A40401- SIGNALS, SYSTEMS AND STOCHASTIC PROCESSES

GPCET

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G. PULLAIAH COLLEGE OF ENGINEERING AND TECHNOLOGY

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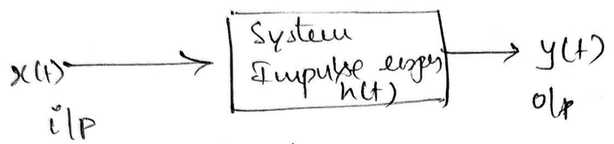
UNIT -I

System :- A system is defined as a set of elements or functional blocks which are connected together & produces an o/p in response of an i/p sgl.



A system

It is represented by a block diagram as shown in figure. An arrow entering the box is the i/p sgl (called excitation, source or driving function) and an arrow leaving the box is an o/p sgl (called response). Generally denoted by x(t) & o/p is y(t)

The relation b/w x(t) & y(t) of system has a form

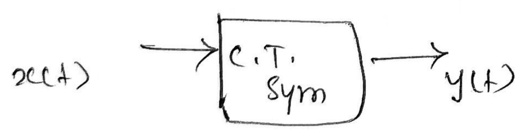
mathematically $y(t) = T[x(t)]$

Transfer function

Systems are broadly classified
1. C.T. syms
2. D.T syms

CT Sym

D.T Sym



Again CT & DT Sym are classified into

1. Lumped & Distributed sym
2. Static (memoryless) & Dynamic Systems
3. Causal & Non Causal sym
4. Linear & Non Linear sym
5. Time Varying & Time-invariant Sym
6. Stable & Unstable system
7. Invertible & Non Invertible System
8. FIR & IIR Systems.

→ 1 Linear & Non Linear Systems :-

A Sym which obeys the principle of Superposition & principle of homogeneity is called a linear system. which doesn't obey the principle of Superposition and homogeneity is called non-linear system.

Homogeneity property means a System which produces an o/p $y(t)$ for an i/p $x(t)$ must produce an o/p $a\,y(t)$ for an i/p $a\,x(t)$.

Superposition means a system which produces an o/p $y_1(t)$ for an input $x_1(t)$ & an o/p $y_1(t)$ & an arbitrary i/p $x_2(t)$ produces an o/p $y_2(t)$ then the weighted sum of i/ps $a x_1(t) + b x_2(t)$ where a, b are Constants, produces an o/p $a y_1(t) + b y_2(t)$ which is the sum of weighted o/p. that is

$$T[a x_1(t) + b x_2(t)] = a\,T[x_1(t)] + b\,T[x_2(t)]$$

Note :- In general, if the describing equation contains square or higher order terms of i/p &/or o/p and/or product of i/p & its derivative or a constant. the system will definitely be non linear.

Ex:- Check whether the following systems are linear or not.

1. $y(n) = n^2 x(n)$

Sol: $y_1(n) = n^2 x_1(n)$

$y_2(n) = n^2 x_2(n)$

Weighted sum of o/ps

$a y_1(n) + b y_2(n) = a[n^2 x_1(n)] + b[n^2 x_2(n)]$

$= n^2[a x_1(n) + b x_2(n)]$

o/p due to weighted sum of i/ps

$y_3(n) = T[a x_1(n) + b x_2(n)]$

$y_3(n) = n^2[a x_1(n) + b x_2(n)]$

∴ System is linear

2. $y(n) = 2x(n) + 4$

Sol: $y_1(n) = 2a\,x_1(n) + 4$

$y_2(n) = 2b\,x_2(n) + 4$

$y_1(n) + y_2(n) = 2a\,x_1(n) + 4 + 2b\,x_2(n) + 4$

$= 2[a x_1(n) + b x_2(n) + 4]$ → (1)

$y_3(n) = 2[a x_1(n) + b x_2(n)] + 4$ → (2)

From (1) & (2)

$y_3(n) \neq y_1(n) + y_2(n)$

∴ System is Non linear system

Time varying & Time invariant system :-

Time Invariance is the property of a system which makes the behaviour of the system independent of time. Means behaviour of the sym doesn't depend on the time at which the i/p is applied.

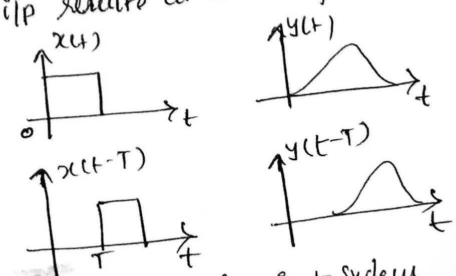
A System is said to be time-invariant (or shift-invariant) if its i/o characteristics do not change with time i.e. if a time shift in the i/p results in a corresponding time shift in the output

If $x(t) \to y(t)$

Then $x(t-T) \longrightarrow y(t-T)$

A System not satisfying the above requirements is called a time-varying System (or shift varying system)

$y(t, T) \neq y(t-T)$



Time invariant System

Ex:- Determine whether the following systems are time-invariant or not.

(a) $y(t) = t^2 x(t)$

$y(t,T) = t^2 x(t-T)$

sol: $y(t-T) = (t-T)^2 x(t-T)$

$y(t,T) \neq y(t-T)$

Time-variant System

(b) $y(t) = x(t)\sin 10\pi t$

$y(t,T) = x(t-T)\sin 10\pi t$

sol:- $y(t-T) = x(t-T)\sin 10\pi(t-T)$

$y(t,T) \neq y(t-T)$

Time-Variant System

(c) $y(t) = x(t^2)$

$y(t,T) = x(t^2-T)$

sol: $y(t-T) = x((t-T)^2)$

$y(t,T) \neq y(t-T)$

Time-Variant System.

(d) $y(t) = x(-2t)$

$y(t,T) = x(-2t-T)$

sol: $y(t-T) = x(-2(t-T))$

$y(t-T) = x(-2t+2T)$

$y(t,T) \neq y(t-T)$

Time-variant system

(e) $y(t) = e^{2x(t)}$

sol: $y(t-T) = e^{2x(t-T)}$

$y(t,T) = e^{2x(t-T)}$

$y(t,T) = y(t-T)$

Time-invariant System

(f) $y(n) = x(n/2)$

$y(n,k) = x(\frac{n}{2}-k)$

sol: $y(n-k) = x(\frac{n-k}{2})$

$y(n-k) \neq x(\frac{n-k}{2})$

$y(n) \neq y(n-k)$

Time-Variant

(g) $y(n) = x(n)$

$y(n,k) = x(n-k)$

sol: $y(n-k) = x(n-k)$

Time-invariant system

(h) $y(n) = x^2(n-2)$

$y(n,k) = x^2(n-k-2)$

sol: $y(n-k) = x^2(n-k-2)$

time-invariant system

(i) $y(n) = x(n) + n\,x(n-2)$

sol: $y(n-k) = x(n-k) + (n-k)\,x(n-k-2)$

$y(n,k) = x(n-k) + n\,x(n-k-2)$

$y(n,k) \neq y(n-k)$

Time-Variant

3. Causal and Non-causal Systems:-

A system is said to be causal (or non-anticipative) if the output of the system at any time t depends only on the present and past values of the input but not on future inputs.

Causal systems are real time systems. They are physically realizable. The impulse response of a causal system is zero for t (or n) < 0 since $\delta(t)$ [or $\delta(n)$] exists only at t (or n) = 0.

$h(t) = 0$ for $t < 0$ and $h(n) = 0$ for $n < 0$.

A system said to be non-causal (anticipative) if the o/p of the system at any time t depends on future i/ps. They do not exist in real time. They are not physically realizable.

Ex:- Check whether the following Systems are causal or not.

(a) $y(t) = x^2(t) + x(t-4)$

sol: $t = -2$ $y(-2) = x^2(-2) + x(-2-4) \Rightarrow y(-2) = x^2(-2) + x(-6)$

$t = 0$ $y(0) = x^2(0) + x(-4)$

$t = 2$ $y(2) = x^2(2) + x(-2)$

The o/p depends only on present & past values of i/p. Therefore the system is causal.

(b) $y(t) = x(2-t) + x(t-4)$

$t = -1$ $y(-1) = x(1) + x(-5)$

$t = 0$ $y(0) = x(2) + x(-4)$

$t = 1$ $y(1) = x(1) + x(-3)$

The o/p depends ~~only~~ on future value of i/p, Therefore the system is non-causal.

4 Stable & Unstable Systems :-

A bounded signal is a signal whose magnitude is always a finite value. for example, a sine wave is a bounded signal. A system is said to be bounded-input, bounded output (BIBO) stable, if and only if every bounded input produces a bounded output ... } a stable system does not diverge & does not grow unreasonably large.

The input signal $x(t)$ be bounded (finite) i.e,

$$|x(t)| \le M_x < \infty \text{ for all } t.$$

$$\left[\because e^{\infty} = \infty, \quad e^{-\infty} = 0\right]$$

M_x is +ve real number

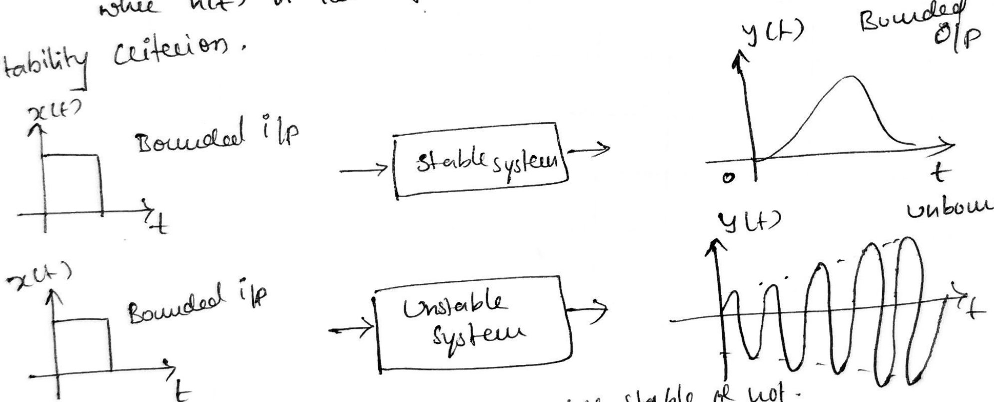
$$|y(t)| \le M_y < \infty$$

$y(t)$ is also bounded, then the system is BIBO stable. otherwise the system is unstable.

BIBO stability criterion :- $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$. This is called BIBO

where $h(t)$ is the impulse response of the system.

stability criterion.



Ex:- Find the whether the following systems are stable or not.

ⓐ $y(t) = e^{x(t)}$; $|x(t)| < 8$

sol:- $|x(t)| < \infty$ i/p Bounded

$e^{-8} \le y(t) \le e^{8}$

o/p also bounded.

$\therefore$ system is stable

ⓑ $y(t) = (t+5)\,u(t)$

sol:- $y(t) = (t+5) \quad t \ge 0$

$t \to \infty$; $y(t) \to \infty$

Hence o/p grows without any bounded & hence the given system is unstable.

ⓒ $h(t) = (2 + e^{-3t})\,u(t)$

$$\int_{-\infty}^{\infty} h(t)\,dt = \int_{-\infty}^{\infty} (2 + e^{-3t})\,u(t)\,dt$$

$$= \int_{0}^{\infty} (2 + e^{-3t})\,dt$$

$$= 2[\infty - 0] + \frac{e^{-3t}}{-3}\Big|_0^{\infty}$$

$$\int_{-\infty}^{\infty} h(t)\,dt = \infty$$

System is unstable

ⓓ $h(t) = e^{2t}\,u(t)$

$$\int_{-\infty}^{\infty} h(t)\,dt = \int_{-\infty}^{\infty} e^{2t}\,u(t)\,dt$$

$$= \int_{0}^{\infty} e^{2t}\,dt$$

$$= \frac{e^{2t}}{2}\Big|_0^{\infty}$$

$= e^{\infty} = \infty$ Hence unstable

ⓔ $h(t) = \frac{1}{RC} e^{-t/RC}\,u(t)$

sol $\int_{-\infty}^{\infty} h(t)\,dt = \int_{-\infty}^{\infty} \frac{1}{RC} e^{-t/RC}\,u(t)\,dt$

$$= \frac{1}{RC}\int_{0}^{\infty} e^{-t/RC}\,dt$$

$$= \frac{1}{RC}(-RC)\left[e^{-\infty} - e^{-0}\right]$$

$= 1$ Hence Stable

5. Static & Dynamic System :-

A System is said to be Static or memoryless if the response is due to present input alone i.e., for a static or memoryless system, the o/p at any instant t (or n) depends only on the i/p applied at that instant t (or n) but not ~~on~~ the past or future values of input.

Ex:- $y(t) = x(t)$ $\quad y(n) = x(n)$

$y(t) = x^2(t)$ $\quad y(n) = 2x^2(n)$

In contrast, a system is said to be dynamic or memory system if the response depends upon past / future inputs. $y(n) = x(n) + x(n-2)$.

Ex:- $y(t) = x(t-1)$ $\quad y(t) = \frac{d^2x(t)}{dt^2} + x(t)$

$y(t) = x(t) + x(t+2)$ $\quad y(n) = x(2n)$

For C.T. Sym → Static System — purely resistive electrical ckt
→ Dynamic system — ele ckt having inductors and/or capacitors.

For D.T. Sym → ~~Static System~~ → Summer / Accumulator
→ Dynamic system → A delay is also D.T Sym with memory

Ex Find whether the following systems are dynamic or not.

(a) $y(t) = x(t-3)$

sol $y(0) = x(0-3) = x(-3)$

$y(1) = x(1-3) = x(-2)$

$y(2) = x(2-3) = x(-1)$

$y(3) = x(3-3) = x(0)$

depends on past i/p

∴ Dynamic System

(b) $y(t) = x(2t)$

sol $y(0) = x(0)$

$y(1) = x(2)$

$y(2) = x(4)$

depends on future i/p's

Dynamic system

(c) $y(t) = \frac{d^2x(t)}{dt^2} + 2x(t)$

sol $y(0) = \frac{d^2}{dt^2}x(0) + 2x(0)$

$y(1) = \frac{\partial^2}{\partial t^2}x(1) + 2x(1)$

$y(2) = \frac{\partial^2}{dt^2}x(2) + 2x(2)$

∴ Dynamic systems

Differential Equations come under

(d) $y(n) = x(n+2)$

Dynamic System

(e) $y(n) = x^2(n)$

Static System

(f) $y(n) = x(n-2) + x(n)$

Dynamic system.

(6) Lumped & Distributed System :-

Lumped parameter systems are the systems in which each component is lumped at one point in space. These systems are described by ordinary differential equations.

Distributed Systems are the systems in which signals are functions of space as well as time, These systems are described by partial differential Equations

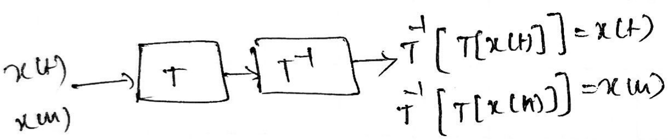
7. Invertible & Non-invertible Systems :-

If a system has a unique relationship b/w its i/p $x(t)$ (or $x[n]$) and output $y(t)$ [or $y[n]$] the system is known as invertible. Therefore, for an invertible system if $y(t)$ [or $y[n]$] is known $x(t)$ [or $x[n]$] can be found out unambiguously and uniquely.

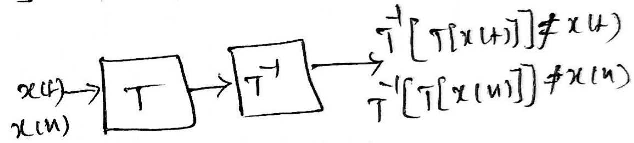
Other hand, if the system does not have a unique relationship b/w its i/p and o/p. the system is said to be non-invertible. In other words a system is known as invertible only if an inverse system exists which when cascaded with the original system produces an o/p equal to the i/p of the first system.

Mathematically :, a system is to be invertible if

$$x(t) = T^{-1}\{T[x(t)]\}$$



Invertible System



Non-Invertible system

8. FIR and IIR Systems :-

LTI syms can be classified according to the type of impulse response. if impulse response sequence is of finite duration - finite impulse response if impulse respons sequence is of infinite duration - - Infinite impulse response

$$\text{FIR} \qquad h(n) = \begin{cases} -2 & n = 2, 4 \\ 2 & n = 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{IIR} \qquad h(n) = 2^n u(n)$$

Ex State whether the following system is linear, causal, time invariant and stable

$y(n) + y(n-1) = x(n) + x(n-2)$

Sol Given $y(n) = -y(n-1) + x(n) + x(n-2)$

1. $a\,y_1(n) + b\,y_2(n) = -[\cancel{a\,y_1(n-1)} + \cancel{b\,y_2(n-1)}]$.

weighted sum of o/p $= -a\,y_1(n-1) + a\,x_1(n) + a\,x_1(n-2) + b\,y_2(n-1)$

$+ b\,x_2(n) + b\,x_2(n-2)$

$= -[a\,y_1(n-1) + b\,y_2(n-1)] + [a\,x_1(n) + b\,x_2(n)]$

$+ [a\,x_1(n-1) + b\,x_2(n-2)] \rightarrow (1)$

2. o/p of weighted sum of i/ps $y_3(n) = -[a\,y_1(n-1) + b\,y_2(n-1)]$

$+ [a\,x_1(n) + b\,x_2(n)] + [a\,x_1(n-1) + b\,x_2(n-2)] \rightarrow (2)$

From Eqn (1) & (2) both are equal then system is linear system

2. The o/p depends only on the present & past i/ps & past o/ps. so the system is causal.

3. All the co-eff of the differential equation are constants. So the system is time invariant.

4. For $x(n) = \delta(n)$ $y(n) = h(n)$

$h(n) = -h(n-1) + \delta(n) + \delta(n-2)$

$h(0) = -h(-1) + \delta(0) + \delta(-2) = 1$

$h(1) = -h(0) + \delta(1) + \delta(-1) = -1$

$h(2) = -h(1) + \delta(2) + \delta(0) = 1 + 0 + 1 = 2$

$h(3) = -h(2) + \delta(3) + \delta(1) = -2 + 0 + 0 = -2$

$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 1 + 2 + 2 + \dots = \infty$

$\therefore$ impulse response is Not absolutely ~~integrable~~ summable so the system is unstable

Eg Comment about the linearity, stability, time invariance and causality for the following filter.

$$y(n) = 2x(n+1) + [x(n-1)]^2$$

sol Given $y(n) = 2x(n+1) + [x(n-1)]^2$

1. There is a square term of delayed i/p [i.e. $x(n-1)^2$] in the difference Equation. So the system is non linear.

2. The o/p depends on the future value of i/p [i.e, $2x(n+1)$] So the system is non-causal.

3. For $x(n) = \delta(n)$, $y(n) = h(n)$

$\therefore$ $h(n) = 2\delta(n+1) + \{\delta(n-1)\}^2$

$h(0) = 2\delta(1) + [\delta(-1)]^2 = 0+0=0$

$h(1) = 2\delta(2) + [\delta(0)]^2 = 0+1=1$

$h(-1) = 2\delta(0) + [\delta(-2)]^2 = 2+0=2$

$h(-2) = 2\delta(-1) + [\delta(-3)]^2 = 0+0+0$

$h(n) = 0$ for any other n

$\sum_{n=-\infty}^{\infty} |h(n)| = 0+1+2+0+0+\ldots = 3 < \infty$

Impulse response is absolutely summable so system is stable.

