - \theta) \longrightarrow ⑦$$

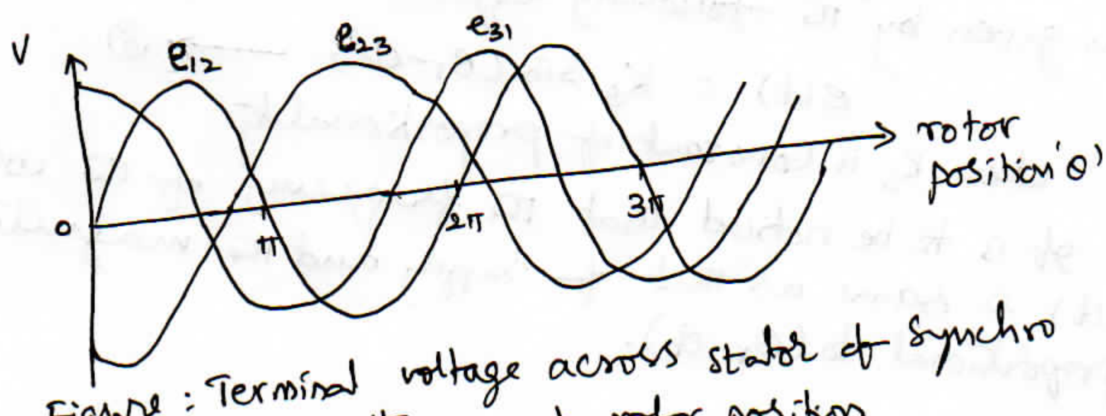


Figure : Terminal voltage across stator of synchro transmitter w.r.t rotor position

The three voltages e_{1n}, e_{2n} and e_{3n} are Connected Consecutively to three Stator windings of the Control transformer and produce a resultant flux in the air gap of the same stator windings, which inturn induces a voltage across the rotor winding of the Control transformer. The magnitude of this Induced voltage depends on the difference $(\theta_r - \theta_0)$. If the difference $(\theta_r - \theta_0)$ is zero, the induced voltage across the rotor winding terminals of the Control transformer is zero, maximum for $\theta_r - \theta_0 = 90°$ and again zero when $\theta_r - \theta_0 = 180°$. After $180°$, the phase of the induced voltage reverses. The magnitude is again maximum with a reversed phase for $\theta_r - \theta_0 = 270°$ and finally zero for $\theta_r - \theta_0 = 360°$. The variation of the amplitude of induced voltage e(t) across the rotor of the Control transformer w.r.t $(\theta_r - \theta_0)$ is shown in figure.

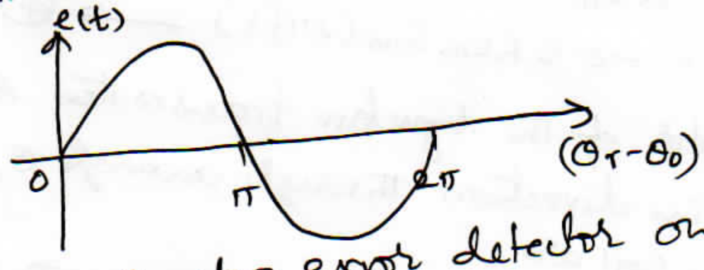


Figure : Synchro error detector output

Therefore, the magnitude of the output induced voltage e(t) developed across the rotor of synchro transformer is given by the following equation

$$e(t) = K_s \sin(\theta_r - \theta_0) \longrightarrow (8)$$

where K_s is Constant of proportionality

It is to be noticed that the frequency of the voltage e(t) is same as that of supply and the magnitude is proportional to $(\theta_r - \theta_0)$.

In general, the angular error $(\theta_r - \theta_o)$ is usually small and $\theta_r - \theta_o$ is expressed in radians, therefore

$$\sin(\theta_r - \theta_o) \simeq \theta_r - \theta_o \longrightarrow (9)$$

$$\therefore e(t) = K_s(\theta_r - \theta_o) \longrightarrow (10)$$

Taking LT on both sides

$$E(s) = K_s[\theta_r(s) - \theta_o(s)] \longrightarrow (11)$$

$$= K_s\,\theta_e(s) \longrightarrow (12)$$

where $\theta_e(s) = \theta_r(s) - \theta_o(s) \longrightarrow (13)$

The block diagram representation of synchro error is shown below

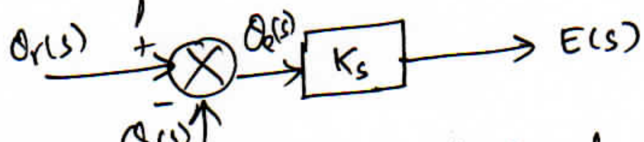


Figure : Block diagram of Synchro error detector

The transfer function of Synchro error detector is

$$\frac{E(s)}{\theta_e(s)} = K_s \quad \text{or} \quad \frac{E(s)}{[\theta_r(s) - \theta_e(s)]} = K_s$$

where K_s is known as the sensitivity or the gain of synchro error detector.

The variation of the magnitude of the output voltage 'e' of synchro error detector is a function of time and shown in figure.

Figure : Variation of error w.r.t time

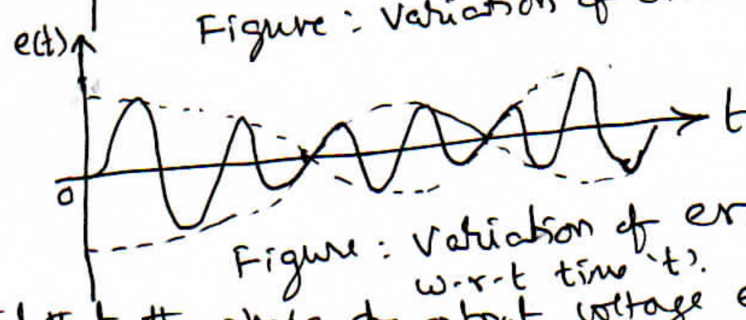


Figure : Variation of error e(t) w.r.t time 't'.

It is to be noted that the phase of output voltage e(t) reverses at points 'a' and 'b' as the error $\theta_e(t)$ changes its sign.

-1001 Systems Engineering

UNIT-1

: INTRODUCTION :

System : System is a coordinated unit of individual elements performing a specific function.

Eg: classroom, Lamp, fan, washing machine etc.

control : It means to regulate, direct, (or) commanding a system, so that a desired output is obtained.

control system : It is combination of elements arranged in a planned manner where in each element causes and effects, produces a desired o/p.

Plant (or) Process :- The portion of the system which is to be controlled (or) regulated is called plant/Process.

Input :- The applied signal (or) excitation signal that is applied to the control system to get required output is known as Input

output :- The actual response that is obtained from a control system with the application of Input is known as output.

controller :- The element of the system itself (or) external in the system which controls the plant (or) Process is known as "controller"

Disturbance :- The signal that has some adverse effect on the value of the output of the system is known as "disturbance".

If a disturbance is produced within the system known as "internal disturbance", otherwise it is know "external disturbance".

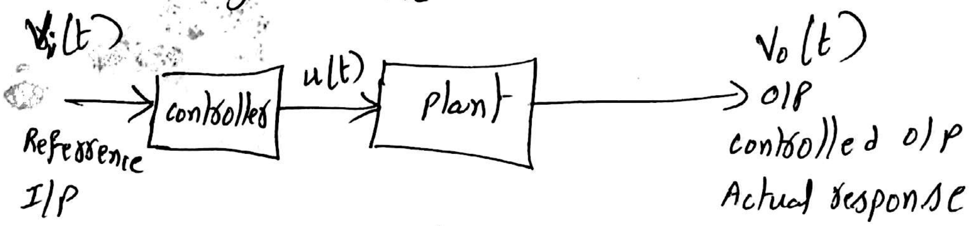
Types of control systems :-

Open loop c.s

closed loop c.s.

Openloop c.s :- A system in which control action doesn't depend on output, is known as OLCS.

Eg:- Washing machine



Block diagram representation of open loop c.s.

In any open loop c.s, the o/p is not compared with the referrence I/P.

The O.L.C.S. are non-feedback c.s.

Advantages :—

- These c.s. are simple in construction and design.
- These are more economic
- These are generally stable
- These are easy for maintainance point of view.

Disadvantages :—

- These systems are inaccurate and unreliable.
- In these systems, that changes in the o/p due to external disturbance are not corrected automatically.

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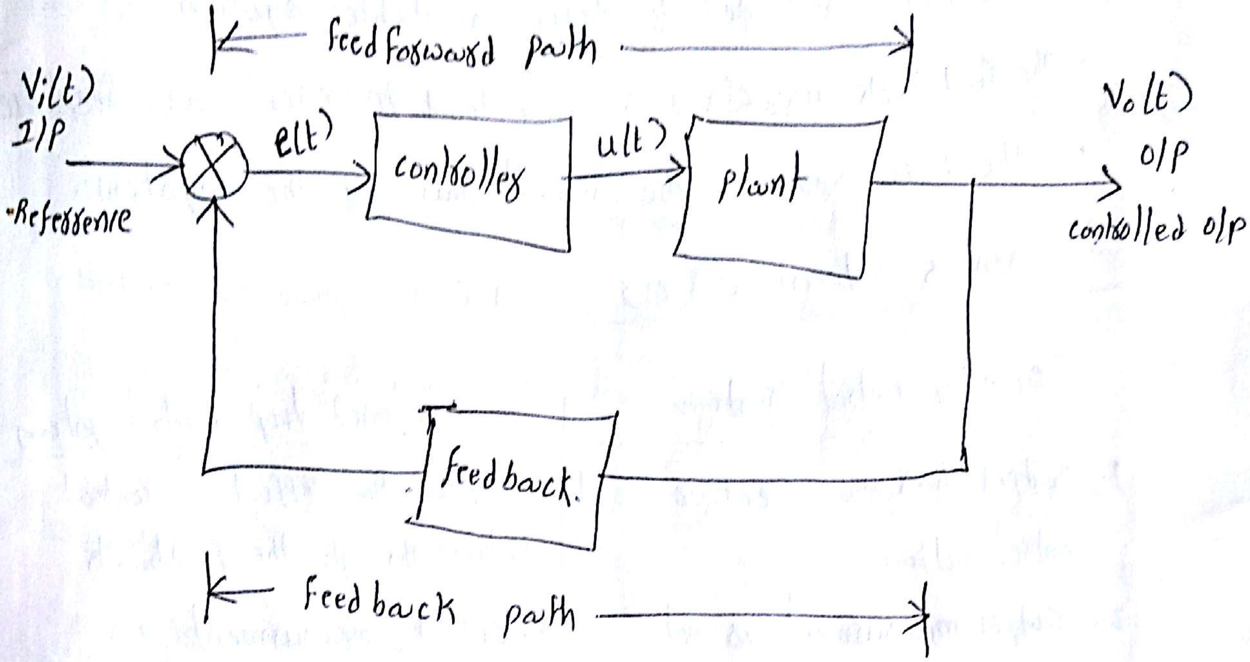
Closed loop c.s :-

The system in which the control action depends on the output through the feedback is known as c.l.c.s.

Eg:- An automatic electric Iron, Voltage stabilizer, a person walking on road in desired path with open eyes.



Block diagram representation of c.l.c.s.

The controller action is activated by the error signal which is the difference b/w I/p signal and feedback signal. The process of comparision b/w the o/p and I/p maintains the output at a desired level through the control action process.

Advantages:-

- These are more accurate due to correction of any arising error.
- The effect of external disturbance signal can be made very small.
- These systems have high bandwidth.
- Speed of the responce can be greatly increased.
- In such systems, there is reduced effect of non-linearities and distortion.

...sitivity of the system may be made small to make the system more stable.

Disadvantages :-

- These are more complex & expensive
- The cost of maintainance is high
- More care is needed to design a stable system
- The feed back in C.L.C.S may lead to oscillatory response.
- The F.B. reduces the overall gain of the system.

Differences between O.L.C.S & C.L.C.S

Open loop control system	closed loop control system
1. Output has no effect on control action.	1. Output has effect on control action through the feedback
2. Output measurment is not required for the system operation	2. Output measurment is necessary.
3. Simple to construct and cheap	3. Complicated to construct and costly.
4. Bandwidth is small	4. Bandwidth is high
5 Inaccurate and unreliable	5. Accurate and reliable
6. These systems are generally stable	6. More care is needed to design a stable system.
7. Error detector is absent	7. Error detector is present
8. changes in the output due to external disturbances are not correct automatically	8. Changes in the o/p due to external disturbances are correct automatically
9. Highly effected by non-linearities	9. Reduced effect of non-linearit...

Classification of control system :-

Control systems may be classified in a no. of ways, depending on the purpose of classification.

→ Natural & Manmade C.S.

Various ~~c.s.~~ control systems that are desined and developed by the man are known as "manmade C.S.s"

Eg: Automobile system with gear, accelarator & breaking arrangement.

The system inside a human being (or) biological system is known as "Natural C.S"

Eg:- Respiration system inside the humanbeing.

The combination of Natural C.S & manmade C.S. is an example of "combinational C.S"

Eg:- The driver driving a car.

→ Time varying and Time invarying C.S.

If the parameters of the C.S. are varying w.r.t. time, then it is called "time varying C.S".

Eg:- Space vehicle during leaving the earth.

If the parameters of the C.S. are not varying w.r.t. time, then it is called "Time invarying C.S".

Eg:- Elements of an electrical network like R, L, C.

→ Linear & non-Linear C.S

If the C.S. is knowns as linear, it satisfies principle of superposition terms of additive, homogeneous property.

Eg:- $f(x) = x$, $f(kx) = k \cdot f(x)$

Those systems which doesn't obey the superposition princi are known as "non-linear c.s"

Eg:- $f(x) = x^2$

Generally all the physical system are non-linear in na If the presence of non-linearity doesn't affect on the performa of the system much, the presence of non-linearity can be negl and the system can be treated as linear system.

→ Continuous & discrete time c.s.

If all the system variables of a c.s are function of t it is known as "continuous time c.s".

Eg:- Speed control of a d.c. motor

If one (or) more system variables of c.s. are known a certain discrete time (samples 0's & 1's)

Eg:- Microprocessor (or) computer.

→ Static & dynamic. c.s

A system is called dynamic, if it's present outp depend on the past Input, where as the static system one whose current output depend only on the current Input

Eg:- Dynamic c.s → lift

Static c.s → Pure d.c. resistive N.W.

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→ Lumped & distributed parameters c.s.

If the c.s. can be represented as a ordinary differential equations, such c.s. are known as "lumped parameter c.s"

Eg:- Electrical network parameters like R, L, C.

A c.s represented by partial differential equations is known as "distributed parameters c.s."

Eg:- In a transmission line, it's parameters like resistance, Inductance are totally distributed along its length.

→ Single Input/output & multi Input/output c.s.

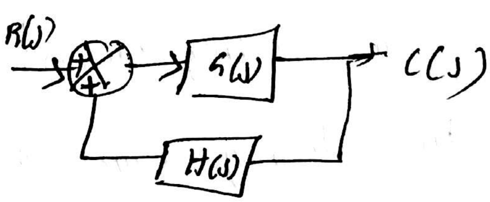
If a c.s. has single I/P and single o/p, it is known as "single I/O c.s."

Eg:- position control system.

If a c.s. has multiple Inputs & multiple outputs, known as "multi I/O c.s".

Eg:- multiplexer, Demultiplexer.

Effect of Positive feedback:—



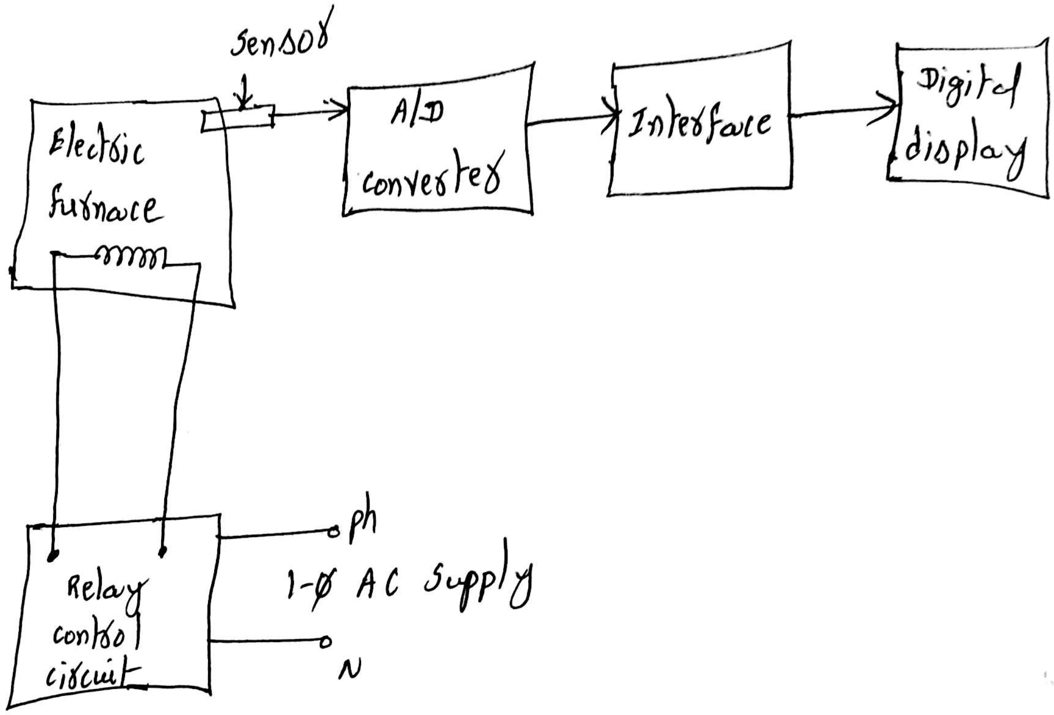
$$T.F = \frac{G(s)}{1 - G(s)H(s)}$$

There is a negative sign in the denominator of eq. which indicates the possibility of denominator becoming equal to zero thereby giving an infinite output for a finite input which is the condition of instability. The positive F.B is sometimes used for increasing the loop gain of feedback system.

Examples of control systems :-

i) Temperature control system :-

Open loop control system :-



The electric furnace shown in fig. is an open loop system. The output in the system is the desired temperature. The temperature of the system is raised by heat generated by the heating element. The output temperature depends on time during which the supply to heater remain 'ON'.

The ON & OFF of the supply is governed by the time setting of the relay. The temperature is measured by a sensor, which gives an analog voltage corresponding to the temperature of the furnace. The analog signal is converted to digital signal by an A/D converter. The digital signal is given to the digital display device to display the temperature. In this system if there is any change in output temperature then the time setting of the relay is ~~not~~ not altered automatically.

Closed loop system :-

The electric furnance shown in fig. a closed loop system. The output of the system is desired temperature and it depends on the time during which supply to heater remain ON.

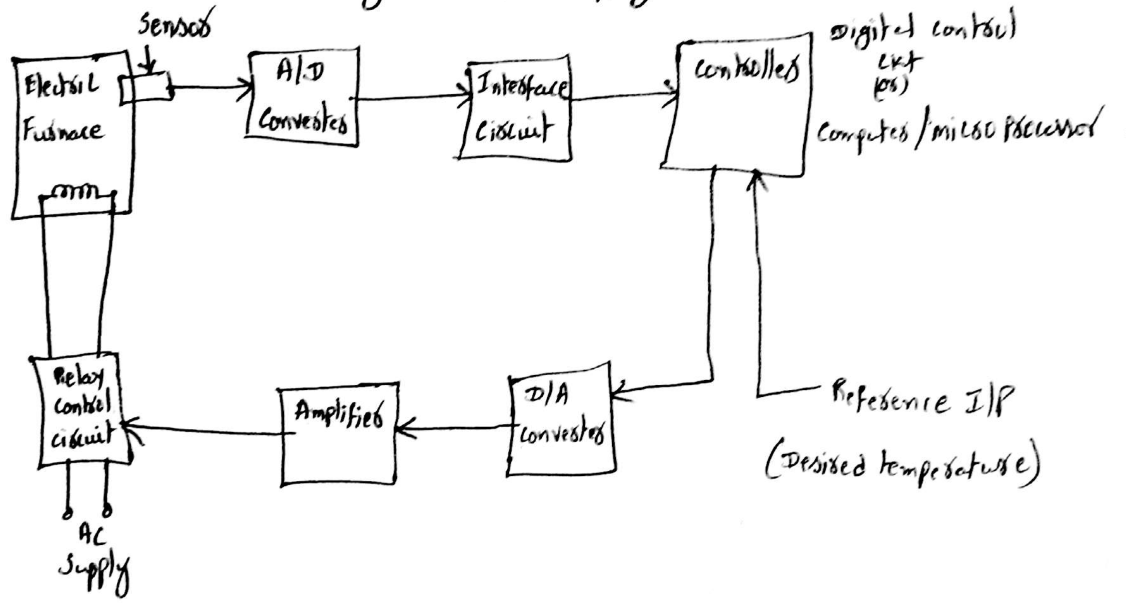


Fig: Closed loop temperature control system.

The switching ON and OFF of the relay is controlled by a controller which is a digital system or computer. The desired temperature is input to the system through keyboard or as a signal corresponding to desired temperature via ports. The actual temperature is sensed by sensor and converted to digital signal by the A/D converter. The computer reads the actual temperature and compares with desired temperature

any difference then it sends signal to switch ON or OFF the

relay through D/A converter and amplifier. Thus the system

automatically corrects any changes in output. Hence it is a

closed loop system.

(e) Traffic control system :-

Open loop System :- Traffic control by means of traffic signals

operated on a time basis constitutes an open loop control system.

The sequence of control signals are based on a time slot given

for each signal. The time slots are decided based on a traffic

study. The system will not measure the density of the

traffic before giving the signals. Since the time slot does not

changes according to traffic density, the system is open loop

system.

closed loop System :- Traffic control system can be made as

a closed loop system if the time slots of the signals are

decided based on the density of traffic. In closed loop

traffic control system, the density of the traffic is measured

on all the sides and the information is fed to a computer. The

timings of the control signals are decided by the computer based

on the density of traffic. Since the closed loop system dynamically

changes the timings, the flow of vehicles will be better than

open loop system.

Feed back characteristics and Effects of feed back :-

"Feedback" generally means connecting the o/p through the

possible blocks back to I/P. This is the basic thing involved

in closed loop control system.

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The main characteristics are:

(1) Sends the o/p signal back to I/p and generated error signals.

(2) It helps in good regulation of the o/p

(3) It doesn't require any manual interruption and errors are minimised automatically

(4) System gain is decreased due to feed back.

These are five major effects of feedback.

(1) Effect of feedback on noise to signal ratio

(2) Effect of feedback on the stability of closed loop system

(3) " " " " sensitivity " "

(4) " " " " " gain

(5) " " " " time constant.

Mathematical model:-

A physical system is a collection of physical objects connected together to serve an object. An idealized physical system is called "physical model."

Once a physical model of a physical system is obtained, the next step is to obtain a "Mathematical model" which is the mathematical representation of the physical model through the used of appropriate physical laws.

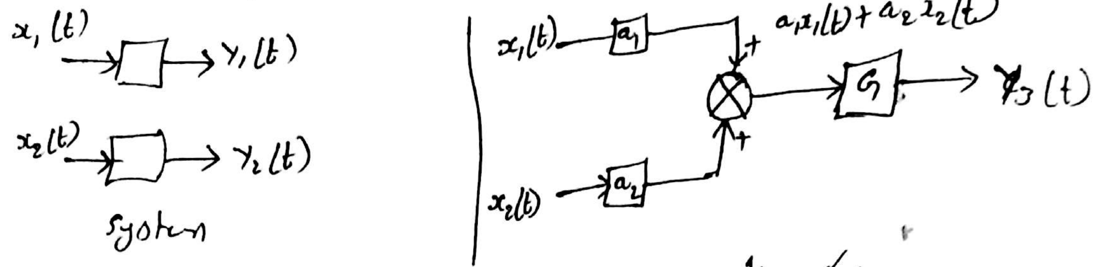
When the mathematical model of a physical system is solved for various input conditions, the result represents the dynamic response of the system.

Mathematical model of a c.s constitues a set of differential equations. The response of output of the system can studied by solving differential equations for various input conditions.

A mathematical model of a system is linear if it obeys the principle of "superposition & homogenity.

Principle of superposition :-

$x_1(t)$, $x_2(t) \Rightarrow$ inputs , $y_1(t)$, $y_2(t)$ - o/p.



If $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$ — linear.

A mathematical model will be linear if the differential equation describing the system has constant coefficients. If the coefficients of differential equations describing the system are constants then the model is linear time invarient. If the coefficient of differential equations governing the system are function of time then the model is linear time varying.

The differential equations of a linear time invariant system can be reshaped into different form for the convenience of analysis. One such model for single I/p and O/p system analysis is a transfer function of the system.

Transfer function :-

Mathematically transfer function is defined as the ratio of Laplace transform of o/p of the system to the Laplace transform of input, under the assumption that all initial conditions are zero.

$G(s)$ is transfer function of the system

$$G(s) = \frac{\text{Laplace transfor of O/P}}{\text{Laplace tran of I/P}} \Bigg|\ \text{initial conditions are zero.}$$

* Transfer function model → applicable for linear time invariant

state space model → applicable to all systems.

Features of Transfer function :-

* Gives mathematical models of all system components and hence of the over all system.
* Converts integral - differential time domain equations using laplace approach to simple algebraic equations.
* Transfer function is the property and the characteristics of the system itself. Its value is dependent on the parameters of the system and independent of the value of inputs.
* It helps in calculating the output by any type of input applied to the system.

Imp • Once transfer function is know, output response for any type of reference input can be calculated.

Imp. It helps in determing the information about the system, i.e. poles, zero, characteristic equation etc.

* It helps in the stability analysis of the system.
* The system differential equation can be obtained by replacing the variable 's' by $\frac{d}{dt}$.

Disadvantages of Transfer function :-

* only applicable to linear time invariant system (LTI)
* It doesn't provide any information concerning the physical structure of the system.
* Effects arising due to initial conditions are totally neglected. Hence initial condition loss there importance.

Procedure to obtain the transfer function :-

* Write down the time domain equations for the system by introducing different variables in the system.

- Take the laplace transform of the system equations assuming all initial conditions are zero.
- Identify the system input and output variables.
- Take the ratio of laplace transform of output variable to laplace transform of input variable to get the transfer function of system.

Impulse response:-

$\delta(t) = A, \quad t = 0$

$\quad\;\; = 0 \quad t \neq 0$

n-order

$$a_0(t)\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} \dots a_{n-1}(t)\frac{dx}{dt} + a_n$$

Laplac $\mathcal{L}$

$F(s) = \int_0^\infty F(t)\, e^{-st}\, dt$, inverse $: f(t) = \frac{1}{2\pi}$

step: $u(t) = 1 \quad t > 0$, $u(t) = 0$, $t < 0$

$U(s) = \int_0^\infty u(t)\, e^{-st} dt = \int_0^\infty 1\, e^{-st} = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = .$

Consider that a linear time-invariant system has the input $r(t)$ and output $c(t)$. The system can be characterized by its impulse response $g(t)$, which is defined as the output when the input is a unit-impulse function $\delta(t)$. Once the impulse response of a linear system is known, the output of the system $c(t)$, with any input ~~r(t)~~ $r(t)$, can be found by using the transfer function.

$R(s) = 1,$

$G(s) = C(s)$

Transfer function (single input, single output systems):

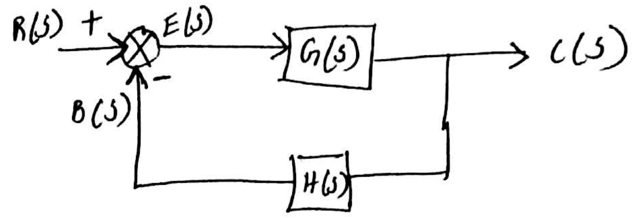
The transfer function of a linear time-invariant system is defined as the laplace transform of the impulse response, with all the initial conditions set to zero.

Transfer function of open loop c.s:

$$\therefore G(s) = \frac{C(s)}{R(s)}$$

Transfer function of closed loop C.S :-

consider negative feedback



R(s) : I/p signal, C(s) : o/p signal, E(s) : Actuating signal

B(s) : Feedback signal, G(s) : Forward path Transfer Function

H(s) : Feedback path transfer function, T(s) : Transfer function of close loop C.S.

$$G(s) = \frac{C(s)}{E(s)} \Rightarrow C(s) = G(s)E(s) \quad \text{--- (1)}$$

$$T(s) = \frac{C(s)}{R(s)}$$

$$H(s) = \frac{B(s)}{C(s)} \Rightarrow B(s) = H(s)C(s) \quad \text{--- (2)}$$

From (1) $C(s) = G(s)E(s)$, $E(s) = R(s) - B(s)$

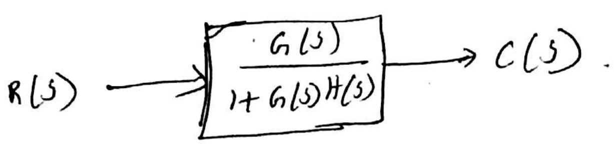
$\therefore E(s) = R(s) - H(s)C(s)$ $\quad (\because (2))$

$\therefore C(s) = G(s)[R(s) - H(s)C(s)]$

$C(s) + G(s)H(s)C(s) = G(s)R(s)$

$C(s)[1 + G(s)H(s)] = G(s)R(s)$

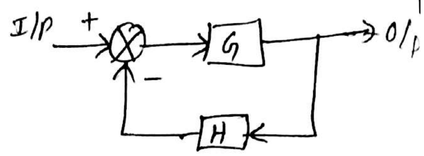
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = T(s)$$



consider Positive feedback :- $T(s) = \dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1 - G(s)H(s)}$

Effects of feedback:

T.F of closed loop = $\frac{G}{1+GH}$.



−ve feedback

① **Effect of Feedback on overall gain:-**

G and H are functions of frequency, so the magnitude of $1+GH$ may be greater than 1 in one frequency range but less than 1 in another.

Therefore, Feedback could increase the gain of system in one frequency range but decrease in another.

② **Effect of feedback on stability:-**

A system is said to be unstable, if its output is out of control.

if $GH = -1$, T.F is infinite,

So, the feedback can improve stability or be harmful to stability, if it is not properly applied.

③ **Effect of feedback on sensitivity:**

Feedback can increase or decrease the sensitivity of a system.

The sensitivity of the system $S^M_G = \frac{\partial M/M}{\partial G/G} = \frac{\text{Percentage change in } M}{\text{Percentage change in } G}$

∂M – incremental change in gain of the overall system

∂G = " " " G.

$\therefore \cancel{\frac{\partial M}{\partial G} \times \frac{G}{M} = \frac{1}{1+GH} \times \frac{G}{\frac{G}{1+GH}}} = \frac{1}{1+GH} = S^M_G = \frac{\partial M}{\partial G}\frac{G}{M}$

(4) Effect of feedback on External disturbance or Noise:-

Feedback can reduce the effect of noise and disturbance on system performance.

In general, feedback ~~also~~ has effect on such performance characteristics as bandwidth, impedance, transient responce, and frequency response.

Translational and Rotational Mechanical Systems:-

In Mechanical systems are two type:

(1) Translational mechanical system - motion taken place along straight line

(2) Rotational mechanical system. - motion about a fixed axis

Mechanical translational system:-

The model of mechanical translational system can be obtained by using three basic elements mass, spring and dash-pot.

The weight of the mechanical system is represented by the element mass. The elastic deformation of the body can be represented by a spring. The friction existing in a rotating mechanical system can be represented by the dash-pot. The dash-pot is a piston moving in side a cylinder filled with viscous fluid.

When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, fricion, and elasticity of the system. The force acting on a mechanical body are governed by Newton's second law of motion. (Newton's second law states that the sum of applied forces is equal to the sum of the opposing forces on a body).

List of Symbols used in mechanical translational system:

x = Displacement m.

$V = \frac{dx}{dt}$ = velocity m/sec.

$a = \frac{dV}{dt} = \frac{d^2x}{dt^2}$ = Acceleration, m/sec^2.

f = Applied force, N (Newtons)

f_m = Opposing force offered by mass of the body, N.

f_K = " " " elasticity of " (spring), N.

f_b = " " " the friction of the body (dash-pot), N.

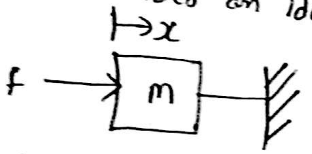
M = Mass kg

K = stiffness of spring, N/m.

B = Viscous friction co-efficient, N-sec/m.

Force balance Equations of Idealized elements:-

Consider an ideal mass element as shown in fig. which has negligible friction and elasticity.



Let a force be applied on it. The mass will an opposing force which is proportional to acceleration of the body.

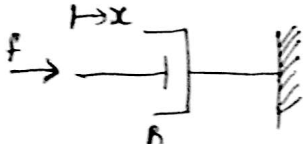
Let f = Applied force

f_m = opposing force due to mass

$\therefore f_m \propto a \Rightarrow f_m \propto \frac{dV}{dt} \Rightarrow f_m \propto \frac{d^2x}{dt^2}$ $(\because F = ma)$

$$\boxed{\therefore f = f_m = m\frac{d^2x}{dt^2}}$$

Consider an ideal frictional element dash-pot as shown in fig.

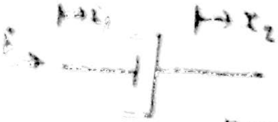


Which has negligible mass and elasticity. Let a force be applied on it. The dash-pot will offer an opposing force which is proportional to the velocity of the body. Let f = Applied force, f_b = opposing force due to friction

$\therefore f_b \propto \frac{dx}{dt} \Rightarrow f_b = B\frac{dx}{dt}$

$\therefore f = f_b = B\frac{dx}{dt}$

When the dash pot has displacement at both ends as shown in fig. The opposing force is proportional to differential velocity.



$$f_b \propto \frac{d}{dt}(x_1 - x_2) \quad \text{or} \quad f_b = B\frac{d}{dt}(x_1 - x_2)$$

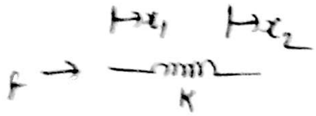
$$\boxed{\therefore\ f = f_b = B\frac{d}{dt}(x_1 - x_2)}$$

Consider an ideal elastic element spring shown in fig. which has negligible mass & friction. Let a force be applied on it. The spring will offer an opposing force which is proportional to displacement of the body.

Let f = Applied force, f_K = opposing force due to elasticity

$f_K \propto x$ or $f_K = Kx$ $\quad \boxed{\therefore\ f = f_K = Kx}$

When the spring has displacement at both ends as shown in fig. the opposing force is proportional to differential displacement.



$$f_K \propto (x_1 - x_2) \quad \text{or} \quad f_K = K(x_1 - x_2)$$

$$\boxed{\therefore\ f = f_K = K(x_1 - x_2)}$$

Procedure to determine the transfer function of mechanical Translational system:-

① In this system, the differential equations governing the system are obtained by writing the force balance equations at nodes in the system. The nodes are meeting point of elements (the nodes are mass elements). In some cases the nodes may be without mass element.

② The linear displacement of the masses (nodes) are assumed as $x_1, x_2, x_3 \cdots$ etc. and assign a displacement of each mass (node).

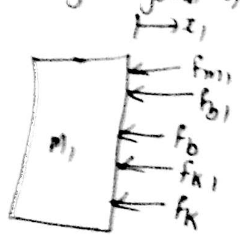
③ Draw the free body diagram of the system. The free body diagram is obtained by drawing each mass seperately and then marking all the forces acting on that mass. The opposing forces acts in a direction of opposite of the applied force.

(4) For each free body diagram, write one differential equation by equating the sum of applied forces to the sum of opposing forces.

(5) Take Laplace transform of differential equations to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the ratio between output variable and I/p variable. This ratio is the transfer function of the system.

Ex (1) Write the differential equations governing the mechanical system as shown in fig. and determine transfer function.

Ans:- Free body diagram of mass m_1



$\therefore$ opposing forces acting mass m_1

$f_{m_1}, f_{b_1}, f_b, f_K, f_{K_1}$

$f_{m_1} = m_1 \dfrac{d^2x_1}{dt^2}$, $f_{b_1} = B_1 \dfrac{dx_1}{dt}$, $f_b = B\dfrac{d}{dt}(x_1 - x)$

$f_{K_1} = K_1 x_1$, $f_K = K(x_1 - x)$.

By newton's second law; $f_{m_1} + f_b + f_{b_1} + f_K + f_{K_1} = 0$

$$\therefore m_1 \frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + K_1x_1 + K(x_1 - x) = 0 \quad (1)$$

Apply Laplace transform.

$$m_1 s^2 X_1(s) + B_1 s X_1(s) + B s[X_1(s) - X(s)] + K_1 X_1(s) + K[X_1(s) -$$

$$X_1(s)[m_1 s^2 + B_1 s + Bs + K_1 + K] - X(s)[Bs + K] = 0$$

$$\therefore X_1(s)[m_1 s^2 + (B_1 + B)s + (K_1 + K)] - X(s)[Bs + K] = 0$$

$$\therefore X_1(s) = \frac{X(s)[Bs + K]}{m_1 s^2 + (B_1 + B)s + (K_1 + K)}$$

Free body diagram of m_2

The opposing forces acting on m_2 are f_{m2}, f_{b2}, f_K, f_b.

$$f_{m_2} = m_2 \frac{d^2x}{dt^2}, \quad f_{b_2} = B_2 \frac{dx}{dt}$$

$$f_b = B\frac{d}{dt}(x - x_1), \quad f_K = K(x - x_1)$$

By Newton's law, $f_{m_2} + f_{b_2} + f_b + f_K = f(t)$

$$m_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B\frac{d}{dt}(x - x_1) + K(x - x_1) = f(t) \quad \text{--- } ②$$

Apply Laplace transform.

$$m_2 s^2 X(s) + B_2 s X(s) + Bs[X(s) - X_1(s)] + K[X(s) - X_1(s)] = F(s)$$

$$X(s)[m_2 s^2 + B_2 s + Bs + K] - X_1(s)[Bs + K] = F(s)$$

$$X(s)[m_2 s^2 + s(B_2 + B) + K] - X_1(s)[Bs + K] = F(s) \quad \text{--- } ③$$

Substituting $X_1(s)$ in equation ③

$$X(s)[m_2 s^2 + s(B_2 + B) + K) - X(s)\frac{(Bs + K)^2}{m_1 s^2 + (B_1 + B)s + (K_1 + K)} = F(s)$$

$$X(s)\left[\frac{[m_2 s^2 + s(B_2 + B) + K][m_1 s^2 + (B_1 + B)s + (K_1 + K)] - (Bs + K)^2}{m_1 s^2 + (B_1 + B)s + (K_1 + K)}\right] = F(s)$$

$\therefore$ Transfer function

$$\frac{X(s)}{F(s)} = \frac{m_1 s^2 + (B_1 + B)s + (K_1 + K)}{[m_1 s^2 + (B_1 + B)s + (K_1 + K)][m_2 s^2 + (B_2 + B)s + K] - (Bs + K)^2}$$

① & ② equation are the differential equations governing the system.

Mechanical Rotational System :-

The model of rotational mechanical system can be obtained by using three elements moment of Inertia (J) of mass, dash-pot with rotational frictional coefficient [B] and torsional spring with stiffness [K].

The weight of the rotational mechanical system is represented the moment of inertia of the mass. The elastic deformation of the body can be represented by spring. (torsional spring). The friction existing in rotational mechanical system can be represented by the dash-pot. The dash-pot is a piston rotating inside a cylinder filled with viscous fluid.

When the torque is applied to the rotational mech. system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torques acting on a rotational mechanical body are governed by Newton's second law of motion for rotational system. (Newton's states that the sum of applied torques is equal to the sum of opposing torques on a body

List of symbols used in mechanical rotational system:-

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec.

$\frac{d^2\theta}{dt^2}$ = Angular acceleration, rad/sec^2

T = Applied torque, N-m

J = Moment of inertia, kg-m^2/rad.

B = Rotational frictional coefficient, N-m/(rad/sec)

K = Stiffness of spring, N-m/rad.

Torque balance equations of idealised elements:-

consider an ideal mass element as shown in fig which is negligible friction & elasticity. The opposing torque due to moment of inertia is proportional to the angular acceleration.

$T_j \propto \frac{d^2\theta}{dt^2}$ (or) $T_j = J\frac{d^2\theta}{dt^2}$

Let T = applied torque

T_j = opposing torque due to moment of inertia of the body

$$\boxed{\therefore T = T_j = J\frac{d^2\theta}{dt^2}}$$

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Consider an ideal frictional element dash-pot as shown in fig.

which has negligible moment of inertia & elasticity. Let a torque be applied to it. The dash-pot will offer an opposing torque which is proportional to the angular velocity of the body.

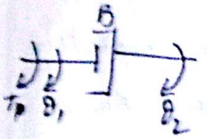
Let T = Applied torque

T_b = opposing torque due to friction

$T_b \propto \frac{d\theta}{dt}$ or $T_b = B\frac{d\theta}{dt}$

$$\boxed{\therefore T = T_b = B\frac{d\theta}{dt}}$$

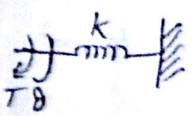
When the dash-pot has angular displacement at both ends as shown in fig. The opposing torque is proportional to the differential angular velocity.



$T_b \propto \frac{d}{dt}(\theta_1-\theta_2) \Rightarrow T_b = B\frac{d}{dt}(\theta_1-\theta_2)$

$$\boxed{\therefore T = T_b = B\frac{d}{dt}(\theta_1-\theta_2)}$$

Consider an ideal elastic element, torsional spring as shown in fig. which has negligible moment of inertia and friction.



Let a torque be applied on it. The torsional spring will offer an opposing torque which is proportional to angular displacement of the body.

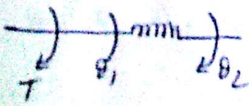
Let, T = applied Torque

T_K = opposing torque due to elasticity

$T_K \propto \theta \Rightarrow T_K = K\theta$.

By Newton's second law $\boxed{T = T_K = K\theta}$

When spring has angular displacement at both ends as shown in fig.



T_K The opposing torque is proportional to differential angular displacement.

$T_K \propto (\theta_1-\theta_2) \Rightarrow T_K = K(\theta_1-\theta_2)$ $\boxed{\therefore T = T_K = K(\theta_1-\theta_2)}$

Procedure to determine the Transfer function of Mechanical Rotational system

(1) In mechanical rotational system, the differential equations governing the system are obtained by writing torque balance equations at nodes in the system. Generally the nodes are mass elements with moment of inertia in the system. In some cases the nodes may be without mass element.

(2) The angular displacement of the moment of inertia of the masses (nodes) are assumed as θ_1, θ_2, θ_3 etc. and assign a displacement to each mass (node).

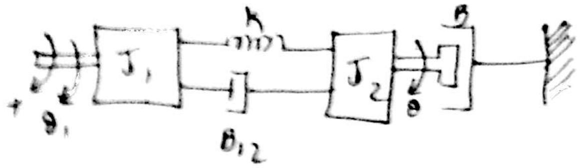
(3) Draw the free body diagram of the system. The free body diagram is obtained by drawing each moment of inertia of mass separately and then marking all the torques acting on that body. Always the opposing torques acts in a direction opposite to applied torque.

(4) The mass has to rotate in the direction of the applied torque. Hence the angular displacement, velocity and acceleration of the mass will be in the direction of the applied torque. If there is no applied torque then the angular displacement, velocity, and acceleration of the mass is in a direction opposite to that of opposing torque.

(5) For each body diagram write one differential equation by equating the sum of applied torques to the sum of opposing torques.

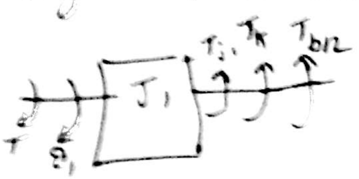
(6) Take laplace transform of differential equation to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the relation between output variable and input variable. This ratio is the Transfer function of the system.

Q) Write the differential equations governing the mechanical rotational system as shown in fig. determine the Transfer Function $\theta(s)/T(s)$. (11) I



Ans: System has 2 nodes, J_1 & J_2.

The free body diagram of J_1 as shown in fig. The opposing torques acting on J_1 are marked as T_{j1}, T_{b12} & T_K



$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2}, \quad T_{b12} = B \frac{d}{dt}(\theta_1 - \theta)$$

$$T_K = K(\theta_1 - \theta)$$

By newton's second law, $T = T_{j1} + T_{b12} + T_K$.

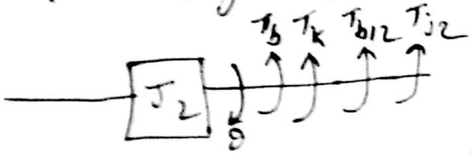
$$J_1 \frac{d^2\theta_1}{dt^2} + B\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T$$

on taking Laplace Transform

$$J_1 s^2 \theta_1(s) + sB_{12}\left[\theta_1(s) - \theta(s)\right] + K\left[\theta_1(s) - \theta(s)\right] = T(s)$$

$$\theta_1(s)\left[J_1 s^2 + sB_{12} + K\right] - \theta(s)\left[sB_{12} + K\right] = T(s) \quad \text{--- (1)}$$

The free body diagram of J_2 as shown in fig. The opposing torques acting on J_2 are marked as T_{j2}, T_{b12}, T_b, T_K.



$$T_{j2} = J_2 \frac{d^2\theta}{dt^2}, \quad T_{b12} = B_{12}\frac{d}{dt}(\theta - \theta_1), \quad T_b = B\frac{d\theta}{dt}, \quad T_K = K(\theta - \theta_1)$$

By Newton's second law, $T_{j2} + T_{b12} + T_b + T_K = 0$

$$\therefore J_2 \frac{d^2\theta}{dt^2} + B_{12}\frac{d}{dt}(\theta - \theta_1) + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

on taking Laplace transform

$$J_2 s^2\theta(s) + B_{12}s\left[\theta(s) - \theta_1(s)\right] + Bs\theta(s) + K\left[\theta(s) - \theta_1(s)\right] = 0$$

$$J_2 s^2\theta(s) - B_{12}s\theta_1(s) + s\theta(s)\left[B_{12} + B\right] + K\theta(s) - K\theta_1(s) = 0$$

$$\theta(s)\left[s^2J_2 + s(B_{12}+B) + K\right] - \theta_1(s)\left[sB_{12}+K\right] = 0$$

$$\theta_1(s) = \frac{\left[s^2J_2 + s(B_{12}+B) + K\right]}{\left[sB_{12}+K\right]}\,\theta(s) \quad \text{--- (2)}$$

Substituting for $\theta_1(s)$ from (2) in equation (1), we get.

$$\left[J_1s^2 + sB_{12} + K\right]\frac{\left[J_2s^2 + s(B_{12}+B) + K\right]\theta(s)}{(sB_{12}+K)} - (sB_{12}+K)\,\theta(s) = T(s)$$

$$\left[\frac{(J_1s^2 + sB_{12} + K)\left[J_2s^2 + s(B_{12}+B) + K\right] - (sB_{12}+K)^2}{(sB_{12}+K)}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{(sB_{12}+K)}{(J_1s^2 + sB_{12} + K)\left[J_2s^2 + s(B_{12}+B) + K\right] - (sB_{12}+K)^2}$$

Electrical Systems:-

The models of electrical system can be obtained by using resistor, capacitor & Inductor. The current-voltage relation of resistor, inductor & capacitor are given in table: 1

Element	Voltage across the element	Current through the element.
$i(t)$ R, $+\ v(t)\ -$	$v(t) = R\,i(t)$	$i(t) = \frac{v(t)}{R}$
$i(t)$ L, $+\ v(t)\ -$	$v(t) = L\frac{d}{dt}i(t)$	$i(t) = \frac{1}{L}\int v(t)$
$i(t)$, $+\ v(t)\ -$	$v(t) = \frac{1}{C}\int i(t)\,dt$	$i(t) = C\frac{dv(t)}{dt}$

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Electrical analogous of Mechanical translational system:-

The electric analogoue of any other kind of system is of greater impostance since it is easier to construct electrical models and analyse them.

Electrical system two types of I/P - voltage or current

There are two types of analogies (1) Force-voltage analogy

(2) Force-current analogy.

Force-voltage analogy & Force-current Analogy.

The force balance equations of mechanical elements and their analogous electrical elements in Force-voltage analogy & Force current Analogy is shown in table.

Mechanical system

I/P - Force

O/P = velocity

Electrical system

Force-current

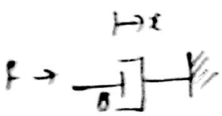
I/P - current source

O/P = V across element

(Force-V)

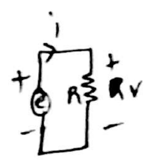
I/P = voltage source

O/P = current through element.

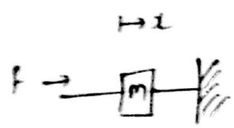


$F = B\frac{dx}{dt} = BV$

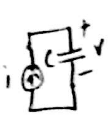
$i = \frac{1}{R}V$



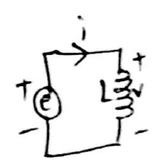
$e = v = Ri = R\frac{dq}{dt}$



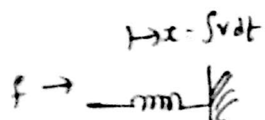
$F = m\frac{d^2x}{dt^2} = m\frac{dV}{dt}$



$i = c\frac{dV}{dt} = \frac{d^2\phi}{dt^2}$

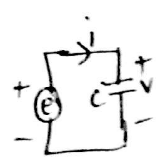


$e = v = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$



$F = kx = k\int vdt$

$i = \frac{1}{L}\int vdt$



$e = v = \frac{1}{c}\int idt = \frac{q}{c}$

Procedure to obtain Electrical analogous of mechanical systems based on force-voltage analogy:-

(1) In electrical system the elements in series will have same current, likewise in mech. system, the elements having same velocity are said to be in series.

(2) The elements having same velocity in mech. system should have analogous same current in electrical analogous system.

(3) Each node in the mech. system corresponds to a closed loop in a electrical system. A mass is considered as a node.

(4) The number of meshes in electrical analogous is same as that of the number of nodes in mech. system. Hence number of mesh currents and system equations will be same as that of the number of velocities of nodes in mech. system.

(5) The mechanical driving (force) sources and passive elements connected to the node in mech. system should be represented by analogous elements in a closed loop in analogous electrical system.

(6) The elements connected between two nodes in mech. system is represented as a common element b/w two meshes in electrical analogous system.

Analogous Quantities in Force - Voltage Analogy & Current Analogy

Item	mech. system	Elect. system ~~(mesh b.~~ voltage	current
Independent variable (I/P)	Force F	voltage V	i
Dependent variable (dP)	velocity (v) Displacement (x)	current i charge q	V flux ϕ
Dissipative element	dash-pot B	Resistance R	$G = \frac{1}{R}$
Storage element	m (mass) K (spring)	L 1/C	C $\frac{1}{L}$
physical law	$\Sigma F = 0$ (Newton's second law)	$\Sigma V = 0$ (KVL)	$\Sigma i = 0$ (K
changing the level of independent variable	lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$	T/F $\frac{i_1}{I_2} =$

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Procedure to obtain electrical analogous of mechanical systems based on Force-current analogy:- (13) I

1) In electrical systems element in parallel will have same voltage, likewise in mech system, the elements having same force are said to be in parallel.

2) The elements having same velocity in mech. system should have ~~corresponds~~ analogous same voltage in electrical analogy system.

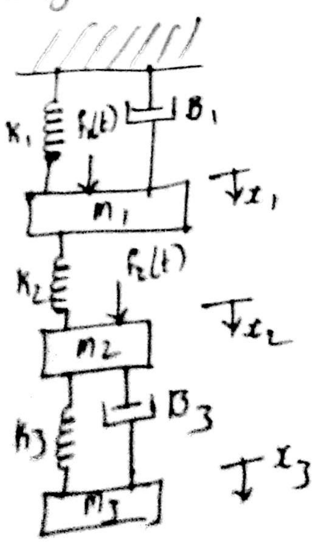
3) Each node in mech. system corresponds to a node in electrical system.

(4) The number of nodes in electrical analogous is same as that of the number of masses in mech. system. Hence the no. of node voltages and system equations will be same as that of the no. of velocities of masses in mech. system.

(5) The mech. force & passive elements connected to the mass in mech. system should be represented by analogous elements connected to a node in electrical system.

6) The element connected b/w two nodes in mech. system is represented as a common element b/w two nodes in electrical analogous system.

(1) Write the differential equations governing the mechanical system shown in fig. Draw the force-voltage and force-current electrical analogous ckts and verify by writing mesh and node equations.



Ans: 3 nodes - 3 masses

Let the displacements of masses m_1, m_2, m_3 be x_1, x_2, x_3 respectively. The corresponding velocities be v_1, v_2 and v_3.

Free body diagram of m_1: the opposing forces are $f_{m1}, f_{b1}, f_{k2}, f_{k1}$

By newton's second law

$$m_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+K_2(x_1-x_2)+K_1x_1=f_1(t) \quad \text{— (1)}$$

Free body diagram of m_2, the opposing forces are $f_{k2}, f_{k3}, f_{b3}, f_{m2}$.

By newton's second law.

$$m_2\frac{d^2x_2}{dt^2}+K_2(x_2-x_1)+K_3(x_2-x_3)+B_3\frac{d}{dt}(x_2-x_3)=f_2(t)$$

Free body diagram of m_3, The opposing forces are f_{m3}, f_{k3}, f_{b3}.

By newton's second law

$$m_3\frac{d^2x_3}{dt^2}+B_3\frac{d}{dt}(x_3-x_2)+K_3(x_3-x_2)=0 \quad \text{— (3)}$$

on replacing the displacements by velocity in the differential equations (1) (2) & (3) governing the mech. system we get,

$$\left(\frac{d^2x}{dt^2}=\frac{dV}{dt},\ \frac{dx}{dt}=v\ \&\ x=\int v\,dt.\right)$$

$$m_1\frac{dv_1}{dt}+B_1v_1+K_1\int v_1\,dt+K_2\int(v_1-v_2)dt=f_1(t) \quad \text{— (4)}$$

$$m_2\frac{dv_2}{dt}+B_3(v_2-v_3)+K_2\int(v_2-v_1)dt+K_3\int(v_2-v_3)dt=f_2(t) \quad \text{— (5)}$$

$$m_3\frac{dv_3}{dt}+B_3(v_3-v_2)+K_3\int(v_3-v_2)dt=0 \quad \text{— (6)}$$

Force voltage analogous ckt:-

nodes - 3 , meshes - 3

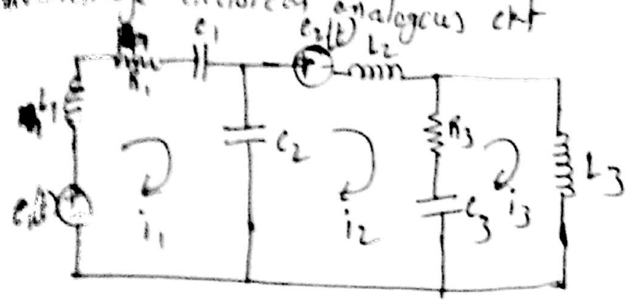
the electrical analogous elements for the elements of mech. system

$f_1(t) \rightarrow e_1(t)$ $\quad v_1 \rightarrow i_1$ $\quad m_1 \rightarrow L_1$ $\quad B_1 \rightarrow R_1$ $\quad K_1 \rightarrow \frac{1}{C_1}$

$f_2(t) \rightarrow e_2(t)$ $\quad v_2 \rightarrow i_2$ $\quad m_2 \rightarrow L_2$ $\quad B_3 \rightarrow R_2$ $\quad K_2 \rightarrow \frac{1}{C_2}$

$v_3 \rightarrow i_3$ $\quad m_3 \rightarrow L_3$ $\quad K_3 - \frac{1}{C_3}$

Force-voltage electrical analogous ckt



The mesh basic equations using KVL for the ckt

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1\, dt + \frac{1}{C_2}\int (i_1 - i_2)\, dt = e_1(t) \quad -(7)$$

$$L_2 \frac{di_2}{dt} + R_3(i_2 - i_3) + \frac{1}{C_3}\int (i_2 - i_3)\, dt + \frac{1}{C_2}\int (i_2 - i_1)\, dt = e_2(t) \quad -(8)$$

$$L_3 \frac{di_3}{dt} + R_3(i_3 - i_2) + \frac{1}{C_3}\int (i_3 - i_2)\, dt = 0 \quad -(9)$$

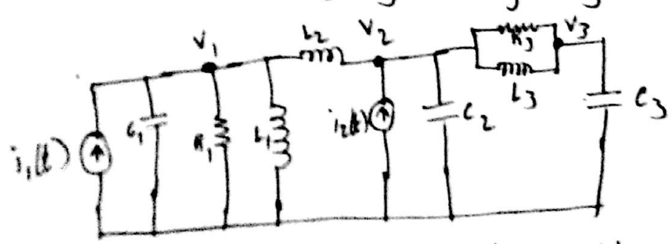
It is observed that the mesh equations (7), (8) & (9) are similar to the differential equation (4), (5) & (6) governing mech. system.

<u>Force current analogous ckt</u> :- masses - 3, nodes - 3

$f_1(t) \rightarrow i_1(t)$ $\quad v_1 = V_1$ $\quad m_1 \rightarrow C_1$ $\quad B_1 \rightarrow 1/R_1$ $\quad K_1 \rightarrow 1/L_1$

$f_2(t) \rightarrow i_2(t)$ $\quad v_2 = V_2$ $\quad m_2 \rightarrow C_2$ $\quad B_3 \rightarrow 1/R_3$ $\quad K_2 \rightarrow 1/L_2$

$v_3 = V_3$ $\quad m_3 \rightarrow C_3$ $\quad K_3 \rightarrow 1/L_3$



Force - current electrical analogous ckt.

From Nodal Analysis:

$$C_1 \frac{dV_1}{dt} + \frac{1}{R_1}V_1 + \frac{1}{L_1}\int V_1\,dt + \frac{1}{L_2}\int (V_1 - V_2)\,dt = i_1(t) \quad \text{--- (10)}$$

$$C_2 \frac{dV_2}{dt} + \frac{1}{R_3}(V_2 - V_3) + \frac{1}{L_3}\int (V_2 - V_3)\,dt + \frac{1}{L_2}\int (V_2 - V_1)\,dt = i_2(t) \quad \text{--- (11)}$$

$$C_3 \frac{dV_3}{dt} + \frac{1}{R_3}(V_3 - V_2) + \frac{1}{L_3}\int (V_3 - V_2)\,dt = 0 \quad \text{--- (12)}$$

It is observed that node basis equations (10), (11) & (12) are similar to the differential equations (4), (5) & (6) governing the mech. system.

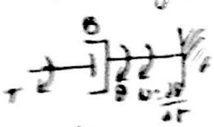
Electrical Analogous of Mechanical rotational system:-

R, L, C $\Rightarrow$ B, J, K.

V, I $\Rightarrow$ T.

Analogous elements of Torque - Force voltage & Force current Analogy

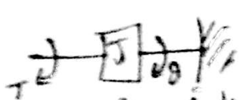
Mechanical rotational	Force voltage	Force current:
I/P = Torque	I/P = voltage source	I/P = current source
O/P = Angular velocity $\omega = \frac{d\theta}{dt}$	O/P: current through element	O/P = V across the element



$T = B\frac{d\theta}{dt} = B\omega$

$e = v = iR$

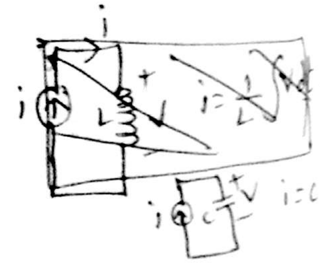
$i = \frac{1}{R}V$



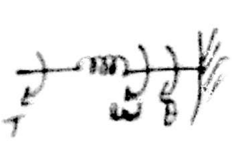
$T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$

$v = L\frac{di}{dt}$

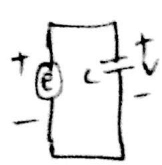
$e = v = L\frac{di}{dt}$



$i = C\frac{dv}{dt}$

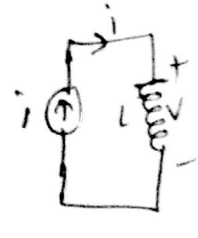


$T = K\theta = K\int \omega\,dt$



$e = v,\ v = \frac{1}{C}\int i\,dt$

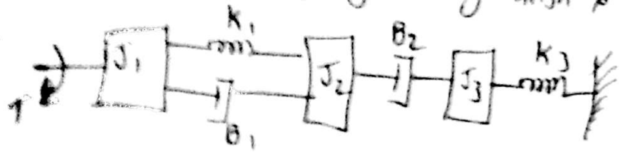
$\therefore e = \frac{1}{C}\int i\,dt$



$i = \frac{1}{L}\int v\,dt$

Procedure is same (replace velocity $\rightarrow$ angular velocity, mass $-$ inertia

Write the differential equations governing the mechanical rotational system shown in fig. Draw the torque voltage & Torque-current electrical analogous ckt & verify by writing mesh & node equations.



Ans: moment of inertia of mass - J,

Angular displacements J_1, J_2 & J_3 be θ_1, θ_2 & θ_3 respectively.
$\omega_1, \omega_2, \omega_3$ velocities.

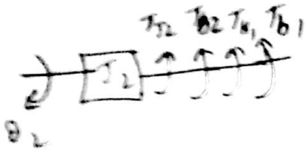
Free body diagram of J_1:

$T_{j_1} = J_1 \frac{d^2\theta_1}{dt^2}$, $T_{K_1} = K_1(\theta_1 - \theta_2)$

$T_{b_1} = B_1 \frac{d}{dt}(\theta_1 - \theta_2)$

By Newton's II law $J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d}{dt}(\theta_1 - \theta_2) + K_1(\theta_1 - \theta_2) = T$ — (1)

Free body diagram of J_2:



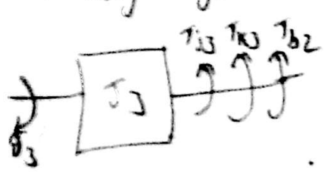
$T_{j_2} = J_2 \frac{d^2\theta_2}{dt^2}$, $T_{b_2} = B_2 \frac{d}{dt}(\theta_2 - \theta_3)$

$T_{K_1} = K_1(\theta_2 - \theta_1)$, $T_{b_1} = B_1 \frac{d}{dt}(\theta_2 - \theta_1)$

By newton's II law, $T_{j_2} + T_{b_2} + T_{b_1} + T_{K_1} = 0$

$\therefore J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d}{dt}(\theta_2 - \theta_3) + B_1 \frac{d}{dt}(\theta_2 - \theta_1) + K_1(\theta_2 - \theta_1) = 0$ — (2)

The freebody diagram of J_3:



$T_{j_3} = J_3 \frac{d^2\theta_3}{dt^2}$, $T_{K_3} = K_3\theta_3$, $T_{b_2} = B_2 \frac{d}{dt}(\theta_3 - \theta_2)$

$\therefore J_3 \frac{d^2\theta_3}{dt^2} + B_2 \frac{d}{dt}(\theta_3 - \theta_2) + K_3\theta_3 = 0$ — (3)

Replace angular displacements by angular velocity $\left(\because \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}, \frac{d\theta}{dt} = \omega, \theta = \int \omega\, dt\right)$

From (1), (2) & (3)

$J_1 \frac{d\omega_1}{dt} + B_1(\omega_1 - \omega_2) + K_1\int(\omega_1 - \omega_2)dt = T$ — (4)

$J_2 \frac{d\omega_2}{dt} + B_2(\omega_2 - \omega_3) + B_1(\omega_2 - \omega_1) + K_1\int(\omega_2 - \omega_1)dt = 0$ — (5)

$J_3 \frac{d\omega_3}{dt} + B_2(\omega_3 - \omega_2) + K_3\int\omega_3\, dt = 0$ — (6)

Torque voltage analogous ckt :-

$T \rightarrow e(t)$ $\quad \omega_1 \rightarrow i_1$ $\quad J_1 - L_1$ $\quad B_1 \rightarrow R_1$ $\quad K_1 - \frac{1}{C_1}$

$\omega_2 \rightarrow i_2$ $\quad J_2 - L_2$ $\quad B_2 \rightarrow R_2$ $\quad K_3 - \frac{1}{C_3}$

$\omega_3 \rightarrow i_3$ $\quad J_3 - L_3$

nodes - 3 meshes - 3

By using KVL

$$L_1 \frac{di_1}{dt} + R_1(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)\, dt = e(t) \quad \text{--- (7)}$$

$$L_2 \frac{di_2}{dt} + R_1(i_2 - i_1) + R_2(i_2 - i_3) + \frac{1}{C_1}\int (i_2 - i_1)\, dt = 0 \quad \text{--- (8)}$$

$$L_3 \frac{di_3}{dt} + R_2(i_3 - i_2) + \frac{1}{C_3}\int i_3\, dt = 0 \quad \text{--- (9)}$$

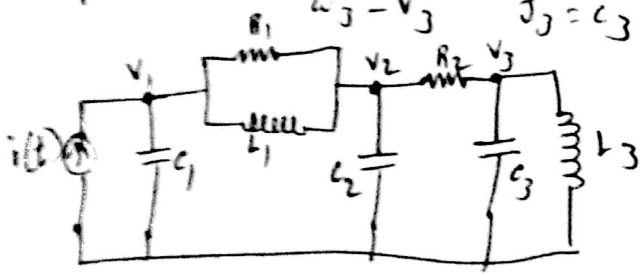
(7), (8), (9) are similar to (4) (5) (6).

Torque current analogous ckt :- nodes - 3

$T \rightarrow i(t)$ $\quad \omega_1 - V_1$ $\quad J_1 = C_1$ $\quad B_1 = \frac{1}{R_1}$ $\quad K_1 - \frac{1}{L_1}$

$\omega_2 - V_2$ $\quad J_2 = C_2$ $\quad B_2 = \frac{1}{R_2}$ $\quad K_3 - \frac{1}{L_3}$

$\omega_3 - V_3$ $\quad J_3 = C_3$



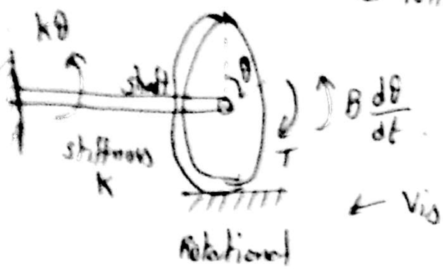
using KCL

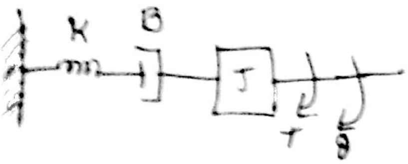
$$C_1 \frac{dV_1}{dt} + \frac{1}{R_1}(V_1 - V_2) + \frac{1}{L_1}\int (V_1 - V_2)\, dt = i(t) \quad \text{--- (10)}$$

$$C_2 \frac{dV_2}{dt} + \frac{1}{R_1}(V_2 - V_1) + \frac{1}{L_1}\int (V_2 - V_1)\, dt + \frac{1}{R_2}(V_2 - V_3) = 0 \quad \text{--- (11)}$$

$$C_3 \frac{dV_3}{dt} + \frac{1}{R_2}(V_3 - V_2) + \frac{1}{L_3}\int V_3\, dt = 0 \quad \text{--- (12)}$$

(10), (11), (12) are similar to (4), (5), (6) Derive the transfer function for the following rotational mech. system shown in fig.





applied torque = T

opposing torques : T_K, T_b, T_j

θ = angular displacement

According to Newton's second law $T = T_K + T_b + T_j$

$$T_K = K\theta \quad , \quad T_b = B\frac{d\theta}{dt} \quad , \quad T_j = J\frac{d^2\theta}{dt^2}$$

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = T$$

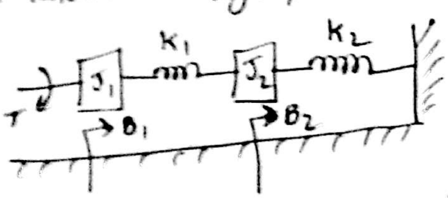
using Laplace Transform

$$Js^2\theta(s) + Bs\theta(s) + K\theta(s) = T(s)$$

$$\theta(s)\left[s^2J + Bs + K\right] = T(s)$$

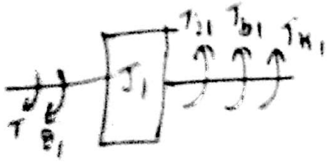
$$\text{Transfer function} = \frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}$$

(2) Write the differential equations governing the mech. rotational system as shown in fig & transfer function of system.



Ans:- 2 nodes (moment of inertia of masses)

The free body diagram of J_1 , Let angular displacements of J_1 & J_2 be θ_1 & θ_2 respectively.



The opposing torques are T_{j1}, T_{b1}, T_{K1}

$$T_{j1} = J_1\frac{d^2\theta_1}{dt^2} \quad , T_{b1} = B_1\frac{d\theta_1}{dt} \quad , T_{K1} = K_1(\theta_1 - \theta_2)$$

According Newton's II law $J_1\frac{d^2\theta_1}{dt^2} + B_1\frac{d\theta_1}{dt} + K_1(\theta_1 - \theta_2) = T$ —— (1)

The free body diagram of J_2 ,

The opposing forces are $T_{j2}, T_{k2}, T_{k1}, T_{b2}$

$$T_{j2} = J_2 \frac{d^2\theta_2}{dt^2}, \quad T_{k2} = K_2\theta_2, \quad T_{k1} = K_1(\theta_2 - \theta_1)$$

$$T_{b2} = B_2 \frac{d\theta_2}{dt}.$$

By Newton's second law, $J_2 \frac{d^2\theta_2}{dt^2} + K_1(\theta_2 - \theta_1) + K_2\theta_2 + B_2\frac{d\theta_2}{dt} = 0$ — (2)

(1) & (2) are differential equation of the given system.

Apply laplace transform to (1) & (2)

$$J_1 s^2\theta_1(s) + B_1 s\theta_1(s) + K_1[\theta_1(s) - \theta_2(s)] = T(s) \quad \text{— (3)}$$

$$J_2 s^2\theta_2(s) + B_2 s\theta_2(s) + K_1[\theta_2(s) - \theta_1(s)] + K_2\theta_2(s) = 0 \quad \text{— (4)}$$

from (3)

$$\theta_1(s)\left[J_1 s^2 + B_1 s + K_1\right] - K_1\theta_2(s) = T(s) \quad \text{— (5)}$$

from (4)

$$\theta_2(s)\left[J_2 s^2 + B_2 s + K_1 + K_2\right] - K_1\theta_1(s) = 0$$

$$\theta_2(s) = \frac{K_1\theta_1(s)}{J_2 s^2 + B_2 s + K_1 + K_2}$$

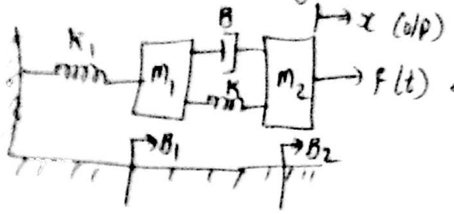
Substitute $\theta_2(s)$ in (5)

$$\theta_1(s)\left[J_1 s^2 + B_1 s + K_1\right] - K_1 \cancel{\theta_1(s)} \left[\frac{K_1\theta_1(s)}{J_2 s^2 + B_2 s + K_1 + K_2}\right] = T(s)$$

$$\theta_1(s)\left[\frac{(J_1 s^2 + B_1 s + K_1)(J_2 s^2 + B_2 s + K_1 + K_2) - K_1K_1}{J_2 s^2 + B_2 s + K_1 + K_2}\right] = T(s)$$

Transfer function: $$\frac{\theta_1(s)}{T(s)} = \frac{J_2 s^2 + B_2 s + K_1 + K_2}{(J_1 s^2 + B_1 s + K_1)(J_2 s^2 + B_2 s + K_1 + K_2) - K_1K_2}$$

obtain the T F of mech. system as shown in fig:



As: mass - 2, nodes) , m_1, m_2 — displacement x_1, x.

Free body diagram m_1

opposing forces are, $f_{m_1}, f_{k_1}, f_{b_1}, f_K, f_b$.

$$m_1 \frac{d^2 x_1}{dt^2} + K_1 x_1 + B_1 \frac{dx_1}{dt} + K(x_1 - x) + B\frac{d}{dt}(x_1 - x) = 0.$$

Laplace Transform.

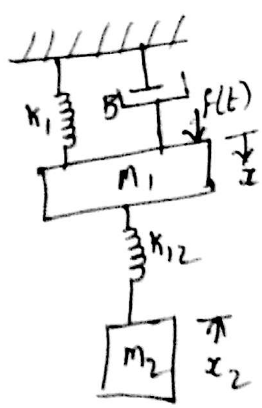
$$m_1 s^2 X_1(s) + K_1 X_1(s) + B_1 s X_1(s) + K\left[X_1(s) - X(s)\right] + B s\left[X_1(s) - X(s)\right] = 0$$

already solved

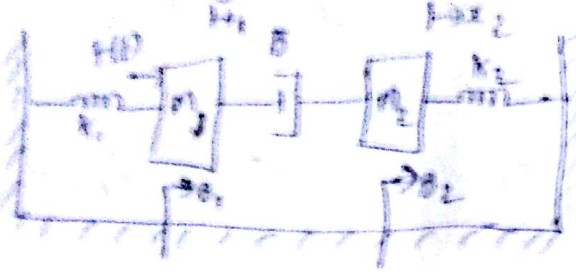
$\cancel{X_1(s)\left[m_1 s^2 + (B_1 + B) s\right.}$

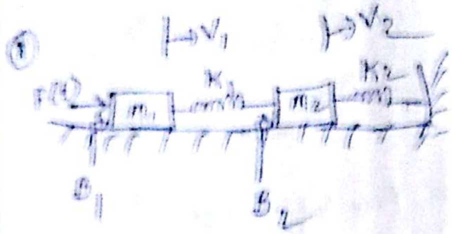
(5) obtain the T.F of mech. system as shown in fig.

(4) Write the differential equations governing the behaviour of the mechanical system shown in fig. Also obtain an analogous electrical ckt base on force current analogy.



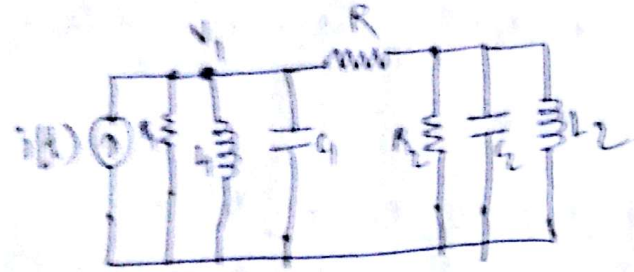
5) Write the differential equations for mech. systems as shown in fig. Also obtain an analogous electrical ckt based on force-current analogy.



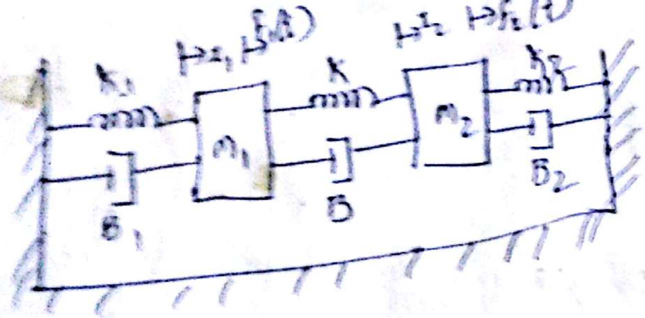


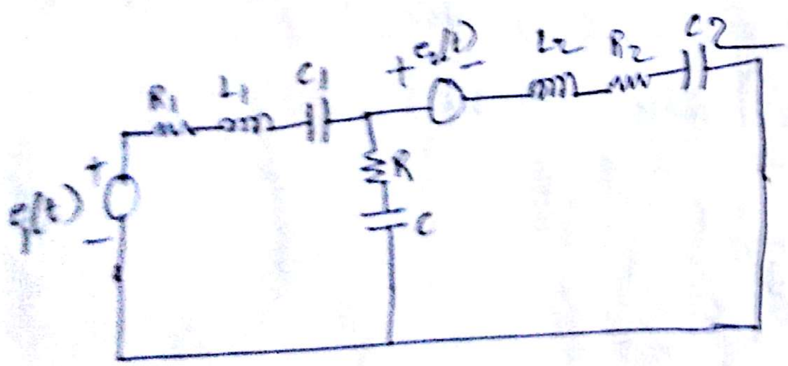
F–V, & F–I

Ans. –

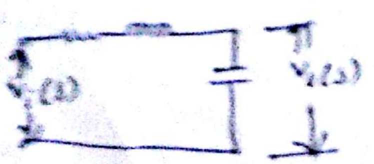


6) Write the differential equation & F–V & F–I ckt.





7)



$$T.F = \frac{1}{s^2LC + RSC + 1}$$

Transfer Function Representation:

Servo Mechanism:— The control systems which are used to control the position of body (or) object is known as "servo mechanism"

Servo motor :- These are the motors which are used in ~~automatic~~ automatic control systems.

- These are very small in size, and produce rapid acceleration.
- Based on the electric supply given to the motor, these are classified into two types.

(1) DC servo motor

(2) AC servo motor.

Control techniques of DC servomotor:

(1) Flux control method

(2) Armature voltage control method.

Transfer function of Field controlled DC Motor:—

{Transfer Function of DC servo motor for flux control method}—

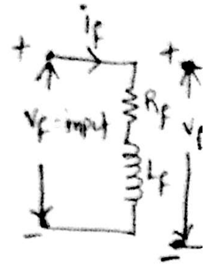
It is a DC shunt motor designed to satisfy the requirement of a servomotor. In this motor, the armature is supplied with a constant current or voltage. When the armature voltage is constant, the torque is directly proportional to the field flux. Since the the field current is proportional to flux, the torque of the motor is controlled by controlling the field current. Reversible operation is possible by reversing the field current.

Difference b/w AC Servomotor & DC Servomotor

DC Servomotor	AC Servomotor
1. High power o/P	(1) Relatively lower o/p power o/p than a DC servomotor of same size
2. characteristics are linear	(2) characteristics are non-linear
3. Fast response due to low electrical & mechanical time constant	(3) The response is relatively let, slower than DC servomotor due to higher values of time constants.
4. Suitable for large power applications	(4) Suitable for low power applications.
5. ~~low~~ higher cost	(5) ~~higher~~ lower cost compare with DC servomotor
(6) maintaince is high.	(6) less maintaince since there is no commutator & brush
7. used in robotics & machine tools.	(7) used in computer related equipment e.g Disk drives, Tape drives printers.

E_b =

$N \propto$



Field contro

V_f is I

$R_f \rightarrow$ Re

$L_f \rightarrow$ I

$i_f \rightarrow$

$V_a \rightarrow$

$R_a \rightarrow$

$L_a \rightarrow$

$E_b \rightarrow$

$i_a \rightarrow$

T_m —

J —

B —

In this me
s kept cons

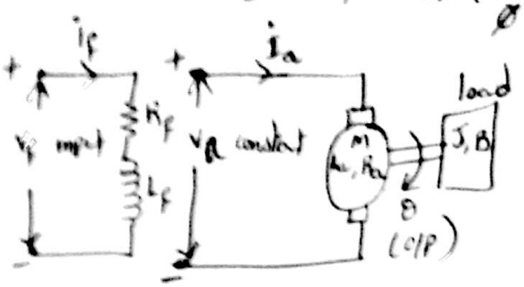
Flux
to field cur

Torque dev

Product of

$E_b = \frac{\phi Z N P}{60 A}$, $E_b = V - I_a R_a$

$N \propto E_b$ (or) V , $N \propto \frac{1}{\phi}$



Field controlled DC motor

Let
V_f is the supply voltage of field winding
$R_f \rightarrow$ Resistance of field winding
$L_f \rightarrow$ Inductance " "
$i_f \rightarrow$ Field current
$V_a \rightarrow$ Armature voltage
$R_a \rightarrow$ Resistance of armature
$L_a \rightarrow$ Inductance of Armature
$E_b \rightarrow$ Back emf
$i_a \rightarrow$ current flowing through the armature
$T_m \rightarrow$ Torque developed by the motor
$J \rightarrow$ Moment of inertia of motor
$B \rightarrow$ Viscous friction offered by motor

In this method of control technique armature current (i_a) is kept constant and field current (i_f) is variable.

Flux produce by field winding is directly proportional to field current

$$\phi \propto i_f \Rightarrow \phi = K_f i_f$$

Torque developed by a motor is directly proportional to the Product of flux and armature current

$$T_m \propto \phi I_a$$

$$T_m = K_1 \phi\, i_a$$

$$T_m = K_1 K_f\, i_f\, i_a$$

$$T_m = K_T\, i_f \qquad \text{where } K_T = K_1 K_f\, i_a \text{ (constant)}$$

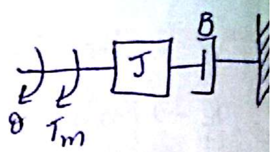
K_T = motor torque constant.

Apply the laplace transform

$$T_m(s) = K_T\, I_f(s) \quad \text{——— (1)}$$

From the rotor circuit, Torque developed by the motor $T_m = T_J +$

$$T_m = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}$$



Apply laplace transform

$$T_m(s) = Js^2\theta(s) + Bs\,\theta(s)$$

$$T_m(s) = \left[Js^2 + Bs\right]\theta(s) \quad \text{——— (2)}$$

Apply the KVL to the field circuit

$$V_f = R_f\, i_f + L_f\frac{di_f}{dt}$$

Apply laplace transform

$$V_f(s) = R_f\, I_f(s) + sL_f\, I_f(s)$$

$$V_f(s) = \left[R_f + sL_f\right] I_f(s)$$

$$I_f(s) = \frac{V_f(s)}{R_f + sL_f} \quad \text{——— (3)}$$

From the equations (1) & (2) (equating)

$$K_T\, I_f(s) = \left[Js^2 + Bs\right]\theta(s)$$

From (3)

$$K_T\left[\frac{V_f(s)}{R_f + sL_f}\right] = \left[Js^2 + Bs\right]\theta(s)$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_T}{s\left[Js + B\right]\left[R_f + sL_f\right]}$$

Further it can be simplified by taking R_f and B common

$$\frac{\theta(s)}{V_f(s)} = \frac{K_T}{sB\left[1+\frac{sJ}{B}\right]R_f\left[1+\frac{sL_f}{R_f}\right]}$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_T/BR_f}{sB\left[1+\frac{sJ}{B}\right]\left[1+\frac{sL_f}{R_f}\right]}$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s\left[1+sT_{m1}\right]\left[1+sT_{m2}\right]}$$

where $K_m = \frac{K_T}{R_f B}$ (motor gain constant)

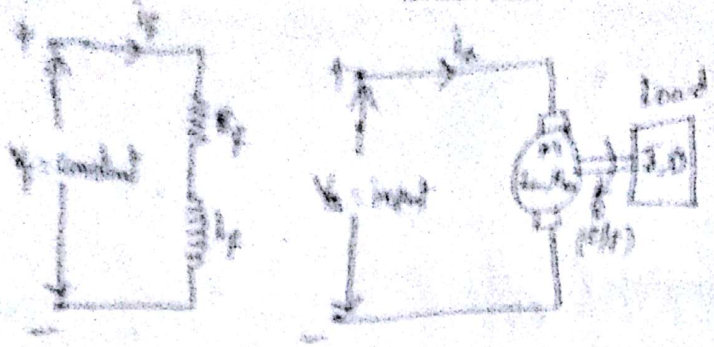
$T_{m1} = \frac{J}{B}$ = mechanical time constant

$T_{m2} = \frac{L_f}{R_f}$ = field time constant.

<u>Transfer function of DC servo motor for armature voltage</u>

<u>control method</u>

It is a DC shunt motor designed to satisfy the requirement of a servomotor. The field is excited by a constant DC supply. If the field current is constant then the speed is directly proportional to armature voltage and torque is directly proportional to armature current. Hence the torque and speed can be controlled by armature voltage. Reversible rotation is possible by reversing the armature voltage.



Armature controlled DC motor

Let, R_a = Armature resistance, Ω

L_a = " inductance, H

i_a = " current, A

V_a = " voltage, V

E_b = Back emf, V

K_T = Torque constant, N-m/A

T_m = Torque developed by motor, N-m

J = Moment of inertia of motor, kg-m²/rad

B = Frictional coefficient of motor, N-m/(rad/sec)

K_b = Back emf constant, (V/(rad/sec))

The equivalent ckt of armature is shown in fig.

By KVL,

$$i_a R_a + L_a \frac{di_a}{dt} + E_b = V_a$$

Apply Laplace transform

$$I_a(s) R_a + L_a s I_a(s) + E_b(s) = V_a(s) \quad (1) \quad (R_a + sL_a) I_a(s) + E_b($$

~~$I_a(s) = [V_a(s) - E_b(s)] / [R_a + sL_a]$~~

Torque developed by the motor is proportional to product of flux and armature current

$$T_m \propto \phi I_a \quad , \quad \phi \propto i_f \Rightarrow \phi = K_f i_f$$

$$T_m = K_1 \phi I_a$$

$$T_m = K_1 K_f i_f i_a$$

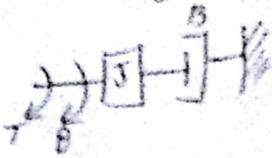
$$T_m = K_T i_a \qquad \text{where } K_1 K_f i_f = K_T \text{ (constant)}$$

K_T = motor torque constant

Apply laplace transform

$T_m(s) = K_T I_a(s)$ —— (2)

The differential equation governing the mechanical system of motor is



$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T_m$$

Apply laplace transform

$$Js^2\theta(s) + Bs\,\theta(s) = T(s)$$

$$\theta(s)\left[Js^2+Bs\right] = T_m(s) \quad \text{—— (3)}$$

From (2) & (3) $\quad K_T I_a(s) = \theta(s)\left[Js^2+Bs\right]$ —— (5)

$I_a(s) = \theta(s)[Js^2+Bs]/K_T$ —— (5)

The back emf of DC motor is proportional to speed (angular velocity) of shaft.

$$\therefore e_b \propto \frac{d\theta}{dt} \Rightarrow e_b = K_b\frac{d\theta}{dt}$$

Apply laplace transform

$$E_b(s) = K_b s\theta(s) \quad \text{—— (4)}$$

From (1) & (4) $\quad I_a(s) = \dfrac{V_a(s) - K_b s\theta(s)}{R_a+sL_a}$ —— (6)

From (5) & (6) $\quad K_T\dfrac{\left[V_a(s) - K_b s\theta(s)\right]}{\left[R_a+sL_a\right]} = \left[Js^2+Bs\right]\theta(s)$

From (5) & (1) & (4) $\quad (R_a+sL_a)\dfrac{(Js^2+Bs)}{K_T}\theta(s) + K_b s\theta(s) = V_a(s)$

$$\left[\frac{(R_a+sL_a)(Js^2+Bs) + K_bK_Ts}{K_T}\right]\theta(s) = V_a(s)$$

$$\frac{\theta(s)}{V_a(s)} = \frac{K_T}{(R_a+sL_a)(Js^2+Bs)+K_bK_Ts} = \frac{K_T}{s[(R_a+sL_a)(Js+B)+K_bK_T]}$$

Further it can be simplified by taking R_a & B common

$$\frac{\theta(s)}{V_a(s)} = \frac{K_T}{R_a\left(1+\frac{sL_a}{R_a}\right)Bs\left(1+\frac{Js}{B}\right)+K_bK_Ts}$$

$$= \frac{K_T/R_aB}{s\left[(1+s\tau_a)(1+s\tau_{m1})+\frac{K_bK_T}{R_aB}\right]}$$

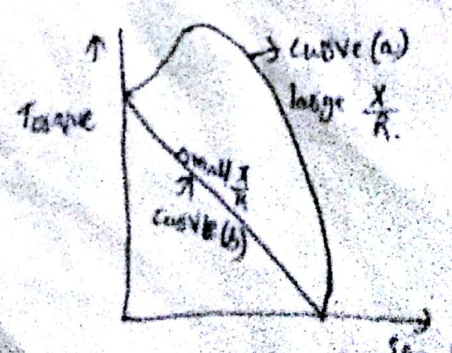
where, $\frac{L_a}{R_a} = \tau_a$ = Electrical time constant.

$\frac{J}{B} = \tau_{m1}$ = mechanical time constant.

AC Servo motor :-

An ac servomotor is basically a two-phase induction motor except for certain special design features. It works on the principle of induction motor and it exhibits non-linear characteristics because of having large $\frac{X}{R}$ ratio.

In order to get the linear characteristics and in[illegible] the speed of the motor, we must use smaller diameter of [illegible] and lower X/R ratio



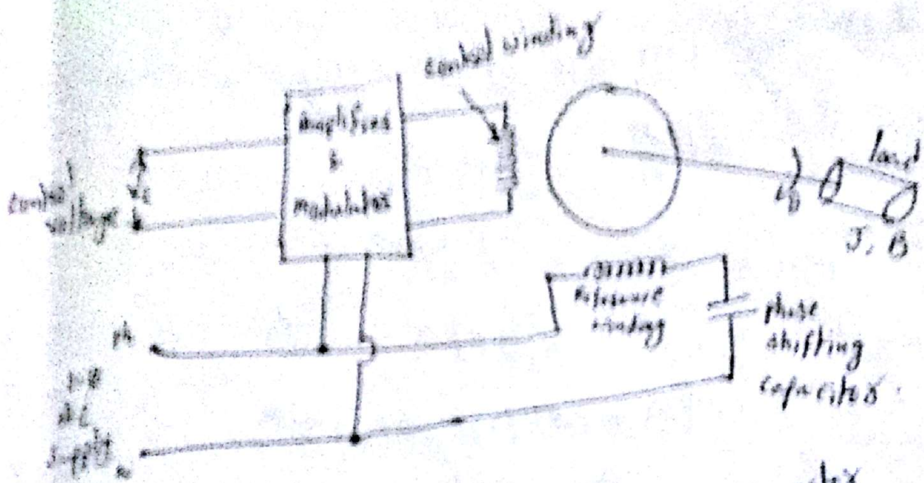
curve (a) – Normal Induction [illegible]

curve (b) – AC Servo motor

Speed - torque characteristics of induction motor & ac servo m[illegible]

Transfer function of AC servo motor :-

The symbolic representation of an ac servomotor as a control system component is shown in fig.



Symbolic representation of an ac servo motor.

Constructionally it is a single phase AC servomotor and operationally it act as a two phase induction motor.

It has two windings namely (1) control winding
(2) Reference winding.

and these are displaced by 90° electrically.

Let T_m = Torque developed by servomotor

θ = Angular displacement of rotor

$\omega = \frac{d\theta}{dt}$: Angular speed

~~T_L = Torque required by the load~~

J = Moment of inertia of load and the rotor

B = Viscous-frictional coefficient of load and the motor

K_1 = slop of control phase voltage ~~Versus~~ Torque characteristics

K_2 = slop of speed-torque characteristics.

Torque developed by AC servomotor is proportional to difference of control voltage and back emf

$$T_m = K_1 V_c - K_2 \frac{d\theta}{dt} \quad \text{——} \ (1)$$

From the shaft, torque developed by AC servomotor equal to

$$T_m = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \quad \text{——} \ (2)$$

Taking laplace transform to (1) & (2)

$$T_m(s) = K_1 V_c(s) - K_2 s\theta(s) \quad \text{——} \ (3)$$

$$T_m(s) = Js^2(\theta) + Bs\theta(s) = \theta(s)\left[Js^2 + Bs\right] - (4)$$

From (3) & (4)

$$K_1 V_c(s) - K_2 s(\theta(s)) = \theta(s)\left[Js^2 + Bs\right]$$

$$K_1 V_c(s) = \theta(s)\left[Js^2 + Bs + K_2 s\right]$$

$$\therefore \frac{\theta(s)}{V_c(s)} = \frac{K_1}{s\left[Js + B + K_2\right]}$$

$$= \frac{K_1/(B+K_2)}{s\left[s\frac{J}{B+K_2} + 1\right]} = \frac{K_m}{s(1 + s\tau_m)}$$

where $K_m = \frac{K_1}{B+K_2}$ = motor gain constant

τ_m = motor time constant

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Speed-Torque characteristics of AC servomotor :-

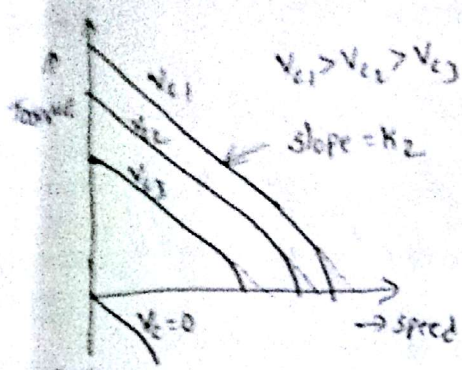


Fig (a) Speed-torque curves of an ac servo motor

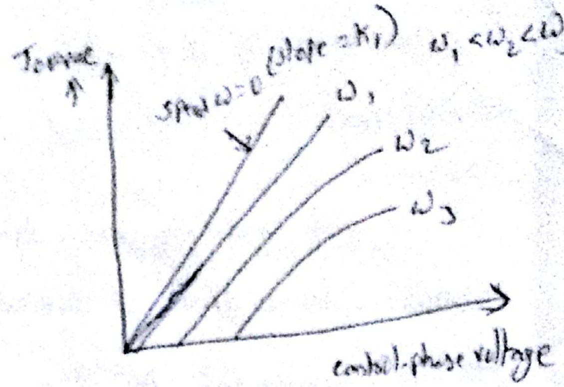


Fig (b) control voltage Vs Torques curves of an ac servo motor.

The speed-torque curves of a typical ac servomotor plotted for fixed reference phase voltage and different values of control voltages as shown in fig (a). When the control phase voltage becomes zero, the motor develop a decelerating torque and so the motor stops. The curves show a large torque at zero speed.

The speed-torque curves of ac servomotor are nonlinear except in the low speed region. For constant speed, except near zero speed, the torque does not vary linearly with respect to input voltage V_c, as shown in fig (b).

Synchro transmitter and Receiver :-

Synchro is a 1-ϕ AC device and also having pair of devices. Such as 1. Synchro Transmitter
2. Synchro Receiver

Synchro transmitter produces 3-ϕ voltages by giving excitation to the rotor of synchro transmitter

Synchro Receiver produces the displacement (angular) proportional to the 3-ϕ voltages which are developed by Syn transmitter.

Synchro act as error detector, it is formed by interconn of synchro transmitter and synchro receiver (Synchro control transfor In this arrangement, the stator leads of the transmitter are directly connected to the stator leads of control transformer. The an position of the transmitter-rotor is the reference input and the rotor is excited by a AC supply.

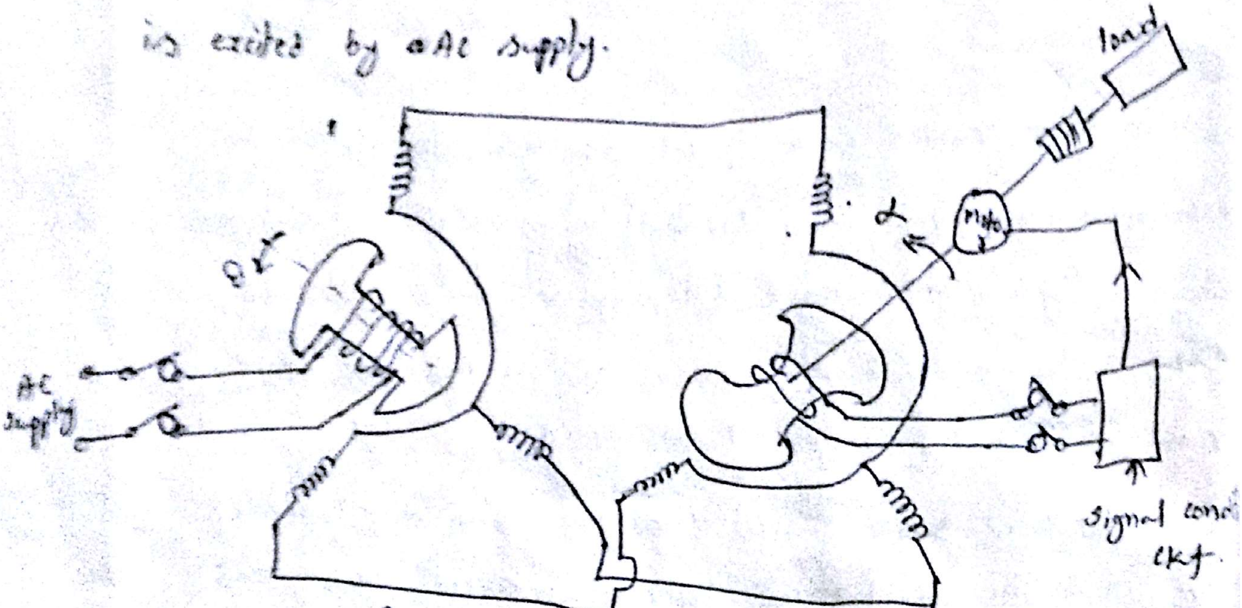


Fig: Servo system using synchro error detector.

The control transformer rotor is connected to a servomoto and to the shaft of the load, whose position is desired output. The induced emf is available across the rotor slip rings of contro transformer is measured by a signal conditioning ckt. The o/p of signal conditioning ckt is used to drive motor so that desired load position is achieved. The simple schematic diagram of syn as error detector as shown in fig.

Initially the shafts of transmitter and control transfo are assumed to be in aligned position. In this position the

Transmitter rotor will be in electrical zero position. and the control transformer will be in null position and the angular separation of both rotor axis in aligned position is $90°$. The null position of a control transformer in a servo system is defined as position of its rotor for which the o/p voltage on the rotor winding is zero with the transmitter in its electrical zero position.

Let the rotor of the transmitter rotate through an angle θ from its electrical zero position. Now the rotor of the control transformer will be rotate in the same direction through an angle α from its null position. The net angular separation of the two rotors is equal to $(90-\theta+\alpha)$ and the voltage induced in the control transformer rotor is proportional to the cosine of this angle. The error voltage is amplified and used to drive a servomotor. The motor drives the shaft of the synchro control transformer until it comes to a new aligned position at which the error voltage is zero.

∴ Voltage across slip rings of control transformer $e_m(t) = KV_r \cos(90-\theta+\alpha)\sin\omega t$

$(\because 90-(\theta-\alpha))$

$= KV_r \sin(\theta-\alpha)\sin\omega t$

For small angular displacement b/w two rotor positions,

$e_m(t) = KV_r(\theta-\alpha)\sin\omega t$. let $\phi(t) = \theta-\alpha$

$e_m(t) = KV_r\,\phi(t)\sin\omega t$ —— ① (K proportional constant)

From ①, the o/p voltage of synchro error detector is a modulated signal with carrier frequency ω (which is same as supply frequency of the transmitter rotor). The magnitude of the modulated carrier wave is proportional to $\phi(t)$ and the phase reversal of the modulated wave depend on the sign of ϕ. the signal conditioning ckt demodulates the voltage available across slip rings and develops a demodulated and amplified error voltage to drive the motor.

The demodulated error voltage, $e = K_s \phi(t)$ —— (2)

where K_s : Sensitivity of synchro error detector in V/

on taking laplace transform of (2)

$E(s) = K_s \phi(s)$ $\quad\therefore\quad \frac{E(s)}{\phi(s)} = K_s$ —— (3)

eq (3) is the transfer function of the synchro error detector.

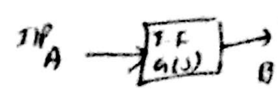
~~Components of Automatic control system~~ :-

Block diagram algebra :-

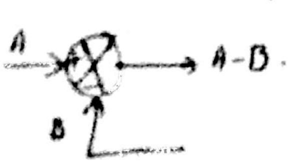
A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals.

The elements of a block diagram are block, branch point and summing point

- Block is symbol for the mathematical operation on the I/P signal to the block that produces the o/p.

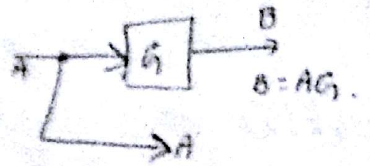


- Summing points are used to add two or more signals in the system



A circle with a cross is the symbol that indicates a summing operation.

- Branch point is a point from which the signal from a block goes concurrently to other blocks or summing points.



Constructing block diagram for control systems:-

A control system can be represented diagramatically by block diagram. The differential equations governing the system are used to construct the block diagram. By taking Laplace transform the differential equations are converted to algebraic equations. The system I/P & O/P variables are identified and the block diagram for each equation can be drawn. Each equation gives one section of block diagram. The O/P of one section will be I/P of another section. The various sections are interconnected to obtain the overall block diagram of the system.

Construct the block diagram of field controlled dc motor.

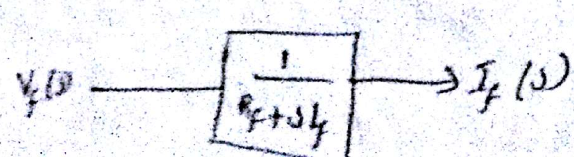
differential equations $V_f = i_f R_f + L_f \frac{di_f}{dt}$ —— ①

$T = K_T i_f$ —— ②

$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}$ —— ③

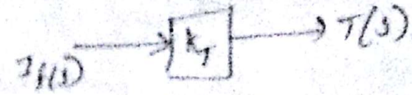
From ①, taking L.T $V_f(s) = R_f I_f(s) + L_f s I_f(s)$

$$I_f(s) = \frac{1}{R_f + sL_f} V_f(s)$$



From ① take L.P.T

$T(s) = K_T\, I_f(s)$



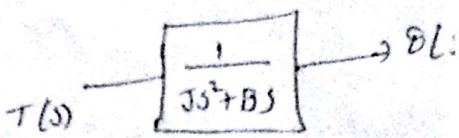
fig(2)

From ③ take L.P.T

$T(s) = Js^2\theta(s) + Bs\,\theta(s)$

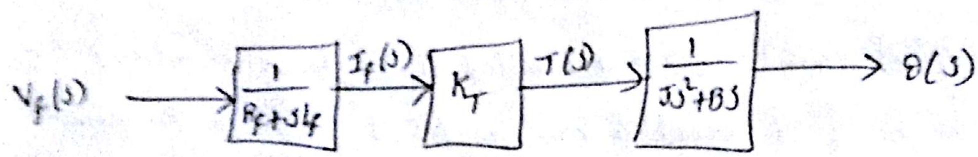
$T(s) = \theta(s)\left[Js^2 + Bs\right]$

$\theta(s) = \dfrac{T(s)}{(Js^2+Bs)}$



fig(3).

from fig ①, ② & ③



block diagram of field controlled dc motor.

it is open loop c.s.

② construct the block diagram of armature controlled d.c motor

Ans:- differential equations:

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \quad \text{—①}$$

$$T = K_T\, i_a \quad \text{—②}$$

$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \quad \text{—③}$$

$$e_b = K_b \frac{d\theta}{dt}. \quad \text{—④}$$

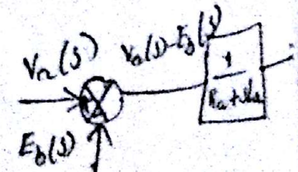
$$\frac{d\theta}{dt} = \omega$$

Taking L.P.T to ①

$V_a(s) = R_a I_a(s) + L_a s I_a(s) + E_b(s)$

$V_a(s) - E_b(s) = I_a(s)\left[R_a + sL_a\right]$

$I_a(s) = \dfrac{V_a(s) - E_b(s)}{R_a + sL_a}$



fig(1)

Take L.T to ②

$T(s) = K_T\, I_a(s)$

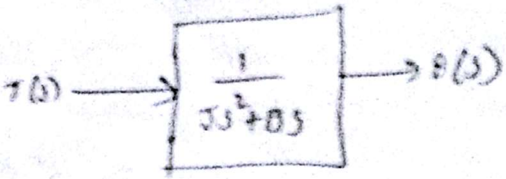
fig(2)

Take L.T to ③

$T(s) = Js^2\theta(s) + Bs\theta(s)$

$T(s) = (Js^2 + Bs)\theta(s)$

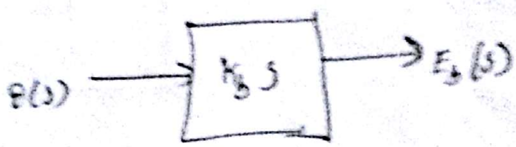
$$\theta(s) = \frac{T(s)}{Js^2 + Bs}$$



fig(3)

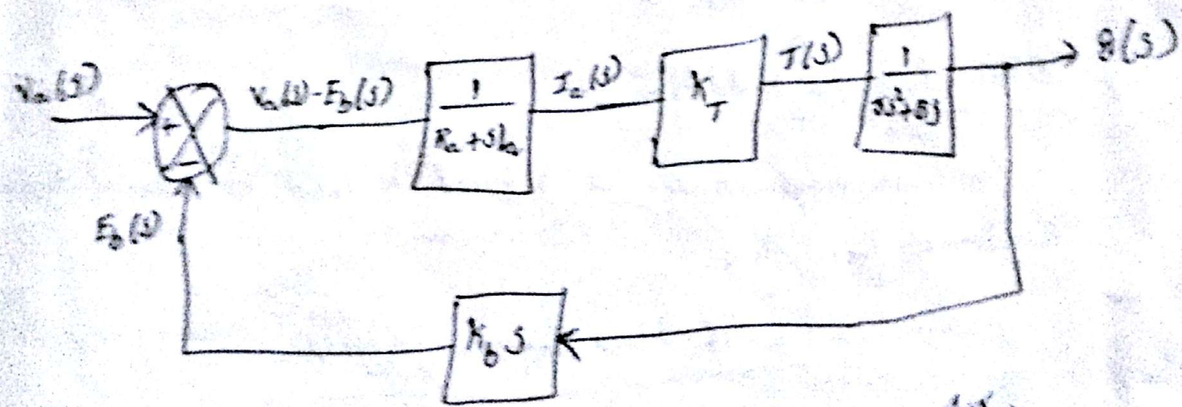
Take L.T to ④

$E_b(s) = K_b\, s\,\theta(s)$



fig(4).

From ① to ④



Block diagram of Armature controlled DC motor.

It is closed loop C.S.

Construct the block diagram of AC servo motor.

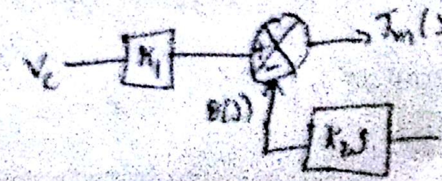
differential equations

$T_m = K_1 V_c - K_2 \frac{d\theta}{dt}$ — ①

$T_m = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}$ — ②

Take L.T ①

$T_m(s) = K_1 V_c(s) - K_2 s\theta(s)$



fig(1)

Taking L.T in ③

$$T_m(s) = Js^2\theta(s) + Bs\theta(s)$$

$$T_m(s) = \theta(s)\left[Js^2 + Bs\right]$$

$$\theta(s) = \frac{T_m(s)}{(Js^2 + Bs)}$$

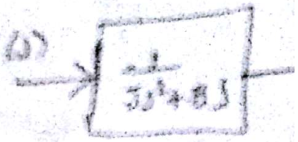
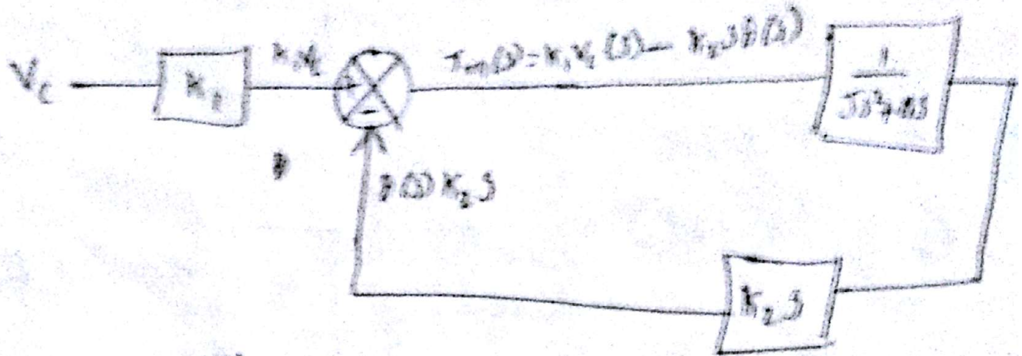


Fig (2)

From ① & ②



Block diagram of AC servo motor.

It is closed loop C.S.

Block diagram Reduction :-

The block diagram can be reduced to find the overall transfer function of the system.

Rules of block diagram algebra

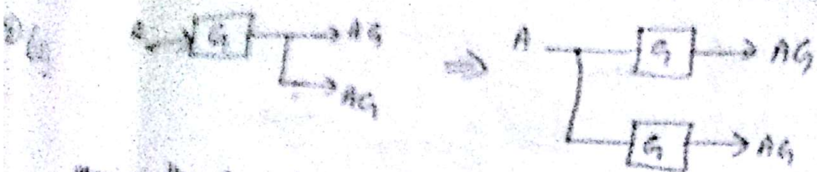
Combining the blocks in cascade

Combining parallel blocks

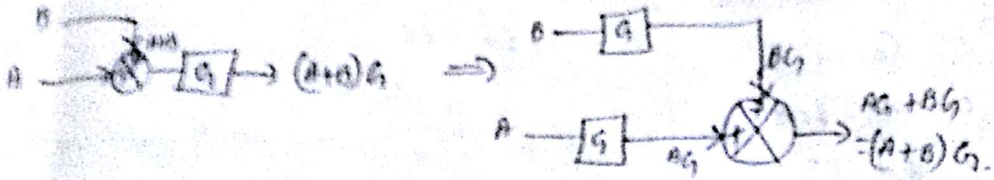
Moving the branch point ahead of the block.

Scanned by CamScanner

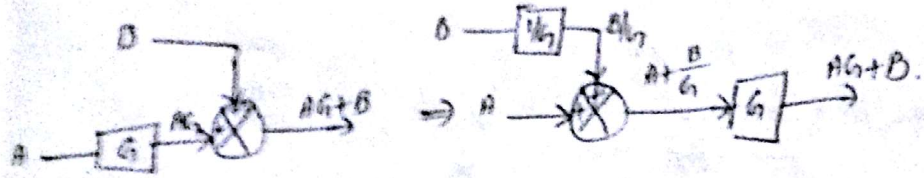
moving the branchpoint before the block



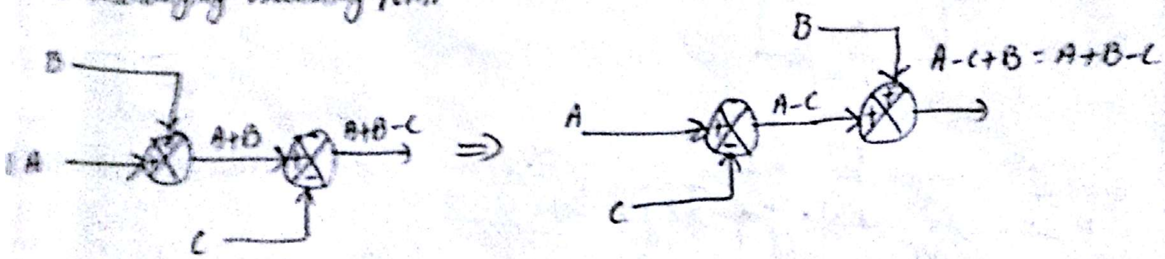
moving the summing point ahead of the block.



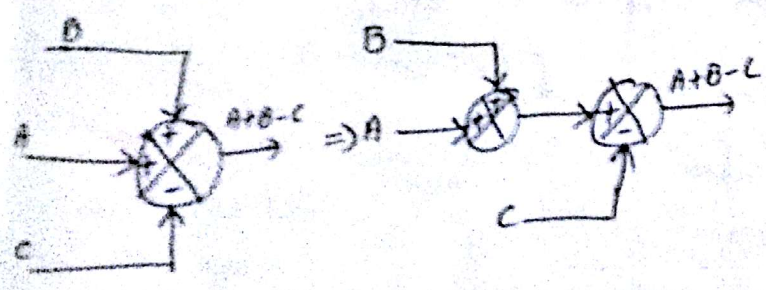
Moving the summing point before the block.



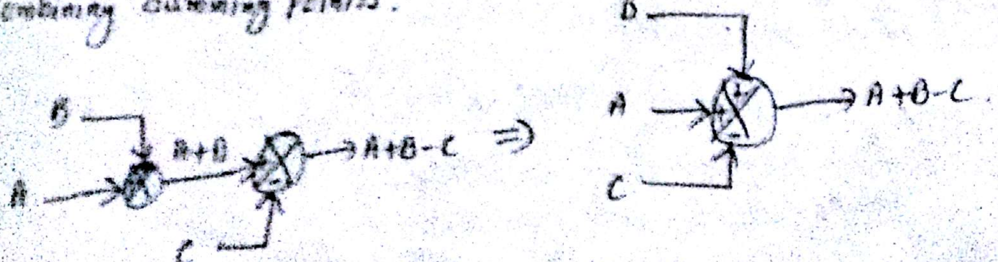
Interchanging summing point



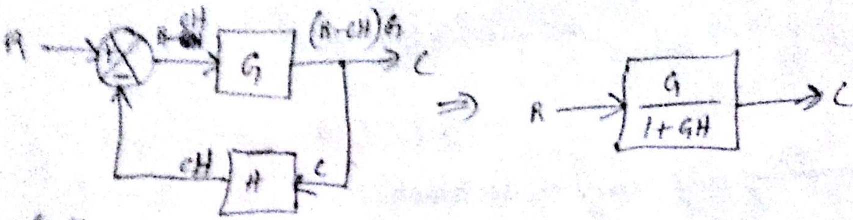
Splitting summing points:



Combining summing points:



Elimination of feedback:



Proof:

$c = (R - cH)G$

$c = RG - GcH$

$c + GcH = RG$

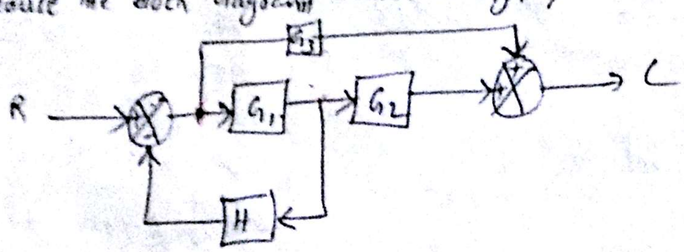
$c(1 + GH) = RG$

$\frac{c}{R} = \frac{G}{1+GH}$ ⟹ −ve feedback.

$\frac{c}{R} = \frac{G}{1-GH}$ ⟹ +ve feedback.

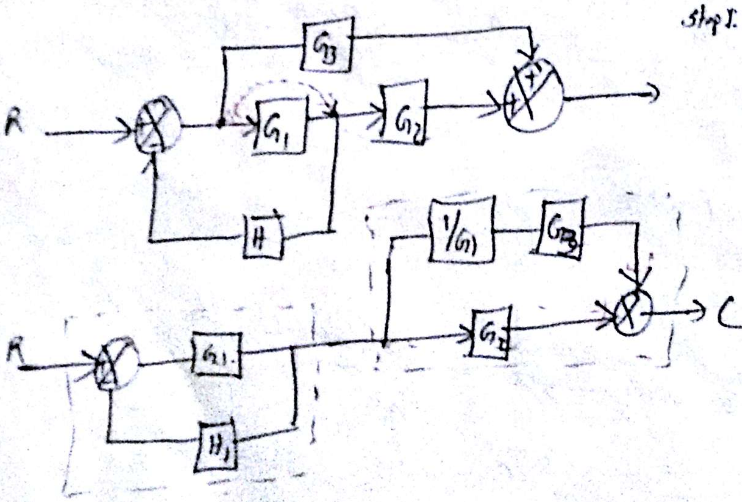
Problems

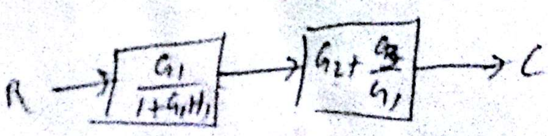
(1) Reduce the block diagram shown in fig. & find C/R.



Ans:

step I: move the branch point after the





$$\frac{C}{R} = \left(\frac{G_1}{1+G_1H_1}\right)\left(G_2 + \frac{G_3}{G_1}\right) = \left(\frac{G_1}{1+G_1H_1}\right)\left(\frac{G_1G_2+G_3}{G_1}\right) = \frac{G_1G_2+G_3}{1+G_1H}$$

$$\text{overall T.F} = \frac{C}{R} = \frac{G_1G_2+G_3}{1+G_1H}$$

Signal flow gr

The signal

graphically and

The time

system can be

The SFG of the

The

gain formula the

This method is

In SI

f signal flow

gain is indicate

Terms used

Node: A node

Branch: A bra

I/P node (sou

O/P node (sin

mixed node

Path

open pa

closed pa

Forward

Forward P

Signal flow graph: (SFG)

The signal flow graph is used to represent the control system graphically and it was developed by S.J. Mason.

The time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. the SFG of the system can be constructed using these equations.

The advantage in SFG method is that, using Mason's gain formula the overall gain of the system can be computed easily. This method is simpler than the block diagram reduction method.

In SFG, the signal flows in only one direction. The direction of signal flow is indicated by an arrow placed on the branch and gain is indicated along the branch.

Terms used in SFG:-

Node: A node is a point representing a variable or signal.

Branch: A branch is directed line segment joining two nodes.

I/P node (source): It is a node that has only outgoing branches.

O/P node (sink): It is a node that has only incoming branches.

Mixed node : It is a node that has both incoming & outgoing branches.

Path : A path is a traversal of connected branches in the direction of the branch arrows.

Open path : A open path starts at a node and ends at another node.

closed path : closed path starts & ends at same node.

Forward path : It is a path from an input node to an output node that does not cross any node more than once.

Forward path gain : It is the product of the branch gains of a forward path.

Individual loop: It is a closed path starting from a node and after passing through a certain part of a graph arrives at same node without crossing any node more than once.

Loop gain: It is the product of the branch gains of a loop.

Non-touching loops: If the loops does not have a common node, they are said to be non-touching loops.

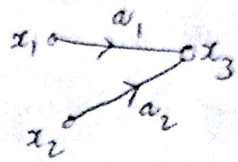
~~Signal flow graph alge~~

SFG algebra:

SFG for a system can be reduced to obtain the transfer f. of the system using the following rules.

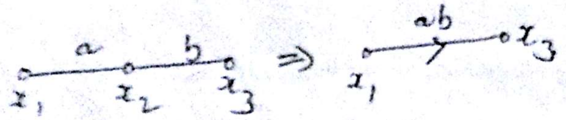
(1)

$x_2 = ax_1$

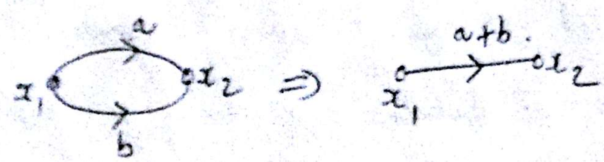


$x_3 = a_1x_1 + a_2x_2$

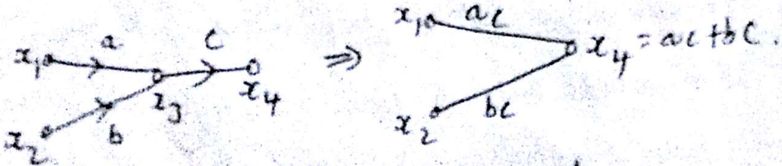
(2)



(3)

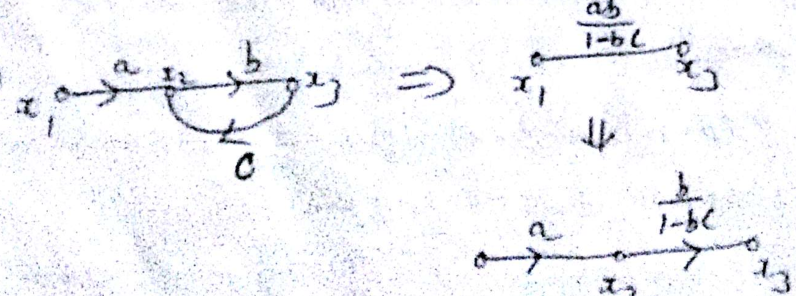


(4)



$x_4 = ac + bc$.

(5)



SFG Reduction:

The SFG of a system can be reduced either by using the rules of a SFG algebra or by using Mason's gain formula.