4. The o/p due to delayed input is given by

$y(n,k) = 2x(n+1-k) + \{x(n-1-k)\}^2$

The delayed output is

$y(n-k) = 2x(n+1-k) + \{x(n-k-1)\}^2$

$y(n,k) = y(n-k)$

$\therefore$ The System is time-invariant.

Convolution :- it is a mathematical way of combining two signals to form a third signal. it is important because it relates the i/p sgl & impulse response of the system to the o/p of the system.

Correlation :- it compares two signals in order to determine the degree of similarities b/w them.

Auto correlation - when one signal is correlated with itself to form ~~another sign~~ another signal.

Cross correlation - when one signal is correlated with another signal to form a third signal

Concept of convolution :-

Convolution is a mathematical operation which is used to express the i/p & o/p relationship of an LTI system.

Let us consider an LTI system which is initially relaxed at $t=0$ if the i/p to the system is an impulse, then the o/p of the system is denoted by $h(t)$ and is called Impulse response of the system.

impulse response $h(t) = T[\delta(t)] \rightarrow (1)$

For any arbitrary signal $x(t)$ can be represented as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \rightarrow (2)$$

system o/p $y(t) = T[x(t)] = T\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right]$

$$= \int_{-\infty}^{\infty} x(\tau)\, T[\delta(t-\tau)]\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \rightarrow (3)$$

This is called Convolution integral, or simply convolution

The Convolution of two signals can be represented as

$$y(t) = x(t) * h(t)$$

$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$ - when $x(t)$, $h(t)$ are non causal

$y(t) = \int_{-\infty}^{t} x(\tau)\,h(t-\tau)\,d\tau$ - when $h(t)$ is causal $x(t)$ - non causal

$y(t) = \int_{0}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$ - when $h(t)$ is non causal $x(t)$ is causal

$y(t) = \int_{0}^{t} x(\tau)\,h(t-\tau)\,d\tau$ - when both $x(t)$, $h(t)$ are causal properties of convolution :-

1. Commutative property

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

2. Distributive property

$$x_1(t) * [x_2(t) + x_3(t)] = [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)]$$

3. Associative property

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

4. Shift property - $x_1(t) * x_2(t) = z(t)$ $\quad x_1(t-T) * x_2(t) = z(t-T)$

$x_1(t) * x_2(t-T) = z(t-T)$ $\quad x_1(t-T_1) * x_2(t-T_2) = z(t-T_1-T_2)$

5. Convolution with impulse

$$x(t) * \delta(t) = x(t)$$

6. wide property - let duration of $x_1(t)$ & $x_2(t)$ be T_1 & T_2 respectively then the duration of the signal obtained by convoluting $x_1(t)$ & $x_2(t)$ is $T_1 + T_2$.

Ex Find convolution of the following signals.

$x_1(t) = e^{-2t}u(t)$ $\quad x_2(t) = e^{-4t}u(t)$

sol

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\tau} u(\tau)\, e^{-4(t-\tau)} u(t-\tau)\, d\tau$$

$u(\tau) = 1$ for $\tau > 0$

$u(t-\tau) = 1$ for $t > \tau$

$$= \int_0^t e^{-2\tau} e^{-4t} \cdot e^{4\tau}\, d\tau$$

$$= e^{-4t} \int_0^t e^{2\tau}\, d\tau$$

$$= e^{-4t}\left[\frac{e^{2\tau}}{2}\right]_0^t = \frac{e^{-4t}}{2}\left[e^{2t} - e^0\right]$$

$$= \frac{e^{-2t}}{2} - \frac{e^{-4t}}{2} = \frac{e^{-2t} - e^{-4t}}{2}\, u(t).$$

Correlation Concept :- it compares two signals in order to determine the degree of similarities b/w them.

The integral $\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt$ forms the basis of comparison of the two signals $x_1(t)$ & $x_2(t)$ over the interval (t_1, t_2) over the interval (t_1, t_2) then

if $x_1(t)$ & $x_2(t)$ have no similarities

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0 .$$

→ Auto correlation - it gives the measures of match similarities & relatedness or coherence b/w a signal & its time delayed version. it is defined as the correlation of a signal with itself.

it is represented with $R(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$.

τ is delay parameter ~~with +ve direction~~

(or)

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\, dt .$$

① $R(\tau) = R^*(-\tau)$ - Conjugate symmetry

② $R(0) \geq R(\tau)$

③ $FT[R(\tau)] \longleftrightarrow S(\omega)$

→ Cross correlation - (it measures the similarities of two different signals in that one signal, another signal delayed version of) $x(t)$

The cross correlation b/w two different waveforms or signals. is a measure of similarity or match or relatedness or coherence b/w one signal and time delayed version of another signal. that means this indicates cross correlation b/w two signals indicated how much one signal is related to the time delayed version of another signal.

it is represented with $R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^*(t-\tau)\, dt$$

(or)

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2^*(t)\, dt .$$

① Conjugate symmetry - $R_{12}(\tau) = R_{21}^*(-\tau)$

② if $\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t) = 0$ means two signals are orthogonal signals.

③ $R_{12}(\tau) \longleftrightarrow X_1(\omega)\, X_2^*(\omega)$, Multiplication of the FT of one & complex conjugate of FT of second signal

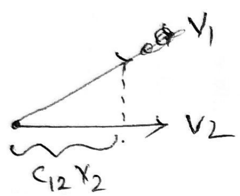
Analogy b/w vectors & Signals :- (Summary)

Vector

1. The dot product b/w the vectors V_1 & V_2 is $V_1 \cdot V_2$

$$V_1 \cdot V_1 = V_1^2$$

2. Consider two vectors V_1 & V_2



The Component V_1 along V_2 $= C_{12} V_2$

3. The value of C_{12} is $C_{12} = \frac{V_1 \cdot V_2}{V_2^2}$

4. If two vectors V_1 & V_2 are orthogonal to each other $V_1 \cdot V_2 = 0$

5. If $x_1, x_2 \ldots x_n$ unit vectors along 'n' mutually ⊥lar axes then any vector A which has components $C_1, C_2, \ldots C_n$ along these mutually ⊥lar can be written as

$$A = C_1 x_1 + C_2 x_2 + \ldots + C_n x_n$$

6. The Component 'C_r' is

$$C_r = \frac{A \cdot x_r}{x_r \cdot x_r}$$

Signals

1. The dot product is Equivalent to

$$V_1 \cdot V_2 \sim \int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt$$

$$V_1 \cdot V_2 \sim \int_{t_1}^{t_2} x_1^2(t)\, dt$$

2. Consider two Signals $x_1(t)$, $x_2(t)$ in the interval $t_1 < t < t_2$

Then $x_1(t)$ can be approximated using $x_2(t)$ as

$$x_1(t) \simeq C_{12}\, x_2(t). \quad t_1 < t < t_2$$

3. The value of C_{12} at which mean Square Error will be minimum is

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$

4. If two signals $x_1(t)$ & $x_2(t)$ are orthogonal in interval $t_1 < t < t_2$

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0.$$

5. If $g_1(t), g_2(t), \ldots g_n(t)$ are n mutually orthogonal signal in interval $t_1 < t < t_2$ then any function $x(t)$ can be approximated

$$x(t) \simeq C_1 g_1(t) + C_2 g_2(t) + \ldots + C_n g_n(t)$$
$$t_1 < t < t_2$$

6. $$C_r = \frac{\int_{t_1}^{t_2} x(t)\, g_r(t)\, dt}{\int_{t_1}^{t_2} g_r^2(t)\, dt}$$

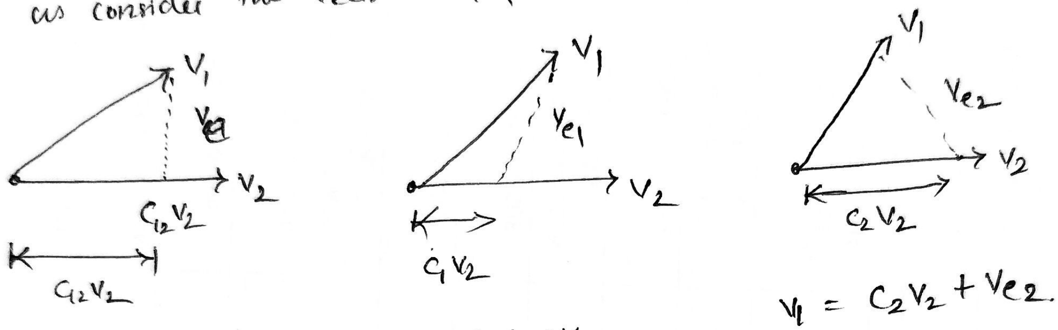
Analogy b/w vectors & signals :-

↑
(Similarities)

Vector :- Any physical quantity which has both magnitude & direction is called a "Vector".

Ex: Velocity, Acceleration, force, Ele fld, Etc.

Let us consider two vectors V_1 & V_2

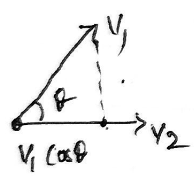


$V_1 = C_{12}V_2 + V_e$ $\quad$ $V_1 = C_1V_2 + V_{e1}$ $\quad$ $V_1 = C_2V_2 + V_{e2}$.

$$\boxed{\cos\theta = \frac{V_2}{V_1} \qquad V_2 = V_1\cos\theta}$$

where V_e, V_{e1}, V_{e2} are Error vectors.

The component of V_1 along V_2 = $C_{12}V_2$



$V_1\cos\theta = C_{12}V_2$

From dot product definition $V_1 \cdot V_2 = V_1V_2\cos\theta$

$$\frac{V_1 \cdot V_2}{V_2} = V_1\cos\theta$$

$$\frac{V_1 \cdot V_2}{V_2} = C_{12}V_2$$

$$C_{12} = \frac{V_1 \cdot V_2}{V_2^2}$$

If $C_{12} = 0$ then $0 = \frac{V_1 \cdot V_2}{V_2^2}$

$$\boxed{V_1 \cdot V_2 = 0}$$

i.e, two vectors V_1 & V_2 are mutually $\perp$lar to each other.

In other words there is no component of V_1 along V_2

Ex:- $V_1 = 3i + 3j + 3k$

$V_2 = i + j + k$

$V_1 = C_{12}V_2$

$$C_{12} = \frac{V_1 \cdot V_2}{V_2^2} = \frac{(3i+3j+3k)\cdot(i+j+k)}{(i+j+k)^2} = \frac{3+3+3}{(\sqrt{1^2+1^2+1^2})^2} = \frac{9}{3} = 3.$$

$$\boxed{C_{12} = 3}$$

$V_1 = 3V_2$

Signals :-

Let us consider two signals $x_1(t)$ & $x_2(t)$ interval $t_1 < t < t_2$
Approximate $x_1(t)$ in terms of $x_2(t)$ over a certain interval ($t_1 < t < t_2$) as follows.

$$x_1(t) = C_{12}\, x_2(t) \quad \text{for } t_1 < t < t_2.$$

To minimize the error b/w the actual function & the approximated function over the interval ($t_1 < t < t_2$)

The error function $x_e(t)$ is defined as:

$$x_e(t) = x_1(t) - C_{12}\, x_2(t)$$

To minimize the error $x_e(t)$ over the interval t_1 to t_2 by minimizing over this interval. i.e, by minimizing the average value of $x_e(t)$ over

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x_e(t)\,dt \quad \text{i.e.} \quad \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[x_1(t) - C_{12}\, x_2(t)\right]dt$$

It may give wrong result bcoz large +ve & -ve errors present in t_1 to t_2 that may cancel one another in the process of averaging and give the false indication that the error is zero.

To overcome this we choose to minimize the mean square of the error instead of the error itself.

Let us designated the average of squared error $x_e^2(t)$ by $\mathcal{E}$

$$\mathcal{E} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x_e^2(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[x_1(t) - C_{12}\, x_2(t)\right]^2 dt$$

The value of C_{12} which minimises $\mathcal{E}$ can be found from $\frac{d\mathcal{E}}{dC_{12}} = 0$

$$\frac{d\mathcal{E}}{dC_{12}} = \frac{d}{dC_{12}}\left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[x_1(t) - C_{12}\, x_2(t)\right]^2 dt\right] = 0.$$

$$\left.\left[x_1^2(t) + C_{12}^2 x_2^2(t) - 2\, x_1(t)\, C_{12}\, x_2(t)\right]\right] = 0$$

$$\frac{1}{t_2 - t_1}\left[\frac{d}{dC_{12}}\int_{t_1}^{t_2} \ldots\right.$$

$$\frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2}\frac{d}{dC_{12}}\left[x_1^2(t)\right]dt + \int_{t_1}^{t_2}\frac{d}{dC_{12}}\left(C_{12}^2\, x_2^2(t)\right)dt - 2\int_{t_1}^{t_2}\frac{d}{dC_{12}}\left[C_{12}\, x_1(t)\, x_2(t)\right]dt\right] = 0$$

$$\frac{1}{t_2 - t_1}\left[0 + \int_{t_1}^{t_2} 2C_{12}\, x_2^2(t)\,dt - 2\int_{t_1}^{t_2} x_1(t)\, x_2(t)\,dt\right] = 0$$

$$C_{12}\int_{t_1}^{t_2} x_2^2(t)\,dt = \int_{t_1}^{t_2} x_1(t)\, x_2(t)\,dt$$

$$\boxed{C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\,dt}{\int_{t_1}^{t_2} x_2^2(t)\,dt}}$$

By Analogy with vectors ; ~~we say that $x_1(t)$ has com~~

$$C_{12} = \frac{V_1 V_2}{V_2^2} \qquad C_{12} = \frac{\int_{t_1}^{t_2} x_1(t) x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$

if $C_{12} = 0$ then the signal $x_1(t)$ obtains no components of the signal $x_2(t)$

The two functions are orthogonal over the interval (t_1, t_2)

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0$$

A40401- SIGNALS, SYSTEMS AND STOCHASTIC PROCESSES

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Fouriel Series

A sgl is said to be a continuous-time signal if it is available at all instants of time. A real time naturally available signal is in the form of time domain.

The Analysis of a signal is far more convenient in the frequency domain. There are three important classes of transformation methods available for C.T. Sym they are.

1. Fourier Series
2. Fourier Transform
3. Laplace Transform

Fourier Series :- The representation of signals over a certain interval of time in terms of a linear combination of orthogonal functions is called Fourier Series.

It is applicable only for periodic signals. A periodic sgl is one which repeats itself at regular intervals of time periodically over $-\infty$ to ∞.

Three methods
- → Trigonometric form
- → Cosine form
- → Exponential form

F.S to Exist for a periodic sgl, it must satisfy certain Conditions

1. The function $x(t)$ must be a Single Valued function
2. $x(t)$ has only a finite number of maxima & minima } Dirichlet Condition
3. $x(t)$ has finite number of discontinuities
4. The $x(t)$ is absolute integrable over one period, that is } weak Dirichlet Condition

$$\int_0^T |x(t)|\, dt < \infty$$

Trigonometric form :-

A Sinusoidal sgl $x(t) = A \sin w_0 t$ is a periodic Signal with period $T = \frac{2\pi}{w_0}$.

Also sum of two sinusoidal is periodic provided that their frequencies are integral multiples of a fundamental frequency w_0.

The signal $x(t)$, a sum of sine & cosine functions whose frequencies are integral multiples of ω_0 is a periodic signal.

Let the signal $x(t)$ be

$$x(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + \dots + a_k \cos k\omega_0 t + b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + \dots + b_k \sin k\omega_0 t$$

$$x(t) = a_0 + \sum_{n=1}^{K} a_n \cos\omega_0 n t + b_n \sin\omega_0 n t.$$

For periodic signal must satisfy the condition $x(t) = x(t+T)$ for all 't'

$$x(t+T) = a_0 + \sum_{n=1}^{K} a_n \cos\omega_0 n(t+T) + b_n \sin\omega_0 n(t+T)$$

$$= a_0 + \sum_{n=1}^{K} a_n \cos\omega_0 n\left(t+\frac{2\pi}{\omega_0}\right) + b_n \sin\omega_0 n\left(t+\frac{2\pi}{\omega_0}\right)$$

$$= a_0 + \sum_{n=1}^{K} a_n \cos(\omega_0 n t + 2n\pi) + b_n \sin(\omega_0 n t + 2n\pi).$$

$$= a_0 + \sum_{n=1}^{K} (a_n \cos\omega_0 n t + b_n \sin\omega_0 n t)$$

$x(t+T) = x(t)$

The infinite series of sine & cosine terms of frequencies $0, \omega_0, 2\omega_0, \dots k\omega_0$ is known as trigonometric form of F.S & can be written as

$$x(t) = \sum_{n=0}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \quad \rightarrow (1)$$

a_n, b_n are constants. a_0 is called dc component.

$a_1 \cos\omega_0 t + b_1 \sin\omega_0 t$ is first harmonic
$a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t$ is second harmonic.
$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$ is nth harmonic

The constant $b_0 = 0$ becoz $\sin n\omega_0 t = 0$ for $n = 0$,

$a_0, a_1, a_2 \dots, b_0, b_1, b_2 \dots b_n$ are called Fourier Co-eff.

<u>To find a_0:-</u> integrate both sides of Eqn (1) over one period t_0, t_0+T at an arbitrary time t_0.

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 \int_{t_0}^{t_0+T} dt + \int_{t_0}^{t_0+T} \left[\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \right] dt \quad (3)$$

$$= a_0 \int_{t_0}^{t_0+T} dt + \sum_{n=1}^{\infty} \left[\int_{t_0}^{t_0+T} a_n \cos n\omega_0 t \, dt + \int_{t_0}^{t_0+T} b_n \sin n\omega_0 t \, dt \right]$$

$$= a_0 T + 0 + 0$$

$\left[\because \int_{t_0}^{t_0+T} a_n \cos n\omega_0 t = 0 \;\&\; \int_{t_0}^{t_0+T} b_n \sin n\omega_0 t = 0 \right.$ for net area under any sinusoidals over any complete period is zero.