S. J Mason has developed a simple procedure to determine the transfer function of the system represented as a SFG.

Mason's gain formula:—

Let $R(s)$ = I/P of system
$C(s)$ = O/P " "

$$T(s) = \frac{C(s)}{R(s)}$$

Mason's gain formula states the overall gain of the system as follows,

$$\text{overall gain, } T = \frac{1}{\Delta} \sum_K P_K \Delta_K .$$

where $T = T(s)$ = Transfer function of the system

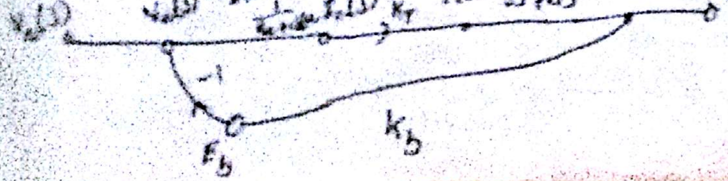
P_K = Forward path gain of K^{th} forward path.

K = Number of forward paths in SFG.

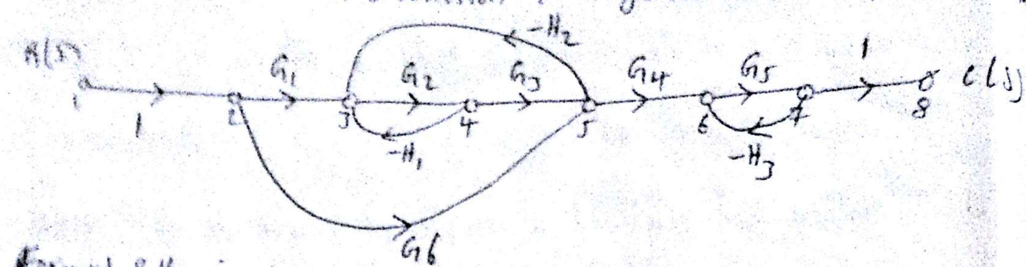
Δ = 1 − (Sum of individual loop gains)
+ (Sum of gain products of all possible combination of two non-touching loops)
− (Sum of gain products of all possible combination of three non-touching loops)
+ - - - -

Δ_K = Δ for that part of the graph which is not touching K^{th} forward path.

Draw SFG to armature voltage controlled DC servo motor



(3) Find the overall transfer function of the system whose SFG is shown.

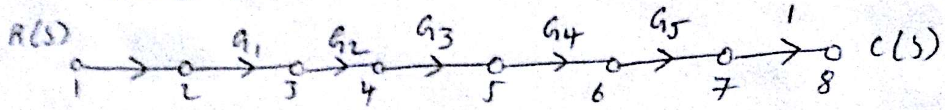


I. Forward Path gain:

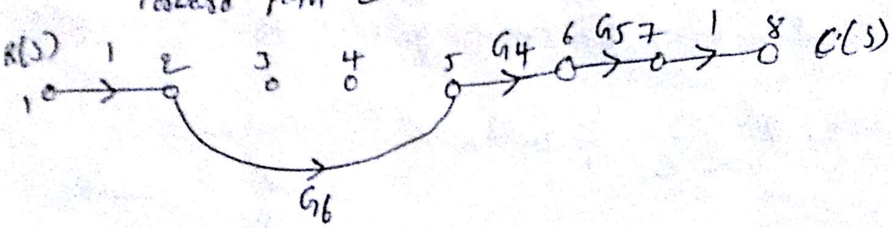
Ans:- There are two forward paths. $\therefore K = 2$.

Let forward path gains be P_1 & P_2.

Forward path 1



Forward path 2



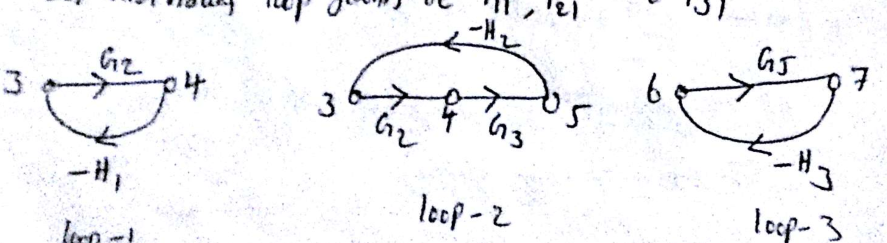
Gain of forward path-1, $P_1 = G_1G_2G_3G_4G_5$

" " " -2, $P_2 = G_4G_5G_6$.

II Individual loop gain:

There ~~are~~ three individual loops.

Let individual loop gains be P_{11}, P_{21} and P_{31}



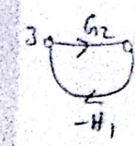
loop gain of individual loop-1, $P_{11} = -G_2H_1$

" " " " -2, $P_{12} = -G_2G_3H$

" " " " -3, $P_{13} = -G_5H_3$.

III. Gain products of two non-touching loops:

There are two combinations of two non-touching loops. Let the gain products of two non-touching loops be P_{121} and P_{22}.



First co

Gain produ

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IV. Calcul

$\Delta = 1$

$= 1-$

$= 1$

$\Delta_1 = 1,$

The part

$\Delta_2 = 1$

V. Transfe

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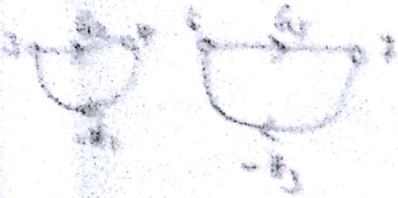
$T =$



First combination of 2 non-touching loops

Second combination of 2 non-touching loops.

Gain products of first combination of 2 non touching loops $= P_{21} = P_{11} P_{13} = G_2 G_5 H_1 H_3$

" " second " " " $= P_{22} = P_{12} P_{13} = G_2 G_3 G_5 H_2 H_3$.

IV Calculation of Δ & Δ_K:

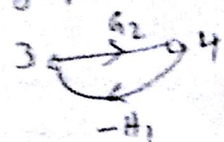
$$\Delta = 1 - (P_{11} + P_{12} + P_{13}) + (P_{21} + P_{22})$$

$$= 1 - (-G_2 H_1 - G_2 G_3 H_2 - G_5 H_3) + (G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3)$$

$$= 1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3.$$

$\Delta_1 = 1$, Since there is no part of graph which is not touching with first forward path.

The part of the graph which is non touching with second forward path as shown in fig.



$$\Delta_2 = 1 - P_{11} = 1 - (-G_2 H_1) = 1 + G_2 H_1$$

V Transfer function T

By Mason's gain formula the transfer function T, is given by,

$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K = \frac{1}{\Delta}(P_1 \Delta_1 + P_2 \Delta_2) \qquad (\because K = 2)$$

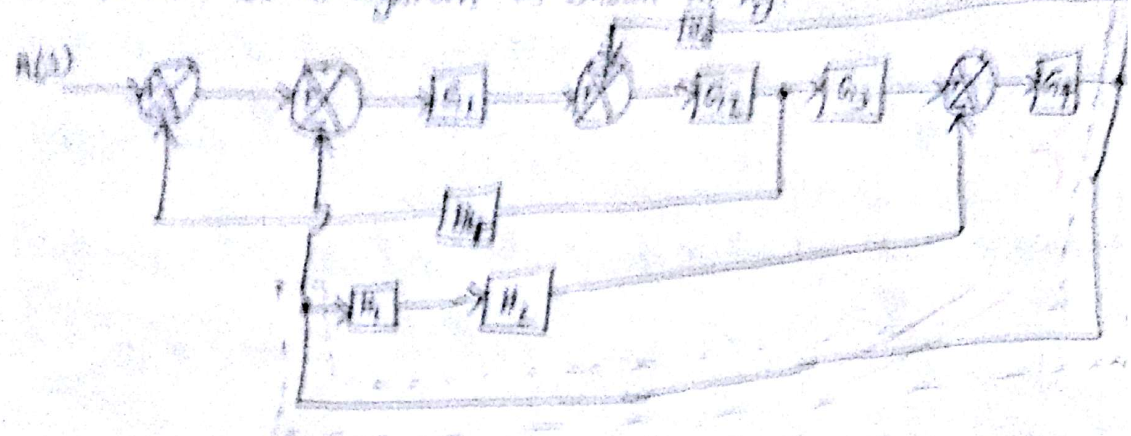
$$= \frac{G_1 G_2 G_3 G_4 G_5 + G_4 G_5 G_6 (1 + G_2 H_1)}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$$

$$= \frac{G_1 G_2 G_3 G_4 G_5 + G_4 G_5 G_6 + G_2 G_4 G_5 G_6 H_1}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$$

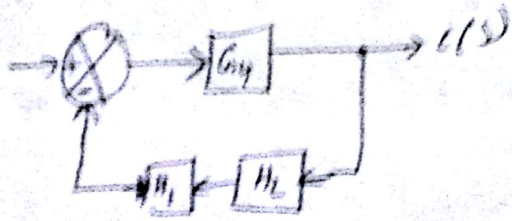
$$= \frac{G_2 G_4 G_5 (G_1 G_3 + G_6 / G_2 + G_6 H_1)}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}.$$

Problems:

(1) Using block diagram reduction technique find the transfer function $C(s)/R(s)$ for the system as shown in fig.



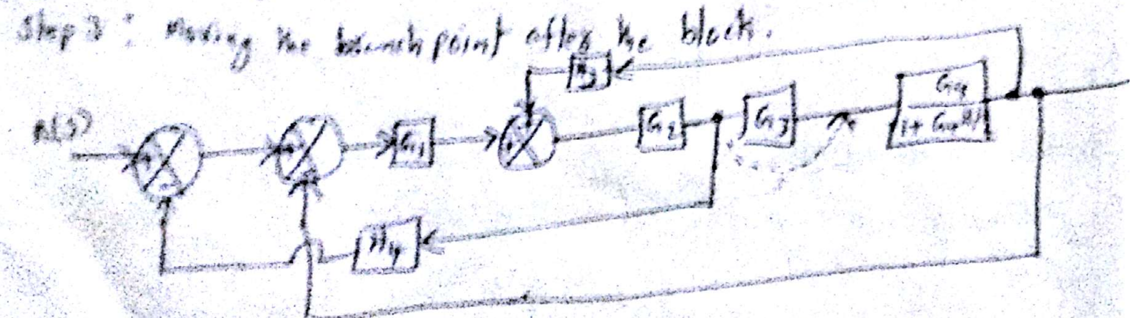
Ans:- Step 1: Rearranging the branch points.

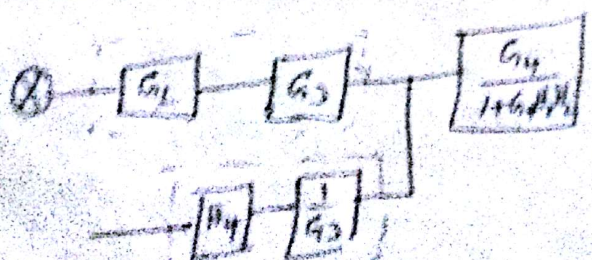


$$= \frac{G_4}{1+G_4H_1H_2}$$

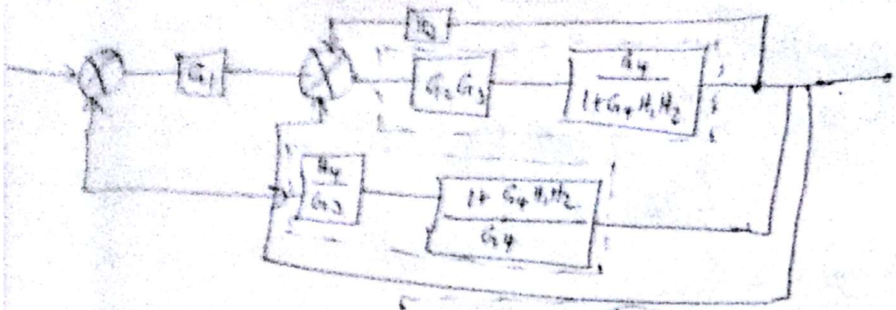
Step 2: combining the blocks in cascade and eliminating the feedback

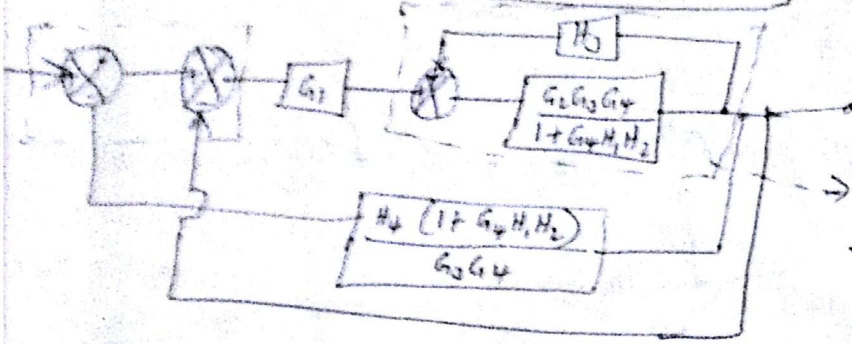
Step 3: Moving the branch point after the block.





Step 4: Moving the branch point and combining the blocks in cascade.



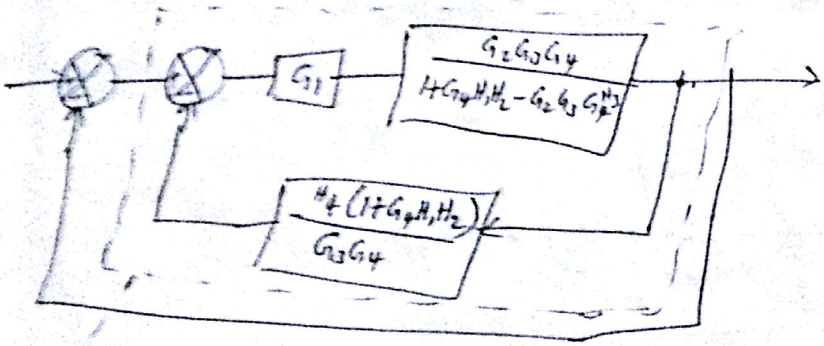


$$\rightarrow \frac{\dfrac{G_2G_3G_4}{1+G_4H_1H_2}}{1-\dfrac{G_2G_3G_4H_3}{1+G_4H_1H_2}}$$

$$= \frac{G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}$$

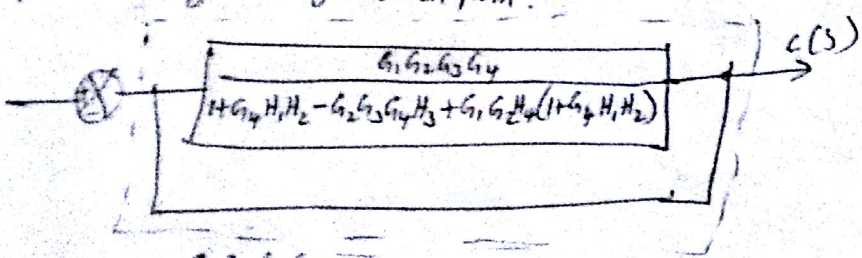
step 6: Eliminating feedback paths & interchanging summing points.

step 7: combining the blocks in cascade & eliminating feedback path.



$$\Rightarrow \frac{G_1G_2G_3G_4/(1+G_4H_1H_2-G_2G_3G_4H_3)}{1+\left(\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}\right)\left(\dfrac{H_4(1+G_4H_1H_2)}{G_3G_4}\right)} = \frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}$$

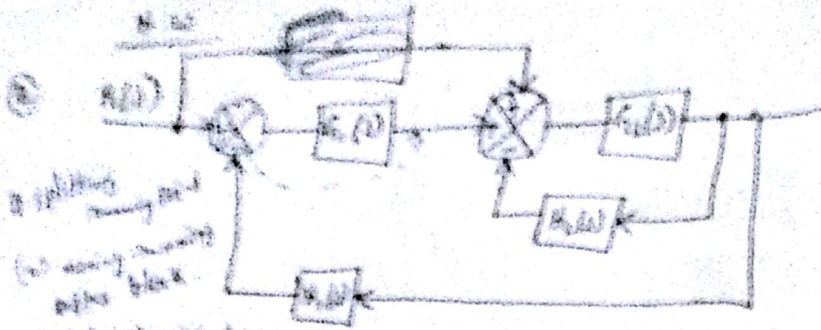
step 8: Eliminating the unity feedback path.



$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}{1+\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$$

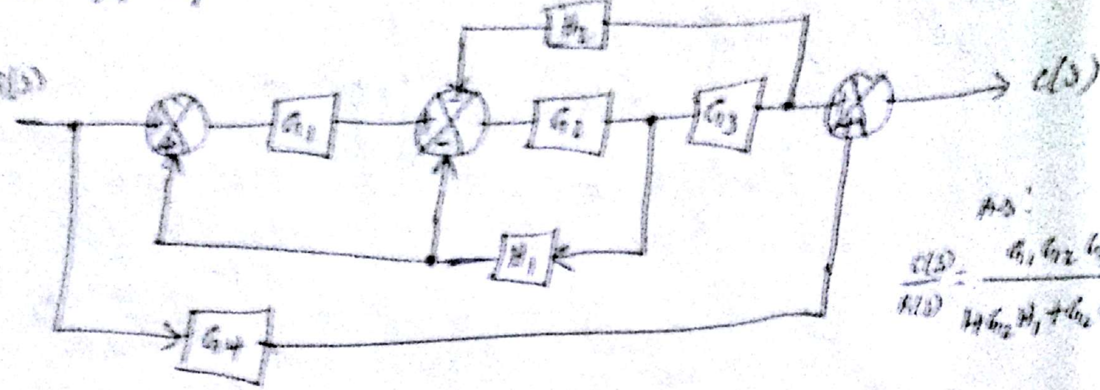
③



(i) splitting summing point
(ii) moving summing before block
(c) interchanging summing

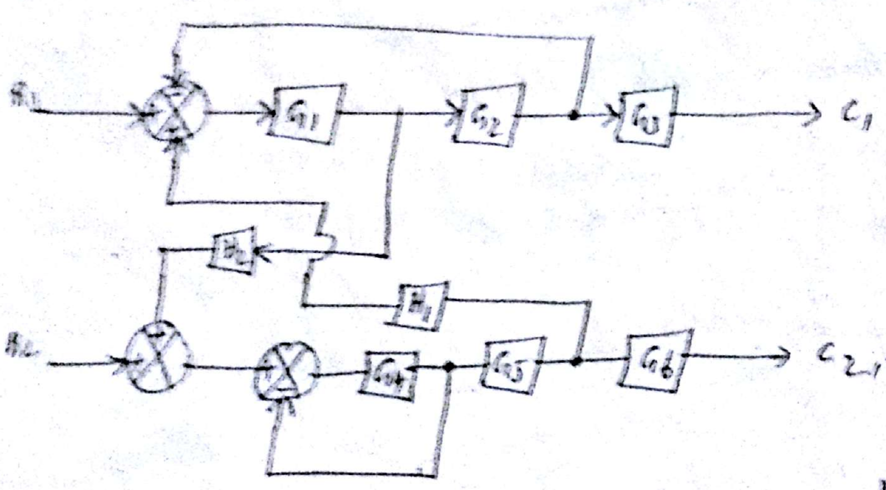
Ans:
$$\frac{C(s)}{R(s)} = \frac{G_2(G_1+1)}{1+G_2H_2+G_1G_2H_1}$$

④



Ans:
$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1}$$

④

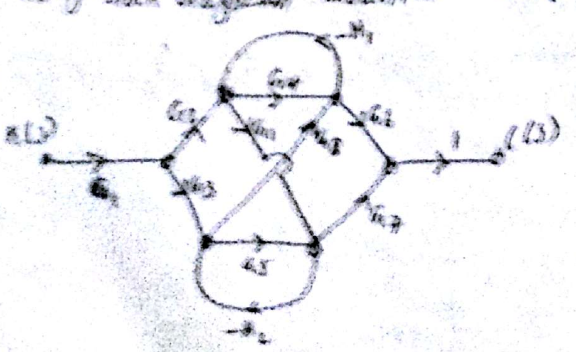


$$C_1/R_1 = \frac{G_1G_2G_3(1+G_4)}{(1+G_1G_2)(1+G_4)-G_1G_4G_5H_1H_2}$$

$$C_2, \frac{C_2}{R_1} = \frac{G_1G_4G_5G_6H_2}{(1+G_4)(1+G_1G_2)-G_1G_4G_5H_1H_2}$$

Using block diagram reduction technique find out T.F C_1/R_1 and C_2/R_1

⑤



Find the overgain of the system.

Ans:- No. of forward paths $K = 6$

$P_1 = G_2G_4G_6$

$P_2 = G_3G_5G_7$

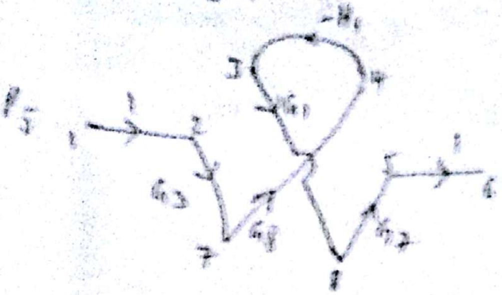
P_3

P_4

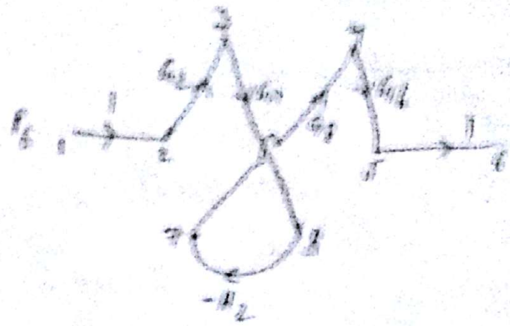
$P_3 = G_1 G_2 G_3$

$P_4 = G_3 G_8 G_4$

P_5



P_6

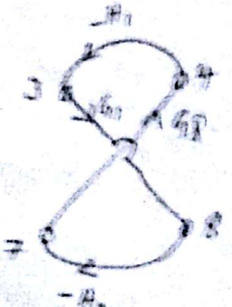


$P_5 = -G_1 G_3 G_7 G_8 H_1$

$P_6 = -G_1 G_2 G_8 G_4 H_2$

Individual loop gain :-

individual loops = 3



$P_{11} = -G_4 H_1$

$P_{21} = -G_5 H_2$

$P_{31} = G_1 G_8 H_1 H_2$

Gain products of Two Non-touching loops :-

only one combination of two non-touching loops.

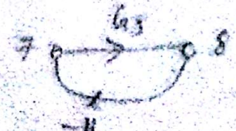
$\therefore P_{12} = P_{11} P_{21} = G_4 G_5 H_1 H_2$

calculation of Δ and Δ_k

$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + P_{12} = 1 - (-G_4 H_1 - G_5 H_2 + G_1 G_8 H_1 H_2) + G_4 G_5 H_1 H_2$

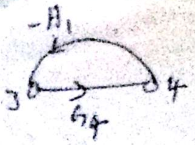
$= 1 + G_4 H_1 + G_5 H_2 - G_1 G_8 H_1 H_2 + G_4 G_5 H_1 H_2$

the part of the graph non-touching forward path-1



$\therefore \Delta_1 = 1 - (-G_5 H_2) = 1 + G_5 H_2$

The part of the graph non-touching forward path-2



$\therefore \Delta_2 = 1-(-G_4H_1) = 1+G_4H_1$

There is no part of the graph which is non-touching with forward paths 3, 4, 5, and 6.

Transfer Function: T

By Mason's gain formula the T.F.

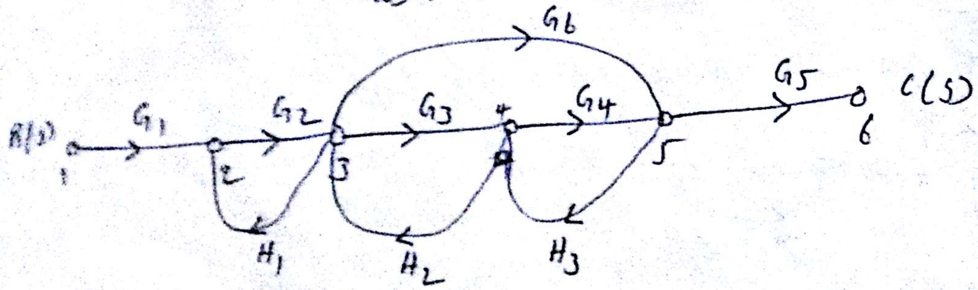
$$T = \frac{1}{\Delta}\left(\sum_K P_K \Delta_K\right)$$ (Number of forward paths are 6, K=6)

$$= \frac{1}{\Delta}\left(P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6\right)$$

$$= \frac{G_2G_4G_6(1+G_5H_2) + G_3G_5G_7(1+G_4H_1) + G_1G_2G_7 + G_3G_6G_8 - G_1G_3G_7G_8H_1 - G_1G_2G_6G_8}{1+G_4H_1 + G_5H_2 - G_1G_8H_1H_8 + G_4G_5H_1H_2}$$

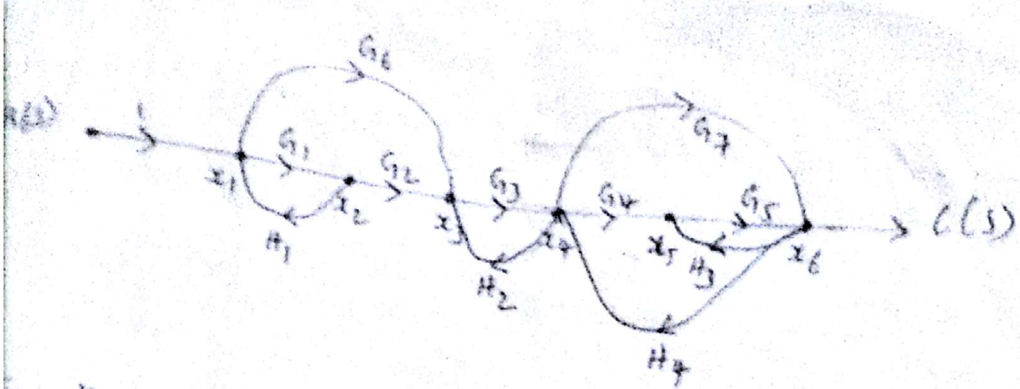
H.W

(7) The SFG for a feedback c.s as shown in fig. Determine the closed loop T.F $\frac{C(s)}{R(s)}$.



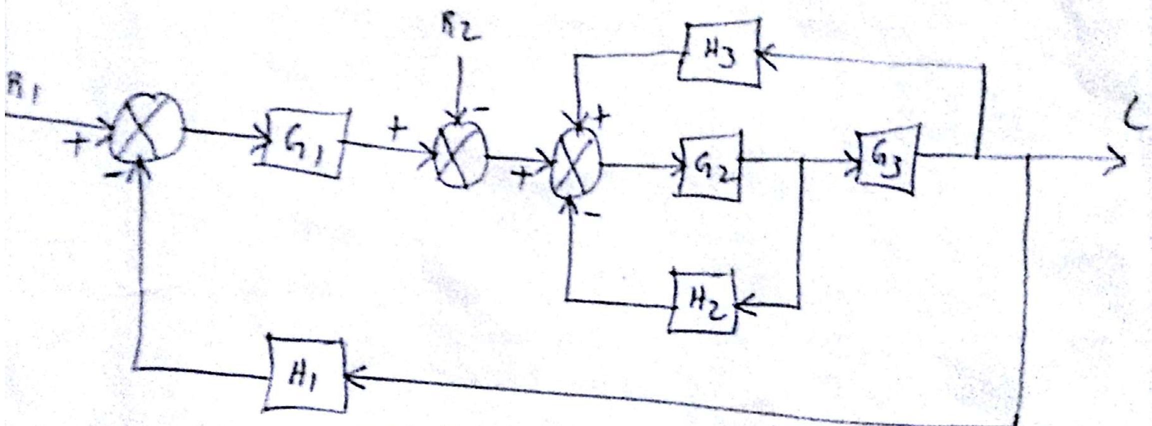
Ans:- $$T.F = \frac{G_1G_2G_3G_4G_5 + G_1G_2G_5G_6}{1 - G_2H_1 - G_3H_2 - G_4H_3 - G_6H_2H_3 + G_2G_4H_1H_3}$$

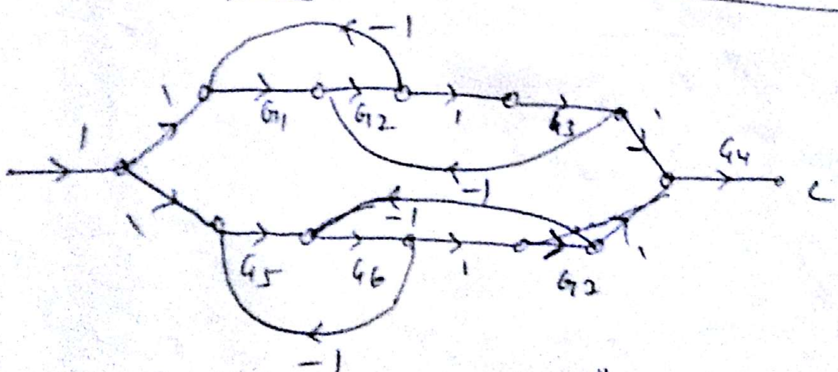
...ain the T.F using mason gain formula for the SFG as shown in fig. (16)

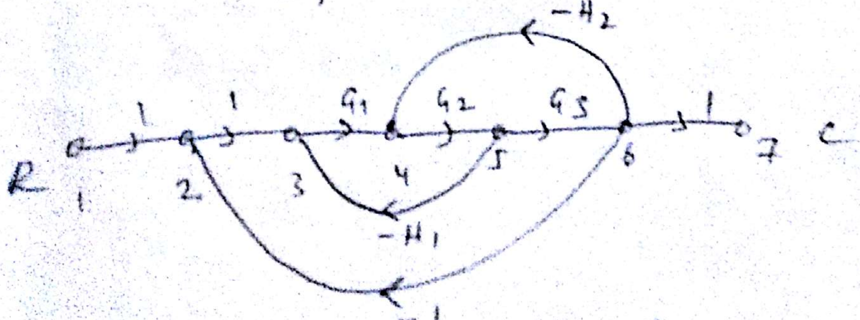


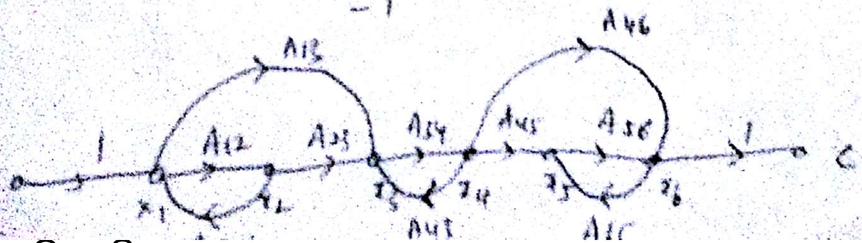
For the system represented in the given fig. obtain T.F.

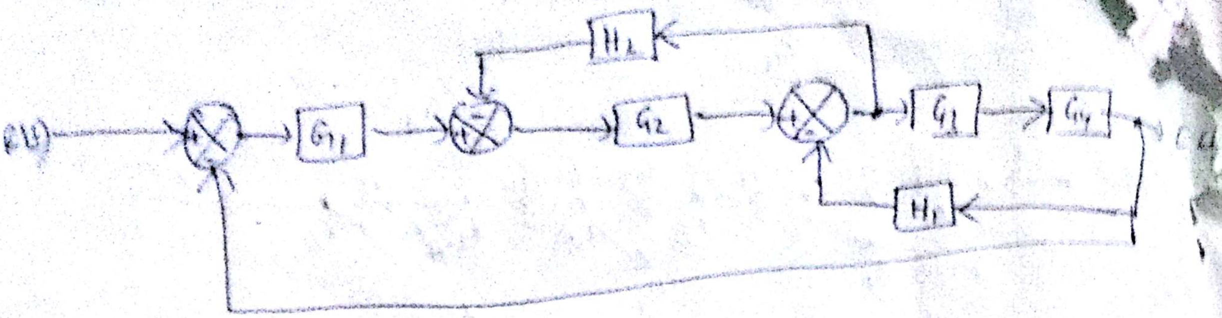
(a) C/R_1 (b) C/R_2











R(s)
G1
G2
G3
G4
H1
H2
C(s)

TIME RESPONSE ANALYSIS

The transient response and steady state behaviour of a system are together referred to as time response analysis.

The behaviour of a system from initial state to final state is referred to as transient response. The behaviour of a system as time 't' tends to infinity is referred as steady-state response. Thus, the system response c(t) may be written as

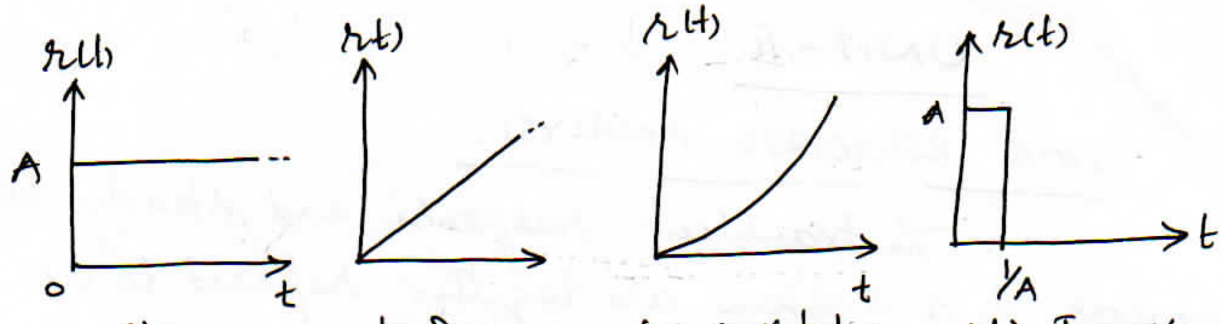
$$c(t) = c_{tr}(t) + c_{ss}(t)$$

The transient response is the response of the system when the input changes from one state to another. The steady state response is the response as the time 't' approaches infinity.

Standard Test Signals: The knowledge of input signal is required to predict the response of a system. The characteristics of actual input signals are sudden shock, sudden change, constant velocity and constant acceleration. Test signals with these characteristics are used as input signals to predict the performance of the system. The commonly used test input signals are step, ramp, parabolic and impulse.

(1) Step Signal: The step is a signal whose value changes from one level to another level in zero time. The mathematical representation of the step function is

$$r(t) = A\,u(t)\;;\quad \text{where}\quad u(t) = 1 \quad t>0$$
$$= 0 \quad t<0$$



(a) Step (b) Ramp (c) parabolic (d) Impulse

Figure : Standard test Signals

In laplace transform form $R(s) = \frac{A}{s}$

(2) Ramp Signal : The ramp is a signal which starts at a value of zero and increases linearly with time.

$$r(t) = At \; ; \; t > 0$$
$$= 0 \quad t < 0$$

In the laplace transform form $R(s) = \frac{A}{s^2}$

The ramp is integral of step signal.

(3) parabolic Signal : The parabolic signal is the integral of ramp signal and is given by

$$r(t) = \frac{At^2}{2} \; ; \; t > 0$$
$$= 0 \quad t < 0$$

In the laplace transform form $R(s) = A/s^3$

(4) Impulse Signal : An impulse is a signal whose value is zero every where except at $t = 0$. At $t = 0$ it has an infinite magnitude

$$\delta(t) = 0 \; ; \; t \neq 0$$
$$\underset{\epsilon \to 0}{Lt} \int_{\epsilon^-}^{\epsilon^+} \delta(t)\,dt = 1$$

Since a perfect impulse can not be achieved in practice, it is usually approximated by a pulse of small width but unit area.

Impulse is the derivative of step signal ②

$$\delta(t) = \frac{d}{dt}u(t) = \dot{u}(t)$$

$$L[\delta(t)] = 1$$

Let us consider a system with transfer function

$\frac{C(s)}{R(s)} = G(s)$: if input $r(t) = \delta(t)$ then $R(s) = 1$

$$\therefore C(s) = G(s)R(s) = G(s)$$

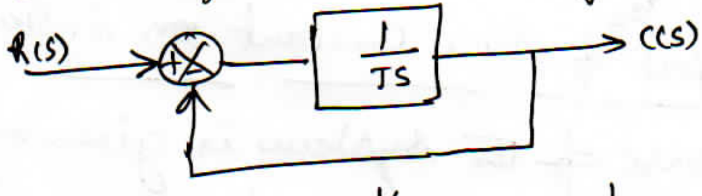
taking inverse LT on both sides

$$\boxed{c(t) = g(t)}$$

Thus, the impulse response of a system, indicated by $g(t)$ is the inverse Laplace transform of its transfer function $G(s)$. This is sometimes referred to as weighting function of the system. The weighting function of a system can be used to find the system's response to any input $r(t)$ by means of the convolution integral. Thus

$$c(t) = \int_0^t g(t-\tau)\, r(\tau)\, d\tau$$

<u>Time Response of First-order Systems</u>: Let us consider a first order system with unity feedback shown in figure.



$$\therefore \frac{C(s)}{R(s)} = \frac{1/Ts}{\frac{1}{Ts}+1} = \frac{1}{1+Ts}$$

(i) <u>Response to the unit-step input</u>: if $r(t) = u(t)$, then $R(s) = \frac{1}{s}$

$$\therefore C(s) = R(s)\cdot\frac{1}{1+Ts} = \frac{1}{s(1+Ts)} = \frac{1}{s} - \frac{T}{Ts+1}$$

Taking inverse Laplace transform

$$c(t) = 1 - e^{-t/T}$$

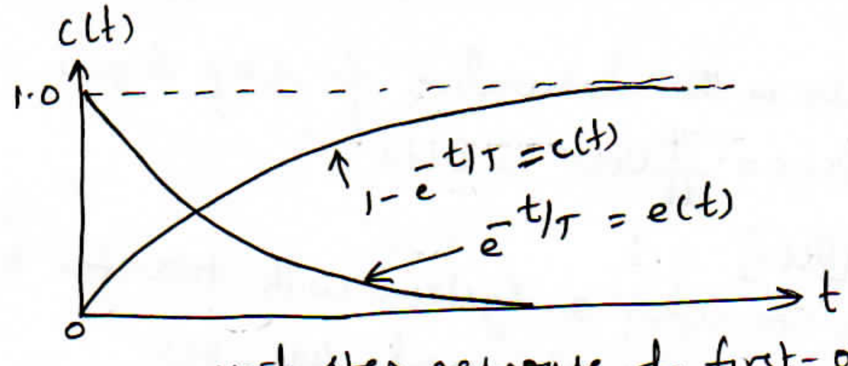


Figure : Unit-step response of first-order system

It is seen that the output rises exponentially from zero value to final value of unity.

The initial slope of the curve at $t=0$ is given by

$$\frac{d}{dt}c(t)\Big|_{t=0} = \frac{1}{T}e^{-t/T}\Big|_{t=0} = \frac{1}{T}$$

where 'T' is known as the time constant of the system

The time constant is indicative of how fast the system tends to reach the final value. A large time constant corresponds to a sluggish system and a small time constant corresponds to a fast response

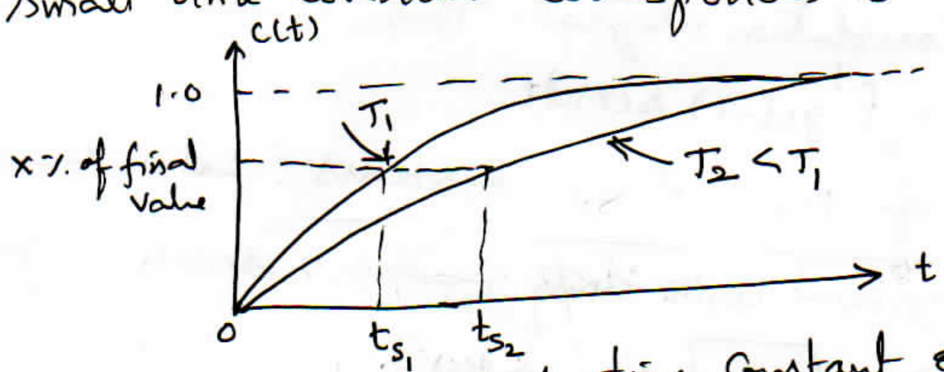


Figure : Effect of time constant on system response

The error response of the system is given by

$$e(t) = r(t) - c(t) = e^{-t/T}$$

The steady state error e_{ss} is given by

$$e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, sE(s) = 0$$

Thus this first order system tracks the unit-step input with zero steady state error.

(2) <u>Response to the unit-ramp input</u>:

if $r(t) = t$; then $R(s) = 1/s^2$

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1} \quad \therefore C(s) = R(s)\cdot\frac{1}{Ts+1}$$

or $$C(s) = \frac{1}{s^2(Ts+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{sT+1}$$

$$= -\frac{T}{s} + \frac{1}{s^2} + \frac{T^2}{Ts+1}$$

taking inverse Laplace transform

$$c(t) = -T + t + Te^{-t/T}$$

$\therefore$ The error signal $e(t) = r(t) - c(t) = T(1 - e^{-t/T})$

The steady state error $e_{ss} = \lim_{t\to\infty} e(t) = T$

$$= \lim_{s\to 0} sE(s) = T$$

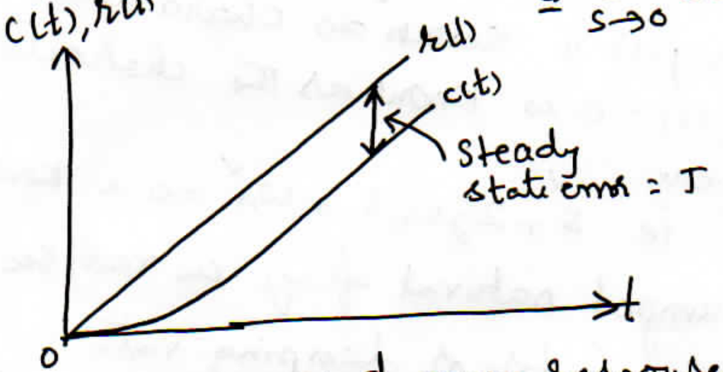


Figure: Unit ramp response of first order system

Thus the first order system will track the unit-ramp input with a steady state error T, which is equal to the time constant of the system.

By reducing the system time constant, we can improve the speed of the response but also reduces the steady-state error to a ramp input.

<u>Time Response of 2nd order System</u> :

Let us consider a second order system shown in figure.

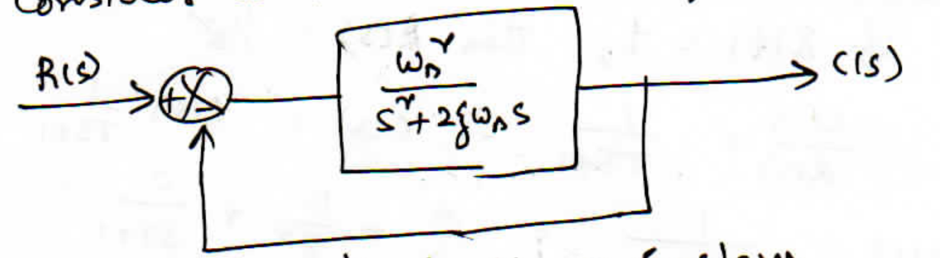


Figure : unity feedback system

The transfer function $\frac{C(s)}{R(s)} = \frac{\left(\omega_n^2 / s^2+2\xi\omega_n s\right)}{1+\frac{\omega_n^2}{s^2+2\xi\omega_n s}(1)}$ $\longrightarrow$ ①

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{p(s)}{q(s)}$$

where the denominator $q(s)$ is known as characteristic polynomial and $q(s)=0$ is known as the characteristic equation of the system. ie $s^2+2\xi\omega_n s+\omega_n^2=0$ is known as CE

where ω_n = undamped natural freq in rad/sec

ξ = damping factor or damping ratio

if $\xi=0$, undamped system

$0<\xi<1$, under damped system

$\xi=1$, critically damped system

$\xi>1$, over damped system

$\omega_d=\omega_n\sqrt{1-\xi^2}$ is damped natural frequency.

<u>Unit step response of 2nd order system</u>: If $r(t)=u(t)$,

then $R(s)=\frac{1}{s}$ $\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

and $C(s) = R(s)\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

or $C(s) = \frac{1}{s} \frac{\omega_n^2}{(s^2+2\xi\omega_n s+\omega_n^2)}$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}$$

let $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\therefore C(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

Taking inverse LT on both sides

$$L^{-1}C(s) = c(t) = L^{-1}\left\{\frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right\}$$

$$\therefore c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}e^{-\xi\omega_n t}\sin\omega_d t$$

put $= \omega_d = \omega_n\sqrt{1-\xi^2}$

$$\therefore c(t) = 1 - e^{-\xi\omega_n t}\left\{\cos\omega_d t + \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\sin\omega_d t\right\}$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left\{\sqrt{1-\xi^2}\cos(\omega_d t) + \xi\sin(\omega_d t)\right\}$$

let $\xi = \cos\phi$ $\therefore$ $\sin\phi = \sqrt{1-\cos^2\phi} = \sqrt{1-\xi^2}$

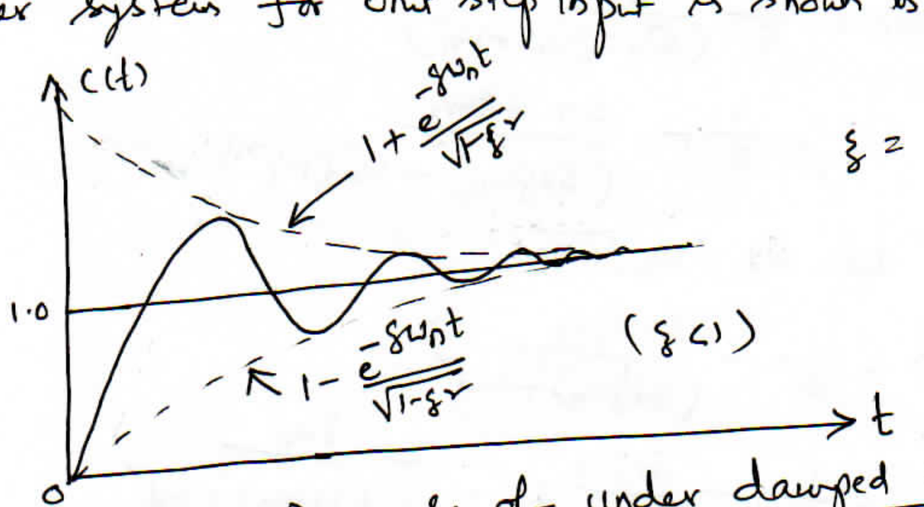
$$\therefore c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left\{\sin\phi\cos(\omega_d t) + \cos\phi\sin(\omega_d t)\right\}$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\phi)$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left[(\omega_n\sqrt{1-\xi^2})t + \tan^{-1}\sqrt{\frac{1-\xi^2}{\xi}}\right]$$

The steady state value of $c(t) = \lim_{t\to\infty} c(t) = 1$

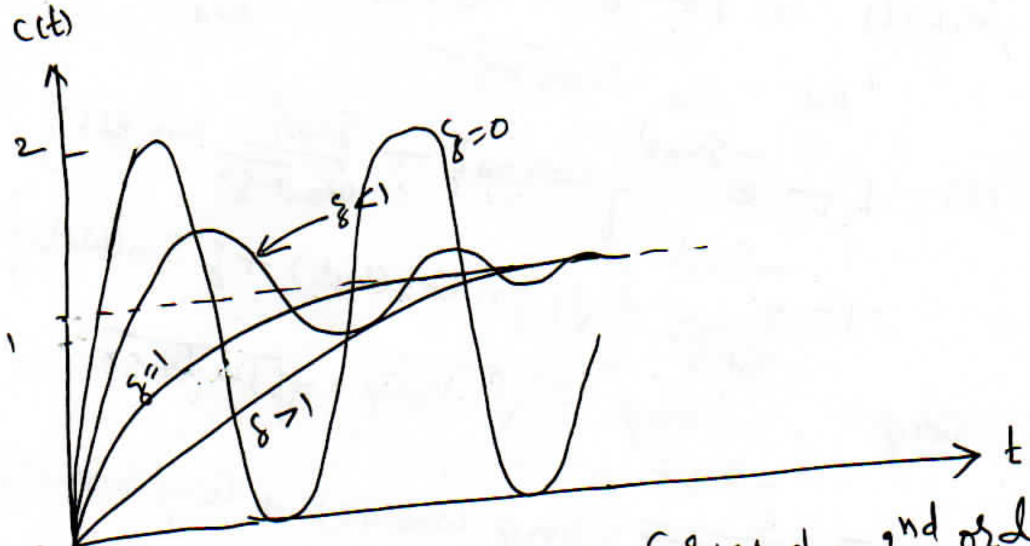
The time response of under damped ($\xi<1$) second order system for unit step input is shown in figure



Figure(1): Time Response of under damped second order system for unit step input.

If $\xi=0$, $c(t) = 1-\cos\omega_n t$

$\xi = 1$ $\quad c(t) = 1-e^{-\omega_n t}-\omega_n t\, e^{-\omega_n t}$



Figure(2): Unit step response curves of 2nd order system for different values of 'ξ'.

The characteristic equation of 2nd order systems is

$$s^2+2\xi\omega_n s+\omega_n^2=0$$

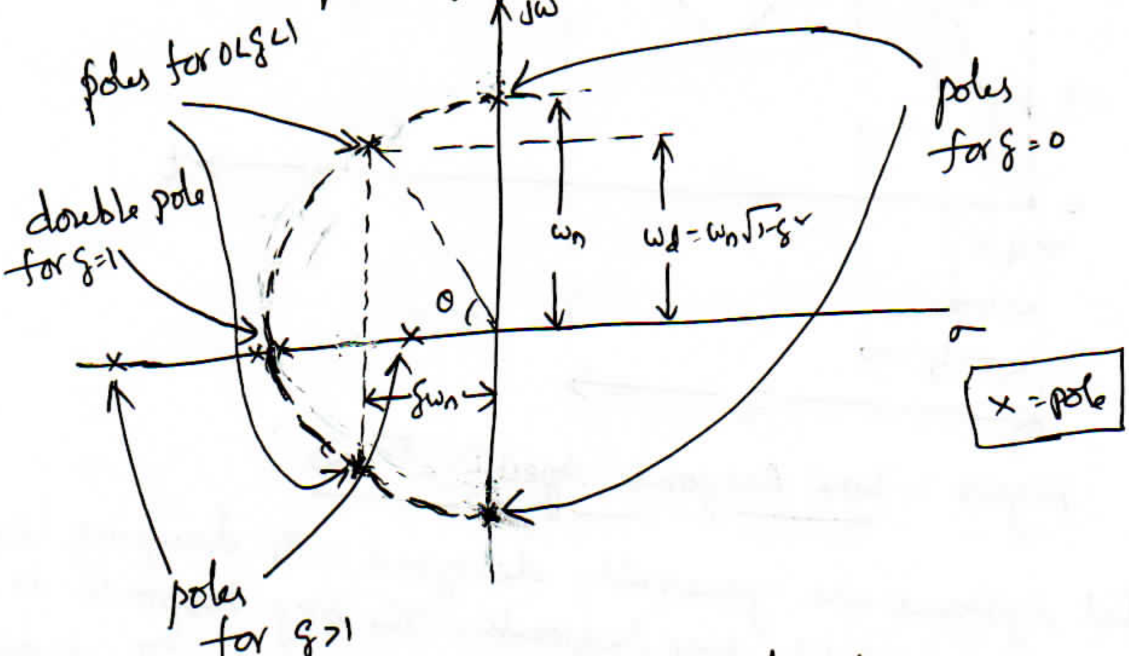
The roots of CE are given by

$$s_1, s_2 = \frac{-2\xi\omega_n \pm\sqrt{(2\xi\omega_n)^2-4\omega_n^2}}{2}$$

$$s_1, s_2 = -\frac{2\zeta\omega_n \pm \sqrt{-1}\sqrt{4}\sqrt{\omega_n^2 - \zeta^2\omega_n^2}}{2}$$

$$= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Important: The roots of the characteristic equation are the poles of closed loop system



Figure(3): pole locations of 2nd order system for different values of 'ζ'

For $\zeta=0$, the poles lie on the imaginary axis

$0<\zeta<1$, the poles are complex conjugate and lie in LHS

$\zeta=1$, Double pole on the real axis in LHS

for $\zeta>1$, the poles move in opposite direction on the real axis

Time - Domain Specifications :

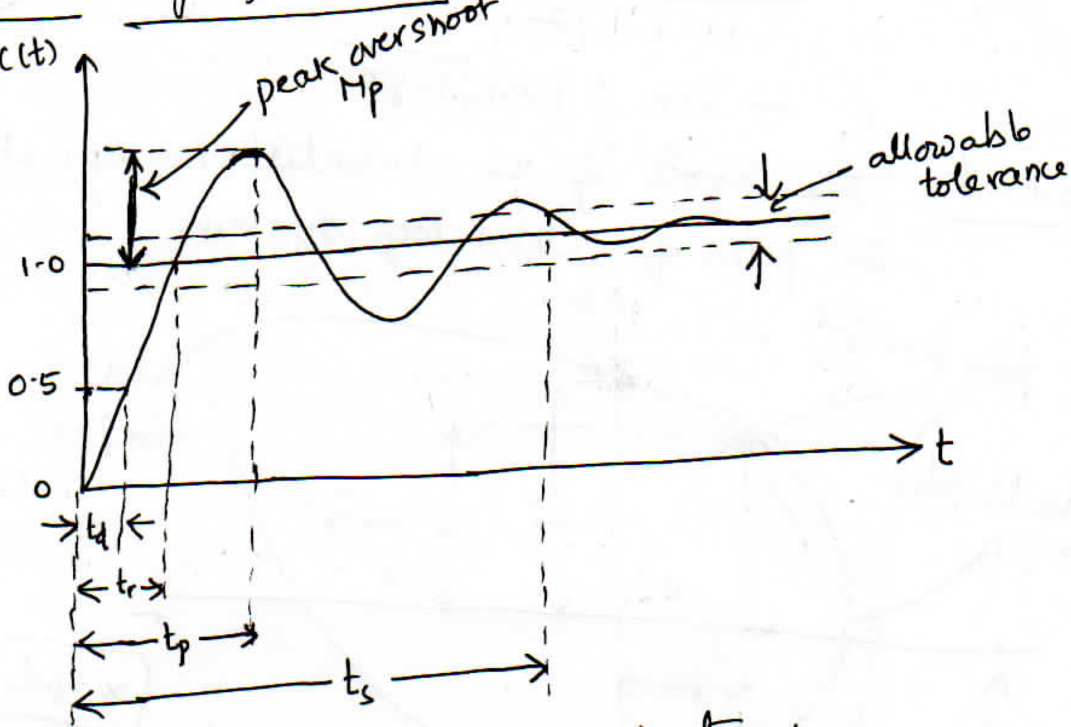


Figure : Time Response Specifications

Control systems are generally designed with damping less than one; ie, oscillatory step response. The step response is characterized by the following performance indices. The indices are qualitatively related to

(i) How fast the system moves to follow the input ?

(ii) How oscillatory it is ($\xi = ?$) ?

(iii) How long does it take to practically reach the final value?

It may be noted that various indices are not independent of each other.

(1) Delay time(t_d): It is the time required for the response to reach 50% of the final value in first attempt.

(2) Rise time (t_r) : It is the time required for the response to rise from 10% to 90% of the final value for over damped systems and 0 to 100% of the final value for under damped systems

(3) Peak time (t_p) : It is the time required for the response to reach the peak of time response or the peak over shoot.

(4) Peak overshoot (M_p): It indicates the normalized difference between the time response peak and the steady output and is defined as

$$\%\ \text{peak over shoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

(5) Settling time (t_s): It is the time required for the response to reach and stay within a specified tolerance band (usually 2% or 5%) of its final value.

(6) Steady state error (e_{ss}): It indicates the error between the actual output and desired output as 't' tends to infinity

ie $e_{ss} = \lim_{t\to\infty} [r(t) - c(t)]$

Time Response Specifications of Second-order Systems:

The unit step response of 2^{nd} order system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin[\omega_d t + \phi] \longrightarrow (1)$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$; $\phi = \tan^{-1}\sqrt{\frac{1-\xi^2}{\xi}}$

(1) Rise time (t_r): The rise-time t_r is obtained when $c(t)$ reaches unity ie $c(t)\big|_{t=t_r} = 1$

$$\Rightarrow c(t_r) = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin[\omega_d t_r + \phi] = 1$$

$$\Rightarrow \sin(\omega_d t_r + \phi) = 0$$

$\sin\theta = 0$; for $\theta = n\pi$, $n = 0, 1, 2 \ldots$

$\therefore \omega_d t_r + \phi = \pi$; (before completing one cycle)

$$\text{or } t_r = \frac{\pi - \phi}{\omega_d} = \frac{\pi - \tan^{-1}\sqrt{1-\xi^2}/\xi}{\omega_n\sqrt{1-\xi^2}} \longrightarrow (i)$$

(2) peak time (t_p): up to peak time 't_p' the response $c(t)$ increases, then decreases.

$$\therefore \frac{d}{dt} c(t)\Big|_{t=t_p} = 0$$

$$\frac{d}{dt}\left\{1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)\right\}\Big|_{t=t_p} = 0$$

$$\left[-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos(\omega_d t + \phi)\,\omega_d - \sin(\omega_d t + \phi) \cdot \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cdot (-\xi\omega_n)\right]\Big|_{t=t_p} = 0$$

or $\sin(\omega_d t_p + \phi)\,\xi - \cos(\omega_d t_p + \phi)\sqrt{1-\xi^2} = 0$

where $\xi = \cos\phi \quad \therefore \sin\phi = \sqrt{1-\xi^2}$

$$\sin(\omega_d t_p + \phi)\cos\phi - \cos(\omega_d t_p + \phi)\sin\phi = 0$$

$$\sin(\omega_d t_p + \phi - \phi) = 0$$

or $\sin(\omega_d t_p) = 0$

or $\omega_d t_p = n\pi$

since the first peak occurs before 2π

$$\therefore \omega_d t_p = \pi \longrightarrow \text{(ii)}$$

$$\text{or } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

This is the time required to reach first peak overshoot

The first undershoot occurs at $t = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$ and the 2nd over shoot occurs at $t = \frac{3\pi}{\omega_n\sqrt{1-\xi^2}}$

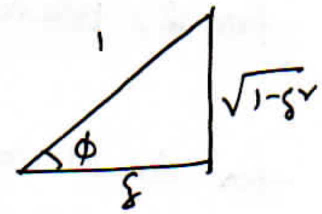
(3) peak over shoot (M_p): $M_p = c(t_p) - 1$

$$= 1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \phi) - 1$$

$$= -\frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin\left[\omega_n\sqrt{1-\xi^2}\cdot\frac{\pi}{\omega_n\sqrt{1-\xi^2}}+\phi\right] \quad (7)$$

$$= -e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \sin(\pi+\phi)\frac{1}{\sqrt{1-\xi^2}}$$

$$= -e^{\frac{-\pi\xi}{\sqrt{1-\xi}}}\left[-\sin\phi\right]$$



$$= +e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}\sqrt{1-\xi^2}\cdot\frac{1}{\sqrt{1-\xi^2}}$$

$$= e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

% peak overshoot $= 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$

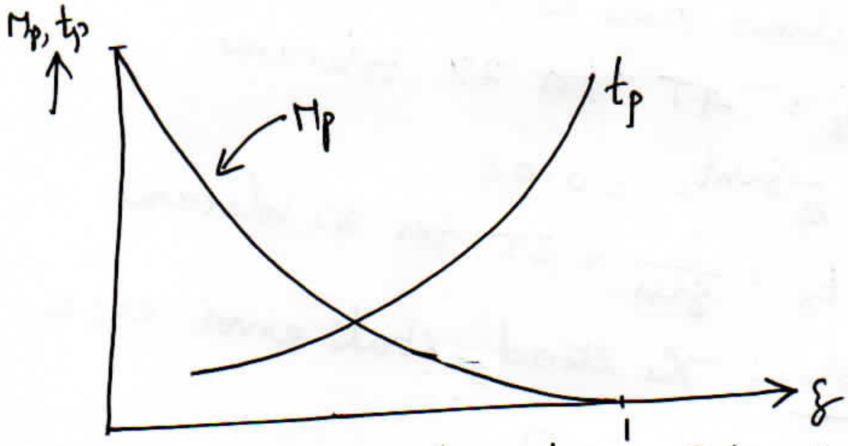


Figure : variation of M_p & t_p w.r.t ξ

From the figure, it is observed that M_p decreases and 't_p' increases as ξ increases.