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 T$$

$$\boxed{a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\,dt}$$

To find a_n :-

Multiply eqn (1) with $\cos m\omega_0 t$ & integrate over one period.

$$\int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t \, dt = a_0 \int_{t_0}^{t_0+T} \cos m\omega_0 t \, dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t \cdot \cos m\omega_0 t \, dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t \cos m\omega_0 t \, dt$$

first & third integrals are zero

if $m = n$

$$\int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t \, dt = a_n \frac{T}{2}$$

$$\boxed{a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t}$$

To find b_n :- multiply eqn (1) with $\sin m\omega_0 t$ & integrate over one period

$$\int_{t_0}^{t_0+T} x(t) \sin m\omega_0 t \, dt = a_0 \int_{t_0}^{t_0+T} \sin m\omega_0 t \, dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t \sin m\omega_0 t \, dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t \sin m\omega_0 t \, dt$$

$$\boxed{b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t}$$

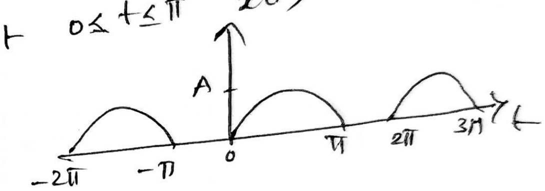
a_0, a_n, b_n are called trigonometric fourier series coeff

$$\int_{t_0}^{t_0+T} \cos n\omega_0 t \cos m\omega_0 t \, dt = \begin{cases} 0 & \text{for } m \neq n \\ T/2 & \text{for } m = n \neq 0 \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin n\omega_0 t \, \sin m\omega_0 t \, dt = \begin{cases} 0 & \text{for } m \neq n \\ T/2 & \text{for } m = n \neq 0. \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin n\omega_0 t \cos m\omega_0 t \, dt = 0 \quad \text{for all } m \,\&\, n.$$

Ex:- Find the fourier series expansion of the half wave rectified sine wave shown below



Sol $x(t) = \begin{cases} A \sin \omega t = A \sin \frac{2\pi}{2\pi} t = A \sin t & 0 \leq t \leq \pi \\ 0 & \pi \leq t \leq 2\pi. \end{cases}$

fundamental period $T = 2\pi$.

Exponential Fourier Series :-

The function $x(t)$ is expressed as a weighted sum of the complex Exponential functions.

A set of complex Exponential functions $e^{jn\omega_0 t}$, $n = 0, \pm 1, \pm 2, \pm 3 \ldots$ form a closed orthogonal set over an interval (t_0, t_0+T) where $T = 2\pi/\omega_0$. for any value of t_0 and therefore it can be used as a fourier Series.

Using Euler's Identity,

$$A_n \cos(n\omega_0 t + \theta_n) = A_n \left[\frac{e^{j(n\omega_0 t+\theta_n)} + e^{-j(n\omega_0 t + \theta_n)}}{2} \right] \rightarrow (1)$$

Cosine representation of $x(t) = A_0 + \sum_{n=1}^{\infty} A_n [\cos(n\omega_0 t + \theta_n)] \rightarrow (2)$

sub Eqn (2) in Eqn (1)

$$x(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} \left[e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)} \right]$$

$$= A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} \left[e^{jn\omega_0 t} e^{j\theta_n} + e^{-jn\omega_0 t} e^{-j\theta_n} \right]$$

$$= A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} e^{j\theta_n} \left[e^{jn\omega_0 t} \right] + \sum_{n=1}^{\infty} \frac{A_n}{2} e^{-j\theta_n} \left[e^{-jn\omega_0 t} \right]$$

Let $n = -k$ in second summation of above Eqn, we have

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(\frac{A_n}{2} e^{j\theta_n} \right) e^{j\omega_0 n t} + \sum_{k=-1}^{-\infty} \left(\frac{A_k}{2} e^{j\theta_k} \right) e^{jk\omega_0 t}$$

on Comparing above Eqns we get

$A_n = A_k$; $(-\theta_n) = \theta_k$ $\quad n > 0$, $k < 0$

$C_0 = A_0$ $\quad C_n = \frac{A_n}{2} e^{j\theta_n}$ $\quad n > 0$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} e^{j\theta_n} e^{j\omega_0 n t} + \sum_{n=-1}^{-\infty} \frac{A_n}{2} e^{j\theta_n} e^{jn\omega_0 t}$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}.$$

This is ~~called~~ known as Exponential form of Fourier Series. The above Eqn is called Synthesis Equation.

Exponential Series from Cosine Series is $\quad C_0 = A_0$

$$C_n = \frac{A_n}{2} e^{j\theta_n}$$

Coeff of Exponential series C_0, & C_n

Ans $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$ where $\omega_0 = \frac{2\pi}{T}$

multiplying both sides by $e^{-jk\omega_0 t}$ & integrating over one period, we get

$$\int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt = \int_{t_0}^{t_0+T} \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \right) e^{-jk\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} C_n \int_{t_0}^{t_0+T} e^{jn\omega_0 t} e^{-jk\omega_0 t} dt$$

we know $$\int_{t_0}^{t_0+T} e^{jn\omega_0 t} e^{-jk\omega_0 t} dt = \begin{cases} 0 & \text{for } k \neq n \\ T & \text{for } k = n \end{cases}$$

$$\int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt = T C_k$$

$$\boxed{C_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt}$$

(or) $$\boxed{C_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt}$$

$$C_0 = A_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$$

C_n exist only for discrete values of n. As represent a complex spectrum, it has both magnitude & phase spectra. The following points may be noted.

1. magnitude line spectrum is always an even function of n.
2. phase line spectrum is always an odd function of n.

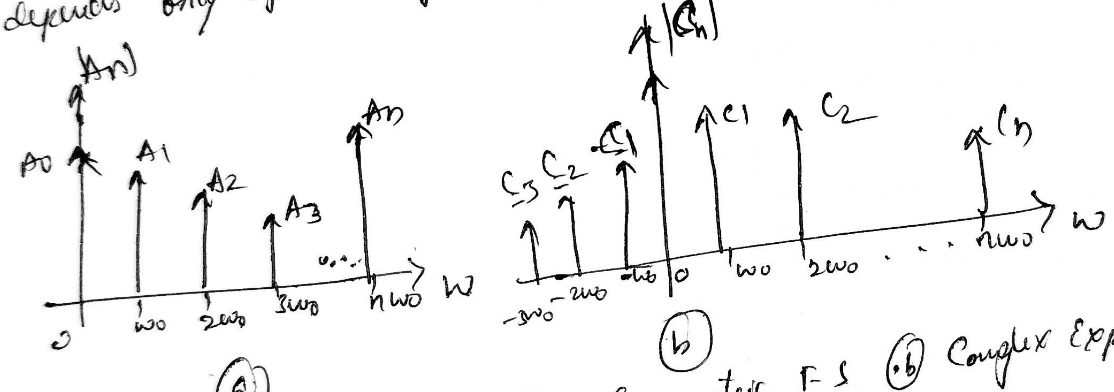
Fourier Spectrum :-

Fourier spectrum of a periodic signal x(t) is a plot of its Fourier Co-Eff versus frequency w.

it has two parts → Amplitude Spectrum – Amp of F. Coeff Vs Frequency
→ phase Spectrum – phase of F-Coeff Vs Frequency

Both plot together is called Fourier frequency spectra of x(t) this type of representation is called frequency domain representation.

F. Spectrum Exists only at discrete frequencies nw_0 where $n=0,1,2..$ Hence known as discrete Spectrum / Line Spectrum. The Envelope of the spectrum depends only upon the pulse shape, but not upon the period of repetition.



Complex frequency spectrum for ⓐ Trigonometric F-S ⓑ Complex Exponential F-S

$$C_n = |C_n| e^{j\theta_n}$$

$$C_{-n} = |C_n| e^{-j\theta_n}$$

$$|C_n| = |C_{-n}|$$

Even symmetry

The magnitude spectrum is symmetrical about the vertical axis passes through origin & phase spectrum is Asymmetric about the vertical axis passes.

odd symmetry

Properties :-

1. Linearity property

$x_1(t) \xleftrightarrow{FS} C_n$ & $x_2(t) \xleftrightarrow{FS} D_n$

$A x_1(t) + B x_2(t) \xleftrightarrow{FS} A C_n + B D_n$

2. Time shifting property

$x(t) \xleftrightarrow{FS} C_n$

$x(t - t_0) \xleftrightarrow{FS} e^{-jn\omega_0 t_0} C_n$

3. Time Reversal property

$x(t) \xleftrightarrow{FS} C_n$

$x(-t) \xleftrightarrow{FS} C_{-n}$

4. Time scaling property

$x(t) \xleftrightarrow{FS} C_n$

$x(\alpha t) \xleftrightarrow{F.S} C_n$ with fundamental frequency of $\alpha\omega_0$

5. Time differentiation property

$x(t) \xleftrightarrow{FS} C_n$

$\frac{d}{dx} x(t) \xleftrightarrow{FS} jn\omega_0 C_n$

6. Time integration property

$x(t) \xleftrightarrow{FS} C_n$

$\int_{-\infty}^{t} x(t)\, dt \xleftrightarrow{FS} \frac{C_n}{jn\omega_0}$ [if $C_0 = 0$)

7. Convolution property

$x_1(t) \xleftrightarrow{FS} C_n$ & $x_2(t) \xleftrightarrow{FS} D_n$

$x_1(t) * x_2(t) \xleftrightarrow{FS} T C_n D_n$

8. modulation | multiplication property

$x_1(t) \xleftrightarrow{FS} C_n$ & $x_2(t) \xleftrightarrow{FS} D_n$

$x_1(t)\, x_2(t) \xleftrightarrow{FS} \sum_{l=-\infty}^{\infty} C_l D_{n-l}$

9. Conjugation & Conjugate Symmetry property

$$x(t) \xleftrightarrow{FS} C_n$$

Conjugate $\quad x^*(t) \xleftrightarrow{FS} C^*_{-n}$ [for complex $x(t)$]

Conjugate Symmetry $C_{-n} = C_n^*$ [for real $x(t)$].

10. Parseval's theorem / property / relation

$$x_1(t) \xleftrightarrow{FS} C_n$$

$$x_2(t) \xleftrightarrow{FS} D_n$$

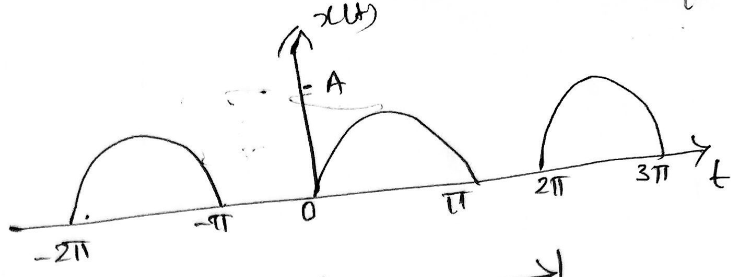
The parseval's relation states that

$$\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\, x_2^*(t)\, dt = \sum_{n=-\infty}^{\infty} C_n D_n^*$$

parseval's identity

$$\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2 \quad \text{if} \quad x_1(t) = x_2(t) = x(t)$$

(ii) Find the F.S expansion of the half wave rectified sine wave is shown in figure (10)



sol From figure

it is a periodic signal

with period $T = 2\pi$

$$x(t) = \begin{cases} A\sin\omega t = A\sin\frac{2\pi}{T}t = A\sin t & \text{for } 0 \le t \le \pi \\ 0 & \pi \le t \le 2\pi \end{cases}$$

fundamental time period $T = 2\pi$.

fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

Let

$$a_0 = \frac{1}{T}\int_0^{2\pi} x(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} A\sin t\,dt = \frac{A}{2\pi}\int_0^{\pi}\sin t\,dt$$

$$= \frac{A}{2\pi}[-\cos\pi + \cos 0]$$

$$= \frac{A}{2\pi}[-(-1)+1]$$

$$= \frac{2A}{2\pi} = \frac{A}{\pi}$$

$$\boxed{a_0 = \frac{A}{\pi}}$$

$$a_n = \frac{2}{T}\int_0^{2\pi} x(t)\cos nt\,dt$$

$$= \frac{2}{2\pi}\int_0^{\pi} A\sin t\cos nt\,dt = \frac{1}{2\pi}\int_0^{\pi}[\sin(t+nt) + \sin(t-nt)]\,dt$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(1+n)t}{1+n}\Big|_0^{\pi} - \frac{\cos(1-n)t}{1-n}\Big|_0^{\pi}\right]$$

$$= \frac{-1}{2\pi}\left[\frac{\cos(1+n)\pi - \cos 0}{1+n} + \frac{\cos(1-n)\pi - \cos 0}{1-n}\right]$$

$$= \frac{-1}{2\pi}\left[\frac{(-1)^{1+n} - 1}{1+n} + \frac{(-1)^{n+1} - 1}{1-n}\right]$$

For odd 'n'

$$= \frac{-1}{2\pi}\left[\frac{+1-1}{1+n} + \frac{1-1}{1-n}\right]$$

$$= 0$$

$a_n = 0$

$$b_n = \frac{2}{T}\int_0 x(t)\ \sin nt\ dt$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} A\ \sin t\ \sin nt\ dt$$

$$= \frac{A}{2\pi}\int_0^{\pi}\left[\cos(n-1)t - \cos(n+1)t\right]dt$$

$$= \frac{A}{2\pi}\left[\frac{\sin(n-1)t}{n-1}\Big|_0^{\pi} - \frac{\sin(n+1)t}{n+1}\Big|_0^{\pi}\right]$$

$$= \frac{A}{2\pi}\left[\frac{\sin(n-1)\pi - \sin 0}{(n-1)} - \frac{\sin(n+1)\pi - \sin 0}{(n+1)}\right]$$

This in zero for all values of n Except for n=1

~~for n=1~~

$$\cancel{b_n = \frac{A}{2\pi}\left[\frac{\sin 0 - \sin 0}{n-1} - \left[\frac{\sin 2\pi - \sin 0}{(n+1)}\right]\right]}$$

$$= \frac{A}{2\pi}\left[\frac{\pi(-1)^n - 0}{0} - \cancel{(n+1)}^n\right]$$

$$= \frac{A}{2\pi}$$

$$\cancel{b_n = \frac{A}{2\pi}\left[0\right.}$$

$$a_n = \frac{A}{2\pi}\left[\frac{-1-1}{1+n} + \frac{-1-1}{1-n}\right] = \frac{A}{2\pi}\left[\frac{2}{n+1} - \frac{2}{n-1}\right]$$

$$= \frac{-2A}{\pi(n^2-1)}.$$

$$a_n = -\frac{2A}{\pi(n^2-1)} \quad (\text{for Even } n)$$

∴ Trignometric fourier series is

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= \frac{A}{\pi} + \frac{A}{2\pi}\sin t - \sum_{n=\text{even}}^{\infty}\frac{2A}{\pi(n^2-1)}\cos nt$$

//

A40401- SIGNALS, SYSTEMS AND STOCHASTIC PROCESSES

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Fourier Transform

Defination :- it is a transformation technique which transforms signals from the continuous-time domain to the corresponding frequency domain & vice-versa. and which applies for both periodic as well as aperiodic signals. it is Extremely useful technique used in analysis of linear time-invariant (LTI) System.

Derivation of the F.T of Non periodic Signal from the F.S of periodic signal

Let $x(t)$ be a non-periodic function & $x_T(t)$ be periodic with period T. & Let their relation is given by

$$x(t) = \lim_{t \to \infty} x_T(t)$$

F.S of a periodic signal $x_T(t)$ is

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) e^{-jn\omega_0 t} dt \quad \& \quad \omega_0 = \frac{2\pi}{T}$$

$$T C_n = \int_{-T/2}^{T/2} x_T(t) e^{-jn\omega_0 t} dt$$

Let $n\omega_0 = \omega_0$ Let $T \to \infty$. As $T \to \infty$ we have $\omega_0 = 2\pi/T \to 0$ and discrete fourier Spectrum becomes Continuous ; the summation becomes integration $x_T(t) \to x(t)$.

thus at $T \to \infty$

$$T C_n = \lim_{T \to \infty} \int_{-T/2}^{T/2} x_T(t) e^{-j\omega t} dt = \int_{-T/2}^{T/2} \left[\lim_{T \to \infty} x_T(t)\right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= X(\omega)$$

$$\boxed{\therefore X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt} \quad \text{F.T}$$

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$\because$ $T C_n = X(\omega)$

$C_n = \frac{X(\omega)}{T}$

$$= \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{T} e^{jn\omega_0 t}$$

$T = \frac{2\pi}{\omega_0}$

$$= \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{2\pi} e^{jn\omega_0 t} \omega_0$$

$$x(t) = \lim_{T\to\infty} x_T(t) = \lim_{T\to\infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(\omega) e^{jn\omega_0 t} dt.$$

As $T \to \infty$; $\omega_0 = \frac{2\pi}{T}$ becomes infinitesimally small & may be represented by $d\omega$ Also the summation becomes integration

$$\cancel{x(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(\omega) e^{jn\omega_0 t} d\omega_0}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Here $x(t)$ is called inverse F.T of $X(\omega)$.

The equations

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$X(\omega)$ & $x(t)$ are known as F.T pair & can be denoted as

$$X(\omega) = F[x(t)]$$

$$x(t) = F^{-1}[X(\omega)]$$

Properties :-

1. Linearity property

$x_1(t) \xleftrightarrow{FT} X_1(\omega)$ $\quad$ $x_2(t) \xleftrightarrow{FT} X_2(\omega)$

$a x_1(t) + b x_2(t) \xleftrightarrow{FT} a X_1(\omega) + b X_2(\omega)$

2. Time shifting property.

$x(t) \xleftrightarrow{FT} X_1(\omega)$

$x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(\omega)$

3. Frequency shifting property

$x(t) \xleftrightarrow{FT} X(\omega)$

$e^{j\omega_0 t} x(t) \xleftrightarrow{FT} X(\omega - \omega_0)$

4. Time reversal property

$x(-t) \xleftrightarrow{FT} X(-\omega)$

5. Time Scaling property

$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

6. Differentiation in time domain property

$\frac{d}{dt} x(t) \xleftrightarrow{FT} j\omega X(\omega)$

7. Differentiation in frequency domain property

$t x(t) \xleftrightarrow{FT} j \frac{d}{dt} X(\omega)$

8. Time integration property

$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(j\omega)$ if $X(0) = 0$

9. Convolution property $\quad$ $x_1(t) \xleftrightarrow{FT} X_1(\omega)$ $\quad$ $x_2(t) \xleftrightarrow{FT} X_2(\omega)$

$x_1(t) * x_2(t) \xleftrightarrow{FT} X_1(\omega) X_2(\omega)$

10. Multiplication property

$x_1(t)\, x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

11. Duality (symmetry) property if $x(t) \xleftrightarrow{FT} X(\omega)$ then $X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$

12. Modulation property $x(t) \cos \omega_c t \xleftrightarrow{FT} \frac{1}{2}[X(\omega - \omega_c) + X(\omega + \omega_c)]$

13. Conjugation Property $x^*(t) \xleftrightarrow{FT} X^*(-\omega)$

14. Autocorrelation property $R(\tau) \xleftrightarrow{FT} |X(\omega)|^2$

15. $x_1(t) \xleftrightarrow{FT} X_1(\omega)$ $\quad x_2(t) \xleftrightarrow{FT} X_2(\omega)$ Parseval's relation

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, X_2^*(\omega)\, d\omega$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega .$$

16. Area under the curve

$$\int_{-\infty}^{\infty} x(t)\, dt = \frac{1}{2\pi} X(0)$$

$$\int_{-\infty}^{\infty} X(\omega)\, d\omega = X(0)$$

17. FT of complex & real function

$$x(t) = x_R(t) + j\, x_I(t)$$

$$x_R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [X_R(\omega) \cos\omega t - X_I(\omega) \sin\omega t]\, d\omega$$

$$x_I(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [X_R(\omega) \sin\omega t + X_I(\omega) \cos\omega t]\, d\omega$$

$$\cancel{X_I(\omega) = -\int^{\infty} x(t) \sin \omega t}$$

* FT of Standard Signals :-

Impulse function $\delta(t)$

$$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\Big|_{t=0} = 1$$

$$F[\delta(t)] = 1$$

$$\delta(t) \xleftrightarrow{FT} 1$$

$$\delta(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0}$$

* **Single-sided Real Exponential function $e^{-at}u(t)$ is**

Given $x(t) = e^{-at}u(t)$ $\quad u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \Bigg|_0^{\infty} = \frac{e^{-\infty} - e^{0}}{-(a+j\omega)} = \frac{1}{a+j\omega}$$

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$$

* **Unit step function is $u(t)$.**

Unit step function not absolutely integrable, we can't directly find its fourier transform so express the unit step function in terms of signum function

$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}$

$$u(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$X(\omega) = F[u(t)] \Rightarrow F\left[\frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)\right]$$

$$= \frac{1}{2} F[1] + \frac{1}{2} F[\mathrm{sgn}(t)]$$

$$F[u(t)] = \frac{1}{2}\left[2\pi\delta(\omega) + \frac{2}{j\omega}\right] = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\boxed{u(t) \xleftrightarrow{FT} \pi\delta(\omega) + \frac{1}{j\omega}}$$

* **Signum function [Sgn(t)] is**

This is not absolutely integrable so we cannot directly find its FT but we consider $e^{-a|t|}\,\mathrm{sgn}(t)$ & sub $a \to 0$, to obtain $\mathrm{Sgn}(t)$

$$\mathrm{Sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

$$x(t) = \mathrm{Sgn}(t) = \lim_{a \to 0} e^{-a|t|}\,\mathrm{Sgn}(t)$$

$$= \lim_{a \to 0} \left[e^{-at}u(t) - e^{at}u(-t)\right]$$

$$X(\omega) = \int_{-\infty}^{\infty} \lim_{a \to 0} \left[e^{-at}u(t) - e^{at}u(-t)\right] dt$$

$$= \lim_{a \to 0} \left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right] = \frac{1}{j\omega} - \frac{1}{-j\omega} = \frac{2}{j\omega}$$

Sampling :- The process of converting C.T sgl into D.T. sgl is called Sampling
After Sampling the Signal is defined at discrete instants of time & the time interval b/w two successive sampling instants is called Sampling period of sampling interval.
important factor in sampling is 'Sampling rate (f_s) must be kept sufficiently high so that the original signal $x(t)$ can be reconstructed from its samples.