(4) <u>Settling time (t_s)</u>: The response of 2nd order system has two components (i) Decaying exponential Component $\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$

(ii) Sinusoidal Component $\sin(\omega_d t+\theta)$

The decaying exponential term reduces the oscillations produced by sinusoidal component. Hence, the settling is decided by exponential component. The settling time can be found by equating exponential component to percentage of tolerance.

for 2% tolerance, $\left.\dfrac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\right|_{t=t_s} = 0.02$

for small values of ξ $\sqrt{1-\xi^2} \simeq 1$

$\therefore e^{-\xi\omega_n t_s} = 0.02$

taking 'ln' on both sides

$-\xi\omega_n t_s = \ln(0.02) = -4$

or $t_s = \dfrac{4}{\xi\omega_n}$ for 2% tolerance

for second order systems time constant $T = \dfrac{1}{\xi\omega_n}$

$\therefore$ settling $t_s = 4T$ for 2% tolerance

For 5% tolerance $e^{-\xi\omega_n t_s} = 0.05$

$\therefore t_s = \dfrac{3}{\xi\omega_n} = 3T$ for 5% tolerance

(5) Steady state Error : The steady state error e_{ss} is given by $e_{ss} = \lim_{t\to\infty}[r(t) - c(t)]$

(i) for unit step input $r(t) = u(t) = 1$

$\therefore e_{ss} = \lim_{t\to\infty}[1 - c(t)] = 0$

Thus, the second order system has zero steady state error to unit step input.

(ii) for ramp input $r(t) = t$

$\therefore$ steady state error $= \lim_{t\to\infty}[t - c(t)] = \lim_{s\to 0} s\left[\frac{1}{s^2} - C(s)\right]$

$= \dfrac{2\xi}{\omega_n}$

Steady state Errors & Error Constants: ⑧

The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non-linearities of system components such as static friction, backlash etc.

Let us consider a feed back system shown in figure.

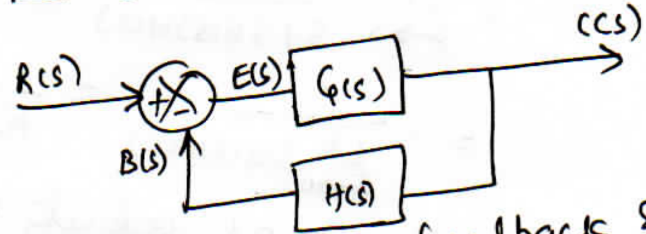


Figure: Negative feedback system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \quad \text{and} \quad C(s) = E(s)G(s)$$

$E(s)$ = Error signal

$$\therefore E(s) = \frac{C(s)}{R(s)} = \frac{R(s)}{1+G(s)H(s)}$$

$\therefore$ The steady state error $e_{ss} = \underset{s\to 0}{Lt}\ sE(s)$

$$= \underset{s\to 0}{Lt}\ s\frac{R(s)}{1+G(s)H(s)} \longrightarrow ①$$

The above expression shows that the steady state error depends upon type of input and forward path transfer function G(s). The steady state errors for various types of standard input signals are derived below

(1) Unit step input: $r(t) = u(t)$ $\therefore R(s) = \frac{1}{s}$

$\therefore$ For unit step input, $e_{ss} = \underset{s\to 0}{Lt}\ \frac{s\cdot\frac{1}{s}}{1+G(s)H(s)}$

$$= \underset{s\to 0}{Lt}\ \frac{1}{1+G(s)H(s)}$$

$$= \frac{1}{1+\underset{s\to 0}{Lt}\ G(s)H(s)} = \frac{1}{1+K_p}$$

where $K_p = \underset{s\to 0}{Lt}\ G(s)H(s)$ is defined as position error constant

(2) For unit-ramp (velocity) input:

$$r(t) = t \quad \therefore R(s) = \frac{1}{s^2}$$

steady state error $e_{ss} = \underset{s\to 0}{Lt} \dfrac{s R(s)}{1+G(s)H(s)}$

$$= \underset{s\to 0}{Lt} \frac{s \cdot \frac{1}{s^2}}{1+G(s)H(s)}$$

$$= \underset{s\to 0}{Lt} \frac{1}{s+sG(s)H(s)}$$

$$= \frac{1}{\underset{s\to 0}{Lt}\, sG(s)H(s)} = \frac{1}{K_v}$$

where $K_v = \underset{s\to 0}{Lt}\, sG(s)H(s)$ is known as velocity error constant

(3) unit-parabolic (acceleration) input:

For parabolic input $r(t) = t^2/2 \quad \therefore R(s) = \dfrac{1}{s^3}$

$\therefore$ The steady state error $e_{ss} = \underset{s\to 0}{Lt} \dfrac{s\,(1/s^3)}{1+G(s)H(s)}$

$$= \underset{s\to 0}{Lt} \frac{1}{s^2+s^2G(s)H(s)}$$

$$= \frac{1}{\underset{s\to 0}{Lt}\, s^2G(s)H(s)} = \frac{1}{K_a}$$

where $K_a = \underset{s\to 0}{Lt}\, s^2G(s)H(s)$ is known as acceleration error constant.

Types of feedback Control systems: The openloop transfer function $G(s)$ can be written in two standard forms namely time-constant form and pole-zero form.

$$G(s) = \frac{K(T_{z_1}s+1)(T_{z_2}s+1)\cdots}{s^n(T_{p_1}s+1)(T_{p_2}s+1)\cdots} \quad \text{Time-Constant form}$$

$$= \frac{K'(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots} \quad \text{pole-zero form}$$

The term s^n corresponds to number of integrations in the system. s^n also represents number of poles at the origin. The number of poles at the origin is also known as the type of system. Now, we can determine steady state errors for different types of systems.

(1) Type – 0 Systems: If $n = 0$, the steady state errors to various inputs are as follows $G(s) = \dfrac{K'(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots}$

(i) $$e_{ss}(\text{position}) = \frac{1}{1+\underset{s\to 0}{Lt}\, G(s)H(s)} = \frac{1}{1+K_p} \quad \boxed{\text{If } H(s) = 1}$$

(ii) $$e_{ss}(\text{velocity input}) = \frac{1}{\underset{s\to 0}{Lt}\, sG(s)H(s)}$$
$$= \frac{1}{\underset{s\to 0}{Lt}\, sG(s)H(s)} = \infty \quad \boxed{\text{If } H(s) = 1}$$

(iii) $$e_{ss}(\text{acceleration}) = \frac{1}{\underset{s\to 0}{Lt}\, s^2G(s)H(s)} = \frac{1}{0} = \infty \quad \text{If } H(s) = 1$$

Thus a type – 0 system has constant position error, infinite velocity and acceleration errors.

(2) Type – 1 System: If $n = 1$; $G(s) = \dfrac{K'(s+z_1)(s+z_2)\cdots}{s(s+p_1)(s+p_2)\cdots}$

(i) $$e_{ss}(\text{for position input}) = \frac{1}{1+\underset{s\to 0}{Lt}\, G(s)H(s)} \quad \text{if } H(s) = 1$$

$$e_{ss} = \frac{1}{1+\underset{s\to 0}{Lt}\, \dfrac{(s+z_1)(s+z_2)\cdots}{s(s+p_1)(s+p_2)\cdots}}$$

$$= \frac{1}{1+\infty} = 0$$

(ii) $e_{ss}(\text{velocity input}) = \dfrac{1}{\underset{s\to 0}{Lt}\, sG(s)H(s)}$

for unity feedback system $H(s) = 1$

$\therefore e_{ss}(\text{velocity}) = \dfrac{1}{\underset{s\to 0}{Lt}\, \cancel{s}\, \dfrac{K'(s+z_1)(s+z_2)\cdots}{\cancel{s}(s+p_1)(s+p_2)\cdots}}$

$= \dfrac{1}{K'} = \dfrac{1}{K_v}$

(iii) $e_{ss}(\text{acceleration input}) = \dfrac{1}{\underset{s\to 0}{Lt}\, s^2 G(s)H(s)}$

for unity feedback system $H(s) = 1$

$\therefore e_{ss}(\text{acceleration}) = \dfrac{1}{\underset{s\to 0}{Lt}\, s^{\cancel{2}} \cdot \dfrac{K'(s+z_1)(s+z_2)\cdots}{\cancel{s}(s+p_1)(s+p_2)\cdots}}$

$= \dfrac{1}{0} = \infty$

(3) <u>Type-2 System</u> : If $n = 2$; $G(s) = \dfrac{K'(s+z_1)(s+z_2)\cdots}{s^2(s+p_1)(s+p_2)\cdots}$

$\therefore$ (i) $e_{ss}(\text{position input}) = \dfrac{1}{1 + \underset{s\to 0}{Lt}\, G(s)H(s)}$

if $H(s) = 1$ $\therefore e_{ss} = \dfrac{1}{1 + \underset{s\to 0}{Lt}\, \dfrac{K'(s+z_1)(s+z_2)\cdots}{s^2(s+p_1)(s+p_2)\cdots}}$

$= \dfrac{1}{1+\infty} = \dfrac{1}{\infty} = 0$

(ii) $e_{ss}(\text{velocity}) = ?$

$e_{ss}(\text{velocity}) = \dfrac{1}{\underset{s\to 0}{Lt}\, sG(s)H(s)}$ and if $H(s) = 1$

$e_{ss}(\text{velocity}) = \dfrac{1}{\underset{s\to 0}{Lt}\, s \cdot \dfrac{K'(s+z_1)(s+z_2)\cdots}{s^2(s+p_1)(s+p_2)\cdots}} = \dfrac{1}{\infty} = 0$

(iii) $e_{ss}(\text{velocity}) = \dfrac{1}{\underset{s\to 0}{Lt}\, \cancel{s^2} \cdot \dfrac{K'(s+z_1)(s+z_2)\cdots}{\cancel{s^2}(s+p_1)(s+p_2)\cdots}} = \dfrac{1}{K'} = \dfrac{1}{K_a}$

Thus a type-2 system has zero position error, zero velocity error and a constant acceleration error.

Type of Input	Steady-state Error: Type-0 System	Type-1 System	Type-2 System
unit step input	$\frac{1}{1+K_p}$	0	0
unit-ramp	∞	$\frac{1}{K_v}$	0
unit-parabolic	∞	∞	$\frac{1}{K_a}$

Table: Steady-state Errors for various Inputs and systems Types.

The error Constants K_p, K_v and K_a describe the ability of a system to reduce or eliminate steady-state errors. As the type of system becomes higher, progressively more errors (steady-state) are eliminated. In general, type-0, -1 and -2 are the most commonly employed systems in practice. Systems with type higher than 2 are not employed in practice because of two reasons.

(i) These are more difficult to stabilize

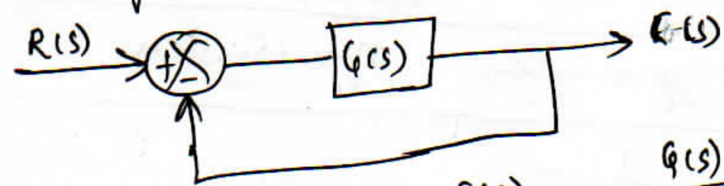
(ii) The dynamic errors for such systems tend to be larger than those for type-0, -1 and -2, although their steady state performance is desirable.

One of the disadvantages of error constants is that they do not give information on the steady-state error when inputs are other than the three basic types - step, ramp and parabolic. Another difficulty is that the error constants fail to indicate the exact manner in which error function change with time.

Problems

① obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$, when the input is unit step

(Sol)



The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{\frac{4}{s(s+5)}}{1+\frac{4}{s(s+5)}(1)} = \frac{4}{s(s+5)+4} = \frac{4}{s^2+5s+4} \rightarrow ①$$

given that $r(t) = u(t)$ $\therefore R(s) = \frac{1}{s} \longrightarrow ②$

from eqs ① & ②

$$C(s) = R(s)\,\frac{4}{s^2+5s+4} = \frac{4}{s(s^2+5s+4)} = \frac{4}{s(s+1)(s+4)}$$

The time response is obtained by taking inverse laplace transform of C(s).

$$\therefore c(t) = L^{-1}C(s) = L^{-1}\left\{\frac{4}{s(s+1)(s+4)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s} - \frac{4}{3}\cdot\frac{1}{s+1} + \frac{1}{3}\cdot\frac{1}{s+4}\right\}$$

$$= 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t}$$

② The response of a servo mechanism is $c(t) = 1+0.2e^{-60t} - 1.2e^{-10t}$ when subject to a unit step input. obtain an expression for closed loop transfer function. Determine undamped natural frequency and damping ratio.

(Sol) given that $c(t) = 1+0.2e^{-60t} - 1.2e^{-10t}$

taking LT on both sides

$$C(s) = 1 + 0.2\,\frac{1}{s+60} - 1.2\,\frac{1}{s+10}$$

$$= \frac{(s+60)(s+10) + 0.2(s+10) - 1.2(s+60)}{(s+60)(s+10)}$$

$$\Rightarrow C(s) = \frac{s^2+70s+600+0.2s^2+2s-1.2s^2-72s}{s(s+60)(s+10)} \quad (11)$$

$$= \frac{600}{s(s+10)(s+60)}$$

$\therefore$ The closed loop transfer function $= \dfrac{C(s)}{R(s)} = \dfrac{600}{(s+10)(s+60)}$

where $r(t) = u(t)$

$\therefore R(s) = 1/s$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{600}{s^2+70s+600} \longrightarrow (1)$$

The general form of 2nd order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \longrightarrow (2)$$

from eq (1) & (2) $\quad 2\zeta\omega_n = 70 \;;\; \omega_n^2 = 600$

$\therefore$ undamped natural frequency $\omega_n = \sqrt{600}$ rad/sec

The damping factor $\zeta = \dfrac{70}{2\omega_n} = 1.43$

(3) A unity feedback system is characterized by an open loop transfer function $\dfrac{K}{s(s+10)}$. Determine the gain 'K' so that the system will have a damping ratio of 0.5. For this value of K determine peak over shoot and time at peak overshoot.

(Sol)

The closed loop transfer function $\dfrac{C(s)}{R(s)} = \dfrac{K}{s(s+10)+K}$

$$= \frac{K}{s^2+10s+K}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$2\zeta\omega_n = 10 \;;\; \omega_n^2 = K \Rightarrow K = 100$

$2(0.5)\omega_n = 10$

$\Rightarrow \omega_n = 0$

$\% M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100 = 16.3\%$

$t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{10\sqrt{1-0.5^2}} = 0.363$ sec

(4) A closed loop servo is represented by the differential equation $\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64e$. where c is the displacement of the output shaft and 'r' is the displacement of input shaft and $e = r - c$. Determine undamped natural frequency, damping ratio and percentage maximum overshoot for unit step input

(Sol) The system is represented by $\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64e$

where $e = r - c$ $\quad r$ = input ; c = output

$$\therefore \frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64(r-c)$$

taking LT on both sides

$$s^2C(s) + 8s\,C(s) = 64[R(s) - C(s)]$$

$$\text{or } C(s)[s^2+8s+64] = 64R(s) \longrightarrow (1)$$

$$\therefore TF = \frac{C(s)}{R(s)} = \frac{64}{s^2+8s+64}$$

The general form of 2nd order system TF is $\longrightarrow (2)$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

from eqs (1) & (2) $\quad 2\xi\omega_n = 8$; $\omega_n^2 = 64$

$\therefore \omega_n = 8$ rad/sec

$$\therefore \xi = \frac{8}{2\omega_n} = \frac{8}{2(8)} = 0.5$$

$\therefore$ undamped natural frequency $= \omega_n = 8$ rad/sec

damping factor $\xi = 0.5$

$$\text{\% Max peak overshoot } M_p = 100\,e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

$$= 100\,e^{\frac{-\pi\times 1/2}{\sqrt{1-(1/2)^2}}}$$

$$= 100\,e^{-\frac{\pi}{2\sqrt{3}/2}} = 100\,e^{-\pi/\sqrt{3}} = 16.37\%$$

(5) A system has the closed loop transfer function $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$ It is required that the unit step response of the system should be have a settling time of 2 sec according to 2% criterion and the overshoot should be approximately 5%. what should be the closed loop pole locations.

(Sol) Given that $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

Settling time $t_s = \frac{4}{\xi\omega_n}$ for 2% tolerance

$= 2$ sec

$\therefore \xi\omega_n = \frac{4}{2} = 2 \longrightarrow (1)$

% peak overshoot $= 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = 5$

$\therefore e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = \frac{5}{100}$

taking 'ln' on both sides

$$\frac{-\pi\xi}{\sqrt{1-\xi^2}} = \ln(0.05) = -3$$

$$\therefore \frac{\pi^2\xi^2}{1-\xi^2} = 9 \Rightarrow \xi^2(\pi^2+9) = 9$$

$$\therefore \xi = \sqrt{\frac{9}{(9+\pi^2)}} = 0.69 \longrightarrow (2)$$

substituting eq (2) in (1) $\omega_n = \frac{2}{\xi} = 2.895$ rad/sec

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{8.39}{s^2+4s+8.39}$$

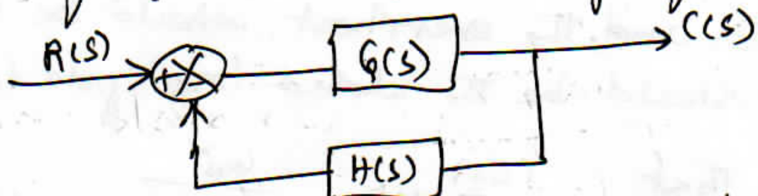
$$\therefore s = \frac{-4 \pm \sqrt{4^2-4(1)(8.39)}}{2} = \frac{-4 \pm j4.19}{2}$$

$$= -2 \pm j2.09$$

$\therefore$ The poles at $s_1 = -2+j2.09$ and $s_2 = -2-j2.09$

(6) For a unity feedback system, the open loop transfer function $G(s) = \frac{10}{s(s+2)}$; find the time domain specifications for a step input of 12 units.

(Sol)



Given that $G(s) = \frac{10}{s(s+2)}$; $H(s) = 1$

$\therefore$ The closed loop TF $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{10}{s(s+2)+10}$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{10}{s^2+2s+10} \longrightarrow ①$$

The ... standard form of 2nd order system TF is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \longrightarrow ②$$

Comparing eq ① & ②

$$\omega_n^2 = 10 \Longrightarrow \omega_n = \sqrt{10} \text{ rad/sec}$$

$$2\xi\omega_n = 2 \Rightarrow \xi = \frac{2}{2\omega_n} = \frac{1}{\sqrt{10}}$$

The time domain specifications are

(i) Rise time $t_r = \frac{\pi - \phi}{\omega_d} = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$

$$= \frac{\pi - \tan^{-1}\left(\sqrt{1-1/10} / 1/\sqrt{10}\right)}{\sqrt{10}\sqrt{9/10}} = \frac{\pi - \tan^{-1} 3}{3}$$

$$= 0.63 \text{ sec}$$

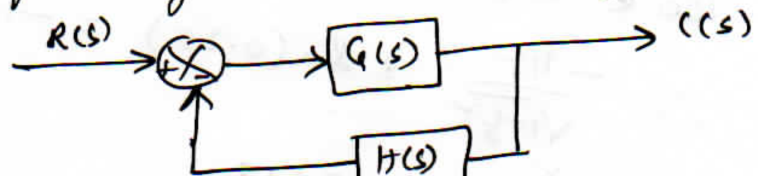
(ii) peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{3} = 1.05 \text{ sec}$

(iii) $\% M_p = 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = 100\, e^{\frac{-\pi \cdot \frac{1}{\sqrt{10}}}{3/\sqrt{10}}} = 100\, e^{-\pi/3} = 35\%$.

(iv) $t_s = \frac{4}{\xi\omega_n} = \frac{4}{\frac{1}{\sqrt{10}}\sqrt{10}} = 4 \text{ sec}$ for 2% tolerance

1) The open loop transfer function of a unity feedback system is given by $G(s) = \frac{K}{s(Ts+1)}$. where K and T are constants. By what factor should the amplifier gain be reduced so that the peak overshoot of unit ramp response of the system is reduced from 75% to 25%.

(Sol)



Given that $G(s) = \frac{K}{s(Ts+1)}$; $H(s) = 1$

$\therefore$ The closed loop TF $= \frac{C(s)}{R(s)} = \frac{K}{s(Ts+1)+K} = \frac{K}{s^2T+s+K}$

or $\frac{C(s)}{R(s)} = \frac{(K/T)}{s^2+\frac{1}{T}s+K/T} \longrightarrow ①$

Also $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \longrightarrow ②$

from eq ① & ② $\omega_n = \sqrt{K/T}$ and $2\xi\omega_n = \frac{1}{T}$

$\therefore \xi = \frac{1}{2T\omega_n} = \frac{1}{2T\sqrt{K/T}}$

$= \frac{1}{2\sqrt{KT}} \longrightarrow ①$

For 75% peak over shoot let $K = K_1$

$\therefore \% M_p = 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = 75$

$e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} = 0.75$

taking 'ln' on both sides

$\frac{-\pi\xi}{\sqrt{1-\xi^2}} = \ln(0.75) = -0.2877$

$\frac{\pi^2\xi^2}{1-\xi^2} = 0.082$

$\Rightarrow \xi^2(\pi^2+0.082) = 0.082$

$\therefore \xi = \frac{\sqrt{0.082}}{\sqrt{\pi^2+0.082}} = \sqrt{\frac{0.082}{9.8764}} = 0.09 \longrightarrow ②$

from eqs ① & ②

$$\frac{1}{2\sqrt{KT}} = 0.09 \quad \therefore \quad K_1 = \frac{1}{T}\cdot\frac{1}{4(0.09)^2}$$

$$= \frac{30.86}{T} \rightarrow (i)$$

for 25% peak over shoot let $K = K_2$

$$\therefore \quad 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = 25$$

$$\therefore \quad \frac{-\pi\xi}{\sqrt{1-\xi^2}} = \ln(0.25) = -1.386$$

$$\frac{\pi^2\xi^2}{1-\xi^2} = 1.9218$$

$$\Rightarrow \xi^2(\pi^2 + 1.9218) = 1.9218$$

$$\text{and } \xi = \sqrt{\frac{1.9218}{\pi^2+1.9218}} = 0.4037 \rightarrow ③$$

from eqs ① & ③ $\quad \frac{1}{2\sqrt{K_2 T}} = 0.4037$

$$\therefore \quad K_2 = \frac{1}{T}\cdot\frac{1}{4(0.4037)^2} = 1.53/T \rightarrow (ii)$$

$$\therefore \quad \frac{K_1}{K_2} = \frac{(30.86/T)}{(1.53/T)} = 20$$

$\therefore$ The gain should be reduced by a factor 20

⑥ The open loop transfer function of a unity feedback system is given by $G(s) = \frac{K}{s(s+1)(s+2)}$. Find the minimum value of 'K' for which the steady state error is less than 0.1 for unit ramp input

(Sol) Given that $G(s) = \frac{K}{s(s+1)(s+2)}$

$$H(s) = 1$$

$$r(t) = t \quad \therefore R(s) = \frac{1}{s^2}$$

The steady state error $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, sE(s)$

$$= \underset{s\to 0}{Lt} \frac{s\,R(s)}{1+G(s)H(s)}$$

$$0.10 = \underset{s\to 0}{Lt} \frac{s\cdot\frac{1}{s^2}}{1+\frac{K}{s(s+1)(s+2)}}$$

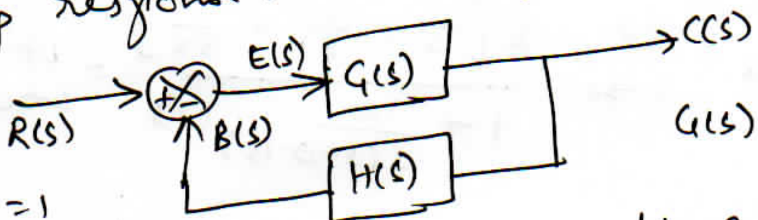
$$\Rightarrow 0.1 = \underset{s\to 0}{Lt} \frac{\left(\frac{1}{s}\right)}{\frac{s(s+1)(s+2)+K}{s(s+1)(s+2)}}$$

$$0.1 = \underset{s\to 0}{Lt} \frac{\left(\frac{1}{\cancel{s}}\right)\cancel{s}(s+1)(s+2)}{s(s+1)(s+2)+K} = \frac{2}{K}$$

$$\boxed{\therefore K = \frac{2}{0.1} = 20}$$

(7) A unity feedback control system has the closed loop transfer function $\frac{Ks+b}{s^2+as+b}$. Determine the steady state error in the unit ramp response in terms of K, a, and b.

(Sol)



$G(s) = \frac{Ks+b}{s^2+as+b}$

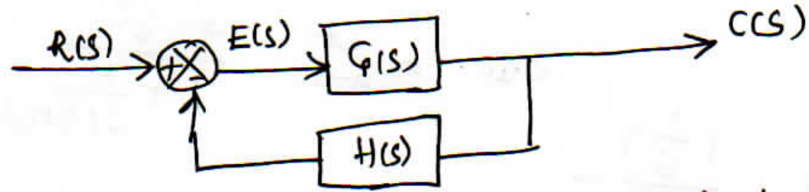
$r(t) = t;\ H(s) = 1$

Steady state error $e_{ss} = \underset{s\to 0}{Lt}\, sE(s) = \underset{s\to 0}{Lt} \frac{s\,R(s)}{1+G(s)H(s)}$

$$\Rightarrow e_{ss} = \underset{s\to 0}{Lt} \frac{s\cdot\frac{1}{s^2}}{1+\frac{(Ks+b)}{s^2+as+b}(1)} = \underset{s\to 0}{Lt} \frac{(1/s)(s^2+as+b)}{s^2+as+b+Ks+b} = \infty$$

① Find the steady state error as a function of time for the unity feedback system $G(s) = \frac{100}{s(1+0.1s)}$ for the input $r(t) = 1 + 2t + t^2/2$

(Sol)



Given that $G(s) = \frac{100}{s(1+0.1s)}$; $H(s) = 1$

Steady state Error $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, sE(s)$

$$= \underset{s\to 0}{Lt}\; s\frac{R(s)}{1+G(s)H(s)}$$

where $R(s) = L[r(t)] = L[1 + 2t + t^2/2]$

$$= \frac{1}{s} + \frac{2}{s^2} + \frac{1}{s^3}$$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\; \frac{s\left[\frac{1}{s} + \frac{2}{s^2} + \frac{1}{s^3}\right]}{1 + \frac{100}{s(\ +0.1s)}(1)} = \underset{s\to 0}{Lt}\; \frac{s\left[\frac{1}{s} + \frac{2}{s^2} + \frac{1}{s^3}\right]}{\frac{s(\ +0.1s) + 100}{s(1+0.1s)}}$$

$$= \underset{s\to 0}{Lt}\; \frac{s\cdot\frac{1}{s}\, s(1+0.1s)}{s(1+0.1s)+100} + \underset{s\to 0}{Lt}\; \frac{s\cdot\frac{2}{s^2}\, s(0.1s+1)}{s(0.1s+1)+00} +$$

$$\underset{s\to 0}{Lt}\; \frac{s\cdot\frac{1}{s^3}\, s(0.1s+1)}{s(0.1s+1)+100}$$

$$e_{ss} = \underset{s\to 0}{Lt}\; \frac{s(1+0.1s)}{s(0.1s+1)+100} + \underset{s\to 0}{Lt}\; \frac{2(0.1s+1)}{s(0.1s+1)+00} + \underset{s\to 0}{Lt}\; \frac{(0.1s+1)}{s[s(0.1s+1)+00]}$$

$$= 0 + \frac{2}{100} + \frac{1}{0}$$

$$= 0 + \frac{2}{100} + \infty = \infty$$

UNIT - III

STABILITY

A linear-time invariant system is stable, if the following two notions of system stability are satisfied

(1) when the system is excited by a bounded input, the output is bounded.

(2) In the absence of the input, the output tends towards zero irrespective of initial conditions. This stability concept is known as asymptotic stability.

Let us observe the implication of the two notions of stability defined, by considering a single-input, single-output systems with transfer function

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n} \quad ; \quad m < n$$

with zero initial conditions, the output of the system is given by $c(t) = L^{-1}[G(s)R(s)] = \int_0^\infty g(\tau) r(t-\tau) d\tau$

where $g(t) = L^{-1}[G(s)]$ is the impulse response of the system

Taking the absolute value on both sides, we get

$$|c(t)| = \left| \int_0^\infty g(\tau) r(t-\tau) d\tau \right|$$

Since the absolute value of integral is not greater than the integral of the absolute value of the integrand

$$|c(t)| \leq \int_0^\infty |g(\tau)| |r(t-\tau)| d\tau$$

Since, the first notion of stability is satisfied if for every bounded input ($|r(t)| \leq M_1 < \infty$), the output is bounded ($|c(t)| \leq M_2 < \infty$); thus for bounded input, the bounded output condition is

$$|c(t)| \leq M_1 \int_0^\infty |g(\tau)| d\tau \leq M_2 \longrightarrow (1)$$

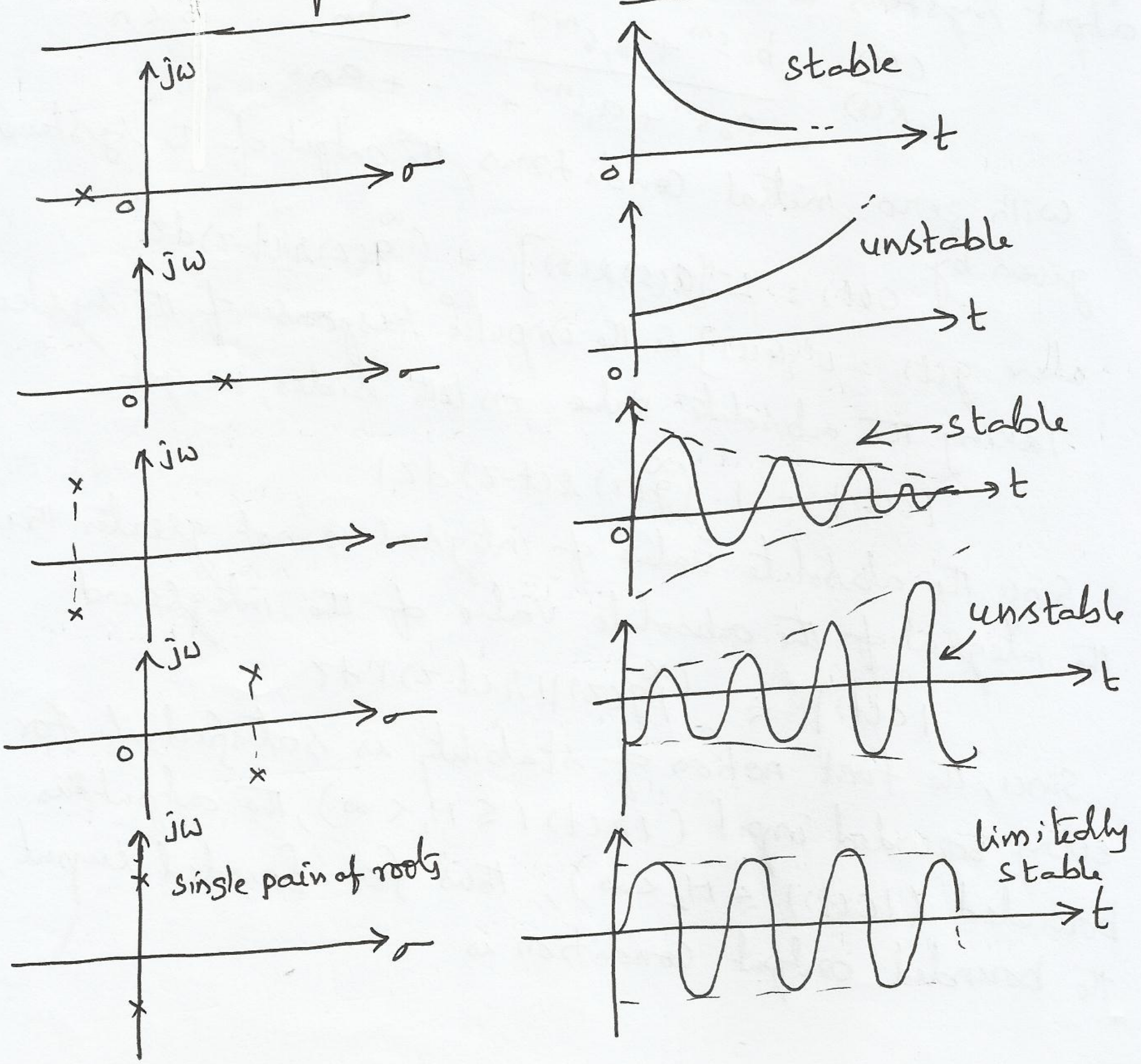
Thus the first notion of stability is satisfied if the impulse response $g(t)$ is absolutely integrable.

ie $\int_0^\infty |g(\tau)| d\tau$ is finite.

The nature of $g(t)$ depends upon the poles of the transfer function, which are the roots of the characteristic equation.

The nature of response terms contributed by all possible types of roots are shown in figure below.

Roots in the s plane — Corresponding Impulse Response



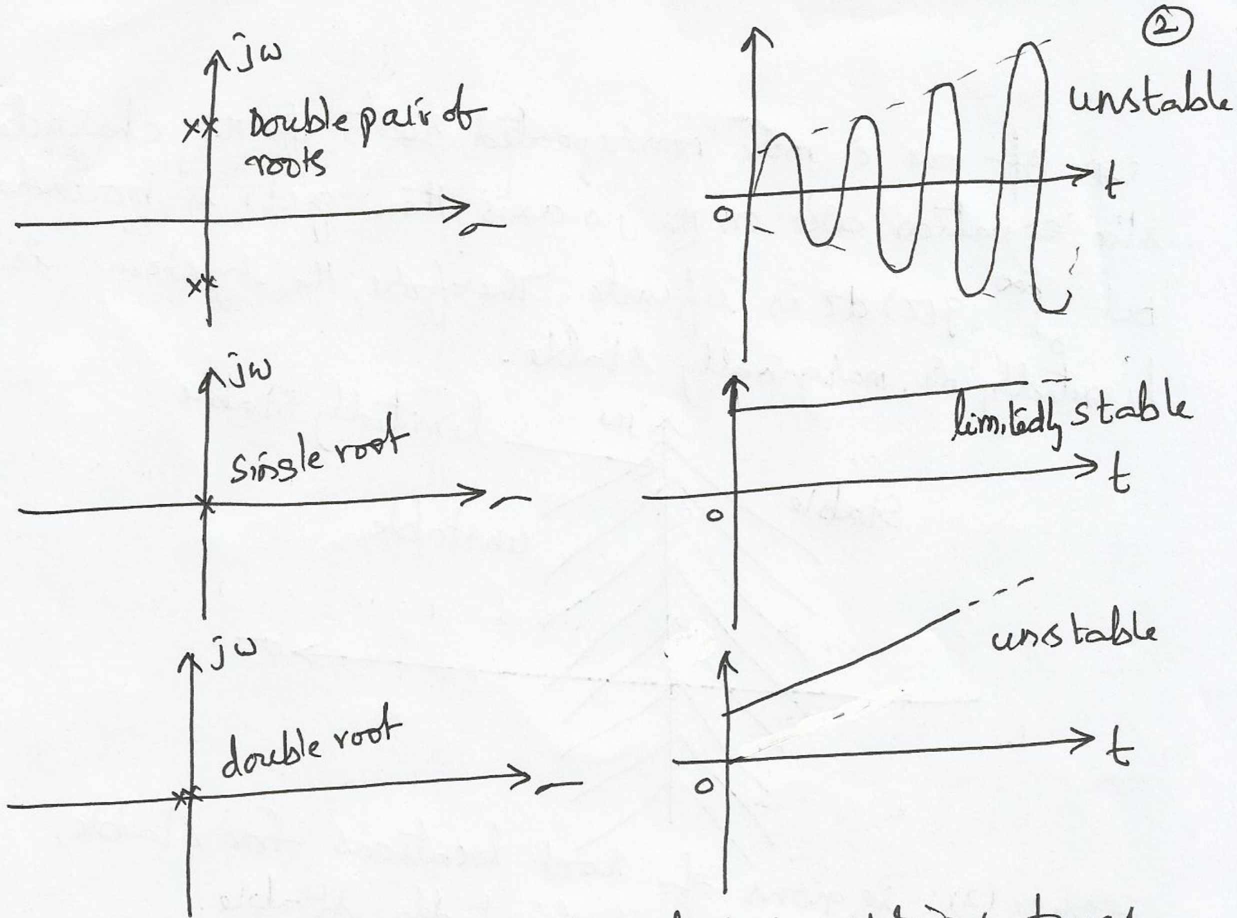


Figure: Response terms contributed by various types of roots

The above observations lead to the following general conclusions regarding system stability

(1) If all the roots of characteristic equation have negative real parts, then the impulse response $g(t)$ is bounded and $\int_0^\infty |g(\tau)|d\tau$ is finite. Therefore the system is stable.

(2) If any root of the characteristic equation has a positive real part, $g(t)$ is unbounded and $\int_0^\infty |g(\tau)|d\tau$ is infinite. Therefore, the system is unstable.

(3) If the characteristic equation has repeated roots on the $j\omega$-axis, $g(t)$ is unbounded and $\int_0^\infty |g(\tau)|d\tau$ is infinite. Therefore, the system is unstable.

(4) If one or more nonrepeated roots of the characteristic equation are on the $j\omega$-axis, then $g(t)$ is bounded but $\int_0^\infty |g(\tau)|\,d\tau$ is infinite. Therefore, the system is limitedly or marginally stable.

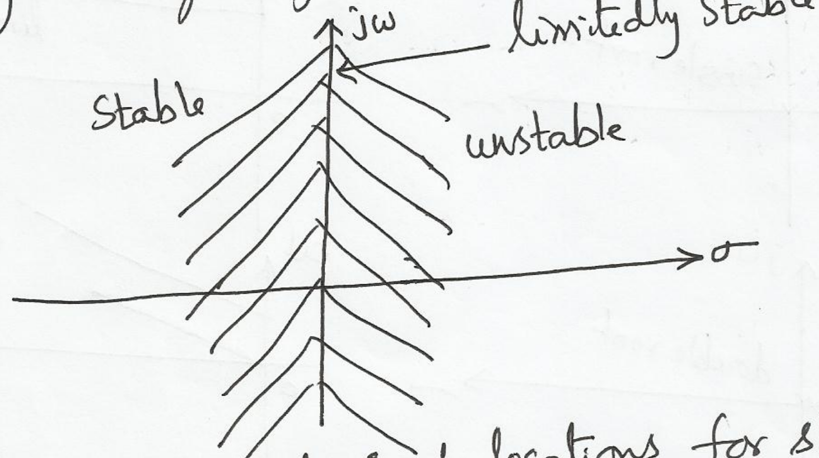


Figure (2): Regions of root locations for stable, unstable and limitedly stable.

In a vast majority of practical systems, the following statements on stability are quite useful.

(1) If all the roots of the characteristic equation have negative real parts, the system is stable.

(2) If any root of the characteristic equation has a positive real part or if there is a repeated root on the $j\omega$-axis, the system is unstable.

(3) If the condition (1) is satisfied except for the presence of one or more nonrepeated roots on the $j\omega$-axis, the system is limitedly or marginally stable.

A linear system is characterized as

(1) Absolutely stable with respect to a parameter of the system if it is stable for all values of this parameter.

(2) Conditionally stable with respect to a parameter, if the system is stable for only certain bounded ranges of values of this parameter.

<u>Routh Stability criterion</u>: This criterion is based on ordering the coefficients of the characteristic equation into an array, called Routh array as given below

Let us consider a characteristic equation given by

$$q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

The Routh array of $q(s)$ is as follows.

$$\begin{array}{lllll}
s^n & a_0 & a_2 & a_4 & a_6 \; - \\
s^{n-1} & a_1 & a_3 & a_5 & - - - \\
s^{n-2} & b_1 & b_2 & b_3 & - - - \\
s^{n-3} & c_1 & c_2 & c_3 & - - - \\
s^{n-4} & d_1 & d_2 & \cdots & \\
\vdots & \vdots & \vdots & & \\
s^2 & e_1 & a_n & & \\
s^1 & f_1 & & & \\
s^0 & a_n & & &
\end{array}$$

The coefficients $b_1, b_2 ---$ are evaluated as follows

$$b_1 = \frac{(a_1 a_2 - a_0 a_3)}{a_1}; \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

This process is continued till we get a zero as the last coefficient in the third row. In a similar way, the coefficients of $4^{th}, 5^{th} --$ n^{th} and $(n+1)^{th}$ rows are evaluated.

$$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}; \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}; ---$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}; \quad d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}; ---$$

In the process of generating Routh array the missing terms are regarded as zero. Also all the elements of any row can be divided by a positive Constant during the process to simplify the computational work.

The Routh stability criterion is stated as below

" For a system to be stable, it is necessary and sufficient that each term of first column of Routh array of its characteristic equation be positive if $a_0 > 0$. If this condition is not met, the system is unstable and number of sign changes of the terms of the first column of Routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane".

① The characteristic equation of a system is given by $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$; check, whether the system is stable or not.

(Sol)

$$
\begin{array}{llll}
s^4 & 1 & 18 & 5 \\
s^3 & 8 & 16 & 0 \\
s^2 & \frac{8\times18 - 16\times1}{8} = 16 & \frac{8\times5 - 1\times0}{8} = 5 & \\
s^1 & \frac{16\times16 - 8\times5}{16} = 135 & 0 & \\
s^0 & 5 & &
\end{array}
$$

Since all the terms in the first column are positive hence the system is stable.

② The CE of a system is $3s^4 + 10s^3 + 5s^2 + 5s + 2 = 0$. check, whether the system is stable or not.

(Sol) The Routh array is

$$
\begin{array}{llll}
s^4 & 3 & 5 & 2 \\
s^3 & 10 & 5 & 0 \\
s^3 & 2 & 1 & 0
\end{array}
$$

$s^2 \quad \frac{7}{2} \quad 2$

$s^1 \quad -\frac{1}{7}$

$s^0 \quad 2$

It may be noted that in order to simplify computational work, the s^3-row is modified by dividing it by 5.
Examining the first column, there are two sign changes.
Therefore, the system is unstable having two poles in the right-half s-plane.

(3) The characteristic equation of a system in differential equation form is $\ddot{x} - (K+2)\dot{x} + (2K+5)x = 0$.

(a) Find the value of 'K' for which the system is
(i) stable (ii) limitedly stable (iii) unstable

(b) For the stable case for what values of K, The system is
(i) critically damped (ii) under damped (iii) over damped

(Sol) Given that $\ddot{x} - (K+2)\dot{x} + (2K+5)x = 0$

taking Laplace transform with zero initial conditions

$s^2 X(s) - (K+2)sX(s) + (2K+5)X(s) = 0$

$X(s)[s^2 - (K+2)s + (2K+5)] = 0$

or $s^2 - (K+2)s + (2K+5) = 0$

The Routh array is

$s^2 \quad 1 \quad (2K+5)$

$s^1 \quad -(K+2) \quad 0$

$s^0 \quad (2K+5)$

(a)(i) For the system to be stable, all the terms must be +ve

$-(K+2) > 0$ and $2K+5 > 0$

or $K+2 < 0$ & $2K > -5$

$K < -2$ and $K > -2.5$

or $\boxed{-2.5 < K < -2}$

$\boxed{-2 > K > -2.5}$

(ii) For limitedly stable system

$K = -2$ and $K = -2.5$

(iii) For the system to be unstable

$K > -2$ and $K < -2.5$

(b) The roots of characteristic equation are

$$S_1, S_2 = \frac{1}{2}\left\{ K+2 \pm \sqrt{(K+2)^2 - 4(2K+5)} \right\}$$

(i) For critically damped system, the imaginary part is zero.

$$\therefore (K+2)^2 - 4(2K+5) = 0$$

$$K = 6.47, \; -2.47$$

For $K = 6.47$, the system is unstable. Hence, for the stable critically damped system $K = -2.47$

(ii) For under damped case $\quad -2 > K > -2.47$
(Larger than critically damped)

(iii) For over damped case $\quad -2.47 > K > -2.5$
(Smaller than critically damped)

<u>Special Cases</u>: The following difficulties arise in Routh array formation.

<u>Difficulty (1)</u>: When the first term in any row of Routh array is zero while rest of the row has at least one nonzero term. Because of this zero term, the terms in the next row become infinite and the Routh's test breaks down. The following methods can be used to overcome this difficulty.

(a) Substitute a small positive number 'ϵ' for the zero and proceed to evaluate the rest of the Routh array. Then examine the signs of the first column of Routh array by letting $\epsilon \to 0$

or

(b) Modify the original characteristic equation by replacing 's' by $\frac{1}{z}$. Apply the Routh's test on the modified equation interms of 'z'. This transformation maps the left half of the s-plane into the left half of the z-plane and the right half of the s-plane into right half of the z-plane. The number of z-roots with positive real parts are the same as the number of s-roots with positive real parts. This method works in most but not all cases.

(3) The characteristic equation of a system is given by $s^5+s^4+2s^3+2s^2+3s+5=0$. Check the stability of the system.

(Sol) The Routh array is

s^5	1	2	3
s^4	1	2	5
$\boxed{s^3}$	$\boxed{0}$	$\boxed{-2}$	
s^3	ϵ	-2	
s^2	$\frac{2\epsilon+2}{\epsilon}$	5	
s^1	$\frac{-4\epsilon-4-5\epsilon^2}{2\epsilon+2}$		
s^0	2		

when $\epsilon \to 0$ $\frac{-4\epsilon-4-5\epsilon^2}{2\epsilon+2} \to -2$

The first element in the third row is zero. It is replaced by ϵ, a small positive number. The first element in the 4th row is now $\frac{2\epsilon+2}{\epsilon}$ which has a positive sign as $\epsilon \to 0$. The first term in 5th row is $-ve$ as $\epsilon \to 0$. Examining the terms in the first column of Routh array, it is found that there are two changes in sign and hence the system is unstable having two poles in the right half s-plane.

II Method: Replacing 's' by $\frac{1}{z}$ in the characteristic equation, we will get

$$\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^4 + 2\left(\frac{1}{z}\right)^3 + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right) + 5 = 0$$

or $5z^5 + 3z^4 + 2z^3 + 2z^2 + z + 1 = 0$

The Rouths array for this equation is

z^5	5	2	1
z^4	3	2	1
z^3	$-4/3$	$-2/3$	
z^2	$1/2$	1	
z^1	2		
z^0	1		

There are two changes of signs in the first column of Rouths array, which indicates that there are 2 z-roots in RHS plane of 'z'. Therefore, the number of 's' roots in RHS plane of 's' is also 2.

Difficulty 2: when all the elements in any one row of the Rouths array are zero. This condition indicates that there are symmetrically located roots in the s-plane. The polynomial whose coefficients are the elements of the row just above the row of zeros in the Rouths array is called an auxiliary polynomial. This polynomial gives the number and location of root pairs of the characteristic equation which are symmetrically located in the s-plane. The order of the auxiliary polynomial is always even.

Because of a zero row in the array, the Routh's test breaks down. This situation is overcome by replacing the row of zeros in the Routh array by a row of coefficients of the polynomial generated by taking the first derivative of the auxiliary polynomial.

1) Check the system with CE $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$ is stable or not.

(Sol) Routh array is

s^6	1	8	20	16
s^5	2	12	16	
s^4	2	12	16	
s^3	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	
s^3	8	24		
s^3	1	3		
s^2	6	16		
s^2	3	8		
s^1	1/3			
s^0	8			

The auxiliary polynomial is

$2s^4+12s^2+16=A(s)$

$\frac{d}{ds}A(s) = 8s^3+24s$

There are no sign changes in the first column. Therefore, the number of roots in RHS of s-plane are zero.

The roots of auxiliary equation are

$2s^4+12s^2+16=0$ or $s^4+6s^2+8=0$

$\therefore s^2 = \frac{-6\pm\sqrt{36-32}}{2} = \frac{-6\pm 2}{2} = -4, -2$

if $s^2=-4$; then $s=\pm\sqrt{-4} = \pm j2$

$s^2=-2$; then $s=\pm\sqrt{-2} = \pm j\sqrt{2}$

There are also the roots of CE, and the system is limitedly stable.

① The open loop transfer function of a unity feedback system is given by $G(s) = \frac{K}{s(s^2+s+1)(s+4)}$. Determine the value of for the system to be stable.

(Sol) $$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s(s^2+s+1)(s+4)+K}$$

$$= \frac{K}{s^4+5s^3+5s^2+4s+K}$$

The characteristic equation is $s^4+5s^3+5s^2+4s+K=0$

The Routh array is:

s^4	1	5	K
s^3	5	4	
s^2	$\frac{21}{5}$	K	
s^1	$\frac{\frac{84}{5}-5K}{(21/5)}$		
s^0	K		

For the system to be stable, all the terms in the first column must be greater than zero

ie $\frac{\frac{84}{5}-5K}{(21/5)} > 0$ or $\frac{84}{5} - 5K > 0$ or $84 > 25K$

or $K < \frac{84}{25}$ and $K > 0$ $\boxed{0 < K < \frac{84}{25}}$

For the system to be stable

$$\boxed{0 < K < \frac{84}{25}}$$

(1) The open loop transfer function of a unity feedback control system is given by $G(s) = \dfrac{K}{(s+2)(s+4)(s^2+6s+25)}$. By applying Routh criterion, discuss the stability of the closed loop system as a function of 'K'. Determine the value of 'K' which will cause sustained oscillations in the closed loop system. What are the corresponding oscillation frequencies.

Sol) The characteristic equation of given control system is $1+G(s)H(s) = 0$

$$\Rightarrow 1 + \frac{K}{(s+2)(s+4)(s^2+6s+25)} = 0$$

$$\Rightarrow (s+2)(s+4)(s^2+6s+25) + K = 0$$

$$(s^2+6s+8)(s^2+6s+25) + K = 0$$

or $s^4 + 12s^3 + 69s^2 + 198s + (200+K) = 0$

The Routh array of given CE is

s^4	1	69	$(200+K)$
s^3	12	198	0
s^2	52.5	$(200+K)$	
s^1	$\dfrac{7995-12K}{52.5}$		
s^0	$(200+K)$		

(i) For the system to be stable, all the terms in the first column of Routh array of CE must be positive

ie $\dfrac{7995-12K}{52.5} > 0$ or $K < \dfrac{7995}{12}$

or $K < 666.25$

also $(200+K) > 0$ or $K > -200$

$\therefore$ For the system to be stable $\underline{-200 < K < 666.25}$

(ii) To sustain oscillations $\frac{7995-K}{52.5} = 0$

or $K = 666.25$

$\therefore$ The Auxiliary equation is

$$(52.5)s^2 + (200+K) = 0$$

or $(52.5)s^2 + (200 + 666.25) = 0$

or $52.5 s^2 = -866.25$

or $s^2 = -16.5$

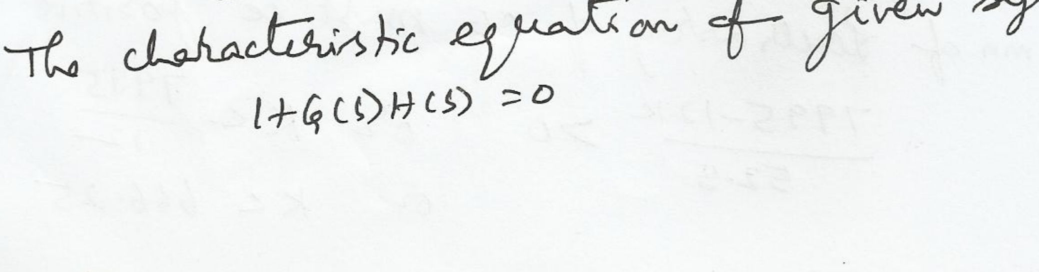
where $s = j\omega$

$\therefore (j\omega)^2 = -16.5$

$\therefore \omega = \sqrt{16.5} = 4.06$ rad/sec

The frequency of oscillations is $\omega = 4.06$ rad/sec

② A system oscillates with frequency 'ω' if it has poles at $s = \pm j\omega$ and no poles in the right half s-plane. Determine the values of 'K' and 'a' so that the system shown in figure oscillates at a frequency 2 rad/sec.



(Sol) The characteristic equation of given system is

$$1+G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{K(s+1)}{s^3+as^2+2s+1} = 0$$

$$s^3+as^2+2s+1+K(s+1) = 0$$

or $s^3+as^2+s(2+K)+(K+1) = 0$

The Routh array of given CE is

$$\begin{array}{lll} s^3 & 1 & (2+K) \\ s^2 & a & (K+1) \\ s^1 & \frac{a(2+K)-(K+1)}{a} & \\ s^0 & (K+1) & \end{array}$$

To sustain oscillations $\frac{a(2+K)-(K+1)}{a} = 0$

$$\Rightarrow a = \frac{K+1}{2+K}$$

The auxiliary polynomial is

$$as^2+(K+1) = 0$$

$$\left(\frac{K+1}{K+2}\right)s^2+(K+1) = 0$$

$$\Rightarrow s^2 = -(K+2)$$

Given that the frequency of oscillations is

$$\omega = 2 \text{ rad/sec.}$$

$$\therefore (j\omega)^2 = -(K+2)$$

$$(j2)^2 = -(K+2)$$

or $K+2 = 4$

$$\therefore K = 2$$

and $a = \frac{K+1}{K+2} = \frac{2+1}{2+2} = \frac{3}{4} = 0.75$

(3) A feedback system has an open-loop TF

$G(s)H(s) = \dfrac{Ke^{-s}}{s(s^2+5s+9)}$. Determine the maximum value of 'K' for the system to be stable.

(Sol) Note: For low frequencies $e^{-s} = 1-s$

$\therefore$ The characteristic equation of the system is given by $1+G(s)H(s) = 0$

ie $1 + \dfrac{K(1-s)}{s(s^2+5s+9)} = 0$

$$s(s^2+5s+9) + K(1-s) = 0$$

$$s^3+5s^2+9s+K-Ks = 0$$

$$s^3+5s^2+s(9-K)+K = 0$$

The Routh array of CE is

s^3	1	$9-K$
s^2	5	K
s^1	$\dfrac{5(9-K)-K}{5}$	0
s^0	K	

For the system to be stable, all the terms in the first column of Routh array must be positive.

ie $\dfrac{45-6K}{5} > 0$ or $45 > 6K$ or $K < \frac{45}{6}$

also $K > 0$ ie $0 < K < \frac{45}{6}$

$\therefore$ The maximum value of 'K' for the system to be stable is $K = \frac{45}{6}$

<u>Relative stability</u>: once the System is said to be stable, the relative stability quantitatively determined by finding the settling time of the dominant roots of the characteristic equation. The roots nearer to imaginary axis in left hand side of the s-plane are known as dominant roots. The settling time is inversely proportional to the real part of the dominant roots.

The relative stability can be specified by requiring that all the roots of the characteristic equation are more negative than a certain value. That is, all the roots must lie to the left of the lines $s = -s_1$ $(s_1 > 0)$. Then the characteristic equation is modified by shifting the origin of the plane to $s = -s_1$, ie, by substituting $s = z - s_1$ as shown in figure.

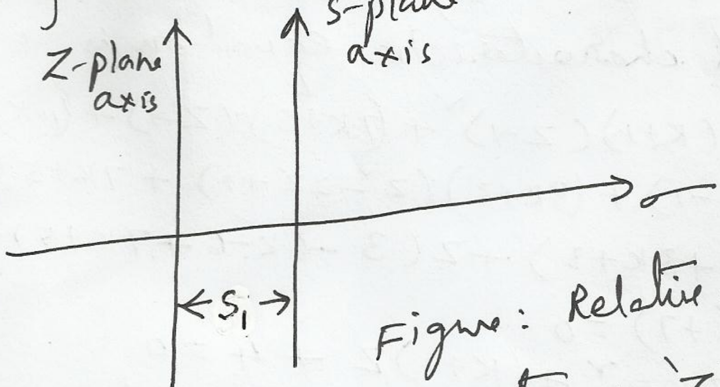


Figure: Relative stability

If the new characteristic equation in 'z' satisfies the Routh criterion, it implies that all the roots of the original characteristic equation are more negative than '$-s_1$'

① Show that the roots of given CE are more negative than -1. $s^3+7s^2+25s+39=0$

(Sol) put $s=z-1$ in the given CE

$$\therefore (z-1)^3+7(z-1)^2+25(z-1)+39=0$$

$$z^3+4z^2+14z+20=0$$

The Routh array is

$$\begin{array}{c|cc} z^3 & 1 & 14 \\ z^2 & 4 & 20 \\ z^1 & 9 & 0 \\ z^0 & 20 & \end{array}$$

Since all the terms in the first column are positive, hence all the roots have negative real parts more than -1.

② Determine the range of values of K ($K>0$) such that the characteristic equation $s^3+3(K+1)s^2+(7K+5)s+(4K+7)=0$ has roots more negative than $s=-1$.

(Sol) If the roots are more negative than $s=-1$, shift the origin to $s=-1$ by substituting $s=z-1$. Therefore, the modified characteristic equation is

$$(z-1)^3+3(K+1)(z-1)^2+(7K+5)\cdot(z-1)+(4K+7)=0$$

$$(z^3-3z^2+3z-1)+(3K+3)(z^2-2z+1)+7K+5(z-1)+4K+7=0$$

$$z^3+z^2(-3+3K+3)+z(3-6K-6+7K+5)+(-1+\cancel{3K}+3$$

$$-7\cancel{K}-5+4\cancel{K}+7)=0$$

or $z^3+3Kz^2+(K+2)z+4=0$

The Routh array is

$$\begin{array}{c|cc} z^3 & 1 & (K+2) \\ z^2 & 3K & 4 \\ z^1 & \dfrac{3K(K+2)-4}{3K} & 0 \\ z^0 & 4 & \end{array}$$

For the roots to have $-$ve real parts more than -1

$\dfrac{3K(K+2)-4}{3K}>0$ or $3K^2+6K-4>0$; $K=\dfrac{-6\pm9.17}{6}$

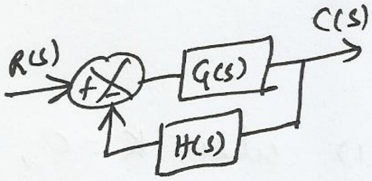
$\therefore K>-2.53$ & $K>\underline{\underline{0.53}}$

UNIT - III PART-B

THE ROOT LOCUS CONCEPT

The root locus concept introduced by W.R. Evans, provide a graphical method of plotting the locus of the roots in the s-plane as a given system parameter (open loop gain K) is varied over the complete range of values (from '0' to ∞). The root locus technique is a powerful tool for adjusting the location of closed loop poles to achieve the desired system performance by varying one or more system parameters.

From the Mason's gain formula, the transfer function of a system is $\frac{C(s)}{R(s)} = \frac{1}{\Delta} \sum_K P_K \Delta_K \longrightarrow (1)$



Also $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \longrightarrow (2)$

The characteristic equation of the system is

$$1+G(s)H(s) = 0 \quad \text{or} \quad \Delta(s) = 0$$

let $G(s)H(s) = P(s)$

$\therefore \quad 1+P(s) = 0$ is the CE of the system

$\therefore \quad P(s) = -1 \longrightarrow (3)$

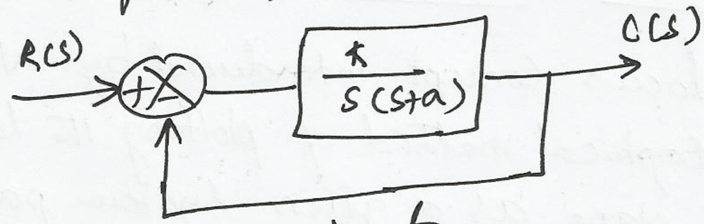
Since, 's' is a complex variable, P(s) has magnitude and phase angle given by $\longrightarrow$ (I)

$$|P(s)| = 1$$

$$\angle P(s) = \pm(2q+1)180^\circ, \quad q = 0,1,2 \ldots \longrightarrow (II)$$

Therefore, a plot of the points satisfying the angle criterion equation II in the s-plane is the root locus. A point on the root locus can be determined from magnitude equation.

① For the System shown below, Sketch the locus of the roots, when open loop gain 'K' is varied from 0 to ∞:



where 'K' and 'a' are Constants.

(Sol) The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{K}{s(s+a)+K}$

$$\frac{C(s)}{R(s)} = \frac{K}{s^2+as+K}$$

The characteristic equation is $s^2+as+K=0$

The roots of CE are $s_1, s_2 = \frac{-a \pm \sqrt{a^2-4K}}{2}$

$$= -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - K}$$

(1) when $K=0$, the two roots are $s_1, s_2 = -a/2 \pm \frac{a}{2}$

$= 0, -a$

which are same as the open loop poles.

(2) If $K = \left(\frac{a}{2}\right)^2$ the roots of CE are $s_1, s_2 = -\frac{a}{2}$

(3) For $K > \frac{a^2}{4}$, the roots are imaginary with real part equal to $-\frac{a}{2}$.

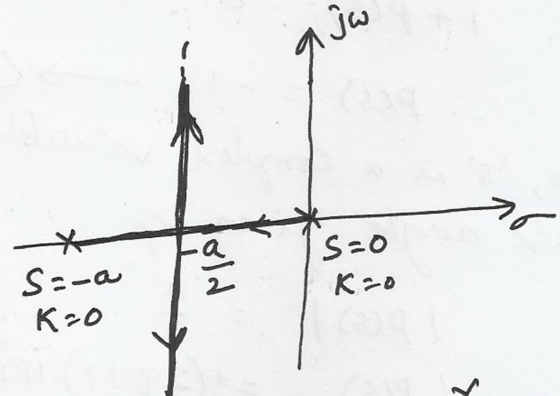


Figure: Root locus of $s^2+as+K=0$ as a function of 'K'.

Rules to Construct Root locus:

A set of rules have been developed to reduce the task involved in sketching root locus and to develope a quick approximate sketch. To develope root locus, the open loop transfer function is required.

Rule 1: The root locus is symmetrical about the real axis (σ-axis).

Since, the roots of the characteristic equation are either real or complex conjugate or combinations of both. Therefore, their locus must be symmetrical about the σ-axis of the s-plane.

Rule 2: As 'K' increases from zero to infinity, each branch of the root locus originates from an open loop pole with $K=0$ and terminates either on an open-loop zero or on an infinity with $K=\infty$. The number of branches terminating on infinity equals the number of open-loop poles minus zeros.

In general, the characteristic equation in pole-zero form can be represented as

$$1+G(s)H(s) = 1 + \frac{K\prod_{i=1}^{m}(s+Z_i)}{\prod_{j=1}^{n}(s+P_j)} = 0 \longrightarrow ①$$

where m = number of zeros; n = number of poles (open loop)

equation ①, can also be represented as

$$\prod_{j=1}^{n}(s+P_j) + K\prod_{i=1}^{m}(s+Z_i) = 0 \longrightarrow ②$$

if $K=0$, The roots of CE are $-P_j$, which are same as open loop poles.

equation ① can also be represented as

$$\frac{1}{K}\prod_{j=1}^{n}(s+p_j) + \prod_{i=1}^{m}(s+z_i) = 0$$

if $K \to \infty$, $\frac{1}{K} = 0$; therefore, the roots of CE are same as open loop zeros, $-z_i$.

<u>Rule 3</u>: A point on the real axis lies on the locus if the number of open-loop poles plus zeros on the real axis to the right of this points is odd.

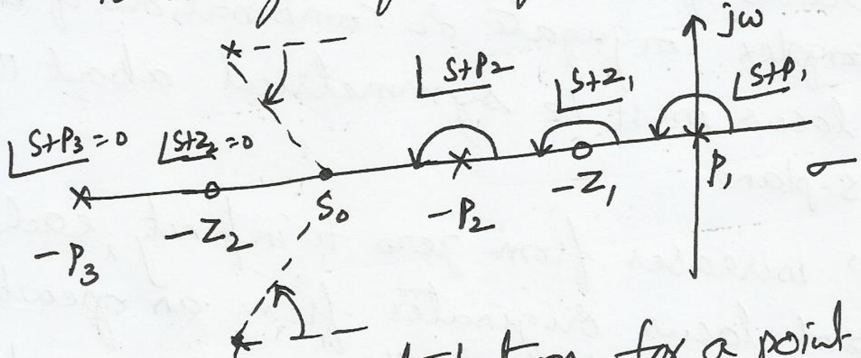


Figure: Angle <u>contribution</u> <u>for</u> a point <u>on the real axis</u>

As shown in the figure, (i) the poles and zeros on the real axis to the right of point s_0 contribute an angle of $180°$ each (ii) The poles and zeros to the left of this point contribute an angle of $0°$ each

(iii) The net angle contribution of a complex conjugate pole or zero pair is always zero.

∴ The angle criterion equation becomes

$$\angle G(s)H(s) = (m_r - n_r)\,180° = \pm(2q+1)180° \;;\; q = 0, 1, 2, \ldots$$

where m_r = right side zeros

n_r = right side poles

Therefore for a point s_0 on the real axis, the angle criterion is only met if $(m_r - n_r)$ or $(m_r + n_r)$ is odd, hence the rule.

Rule 4: The $(n-m)$ branches of the root locus which tend to infinity, do so along straight line asymptotes whose angles are given by

$$\phi_A = \frac{(2q+1)180^\circ}{(n-m)} \; ; \quad q = 0, 1, 2 \cdots (n-m-1)$$

Rule 5: The asymptotes cross the real axis at a point known as centroid, determined by

$$-\sigma_A = \frac{\text{Sum of real parts of poles} - \text{Sum of real parts of zeros}}{(\text{number of poles} - \text{number of zeros})}$$

$$= \frac{\Sigma\, \text{real parts of poles} - \Sigma\, \text{real parts of zeros}}{\text{number of poles} - \text{number of zeros}}.$$

Rule 6: The breakaway points (points at which multiple roots of the characteristic equation occur) of the root locus are the solutions of $\frac{dK}{ds} = 0$.

Rule 7: The angle of departure from an open-loop pole is given by $\phi_p = \pm 180^\circ(2q+1) + \phi \; ; \; q = 0, 1, 2, \cdots$

where 'ϕ' is the net angle contribution of all other open-loop poles and zeros at this point.

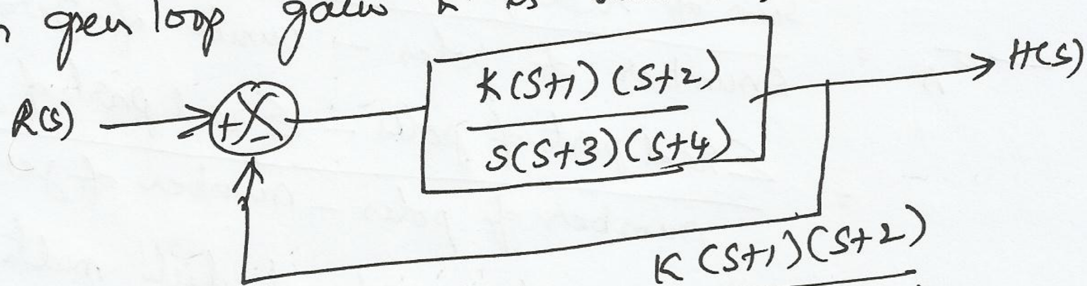
Similarly the angle of arrival at an open-loop zero is given by $\phi_z = \pm 180^\circ(2q+1) - \phi \; ; \; q = 0, 1, 2 \cdots$

Rule 8: The intersection of root locus branches with the imaginary axis can be determined by use of Routh criterion

(9) The open-loop gain 'K' in pole-zero form at any point 'S_0' on the root locus is given by

$$K = \frac{\prod_{j=1}^{n} (S_0 + P_j)}{\prod_{i=1}^{m} (S_0 + Z_i)} = \frac{\text{product of phasor lengths from } S_0 \text{ to open loop poles}}{\text{product of phasor lengths from } S_0 \text{ to open loop zeros.}}$$

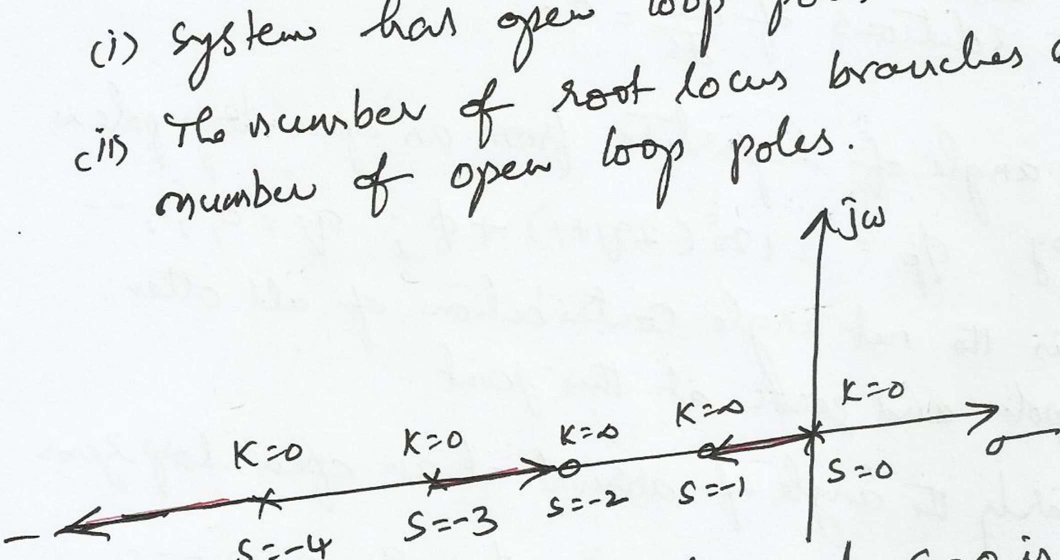
(1) For the system shown below, sketch the root locus when open loop gain 'K' is varied from 0 to ∞.



(Sol) The open loop TF $G(s) = \frac{K(S+1)(S+2)}{S(S+3)(S+4)}$

(i) System has open loop poles at $s = 0, -3, -4$

(ii) The number of root locus branches are equal to number of open loop poles.



(iii) The angle of departure at $S=0$ is 180°

The angle of departure at $S=-3$ is

$180 - 180 - 180 + 180 = 0$

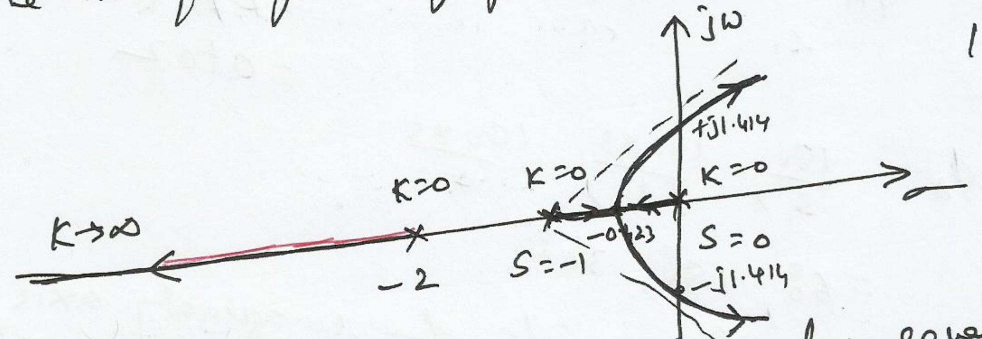
The angle of departure at $S=-4$ is

$180 - \cancel{180} + \cancel{180} + \cancel{180} - \cancel{180} = 180^\circ$

② Consider a feedback system with characteristic equation $1+\frac{K}{S(S+1)(S+2)}=0$. Sketch the root locus when open loop gain 'K' is varied from 0 to ∞.

(Sol) The open loop TF $G(s)H(s)=\frac{K}{S(S+1)(S+2)}$

(i) The no of open loop poles are '3' at $S=0,-1,-2$



(ii) The number of root locus branches equal to '3'

(iii) (a) The angle of departure at $S=0$ is given by
$180+0=180^\circ$

(b) The angle of departure at $S=-1$ is $180^\circ-180^\circ=0$

(c) The angle of departure at $S=-2$ is $180-180-180^\circ$
$=-180^\circ$

(iv) The root locus branches from $S=0$ and $S=-1$ are moving in opposite direction, therefore the break away points are the solutions of $\frac{dK}{ds}=0$ from the characteristic equation

From the CE, $K=-S(S+1)(S+2)$

$$=-(S^2+S)(S+2)$$
$$=-(S^3+2S^2+S^2+2S)$$
$$=-(S^3+3S^2+2S)$$

$$\frac{dK}{ds}=-3S^2+6S+2=0$$

$$\therefore\ S_1,S_2=\frac{-6\pm\sqrt{36-24}}{6}=-0.423,-1.577$$

Since the break away point must be in between 0 and -1, $s = -0.423$ is the actual breakaway point

(V) The centroid $-\sigma_A = \dfrac{S.R.P - S.R.Z}{n-m} = \dfrac{-1-2-0}{3-0} = -1$

(Vi) The angles of asymptotes are given by

$$\phi_A = \frac{(2q+1)\,180^\circ}{n-m} \; ; \quad q = 0, 1, 2 \ldots (n-m-1)$$

$= 0$ to $(3-0-1)$

$= 0$ to 2

$$\phi_A = \frac{180}{3}, \frac{180 \times 3}{3}, \frac{180 \times 5}{3}$$

$= 60^\circ, 180^\circ, 300^\circ$

(Vii) The intersection points of imaginary axis and root locus can be determined from R-H criterion

The CE of the systems is $\Rightarrow$ $s^3 + 3s^2 + 2s + K = 0$

$$1 + \frac{K}{s(s+1)(s+2)} = 0$$

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	0
0	K	

To have roots on the imaginary axis

$\frac{6-K}{3} = 0$ or $K = 6$

The Auxiliary equation is

$3s^2 + K = 0$ or $3s^2 + 6 = 0$

$s^2 = -2$

$s = \pm j\sqrt{2} = \pm j1.414$

(3) The open loop transfer function of a system is given by

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

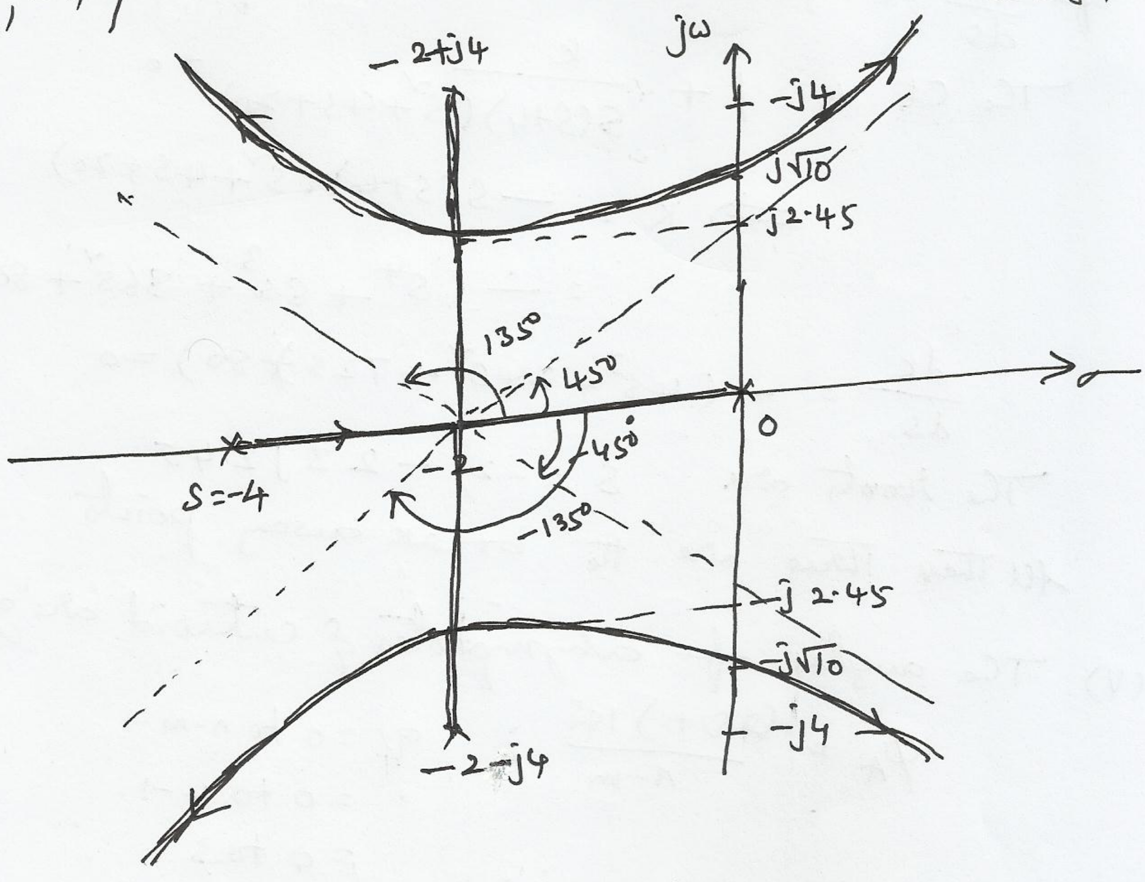
(Sol) The roots of $s^2+4s+20$ are $s = \frac{-4 \pm \sqrt{16-80}}{2}$

$= \frac{-4 \pm \sqrt{-64}}{2}$

$= \frac{-4 \pm j8}{2} = -2+j4$ & $-2-j4$

(i) The open loop poles are at

$s = 0, -4, -2+j4$ and $-2-j4$



(ii) The root locus branches are '4'

(iii)(a) The angle of departure at $s=0$ is given by

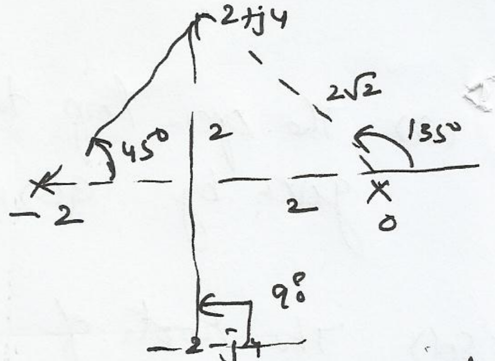
$180° + 0 = 180°$

(b) The angle of departure at $s=-4$ is given by

$180 - 180° = 0°$

(c) The angle of departure at $s = -2+j4$ is

$$\phi_P = 180^\circ - 135^\circ + 90 - 45^\circ$$
$$= 180^\circ - 270^\circ = -90^\circ$$



∴ The angle of departure at $s = -2 - j4$ is $+90^\circ$

(IV) All the root locus branches are moving in opposite direction, therefore the break away points are the solutions of $\frac{dK}{ds} = 0$ from the characteristic equation

The CE is $1 + \frac{K}{s(s+4)(s^2+4s+20)} = 0$

$$\Rightarrow K = -s(s+4)(s^2+4s+20)$$
$$= -(s^4 + 8s^3 + 36s^2 + 80s)$$

$$\frac{dK}{ds} = -(4s^3 + 24s^2 + 72s + 80) = 0$$

The roots are $s = -2, -2 \pm j2.45$

All these three are the break away points

(V) The angles of asymptotes & centroid are given by

$$\phi_A = \frac{(2q+1)180^\circ}{n-m}; \quad q = 0 \text{ to } n-m-1$$
$$= 0 \text{ to } 4-1$$
$$= 0 \text{ to } 3$$

$$\phi_A = \frac{180^\circ}{4}, \frac{3\times180^\circ}{4}, \frac{5\times180^\circ}{4}, \frac{7\times180^\circ}{4} = 45^\circ, 135^\circ, 225^\circ, 315^\circ$$

centroid $\sigma_A = \frac{-4-2-2}{4-0} = -\frac{8}{4} = -2$

(Vi) The intersection points of root locus & imaginary axis are the solutions of $\frac{dK}{ds} = 0$

The characteristic equation of the system is given by

$$1 + \frac{K}{s(s+4)(s^2+4s+20)} = 0$$

$$\Rightarrow \quad s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

s^4	1	36	K
s^3	8	80	
s^3	1	10	
s^2	26	K	
s	$\frac{260-K}{26}$	0	
s^0	K		

$$\frac{36-10}{1} = 26$$

To have roots on the imaginary axis, $\frac{260-K}{26} = 0$

$\therefore K = 260$

The AE is $26s^2 + K = 0$ or $26s^2 + 260 = 0$

$s^2 + 10 = 0$

$s^2 = -10$ or $s = \pm j\sqrt{10}$

④ Sketch the root locus plot of the system with the characteristic equation $1 + \frac{K(s+2)}{(s^2+2s+2)} = 0$

(Sol) The open loop TF is $G(s)H(s) = \frac{K(s+2)}{(s^2+2s+2)}$

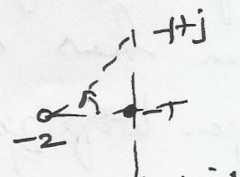
(i) The system has open loop zero at $s = -2$

(ii) The open loop poles are $s = \frac{-2 \pm \sqrt{2^2 - 4\times 2}}{2}$

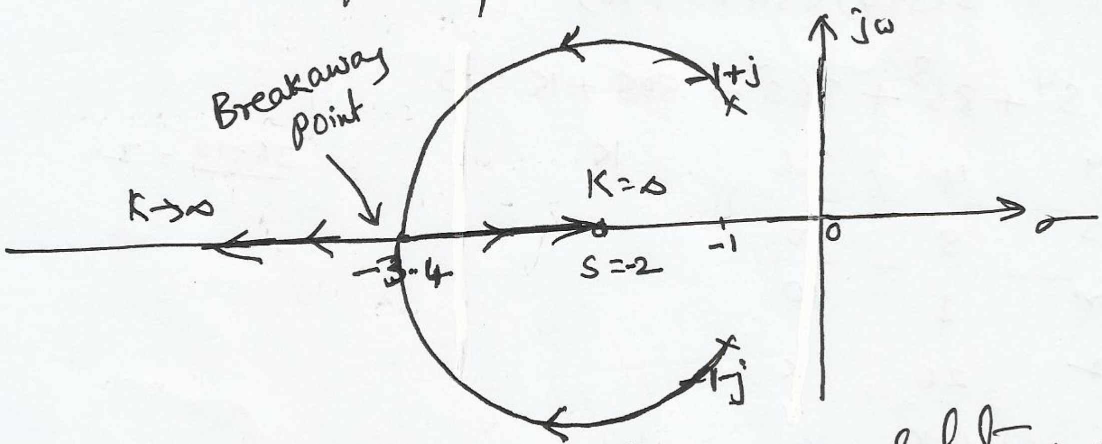
$= -\frac{2 \pm j2}{2} = -1 \pm j1$

(iii) The angle of departure at $s = -1+j$ is

$\phi_p = 180 - 90^\circ + 45^\circ = 135^\circ$



∴ The angle of departure at $s=-1-j$ is $-135°$



(iii) The break away points are the solutions of $\frac{dK}{ds}=0$

From the characteristic equation

$$K = \frac{-(s+2)}{s^2+2s+2}$$

$$\frac{dK}{ds} = \frac{-(s^2+2s+2)+(s+2)(2s+2)}{(s^2+2s+2)^2}=0$$

$$\Rightarrow -s^2-\cancel{2s}-2+2s^2+\cancel{2s}+4s+4=0$$

$$s^2+4s+2=0$$

$$s = \frac{-4\pm\sqrt{4^2-4(2)}}{2} = \frac{-4\pm\sqrt{16-8}}{2}$$

$$= \frac{-4\pm\sqrt{8}}{2} = \frac{-4\pm2\sqrt{2}}{2}$$

$= -3.4$ is the break away point

(5) The open loop transfer function of a unity feedback system is given by

$G(s)H(s) = \frac{K(s+1)(s+2)}{(s+0.1)(s-1)}$. Sketch the root locus when the open loop gain 'K' is varied from 0 to ∞.

(sol) (i) System has open loop zeros at $s=-1$ and -2 and open loop poles at $s=-0.1$ and $s=1$

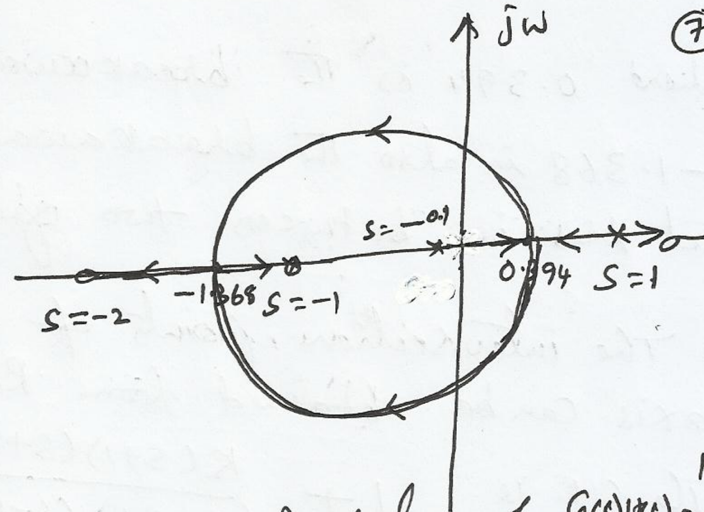


Figure: Root locus of $G(s)H(s) = \frac{K(s+1)(s+2)}{(s+0.1)(s-1)}$

(ii) (a) The angle of departure at pole $s = -0.1$ is given by

$$\phi_p = 180 - 180^\circ = 0$$

(b) The angle of departure at pole $s = +1$ is given by

$$\phi_p = 180 - 0 = 180$$

∴ Both the branches move in opposite direction.

(iii) The break away points are the solutions of $\frac{dK}{ds} = 0$. From the characteristic equation

$$1 + \frac{K(s+1)(s+2)}{(s+0.1)(s-1)} = 0$$

$$\therefore K = \frac{-(s+0.1)(s-1)}{(s+1)(s+2)} = \frac{-s^2 + 0.9s + 0.1}{s^2 + 3s + 2}$$

$$\frac{dK}{ds} = \frac{(s^2+3s+2)(-2s+0.9) - (-s^2+0.9s+0.1)(2s+3)}{(s^2+3s+2)^2} = 0$$

$$\Rightarrow 3.9s^2 + 3.8s - 2.1 = 0$$

$$s = \frac{-3.8 \pm \sqrt{(3.8)^2 - 4(3.9)(-2.1)}}{2(3.9)}$$

$$= -1.368,\ 0.394$$

where 0.394 is the breakaway point and -1.368 is also the breakaway point because it lies in between two open loop zeros.

(iv) The intersection points of root locus & imaginary axis can be obtained from Routh criterion.

The CE is $\quad 1 + \dfrac{K(s+1)(s+2)}{(s+0.1)(s-1)} = 0$

$$\Rightarrow (s^2 - 0.9s - 0.1) + K(s^2 + 3s + 2) = 0$$

$$s^2(1+K) + s(3K - 0.9) + (2K - 0.1) = 0$$

s^2	$(1+K)$	$(2K-0.1)$
s	$(3K-0.9)$	
s^0	$(2K-0.1)$	

To have roots on the imaginary axis $3K - 0.9 = 0$

$\therefore K = 0.3$

The Auxiliary equation is

$$(1+K)s^2 + (2K - 0.1) = 0 \qquad (K = 0.3)$$

$$1.3s^2 + 0.5 = 0 \Rightarrow s^2 = \frac{-0.5}{1.3} = -0.3846$$

$$\therefore s = \pm j\sqrt{0.3846}$$

① The characteristic equation of a system is given by $1+\frac{Ke^{-s}}{s(s+2)}=0$. Sketch the root locus, when the open loop gain 'K' is varied from 0 to ∞.

(Sol) The O.L.T.F $G(s)H(s)=\frac{Ke^{-s}}{s(s+2)}$

For small values of frequency $G(s)=\frac{K(1-s)}{s(s+2)}$

i) The system has open loop poles at $s=0$ and $s=-2$

The system has open loop zero at $s=1$

$\therefore$ one branch of root locus terminates $s=1$ and the other branch terminates on infinity as $K\to\infty$.

(ii) Note: In this case the CE is $1+\frac{K(1-s)}{s(s+2)}=0$

or $1-\frac{K(s-1)}{s(s+2)}=0$

or $1-p(s)=0$

$\therefore$ The angle of departure at open loop poles is given by $\phi_p=\pm180(2q)$; $q=0,1,2$

(a) The angle of departure at $s=0$ is given by

$\phi_p=0+(+180)=180^\circ$

(b) The angle of departure $s=-2$ is given by

$\phi_p=0+(-180+180)=0^\circ$

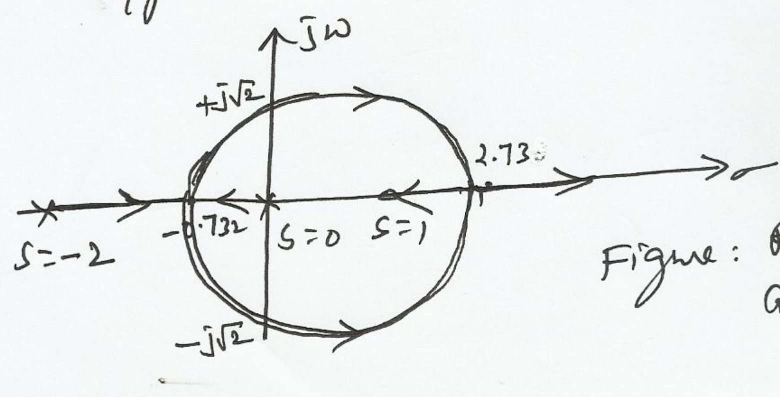


Figure: Root locus of $G(s)H(s)=\frac{Ke^{-s}}{s(s+2)}$

$\therefore$ Both the branches of root locus move in opposite direction.

(iii) The breakaway points are the solutions of $\frac{dK}{ds}=0$

From CE, $1+\frac{K(1-s)}{s(s+2)}$, $K = \frac{-s(s+2)}{1-s}$

$$\therefore \frac{dK}{ds} = \frac{(1-s)(-2s-2)+s(s+2)(-1)}{(1-s)^2} = 0$$

or $-\cancel{2s}+2s^2+\cancel{2s}-2-s^2-2s=0$

$$s^2-2s-2=0$$

$$\therefore s = \frac{2\pm\sqrt{4+8}}{2} = \frac{2\pm\sqrt{12}}{2} = 1\pm\sqrt{3}$$

$$= 1\pm 1.732 = 2.732 \;\&\; -0.732$$

Both are the breakaway points

(IV) The intersection points of root locus and imaginary axis can be obtained from Routh criterion.

The CE is $s(s+2)+K(1-s)=0$

$s^2+2s+K-Ks=0$ or $s^2+s(2-K)+K=0$

s^2	1	K
s	2 − K	
s^0	K	

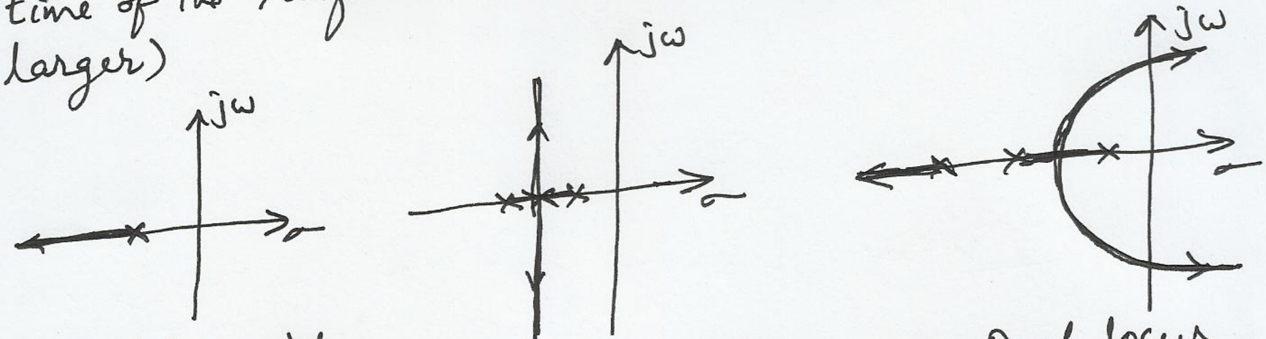
To have roots on imaginary axis, $2-K=0$ or $K=2$

$\therefore$ The Auxiliary equation is $s^2+K=0$; where $K=2$

$\therefore s^2+2=0$ or $s^2=-2$ or $s=\pm j\sqrt{2}$

Effect of Adding poles and Zeros to G(s)H(s) on the root loci:

(1) **Effect of Addition of poles**: The addition of pole to the open loop transfer function G(s)H(s) has the effect of pulling the root locus to the right, tending to lower the system relative stability and to slow down the settling time of the response (ie the value of settling time becomes larger)



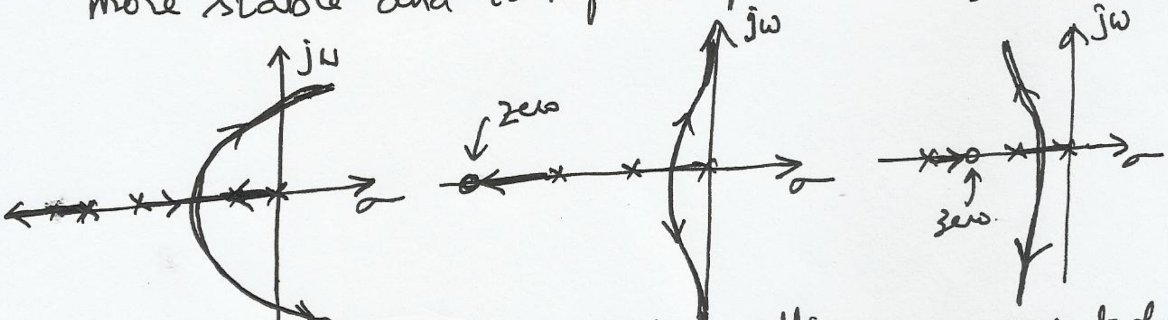
(a) Root locus plot for single pole system

(b) Root locus plot of two-pole system

(c) Root locus plot of three-pole system.

Figure (1): Effect of adding poles to G(s)H(s) on root locus

(2) **Effects of Addion of Zeros**: The addition of zero to the open loop transfer function has the effect of pulling the root locus to the left, tending to make the system more stable and to speed up the settling of the response



(a) Root locus plot of three-pole system.

(b) Effect of Adding zero to the three-pole root locus

(c) Effect of adding zero between two poles.

Procedure to Construct Root locus:

(1) Locate open loop poles and zeros in the s-plane (Note: we need the open loop transfer function to construct the root locus.

(2) Determine the angle of departure at each open loop pole.

(3) If the root locus branches are moving in opposite direction, determine the break away points.

(4) If the number of open loop zeros are less than the number of open loop poles, determine the centroid.

(5) Find the angles of asymptotes.

(6) If the asymptotes cross the imaginary axis, find the intersection points of imaginary axis and root locus.

Note (1) poles are in the denominator, hence the angle contributed by poles at a particular point is -ve of angle contributed at that point.

(2) If the angle of departure is $\pm 180^\circ$, the root locus branch moves towards left on the real axis.

(3) If the angle of departure is 0 or 360°, the root locus branch moves towards right on the real axis.

UNIT - IV

FREQUENCY RESPONSE ALALYSIS

Consider a linear system with a sinusoidal input $r(t) = A \sin \omega t \longrightarrow (1)$

Under steady-state, the system output as well as the signals at other points in the system are sinusoidal. The steady-state may be written as

$$c(t) = B \sin(\omega t + \phi) \longrightarrow (2)$$

The magnitude and phase relationship between the sinusoidal input and the steady-state output of a system is termed as frequency response.

In linear time-invariant systems, the frequency response is independent of the amplitude and phase of the input signal.

The frequency response test on a system is normally performed by keeping the amplitude 'A' fixed and determining 'B' and 'φ' for a suitable range of frequencies.

The frequency response is easily evaluated from the sinusoidal transfer function which can be obtained by replacing 's' by 'jω' in the system transfer function T(s). The transfer function thus obtained T(jω) is a complex function of frequency and has both magnitude and phase angle. These characteristics are conveniently represented by graphical plots.

Advantages of Frequency Response Analysis

The ease and accuracy of measurements are some of the advantages of the frequency response method.

(1) wherever it is not possible to obtain the form of the transfer function of a system through analytical techniques, the necessary information to compute its transfer function can be extracted by performing the frequency response test on the system.

The step response test can also be performed easily but the extraction of transfer function from step response data is quite a laborious procedure.

(2) The design and parameter adjustment of the open-loop transfer function of a system for a specified closed loop performance is carried out somewhat more easily in frequency domain than in time domain.

(3) The effects of noise disturbance and parameter variations are relatively easy to visualize and assess through frequency response.

(4) The Absolute and Relative stability of the closed loop systems can be estimated from the knowledge of their open loop frequency response.

(5) The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipments.

(6) The frequency response analysis and designs can be extended to certain non linear control systems.

Disadvantages of Frequency Response Analysis:

(1) For systems with very large time-constants, the frequency response test is cumbersome to perform as the time required for the output to reach steady-state for each frequency of the test signal is excessively long. Therefore, the frequency response test is not recommended for systems with very large time constants.

(2) Frequency response obviously can not be performed on non-interruptable systems. under such circumstances a single shot test (step or impulse) is more convenient.

Frequency Domain Specifications: The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications. The requirements of a system to be designed are usually specified in terms of these specifications.

The frequency domain specifications are

(1) Resonant peak, M_r (2) Resonant Frequency ω_r

(3) Bandwidth ω_b (4) cut-off rate

(5) Gain Margin (6) phase margin

Let us consider a second order system shown in figure.

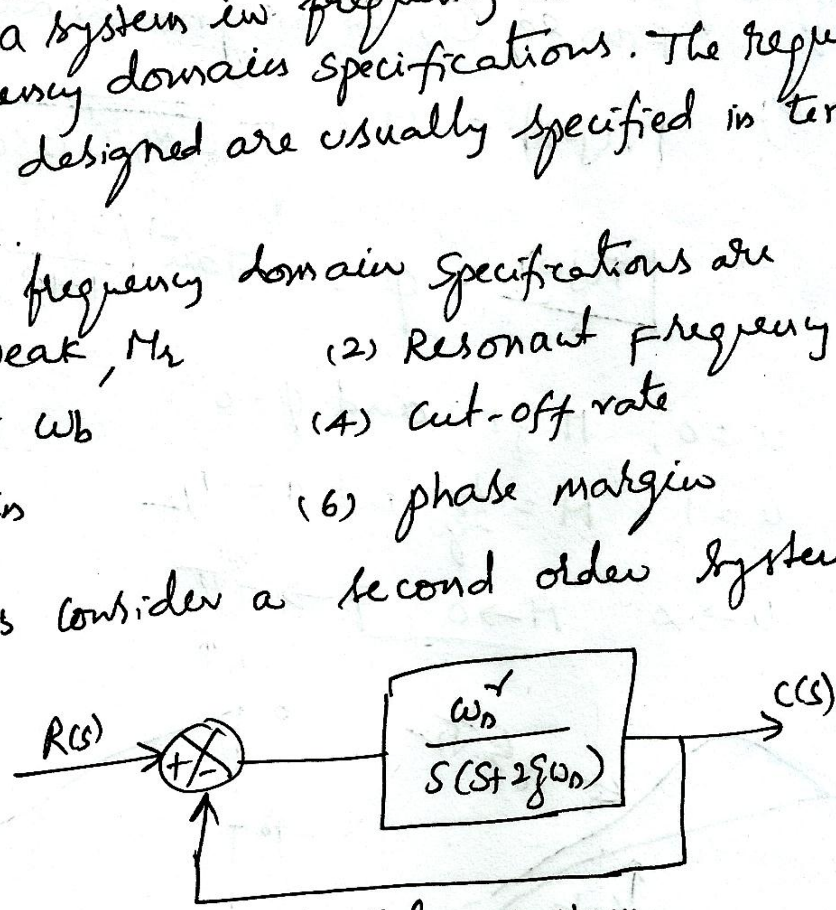


Figure: Second order system

∴ The closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s(s+2\xi\omega_n)+\omega_n^2} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

∴ The sinusoidal transfer function of the system is obtained by replace 's' by 'jω'.

$$\therefore T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2+2\xi\omega_n(j\omega)+\omega_n^2}$$

$$= \frac{\omega_n^2}{\omega_n^2+j2\xi\omega\omega_n-\omega^2}$$

$$= \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+j2\xi\left(\frac{\omega}{\omega_n}\right)} = \frac{1}{(1-u^2)+j2\xi u} \longrightarrow (1)$$

where $u = \frac{\omega}{\omega_n}$ is the normalized driving signal frequency. From eq (1), the magnitude and phase angle are given by

$$|T(j\omega)| = M = \frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}} \longrightarrow (2)$$

$$\angle T(j\omega) = \phi = -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right) \longrightarrow (3)$$

if $u=0$; $M=1$ and $\phi=0$

$u=1$ $M=\frac{1}{2\xi}$ and $\phi=-\pi/2$.

$u\to\infty$ $M\to 0$ $\phi\to-\pi$

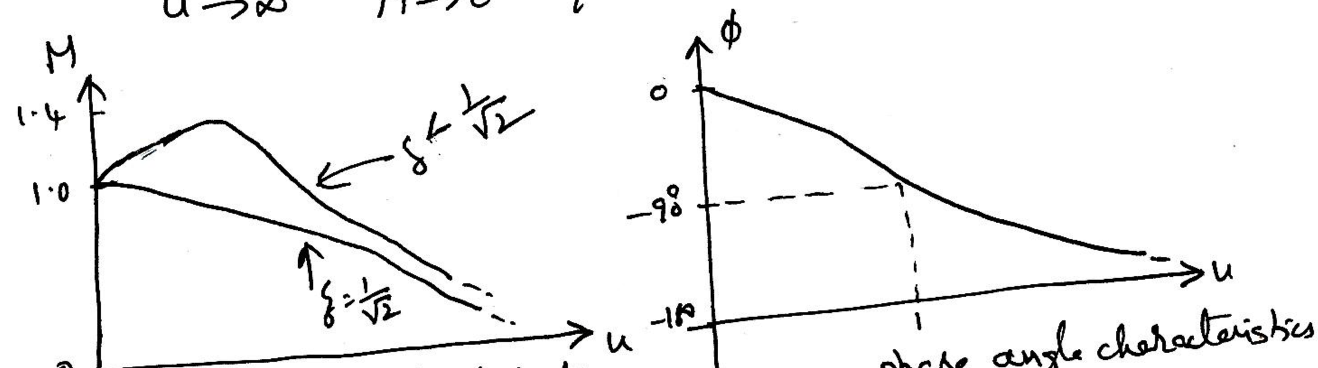


Figure: Magnitude characteristics

Figure: phase angle characteristics

(1) Resonant Frequency : The frequency where magnitude M has a peak value is known as the resonant frequency. At this frequency, the slope of the magnitude curve is zero. Let ω_r be the resonant frequency and $u_r = \frac{\omega_r}{\omega_n}$ be the normalized resonant frequency. Then

$$\left.\frac{dM}{du}\right|_{u=u_r} = 0 \Rightarrow \frac{d}{du}\left\{\frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}}\right\}\Bigg|_{u=u_r} = 0$$

$$\Rightarrow -\frac{1}{2}\left[(1-u^2)^2+(2\xi u)^2\right]^{-3/2}\left\{2(1-u^2)(-2u) + 4\xi u(2\xi)\right\}\Bigg|_{u=u_r} = 0$$

$$\Rightarrow -\frac{1}{2}\left\{\frac{-4(1-u_r^2)u_r + 8\xi^2 u_r}{\left[(1-u^2)^2+(2\xi u)^2\right]^{3/2}}\right\} = 0$$

$$\therefore \quad 2(1-u_r^2)u_r - 4\xi^2 u_r = 0$$

or $\quad -u_r^3 - u_r - 2\xi^2 u_r = 0$

or $\quad u_r^2 - 1 - 2\xi^2 = 0$

or $\quad u_r^2 = 1 - 2\xi^2 \quad \therefore u_r = \sqrt{1-2\xi^2} \longrightarrow$ (i)

$\therefore$ Denormalized resonant frequency $\omega_r = u_r \cdot \omega_n = \omega_n\sqrt{1-2\xi^2}$

(2) Resonant peak : The maximum value of the magnitude of the closed loop transfer function is known as resonant peak. The magnitude is maximum at resonant frequency ω_r

$$\therefore M_r = M\Big|_{\omega=\omega_r} = \frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}}\Bigg|_{u=u_r}$$

$$= \frac{1}{\sqrt{\left[1-(1-2\xi^2)\right]^2+(2\xi)^2(1-2\xi^2)}}$$

$$M_r = \frac{1}{\sqrt{(2\xi^2)^2 + 4\xi^2(1-2\xi^2)}}$$

$$= \frac{1}{\sqrt{4\xi^2(\xi^2) + 4\xi^2(1-2\xi^2)}} = \frac{1}{\sqrt{4\xi^2(1-2\xi^2+\xi^2)}}$$

$$= \frac{1}{\sqrt{4\xi^2(1-\xi^2)}} = \frac{1}{2\xi\sqrt{1-\xi^2}} \longrightarrow (i)$$

The phase angle at resonant frequency ω_r is given by

$$\phi_r = \tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)\Bigg|_{u=u_r} = \tan^{-1}\left(\frac{2\xi u_r}{1-u_r^2}\right)$$

$$= \tan^{-1}\left(\frac{2\xi\sqrt{1-2\xi^2}}{1-1+2\xi^2}\right) = \tan^{-1}\left(\frac{2\xi\sqrt{1-2\xi^2}}{2\xi^2}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1-2\xi^2}}{\xi}\right)$$

(3) Bandwidth: The range of frequencies over which magnitude is equal to or greater than $\frac{1}{\sqrt{2}}$ is defined as bandwidth ω_b. The frequency at which magnitude M has a value of $\frac{1}{\sqrt{2}}$ is called cut-off frequency ω_c.

In general, the bandwidth of a control system indicates the noise-filtering characteristics of the system. Also, bandwidth gives a measure of transient response properties.

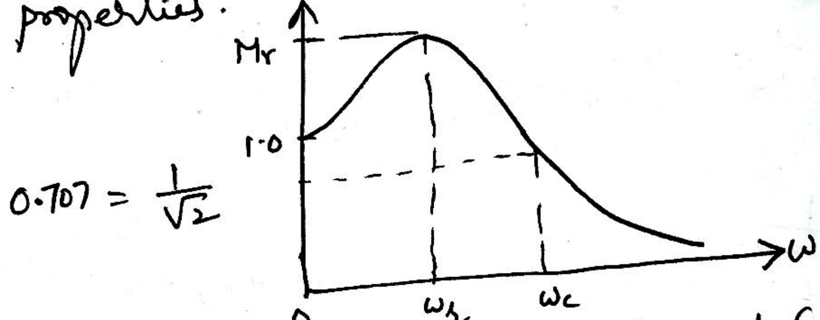


Figure: Bandwidth and Cut-off frequency.

∴ The normalized bandwidth $u_b = \frac{\omega_b}{\omega_n}$ of the second-order systems can be determined as follows.

$$M\Big|_{u=u_b} = \frac{1}{\sqrt{(1-u_b)^2 + (2\zeta u_b)^2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow (1-u_b)^2 + (2\zeta u_b)^2 = 2$$

$$(1-2u_b^2+u_b^4) + 4\zeta^2 u_b^2 = 2$$

$$u_b^4 - 2(1-2\zeta^2)u_b^2 - 1 = 0$$

$$\therefore u_b^2 = \frac{2(1-2\zeta^2) \pm \sqrt{[2(1-2\zeta^2)]^2 - 4(1)(-1)}}{2}$$

$$= \frac{2(1-2\zeta^2) \pm \sqrt{4-16\zeta^2+16\zeta^4+4}}{2}$$

$$= 1-2\zeta^2 \pm \sqrt{2-4\zeta^2+4\zeta^4}$$

∴ Normalized Bandwidth $u_b = \left[1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right]^{1/2}$

Denormalized Bandwidth $\omega_b = \omega_n u_b$

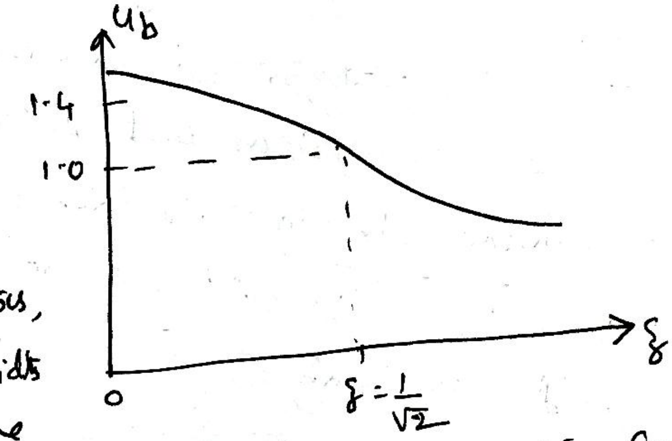


Figure: Bandwidth Vs Damping factor

The bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics, and rise time. A large bandwidth corresponds to a small rise time or fast response.

(4) Cut-off rate: The slope of log-magnitude curve near the cut-off frequency is called cut-off rate. The cut-off rate indicates the ability of a feedback system to distinguish the signal from noise.

(5) Gain Margin (GM): The gain margin is defined as the value of gain, to be added to the system, in order to bring the system to the verge of instability.

The gain margin is given by the reciprocal of the magnitude of open loop transfer function at phase cross over frequency.

The frequency at which the phase of the open loop transfer function is $-180°$ is called the phase cross-over frequency.

$\therefore$ Gain Margin $GM = \frac{1}{|G(j\omega)|}\Big|_{\omega=\omega_p}$

The gain margin in dB can be expressed as

$$GM = 20\log\left|\frac{1}{|G(j\omega)|}\right|_{\omega=\omega_p}$$

where $\angle G(j\omega)\big|_{\omega=\omega_p} = -180°$

The gain margin indicates the additional gain that can be provided to system without affecting the stability of the system.

(6) Phase Margin: The phase margin is defined as the additional phase lag to be added at the gain cross over frequency in order to bring the system to the verge of instability.

The gain cross over frequency ω_g is the frequency at which the magnitude of the open loop transfer function is unity or 0dB.

phase Margin $PM = -(180 + \phi_{gc})$

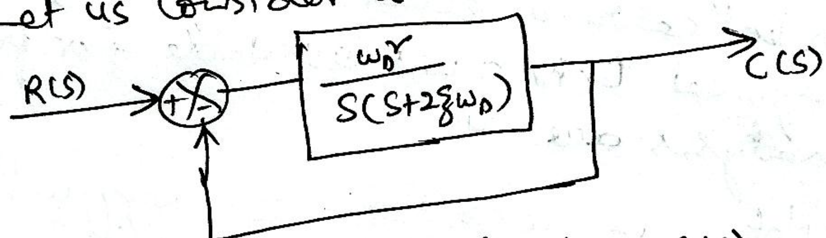
where $\phi_{gc} = \angle G(j\omega)\Big|_{\omega=\omega_{gc}}$ & $|G(j\omega)|\Big|_{\omega=\omega_{gc}} = 1$

The margin indicates the additional phase lag that can be provided to the system without affecting stability.

For stable systems both gain margin and phase margin are positive.

Correlation between Time and Frequency Response:

Let us consider a second order system given by



$\therefore$ The closed loop transfer function $\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}$

For an under damped system ($\xi<1$),

The damped natural frequency $\omega_d = \omega_n\sqrt{1-\xi^2} \rightarrow (1)$

peak overshoot $M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \rightarrow (2)$

Resonant peak $M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$

Resonant frequency $\omega_r = \omega_n\sqrt{1-2\xi^2}$

For $\xi > \frac{1}{\sqrt{2}}$, the resonant peak does not exist and the correlation breaks down.

$$\frac{\omega_r}{\omega_d} = \sqrt{\frac{1-2\xi^2}{1-\xi^2}} \rightarrow (I)$$

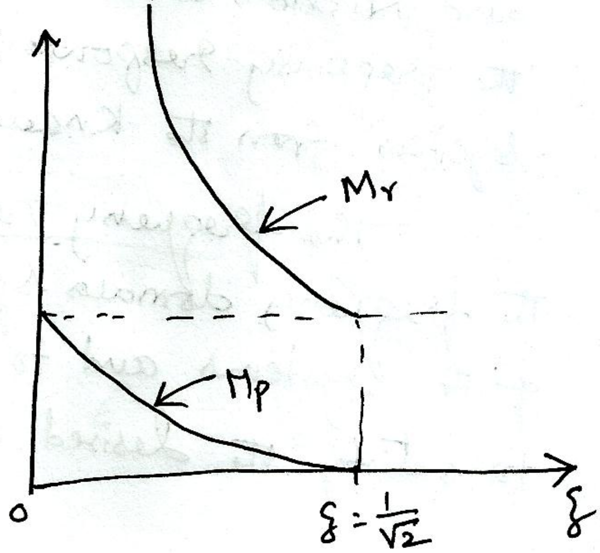


Figure: M_r, M_p Vs ξ

From the figure, it is clear that the correlation breaks down for $\xi > \frac{1}{\sqrt{2}}$.

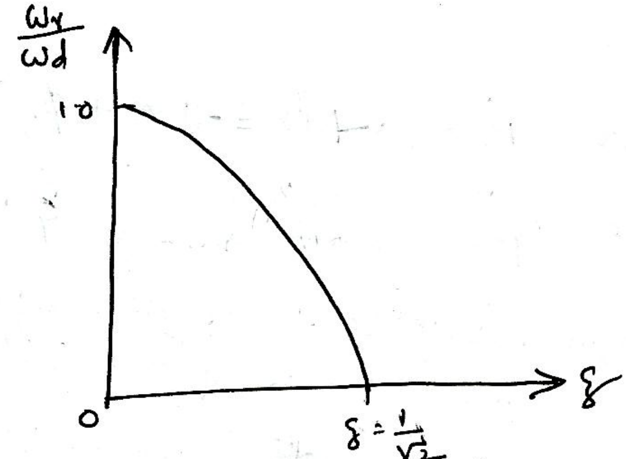


Figure: Correlation between ω_r & ω_d

Frequency Response plots

The frequency response analysis of control systems can be carried either analytically or graphically. The various graphical techniques available for frequency response analysis are

(1) Bode plot (2) polar plot (3) Nyquist plot
(4) Nichols chart (5) M and 'N' circles.

The Bode plot, polar plot and Nyquist plot are usually drawn for open loop systems. From open loop response plot, the performance and stability of closed loop system are estimated. The M and N circles and Nichols chart are used to graphically determine the frequency response of unity feedback closed loop system from the knowledge of open loop response.

The frequency response plots are used to determine the frequency domain specifications, to study the stability of the systems and to adjust the gain of the system to satisfy the desired specifications.

BODE PLOTS: The Bode plot is a frequency response plot of the sinusoidal transfer function of a system. Bode plot consists of two graphs. One is the plot of the magnitude in dB versus $\log\omega$. The other is a plot of the phase angle of sinusoidal transfer function versus $\log\omega$. These plots are called Bode plots in honour of H.W. Bode, who did the basic work in this area.

Let us consider a system with open loop transfer function

$$G(s) = \frac{K s(1+sT_1)}{(1+sT_2)(s^2+2\xi\omega_n s+\omega_n^2)}$$

The sinusoidal transfer function $G(j\omega)$ can be obtained by replacing 's' by $j\omega$. The main advantage of the Bode plot is that multiplication of magnitudes can be converted into addition.

$$\therefore G(j\omega) = \frac{K j\omega(1+j\omega T_1)}{(1+j\omega T_2)((j\omega)^2+j2\xi\omega_n\omega+\omega_n^2)}$$

$$= \frac{K j\omega(1+j\omega T_1)}{(1+j\omega T_2)\,\omega_n^2\left[1-\left(\frac{\omega}{\omega_n}\right)^2+j2\xi\frac{\omega}{\omega_n}\right]}$$

$$= \frac{K j\omega(1+j\omega T_1)}{(1+j\omega T_2)\,\omega_n^2\left[1-u^2+j2\xi u\right]} \quad ; \text{ where } u = \frac{\omega}{\omega_n}$$

(i) If $G_1(j\omega) = K$; $\quad |G_1(j\omega)| = K \quad \angle G_1(j\omega) = 0^\circ$

$\therefore 20\log|G_1(j\omega)| = 20\log K$

(ii) If $G_2(j\omega) = j\omega$; $\quad |G_1(j\omega)| = \omega$; $\quad \angle G_1(j\omega) = 90^\circ$

$\therefore$ Magnitude in dB is $20\log\omega$.

(iii) If $G_3(j\omega) = (1+j\omega T_1)$

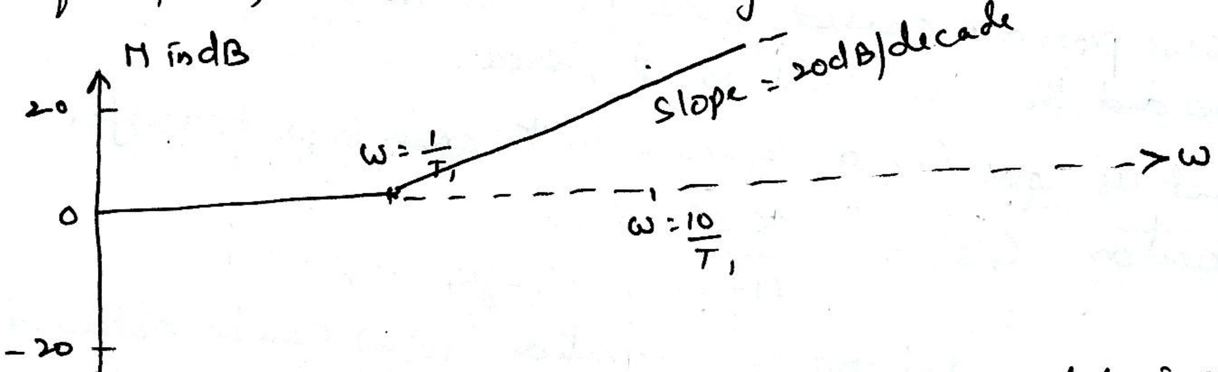
$|G_3(j\omega)| = \sqrt{1+\omega^2T_1^2}$; $\quad \phi = \tan^{-1}\left(\frac{\omega T_1}{1}\right)$

Magnitude in dB is

$$20\log|G_3(j\omega)| = 20\log\sqrt{1+\omega^2T_1^2}$$

If $\omega T>>1$, M in dB $= 20\log\sqrt{\omega^2 T_1^2}$
$= 20\log(\omega T_1)$

If $\omega T_1 << 1$, M in dB $= 20\log 1 = 0$ dB



Therefore, the log magnitude versus $\log\omega$ curve of $(1+j\omega T_1)$ can be approximated by two straight line asymptotes, one a straight line at 0 dB for the frequency range $0<\omega\leq 1/T$ and the other a straight line with a slope 20 dB/decade for the frequency $1/T\leq\omega<\infty$. The frequency $\omega = 1/T$ at which the two asymptotes meet is called the corner frequency or the break frequency.