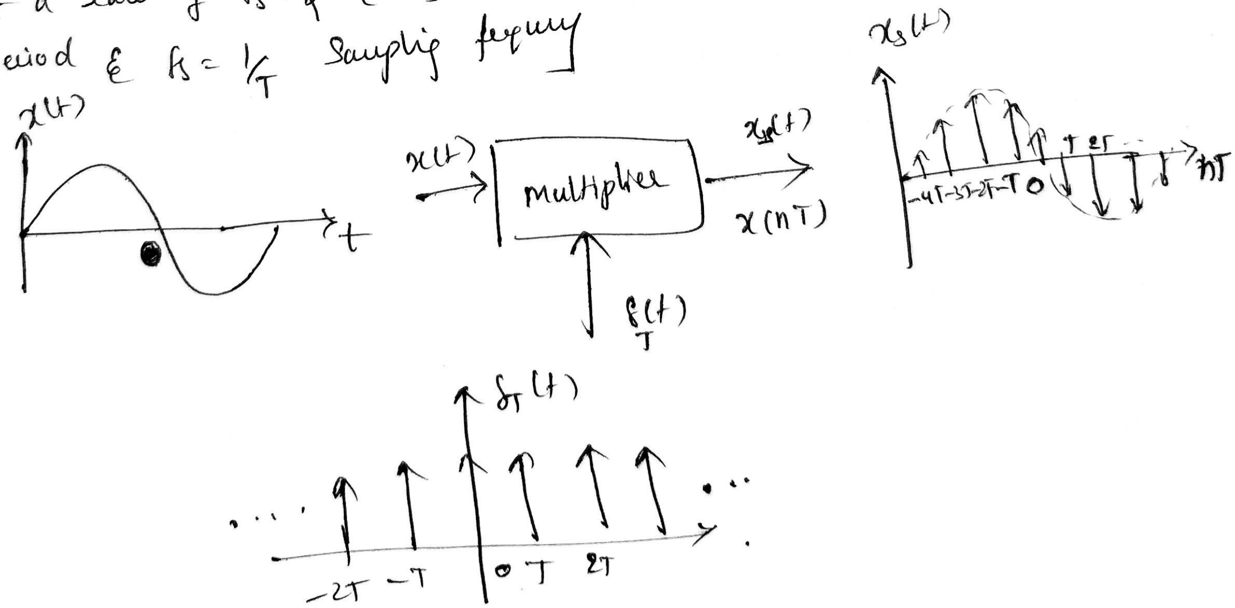
Sampling theorem :- This theorem states that 'a band limited signal $x(t)$ with $X(\omega) = 0$ for $|\omega| \geq \omega_m$ [i.e, $x(t) = 0$ for $f \geq f_m$] can be represented into a uniquely determined from its samples $x(nT)$ if the sampling frequency $f_s \geq 2f_m$.

where f_m - highest frequency component present in it
f_s - Sampling frequency

Sampling frequency must be atleast twice the highest frequency present in the Signal for signal recovery.

it is also called as uniform sampling theorem & low pass sampling theorem.
(uniform intervals $\frac{1}{2f_m}$ sec) ($X(f) = 0$ for all frequencies)

proof :- Let us consider a C.T band limited signal $x(t)$ as shown in figure. which has no spectral components above f_m cycles per sec
i.e $X(\omega) = 0$ for $\omega > \omega_m$. $\delta(t)$ is impulse train which samples at a rate of f_s Hz & $x_s(t)$ is the sampling signal. T is sampling period & $f_s = \frac{1}{T}$ Sampling frequency



Sampling operation

$$x_s(t) = x(t)\,\delta_T(t) \rightarrow (1)$$

where $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$

Exponential form of F.S of $\delta_T(t)$ is

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_s t}$$

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jn\omega_s t} = \frac{1}{T}$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_s t} = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \rightarrow (2)$$

sub (2) in (1)

$$x_s(t) = x(t)\,\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)\, e^{jn\omega_s t}.$$

Taking F.T on both sides

$$FT[x_s(t)] = FT\left[\frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)\, e^{jn\omega_s t}\right]$$

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} FT[x(t)\, e^{jn\omega_s t}]$$

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T}n\right)$$

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f-nf_s)$$

where $X_s(\omega)$ & $X_s(f)$ are spectrums of ~~impt~~ sampled signal.

Thus, the FT of a sampled signal is given by an infinite sum of shifted replicas of the FT of the original sgl.

Two basic conditions to be satisfied if $x(t)$ is to be recovered back from its samples

→ $x(t)$ should be band limited to some frequency ω_m.

→ The sampling frequency f_s should at least twice the band limited frequency ω_m (i.e $\omega_s \geq 2\omega_m$)

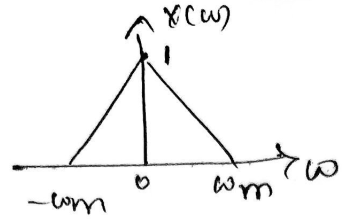
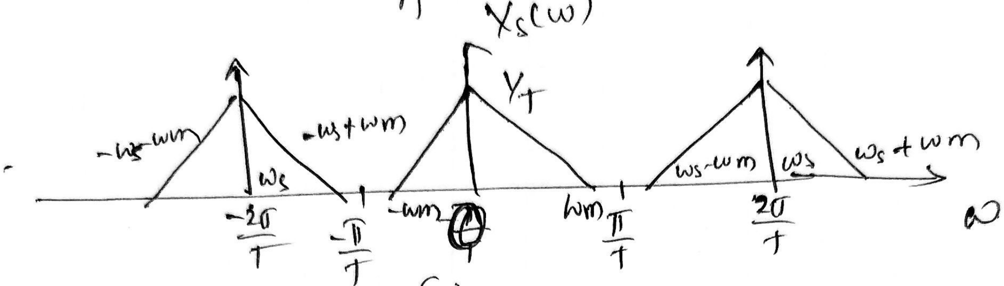
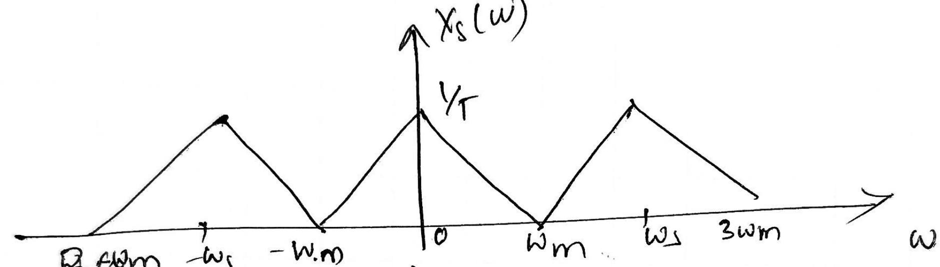


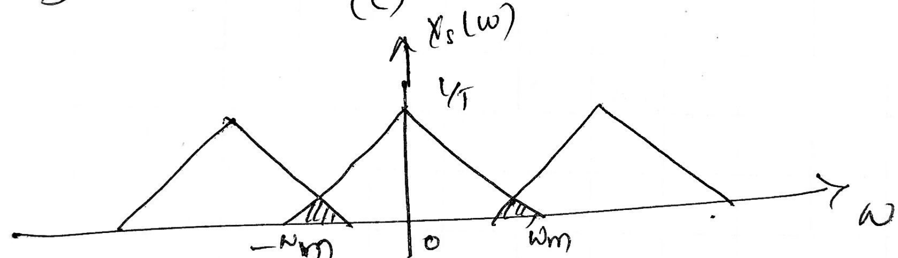
Fig (a)



(b)



(c)



D

Frequency Spectrum of C.T. Sgl $x(t)$

b, c, d frequency spectrum of Sampled signals $x_s(t)$ for $\left(\frac{\pi}{T}\right) > w_m$, $\left(\frac{\pi}{T}\right) = w_m$, $\left(\frac{\pi}{T}\right) < w_m$ respectively

From figure we observe

1. $w_s > 2w_m$ - The spectral replicates have a large separation b/w them known as guard band it makes the process of filtering much easier & effective. not have sharp cutoff.

2. $w_s = 2w_m$ - no separation b/w the replicates. So no guard band exist & $X(w)$ can be obtained from $X_s(w)$ by using only an ideal low pass filter with sharp cutoff.

3. $w_s < 2w_m$ - Low frequency Components are overlap on the high frequency components of $X(w)$, there is distortion & $X(w)$ can not be recovered from $X_s(w)$ by using filters. this type of distortions is called aliasing.

Nyquist rate $f_N = 2f_m$ Hz

Nyquist interval $= \frac{1}{f_N} = \frac{1}{2f_m}$ sec

Effects of Under sampling :-

When $\omega_s < 2\omega_m$ i.e, when the signal is under sampled $X(\omega)$, the spectrum of $x(t)$ is no longer replicated in $X_s(\omega)$ & and thus is no longer recoverable by LPF. This effect in which the individual terms in eqn $X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$ overlap is referred to as aliasing. This process of spectral overlap is also called frequency folding effect.

Aliasing can occur if either of the following conditions exists

1. The sgl is not band-limited to finite range
2. The sampling rate is too low.

To avoid aliasing, it should be ensured that

1. $x(t)$ is strictly band-limited (Using anti aliasing filter)
2. f_s is greater than $2f_m$.

Anti-Aliasing filter :-

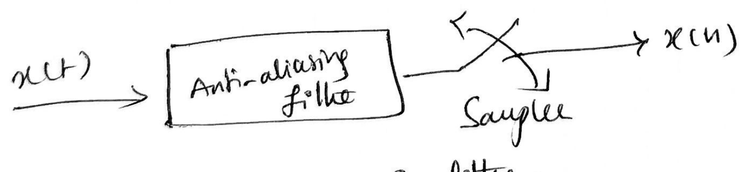


fig: Anti-Aliasing filter.

Sampling theorem states that a signal can be perfectly reconstructed from its samples only if it is a band limited. In practice no signal is strictly band limited. i.e sgl has low & high noise frequency. When such type of sgl sampled it may create aliasing error caused by high frequency sgl. It is aliasing. To avoid aliasing it is necessary to first band limit $x(t)$ to some appropriate frequency f_m using LPF such that most part of energy is retained. The LPF before sampling is referred as an anti-aliasing filter it is used primarily for preventing aliasing

Reconstruction :-

The process of obtaining the analog signal $x(t)$ from the sampled signal $x_s(t)$ is called data reconstruction or interpolation. We know that

$$x_s(t) = x(t)\,\delta_T(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

(or) $$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

Since $\delta(t-nT)$ is zero except at the sampling instants $t=nT$.
The reconstruction filter, which is assumed to be linear & time invariant has unit impulse response $h(t)$. The reconstruction filter output, $y(t)$ is given by the convolution.

$$y(t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT)\,\delta(\lambda - nT)\,h(t-\lambda)\,d\lambda$$

or upon changing the order of summation & integration

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(\lambda - nT)\,h(t-\lambda)\,d\lambda$$

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\,h(t-nT)$$

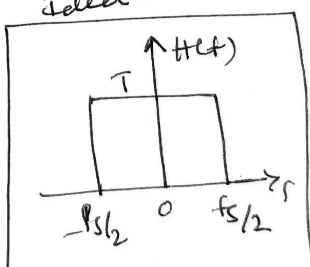
$x_s(t) \rightarrow$ [Impulse response $h(t)$] $\rightarrow y(t)$

Ideal Reconstruction filter :-

If $x(t)$ is sampled at the frequency exceeding the Nyquist rate & if the sampled signal $x_s(t)$ is passed through an ideal LPF, with B.W greater than f_m but less than $f_s - f_m$ & pass band amplitude response of T, the filter o/p is $x(t)$. We choose the B.W of the ideal reconstruction filter to be $0.5 f_s$.

$\sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT) \longrightarrow$

Ideal reconstruction filter



$\longrightarrow x(t)$

Reconstruction filtering

$$T\text{-}F \quad H(f) = \begin{cases} T & |f| < 0.5 f_s \\ 0 & \text{otherwise} \end{cases}$$

The impulse response of the ideal reconstruction filter is given by

$$h(t) = \int_{-f_s/2}^{f_s/2} T e^{j2\pi f t}\, df$$

which is

$$h(t) = T \left[\frac{e^{j2\pi f t}}{j2\pi t} \right]_{-f_s/2}^{f_s/2}$$

$$= \frac{T}{j2\pi t} \left[e^{j\pi f_s t} - e^{-j\pi f_s t} \right]$$

$$= \frac{1}{\pi f_s t} \left[\frac{e^{j\pi f_s t} - e^{-j\pi f_s t}}{2j} \right]$$

$$h(t) = \frac{\sin \pi f_s t}{\pi f_s t}$$

$$h(t) = \operatorname{sinc} f_s t \rightarrow (2)$$

o/p $\quad y(t) = x(t) = \sum_{n=-\infty}^{\infty} x(nT) \sin f_s(t-nT)$

A more convenient form for this expression, which is often referred to as a interpolation formula is

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \operatorname{sinc}\left(\frac{t}{T} - n\right)$$

The reconstruction filter is non-causal & impulse response is not limited. So it cannot be used for real time applications.

Other practical methods are

1. Zero order hold
2. First order hold
3. Linear interpolator

* Zero order hold :-

Most widely used interpolator is the (ZOH). The ZOH reconstructs the CT sgl from its samples by holding the given sample for an interval until the next sample is received.

$$\tilde{x}_a(t) = x(n) \quad \text{for} \quad nT \le n \le (n+1)T$$

$$\tilde{x}_a(t) = x(0) \quad \text{for} \quad 0 \le t \le T$$

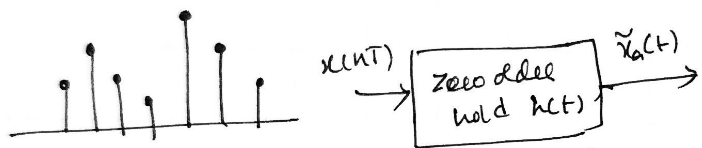
$$= x(T) \qquad T \le t \le 2T$$

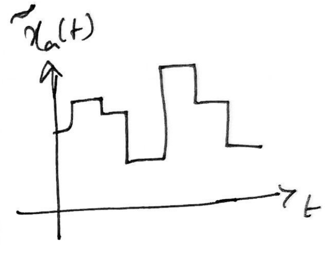
$$= x(2T) \qquad 2T \le t \le 3T$$

$$\vdots$$

Impulse response of a zero order hold is given by

$$h(t) = 1 \quad 0 \le t \le T$$
$$= 0 \quad \text{otherwise}$$





* Transfer function of a zero order hold :-

The o/p of ZOH is convolution of its i/p & its impulse response

h(t) i.e. $\tilde{x}_a(t) = x(nT) * h(t)$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, h(t-nT).$$

For zero order hold

$$h(t) = u(t) - u(t-T)$$

$$h(t-nT) = u(t-nT) - u(t-(n+1)T)$$

$$\tilde{x}_a(t) = \sum_{n=-\infty}^{\infty} x(nT)\,[u(t-nT) - u[t-(n+1)T]]$$

Taking Laplace Transform on both sides.

$$L[\tilde{x}_a(t)] = \left[\frac{1-e^{-Ts}}{s}\right] X^*(s)$$

T.F of zero order hold $= \dfrac{\tilde{X}_a(s)}{X^*(s)} = \dfrac{1-e^{-Ts}}{s}$

Since the o/p of the ZOH consists of steps, it consists of higher order harmonics. To remove these harmonics, the o/p of ZOH is applied to an LPF. This filter tends to smooth the corners on the step approximations created by ZOH. ∴ Hence filter is often called a smoothing filter.

(P) Determine the Nyquist rate corresponding to each of the following signals.

(a) $x(t) = 1 + \cos 2000\pi t + \sin 4000\pi t$

(b) $x(t) = \sin(4000\pi t)/\pi t$

(c) $x(t) = \left(\dfrac{\sin(4000\pi t)}{\pi t}\right)^2$

$x(t) = 1 + \cos 2000\pi t + \sin 4000\pi t$.

Sol: (a) Given $x(t) = 1 + \cos 2000\pi t$

highest frequency component $2\pi f_m = 4000\pi$

$f_m = \dfrac{\cancel{4000}^{2000}\,\cancel{\pi}}{\cancel{2\pi}} = 2000\ Hz$

Nyquist rate $f_N = 2f_m = 2 \times 2000\ Hz = 4000\ Hz$

Nyquist interval $= \dfrac{1}{f_N} = \dfrac{1}{4000\ Hz} = 0.25\ msec$

(b) highest frequency component is $\omega_m = 4000\pi$

$2\cancel{\pi} f_m = 4000\cancel{\pi}$

$f_m = 2000\ Hz$

Nyquist rate $f_N = 2f_m = 4000\ Hz$

Nyquist interval $= \dfrac{1}{f_N} = \dfrac{1}{2f_m} = \dfrac{1}{4000} = 0.25\ ms$

(c) highest frequency component $\omega_m = 8000\pi$

$f_m = 4000\ Hz$

$x(t) = \left(\dfrac{\sin 4000\pi t}{\pi t}\right)\left(\dfrac{\sin 4000\pi t}{\pi t}\right) = \dfrac{\sin^2 4000\pi t}{(\pi t)^2} = \dfrac{1 - \cos 8000\pi t}{(\pi t)^2}$

Nyquist rate $f_N = 2f_m = 8000\ Hz$

Nyquist interval $= \dfrac{1}{f_N} = \dfrac{1}{2f_m} = \dfrac{1}{8000\ Hz} = 0.125\ ms.$

(d) $x(t) = \text{sinc}(80\pi t)\ \text{sinc}(120\pi t)$

$\omega_{m1} = 80\pi$ $\qquad \omega_{m2} = 120\pi$

$\omega_m = \omega_{m1} + \omega_{m2} = 200\pi$ rad/sec

$f_m = \dfrac{\omega_m}{2\pi} = \dfrac{200\pi}{2\pi} = 100\ \text{Hz}$

Nyquist sampling rate $f_N = 2f_m = 200\ \text{Hz}$

Nyquist interval $= \dfrac{1}{f_N} = \dfrac{1}{2f_m} = \dfrac{1}{200\ \text{Hz}} = 5\ \text{ms}$

Laplace Transforms :-

A Linear time invariant (LTI) system is described by differential Equations. The response of a system for a given i/p is obtained by solving the differential Equations is quite tedious and time consuming compared to the solution of algebraic Equations. So the Laplace transform is used to solve the differential Equations.

Laplace transform is powerful mathematical tool used to convert the differential Equations into algebraic Equations. It is a simple & systematic method which provides the complete solution in one stroke by taking into account the initial conditions in a natural way at the beginning of the process itself.

The Laplace transform of a time function $x(t)$ is defined as

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt.$$

where s is complex variable & Equal to $s = \sigma + j\omega$. the operator L. is called the LT operator which transforms the time domain function $x(t)$ into frequency domain function $X(s)$.

$$ILT[X(s)] = \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} x(s) e^{st} ds.$$

Unilateral / one sided Laplace Transform $L[x(t)] = X(s) = \int_{0}^{\infty} x(t) e^{-st} dt$

bilateral / two sided Laplace Transform $L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$$\boxed{x(t) \xleftrightarrow{LT} X(s)}$$ Laplace Transform pair.

* Existence of Laplace Transform :-

The necessary and sufficient conditions for the existence of the Laplace transform are.

1. $x(t)$ should be continuous in the given closed interval.
2. $x(t) e^{-\sigma t}$ must be absolutely integrable.

$X(s)$ Exists only if $\int_{-\infty}^{\infty} |x(t) e^{-\sigma t}| dt < \infty$

& only if $\underset{t \to \infty}{dt}\ e^{-st} x(t) = 0$

The range of σ for which the LT converges is known as the region of convergence (ROC).

A40401- SIGNALS, SYSTEMS AND STOCHASTIC PROCESSES

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UNIT - III

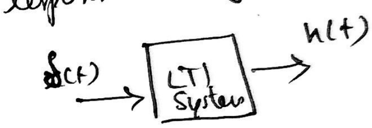
Signal Transmission through Linear Systems

Linear System :- which obeys principle of superposition & principle of homogeneity are called a linear system.

Superposition – weighted sum of i/p sgls = corresponding weighted sum of o/ps of the system to each of the i/p sgl.

Homogeneity – $x_1(t)$ produces $y_1(t)$ then $a x_1(t)$ produce $a y_1(t)$.

Impulse Response :- Impulse response is the o/p of system for a unit impulse i/p. For understanding system behaviour, unit impulse response is very important.



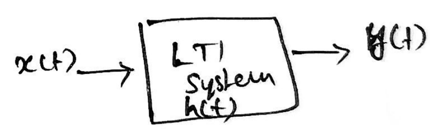
if $x(t)$ is i/p $= \delta(t)$, o/p $y(t) = h(t)$.

$$L[\delta(t)] = 1 \;\&\; F[\delta(t)] = 1$$

Transfer function $H(s)$ of an LTI system is known in s-domain. the impulse response of sym can be found by taking inverse L.T of $H(s)$

$$\boxed{h(t) = L^{-1}[H(s)]}$$

Response of a linear system :-



$y(t)$ is obtained by convoluting i/p $x(t)$ with impulse response of the system $h(t)$.

$$y(t) = h(t) * x(t) = x(t) * h(t).$$

→ If a non causal signal is applied to a non-causal system then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

→ If a causal sgl is applied to a non-causal system then

$$y(t) = \int_{-\infty}^{t} h(\tau)\, x(t-\tau)\, d\tau = \int_{0}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

→ If a non-causal sgl is applied to a causal system, then

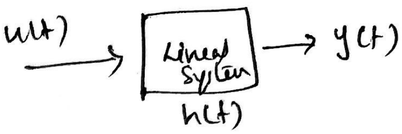
$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

→ If a causal signal is applied to a causal system, then

$$y(t) = \int_{0}^{t} h(\tau)\, x(t-\tau)\, d\tau = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau.$$

Step response :-

$$s(t) = h(t) * u(t)$$



For system is non-causal system

$$s(t) = \int_{-\infty}^{t} h(\tau)\, u(t-\tau)\, d\tau = \int_{-\infty}^{t} h(\tau)\, d\tau$$

For system is causal, then $s(t) = \int_{0}^{t} h(\tau)\, d\tau$

∴ The ~~imp.~~ unit step response of a C.T LTI system is running integral of its impulse response

Linear Time invariant (LTI) – system is Linear as well as Time invariant

$$y(t, T) = y(t-T)$$

o/p due to delayed i/p = delayed o/p

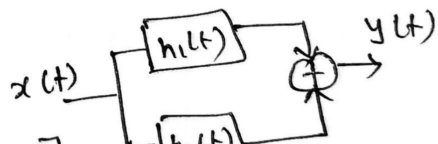
Linear Time variant (LTV) – system is Linear & Time variant

$$y(t, T) \neq y(t-T)$$

Properties of LTI system :-

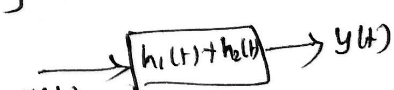
1. Commutative property

$$x(t) * h(t) = h(t) * x(t)$$



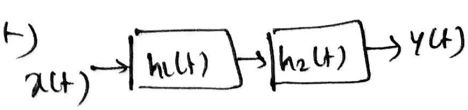
2. Distributive property

$$x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$$



3. Associative property

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$



4. Memoryless & Memory system

Memoryless $h(t) = 0$ for $t \neq 0$ then $y(t) = K\,x(t)$ K-Constant

then $h(t) = K\delta(t)$

Memory $h(t) \neq 0$ for $t \neq 0$

if $K = 1$ then $h(t) = \delta(t)$

Causality :-

→ For causal LTI system $h(t) = 0$ for $t < 0$

For Noncausal sgl o/p of LTI System is

$$y(t) = \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

For Causal sgl o/p of LTI System is

$$y(t) = \int_{0}^{t} h(\tau)\, x(t-\tau)\, d\tau = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

→ For Non causal LTI systems $h(t) \neq 0$ for $t < 0$.

Stability :-

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Invertibility :-

$$h(t) * h_1(t) = \delta(t)$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow \boxed{h_1(t)} \rightarrow x(t) \quad \equiv \quad x(t) \rightarrow \boxed{\begin{array}{c}\text{Identity System} \\ \delta(t)\end{array}} \rightarrow x(t)$$

Unit step Response :-

The unit step response $s(t)$ of an LTI system is the o/p of the system for a unit step i/p $u(t)$

The response can be obtained by convolving the i/p unit step $u(t)$ with the impulse response $h(t)$ of the system.

$$s(t) = h(t) * u(t) = u(t) * h(t)$$

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau .$$

∴ Unit step response of a C.T. LTI System is the running integral of its impulse response

$$h(t) = \frac{d\,s(t)}{dt}$$

The unit impulse response is the first derivative of the unit step response.

Transfer function of an LTI system :-

A C.T system is shown in figure below

i/p sgl $\qquad\qquad$ o/p sgl

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{system}\\ h(t)\end{matrix}} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

Apply L.T on both side

$$Y(s) = X(s)\, H(s)$$

$$\frac{Y(s)}{X(s)} = H(s) = \frac{Y(s)}{X(s)}.$$

& $H(s) = L[h(t)]$ & $h(t) = L^{-1}[H(s)]$

The T.F of a C.T LTI system may be defined using Fourier transform of L.T. The T.F is defined only under zero initial conditions.

The T.F of a LTI System $H(\omega)$ is defined as the ratio of the F.T of the o/p sgl to the F.T of the i/p sgl when the initial conditions are zero.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}.$$

$$H(\omega) = |H(\omega)|\, e^{j\theta(\omega)}$$

The T.F in frequency domain $H(\omega)$ is also called the frequency response of the system. The frequency response is amplitude response plus phase response.

$H(\omega)$ = Amp response of the system

$\theta(\omega) = \angle H(\omega)$ = phase response of the system

$$Y(\omega) = H(\omega)\, X(\omega)$$

$$|Y(\omega)| = |H(\omega)|\,|X(\omega)|$$

$$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$$

$H(\omega)$ has conjugate symmetry property

$$H(-\omega) \equiv H^{*}(\omega)$$

$$|H(-\omega)| = |H(\omega)|$$

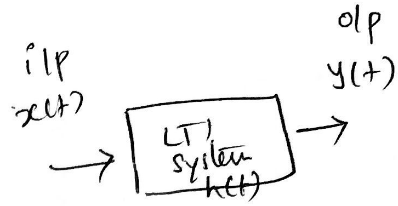
$$\angle H(-\omega) = -\angle H(\omega)$$

The impulse response $h(t)$ of a system is the inverse F.T of its T.F.

$$H(\omega) = F[h(t)]$$

$$h(t) = F^{-1}(H(\omega))$$

Filter Characteristics of linear system :-



$y(t) = x(t) * h(t)$

Apply F.T. O.B.S.

$Y(\omega) = X(\omega)\, H(\omega)$

Here $H(\omega)$ is T.F of sym function of the system

$X(\omega)$ - Spectral density function of the i/p sgl $x(t)$

$Y(\omega)$ - Spectral density function of the response signal $y(t)$

∴ The sym modifies the spectral density function of the i/p.

So it acts ~~like~~ as a kind of filter for various frequency components.

Some freq components are boosted in strength i.e. they are Amplified

Some freq components are weekend in strength i.e. they are attenuated

Some may remain unaffected.

Ily ~~Each~~ Each frequency Component suffers a different amount of phase shift in the process of transmission system.

modifies the spectral ~~characteristics~~ density function of the i/p according to its filter characteristics. The modification is carried out according to the transfer function $H(\omega)$ which represents the response of the System to various frequency components.

$H(\omega)$ acts as a weighting function | Spectral shaping function to the different frequency components in the i/p sgl. An LTI system therefore acts as a filter. A filter is basically a frequency selective n/w.

→ Some LTI sym allow the transmission of only Low freq. comp & stop high freq — LPF

→ Some LTI sym allow the transmission of only high freq & stop low freq — HPF.

→ Some LTI sym allows particular band of freq & ~~rejects~~ stops other freq — BPF

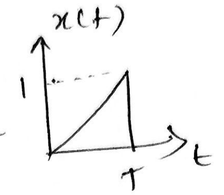
→ Some LTI sym ~~allows~~ reject particular band of freq & stop other freq comp — BRF.

Distortionless Transmission through system :-

The change of shape of the signal when it is transmitted through a system is called distortion.

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

Transmission of a sgl through a system is said to be distortionless if the o/p is an exact replica of the i/p sgl. This replica may have different magnitude & also it may have different time delay.



$y(t) = K x(t - t_d)$

Distortionless Sym

Mathematically $y(t) = K x(t - t_d)$

Where K is a constant representing the change in amplitude & t_d is delay time. A distortionless system and typical i/p & o/p waveforms are shown in figure.

Taking F.T. O.B.S.

$$Y(\omega) = K e^{-j\omega t_d} X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = K e^{-j\omega t_d}$$

$$\boxed{H(\omega) = K e^{-j\omega t_d}}$$

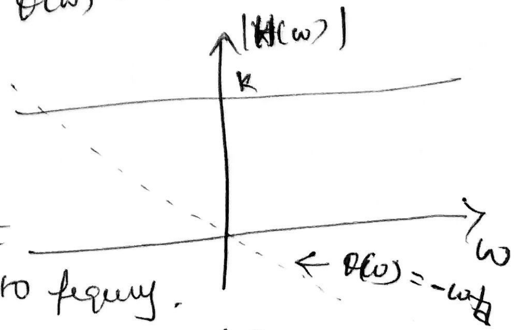
By taking inverse F.T. the corresponding impulse response must be

$$\boxed{h(t) = K\,\delta(t - t_d)}$$

$|H(\omega)| = K$; $\theta(\omega) = \angle H(\omega) = -\omega t_d$ it varies linearly with frequency

$\theta(\omega) = n\pi - \omega t_d$ (n integer)

So for distortionless transmission of a signal through a system, the magnitude $|H(\omega)|$ should be a constant. i.e all the frequency components of the i/p sgl must undergo the same amount of amplification or attenuation ie, the system B.W is infinite and the phase spectrum should be proportional to frequency.



But in practice - no system can have ∞ B.W so distortionless conditions never met.

Linear phase system :- For distortionless transmission, there should not be any phase distortion. No phase distortion means the phase should be linear. So for distortionless transmission, the system must be of linear phase type.

For linear phase systems; the impulse response is symmetrical about $t = t_d$ this can be proved as follows.

For linear phase system $H(\omega) = |H(\omega)| e^{-j\omega t_d}$

The impulse response of such a system is obtained by finding the inverse F.T.

$$h(t) = F^{-1}[H(\omega)] = F^{-1}\left[|H(\omega)| e^{-j\omega t_d} \right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)| e^{-j\omega t_d} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)| e^{j\omega(t-t_d)} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{0} |H(\omega)| e^{j\omega(t-t_d)} d\omega + \int_{0}^{\infty} |H(\omega)| e^{j\omega(t-t_d)} d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{0}^{\infty} |H(\omega)| e^{-j\omega(t-t_d)} d\omega + \int_{0}^{\infty} |H(\omega)| e^{j\omega(t-t_d)} d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{0}^{\infty} |H(\omega)| \left[e^{-j\omega(t-t_d)} + e^{j\omega(t-t_d)} \right] d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{0}^{\infty} |H(\omega)| 2\cos \omega(t-t_d) d\omega \right]$$

$$h(t) \cancel{h(t_d+t)} = \frac{1}{\pi} \int_{0}^{\infty} |H(\omega)| \cos \omega (t-t_d) d\omega$$

$$h(t_d + t) = \frac{1}{\pi} \int_{0}^{\infty} |H(\omega)| \cos \omega t \, d\omega$$

$$h(t_d - t) = \frac{1}{\pi} \int_{0}^{\infty} |H(\omega)| \cos \omega t \, d\omega$$

$$h(t_d + t) = h(t_d - t)$$

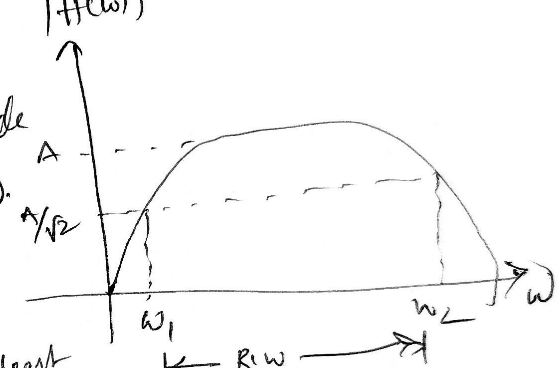
$\therefore$ $h(t)$ for linear phase system is symmetrical about t_d, and it is non-causal (non-zero for $t<0$)

$$h_{max} = h(t_d) = \frac{1}{\pi} \int_{0}^{\infty} |H(\omega)| d\omega$$

Signal Bandwidth :-

Spectral Components of a Signal Extend from $-\infty$ to ∞. Any practical signal has finite Energy. As a result, the spectral components approach zero as ω tends to ∞. Therefore, we neglect the components which have negligible Energy & select only a band of spectral components which have most of the signal energy. This band of frequency components which have most of the signal energy is known as the B.W of frequencies that contain most of the signal energy of the signal. Normally, the band is selected such that it contains around 90% of total Energy depending on the precision.

System Bandwidth :-



The B.W of a system is defined as the range of frequencies over which the magnitude $|H(\omega)|$ remains within $1/\sqrt{2}$ times (with in 3dB) of its value at midband.

The Band limited Sgl can be transmitted without distortion if the system B.W is atleast Equal to the signal B.W.

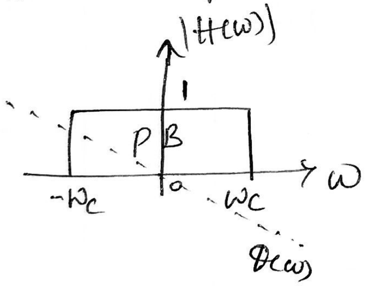
Ideal filter characteristics :-

A filter is a frequency selective n/w. It allows transmission of signal of certain frequencies with no attenuation or with very little attenuation, and it rejects or heavily attenuates signals of all other frequencies.

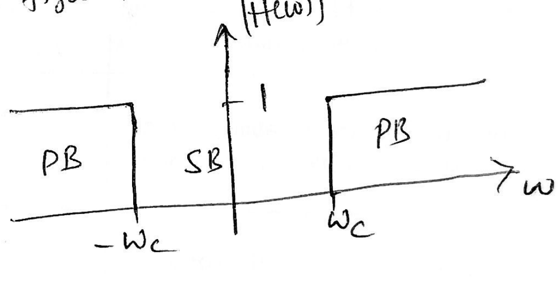
An ideal filter has very sharp cutoff characteristics, and it passes signals of certain specified band of frequencies exactly and totally rejects signals of frequencies outside this band. its phase spectrum is linear.

Filter are usually classified according to their frequency response characteristics as low-pass filter (LPF), high-pass filter (HPF) band-pass filter (BPF) and band-elimination or band-stop or band-reject filter (BEF, BSF, BRF).