(iv) If $G_4(j\omega) = \frac{1}{(1+j\omega T_2)}$

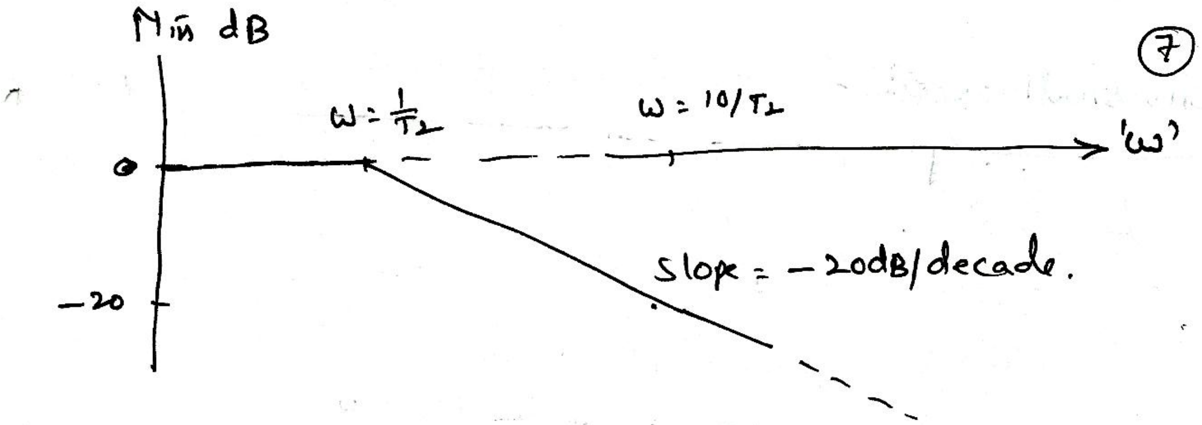
$$|G_4(j\omega)| = \frac{1}{\sqrt{1+\omega^2T_2^2}} \; ; \quad \angle G_4(j\omega) = -\tan^{-1}(\omega T_2)$$

Magnitude in dB $= 20\log|G_4(j\omega)| = 20\log\frac{1}{\sqrt{\omega^2T_2^2+1}}$

$$= -20\log\sqrt{1+\omega^2T_2^2}$$

for $\omega T_2 << 1$, Magnitude $= -20\log(1) = 0$ dB

for $\omega T_2 \quad 1$, Magnitude $= -20\log\sqrt{\omega^2T_2^2} = -20\log(\omega T_2)$



where $\omega = \frac{1}{T_2}$ is known as corner frequency or Break over frequency.

where $\boxed{\omega/\omega_n = u}$

(iv) If $G_5(j\omega) = \frac{1}{(1-u^2+j2\xi u)}$

$$|G_5(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}} \; ; \; \phi = -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)$$

$\therefore$ Magnitude in dB $= 20\log|G_5(j\omega)| = 20\log\frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}}$

By making two assumptions, we can plot the bode plot

(i) if $u^2 \ll 1$; $\quad 20\log\frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}} = 20\log 1 = 0$

if $u^2 \gg 1$; $\quad 20\log\frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}} = 20\log\frac{1}{\sqrt{u^4}}$

$$= -40\log u$$

$\therefore$ Magnitude is zero up to $u=1$, after that magnitude curve is a line with slope -40dB/decade

> Note: Decade : $\omega_2 = 10\omega_1$
> octave : $\omega_2 = 2\omega_1$

In the above case $u = \frac{\omega}{\omega_n} = 1$ is the break over or corner frequency.

Error in Magnitude at corner frequency

(1) If $G(s) = (1+sT_1)$

$G(j\omega) = (1+j\omega T_1)$

$|G(j\omega)| = \sqrt{1+\omega^2 T_1^2}$

$\therefore$ for $\omega T_1 << 1$ Magnitude in dB $= 0$

$\omega T_1 >> 1$ Magnitude in dB $= 20\log(\omega T_1)$

Actually at $\omega T_1 = 1$

$$M \text{ in dB} = 20\log\sqrt{1+1^2} = 20\log\sqrt{2}$$

$$= 10\log 2 = 10(0.3010) = \underline{\underline{3\,dB}}$$

(2) If $G(j\omega) = \frac{1}{1+j\omega T_2}$

Magnitude in dB $= 20\log\frac{1}{\sqrt{1+\omega^2 T_2^2}}$

where $\omega = \frac{1}{T_2}$ is the corner frequency. and

if $\omega T_2 << 1$; $\quad M$ in dB $= 20\log 1 = 0$

for $\omega T_2 >> 1$, $\quad M$ in dB $= 20\log\frac{1}{\sqrt{\omega^2 T_2^2}} = -20\log(\omega T_2)$

At $\omega T_2 = 1$ or at $\omega = 1/T_2$

$$M \text{ in dB} = 20\log\frac{1}{\sqrt{1+1}} = -20\log\sqrt{2}$$

$$= -10\log 2 = \underline{\underline{-3\,dB}}$$

(3) $G(s) = \frac{1}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{1}{\omega_n^2}\cdot\frac{1}{[(1-u^2)+j2\xi u]}$

If $G(ju) = \frac{1}{(1-u^2)+j2\xi u}$

where $u=1$ is the corner or break over frequency.

at $u=1$ or at $\frac{\omega}{\omega_n}=1$ or at $\omega=\omega_n$

Magnitude in dB (at $u=1$) $= 20\log\left[\frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}}\right]_{u=1}$ ⑧

$= 20\log \frac{1}{\sqrt{(1-1)^2+[2\xi(1)]^2}}$

$= -20\log\sqrt{(2\xi)^2}$

$= \underline{-20\log 2\xi}$ for Complex Conjugate poles.

where 'ξ' is the damping factor.

(4) If $G(s) = (s^2+2\xi\omega_n s+\omega_n^2)$, the error at corner frequency $u = \frac{\omega}{\omega_n} = 1$ is $\underline{(20\log 2\xi)}$

<u>Note</u>: In the Construction of the Bode plot, the following factors may appear

(1) Constant gain 'K'

(2) poles at the origin $\frac{1}{(j\omega)^r}$

(3) zeros at the origin $(j\omega)^r$

(4) poles on the real axis $\frac{1}{(1+j\omega T_1)}$

(5) Zeros on the real axis $(1+j\omega T_2)$

(6) Complex Conjugate poles $\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}$

(7) Complex Conjugate zeros $(s^2+2\xi\omega_n s+\omega_n^2)$

<u>Imp</u>: To plot the Bode diagram, we need the transfer function in time-constant form.

ie $G(j\omega) = \frac{K(1+j\omega T_1)(1+j\omega T_2)}{(j\omega)^r((1-u^2)+j2\xi u)}$

procedure to Construct the Bode plot:

(1) obtain the sinusoidal transfer function from the given transfer function.

(2) Identify the corner frequencies of poles and zeros from the time-Constant form of $G(j\omega)$.

(3) sketch the asymptotic bode plot, Then make corrections at corner frequencies.

① Draw the Bode plot for the transfer function

$$G(s) = \frac{64(s+2)}{s(s+0.5)(s^2+3.2s+64)}$$

(Sol) To draw the Bode plot, we need the transfer function in time-Constant form.

$$G(s) = \frac{64 \times 2(1+s/2)}{s \times 0.5(1+s/0.5)64(1+\frac{3.2}{64}s+s^2/64)}$$

Therefore, the sinusoidal transfer function is given by

$$G(j\omega) = \frac{4(1+j\omega/2)}{j\omega(1+j2\omega)(1-u^2+j2\xi u)}$$; where $u = \frac{\omega}{8}$, $\xi = 0.2$

Factor	Corner frequency	Asymptotic log magnitude characteristic
$4/j\omega$	None	Magnitude = $20\log 4/\omega$. straight line of slope -20dB/decade, with magnitude 0 dB at $\omega=4$.
$\frac{1}{1+j2\omega}$	$\omega_{c_1} = \frac{1}{2} = 0.5$ rad/sec	straight line of 0dB up to $\omega<\omega_{c_1}$ and straight line of slope -20dB/decade for $\omega>\omega_{c_1}$.

$1 + j\frac{\omega}{2}$	$\omega_{c_2} = 2$ rad/sec	straight line of 0dB for $\omega < \omega_{c_2}$ and a straight line with slope +20 dB/decade for $\omega > \omega_{c_2}$
$1 + j2(0.2)\left(\frac{\omega}{8}\right) - \left(\frac{\omega}{8}\right)^2$	$\omega_{c_3} = 8$ rad/sec $\xi = 0.2$;	straight line of 0dB for $\omega < \omega_{c_3}$ and straight line of slope −40dB/decade for $\omega > \omega_{c_3}$

$$\phi = \tan^{-1}(\omega/2) - 90^\circ - \tan^{-1}(2\omega) - \tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)$$

where $u = \frac{\omega}{8}$; $\xi = 0.2$

<u>Note</u> : The least corner frequency is 0.5. So that, we can choose the frequency scale from 0.1 onwards.

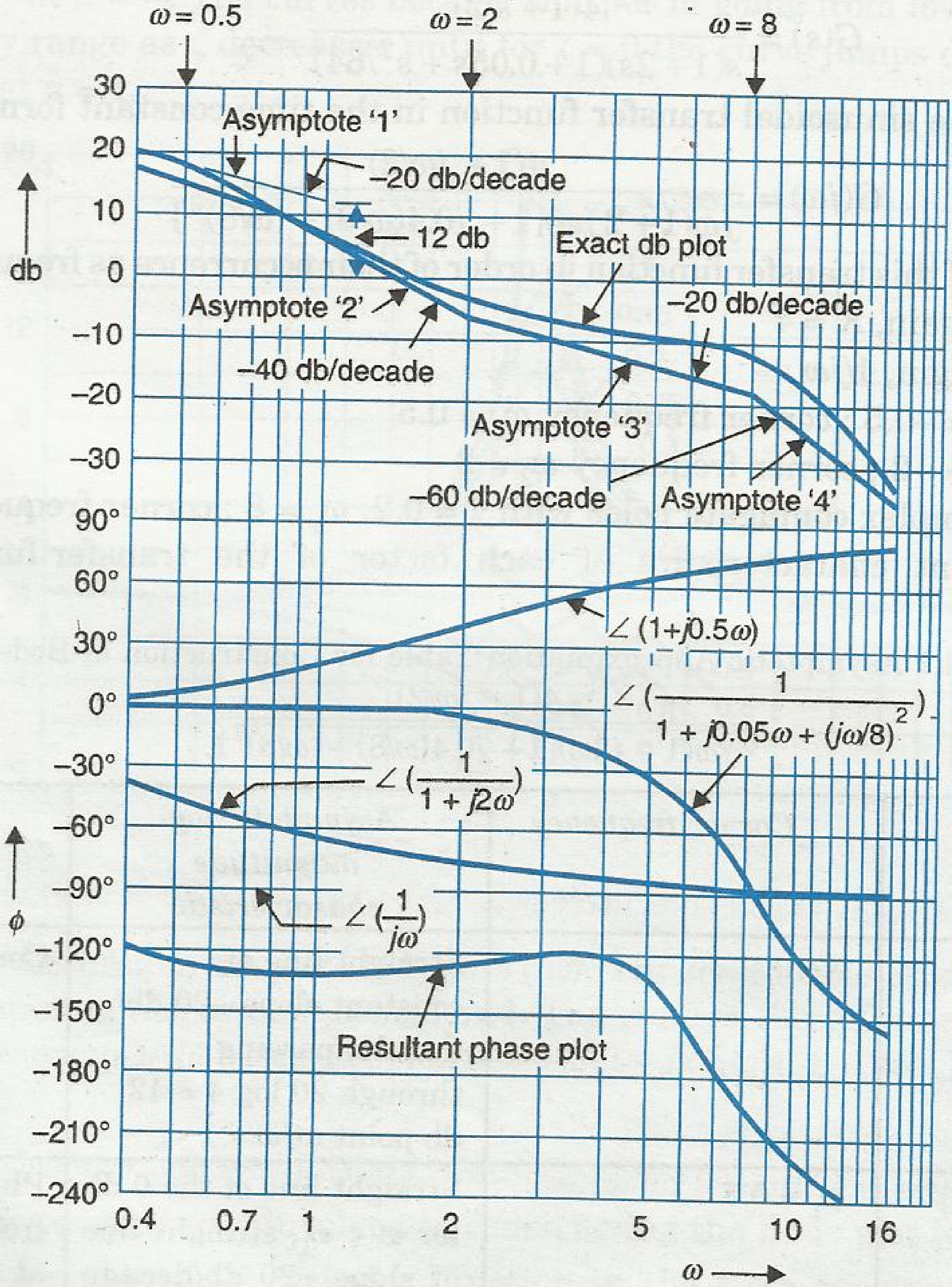


Fig. 8.16. Bode plot of $\dfrac{4(1+j\omega/2)}{j\omega(1+j2\omega)[1+j0.4(\omega/8)-(\omega/8)^2]}$.

Problem 8.1 Sketch the Bode Plots and determine the gain cross-over and phase cross-over frequencies.

$$G(s) = \frac{10}{s(1+0.5s)(1+0.1s)}$$ **(Pune University)**

Solution

Corner frequencies The corner frequencies are 2 and 10.

Magnitude Plot

Ser. No.	*Factor*	*Corner frequency*	*Asymptotic log-magnitude Characteristic*
1	$\frac{1}{s}$	None	Straight line of constant slope (–20 db/dec) passing through at $\omega = 1$
2	$\frac{1}{(1+0.5s)}$	$\omega_1 = 2$	Straight line of constant slope (–20 db/dec) originating from $\omega_1 = 2$
3	$\frac{1}{(1+0.1s)}$	$\omega_2 = 10$	Straight line of constant slope (–20 db/dec) originating from $\omega_2 = 10$
4	10	None	Straight line of constant slope of 0 db/dec starting from 20 log 10 = 20 db point

Magnitude plots for individual factors are shown by dotted lines. Resultant line is shown by a firm line (Fig. 8.1).

Phase Plot $\phi = -90° - \tan^{-1} 0.5\omega - \tan^{-1} 0.1\omega$

Ser. No.	ω	ϕ
1	0	–90°
2	0.1	–93.43°
3	1	–122.3°
4	2	–146.31°
5	5	–184.76°
6	10	–213.7°
7	15	–228.7°

Magnitude and phase plots are shown in Fig. 8.1. From the plots

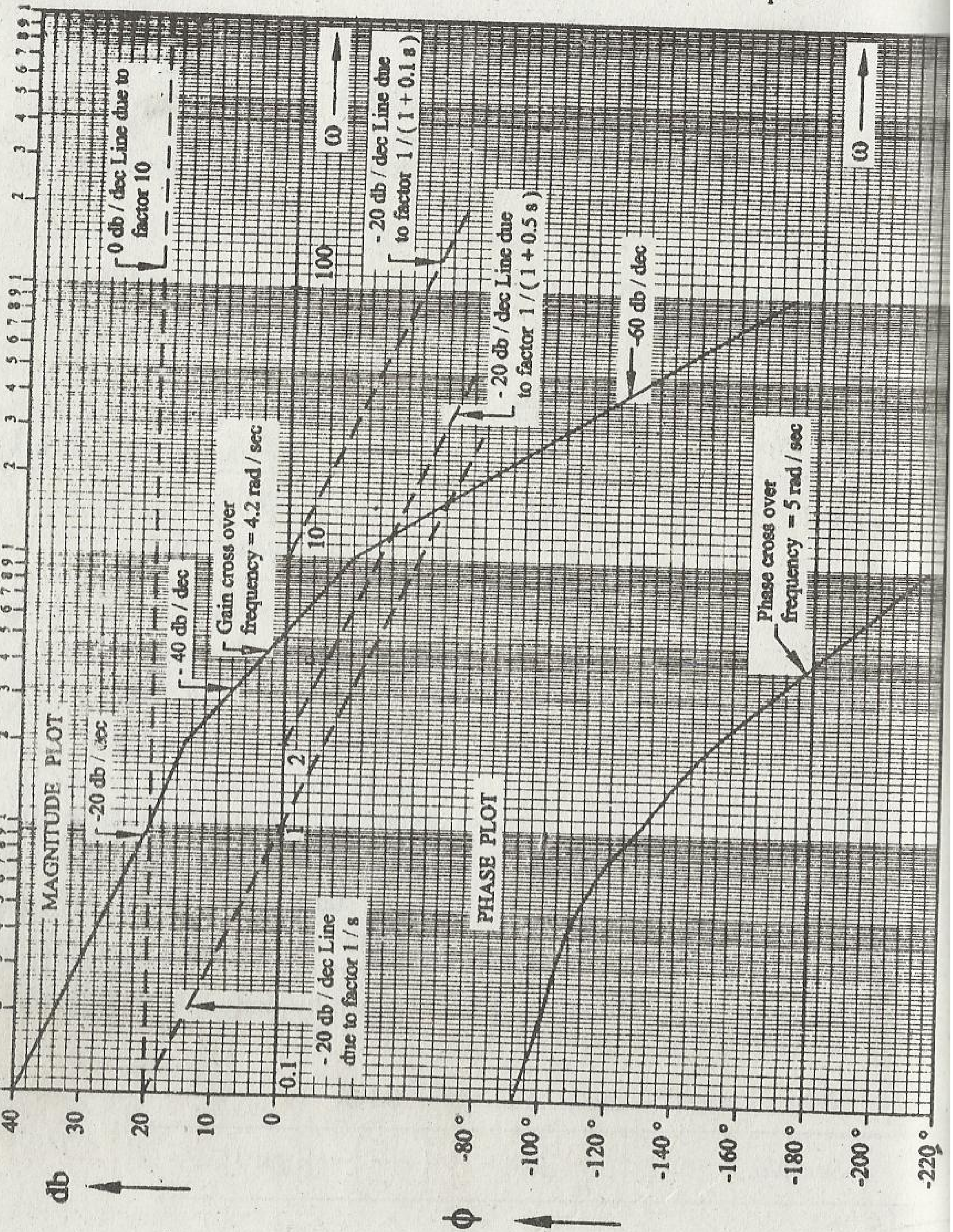


Fig. 8.1

1. Gain crossover frequency = 4.2 rad/sec **Ans.**
2. Phase crossover frequency = 4.5 rad/sec. **Ans.**

Problem 8.2 Sketch the Bode plot for the transfer function

$$G(s) = \frac{K s^2}{(1+0.2s)(1+0.02s)}$$

Determine the system gain K for the gain cross-over frequency to be 5 rad/sec.

Solution

Let, $K = 1$, then

$$G(s) = \frac{s^2}{(1+0.2s)(1+0.02s)}$$

Corner frequencies The corner frequencies are 5 and 50 rad/sec.

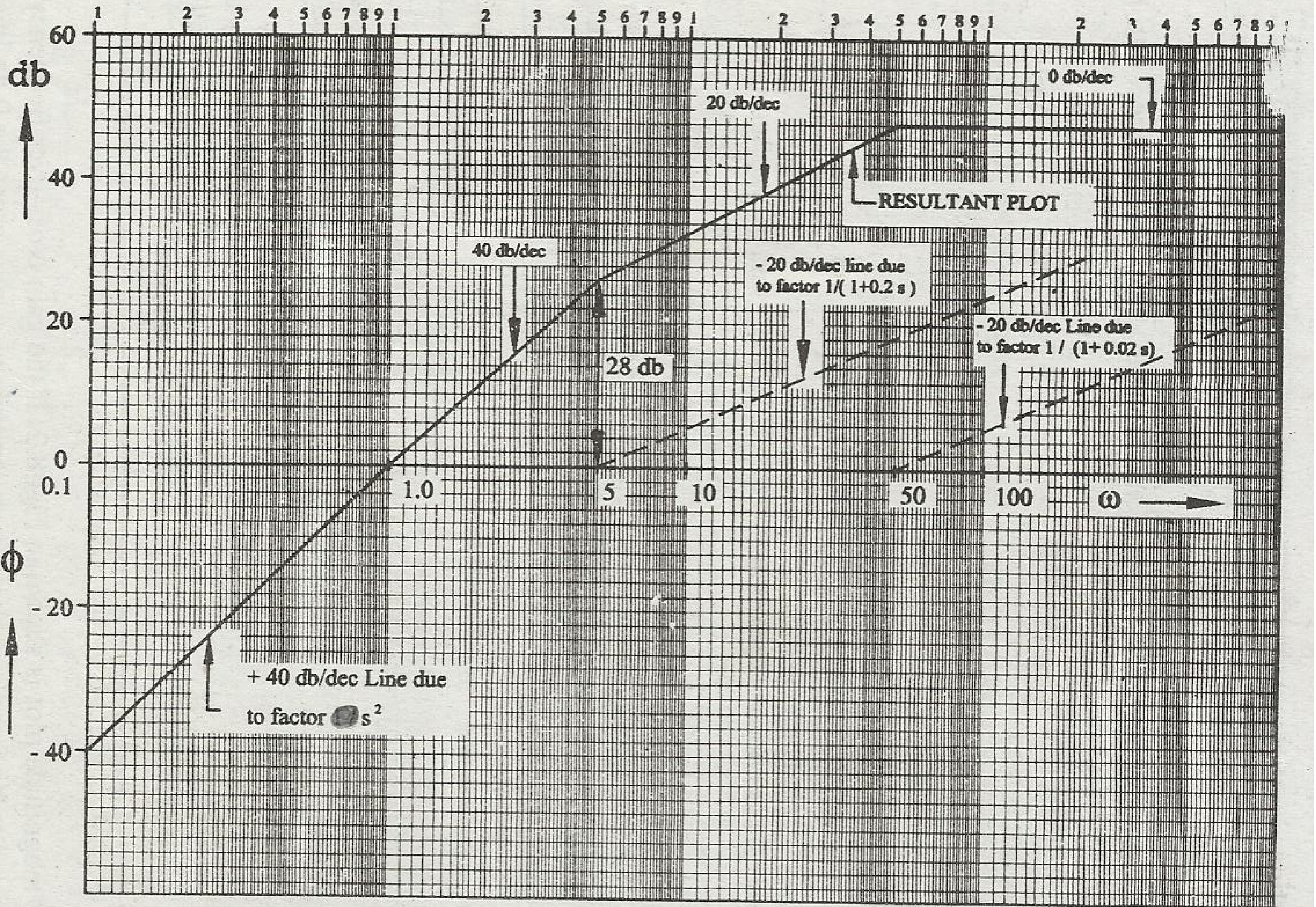
Magnitude Plot

Ser. No.	*Factor*	*Corner frequency*	*Asymptotic log-magnitude Characteristic*
1	s^2	None	Straight line of constant slope 40 db/dec passing through $\omega = 1$
2	$\frac{1}{(1+0.2s)}$	$\omega_1 = 5$	Straight line of constant slope –20 db/dec originating from $\omega = 5$
3	$\frac{1}{(1+0.02s)}$	$\omega_2 = 50$	Straight line of constant slope –20 db/dec originating from $\omega = 50$

The magnitude plot is shown in Fig. 8.2. If the gain cross-over frequency is required to be 5 rad/sec, then the magnitude plot must cross the 0 db line at 5 rad/sec. For this, the plot has to be brought down by 28 db. Hence

$$20 \log K = -28$$

$$\therefore \quad K = 0.04 \qquad \textbf{Ans.}$$



db
φ
60
40
20
0
0.1
- 20
- 40
1.0
5
10
50
100
ω
0 db/dec
20 db/dec
RESULTANT PLOT
40 db/dec
28 db
- 20 db/dec line due
to factor 1/(1+0.2 s)
- 20 db/dec Line due
to factor 1 / (1+0.02 s)
+ 40 db/dec Line due
to factor s 2

Problem 8.4 Draw the Bode plot for a system having

$$G(s)\,H(s) = \frac{100}{s(s+1)(s+2)}.\ \text{Find}$$

(a) Gain Margin

(b) Phase margin

(c) Gain cross over frequency

(d) Phase cross over frequency **(Pune University)**

Solution

$$G(s)\,H(s) = \frac{50}{s(s+1)(1+0.5s)}$$

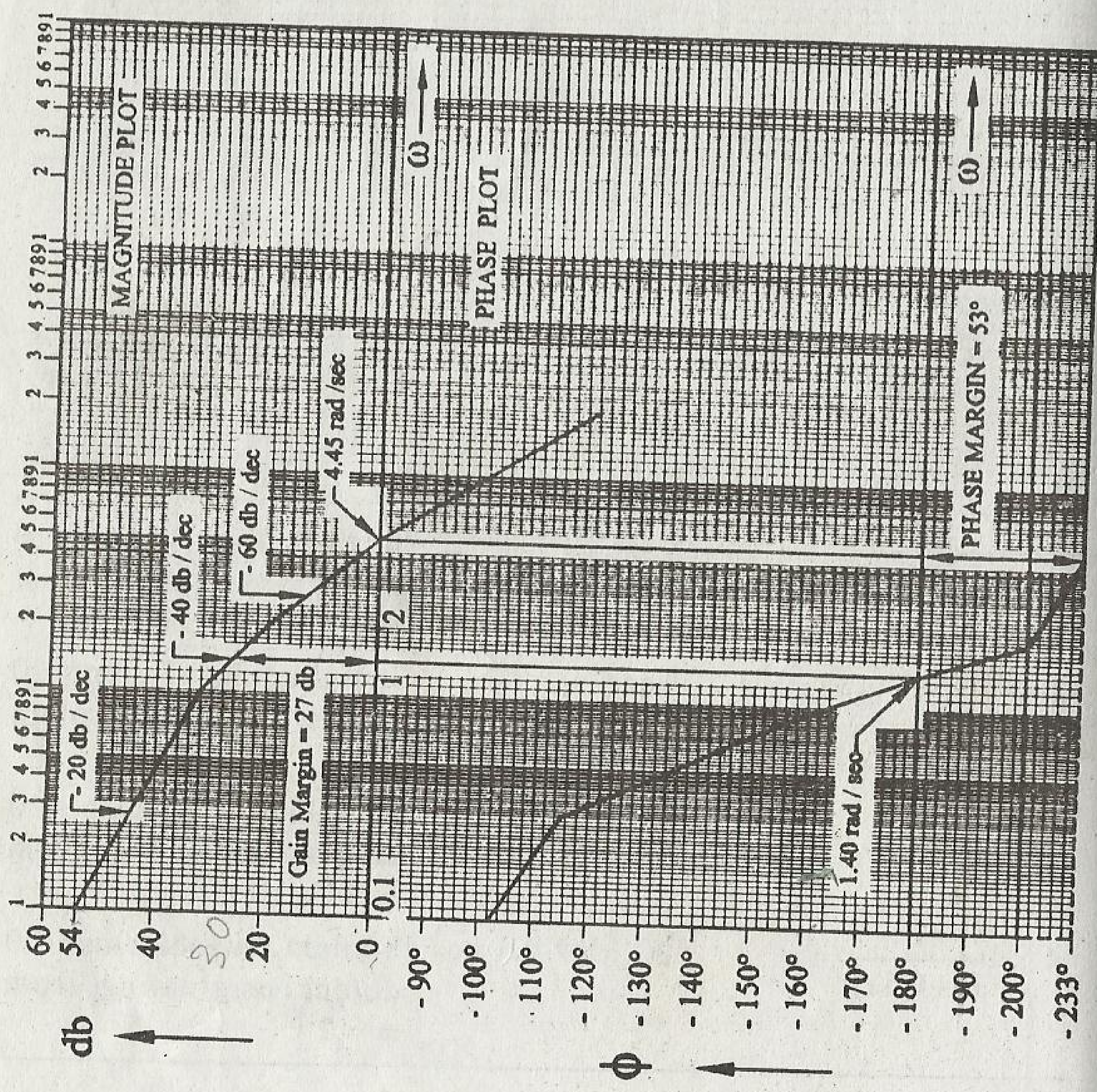
Magnitude Plot

Ser. No.	*Factor*	*Corner frequency rad/sec*	*Asymptotic log-magnitude Characteristic*
1	50	None	Straight line of slope 0 db/dec starting from point 20 log 50 = 34 db
2	$\frac{1}{s}$	None	Straight line of slope 20 db/dec passing through $\omega = 1$
3	$\frac{1}{(1+s)}$	1	Straight line of slope–20 db/dec originating from $\omega = 1$
4	$\frac{1}{(1+0.5s)}$	2	Straight line of slope–20 db/dec originating from $\omega = 2$

Phase Plot $\phi = -90° - \tan^{-1}\omega - \tan^{-1} 0.5\omega$

Ser. No.	ω *rad/sec*	ϕ
1	0	−90°
2	0.1	−98.6°
3	0.2	−107°
4	0.5	−130.6°
5	1	−161.6°
6	1.3	−175.5°
7	1.4	−179.5°
8	1.5	−183.2°
9	2	−198.4°
10	4.45	−233°

Magnitude and Phase plots as shown in Fig. 8.4



Result

1. Gain Crossover frequency : 4.45 rad/sec
2. Phase Crossover frequency : 1.40 rad/sec
3. Gain Margin : 27 db
4. Phase Margin : 53°.

Problem 8.4 The Open-loop transfer function of a certain unity feedback system is

$$G(s) = \frac{K}{s(s+2)(s+20)}$$

Construct Bode plots and determine.

(a) Limiting value of K for system to be stable
(b) Value of K for gain margin to be 10 db
(c) Value of K for phase margin to be 50° **(Pune University)**

Solution $$G(s) = \frac{0.025\,K}{s(1+0.5s)(1+0.05s)}$$

2 20

Let = 0.025 K = 1, then $G(s) = \dfrac{1}{s(1+0.5s)(1+0.05s)}$

Magnitude Plot

Ser. No.	*Factor*	*Corner frequency rad/sec*	*Asymptotic log-magnitude Characteristic*
1	$\frac{1}{s}$	None	Straight line of 0 db/dec passing through $\omega = 1$
2	$\frac{1}{(1+0.5s)}$	2	Straight line of –20 db/dec originating from $\omega = 2$
3	$\frac{1}{(1+0.05s)}$	20	Straight line of –20 db/dec originating from $\omega = 4$

Phase Plot $\phi = -90° - \tan^{-1} 0.5\,\omega - \tan^{-1} 0.05\,\omega$

Ser. No.	*ω rad/sec*	ϕ
1	0	–90°
2	1	–119°
3	2	–141°
4	2.5	–148.5°
5	3	–155°
6	4	–165°
7	4.5	–168.7°
8	5	–172°
9	6	–178.3°
10	6.5	–181°
11	10	–195°

Bode plots are shown in Fig. 8.5

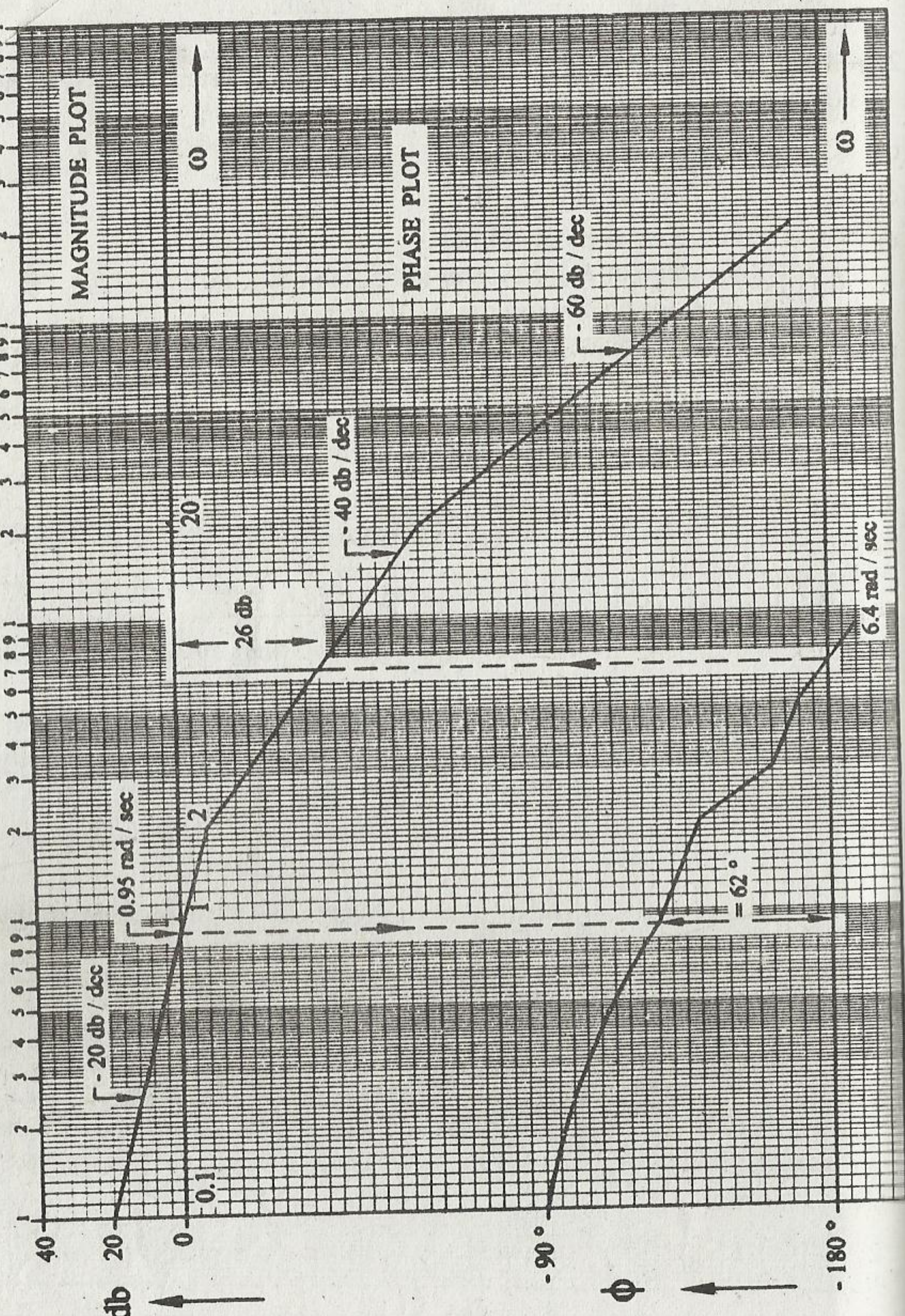


Fig. 8.5

(a) From the curves, the gain margin is 26 db

$$\therefore \qquad 20 \log K_1 = 26$$

$$\text{or} \qquad K_1 = 19.95$$

or $\qquad 0.025\ K = 19.95$

or $\qquad K = 798$ **Ans.**

(b) For the gain margin to be 10 db, the graph has to be lifted up by 26 – 10 = 16 db

$\therefore \qquad 20 \log K_1 = 16$

or $\qquad K_1 = 6.3$

or $\qquad 0.025\ K = 6.3$

or $\qquad K = 252$ **Ans.**

(c) For the phase margin to be 50°, the value of ω at $-180° + 50° = -130°$ is 1.9 rad/sec. Gain Margin at 1.9 rad/sec is 5.5 db. Therefore, to have phase margin of 50°, magnitude plot has to be lifted up by 5.5 db, so that gain cross over frequency is 1.9 rad/sec

$\therefore \qquad 20 \log K_1 = 5.5$

or $\qquad K_1 = 1.88$

or $\qquad 0.025\ K = 1.88 \qquad$ or $\qquad K = \dfrac{1.88}{0.025} = \mathbf{75.35}$ **Ans.**

Extraction of Transfer function from Bode Diagrams :

(1) Find the open loop transfer function of a system whose approximate plot is shown in figure

(Sol) The corner frequencies are

$\omega_{c_1} = 2.5$; $\omega_{c_2} = 10$; $\omega_{c_3} = 25$ rad/sec

— change in magnitude in dB

$= \text{slope} \, (\text{Number of decades between two frequencies})$

$= -20 \, (\log 2.5 - \log 1)$

$= -7.95$

$\therefore$ Magnitude $= -12 - (-7.95)$

$= -12 + 7.95$

$= -4.05 \, dB$

$\therefore 20 \log K = -4.05$

$\log K = \frac{-4.05}{20}$

$\therefore K = 10^{-\frac{4.05}{20}}$

$= 0.63$

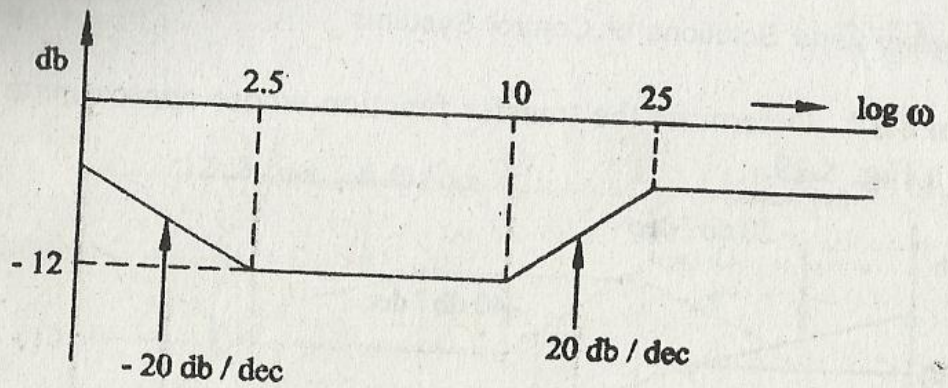


Fig. 8.18

$$= -20 (\log 2.5 - \log 1)$$
$$= -20 \log 2.5 = -7.95$$

Magnitude $= -12 + 7.95$ db $= -4.05$ db

$\therefore$ $20 \log K = -4.05$

or $K = 0.63$

Since first line has a slope of – 20 db/dec and starts from a point – 4.05 db at $\omega = 1$ rad/sec the factor contributing this is

$$= \frac{K}{s} = \frac{0.63}{s}$$

Plot between $\omega = 2.5$ and $\omega = 10$ is having a slope of 0 db/dec. At $\omega = 2.5$ the slope has changed from – = 20 db/dec and this can only happen due to a factor in the numerator and is

$$= \left(\frac{s}{2.5} + 1\right) = (1 + 0.4s)$$

At $\omega = 10$, the slope has changed from 0 db/dec to + 20 db/dec and is due to a factor in the numerator and is

$$= \left(\frac{s}{10} + 1\right) = (1 + 0.1s)$$

At $\omega = 25$, the slope has changed from + 20 db/dec to 0 db/dec and is due to a factor in the denominator and is

$$= \left(\frac{s}{25} + 1\right) = (1 + 0.4s)$$

The open-loop transfer function is this

$$G(s) = \frac{0.63(1 + 0.4s)(1 + 0.1s)}{(1 + 0.04s)}$$

Problem 8.17 Determine the transfer function whose approximate plot is shown in Fig. 8.19.

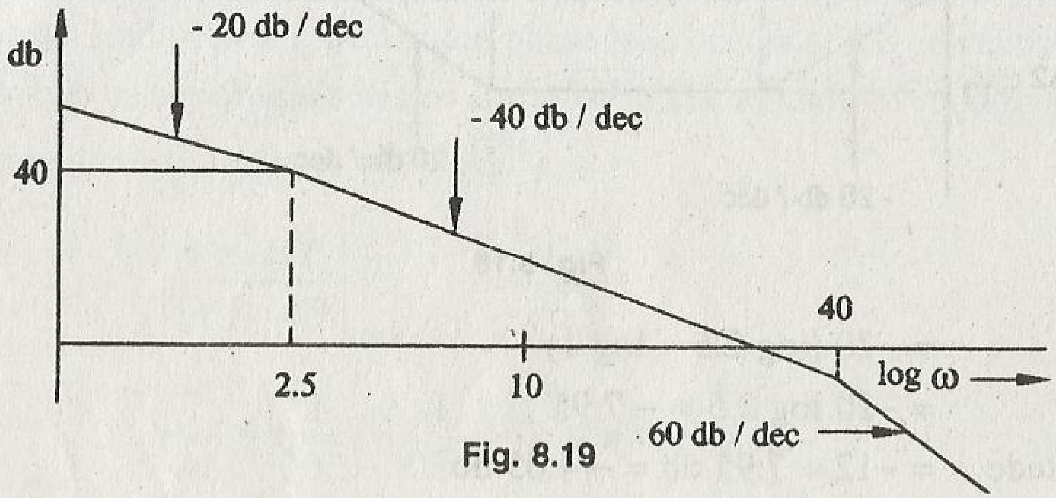


Fig. 8.19

Solution

Corner frequencies are 2.5 and 40 rad/sec

$$20 \log K = 40 + 20 \log 2.5 = 47.95$$

or $$K = 250$$

At $\omega = 2.5$ rad/sec slope changes from -20 db/dec to -40 db/dec due to a factor $\dfrac{1}{\left(1+\dfrac{s}{2.5}\right)}$. At $\omega = 40$ rad/sec slope changes from -40 db/dec to -60 db/dec due to a factor $\dfrac{1}{\left(1+\dfrac{s}{40}\right)}$. Also, since initial slope is -20 db/dec, it is due to factor. $1/s$. Therefore open-loop transfer function is

$$G(s) = \frac{250}{s\left(1+\dfrac{s}{2.5}\right)\left(1+\dfrac{s}{40}\right)} = \frac{250}{s(1+0.4s)(1+0.025s)}$$

Problem 8.18 Determine the open-loop transfer function of a system whose approximate plot is shown in Fig. 8.20.

Solution

First line is having a slope of 12 db/oct (40 db/dec). Therefore, there is a s^2 term in the numerator. At $\omega = 0.5$ rad/sec slope changes to 6 db/oct (20 db/dec) due to a term in the denominator equal to $\left(1+\dfrac{s}{0.5}\right)$

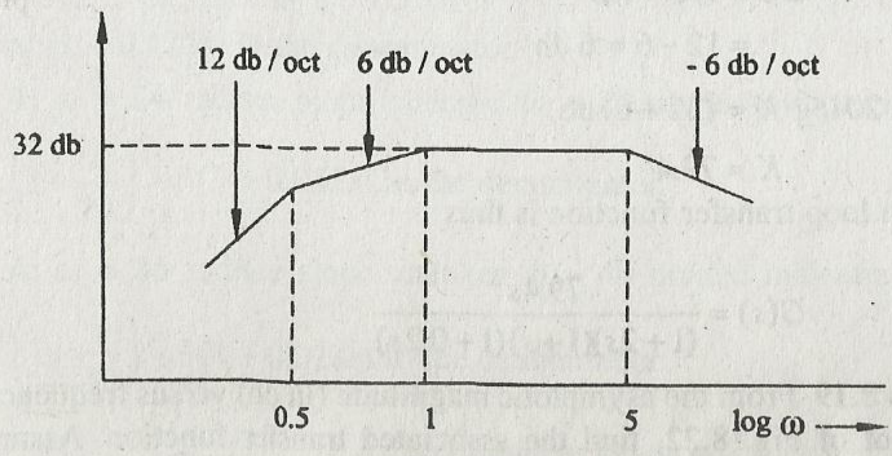


Fig. 8.20

At ω = 1 rad/sec slope becomes 0 db/dec due to a term in the denominator equal to $(1 + s)$.

At ω = 5 rad/sec slope becomes – 6 db/oct (– 20 db/dec) due to a term in the denominator equal to $\left(1+\frac{s}{5}\right)$

$$G(s) = \frac{K s^2}{\left(1+\frac{s}{0.5}\right)(1+s)\left(1+\frac{s}{5}\right)}$$

Calculation of 'K'

Refer Fig. 8.21

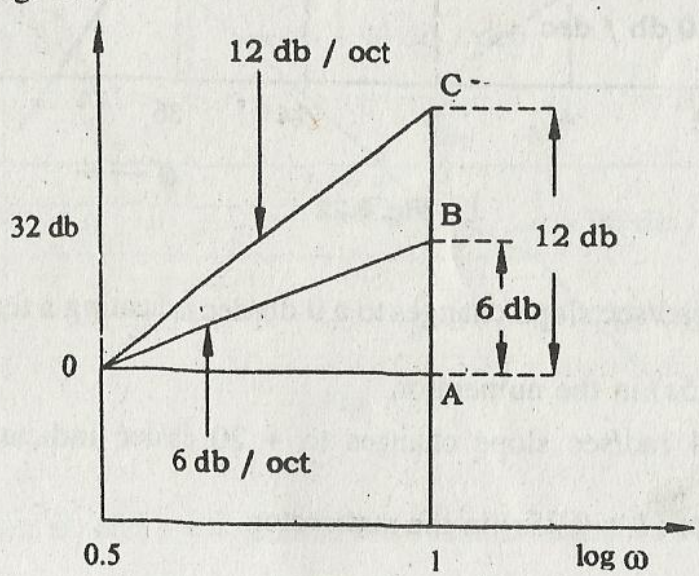


Fig. 8.21

Difference between ω = 0.5 and ω = 1 rad/sec is one octave i.e. $AB = 6$ db since the slope of line OB is 6 db/oct. OC is the extended line having a slope of 12 db/oct.

$\therefore \qquad AC = 12$ db

$$BC = AC - AB$$

$$= 12 - 6 = 6 \text{ db}$$

$\therefore \qquad 20 \log K = (32 + 6)$ db

or $\qquad K = 79.4$

The open loop transfer function is thus

$$G(s) = \frac{79.4s^2}{(1+2s)(1+s)(1+0.2s)}$$

Ans.

Problem 8.19 From the asymptotic magnitude (in db) versus frequency (log scale) plot of Fig. 8.22, find the associated transfer function. Assume no right half plane poles or zeros present. **(Pune University)**

Solution

1. Slope of the first line is – 20 db/dec indicating a term $\frac{1}{s}$

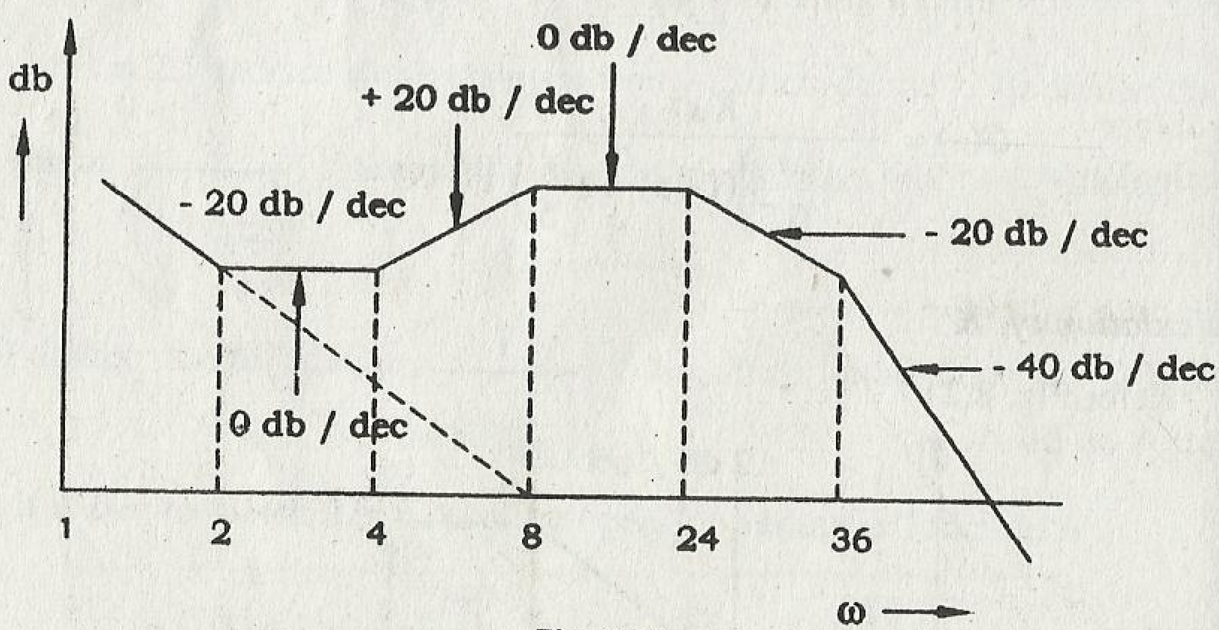


Fig. 8.22

2. At $\omega = 2$ rad/sec slope changes to a 0 db/dec indicating a term $\left(1 + \frac{s}{2}\right)$ or $(1 + 0.5s)$ in the numerator.
3. At $\omega = 4$ rad/sec slope changes to + 20 db/dec indicating a term $\left(1 + \frac{s}{4}\right)$ or $(1 + 0.25s)$ in the numerator.

4. At $\omega = 8$ rad/sec slope changes to 0 db/dec indicating a term $\left(1+\frac{s}{8}\right)$ or $(1 + 0.125s)$ in the denominator.

5. At $\omega = 24$ rad/sec slope changes to $-$ 20 db/dec indicating a term $\left(1+\frac{s}{24}\right)$ or $(1 + 0.042s)$ in the denominator.

6. At $\omega = 36$ rad/sec slope changes to $-$ 40 db/dec indicating a term $\left(1+\frac{s}{40}\right)$ or $(1 + 0.028s)$ in the denominator.

Transfer function is thus $\dfrac{K(1+0.5s)(1+0.25s)}{s(1+0.125s)(1+0.042s)(1+0.028s)}$

Calculation of 'K'

$$20 \log K = 20 \log 8$$

or
$$K = 8$$

$$\therefore \quad G(s) = \frac{8(1+0.5s)(1+0.025s)}{s(1+0.125s)(1+0.042s)(1+0.028s)}$$

Problem 8.20 Derive the transfer function of the system from the data given on the Bode diagram shown in Fig. 8.23 below. (AMIE)

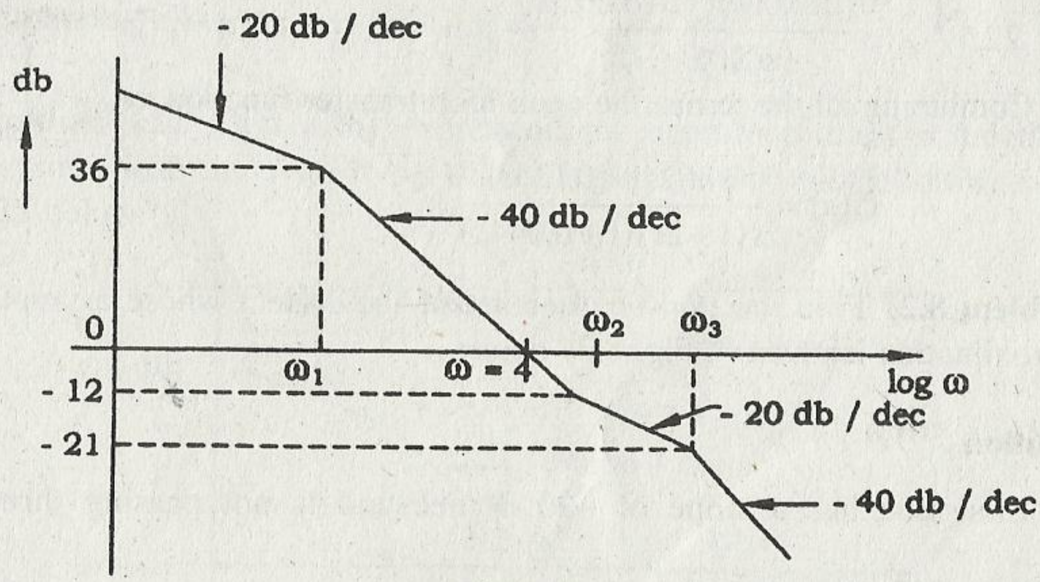


Fig. 8.23

Solution

Between ω_1 and $\omega = 4$ rad/sec there is a decrease of 36 db

$$\therefore \quad -36 = -40(\log 4 - \log \omega_1)$$

or
$$\omega_1 = 0.5036 \cong 0.5 \text{ rad/ sec}$$

Calculation of 'K' $20 \log K = 36 + 20 \log 0.5$

or $K = 31.62$

Calculation of 'ω_2' $-12 = -40 (\log \omega_2 - \log 4)$

or $\omega_2 = 8$ rad/sec

Calculation of 'ω_3' $-21 + 12 = -20 (\log \omega_3 - \log 8)$

or $\omega_3 = 22.5$ rad/sec

First line has a slope of – 20 db/dec indicating a term $\frac{1}{s}$ and since it is not passing through $\omega = 1$ rad/sec, the term is $\frac{K}{s}$ or $\frac{31.62}{s}$.

At $\omega_1 = 0.5$ rad/sec slope changes to –40 db/dec indicating a term $\frac{1}{\left(1+\frac{s}{0.5}\right)}$, or $\frac{1}{(1+2s)}$.

At $\omega_2 = 8$ rad/sec, slope changes to –20 db/dec indicating a term $\left(1+\frac{s}{8}\right)$ or $(1+0.125s)$

At $\omega_3 = 22.5$ rad/sec, slope changes to –40 db/dec indicating a term $\frac{1}{\left(1+\frac{s}{22.5}\right)}$ or $\frac{1}{(1+0.044s)}$

Combining all the terms, the open-loop transfer function is

$$G(s) = \frac{31.62(1+0.125s)}{s(1+2s)(1+0.044s)}$$

Problem 8.21 Find the transfer function of the system whose asymptotic approximation is given in Fig. 8.24 below.

Solution

First line has a slope of –20 db/dec and is not passing through $\omega = 1$ rad/sec. Therefore, it indicates a term $\frac{K}{s}$

$$20 \log K = -9 \quad \text{or} \quad K = 0.35$$

$\therefore$ the term is $\frac{0.35}{s}$

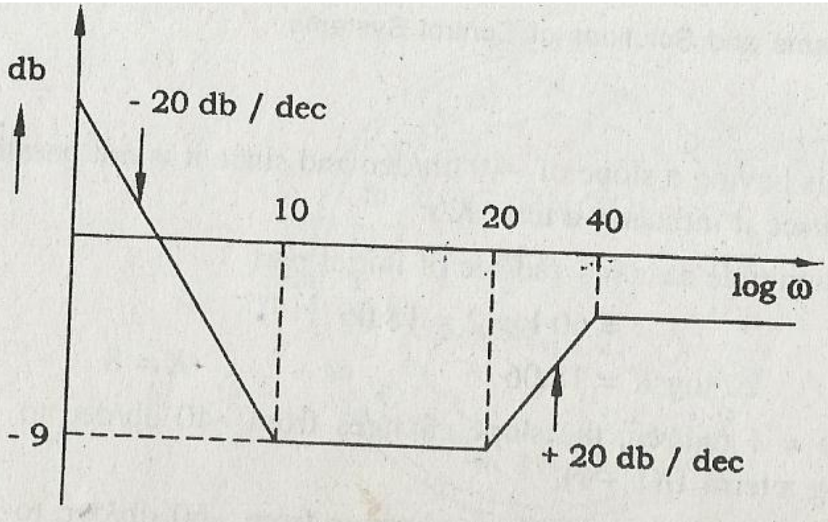


Fig. 8.24

At ω = 1 rad/sec, slope changes to 0 db/dec indicating a term $(1 + s)$.

At ω = 20 rad/sec, slope changes to +20 db/dec indicating a term $\dfrac{1}{\left(1+\dfrac{s}{20}\right)}$ or $(1+0.05s)$.

At ω = 40 rad/sec, slope changes to 0 db/dec indicating a term $\dfrac{1}{\left(1+\dfrac{s}{40}\right)}$ or $\dfrac{1}{(1+0.025s)}$.

Combining all terms, we get $G(s)=\dfrac{0.35(1+s)(1+0.05s)}{s(1+0.025s)}$

Problem 8.22 Obtain the expression for open-loop transfer function for a system with unity feedback whose log-magnitude plot is shown in Fig. 8.25 below:

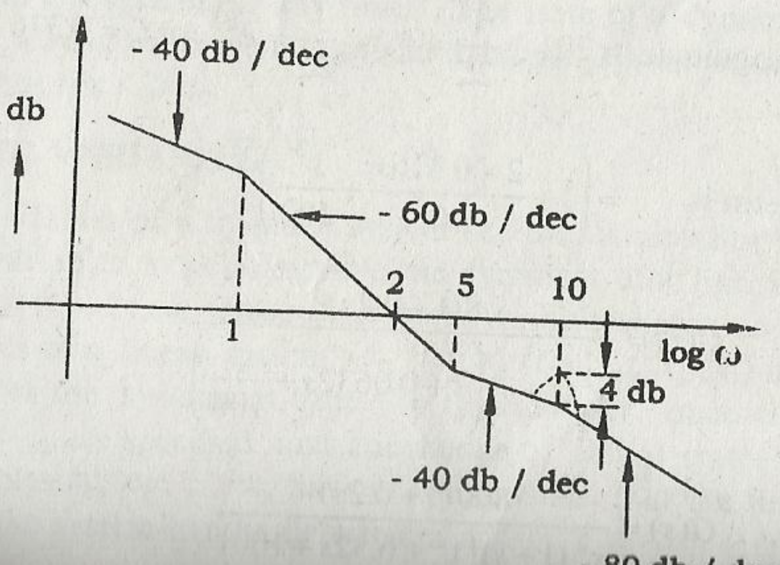


Fig. 8.25

Solution

First line is having a slope of −40 db/dec and since it is not passing through $\omega = 1$ rad/sec it indicates a term K/s^2

Magnitude at $\omega = 1$ rad/sec of initial part

$$= 60 \log 2 = 18.06$$

$$\therefore \quad 20 \log K = 18.06 \qquad \text{or} \qquad K = 8$$

At $\omega = 1$ rad/sec, the slope changes from −40 db/dec to −60 db/dec indicating a term $1/(1 + s)$.

At $\omega = 5$ rad/sec, the slope changes from −60 db/dec to −40 db/dec indicating a term $\left(1+\frac{s}{5}\right)$ or $(1+0.2s)$.

At $\omega = 10$ there is a term of the form $\left\{\left(1+\frac{2\zeta s}{\omega_n}+\frac{s^2}{\omega_n^2}\right)\right\}^{-1}$

because the slope changes from −40 db/dec to −80 db/dec and also a peak of 4 db is shown

$$\omega_n = 10 \text{ rad/sec}$$

Value of $\left\{1+\frac{2\zeta s}{\omega_n}+\left(\frac{s}{\omega_n}\right)^2\right\}^{-1}$ at $\omega = \omega_n$

$$=\left\{\sqrt{\left(1-\frac{10}{10}\right)^2+\left(\frac{2\times\zeta\times 10}{10}\right)^2}\right\}^{-1} = \frac{1}{2\zeta}$$

$$\therefore \quad \text{log magnitude} = 20\log\frac{1}{2\zeta} = 4, \text{ or } \frac{1}{2\zeta} = e^{1/5}, \text{ or } \zeta = 0.316$$

$$\therefore \quad \text{the term is} \quad = \left(1+\frac{2\times 0.316s}{10}+\frac{s^2}{100}\right)^{-1}$$

$$\therefore \quad G(s) = \frac{8(1+0.2s)}{s^2(1+s)\left(1+0.0632s+\frac{s^2}{100}\right)}$$

$$\text{or} \quad G(s) = \frac{800(1+0.2s)}{s^2(1+s)(s^2+6.32s+100)}$$

Minimum phase, All pass and Non-Minimum phase Systems:

(1) All pass System: A system having a pole-zero pattern which is antisymmetric about the imaginary axis, ie, for every pole in left half s-plane, there is a zero in the mirror image position. The transfer function of all pass system is given by

$$G(j\omega) = \left(\frac{1-j\omega T}{1+j\omega T}\right) \longrightarrow (1)$$

$$\text{Magnitude} = |G(j\omega)| = \frac{\sqrt{1+\omega^2T^2}}{\sqrt{1+\omega^2T^2}} = 1$$

$$\text{Phase angle } \phi = -\tan^{-1}(\omega T) - \tan^{-1}(\omega T) = -2\tan^{-1}(\omega T)$$

Thus, the all pass system has a magnitude of unity and phase angle varies from 0 to -180° as ω is increased from 0 to ∞.

(2) Non-Minimum phase System: If a system has poles in the left half s-plane and zeros in both the left and right half s-plane, such a system is said to be non-minimum phase system. The transfer function of such a system is given by

$$G_1(j\omega) = \frac{(1-j\omega T)}{(1+j\omega T_1)(1+j\omega T_2)} \longrightarrow (2)$$

$$\text{Magnitude} = |G_1(j\omega)| = \frac{\sqrt{1+\omega^2T^2}}{\sqrt{1+\omega^2T_1^2}\cdot\sqrt{1+\omega^2T_2^2}}$$

$$\text{phase angle } \phi = -\tan^{-1}(\omega T) - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

Non-minimum phase system: is a combination of both allpass and minimum phase systems. The transfer function of non-minimum phase system is also given by

$$G_1(j\omega) = G_2(j\omega)\, G(j\omega)$$

where $G_2(j\omega)$ is minimum phase system.