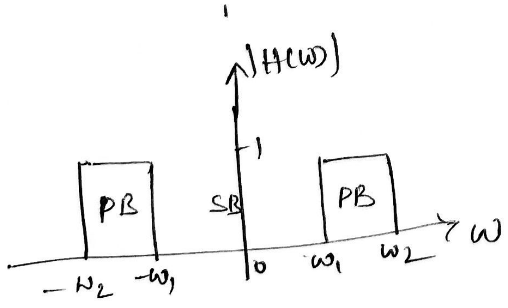
Ideal versions of these filters are described below and their magnitude responses are shown in figure



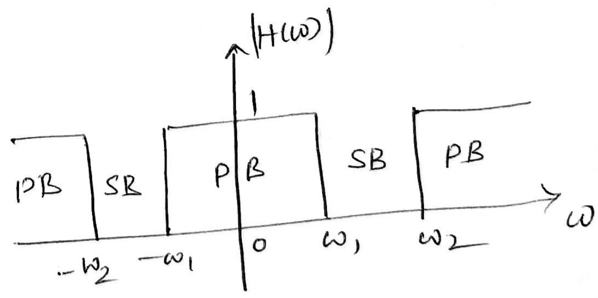
LPF



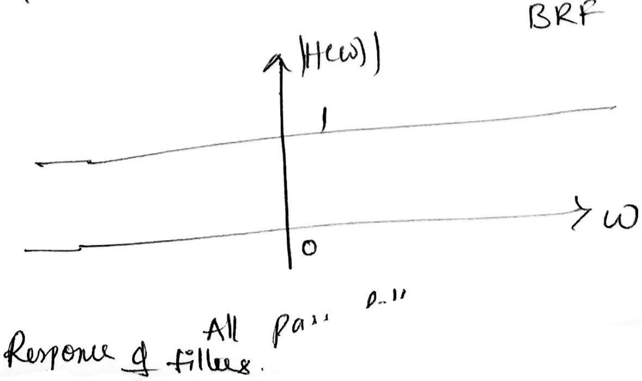
HPF



BPF



BRF



All pass

fig- Frequency Response of filters.

Ideal LPF :- An ideal LPF transmits, without any distortion, all of the signals of frequencies below a certain 10 frequency w_c radians per second. The signals of frequencies above w_c rad/sec are completely attenuated.

TF of ideal LPF

$$|H(w)| = \begin{cases} 1, & |w| < w_c \\ 0, & |w| > w_c \end{cases}$$

The w_c — cut-off frequency

phase function $\theta(w)$ is distortionless transmission is $-wt_d$.

Ideal HPF :- An ideal HPF transmits, without any distortion, all of the signals of frequencies above a certain frequency w_c rad/sec and attenuates completely the signals of frequencies below w_c rad/sec.

$$\text{T.F} \quad |H(w)| = \begin{cases} 0, & |w| < w_c \\ 1, & |w| > w_c \end{cases}$$

Ideal BPF :- An ideal BPF transmit, without any distortion, all of the signals of frequencies within a certain frequency band $(w_2 - w_1)$ rad/sec and attenuates completely the signals of frequencies outside this band $(w_2 - w_1)$ is the B.W of band-pass filter. The corresponding phase function for distortionless transmission is $-wt_d$

$$\text{T.F} \quad |H(w)| = \begin{cases} 1, & |w_1| < w < |w_2| \\ 0, & w < |w_1| \ \& \ w > |w_2| \end{cases}$$

Ideal BRF :- An ideal band rejection filter totally all of the signals of frequencies within a certain freq band $(w_2 - w_1)$ rad/sec and transmits without any distortion all signals of frequencies outside this band $(w_2 - w_1)$ is the rejection band. The corresponding phase function for distortionless transmission is $-wt_d$.

$$\text{T.F} \quad |H(w)| = \begin{cases} 0, & |w_1| < w < |w_2| \\ 1, & w < |w_1| \ \& \ w > |w_2| \end{cases}$$

All pass filter :- All pass filter transmits signals of all frequencies without any distortion, that is, its B.W is ∞ &

T.F $|H(w)| = 1$ for all frequencies.

The corresponding phase function for distortionless transmission is $-wt_d$

All ideal filter are non-causal systems. Hence none of them is physically realizable

* Causality and paley-wiener criterion for physical realization :-

A system is said to be causal if it does not produce an o/p before the input is applied. For LTI system to be causal, the condition to be satisfied is its impulse response must be zero for t less than zero. ie, $h(t) = 0$ for $t < 0$.

physical realization implies that it is physically possible to construct that system in real time.

physically realizable system cannot have a response before i/p is applied. This is known as causality condition.

$h(t)$ must be causal for a physically realizable syms.

In frequency domain, this criterion implies that a necessary & sufficient condition for a magnitude function $H(\omega)$ to be physically realizable is

$$\int_{-\infty}^{\infty} \frac{|\ln H(\omega)|}{1+\omega^2} \, d\omega < \infty .$$

The magnitude function $|H(\omega)|$ must, however, be square integrable before the paley-wiener condition is valid. that is

$$\int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega < \infty .$$

A sym whose magnitude function violates the paley-wiener criterion has a non-causal impulse response.

→ Following conclusions can be drawn from paley-wiener criterion

1. $|H(\omega)|$ may be zero at some discrete frequencies (but it cannot be zero over a finite band of frequencies (it make $\int_{-\infty}^{\infty} [H(\omega)]^2 = \infty$)

2. $|H(\omega)|$ cannot fall off to zero faster than a function of exponential order.

Relationship b/w B.W & Rise time :-

T.F of LPF : $H(\omega) = |H(\omega)| e^{-j\omega t_d}$

where $|H(\omega)| = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$

ω_c - cut off frequency

$H(\omega) = e^{-j\omega t_d}$. $-\omega_c \le \omega \le \omega_c$ i.e $|\omega| \le \omega_c$

$= 0$ $\omega > |\omega_c|$

The impulse response h(t) of the LPF is obtained by taking the inverse fourier Transform of H(w).

$$h(t) = F^{-1}[H(\omega)] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega t_d} e^{j\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_d)} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{+j\omega(t-t_d)}}{j\omega(t-t_d)} \right]_{-\omega_c}^{\omega_c}$$

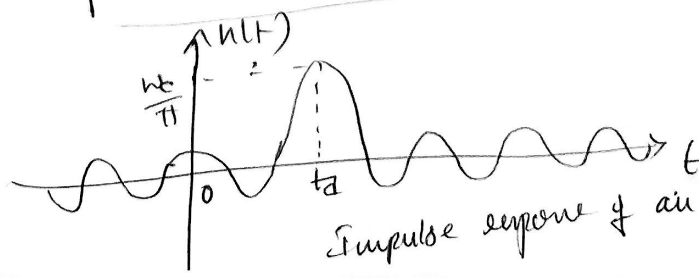
$$= \frac{1}{\pi(t-t_d)} \left[\frac{e^{+j\omega_c(t-t_d)} - e^{-j\omega_c(t-t_d)}}{2j} \right]$$

$$= \frac{1}{\pi(t-t_d)} \sin \omega_c (t-t_d)$$

$$= \frac{1}{\pi} \frac{\sin \omega_c (t-t_d)}{(t-t_d)} \times \frac{\omega_c}{\omega_c}$$

$$= \frac{\omega_c}{\pi} \frac{\sin \omega_c (t-t_d)}{\omega_c (t-t_d)}$$

$$\boxed{h(t) = \frac{\omega_c}{\pi} \operatorname{Sinc} \omega_c (t-t_d)}$$



Impulse response of an ideal LPF

Step response $y(t) = h(t) * u(t)$

$$= \int_{-\infty}^{t} h(t)\, d\tau$$

$$= \int_{-\infty}^{t} \frac{w_c}{\pi} \frac{\sin w_c(t-t_d)}{w_c(t-t_d)}\, d\tau$$

$$x = w_c(t-t_d)$$

$$dx = dw_c\, dt \quad \text{or} \quad d\tau = \frac{dx}{w_c}$$

$$y(t) = \int_{-\infty}^{w_c(t-t_d)} \frac{w_c}{\pi} \frac{\sin x}{x} \frac{dx}{w_c} = \frac{1}{\pi} \int_{-\infty}^{w_c(t-t_d)} \frac{\sin x}{x}\, dx$$

$$= \frac{1}{\pi} \Big[Si(x) \Big]_{-\infty}^{w_c(t-t_d)}$$

where Si is sine integral function

properties 1- $Si(x)$ is an odd function that is $Si(-x) = -Si(x)$

2- $Si(0) = 0$

3- $Si(\infty) = \pi/2$ & $Si(-\infty) = \frac{-\pi}{2}$.

The step response can be expressed as

$$y(t) = \frac{1}{\pi} \left[Si(w_c(t-t_d)) - Si(-\infty) \right]$$

$$= \frac{1}{\pi} \left\{ [Si(w_c(t-t_d))] + \frac{\pi}{2} \right\}$$

$$= \frac{1}{\pi} Si(w_c(t-t_d)) + \frac{1}{2}$$

If $w_c \to \infty$ then the response is:

$$y(t) = \frac{1}{2} + \frac{1}{\pi} Si(\infty) = \frac{1}{2} + \frac{1}{\pi}\left(\frac{\pi}{2}\right) = 1$$

If $w_c \to -\infty$ then the response is

$$y(t) = \frac{1}{2} + \frac{1}{\pi} Si(-\infty) = \frac{1}{2} + \frac{1}{\pi}\left(-\frac{\pi}{2}\right) = 0$$

The step response of LPF is shown in figure

From fig (b) we can observe that $y(t)$ approaches a delay unit step $u(t-t_d)$
But the abrupt rise of i/p corresponds to more gradual rise of the o/p. 14

The rise time t_r is defined as the time required for the response to reach from 0% to 100% of the final value. To find it, draw a tangent at $t = t_d$ with the line $y(t) = 0$ and $y(t) = 1$

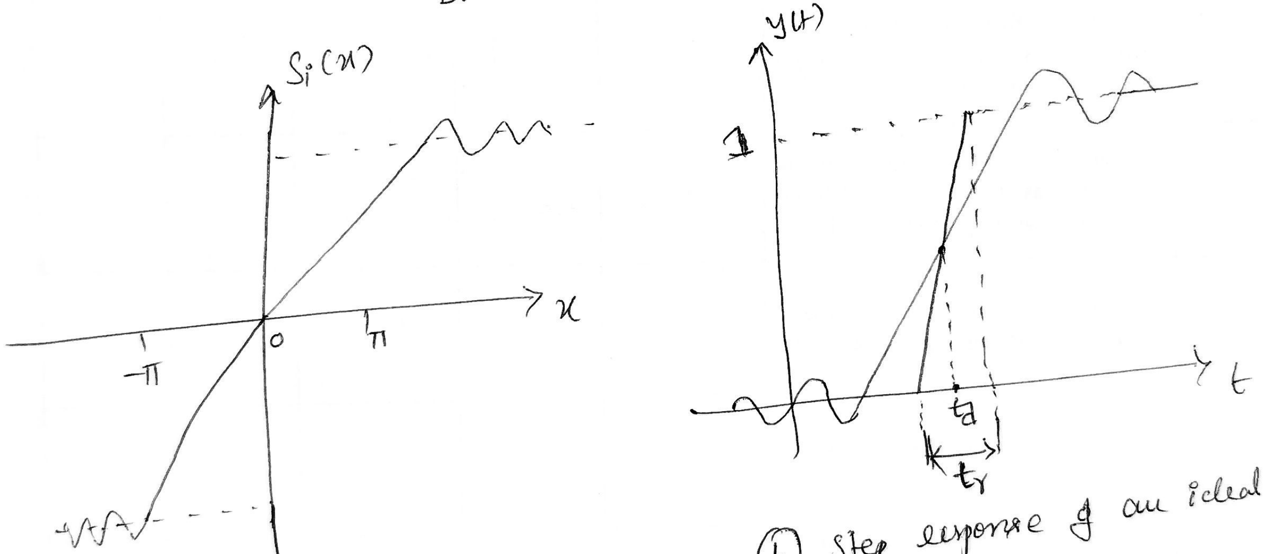
$$\frac{d\,y(t)}{dt} = \frac{1}{t_r} = \frac{w_c}{\pi} \left.\frac{\sin w_c(t-t_d)}{w_c(t-t_d)}\right|_{t=t_d} = \frac{w_c}{\pi}.$$

$$t_r = \frac{\pi}{w_c}$$

For a low-pass filter cut-off frequency = Band width

rise time is inversely proportional to the Bandwidth

Bandwidth X Rise time = Constant.



(a) S_i function

(b) Step response of an ideal LPF.

Energy Density Spectrum :-

Spectral density - it is the distribution of Energy & power of a signal per unit B.W as a function of frequency.

Energy Signals — Signals with finite Energy and zero average power, i.e, $0 < E < \infty$ and $P = 0$ are called Energy signals, E.g: Aperiodic signals like pulse.

. Normalized energy - it defined as the Energy dissipated by a voltage signal applied across 1Ω resistor

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

For Energy signal Exists only if E is finite i.e, $0 < E < \infty$.

Parseval's theorem for Energy signals (Rayleigh Energy theorem)

This Theorem defines the Energy of a signal in terms of its F.T. frequency spectrum $X(\omega)$

Energy of $x(t)$ can directly Evaluated from its time domain version i.e $x(t)$. without the knowledge of its time domain version

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df$$

proof :- Consider a function $x(t)$ Such that

$$x(t) \longleftrightarrow X(\omega)$$

Let $x^*(t)$ be the conjugate of $x(t)$ such that

$$x^*(t) \longleftrightarrow X^*(-\omega).$$

The Energy of signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} x^*(t)\, x(t)\, dt$$

Replace $x(t)$ with IFT of $x(t)$

$$E = \int_{-\infty}^{\infty} x^*(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \right] d\omega$$

interchanging the order of integration

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left[\int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, X^*(-\omega)\, d\omega \,.$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

Let $\omega = 2\pi f$

$d\omega = 2\pi\, df$

Normally $X(2\pi f)$ written as $X(f)$.

$$E = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

This is called parseval's theorem for energy signals

Energy spectral density (ESD) :-

It gives the distribution of energy of a signal in the frequency domain

for an energy signal, the total area under the spectral density curve plotted as a function of frequency is equal to the total energy of the signal.

it is also called energy density spectrum

is designated as $\psi(\omega) = |X(\omega)|^2$

$$Y(\omega) = H(\omega)X(\omega)$$

ESD of i/p $\quad \psi_x(\omega) = |X(\omega)|^2$

ESD of o/p $\quad \psi_y(\omega) = |Y(\omega)|^2$

$$\psi_y(\omega) = |Y(\omega)|^2 = |X(\omega)\, X(\omega)|^2$$

$$= |X(\omega)|^2\, |H(\omega)|^2$$

$$\psi_y(\omega) = |H(\omega)|^2\, \psi_x(\omega)$$

Thus, the ESD of the o/p of a linear system is the product of ESD of i/p (Excitation) & square of the magnitude of the T.F.

Energy of the o/p sgl $E_y = \int_{-\infty}^{\infty} \psi_y(f)\, df$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_y(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 \psi_x(\omega)\, d\omega$$

$$E_y = \frac{1}{\pi}\int_{0}^{\infty} |H(\omega)|^2 \psi_x(\omega)\, d\omega$$

If the LTI system is an ideal LPF with lower & upper cutoff frequency f_L & f_H respectively then $|H(\omega)| = 1$ for $f_L < f < f_H$

$$E_y = \frac{1}{\pi}\int_{f_L}^{f_H} \psi_x(\omega)\, d\omega$$

$$E_y = \frac{1}{\pi}\int_{f_L}^{f_H} \psi_x(2\pi f)\, 2\pi\, df = 2\int_{f_L}^{f_H} \psi_x(f)\, df$$

properties of ESD

1. Total area under the energy density spectrum is equal to the total energy of the signal.

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, d\omega = \int_{-\infty}^{\infty} \psi(f)\, df$$

2. If $x(t)$ is the i/p to an LTI system with impulse response $h(t)$, then the i/p & o/p ESD functions are related as

$$\psi_y(\omega) = |H(\omega)|^2 \psi_x(\omega)$$

or $\psi_y(f) = |H(f)|^2 \psi_x(f)$

3. The auto correlation function $R(\tau)$ & ESD $\psi(\omega)$ form a F.T pair

$$R(\tau) \xleftrightarrow{FT} \psi(\omega)$$

$$R(\tau) \underset{IFT}{\overset{FT}{\rightleftarrows}} \psi(f)$$

Power Density Spectrum :-

Power signals :- signals with finite average power & infinite energy ie, $0 < P < \infty$ and $E = \infty$ are called power signals, e.g. periodic signals.

Average power $$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

Power P is actually mean square value of time average of the squared signal.

$$P = \overline{x^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Parsevals power theorem defines the power of a signal in terms of its F-S coeff ie, in terms of the harmonic components present in the signal.

mathematically it is given by

$$P = \sum_{n=-\infty}^{\infty} |C(n)|^2$$

proof

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, x^*(t) \, dt$$

Exponential Fourier series $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$

$$P = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t} x^*(t) \, dt$$

Interchanging the order of summation & integration

$$P = \sum_{n=-\infty}^{\infty} C_n \frac{1}{T} \int_{-T/2}^{T/2} x^*(t) e^{jn\omega t} \, dt$$

$$= \sum_{n=-\infty}^{\infty} C_n C_n^* = \sum_{n=-\infty}^{\infty} |C_n|^2$$

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2 \quad //$$

Called parsevals power theorem

PSD :- Powee Spectral Density

The distribution of average powee of the Signal in the fequency domain is called power Spectral density or power density or power density spectum (PSD or PD). 19

To derive the PSD, assume the powee signal as a limiting case of the Energy signal. Consider a powee signal $x(t)$. Extending to infinity as shown.

Let us truncated this signal so that it is zero outside the integral $T/2$

Let this truncated Signal be $x_T(t)$

$$x_T(t) = \begin{cases} x(t), & |t| < \frac{T}{2} \\ 0, & \text{Elsewhere} \end{cases}$$

The Signal $x_T(t)$ is of finite duration T & hence it is an Energy Signal with Energy E given by

$$E = \int_{-\infty}^{\infty} |x_T(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega$$

where $x_T(t) \longleftrightarrow X_T(\omega)$.

As $x(t)$ over the interval $\left(-\frac{T}{2} \text{ to } \frac{T}{2}\right)$ is same as $x_T(t)$ over the interval $-\infty$ to ∞, we have

$$\int_{-\infty}^{\infty} |x_T(t)|^2 \, dt = \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 \, dt = \frac{1}{2\pi}\frac{1}{T}\int_{-\infty}^{\infty} |X_T(\omega)|^2 \, d\omega .$$

If $T \to \infty$ the left hand side of above Eqn represents the average powe P of the function $x(t)$.

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T} \, d\omega$$

If $T \to \infty$, $|X_T(\omega)|^2/T$ approaches a finite value.

Let this finite value is denoted by $S(\omega)$ i.e,

$$S(\omega) = \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T}$$

$$P = \overline{x^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} S(f)\, df .$$

where $\overline{x^2(t)}$ is the MSE of $x(t)$.

The average power is

$$P = 2\frac{1}{2\pi}\int_0^\infty S(\omega)\,d\omega = \frac{1}{\pi}\int_0^\infty S(\omega)\,d\omega = 2\int_0^\infty S(f)\,df$$

The PSD of a periodic function is given by

$$S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(\omega - n\omega_0)$$

or alternately $S(f) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0)$

The i/p & o/p relation of a linear system in terms of PSD is given by

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega)$$

$$S_y(f) = |H(f)|^2 S_x(f).$$

<u>Properties of PSD</u> :-

1. The area under the PSD function is equal to the average power of that signal i.e,

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega = \int_{-\infty}^{\infty} S(f)\,df.$$

2. The i/p & o/p PSD of an LTI system are related as

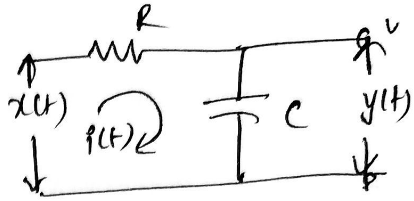
$$S_y(\omega) = |H(\omega)|^2 S_x(\omega)$$

3. The autocorrelation function $R(\tau)$ & PSD $S(\omega)$ form an FT pair ie,

$$R(\tau) \leftrightarrow S(\omega)$$

~~The comparison of ESD & PSD~~

Ex: Find the o/p voltage of the RC low pass filter shown in figure for an i/p voltage of $te^{-t/RC}$ 21



Sol:
$$H(s) = \frac{Y(s)}{X(s)} = \frac{1/cs}{R + 1/cs}$$

$$= \frac{1}{1+RCs}$$

$$x(t) = te^{-t/RC}$$

$$Y(s) = \frac{1}{1+RCs} X(s) \qquad X(s) = \frac{1}{(s+\frac{1}{RC})^2}$$

$$Y(s) = \frac{1}{RC(s+\frac{1}{RC})} \cdot \frac{1}{(s+\frac{1}{RC})^2}$$

$$Y(s) = \frac{1}{RC(s+\frac{1}{RC})^3}$$

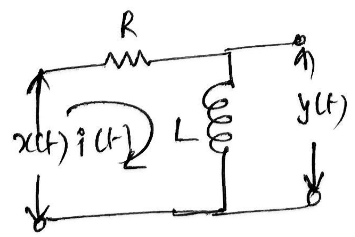
Apply I.L.T on B-sides

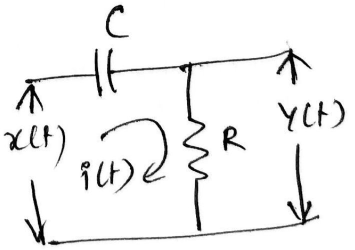
$$y(t) = \frac{1}{RC} t^2 e^{-t/RC} u(t)$$

Ex: $y(t) = e^{-t}u(t)$, $x(t) = e^{-2t}u(t)$ find impulse response & frequency response of the system

Ex: $x(t) = e^{3t}u(t)$, $h(t) = e^{-2t}u(t) + e^{2t}u(-t)$ Find o/p

Ex:





Ex:- Consider a stable LTI system that is characterised by the differential equation $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$ 22

Find its response for i/p $x(t) = e^{-t}u(t)$.

$X(s) = \frac{1}{s+1}$

Sol

$s^2 Y(s) + 4s Y(s) + 3Y(s) = sX(s) + 2X(s)$

$(s^2+4s+3)Y(s) = (s+2)X(s)$

$\frac{Y(s)}{X(s)} = \frac{s+2}{s^2+4s+3} = \frac{s+2}{(s+1)(s+3)}$

$Y(s) = \frac{s+2}{(s+1)(s+2)} \cdot \frac{1}{s+1} = \frac{s+2}{(s+1)^2(s+2)}$

$\frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+2)} = \frac{s+2}{(s+1)^2(s+2)}$

$A(s+1)(s+2) + B(s+1) + C(s+1)^2 = s+2$

$A + C = 0 \qquad 3A + B + 2C = 1 \qquad 2A + B + C = 2$

$3A \qquad +3C=0 \qquad 2A$

Ex $\log\left(\frac{1+s}{s}\right)$

$X(s) = \log\left(\frac{1+s}{s}\right)$

$x(t) = L^{-1}(X(s))$

$X(s) = L(x(t))$

$L[x(t)] = \log\left(\frac{1+s}{s}\right)$

From property of differentiation in s domain $\quad t\,x(t) = -\frac{d}{ds}X(s)$

$L[t\,x(t)] = -\frac{d}{ds}\left[\log\left(\frac{1+s}{s}\right)\right]$

$= -\frac{d}{ds}\left[\log(1+s) - \log s\right]$

$= \frac{-1}{s+1} + \frac{1}{s}$

$t\,x(t) \quad L^{-1}\left(\log\left(\frac{1+s}{s}\right)\right) = -e^{-t}u(t) + u(t)$

$x(t) = \frac{(1 - e^{-t})u(t)}{t}$

Random process - Temporal characteristics

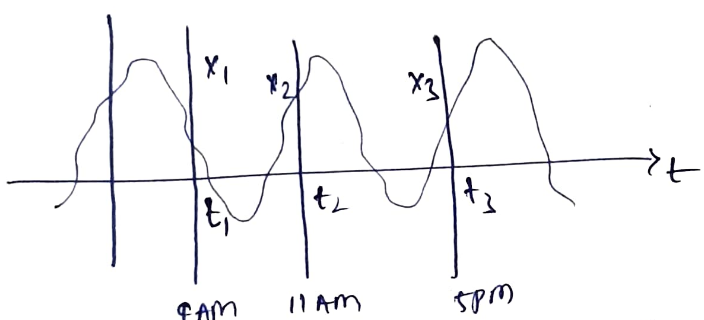
Definition :- Random process is a family of functions $\{X \le x(t, s)$ $s \in S$ & $t \in T\}$ defined with random variables $\{X \le x\}$ on a sample space S and with a function of time T.

Random process is a function of two variables 1. Sample space
2. Time

Ex :- ① Let us consider an experiment which measures the number of calls received in peak hours. The calls received (sample space) is a random process as the number of calls received at different time intervals may be different.

Thus the number of calls is a function of both sample space & time

②



random process is represented $X(t, s) = x(t_1, t_2, t_3; x_1, x_2, x_3)$

Sample space $= \{x_1, x_2, x_3\}$

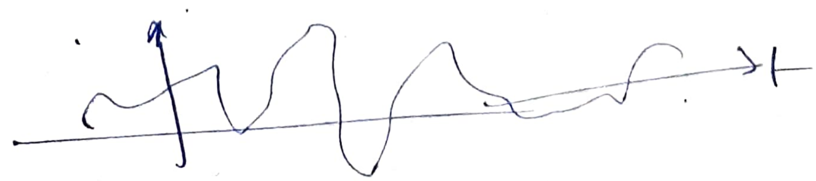
$x(t, s_i) = x_i(t)$

⇒ Deterministic Random process – A process is called deterministic random process if future values of any sample function can be predicted from its past values.

Ex $x(t) = A \sin(\omega_0 t + \theta)$. – predicted A, ω_0 & θ from its known shape

⇒ Non Deterministic Random process – The future values of a sample function cannot be predicted from observed past values.

Ex



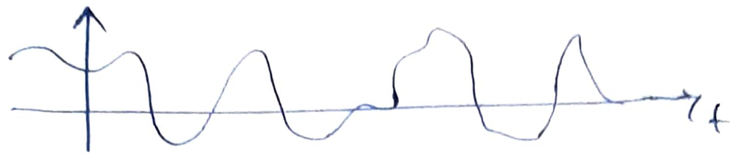
Classification of random process. (2)

Four types based on time t and amplitude of random variable X.

1. Continuous random processes (6)

Both the random variable X & time are continuous over the entire time

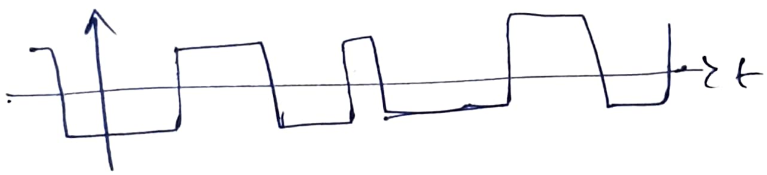
Ex noise in any n/w.



2. Discrete random processes

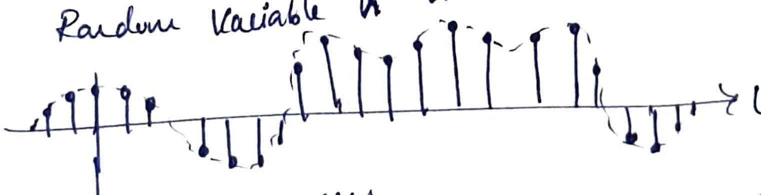
The random variable X has only discrete values while time t is continuous.

Ex: Digital encoded signal has only two discrete values



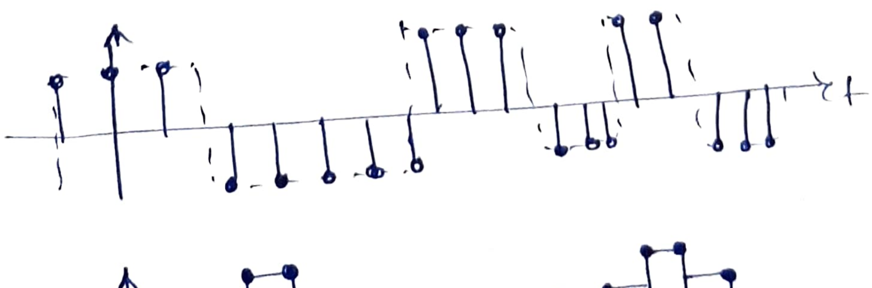
3. Continuous random sequences

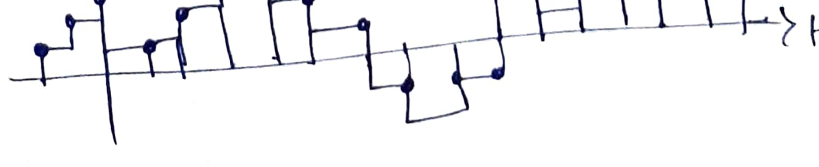
Random variable X is continuous but time t has discrete value



4. Discrete random sequences

Both random variables X and t are discrete - it is obtained by sampling & quantizing a random signal. Called digital process.





Distribution & Density functions of Random process:-

A random variable can be obtained from a random process at a particular instant of time t. The random variable has all the properties that are related to its probability density functions.

If two random variables are obtained from the random process at two different times. They will have all the statistical properties related to their joint probability density functions.

||ly N random variables defined at N time instants will have the statistical properties related to their N-dimensional joint probability density functions.

→ **Joint distribution function of a random process.**

Let a random process $X(t)$, for a single random variable at time t_1, $X_1 = X(t_1)$ the cumulative distribution function is defined as

$$F_X(x_1; t_1) = P\{X(t_1) \le x_1\}$$

where x_1 is any real number. The function $F_X(x_1; t_1)$ is known as the first order distribution function of $X(t)$.

Two random variable at time instants t_1 & t_2 $X(t_1) = X_1(t)$
$X(t_2) = X_2(t)$

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

Called second order Joint distribution function.

N order joint distribution function is

$$F_X(x_1, x_2, x_3, \ldots, x_N; t_1, t_2, \ldots t_N) = P\{X(t_1) \le x_1, X(t_2) \le x_2, \ldots X(t_N) \le x_N\}$$

(4)

Joint Density function: ~~JD~~ JDF of a random process can be obtained from the derivatives of the distribution function.

1. First order density function

$$f_X(x_1; t_1) = \frac{dF_X(x_1; t_1)}{dx_1}$$

2. Second order joint density function

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$

3. N^{th} order joint density function

$$f_X(x_1, x_2, \ldots, x_N; t_1, t_2, \ldots, t_N) = \frac{\partial^N F_X(x_1, x_2, \ldots x_N; t_1, t_2 \ldots t_N)}{\partial x_1 \partial x_2 \ldots \partial x_N}$$

4. If $X(t)$ is a discrete time process, then N^{th} order probability mass function is

$$P_X(x_1, x_2, \ldots x_N; t_1, t_2, \ldots t_N) = P\{x(t_1) = x_1, x(t_2) = x_2, \ldots x(t_N) = x_N\}$$

* Independent random process:

Consider a random process $X(t)$, Let $X(t_i) = x_i$; $i = 1, 2, 3 \ldots N$ with N random variables defined at time instants $t_1, t_2, \ldots, t_N$ with density functions $f_X(x_1; t_1)$; $f_X(x_2, t_2), \ldots, f_X(x_N, t_N)$.

If the random process $X(t)$ is statistically independent, then the N^{th} order joint density function is equal to the product of individual joint function of $X(t)$.

$$f_X(x_1, x_2, x_3, \ldots x_N; t_1, t_2, t_3, \ldots, t_N) = f_X(x_1; t_1) \cdot f_X(x_2; t_2) \cdot f_X(x_3; t_3) \ldots f_X(x_N; t_N)$$

Illar, Let as we consider two random process $X(t)$ and $Y(t)$. Let $X(t)$ have random variables $X(t_1), X(t_2), X(t_3), \ldots X(t_N)$ with joint density function $f_X(x_1, x_2, x_3, \ldots x_N, t_1, t_2, t_3, \ldots t_N)$ and $Y(t)$ have random variables $Y(t_1'), Y(t_2'), \ldots Y(t_M')$. with joint density function &

$$f_Y(y_1, y_2, y_3, \ldots y_M; t_1', t_2', t_3', \ldots t_M').$$

Statistical properties of Random process :-

1. **Mean** :- The mean value of a random process $X(t)$ is Equal to the Expected value of the random process $X(t)$ is denoted as

$$\overline{X(t)} = E[x(t)] = \int_{-\infty}^{\infty} x\, f_X(x:t)\, dx.$$

Where $f_X(x:t)$ is the probability density function of the random process $X(t)$. The mean value $E[X(t)]$ is also called Ensemble average of $X(t)$.

2. **Autocorrelation** :-

Consider a random process $X(t)$. Let X_1 & X_2 be two random variables defined at times t_1 and t_2 respectively with joint Density function $f_X(x_1, x_2; t_1, t_2)$.

The correlation of X_1 and X_2 $\cdot E[X_1 X_2] = E[X(t_1) X(t_2)]$ is called the autocorrelation function of the random process $X(t)$ is denoted as

$$R_{XX}(t_1, t_2) = E[x(t_1) X(t_2)] = E[X_1 X_2]$$

or

$$R_{XX}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)\, dx_1 dx_2.$$

3. **Cross Correlation** :-

Consider two random process $X(t)$ and $Y(t)$ denoted with random variables X and Y at time instants t_1 and t_2 respectively. The joint density function is $f_{X,Y}(x, y; t_1, t_2)$. Then the correlation of X & Y. $E\{X Y\} = E[X(t_1) Y(t_2)]$ is called the cross correlation function of the random processes $X(t)$ and $Y(t)$ it is denoted as

$$R_{XY}(t_1, t_2) = E[X(t_1) Y(t_2)] = E[XY]$$

$$(\text{or})\ R_{XY}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x, y; t_1, t_2)\, dx\, dy$$

Stationary processes :- A random process is said to be stationary if all its statistical properties such as mean, moments, variances etc. donot change with time. (6)

The stationary which depends on the density functions of the random variables of the process has different levels & orders.

Ist order stationary processes :- A random process is said to be stationary to order one & first order stationary if its first order stationary process ~~, then~~ density function doesnot change with time or shift in time value. If $X(t)$ is first order stationary process, then

$$f_X(x_1; t_1) = f_X(x_1; t_1+\Delta t) \text{ for any } t_1.$$

Δt - shift in time value

$\Rightarrow f_X(x_1;t_1)$ is independent of t_1, so the mean value of the process is constant.

proof :-

$$E[X(t_1)] = E[X_1] = \int_{-\infty}^{\infty} x_1 f_X(x_1; t_1)\, dx_1$$

$$E[X(t_2)] = E[X_2] = \int_{-\infty}^{\infty} x_1 f_X(x_1, t_2)\, dx_1$$

sub $t_2 = t_1+\Delta t$

$$E[X(t_1+\Delta t)] = \int_{-\infty}^{\infty} x_1 f_X(x_1, t_1+\Delta t)\, dx_1$$

For stationary random process, we know that

$$f_X(x_1; t_1+\Delta t) = f_X(x_1; t_1)$$

$$E[X(t_1+\Delta t)] = \int_{-\infty}^{\infty} x_1 f_X(x_1; t_1)\, dx$$

$$E[X(t_1+\Delta t)] = E[X(t_1)] = \text{a constant}$$

IInd order stationary process :-

$$f_X(x_1, x_2 : t_1, t_2) = f_X(x_1, x_2; t_1+\Delta t, t_2+\Delta t) \text{ for all } t_1, t_2 \text{ \& } \Delta t.$$

it is a function of time differences (t_2, t_1) and not absolute time t.

Note that a second order stationary process is also a first order stationary processes

$$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

if $\tau = t_2 - t_1$ then

$$R_{XX}(t_1, t_1+\tau) = E[X(t_1)\, X(t_1+\tau)] = R_{XX}(\tau)$$

$R_{XX}(\tau)$ should be independent of time t.

Wide Sense Stationary Processes (WSS)

If a random process $X(t)$ is a second order stationary process, then it is called a wide sense stationary process (WSS) or a weak sense stationary process X. However the converse is not true.

The conditions for a wide sense stationary process are

1. $E[X(t)] = \bar{X}$ = constant
2. $E[X(t)X(t+T)] = R_{XX}(T)$ is independent of absolute time t.

Joint wide sense stationary processes :-

Consider two random processes $x(t)$ & $y(t)$. If they are jointly wide sense stationary, then the cross correlation function of $x(t)$ & $y(t)$ is a function of the time difference $\tau = t_2 - t_1$ only and absolute time.

The cross correlation function

$$R_{XX}(t_1, t_2) = E[x(t_1), y(t_2)].$$

If the time difference $\tau = t_2 - t_1$ & $t = t_1$ then

$$R_{XY}(t, t+T) = E[x(t), y(t+T)] = R_{XY}(\tau).$$

Therefore, the conditions for a process to be a joint wide sense stationary process are

1. $E[X(t)] = \bar{X}$ = constant
2. $E[Y(t)] = \bar{Y}$ = constant
3. $E[X(t)\,Y(t+T)] = R_{XY}(\tau)$ is independent of time t.

Strict sense stationary processes (SSS) :-

A random process $x(t)$ is said to be strictly sense stationary if its Nth order joint density function cannot change with time or shift in time value.

$$f_X(x_1, x_2, x_3 \ldots x_N ; t_1, t_2 \ldots t_N) = f_X(x_1, x_2 \ldots x_N ; t_1+\Delta t, t_2+\Delta t, \ldots t_N+\Delta t).$$

for all $t_1, t_2, t_3 \ldots t_N$ & Δt.