Minimum-phase system: If all the poles and zeros of a system lie in left half s-plane, the system is said to be minimum-phase system. The transfer function of minimum phase system is given by

$$G_2(j\omega) = \frac{(1+j\omega T)}{(1+j\omega T_1)(1+j\omega T_2)} \longrightarrow (3)$$

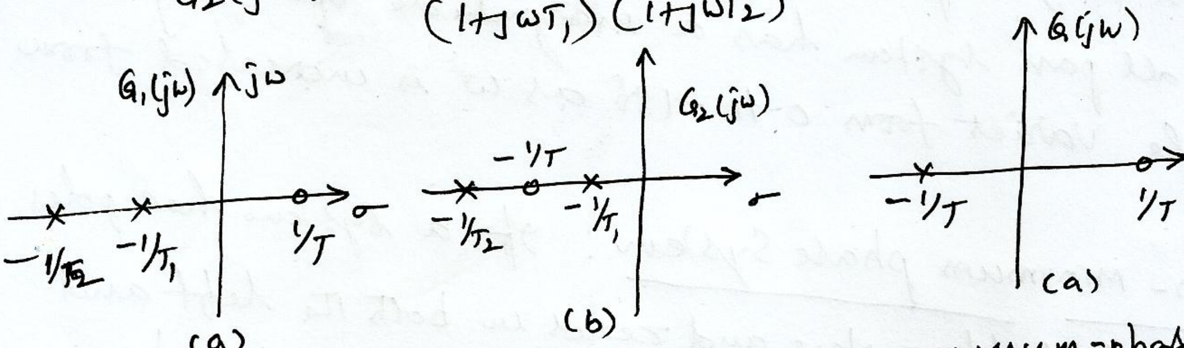


Figure: pole-zero patterns for (a) nonminimum-phase function (b) minimum-phase function (c) all-pass function

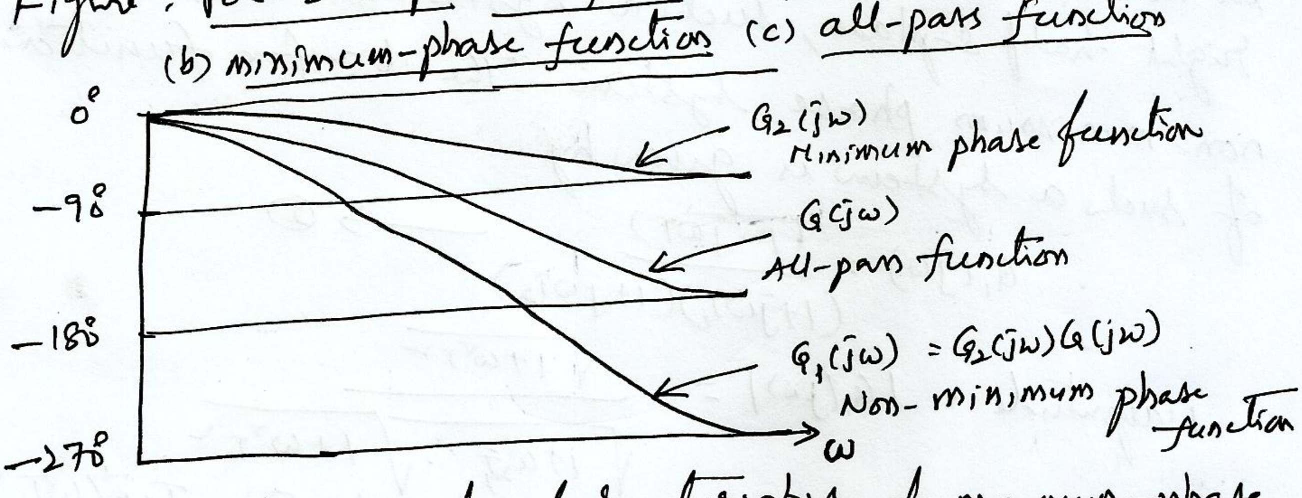


Figure: phase angle characteristics of minimum-phase All-pass and nonminimum-phase functions.

Polar plots: Let us consider a simple RC network shown in figure. (20)

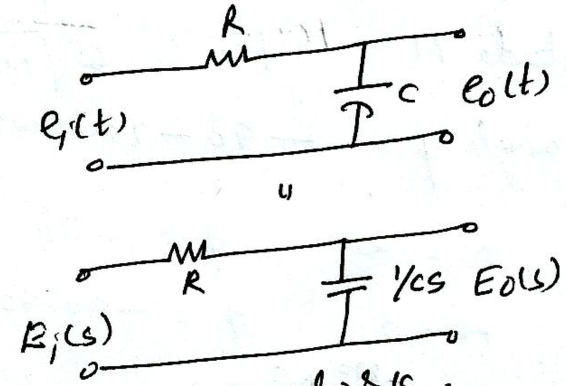


Figure: RC network.

The transfer function $G(s) = \frac{E_0(s)}{E_i(s)} = \frac{1/CS}{R+1/CS} = \frac{1}{1+RCS}$

where $T = RC$ is the time constant

$\therefore$ Transfer function $G(s) = \frac{1}{1+ST}$

The sinusoidal TF $G(j\omega) = \frac{1}{1+j\omega T} \longrightarrow (1)$

Magnitude $|G(j\omega)| = M = \frac{1}{\sqrt{1+\omega^2T^2}}$

phase angle $\angle G(j\omega) = \phi = -\tan^{-1}(\omega T)$

If $\omega = 0$ $\quad M = 1;$ $\quad \phi = 0$

$\omega = \frac{1}{T}$ $\quad M = \frac{1}{\sqrt{2}}$ $\quad \phi = -45^\circ$

$\omega \to \infty$ $\quad M \to 0$ $\quad \phi \to -90^\circ$

As the input frequency ω is varied from 0 to ∞, the magnitude M and phase angle ϕ change and hence the tip of the phase $G(j\omega)$ traces a locus in the complex plane. The locus thus obtained is known as polar plot.

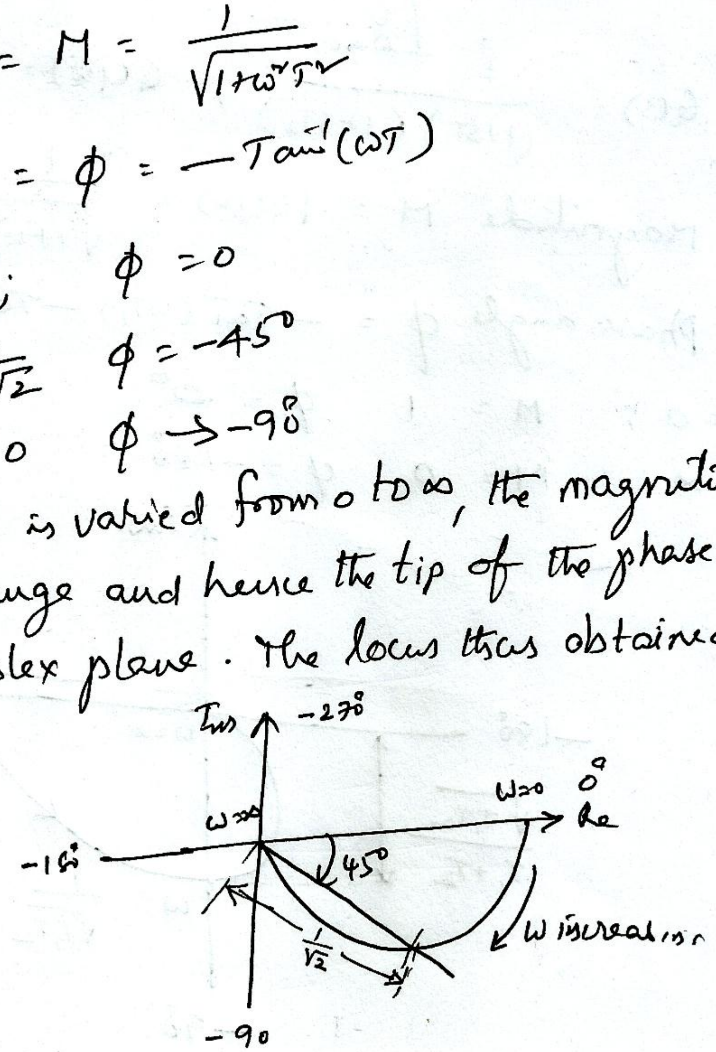


Figure: polar plot of $\frac{1}{1+j\omega T}$

② Sketch the polar plot of $G(j\omega) = \dfrac{1}{j\omega(1+j\omega T)}$

(Sol) Magnitude $M = |G(j\omega)| = \dfrac{1}{\omega\sqrt{1+\omega^2T^2}}$

phase angle $\phi = -90^\circ - \tan^{-1}(\omega T)$

$\omega = 0 \qquad M = \infty \quad ; \ \phi = -90^\circ$

$\omega = \infty \qquad M = 0 \qquad \phi = -90-90 = -180^\circ$

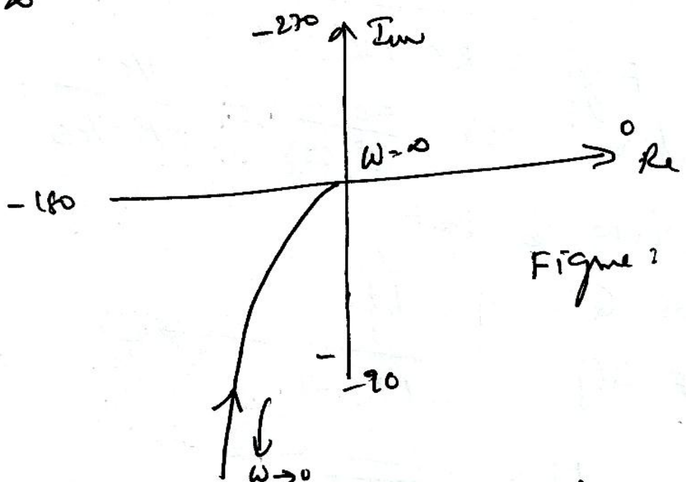


Figure: polar plot of $\dfrac{1}{j\omega(1+j\omega T)}$

(3) $G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)}$; $G(j\omega) = \dfrac{1}{(1+j\omega T_1)(1+j\omega T_2)}$

∴ Magnitude $M = |G(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}$

Phase angle $\phi = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$

$\omega = 0 ; \quad M = 1 \quad \phi = 0^\circ$

$\omega = \infty \quad M = 0 \quad \phi = -180^\circ$

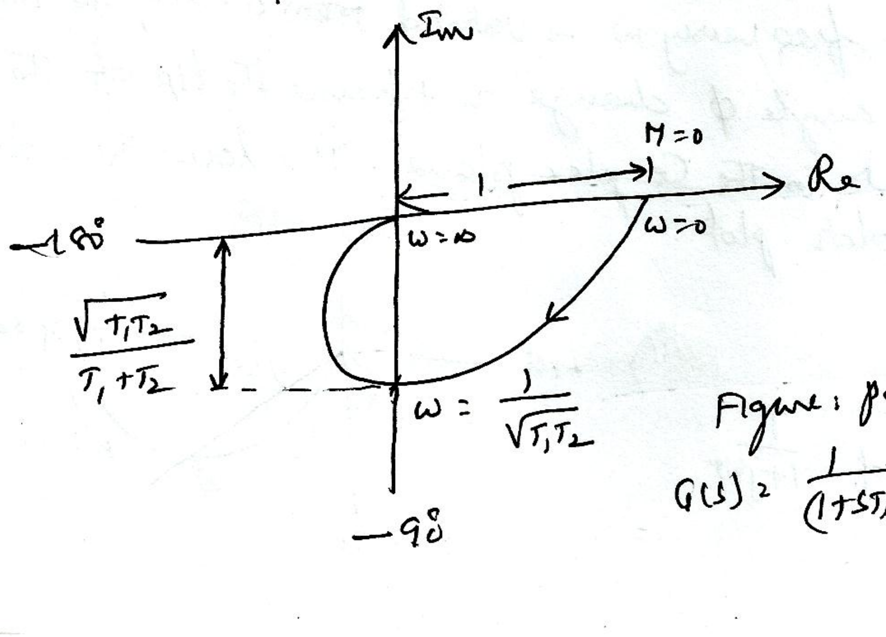


Figure: polar plot of $G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)}$

(3) $G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

Magnitude $M = \dfrac{1}{\sqrt{1+\omega^2 T_1^2}\sqrt{1+\omega^2 T_2^2}\sqrt{1+\omega^2 T_3^2}}$

$$\phi = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_3)$$

$\omega = 0$; $\quad M = 1$; $\quad \phi = 0$

$\omega \to \infty \quad M \to 0 \quad \phi \to -270$

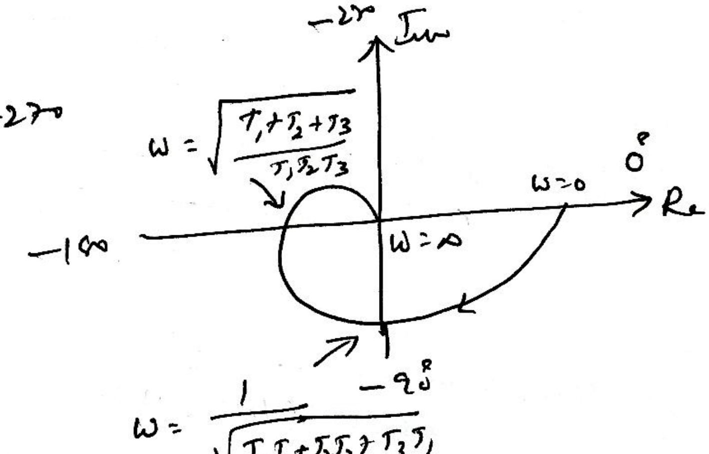


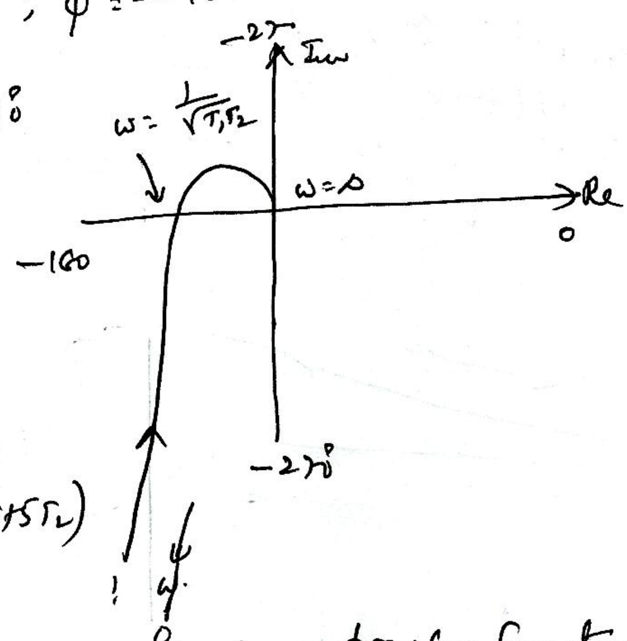
Figure: polar plot of $G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$

(4) $G(j\omega) = \dfrac{1}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$

$M = \dfrac{1}{\omega\sqrt{1+\omega^2 T_1^2}\sqrt{1+\omega^2 T_2^2}}$; $\phi = -90 - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$

$\omega = 0 \quad M = \infty \quad \phi = -90°$

$\omega \to \infty \quad M \to 0 \quad \phi = -270$



polar plot of $G(s) = \dfrac{1}{s(1+sT_1)(1+sT_2)}$

Note: (1) Addition of nonzero pole to a transfer function results in further rotation of the polar plot through an angle of $-90°$ as $\omega \to \infty$.

(2) Addition of a pole at the origin to a transfer function rotates the polar plot at zero and infinite frequencies further by an angle of $-90°$.

(5) $G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)}$ $\therefore G(j\omega) = \dfrac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)}$

$\therefore M = \dfrac{1}{\omega^2\sqrt{1+\omega^2 T_1^2}\sqrt{1+\omega^2 T_2^2}}$; $\phi = -180 - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$

$\omega = 0 \quad M = \infty ; \quad \phi = -180^\circ$

$\omega = \infty \quad M = 0 \quad \phi = -360^\circ$

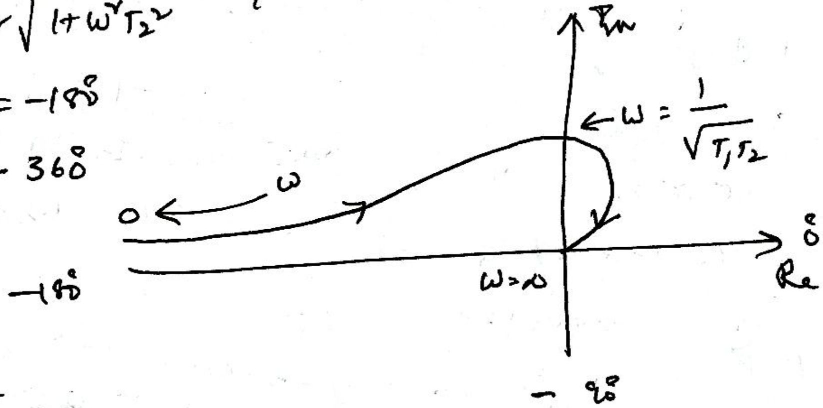


Figure: polar plot of
$G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)}$

(6) $G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$ $\therefore G(j\omega) = \dfrac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$

$M = \dfrac{1}{\omega^2\sqrt{1+\omega^2 T_1^2}\sqrt{1+\omega^2 T_2^2}\sqrt{1+\omega^2 T_3^2}}$; $\phi = -180^\circ - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_3)$

$\omega = 0; \quad M = \infty; \quad \phi = -180^\circ$

$\omega \to \infty \quad M \to 0 \quad \phi \to -450^\circ$

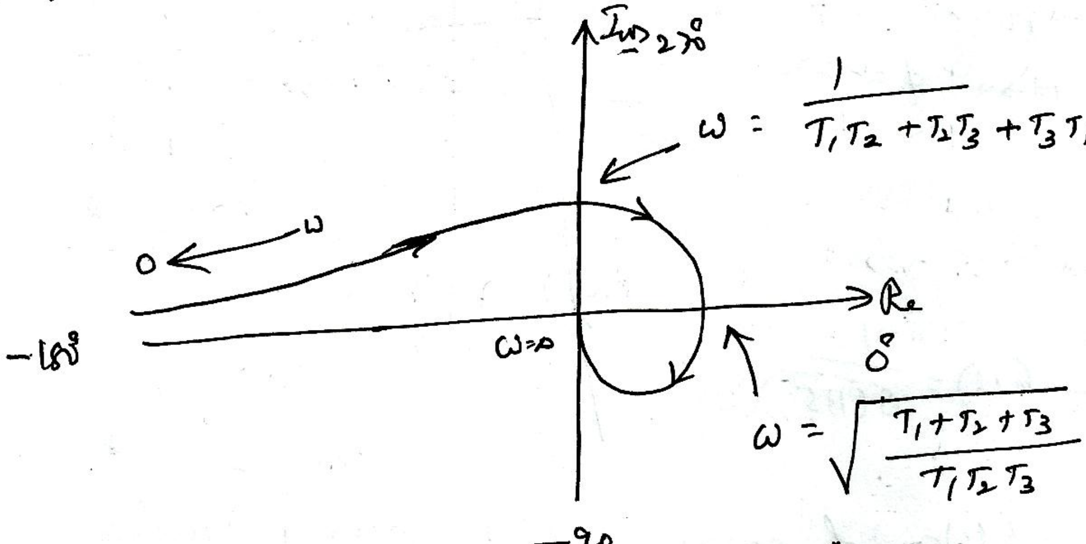


Figure: polar plot of $G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$

Principle of Argument of Cauchy: Let us consider a function $q(s)$ given by
$$q(s) = \frac{(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_m)}{(s-\beta_1)(s-\beta_2)\cdots(s-\beta_n)} \rightarrow ①$$

Let 's' is a complex variable represented by $s = \sigma + j\omega$ on the complex plane. Then $q(s)$ is also complex and may be defined as $q(s) = u + jv$

A function $q(s)$ is analytic in the s-plane provided the function and all the derivatives of it exists. The points in the s-plane where the function or its derivatives does not exist, are called singular points. The poles of a function are singular points.

The equation ① indicates that for every point 's' in the s-plane at which $q(s)$ is analytic, we can find a corresponding point $q(s)$ in the $q(s)$-plane. Alternatively, it can be stated that the function $q(s)$ maps the points in the s-plane into $q(s)$-plane. It follows that for a contour in the s-plane which does not go through any singular point, there corresponds a contour in the $q(s)$-plane as shown in figure. The region to the right of a closed contour is considered enclosed by the contour when the contour is traversed in the clockwise direction. Thus the shaded area is enclosed by the closed contour

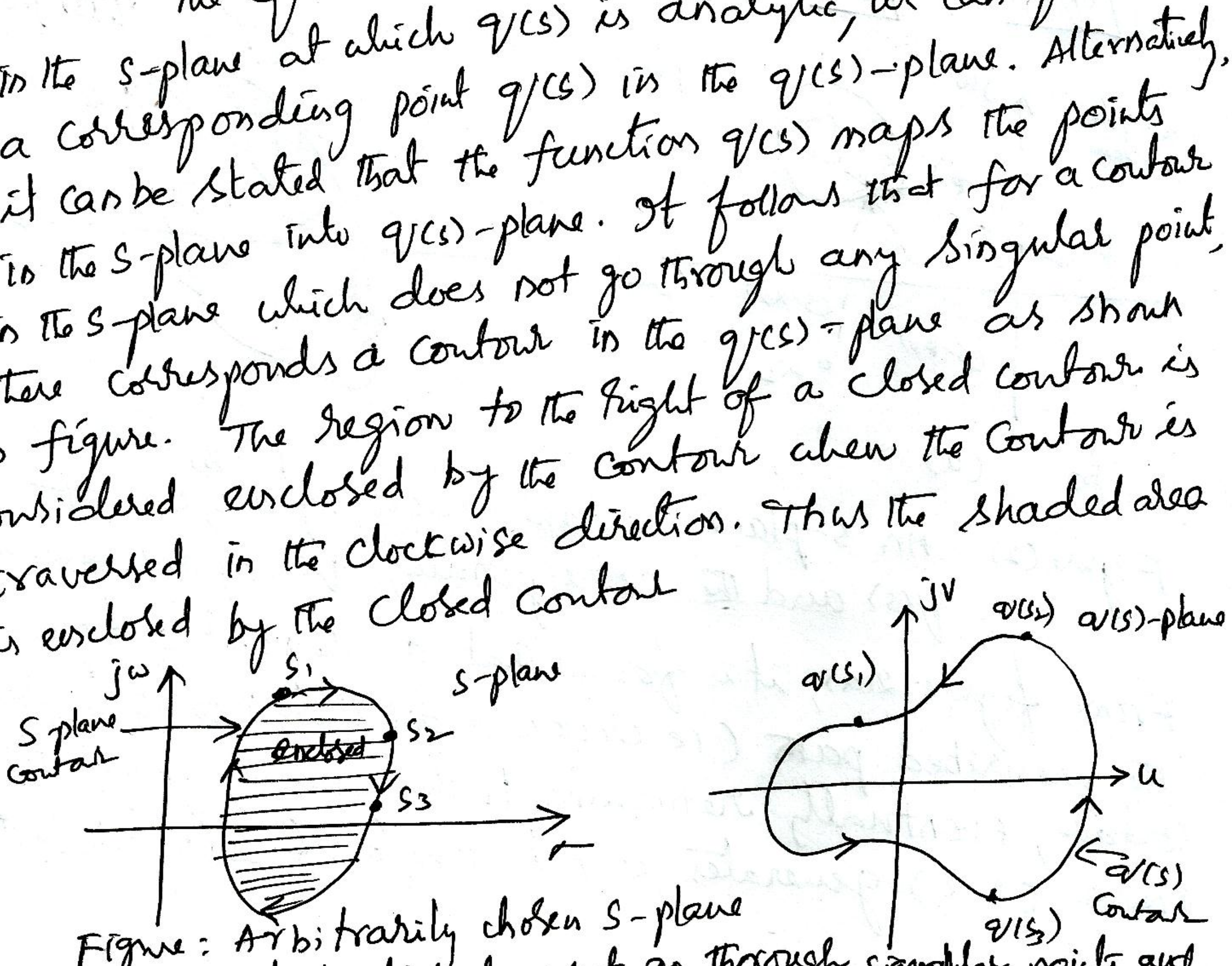


Figure: Arbitrarily chosen s-plane contour which does not go through singular points and the corresponding $q(s)$ plane contour.

we are not interested in the exact shape of the $q(s)$-plane contour. An important fact that concerns is the encirclement of the origin by the $q(s)$-plane contour. To investigate this, consider an s-plane contour which encloses only one of the zeros of $q(s)$, say $s=\alpha_1$, while all the poles and remaining zeros are distributed in the s-plane outside the contour. For any non-singular point 's' on the s-plane contour, there corresponds a point $q(s)$ on the $q(s)$-plane contour, From eq ①, the point $q(s)$ is given by

$$|q(s)| = \frac{|s-\alpha_1|\,|s-\alpha_2|\,---}{|s-\beta_1|\,|s-\beta_2|\,---} \longrightarrow ②$$

$$\angle q(s) = \angle s-\alpha_1 + \angle s-\alpha_2 + -- - \angle s-\beta_1 - \angle s-\beta_2 ---$$

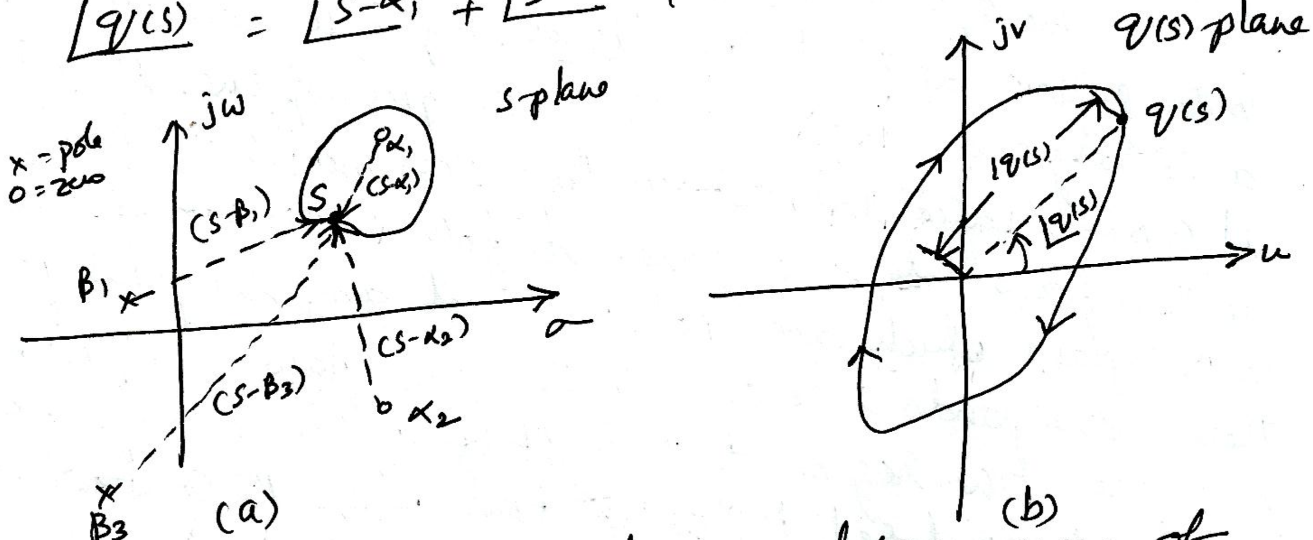


Figure (2): An s-plane contour enclosing a zero of $q(s)$ and the corresponding $q(s)$-plane contour.

From figure 2(a), it is found that as the point s follows the prescribed path (ie clockwise direction) on the s-plane contour, eventually returning to the starting point, the phasor $(s-\alpha_1)$ generates a net angle of -2π, while the

other phasors generate zero net angles. Therefore,
the $q(s)$-phasor undergoes a net phase change of -2π.
This implies that the tip of the $q(s)$-phasor must describe
a closed contour about the origin of the $q(s)$-plane
in the clockwise direction as shown in figure 2(a).

The exact shape of the closed contour in
the $q(s)$-plane is not interest to us, but it is sufficient
for us to observe that this contour encircles the origin
once. If the contour in the s-plane is so chosen that it
does not enclose any zero or pole, the corresponding
contour in $q(s)$-plane then will not encircle the origin.

If the s-plane contour encloses two zeros,
say at $s=\alpha_1$ and $s=\alpha_2$, the $q(s)$-plane contour encircles
the origin twice in the clock wise direction as shown in
figure (3). Generalizing, we can say that for each
zero of $q(s)$ enclosed by the s-plane contour, the
corresponding $q(s)$-plane contour encircles the
origin once in the clockwise direction.

If the s-plane contour encloses a pole at
$s=\beta_1$, then the phasor $(s-\beta_1)$ generates an angle of -2π as
's' traverses the prescribed path. Since $(s-\beta_1)$ is in the denomin-
-ator, the $q(s)$-plane contour experiences an angle change
of $+2\pi$, which means one counter-clockwise encirclement
of the origin.

Thus, if there are 'p' poles and 'z' zeros of
$q(s)$ enclosed by the s-plane contour, then the corresponding
$q(s)$-plane contour must encircle the origin z-times

in the clockwise direction and 'p' times in the counter clockwise direction, resulting in a net encirclement of the origin, (p-z) times in the counter clockwise direction.

For example, in case of 1 zero and 3 poles enclosed by the s-plane contour, the net encirclement of the origin by by the $q(s)$-plane contour is $2\pi(3-1)$ $= 4\pi$ rad, ie, two counterclock wise revolutions as shown below.

This relation between the enclosure of poles and zeros of $q(s)$ by the s-plane contour and the encircle-ments of the origin by the $q(s)$-plane contour is commonly known as the principle of argument.

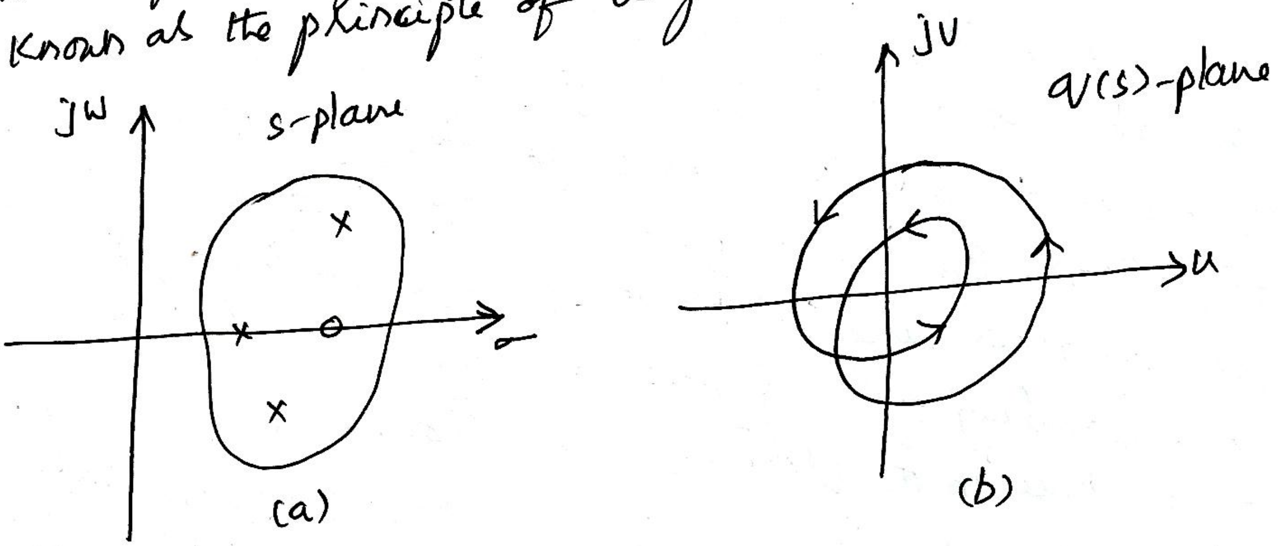


Figure: Mapping of the s-plane contour which encloses 1 zero and 3 poles

Nyquist stability criterion: Consider a feedback system shown in figure:

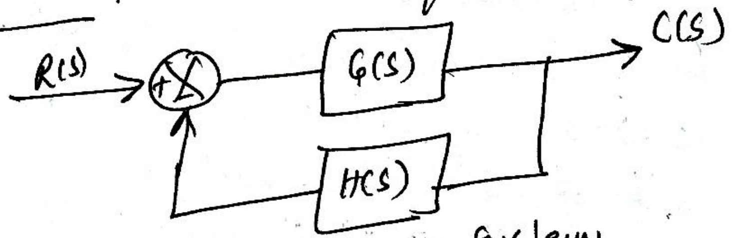


Figure: Feedback system

The characteristic equation of the system is

$$q(s) = 1 + G(s)H(s) = 0.$$

The pole-zero form of the open-loop transfer function is

$$G(s)H(s) = K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)};\quad m \leq n \rightarrow (1)$$

$$\therefore q(s) = 1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \longrightarrow (2)$$

$$= \frac{(s+p_1)(s+p_2)\cdots(s+p_n) + K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= \frac{(s+z_1')(s+z_2')\cdots(s+z_n')}{(s+p_1)(s+p_2)\cdots(s+p_n)} \longrightarrow (3)$$

From the above equation it is seen that the zeros of $q(s)$ at $s = -z_1', -z_2', \cdots -z_n'$ are the roots of the characteristic equation and the poles of $q(s)$ at $-p_1, -p_2 \cdots -p_n$ are same as the open-loop poles of the system.

For the system to be stable, the roots of the characteristic equation and hence the zeros of $q(s)$ must lie in the left half of the s-plane.

It is important to note that even if some of the open-loop poles lie in the right-half s-plane, all the zeros of $q(s)$, ie, the closed loop poles may lie in the left half s-plane. That is even an open-loop unstable system may lead to a closed-loop stable operation.

In order to investigate the presence of any zeros of $q(s) = 1 + G(s)H(s)$ in the right half s-plane, let us choose a contour which completely encloses right half of the s-plane. Such a contour 'C' is called Nyquist contour is shown in figure. The Nyquist contour is directed clockwise and comprises of an infinite line segment C_1 along the $j\omega$-axis and an arc C_2 of infinite radius.

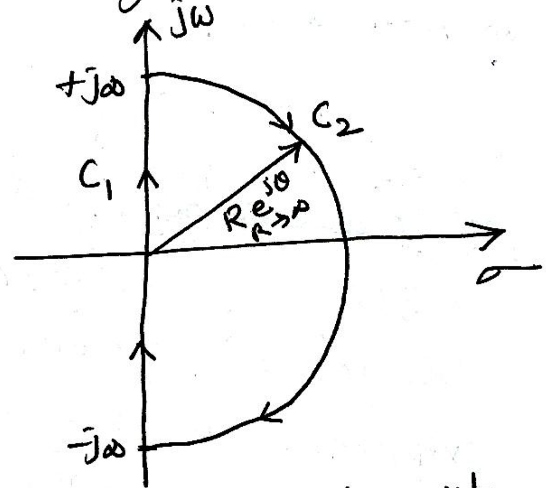


Figure : The Nyquist Contour.

Along C_1,

$s = j\omega$ with s varying from $-j\infty$ to $+j\infty$

Along C_2, $s = \underset{R\to\infty}{Re^{j\theta}}$ with 'θ' varying from $+\frac{\pi}{2}$ to 0 to $-\frac{\pi}{2}$

The Nyquist contour so defined encloses all the right half s-plane zeros and poles of $q(s) = 1 + G(s)H(s)$. Let there are Z zeros and P poles of $q(s)$ in the right half s-plane. As 's' moves along the Nyquist contour in the s-plane, a closed contour Γ_q traversed in the $q(s)$-plane which encloses the origin $N = P - Z$ times in the counter clockwise direction

For the system to be stable, there should be no zeros of $q(s) = 1 + G(s)H(s)$ in the right half s-plane, ie $Z = 0$. This condition is met if $N = P$. That is, for a system (closed-loop) to be stable, the number of counter clockwise encirclements of the origin of the $q(s)$-plane by the contour Γ_q should equal the number of the right half s-plane poles of $q(s)$ which are the poles of open-loop transfer function $G(s)H(s)$.

In the special case of $P=0$, the closed loop system is stable if $N=P=0$

It is easily observed that $G(s)H(s) = [1+G(s)H(s)]-1$

Therefore, it follows that the contour Γ_{GH} of $G(s)H(s)$ corresponding to the Nyquist contour in the s-plane is the same as contour Γ_q of $1+G(s)H(s)$ drawn from the point $(-1+j0)$. Thus the encirclement of the origin by the contour Γ_q is equivalent to the encirclement of the point $(-1+j0)$ by the contour Γ_{GH} as shown in figure.

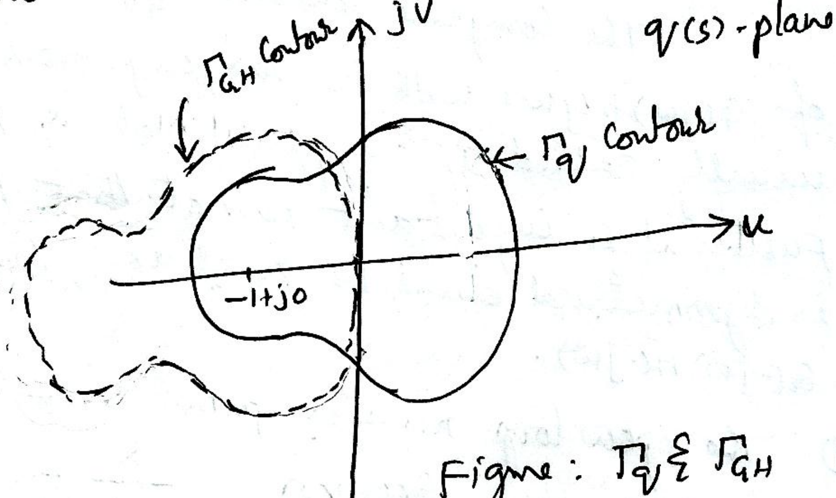


Figure: Γ_q & Γ_{GH} Contours.

The Nyquist stability criterion, now can be stated as;

If the Contour Γ_{GH} of the open-loop transfer function $G(s)H(s)$ corresponding to the Nyquist contour in the s-plane encircles the point $(-1+j0)$ in the Counter clock wise direction as many times as the number of right half s-plane poles of $G(s)H(s)$, the closed loop system is stable.

If the open loop system is stable, then the corresponding closed loop system is stable, if the contour Γ_{GH} of $G(s)H(s)$ does not encircle $(-1+j0)$ point, ie, the net encirclement is zero.

The mapping of the Nyquist contour into the contour Γ_{GH} is carried out as follows:

(1) The mapping of the imaginary axis is carried out by substitution of $s = j\omega$ in $G(s)H(s)$. This converts the mapping function into a frequency function of $G(j\omega)H(j\omega)$

(2) In physical systems $(m \leq n)$, $\lim_{\substack{s = Re^{j\theta} \\ R \to \infty}} G(s)H(s) =$ real constant (it is zero if $m < n$). Thus the infinite arc of the Nyquist contour maps into a point on the real axis.

The complete contour Γ_{GH} is thus the polar plot of $G(j\omega)H(j\omega)$ with 'ω' varying from $-\infty$ to ∞. This is usually called the Nyquist plot or locus of $G(s)H(s)$. Further it is important to note that the Nyquist plot is symmetrical about the real axis, since $G^*(j\omega)H^*(j\omega) = G(-j\omega)H(-j\omega)$.

① The open loop transfer function of a system (feedback) is given by $G(s)H(s) = \dfrac{K}{(1+T_1 s)(1+T_2 s)}$. Sketch the Nyquist plot and comment on stability of closed loop system.

(Sol)

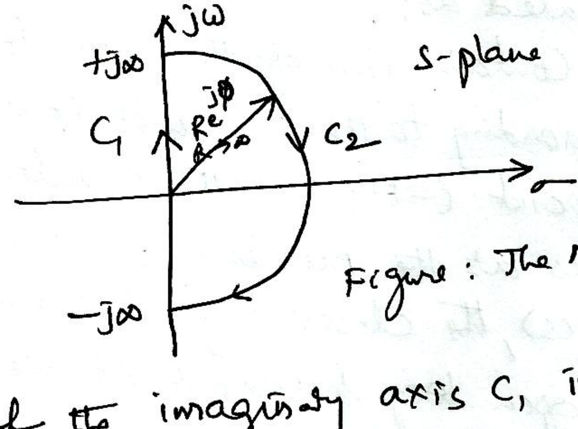


Figure: The Nyquist contour

(1) The mapping of the imaginary axis C_1 into $G(s)$ plane is carried by substituting $s = j\omega$ in $G(s)H(s)$

$$G(j\omega)H(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$$

$$M = |G(j\omega)H(j\omega)| = \frac{K}{\sqrt{1+\omega^2 T_1^2}\sqrt{1+\omega^2 T_2^2}}$$

$$\phi = \angle G(j\omega)H(j\omega) = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

$\omega = 0 \quad M = K; \quad \phi = 0$

$\omega \to \infty \quad M \to 0 \quad \phi \to -180^\circ$

(2) The maping of semicircular arc C_2 is carried out by replacing $s = \underset{R\to\infty}{Re^{j\phi}}$ $(\phi \to +90^\circ \to 0 \;\; 90^\circ)$

$$\therefore \underset{R\to\infty}{Lt} \frac{K}{(1+T_1 Re^{j\phi})(1+T_2 Re^{j\phi})} = 0\, e^{-j2\phi}$$

where semicircular arc is mapped into a point (origin) in $G(s)$-plane

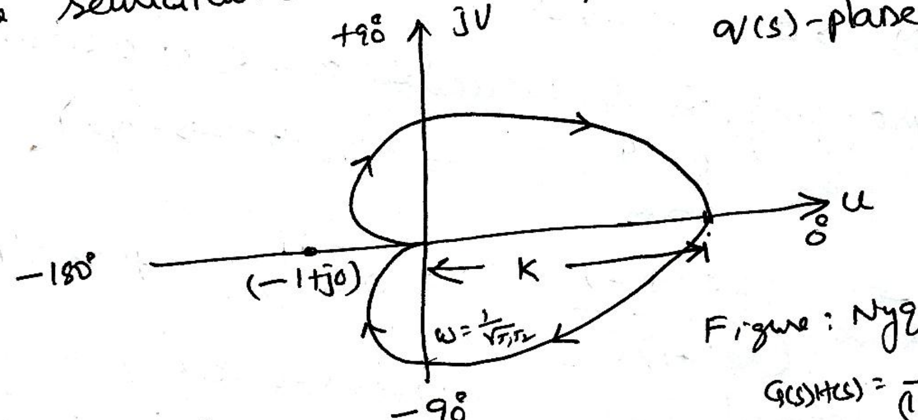


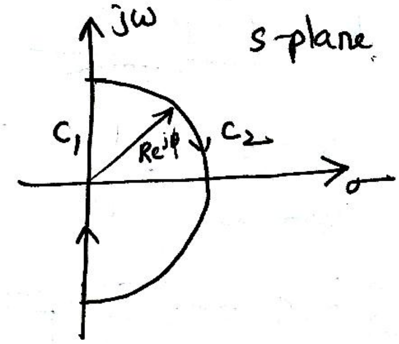
Figure : Nyquist plot of $G(s)H(s) = \frac{K}{(1+sT_1)(1+sT_2)}$

It is seen that the plot of $G(s)H(s)$ does not encircle the point $(-1+j0)$ for any positive values of K, T_1 and T_2. Therefore, the system is stable for all the values of K, T_1 and T_2.

(2) The open loop transfer function of a unity feedback system is given by $G(s)H(s) = \frac{(s+2)}{(s+1)(s-1)}$. Draw the Nyquist plot and determine stability of closed loop system.

(Sol)

$$G(j\omega)H(j\omega) = \frac{(j\omega+2)}{(j\omega+1)(j\omega-1)}$$



$$\text{Magnitude } M = \frac{\sqrt{4+\omega^2}}{\sqrt{1+\omega^2}\sqrt{1+\omega^2}}$$

$$\phi = \tan^{-1}(\omega/2) - \tan^{-1}(\omega) - \tan^{-1}(-\omega)$$

$$= \tan^{-1}(\omega/2) - \cancel{\tan^{-1}(\omega)} - [\pi - \cancel{\tan^{-1}(\omega)}]$$

$$= -\pi + \tan^{-1}(\omega/2)$$

$\omega = 0 \quad M = \sqrt{4} = 2; \quad \phi = -\pi$ or -180°

$\omega \to \infty \quad M \to 0 \quad \phi \to -90^\circ$

(2) The mapping of semicircular arc C_2 is carried out by replacing s by $\underset{R\to\infty}{Re^{j\phi}}$ $(\phi \to +\pi/2 \to 0 \to -\pi/2)$

$$\therefore \lim_{R\to\infty} \frac{(Re^{j\phi}+2)}{(Re^{j\phi}+1)(Re^{j\phi}-1)} = 0e^{-j\phi}; \quad -\phi \to -\pi/2 \to 0 \to \pi/2$$

Thus, the segment C_2 is mapped into origin in $q(s)$-plane

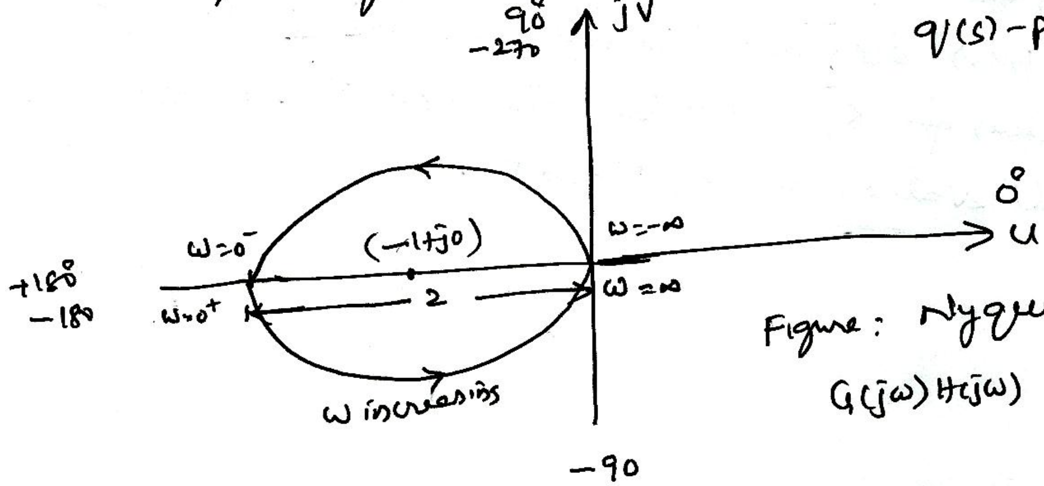


Figure: Nyquist plot of $G(j\omega)H(j\omega) = \frac{(j\omega+2)}{(j\omega+1)(j\omega-1)}$

The contour encircles $(-1+j0)$ point one time in counter clock wise direction $\therefore N = 1$; No of right side open loop poles $P = -1$

$\therefore N = P - Z \Rightarrow Z = P - N = 1 - 1 = 0$. No zeros of $G(s)H(s)$ lies on RHS. Hence the system is stable.

The Nyquist plot encircles the $(-1+j0)$ point one time in counterclockwise direction. Therefore $N = 1$

The number of RHS open loop poles $P = 1$

$\therefore$ The number of zeros of $G(s)H(s)$ on RHS $= Z$

where $N = P - Z$

$\therefore Z = P - N = 1 - 1 = 0$

None of the zeros of $G(s)H(s)$ lie on RHS, Therefore the closed loop system is stable.

Open loop poles on the $j\omega$ axis: If $G(s)H(s)$ and therefore $1+G(s)H(s)$ has any poles on the $j\omega$-axis, the Nyquist contour should not pass through those poles. To study stability in this case, the Nyquist contour must be modified so as to bypass any $j\omega$-axis poles. This is accomplished by indenting the Nyquist contour around the $j\omega$-axis poles along a semicircular of radius ϵ where $\epsilon \to 0$ as shown in figure.

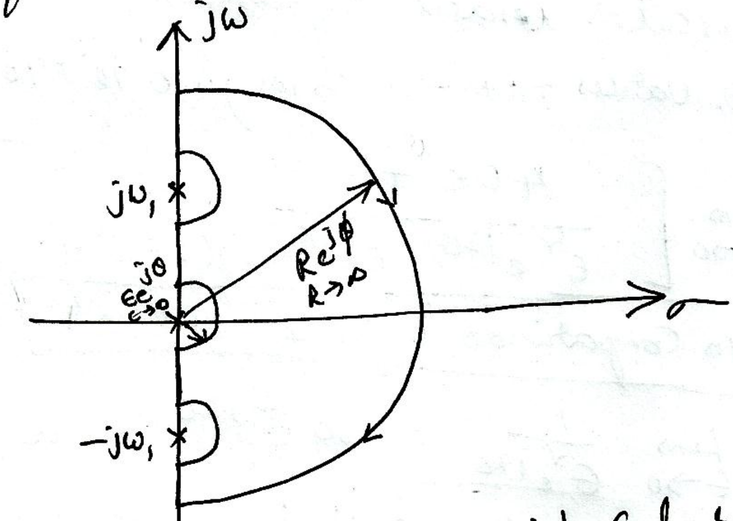


Figure: Indented Nyquist Contour for $j\omega$-axis open loop poles.

① Consider a system with open loop transfer function $G(s)H(s) = \frac{(4s+1)}{s^2(s+1)(2s+1)}$. Determine stability of the system from Nyquist stability criterion

(Sol)

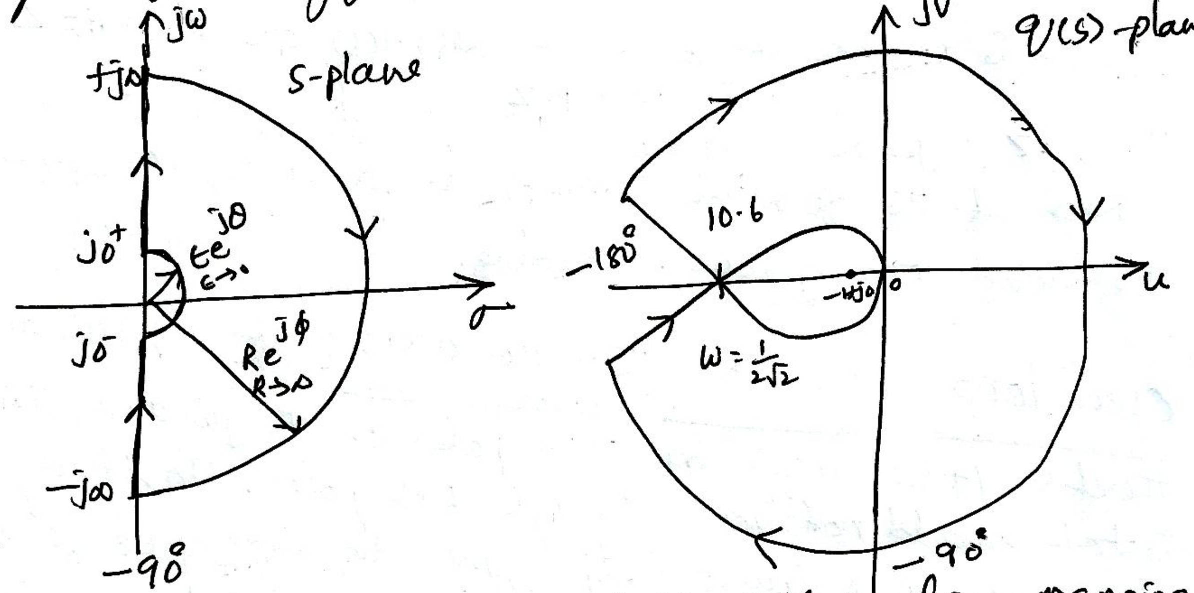


Figure: Nyquist Contour and the corresponding mapping for $G(s)H(s) = (4s+1)/s^2(s+1)(2s+1)$

(1) Semicircular indent represented by $s = \lim_{\epsilon \to 0} \epsilon e^{j\theta}$ (where θ varies from −90° through 0 to +90°) is mapped into

$$\lim_{\epsilon \to 0} \left[\frac{4\epsilon e^{j\theta}+1}{\epsilon^2 e^{j2\theta}(\epsilon e^{j\theta}+1)(2\epsilon e^{j\theta}+1)} \right] = \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon^2 e^{j2\theta}} \right)$$

$\boxed{\text{Note : in Comparison with } 1 \quad 1+\epsilon \simeq 1}$

$$\therefore \lim_{\epsilon \to 0} \frac{1}{\epsilon^2 e^{j2\theta}} = \infty e^{-j2\theta} = \infty(180° \to 0 \to -180°)$$

Thus the semicircular indent is mapped into an infinite circle in q(s)-plane

(2) Along the jω axis $G(j\omega)H(j\omega) = \frac{1+j4\omega}{(j\omega)^2(1+j\omega)(1+j2\omega)}$

$$M = \frac{\sqrt{1+(4\omega)^2}}{\omega^2\sqrt{1+\omega^2}\sqrt{1+(2\omega)^2}}$$

phase angle $\phi = \tan^{-1}(4\omega) - 180^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$ (28)

at $\omega = 0$, Magnitude $M = \infty$ $\phi = -180^\circ$

$\omega = \infty$ $M = 0$; $\phi = 90 - 180 - 90 - 90 = -270^\circ$

(3) The infinite semi circular arc represented by $S = \lim_{R\to\infty} Re^{j\phi}$ (ϕ varies from $+90^\circ$ through 0 to -90°) is mapped into

$$= \lim_{R\to\infty} \frac{(1+4Re^{j\phi})}{R^2 e^{j2\phi}(1+Re^{j\phi})(1+Re^{j\phi})} = 0e^{-j3\phi}$$

$$= 0(-270^\circ \to 0 \to 270^\circ)$$

Thus the infinite semicircular arc is mapped into a point in $G(s)$ plane.

The $G(j\omega)H(j\omega)$ locus intersects the real axis at a point where $\angle G(j\omega)H(j\omega) = -180^\circ$

$$+\tan^{-1}(4\omega) - 180 - \tan^{-1}(\omega) - \tan^{-1}(2\omega) = -180^\circ$$

$$\Rightarrow \tan^{-1}(4\omega) = \tan^{-1}(\omega) + \tan^{-1}(2\omega)$$

Taking tan on both sides

$$\tan(\tan^{-1}4\omega) = \tan(\tan^{-1}\omega + \tan^{-1}2\omega)$$

$$4\omega = \frac{\omega + 2\omega}{1-2\omega^2} = \frac{3\omega}{1-2\omega^2}$$

$$\Rightarrow 4\cancel{\omega}(1-2\omega^2) = 3\cancel{\omega}$$

$$1-2\omega^2 = 3/4 \Rightarrow 2\omega^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\therefore \omega^2 = \frac{1}{8} \text{ and } \omega = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$$

$\therefore$ The magnitude at $\omega = \frac{1}{2\sqrt{2}}$ is

$$|G(j\omega)H(j\omega)|_{\omega=\frac{1}{2\sqrt{2}}} = \left.\frac{\sqrt{1+(4\omega)^2}}{\omega^2\sqrt{1+\omega^2}\sqrt{1+(2\omega)^2}}\right|_{\omega=\frac{1}{2\sqrt{2}}} = 10.6$$

The mapped contour encircles (-1+j0) point '2' times in clock wise direction $\therefore N = -2$ and $P = 0$ $\therefore Z = 0-(-2) = 2$

Therefore, the system is unstable.

(1) Consider a unity feedback system with open loop transfer function $G(s) = \dfrac{1}{s(1+0.2s)(1+0.05s)}$. Sketch the polar plot and determine GM & PM.

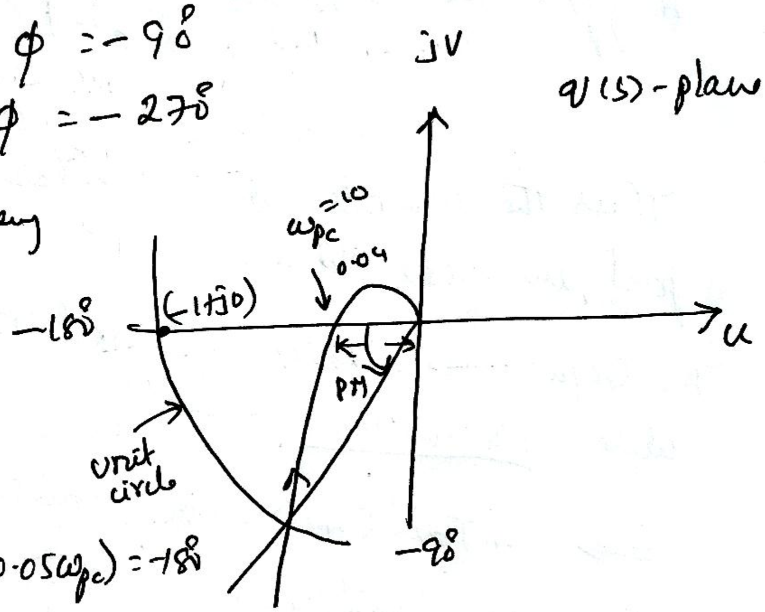
(Sol) $G(j\omega)H(j\omega) = \dfrac{1}{j\omega(1+0.2j\omega)(1+0.05j\omega)}$

$\therefore$ Magnitude $|G(j\omega)H(j\omega)| = M = \dfrac{1}{\omega\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}}$

phase angle $\phi = -90 - \tan^{-1}(0.2\omega) - \tan^{-1}(0.05\omega)$

$\omega = 0 \quad M = \infty \quad \phi = -90^\circ$

$\omega = \infty \quad M = 0 \quad \phi = -270^\circ$



The phase cross over frequency can be determined as follows

$$\angle G(j\omega)H(j\omega)\Big|_{\omega=\omega_{pc}} = -180^\circ$$

$$\Rightarrow -90 - \tan^{-1}(0.2\omega_{pc}) - \tan^{-1}(0.05\omega_{pc}) = -180^\circ$$

$$\Rightarrow \tan^{-1}(0.2\omega_{pc}) = 90 - \tan^{-1}(0.05\omega_{pc})$$

Taking Tan on both sides

$$\tan(\tan^{-1} 0.2\omega_{pc}) = \tan(90 - \tan^{-1} 0.05\omega_{pc})$$

$$0.2\omega_{pc} = \cot\tan^{-1}(0.05\omega_{pc}) = \frac{1}{0.05\omega_{pc}}$$

$$\omega_{pc}^2 = \frac{1}{0.2\times 0.05}$$

$$\Rightarrow \omega_{pc} = 10 \text{ rad/sec}$$

$\therefore$ Gain Margin $= 20\log\left[\dfrac{1}{|G(j\omega)H(j\omega)|}\right]_{\omega=\omega_{pc}} = 20\log\left(\dfrac{1}{0.04}\right)$

$= 28\,dB$

To find phase margin draw a circle with radius '1' and origin as a centre, then identify the intersection of polar plot and circular arc $\therefore$ phase Margin $= 76^\circ$

Compensation Techniques: If the performance of a control system is not upto expectations as per desired specifications, then it is required that some change in the system is needed to obtain the desired performance. The change can be in the form of adjustment of forward path gain or inserting a compensating device in control systems.

For example, the steady state error in a control system can be reduced by increasing forward path gain, but on the other hand this increase in forward path gain results in making the system more oscillatory or sometimes unstable.

Thus the gain adjustment improves the steady state accuracy of the system at the cost of driving the system towards instability. In such cases a compensation network is introduced in the system. The compensation network can be introduced in forward path as shown in figure.

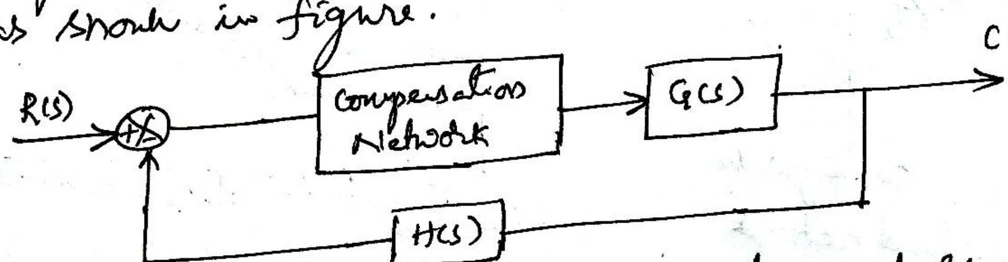


Figure: System with Compensation Network.

There are three types of Compensators

(1) phase lead Compensator

(2) phase lag compensator

(3) Lead-lag Compensator

① phase-lead Compensator : For phase-lead network the output leads the input. Let us consider a phase lead network shown in figure (1)

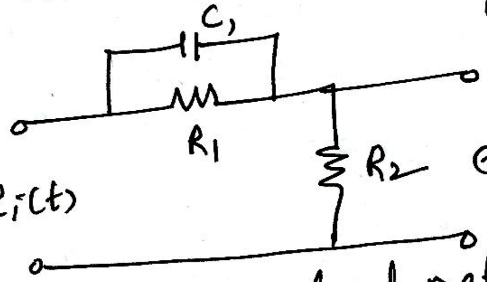


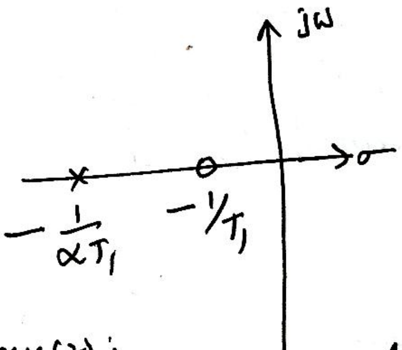
Figure : phase-lead network

The transfer function of phase lead network is given by

$$\frac{E_0(s)}{E_i(s)} = G(s) = \frac{\alpha(1+sT_1)}{(1+\alpha sT_1)}$$

where $\alpha = \frac{R_2}{R_1+R_2} < 1$ and $T_1 = R_1C_1$

The Sinusoidal transfer function $G(j\omega) = \frac{\alpha(1+j\omega T_1)}{(1+j\omega\alpha T)}$



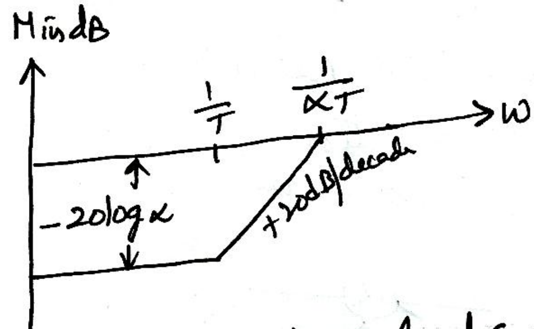


Figure (2) : pole-zero Configuration of phase-lead network

Fig : Bode plot of phase-lead Compensator

The phase-lead network acts as a high pass filter. Thus it attenuates low frequencies and allows high frequencies. The phase-lead compensator increases the phase shift of the system. The phase-lead compensator shifts the gain cross over frequency to a higher value and therefore increases bandwidth, speed of the response and reduces overshoot but the steady state error does not show much improvement.

- **Phase-lag Compensator**: For phase-lag network, the output lags the input. The phase-lag network is shown in figure (1)

The transfer function of phase-lag network is given by

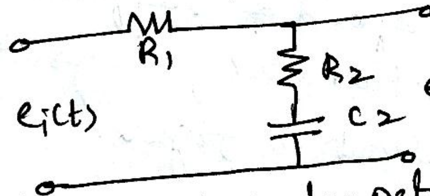


Figure : phase-lag network

$$\frac{E_o(s)}{E_i(s)} = \frac{1+sT_2}{1+s\beta T_2}$$

where $\beta = \frac{R_1+R_2}{R_2} > 1$; Time Constant $T_2 = R_2C_2$

The sinusoidal transfer function is given by

$$\frac{E_o(j\omega)}{E_i(j\omega)} = G(j\omega) = \frac{1+j\omega T_2}{1+j\omega\beta T_2}$$

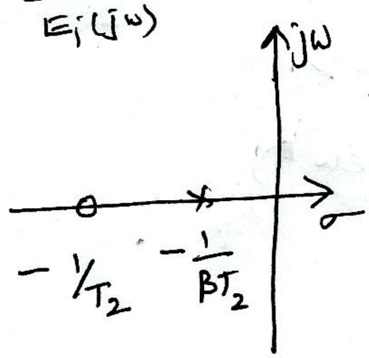


Figure : pole-zero Configuration of phase-lag Compensator

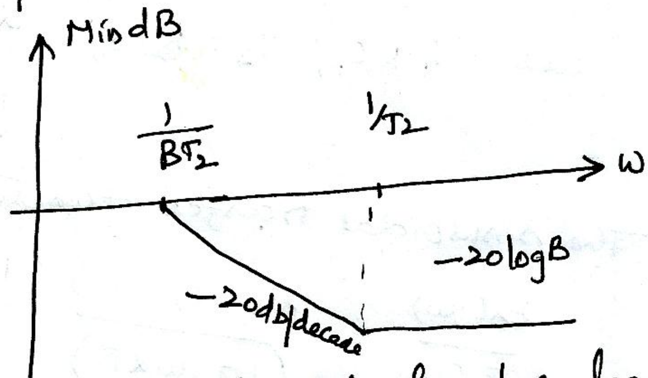


Figure : Bode plot of phase-lag Compensator

When the phase-lag network is introduced in cascade with forward transfer function, the phase-shift will be reduced. The phase-lag compensator shifts the gain cross over frequency to lower value and thus decreases bandwidth and speed but improves the steady state error. The phase-lag compensator acts as a lowpass filter and thus allows low frequency signals and attenuates high frequency signals.

(3) **Lead-lag Compensator**: If phase-lead and phase-lag compensators are simultaneously used, then the speed of response and steady state error are simultaneously improved. The phase lead-lag network is shown in figure.

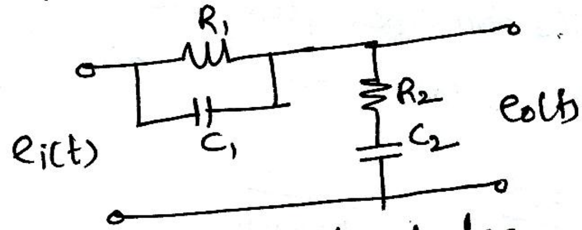


Figure: phase lead-lag network

The transfer function of lead-lag network is given by

$$\frac{E_o(s)}{E_i(s)} = \frac{\alpha(1+sT_1)}{(1+s\alpha T_1)}\,\frac{(1+sT_2)}{(1+s\beta T_2)}$$

where $T_1 = R_1C_1$; $T_2 = R_2C_2$; $\alpha = \frac{R_2}{R_1+R_2} < 1$

$$\beta = \frac{R_1+R_2}{R_2} > 1$$

The sinusoidal transfer function is given by

$$\frac{E_o(j\omega)}{E_i(j\omega)} = \frac{\alpha(1+j\omega T_1)}{(1+j\omega\alpha T_1)}\,\frac{(1+j\omega T_2)}{(1+j\omega\beta T_2)}$$

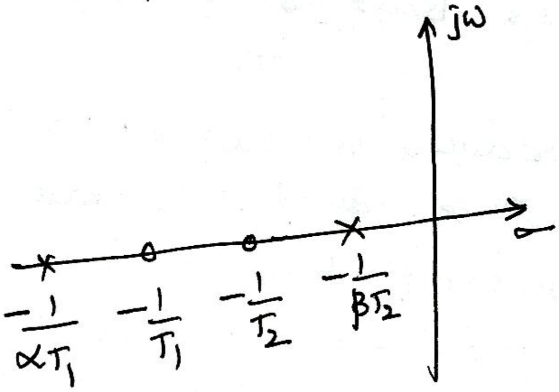


Figure: pole-zero pattern of Lead-lag Compensator

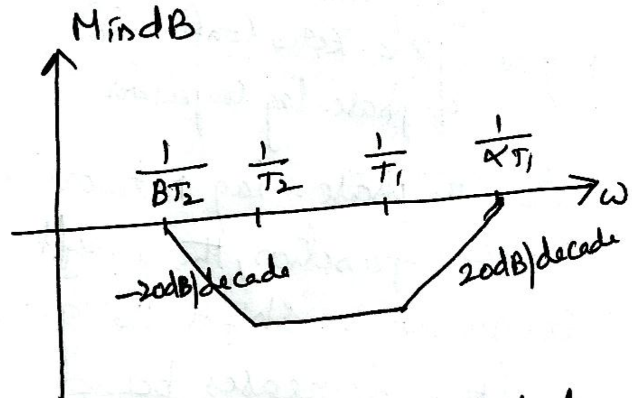


Figure: Magnitude plot of Lead-lag Compensator.

UNIT – V

STATE SPACE ANALYSIS

The root locus and frequency response methods require the physical system in the form of a transfer function. Even though, the transfer function model provides us with simple and powerful analysis and design techniques, it suffers from certain drawbacks such as

(1) The transfer function is only defined under zero initial conditions

(2) The transfer function model is applicable to linear time-invariant systems

(3) The transfer function model is restricted to single-input-single output systems.

(4) The transfer function does not provide any information regarding internal state of the system.

(5) The classical design methods such as root locus and frequency domain methods are essentially trial and error procedures.

To overcome all these drawbacks, the state variable approach is introduced. It is a direct time-domain approach which provides a basis for modern control theory and system optimization. It is a very powerful technique for the analysis and design of linear and non-linear, time-invariant or time-varying multi-input-multi-output systems. The organization of the state variable approach is such that it is easily amenable to solution through digital computers.

Concepts of State, State Variables & State Model

The state of a dynamical System is a minimal set of variables (known as state variables) Such that the knowledge of these variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$ completely determines the behaviour of the System for $t > 0$.

In state variable formation of a system, the state variables are usually represented by $x_1(t)$, $x_2(t)$ ---; the inputs by $u_1(t)$, $u_2(t)$ ---; and the outputs by $y_1(t)$, $y_2(t)$ ---- Let us assume that there are 'm' inputs, 'p' outputs and 'n' State variables.

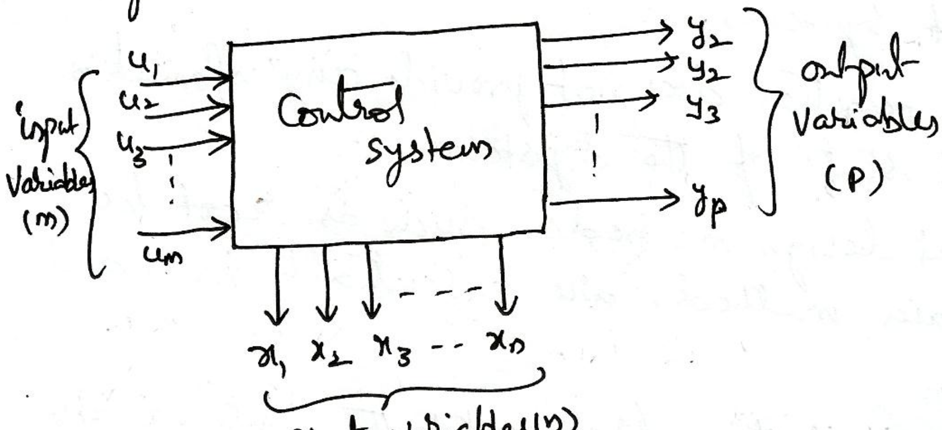


Figure: State Model of a Control System

The input, output and State variables in matrix form are represented as

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}_{m\times 1} ; \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}_{p\times 1} ; \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

For a linear system, the state model is given by

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2m}u_m$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nm}u_m$$

Thus for linear system, the derivative of each state variable is a linear combination of system states and inputs. where a_{ij} and b_{ij} are constants. In vector form, the state equations can be represented as

$$\dot{x}(t) = Ax(t) + Bu(t) \longrightarrow (1)$$

where $x(t)$ is $n\times 1$ state vector, $u(t)$ is $m\times 1$ input vector

A is $n\times n$ system matrix defined by

B is $n\times m$ output matrix defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n} ; \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}_{n\times m}$$

Similarly, the output variables at time 't' are linear combinations of the values of the input and state variables at time 't', ie

$$y_1(t) = c_{11}x_1(t) + c_{12}x_2(t) \cdots + c_{1n}x_n(t) + d_{11}u_1(t) + d_{12}u_2(t) \cdots + d_{1m}u_m(t)$$

$$y_2(t) = c_{21}x_1(t) + c_{22}x_2(t) + \cdots + c_{2n}x_n(t) + d_{21}u_1(t) + d_{22}u_2(t) + \cdots + d_{2m}u_m(t)$$

$$\vdots$$

$$y_p(t) = c_{p1}x_1(t) + c_{p2}x_2(t) \cdots + c_{pn}x_n(t) + d_{p1}u_1(t) + \cdots + d_{pm}u_m(t)$$

where the coefficients c_{ij} and d_{ij} are constants. This set of equations may be put in the vector matrix form as

$$y(t) = Cx(t) + Du(t) \longrightarrow (2)$$

where $y(t)$ is $p\times 1$ output vector, C is $p\times n$ output matrix defined by

$\vec{D}$ is $p\times m$ transmission matrix defined by

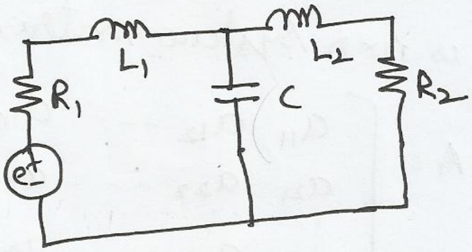
$$\vec{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}_{p\times n} \qquad \vec{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & & & \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}_{p\times m}$$

Thus, the state model of a linear time invariant system is given by

$$\dot{x}(t) = Ax(t) + B\,u(t) \text{ ; State Equation}$$
$$y(t) = C\,x(t) + D\,u(t) \text{ ; output Equation.}$$

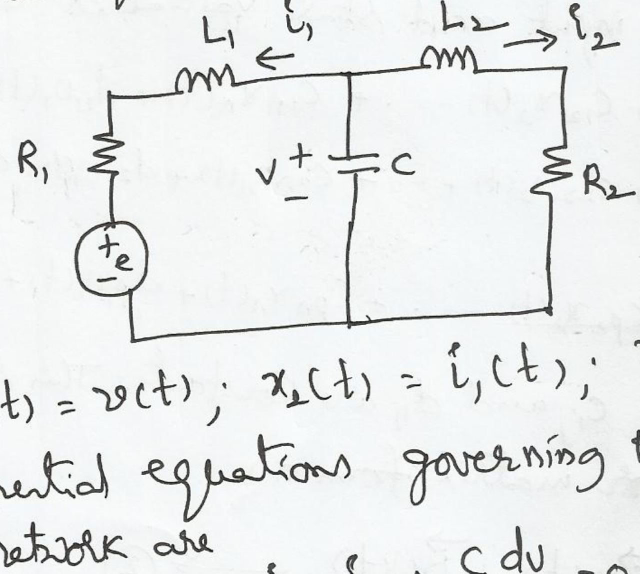
① <u>State space</u> <u>Representation</u> <u>Using</u> <u>physical variables</u>:

① Obtain the state model of an electrical network shown in figure.



(Sol) Let the current i_1 in inductor L_1, current i_2 in inductor L_2 and voltage drop across capacitor 'C' are the state variables

<u>Note</u>: Number of state variables = Number of storage elements



Let $x_1(t) = v(t)$; $x_2(t) = i_1(t)$; $x_3(t) = i_2(t)$

The differential equations governing the behaviour of the RLC network are

$$i_1 + i_2 + C\frac{dv}{dt} = 0 \longrightarrow ①$$

$$L_1 \frac{di_1}{dt} + R_1 i_1 + e - v = 0 \longrightarrow ②$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 - v = 0 \longrightarrow ③$$

we are interested in expressing the variables $\frac{dv}{dt}$, $\frac{di_1}{dt}$ and $\frac{di_2}{dt}$ as linear combination of the variables v, i_1 and i_2 and e

From eq ①,
$$\frac{dv}{dt} = -\frac{1}{c} i_1 - \frac{1}{c} i_2 \longrightarrow \text{I}$$

$$\frac{di_1}{dt} = \frac{1}{L_1} v - \frac{R_1}{L_1} i_1 - \frac{1}{L_1} e \longrightarrow \text{II}$$

$$\frac{di_2}{dt} = \frac{1}{L_2} v - \frac{R_2}{L_2} i_2 \longrightarrow \text{III}$$

where input $u(t) = e(t)$; Now the state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{c} & -\frac{1}{c} \\ \frac{1}{L_1} & -R_1/L_1 & -1/L_1 \\ 1/L_2 & 0 & -R_2/L_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/L_1 \\ 0 \end{bmatrix} u$$

Assume that the voltage across R_2 and current through R_2 are the output variables y_1 and y_2 respectively

$$y_1 = V_2 = R_2 i_2 \; ; \quad y_2 = I = i_2$$

∴ The output equations can be represented as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & R_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

② obtain the state model of armature Controlled DC Motor

(Sol)

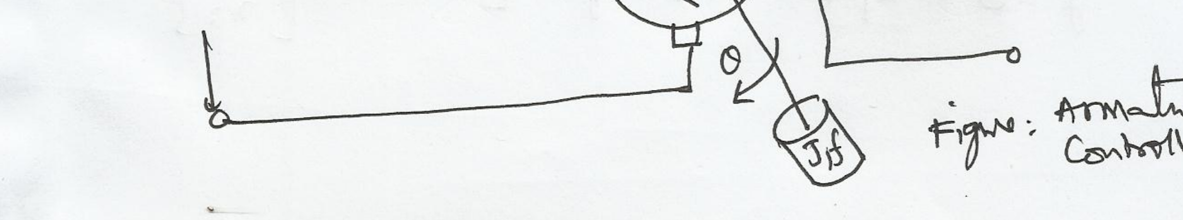


Figure: Armature Controlled DC Motor

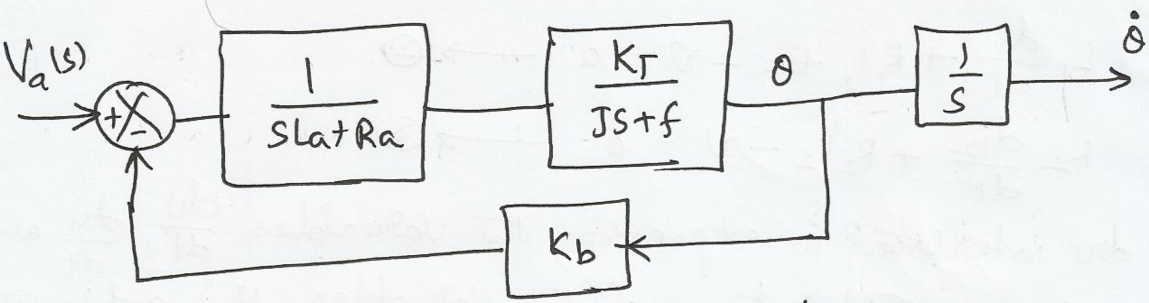


Figure : Block Diagram representation of Armature Controlled DC Motor

The state variables are $x_1 = \theta$; $x_2 = \dot{\theta} = \omega$ and $x_3 = i_a$

Now, we can write the following set of three first-order differential equations relating the inputs and outputs of the first-order factors $\frac{1}{s}$, $\frac{K_T}{JS+f}$ and $\frac{1}{Ra+sLa}$

$$\dot{x}_1 = x_2$$
$$J\dot{x}_2 + f x_2 = K_T x_3$$
$$V_a - K_b x_2 = R_a x_3 + L_a \dot{x}_3$$

$$\left[\begin{array}{l} \frac{d\theta}{dt} = \omega \\ J\frac{d\omega}{dt} + f\omega = K_T i_a \\ V_a - K_b\frac{d\theta}{dt} = R_a i_a + L_a\frac{d}{dt}i_a \end{array}\right]$$

These three first order differential equations can be represented in vector form as

$$\begin{bmatrix} \frac{d\theta}{dt} \\ \frac{d\omega}{dt} \\ \frac{d}{dt}i_a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -f/J & \frac{K_T}{J} \\ 0 & -\frac{K_b}{L_a} & -\frac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ i_a \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} V_a$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -f/J & K_T/J \\ 0 & -K_b/L_a & -R_a/L_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} V_a$$

If the motor angle is regarded as output, then

$$y = \theta = x_1 \qquad \therefore y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

② State space representation using phase variables:

The phase variable state model is easily determined if the system model is already known in differential equation or transfer function form.

The phase variables are defined as those particular state variables which are obtained from one of the system variables and its derivatives. Often the variable used is system output and the remaining state variables are then derivatives of the output.

Case (1): when the transfer function does not have zeros, such a transfer function has the form

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n} \rightarrow ①$$

The corresponding differential equation is

$$y^{(n)} + a y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu \rightarrow ②$$

where $y^{(n)} = \frac{d^n}{dt^n} y$; $\dot{y} = \frac{dy}{dt}$

By letting
$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$
$$\vdots$$
$$x_n = y^{(n-1)} \text{ ; then}$$

$$\dot{x}_1 = \dot{y} = x_2$$
$$\dot{x}_2 = \ddot{y} = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = y^n = -\left[a_n y + a_{n-1}\dot{y} + \cdots + a_1 y^{n-1}\right] + bu$$

$\therefore \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + bu$

The above equations result in the following state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u$$

or The output equation is $y = CX$

where $C = [1\ 0\ 0\ 0 \cdots 0]$

$$\dot{X} = AX + BU$$

It is to be observed that the matrix 'A' has very special form. It has all 1's in the upper off-diagonal, its last row is comprised of the negative of the coefficients of the original differential equation and all other elements are zero. This form of matrix 'A' is known as the Bush form or Companion form.

Also note that the matrix B has the speciality that all its elements except that the last are zero. In fact the matrices A and B can be written directly by inspection of the linear differential equation.

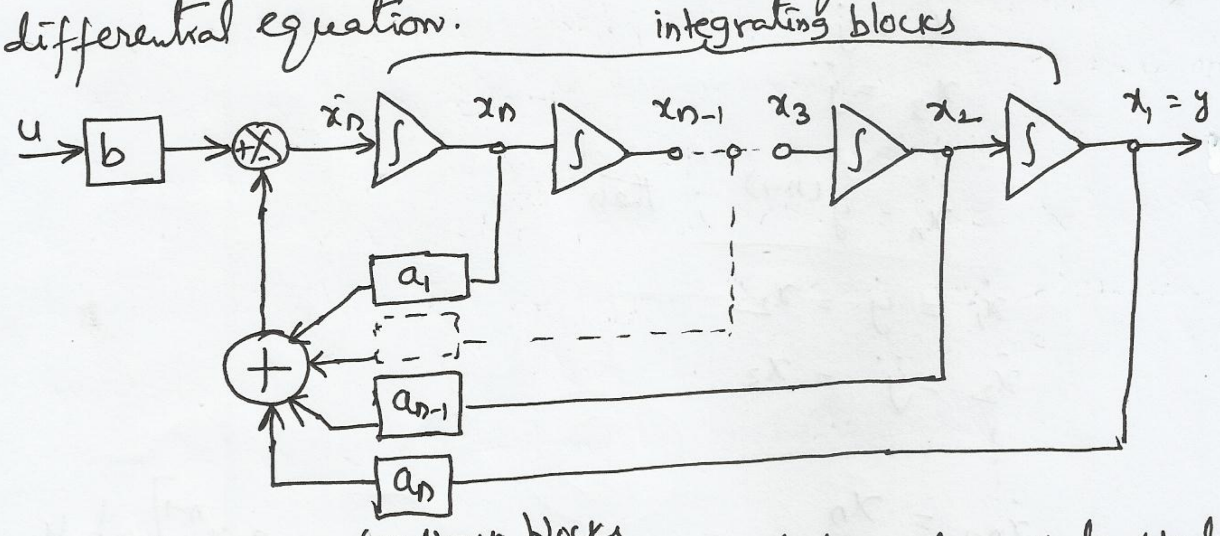


Figure : Block Diagram representation of the State Model

Case (2): phase variable formulation for transfer function with poles and zeros:

Let us consider a third order transfer function

$$\frac{Y(s)}{U(s)} = T(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \longrightarrow ①$$

For this third order transfer function let the state variables are x_1, x_2 and x_3. The above equation may be rearranged as

$$T(s) = \frac{b_0 + b_1/s + b_2/s^2 + b_3/s^3}{1 - (-a_1/s - a_2/s^2 - a_3/s^3)} \longrightarrow ②$$

We have the Mason's gain formula

$$T(s) = \frac{1}{\Delta} \sum_K P_K \Delta_K \longrightarrow ③$$

From equations ① & ③, we observe that a signal flow graph of equation ② may consists of

(i) three feedback loops (touching each other) with gains $-a_1/s$, $-a_2/s^2$ and a_3/s^3;

(ii) Four forward paths which touch the loops and have gains b_0, b_1/s, b_2/s^2 and b_3/s^3

A signal flow graph configuration which satisfies the above requirements is shown in figure.

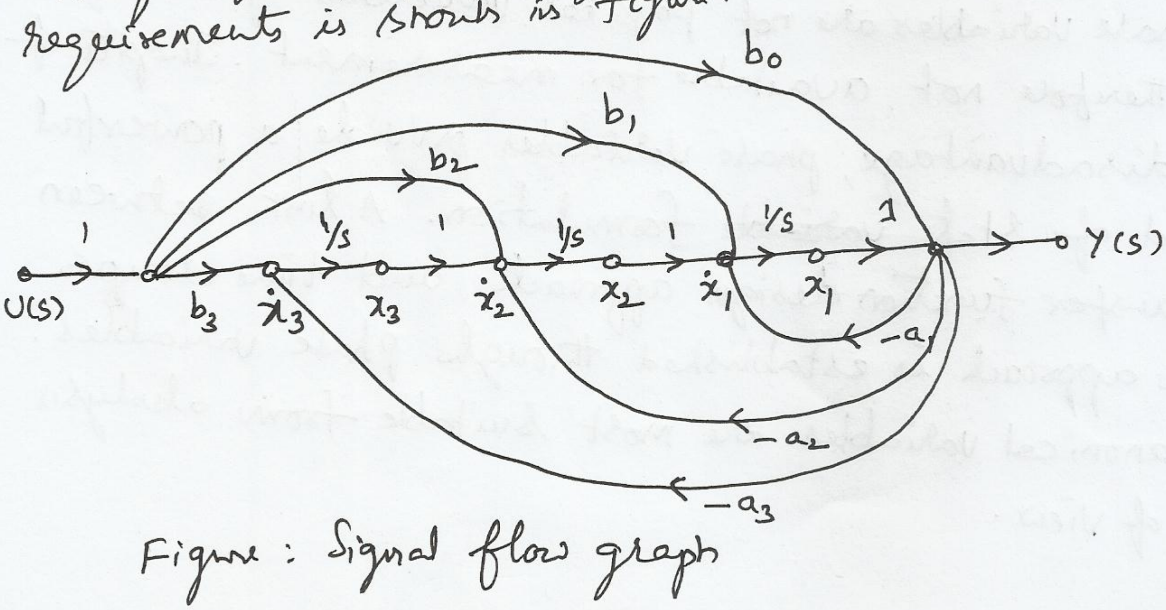


Figure: Signal flow graph

From the Signal flow graph

$$y = x_1 + b_0 u$$

$$\dot{x}_1 = -a_1(x_1 + b_0 u) + x_2 + b_1 u$$

$$= -a_1 x_1 + x_2 + (b_1 - a_1 b_0 u)$$

$$\dot{x}_2 = -a_2 y + x_3 + b_2 u$$

$$= -a_2(x_1 + b_0 u) + b_2 u + x_3 = -a_2 x_1 + x_3 + (b_2 - a_2 b_0)u$$

$$\dot{x}_3 = -a_3 y + b_3 u.$$

$$= -a_3(x_1 + b_0 u) + b_3 u$$

$$= -a_3 x_1 + (b_3 - a_3 b_0)u$$

The above equations can be represented in state model as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \end{bmatrix} u \quad \text{and}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u$$

The disadvantage of phase variable representation is that the phase variables are not physical variables of the system and therefore not available for measurement. Inspite of this disadvantage, phase variables provide a powerful method of state variable formulation. A link between the transfer function design approach and time domain design approach is established through phase variables. The canonical variables are most suitable from analysis point of view.

(3) State space representation using Canonical variables

In canonical-variable or normal-form representation of a system, the system matrix A turns out to be a diagonal matrix. This form of state model plays an important role in control theory. The disadvantage of the canonical form is equally important. The canonical variables, like phase variables are not real physical variables of the system.

Let us consider a transfer function shown below

$$\frac{Y(s)}{U(s)} = T(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \longrightarrow (1)$$

Assume that the denominator is known in factored form and that the poles of the transfer function located at $\lambda_1, \lambda_2 \cdots \lambda_n$ are all distinct. Then the transfer function can then be expanded into partial fractions as

$$\frac{Y(s)}{U(s)} = T(s) = b_0 + \sum_{i=1}^{n} \frac{c_i}{s-\lambda_i} \longrightarrow (2)$$

where c_i are the residues of the poles at $s = \lambda_i$.

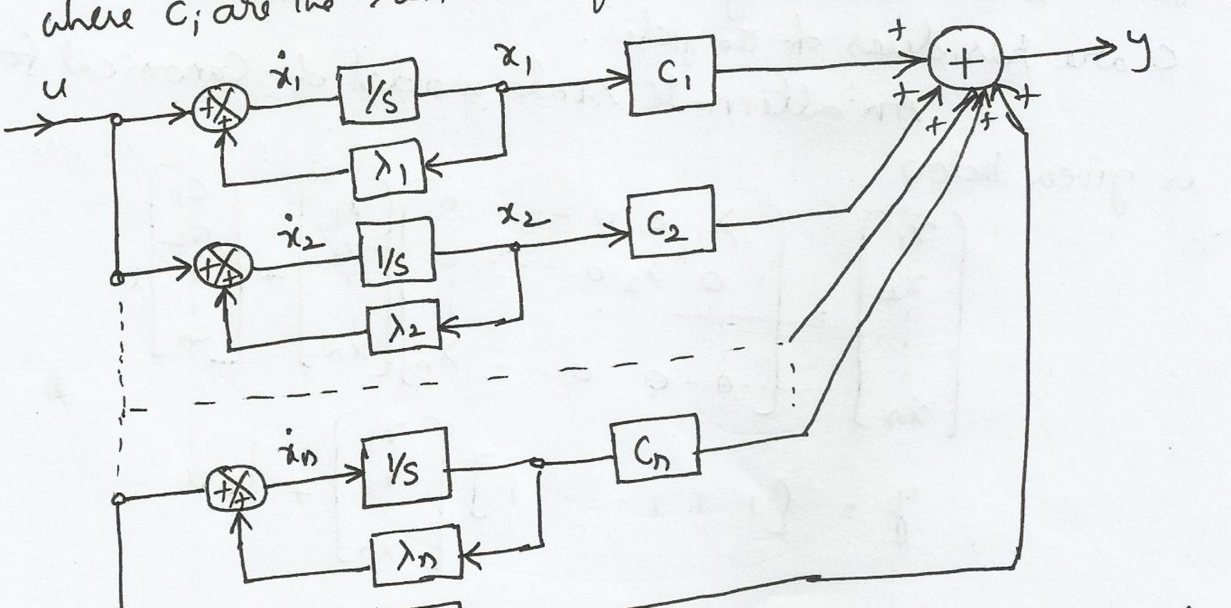


Figure: Block diagram of Canonical model

Defining the output of each integrator to be a state variable, we can write the state equations as

$$\dot{x}_i = \lambda_i x_i + u; \quad i = 1, 2, \ldots n$$

The output y(t) is given by

$$y = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n + b_0 u$$

This state model can be expressed in the vector-matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

In the canonical model, the system matrix 'A' is a diagonal matrix with the poles of T(s) as its diagonal elements. It is also observed that elements of column vector B are all unity and the elements of the row vector C are residues of the poles.

An alternate state model of canonical form is given below

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

(7)

(1) obtain the canonical form & state model of a system described by a differential equation

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = \dddot{u} + 8\ddot{u} + 17\dot{u} + 8u$$

where y is output and u is input.

(Sol) Taking the laplace transform on both sides with zero initial conditions.

$$s^3 Y(s) + 6s^2 Y(s) + 11sY(s) + 6Y(s) = s^3 U(s) + 8s^2 U(s) + 17sU(s) + 8U(s)$$

$$Y(s)\left[s^3 + 6s^2 + 11s + 6\right] = U(s)\left[s^3 + 8s^2 + 17s + 8\right]$$

Therefore, the TF $\frac{Y(s)}{U(s)} = \frac{s^3 + 8s^2 + 17s + 8}{s^3 + 6s^2 + 11s + 6}$

$$= \frac{\left[s^3 + 6s^2 + 11s + 6\right] + \left[2s^2 + 6s + 2\right]}{s^3 + 6s^2 + 11s + 6}$$

$$= 1 + \frac{2s^2 + 6s + 2}{(s+1)(s+2)(s+3)}$$

$$= 1 + \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

where $A = \underset{s \to -1}{Lt} \frac{2s^2 + 6s + 2}{(s+2)(s+3)} = \frac{2 - 6 + 2}{2} = -1$

$B = \underset{s \to -2}{Lt} \frac{2s^2 + 6s + 2}{(s+1)(s+3)} = \frac{8 - 12 + 2}{-1(1)} = 2$

$C = \underset{s \to -3}{Lt} \frac{2s^2 + 6s + 2}{(s+1)(s+2)} = \frac{18 - 18 + 2}{-2(-1)} = 1$

$$\therefore \frac{Y(s)}{U(s)} = 1 - \frac{1}{s+1} + \frac{2}{s+2} + \frac{3}{s+3}$$

The canonical model representation is shown in the block diagram

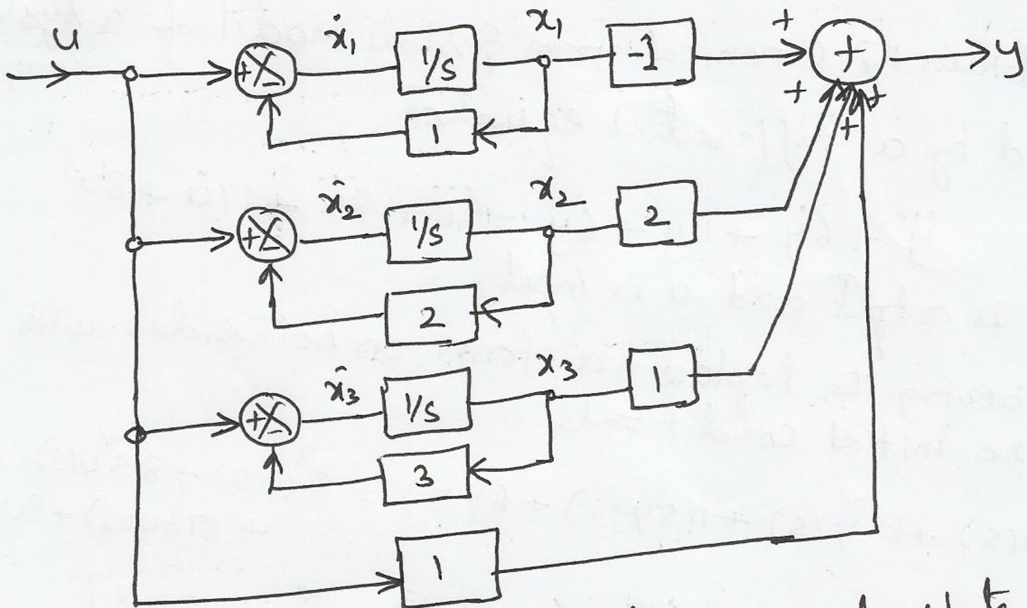


Figure: Block diagram of canonical state model

Therefore, the state equations are

$$\dot{x}_i = \lambda_i x_i + u \; ; \quad i = 1, 2, 3$$

$$\left.\begin{aligned} \therefore \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -2x_2 + u \\ \dot{x}_3 &= -3x_3 + u \end{aligned}\right\}$$

The state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

The output equation is

$$y = -x_1 + 2x_2 + x_3 + u$$

$$= \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + u$$

② obtain the canonical state model of a system, whose transfer function is given by

$$\frac{Y(s)}{U(s)} = \frac{2s^2 + 6s + 7}{(s+1)^2 (s+2)}$$

(Sol) Decomposing the above transfer function by the method of partial fractions yields

$$\frac{Y(s)}{U(s)} = \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{C}{(s+2)}$$

where $A = \underset{s \to -1}{Lt} \dfrac{2s^2+6s+7}{(s+2)} = \dfrac{2-6+7}{1} = 3$

$$B = \underset{s \to -1}{Lt} \frac{d}{ds}\left[\frac{2s^2+6s+7}{s+2}\right] = \underset{s \to -1}{Lt} \frac{(s+2)(4s+6)-(2s^2+6s+7)}{(s+2)^2}$$

$$= \frac{1(2)-(2-6+7)}{1} = \frac{2-3}{1} = -1$$

$$C = \underset{s \to -2}{Lt} \frac{2s^2+6s+7}{(s+1)^2} = 8-12+7 = 3$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{3}{(s+1)^2} - \frac{1}{(s+1)} + \frac{3}{(s+2)}$$

The block diagram representation of the canonical state model is shown in block diagram

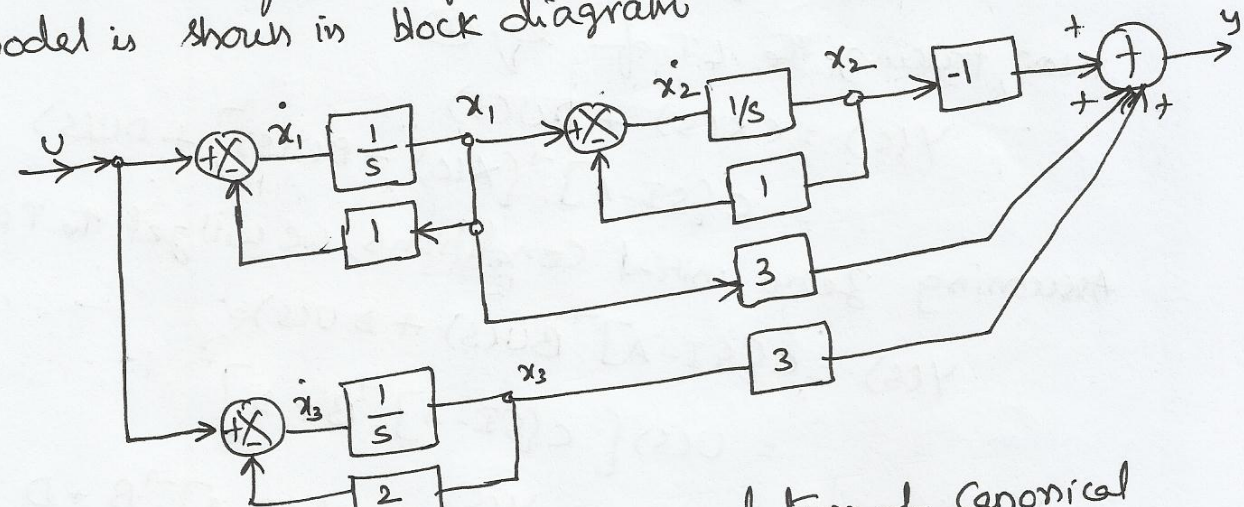


Figure : Block Diagram representation of Canonical state model

The state equations are
$$\dot{x}_1 = -x_1 + u$$
$$\dot{x}_2 = x_1 - x_2$$
$$\dot{x}_3 = -2x_3 + u$$

The output $y = -x_2 + 3x_1 + x_3$

Jordan block

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Because of repeated poles x_1 & x_2 are not in decoupled form.
The dotted block is known as <u>Jordan block</u>.

Derivation of Transfer function from State Model:

Let us consider the state Model of a system given by

$$\dot{X} = AX + BU \longrightarrow (1)$$

$$Y = CX + DU \longrightarrow (2)$$

Taking the LT of eq (1), we will get

$$SX(S) - X(0) = AX(S) + BU(S)$$

$$X(S)[SI-A] = X(0) + BU(S)$$

$$\therefore X(S) = [SI-A]^{-1}[X(0) + BU(S)]$$

Now, taking the LT of eq (2)

$$Y(s) = CX(S) + DU(S)$$

$$= C[SI-A]^{-1}\{X(0) + BU(S)] + DU(S)$$

Assuming zero initial conditions, we will get the TF

$$Y(S) = C\{SI-A]^{-1}BU(S) + DU(S)$$

$$= U(S)[C\{SI-A]^{-1}B + D]$$

$$\therefore \text{Transfer function } T(S) = \frac{Y(S)}{U(S)} = C\{SI-A]^{-1}B + D$$

$$= C\frac{Adj\,[SI-A]}{Det\,[SI-A]}B + D$$

Solving the denominator, we will get the characteristic equation $|SI-A| = 0$

An important observation is that, the state model is not unique, but the transfer function is unique.

Diagonalization : The state model of a system is not unique, since the state model employ physical variable, phase variables and Canonical variables.

From the application point of view, the physical variable representation of state model is more useful because they can be easily measured and used for control purposes. However this state model of physical variables is not convenient for investigation of system properties and evaluation of time response.

The Canonical state model in which the system matrix 'A' is in diagonal form is most suitable for investigation of system properties and evaluation of time response. Therefore it is useful to study techniques for transforming a general state model into Canonical form. These techniques are often referred to as diagonalization techniques.

Let us consider an n^{th} order multi-input-multi-output state model

$$\left.\begin{aligned} \dot{X} &= AX + BU \\ Y &= CX + DU \end{aligned}\right\} \longrightarrow (1)$$

Assume that the matrix 'A' in this model is non diagonal.

Let us define a new state vector V such that

$$X = MV \longrightarrow (2)$$

where M is $n \times n$ nonsingular constant matrix.

Under this transformation, the state model in equation (1) modifies to

$$M\dot{V} = AMV + BU$$

or

$$\dot{V} = M^{-1}AMV + M^{-1}BU$$

$$= \Lambda V + \tilde{B}U \longrightarrow (3)$$

$$Y = CMV + DU \quad \text{or}$$

$$Y = \tilde{C}V + DU \longrightarrow (4)$$

If the matrix M can be selected such that $M^{-1}AM$ is a diagonized matrix Λ, then the model given by eqs ③ & ④ is canonical model. Under this condition, the matrix M is called the diagonalizing matrix or modal matrix.

where $\Lambda = M^{-1}AM$ = diagonal matrix

$\tilde{B} = M^{-1}B$

$\tilde{C} = CM$

The determination of the diagonalizing matrix is facilitated by use of eigen vectors.

<u>Eigenvalues and Eigenvectors</u>:

The eigenvalues corresponding to system matrix 'A' are the solutions of $|\lambda I - A| = 0. \rightarrow ①$

The above equation may be expressed in expanded form as $q(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \rightarrow ②$

The values of λ which satisfy the above equation are called eigenvalues. Equation ② is called the characteristic equation corresponding corresponding to matrix 'A'.

Therefore, it is concluded that the eigen values of the state model and the poles of the system transfer function are the same. Thus a state model is stable if all the eigenvalues have negative real parts.

If all the eigenvalues of matrix 'A' are all distinct, then the rank 'r' of the matrix $(\lambda I - A)$ is $(n-1)$.

The eigen vector m_i associated with the eigenvalue λ_i may be obtained by taking cofactors of matrix $(\lambda_i I - A)$ along any row. ie

$$m_i = \begin{bmatrix} C_{K_1} \\ C_{K_2} \\ \vdots \\ C_{K_n} \end{bmatrix} ; \quad K = 1, 2, \cdots, \text{or } n$$

where C_{K_i} are the co-factors of matrix $(\lambda_i I - A)$

Let $m_1, m_2 \cdots m_n$ be the eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2 \cdots \lambda_n$ respectively. Then the model matrix or diagonalizing matrix M is given by

$$M = [m_1 \vdots m_2 \vdots m_3 \vdots \cdots m_n]$$

Therefore, the diagonalizing matrix is given by

$$\Lambda = M^{-1}AM = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & - & - & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & - - & \lambda_n \end{bmatrix}$$

when 'A' is expressed in the form given below.

$$A = \begin{bmatrix} 0 & 1 & 0 & - & - & - & 0 \\ 0 & 0 & 1 & \cdot & - & - & 0 \\ \vdots & & & & & & 1 \\ 0 & 0 & 0 & - & - & - & -a_1 \\ -a_n & -a_{n-1} & - & - & - & - & \end{bmatrix}$$

then the model matrix 'M' can be shown to be a special matrix called Vander Monde Matrix

$$M = V = \begin{bmatrix} 1 & 1 & - & - & - & 1 \\ \lambda_1 & \lambda_2 & - & - & - & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & - & - & - & \lambda_n^2 \\ \vdots & & & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & - & - & - & \lambda_n^{n-1} \end{bmatrix}$$

① Consider a system matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$.
Find the eigenvalues, eigenvectors and diagonalizing matrix.

(Sol) The characteristic equation is $|\lambda I - A| = 0$

$$\Rightarrow \quad \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 \\ -3 & \lambda & -2 \\ 12 & 7 & \lambda+6 \end{bmatrix} = \lambda I - A$$

$$|\lambda I - A| = \lambda(\lambda^2 + 6\lambda + 14) - (-1)[-3\lambda - 18 + 24] + 0$$
$$= \lambda^3 + 6\lambda^2 + 14\lambda - 3\lambda - 18 + 24$$
$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

The roots of $|\lambda I - A| = 0$ are $\lambda = -1, -2, -3$

$\therefore$ The eigenvalues are $\lambda_1 = -1;\ \lambda_2 = -2;\ \lambda_3 = -3$

The eigen vector m_i associated with eigenvector λ_i is obtained from the Co-factors of the matrix $[\lambda_i I - A]$

For m_1, the matrix $(\lambda_1 I - A)$ is

$$(\lambda_1 I - A) = (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -3 & -1 & -2 \\ 12 & 7 & 5 \end{bmatrix}$$

The Cofactors of 1st row are given by

$$m_1 = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} +[-5+14] \\ -[-15+24] \\ +[-21+12] \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ -9 \end{bmatrix}$$

or $m_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ since the eigenvector has unique direction.

Similarly, the eigen vectors associated with eigen values $\lambda_1 = -2$ and $\lambda_3 = -3$ are given by

$$m_2 = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} ; \quad m_3 = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

Therefore, the modal matrix or diagonalizing matrix is given by $M = [m_1 \vdots m_2 \vdots m_3] = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ -1 & 1 & 3 \end{bmatrix}$

(2) Find the eigenvalues, eigenvectors and modal matrix for a system matrix $A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$

(Sol) The eigenvalues are the solutions of characteristic equation $|\lambda I - A| = 0$.

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-4 & -1 & 2 \\ -1 & \lambda & -2 \\ -1 & 1 & \lambda-3 \end{bmatrix} = \lambda I - A$$

$$\therefore |\lambda I - A| = (\lambda-4)(\lambda^2-3\lambda+2) - (-1)[-\lambda+3-2] + 2[-1+\lambda]$$

$$= [\lambda^3 - 3\lambda^2 + 2\lambda - 4\lambda^2 + 12\lambda - 8] - \lambda + 1 - 2 + 2\lambda$$

$$= \lambda^3 - 7\lambda^2 + 15\lambda - 9$$

$\therefore$ The eigenvalues are the solutions of $|\lambda I - A| = \lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$

$\Rightarrow (\lambda-1)(\lambda-3)^2 = 0$; Therefore, the eigenvalues of the system matrix are $\lambda_1 = 1$; $\lambda_2 = 3$ and $\lambda_3 = 3$

The eigenvector associated with eigenvalue $\lambda = 1$ is obtained from the co-factors of any row of $(\lambda I - A)$, where $\lambda = 1$.

where $(\lambda_1 I - A) = (I - A) = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix}$;

∴ The Co-factors of 1st row are given by

$$m_1 = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Co-factors along first row give a null solution. Let us take co-factors along the second row.

$$m_1 = \begin{bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$$

To obtain eigenvectors associated with the repeated eigenvalue at $\lambda = 3$, we construct the matrix

$$[\lambda_2 I - A] = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_2 - 4 & -1 & 2 \\ -1 & \lambda_2 & -2 \\ -1 & 1 & \lambda_2 - 3 \end{bmatrix}$$

For $\lambda_2 = 3$, the rank of 3×3 matrix $(\lambda_2 I - A)$ is 2. Therefore one independent eigenvector associated with $\lambda = 3$ can be obtained from the Co-factors of 1st row of $(\lambda_2 I - A)$.

$$m_2 = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} \lambda_2(\lambda_2 - 3) + 2 \\ (\lambda_2 - 3) + 2 \\ -1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

The eigenvector m_3 may be generated from the independent eigenvector m_2 as follows

$$m_3 = \begin{bmatrix} \frac{d}{d\lambda_2} c_{11} \\ \frac{d}{d\lambda_2} c_{12} \\ \frac{d}{d\lambda_2} c_{13} \end{bmatrix} = \begin{bmatrix} 2\lambda_2 - 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

The eigenvector m_3 is known as generalized eigenvector.

Therefore, the modal matrix M is given by

$$M = [m_1 \,\vdots\, m_2 \,\vdots\, m_3] = \begin{bmatrix} 0 & 2 & 3 \\ 8 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}.$$

Now, the modal matrix 'M' transform 'A' to the Jordan matrix as follows.

$$M^{-1}AM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} = J = \text{Jordan Matrix}$$

↑ Jordan block

SOLUTION OF STATE EQUATIONS: There are three methods for solution of the state equations from which the system transient response can then be obtained.

(1) classical method (2) Laplace transform method

(3) Cayley - Hamilton method.

(1) Computation of state transition matrix by classical method: Let us consider the non-homogeneous state model given by $\dot{x}(t) = Ax(t) + BU(t)$; $x(0) = x_0$

The above state equation can be rewritten as

$$\dot{x}(t) - Ax(t) = BU(t)$$

multiplying both sides by e^{-At}, we can write

$$e^{-At}[\dot{x}(t) - Ax(t)] = \frac{d}{dt}[e^{-At}x(t)] = e^{-At}BU(t)$$

integrating both sides with respect to 't' between the limits 0 and 't', we get

$$e^{-At} x(t) \Big|_{t=0}^{t} = \int_0^t e^{-A\tau} BU(\tau) d\tau$$

$$e^{-At} x(t) - x(0) = \int_0^t e^{-A\tau} BU(\tau) d\tau$$

Now, pre-multiplying both sides by e^{At}, we have

$$x(t) = e^{At} x(0) + \int e^{A(t-\tau)} BU(\tau) d\tau \longrightarrow \text{I}$$

The above equation represents the solution of non-homo-geneous system.

For homogeneous system $B = 0$

∴ The solution is $x(t) = e^{At} x(0) \longrightarrow \text{II}$

From the above equation, it is observed that the initial state $x(0)$ or x_0 at $t=0$ is driven to a state $x(t)$ at time 't'. This transition in state is carried out by the matrix exponential e^{At}. Because of this property, e^{At} is known as state transition matrix and is denoted by $\phi(t)$.

(2) Computation of State transition Matrix (STM) by Laplace transform method.

Let us consider an unforced system whose state equation is $\dot{x} = Ax$, where 'A' is a constant matrix. Taking the Laplace transform of this equation, we obtain

$$sX(s) - X(0) = AX(s)$$

where $X(s)$ is the Laplace transform of the unforced response and $X(0)$ is the initial condition vector. The above equation may be rearranged as

$$[SI - A] X(s) = X(0)$$

or $X(S) = [SI - A]^{-1} X(0)$

Taking the inverse laplace transform, we get

$$x(t) = L^{-1}[SI-A]^{-1} x(0) \longrightarrow \text{I}$$

where $x(t)$ is the unforced response of the system.

Also we have $x(t) = e^{At} x(0) \longrightarrow \text{II}$

from eqs I & II

The STM $\phi(t) = e^{At} = L^{-1}[SI-A]^{-1} = L^{-1}[\phi(S)]$

where $\phi(S) = (SI-A)^{-1}$ is called the resolvent matrix

Let us now consider the response when the control force U is applied. The state equation for this case is

$$\dot{x} = Ax + BU$$

Taking Laplace transform on bothsides,

$$Sx(S) - X(0) = Ax(S) + BU(S); \quad \text{Let } X(0) = X_0$$

$$\therefore (SI-A)\, x(S) = X_0 + BU(S)$$

Therefore $x(S) = (SI-A)^{-1} X_0 + [SI-A]^{-1} BU(S)$

Taking inverse laplace transform

$$x(t) = L^{-1}\left[(SI-A)^{-1} X_0\right] + L^{-1}\left[[SI-A]^{-1} BU(S)\right]$$

$$= \phi(t) X_0 + L^{-1}[\phi(S) BU(S)]$$

This is the response of forced system model or non-homogeneous system.

Properties of State Transition Matrix (STM):

we have $\phi(t) = e^{At}$ is the STM. Centrally useful properties of STM are given by

(1) $\phi(0) = e^{A0} = I$ = Identity matrix

2) $\phi^{-1}(t) = (e^{At})^{-1} = e^{-At} = \phi(-t)$

3) $\phi(t_1+t_1) = e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}$
$= \phi(t_1)\,\phi(t_2) = \phi(t_2)\,\phi(t_1)$

(1) Let us consider a system with matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.
Find the state transition matrix.

(Sol) The state transition matrix $e^{At} = L^{-1}[SI-A]^{-1}$

where $SI - A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} S-1 & 0 \\ -1 & S-1 \end{bmatrix}$

$$\therefore [SI-A]^{-1} = \frac{1}{\det[SI-A]}\,\mathrm{adj}[A]$$

$$= \frac{1}{(S-1)^2}\begin{bmatrix} S-1 & 0 \\ 1 & S-1 \end{bmatrix}$$

The resolvent matrix is given by

$$\phi(S) = [SI-A]^{-1} = \begin{bmatrix} \frac{1}{S-1} & 0 \\ \frac{1}{(S-1)^2} & \frac{1}{S-1} \end{bmatrix}$$

The state transition matrix $\phi(t) = L^{-1}(\phi(S))$

$\therefore$ The state transition

$$\Rightarrow \phi(t) = \begin{bmatrix} L^{-1}\frac{1}{(S-1)} & 0 \\ L^{-1}\left(\frac{1}{S-1}\right)^2 & L^{-1}\frac{1}{(S-1)} \end{bmatrix} = \begin{bmatrix} e^{t} & 0 \\ te^{t} & e^{t} \end{bmatrix} = e^{At}$$

(2) obtain the time response of the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

where $U(t)$ is a unit step occuring at $t=0$ and $x^T(0) = [1\ 0]$

(sol) The given system is in the form $\dot{X}(t) = AX(t) + BU(t)$

Therefore, the response of non-homogeneous system is

$$x(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}BU(\tau)\,d\tau$$

$$= e^{At}\left[X(0) + \int_0^t e^{-A\tau}BU(\tau)\,d\tau\right]$$

where $e^{At} = \phi(t) = L^{-1}[SI-A]^{-1} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$

From the given data $X(0) = X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Also $e^{-A\tau}B = \begin{bmatrix} e^{-\tau} & 0 \\ -\tau e^{-\tau} & e^{\tau} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-\tau} \\ e^{-\tau}(1-\tau) \end{bmatrix}$

$$\therefore \int_0^t \phi(-\tau)BU(\tau)\,d\tau = \begin{bmatrix} \int_0^t e^{-\tau}\,d\tau \\ \int_0^t e^{-\tau}(1-\tau)\,d\tau \end{bmatrix} = \begin{bmatrix} 1-e^{-t} \\ te^{-t} \end{bmatrix}$$

Therefore, the response of the system is given by

$$x(t) = e^{At} * \left[X(0) + \begin{bmatrix} 1-e^{-t} \\ te^{-t} \end{bmatrix}\right]$$

$$= e^{At}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1-e^{-t} \\ te^{-t} \end{bmatrix}\right\} = e^{At}\begin{bmatrix} 2-e^{-t} \\ te^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}\begin{bmatrix} 2-e^{-t} \\ te^{-t} \end{bmatrix} = \begin{bmatrix} 2e^t - 1 \\ 2te^t \end{bmatrix}$$

(3) Consider a Control systems with state model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}[u];$$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; u = unit step. Compute the STM and state response $x(t)$

(Sol) The given system is in the form $\dot{X} = AX + BU$. The response of this non-homogeneous system is given by

$$x(t) = \phi(t)\left[X(0) + \int_0^t \phi(-\tau) BU(\tau)\, d\tau\right]$$

where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$; $X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The resolvent matrix $\phi(s) = [SI - A]^{-1}$

where $SI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$

$$\therefore \phi(s) = (SI - A)^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & +1 \\ -2 & s \end{bmatrix}$$

$$\therefore \text{The STM } \phi(t) = L^{-1}[SI - A]^{-1} = L^{-1}\begin{Bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{+1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{Bmatrix}$$

where $L^{-1}\dfrac{s+3}{(s+1)(s+2)} = L^{-1}\dfrac{2}{(s+1)} - L^{-1}\dfrac{1}{s+2} = 2e^{-t} - e^{-2t}$

$L^{-1}\dfrac{+1}{(s+1)(s+2)} = L^{-1}\left\{\dfrac{1}{s+1} - \dfrac{1}{s+2}\right\} = e^{-t} - e^{-2t}$

$L^{-1}\dfrac{-2}{(s+1)(s+2)} = L^{-1}\left\{\dfrac{-2}{s+1} + \dfrac{2}{s+2}\right\} = -2e^{-t} + 2e^{-2t}$

$L^{-1}\dfrac{s}{(s+1)(s+2)} = L^{-1}\left\{\dfrac{-1}{s+1} + \dfrac{2}{s+2}\right\} = -e^{-t} + 2e^{-2t}$

$$\therefore \phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\phi(-\tau)BU(\tau) = \begin{bmatrix} 2e^{\tau} - 2e^{2\tau} \\ -2e^{\tau} + 4e^{2\tau} \end{bmatrix}$$

$$\therefore \int_0^t \phi(-\tau)BU(\tau)d\tau = \begin{cases} \int_0^t (2e^{\tau} - 2e^{2\tau})d\tau \\ \int_0^t (-2e^{\tau} + 4e^{2\tau})d\tau \end{cases}$$

$$= \begin{bmatrix} 2e^{t} - e^{2t} - 1 \\ -2e^{t} + 2e^{2t} \end{bmatrix}$$

$\therefore$ The state response $x(t) = \phi(t)\left[x(0) + \int_0^t \phi(-\tau)BU(\tau)d\tau\right]$

$$\Rightarrow x(t) = \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix} + \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 2e^{t} - e^{2t} - 1 \\ -2e^{t} + 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix}$$

The Concepts of Controllability and observability:

The Concepts of Controllability and observability play an important role in Control engineering. These concepts were introduced by Kalman.

Controllability: A System is said to be Completely state Controllable if it is possible to transfer the system from any initial state $x(t_0)$ to any desired system state $x(t)$ in specified finite time by a Control vector $U(t)$.

A general nth order multi-input linear time-invariant System with an m-dimensional Control vector is $\dot{x} = Ax + Bu$ is Completely controllable if and only if the rank of the Composite matrix

$Q_c = [B \vdots AB \vdots \cdots \vdots A^{n-1}B]$ is 'n'.

Since only the matrices A and B are involved, we may say that the pair (A,B) is controlled if rank of the matrix Q_c is n.

<u>observability</u>: A system is said to be completely observable, if every state $x(t_0)$ can be completely identified by measurements of the outputs $y(t)$ over a finite time interval.

A system which is not completely observable, implies that some of its state variables are shielded from observation.

A general nth order multi-input multi-output linear-time invariant systems

$$\dot{x} = AX + BU$$
$$Y = CX$$

is completely observal if and only if the rank of the composite matrix $Q_0 = [C^T \vdots A^T C^T \vdots \cdots (A^T)^{n-1} C^T]$ is n.

This condition is also referred as the pair (A,C) being observable.

<u>Duality property</u>: (1) The pair (AB) is controllable implies that the pair $(A^T B^T)$ is observable

(2) The pair (AC) is observable implies that the pair $(A^T C^T)$ is controllable.

Thus the concepts of controllability and observability are dual concepts.

① Consider a system with state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

check whether the system is completely state controllable or not.

(Sol) The system matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$; $\dot{X} = AX + BU$

The output matrix $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The Kalman's test states that a system is completely state controllable if the rank of the matrix $Q_c = [B \vdots AB \vdots \cdots \vdots A^{n-1}B]$ is 'n', where 'n' is number of state variables. Therefore for the given system $n=3$

$\therefore Q_c = [B \vdots AB \vdots A^2B]$

where $AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$

$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 25 \end{bmatrix}$

$\therefore Q_c = [B \vdots AB \vdots A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$

The determinant of Q_c ie $|Q_c| = -1 \neq 0$;

Therefore the rank of $Q_c = n = 3$;

Therefore the system is completely controllable.

(2) Let us examine the observability of the system given below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(Sol) The given homogeneous system is in the form $\dot{X} = AX$ and $y = CX$; Therefore $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$; $C^T = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

The Kalman's test states that a system is completely observable if the rank of the matrix

$Q_0 = \left[C^T \,\vdots\, A^T C^T \,\vdots\, \cdots\, (A^T)^{n-1} C^T \right]$ is n; where n is number of state variables.

In this case $n = 3$

$$\therefore Q_0 = \left[C^T \,\vdots\, A^T C^T \,\vdots\, (A^T)^2 C^T \right]$$

$$A^T C^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$(A^T)^2 C^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$$

$$\therefore Q_0 = \left[C^T \,\vdots\, A^T C^T \,\vdots\, (A^T)^2 C^T \right] = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$$

The determinant of the matrix Q_0 is given by

$$|Q_0| = \begin{vmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{vmatrix} = 3(-2+2) = 0$$

$\therefore$ The rank of the matrix is less than 3

i.e. the rank of the matrix Q_0 is $r = 2$,

Hence one of the stable variable is unobservable