Ex. A random process $Y(t)$ is given as $Y(t) = X(t)\cos(\omega t + \theta)$, where $X(t)$ is a WSS; ω is a constant & θ is a random phase independent of $X(t)$, uniformly distributed on $(\pi, -\pi)$ find (a) $E[Y(t)]$ (b) $R_{YY}(\tau)$ (8)

Sol for uniformly distributed $f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & \text{other} \end{cases}$

(a) $E[Y(t)] = \int_{-\infty}^{\infty} E[X(t)\cos(\omega t + \theta)]$

$X(t)$ & $\cos(\omega t + \theta)$ are independent so

$$= E[X(t)]\, E[\cos \omega t + \theta]$$

$$= E[X(t)] \int_{-\infty}^{\infty} (\cos \omega t + \theta) f_\theta(\theta)\, d\theta$$

$$= E[X(t)] \int_{-\pi}^{\pi} \cos(\omega t + \theta) \frac{1}{2\pi}\, d\theta$$

$$= E[X(t)] \frac{1}{2\pi} \left[\sin(\omega t + \theta) \right]_{-\pi}^{\pi}$$

$$= E[X(t)] \frac{1}{2\pi} \left[\sin(\omega t + \pi) - \sin(\omega t - \pi) \right]$$

$$= E[X(t)] \frac{1}{2\pi} \left[-\sin \omega t + \sin \omega t \right]$$

$$= E[X(t)] \frac{1}{2\pi} (0)$$

$$E[Y(t)] = 0$$

(b) $R_{YY}(\tau) = E[Y(t)\, Y(t+\tau)]$

$$= E[X(t)\, X(t+\tau)]\, E[\cos(\omega t + \theta)\cos(\omega t + \omega\tau + \theta)]$$

$$= E[X(t)\, X(t+\tau)]\, [0 + \cos \omega \tau]$$

$$= \frac{1}{2} R_{XX}(\tau) + \cancel{\cos \omega t + \theta}$$

$$R_{YY}(\tau) = \frac{1}{2} R_{XX}(\tau) \cos \omega \tau$$

* Time Average of Random process :-

- Time average function :- Let $x(t)$ be a sample function which exist for all time t at a fixed value in the given sample space S.

The average value of $x(t)$ taken over all time is called the time average of $x(t)$: it is also called mean value of $x(t)$.

Expressed as $$\bar{x} = A[x(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

⇒ Time autocorrelation function :-

Consider a random process $x(t)$. The time average of the product $x(t)$ & $x(t+\tau)$ is called time autocorrelation function of $x(t)$ and is denoted as.

$$R_{xx}(\tau) = A[x(t)\,x(t+\tau)]$$

(or) $$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,x(t+\tau)\,dt$$

τ is time difference of time shift in t.

⇒ Time mean square function :-

If $\tau = 0$, the time average of $x^2(t)$ is called time mean square value of $x(t)$. It is denoted as

$$A[x^2(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt.$$

⇒ Time cross correlation function :- Let $x(t)$ & $y(t)$ be two random process with sample function $x(t)$ & $y(t)$ respectively. The time average of the product of $x(t)$ & $y(t+\tau)$ is called time cross correlation function of $x(t)$ & $y(t)$ is denoted as

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,y(t+\tau)\,dt.$$

Note: The ensemble averages & time averages are not always same.

1. $E[A[x(t)]] = E[x(t)] = \bar{X}$
2. $E[R_{xx}(\tau)] = R_{xx}(\tau)$
3. $E[R_{xy}(\tau)] = R_{xy}(\tau)$

*. Ergodic Theorem (or) Ergodic processes :-

It states that for any random ~~variable~~ process $x(t)$, all the time averages of the sample function $x(t)$ are equal to the corresponding statistical or Ensemble averages of $x(t)$

i.e $\bar{x} = \bar{X}$

$$\mathcal{R}_{xx}(\tau) = R_{xx}(\tau)$$

Ergodic process :- Random process that satisfy the Ergodic theorem are called Ergodic processes. Analysis of Ergodicity is extremely complex. In most physical apps, it is assumed that all stationary process are Ergodic processes.

1. **Joint Ergodic processes :-**

Let $X(t)$, $Y(t)$ - R.P. with sample function $x(t)$ & $y(t)$. respectively. ~~then~~ The two R.P ~~jointly~~ called jointly Ergodic process if they are individually Ergodic. & their time cross correlation function are equal to their respective ensembles cross correlation function, that is

1. $\bar{x} = \bar{X}$
2. $\bar{y} = \bar{Y}$
3. $\mathcal{R}_{xx}(\tau) = R_{xx}(\tau)$
4. $\mathcal{R}_{xy}(\tau) = R_{xy}(\tau)$
5. $\mathcal{R}_{yy}(\tau) = R_{yy}(\tau)$.

2. Mean Ergodic process

$$E[X(t)] = \bar{X} = A[x(t)] = \bar{x}$$

3. Auto correlation Ergodic process

$$A[x(t)\, x(t+\tau)] = E[X(t)\, X(t+\tau)]$$

(or) $\mathcal{R}_{xx}(\tau) = R_{xx}(\tau)$

4. Cross correlation Ergodic process

$$A[x(t)\, y(t+\tau)] = E[X(t)\, Y(t+\tau)]$$

$$\mathcal{R}_{xy}(\tau) = R_{xy}(\tau).$$

* Properties of Autocorrelation functions. Consider $x(t)$ is WSS.

ACF $R_{XX}(t, t+T)$ $T = t_2 - t_1$

1. Mean Square value of $X(t)$ is

$$E[X^2(t)] = R_{XX}(0)$$

it is Equal to the power of the process $X(t)$

Proof:- $R_{XX}(\tau) = E[X(t)\, X(t+T)]$

If $\tau = 0$

$$R_{XX}(0) = E[X^2(t)]$$

2. Autocorrelation function is maximum at the origin i.e,

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$

proof:- Let $X(t_1), X(t_2)$ are two R.V of R.P. defined at t_1 & t_2

Consider a +ve quantity $[X(t_1) \pm X(t_2)]^2 \geq 0$

Take Expected value ~~of~~

$$E[X(t_1) \pm X(t_2)]^2 \geq 0$$

$$E[X^2(t_1) + X^2(t_2) \pm 2X(t_1)X(t_2)] \geq 0$$

$$E[X^2(t_1)] + E[X^2(t_2)] \pm 2E[X(t_1)X(t_2)] \geq 0$$

$$R_{XX}(0) + R_{XX}(0) \pm 2R_{XX}(t_1, t_2) \geq 0 \qquad \tau = t_2 - t_1$$

$$2R_{XX}(0) \pm 2R_{XX}(t_1, t_1+T) \geq 0$$

$$2R_{XX}(0) \pm 2R_{XX}(\tau) \geq 0$$

$$\cancel{2}R_{XX}(0) \geq \cancel{2}R_{XX}(\tau)$$

$$R_{XX}(0) \geq R_{XX}(\tau)$$

3. $R_{xx}(\tau)$ is an Even function of τ i.e.

$$R_{xx}(-\tau) = R_{xx}(\tau)$$

Proof:

$$R_{xx}(\tau) = E[x(t)\ x(t+\tau)]$$

let $\tau = -\tau$

$$R_{xx}(-\tau) = E[x(t) \cdot x(t-\tau)]$$

let $u = t-\tau$

$$R_{xx}(-\tau) = E[x(\tau+u)\ x(u)]$$

$$R_{xx}(-\tau) = R_{xx}(\tau)$$

4. If a R.P $x(t)$ has a non-zero mean value, $E[x(t)] \neq 0$ & ergodic with no periodic components, then

$$\lim_{|\tau|\to\infty} R_{xx}(\tau) = \bar{X}^2.$$

proof:

$$R_{xx}(\tau) = E[x(t)\ x(t+\tau)] = E[x(t_1)\ x(t_2)]$$

as $|\tau| \to \infty$, the R.V become independent since process has no periodic components. ∴ that is

$$\lim_{|\tau|\to\infty} R_{xx}(\tau) = E[x(t_1)]\ E[x(t_2)]$$

$x(t)$ is Ergodic so $E[x(t_1)] = E[x(t_2)] = \bar{X}$

$$\lim_{|\tau|\to\infty} R_{xx}(\tau) = E[x(t)]^2 = \bar{X}^2$$ //

5. If $x(t)$ is periodic, then its autocorrelation function is also periodic

$x(t)$ with periodic

$$x(t) = x(t \pm T_0)$$

$$x(t+\tau) = x(t+\tau \pm T_0)$$

$$R_{xx}(\tau) = E[x(t)\ x(t+\tau)]$$

$$R_{xx}(\tau \pm T_0) = E[x(t)\ x(t+\tau \pm T_0)]$$

$x(t)$ is WSS then

$$R_{xx}(\tau \pm T_0) = E[x(t)\ x(t+\tau)]$$

$$R_{xx}(\tau \pm T_0) = R_{xx}(\tau)$$

$R_{xx}(\tau)$ is periodic

7. The autocorrelation function of a random process $R_{XX}(\tau)$ - cannot have any arbitrary shape.

~~proof~~ $R_{XX}(\tau)$ is even function of τ & Max value at origin & also it related to PSD through F.T. The spectrum is not arbitrary shape. Hence $R_{XX}(\tau)$ also cannot have arbitrary shape.

8. If a random process $x(t)$ with zero mean has the DC components A as $Y(t) = A + X(t)$. then $R_{YY}(\tau) = A^2 + R_{XX}(\tau)$.

9. If the random process $z(t)$ is a sum of two random process $X(t)$ & $Y(t)$, that is,

$$Z(t) = X(t) + Y(t)$$

then $R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$

properties of CCF :-

Consider $X(t)$ & $Y(t)$ are two R.P are Jointly WSS. Then $\tau = t_2 - t_1$ is function of only the CCF $R_{XY}(t, t+\tau)$ then the following properties are

1. $R_{XY}(\tau) = R_{YX}(-\tau)$ is a symmetry property

2. If $R_{XX}(\tau)$ & $R_{YY}(\tau)$ are the autocorrelation function of $X(t)$ & $Y(t)$ respectively then CC satisfies the inequality

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0) R_{YY}(0)}$$

3. If $X(t)$ & $Y(t)$ are two random process with ACF $R_{XX}(\tau)$ and $R_{YY}(\tau)$ then the cross correlation function satisfies the inequality

$$|R_{XY}(\tau)| \leq \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]$$

4. If two random processes $x(t)$ & $y(t)$ are statistically independent (iv) and are atleast wide sense stationary

$$R_{XY}(T) = \bar{X}\bar{Y}$$

5. If the two random processes $X(t)$ & $Y(t)$ have zero mean & are jointly WSS. Then $\lim_{T \to \infty} R_{XY}(T) = 0$ //

* Covariance Functions for Random Processes:-

1. Auto covariance Function:-

The autocovariance function is a measure of interdependence b/w two random variables. The same concept of random variables can be extended to random processes.

Consider random process $x(t)$, $x(t+T)$ at two time intervals t and $t+T$. The autocovariance is defined and can be expressed as

$$C_{XX}(t, t+T) = E[(x(t) - E[x(t)]) \cdot (x(t+T) - E[x(t+T)])]$$

$$C_{XX}(t, t+T) = R_{XX}(t, t+T) - E[X(t)]\,E[X(t+T)]$$

(or) $C_{XX}(t, t+T) =$ wide sense stationary random process,

(a) If $x(t)$ is atleast a wide sense stationary

Then $C_{XX}(T) = R_{XX}(T) - \bar{X}^2$

The autocovariance function is not a function of absolute time t.

(b) At $T = 0$

$$C_{XX}(0) = R_{XX}(0) - \bar{X}^2 = E[X^2] - \bar{X}^2 = \sigma_X^2$$

$\therefore$ at $\tau = 0$ the autocovariance function becomes the variance of the random process ie $\sigma_X^2 = C_{XX}(0)$.

(c) The A.C. coeff of the random process $X(t)$ is defined as

$$\rho_{XX}(t, t+T) = \frac{C_{XX}(t, t+T)}{\sqrt{C_{XX}(t, t)\, C_{XX}(t+T, t+T)}}$$

If $T = 0$ $\rho_{XX}(0) = \frac{C_{XX}(0)}{C_{XX}(0)} = 1$

Also $|\rho_{XX}(t, t+T)| \leq 1$.

Cross covariance Function :-

If two random processes $X(t)$ & $Y(t)$ have random variables at $X(t)$ & $Y(t+T)$, then the cross covariance function is defined as

$$C_{XY}(t, t+T) = E[(X(t) - E(X(t)))(Y(t+T) - E[Y(t+T)])]$$

(or) $$C_{XY}(t, t+T) = R_{XY}(t, t+T) - E[X(t)]E[Y(t+T)]$$

1. If the random processes are atleast jointly WSS; then

$$C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\bar{Y}$$

The cross covariance function is not a function of absolute time t.

2. If two random processes $X(t)$ & $Y(t)$ are uncorrelated, then

$$C_{XY}(t, t+T) = 0$$

or $$R_{XY}(t, t+T) = E[X(t)]E[Y(t+T)]$$

This is the condition for two random processes to be statistically independent. Therefore, the independent random processes are uncorrelated. But the ~~covarias~~ converse is not always true.

3. The cross correlation Co-eff of the random processes $X(t)$ and $Y(t)$ is defined as

$$\rho_{XY}(t, t+T) = \frac{C_{XY}(t, t+T)}{\sqrt{C_{XX}(t,t)\, C_{YY}(t+T, t+T)}}$$

if $\tau = 0$

$$\rho_{XY}(0) = \frac{C_{XY}(0)}{\sigma_X \sigma_Y}$$

Ex:- A random process is given as $X(t) = At$ where A is an uniformly distributed random variable on $(0, 2)$ find whether is WSS or not. (16)

Sol:- The condition for WSS is

$$E[X(t)] = \bar{X} = \text{constant}$$

and $R_{XX}(\tau)$ is independent of time 't'.

Given $X(t) = At$

and $f_A(A) = \begin{cases} \frac{1}{2} & 0 \leq A \leq 2 \\ 0 & \text{otherwise} \end{cases}$

ⓐ $E[X(t)]$

$$E[X(t)] = \int_{-\infty}^{\infty} X(t) f_A(A)\, dA$$

$$= \int_0^2 \frac{1}{2} At\, dA$$

$$= \frac{t}{2} \frac{A^2}{2}\Big|_0^2$$

$$= \frac{t}{2}\left[\frac{4}{2} - 0\right]$$

$$E[X(t)] = t$$

ⓑ $R_{XX}(\tau) = E[X(t) X(t+T)]$

$$= E[At\, A(t+T)]$$

$$= t(t+T)\, E[A^2]$$

$$= t(t+T) \cdot \int_0^2 \frac{A^2}{2}\, dA$$

$$= t(t+T) \frac{A^3}{6}\Big|_0^2$$

$$= \frac{t(t+T)}{6}[8-0]$$

$$R_{XX}(\tau) = \frac{4}{3} t(t+T)$$

Hence mean value is not constant and $R_{XX}(T)$ is a function of t.

$X(t)$ is not a WSS.

Ex:- $X(t) = A$, $(0, 1)$

$$E[X(t)] = \int_0^1 A \cdot 1 \cdot dA$$

$$= \frac{A^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$R_{XX}(\tau) = E[X(t)\, X(t+T)]$$

$$= E\int_0^1 [A\, A]$$

$$\cancel{t(t+T)\, E[A^3]}$$

$$\cancel{t(t+T)\left[\frac{A^3}{3}\right]_0^1}$$

$$\cancel{\frac{t(t+T)}{3}}$$

$$= E[A\, A]$$

$$= E[A^2]$$

$$= \frac{A^3}{3}\Big|_0^1 = \frac{1}{3}$$

$\therefore X(t)$ is WSS

$R_{XX}(\tau)$ is independent of time

Ex: If a random process, $X(t) = A\cos\omega t + B\sin\omega t$ is given, where A & B are uncorrelated, zero mean random variables having the variance σ^2. Find (a) Auto correlation & (b) Show that $X(t)$ is WSS. (17)

Sol: Given $x(t) = A\cos\omega t + B\sin\omega t$

A & B are uncorrelated so

$$E[AB] = E[A]E[B] = 0$$

$$\& \quad \sigma_A^2 = E[A^2], \quad \sigma_B^2 = E[B^2]$$

(a) Auto correlation function

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

$$= E[(A\cos\omega t + B\sin\omega t)(A\cos\omega(t+\tau) + B\sin\omega(t+\tau))]$$

$$= E[A^2\cos\omega t\cos\omega(t+\tau) + AB\cos\omega t\sin\omega(t+\tau) + AB\sin\omega t\cos\omega(t+\tau) + B^2\sin\omega t\sin\omega(t+\tau)]$$

$$= E[A^2]E[\cos\omega t\cos\omega(t+\tau)] + E[AB]E[\cos\omega t\cos\omega(t+\tau) + \sin\omega t\sin\omega(t+\tau)] + E[B^2]E[\sin\omega t\sin\omega(t+\tau)]$$

$$= \sigma^2 E[\cos\omega t\cos\omega(t+\tau)] + 0 + \sigma^2 E[\sin\omega t\sin\omega(t+\tau)]$$

$$= \sigma^2[E(\cos\omega t\cos\omega(t+\tau) + \sin\omega t\sin\omega(t+\tau))]$$

$$= \sigma^2 E[\cos(\cancel{\omega t} + \omega\tau - \cancel{\omega t})]$$

$$\therefore \quad R_{XX}(\tau) = \sigma^2\cos\omega\tau$$

(b) The mean value $E[X(t)] = E[A\cos\omega t + B\sin\omega t]$

$$= E[A\cos\omega t] + E[B\sin\omega t]$$

$$= E[A]\cos\omega t + E[B]\sin\omega t$$

$E[X(t)] = 0$ Constant & the ACF.

$R_{XX}(\tau) = \sigma^2\cos\omega\tau$ is independent of time t.

Hence the random process is WSS.

Gaussian Random processes :-

Consider a continuous random process $x(t)$. Let N random variable $X_1 = x(t_1)$, $X_2 = x(t_2)$, ..., $X_N = x(t_N)$ be defined at time instants $t_1, t_2, t_3 \ldots, t_N$ respectively.

If these random variable are jointly Gaussian for any $N = 1, 2 \ldots$ and at anytime instants $t_1, t_2, \ldots t_N$, then the random process $x(t)$ is called a Gaussian random process. The joint density function for a Gaussian random variable is given as.

$$f_X(x_1, x_2, \ldots, x_N; t_1, t_2, \ldots t_N) = \frac{\exp\left\{-\frac{1}{2}(x-\bar{X})^T [C_{XX}]^{-1} [x-\bar{X}]\right\}}{\sqrt{(2\pi)^N [C_{XX}]}}$$

where $\bar{X} = E[X_i] = E[x(t)]$

$[C_{XX}]$ = Covariance matrix and its elements are

$$C_{ik} = C_{X_i X_k} = E[(X_i - \bar{X}_i)(X_k - \bar{X}_k)]$$

$$= E[x(t_i) - \bar{X}(t_i)(x(t_k) - \bar{X}(t_k))]$$

$$C_{ik} = C_{XX}(t_i, t_k)$$

C_{ik} is the autocovariance of $x(t_i)$ & $x(t_k)$. Also by Expanding the above eqn, we can get

$$C_{XX}(t_i, t_k) = R_{XX}(t_i, t_k) - E[x(t_i)]E[x(t_k)].$$

where $R_{XX}(t_i, t_k)$ is the autocorrelation function of X.

If the process is WSS, then it should satisfy the following Conditions.

(1) Mean value will be constant, i.e,

$$\bar{X_i} = E[x(t_i)] = \bar{X} \text{ a constant}$$

(2) The mean autocorrelation & autocovariance functions will depend only on time difference and not on absolute time.

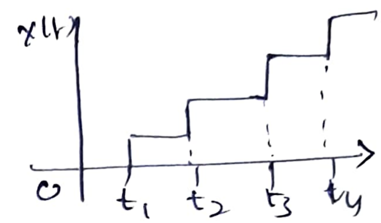
$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$

$$R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$$

Poisson Random Processes :-

The poisson Random process X(t) is a discrete random process which represents the number of time that some event has occurred as a function of time. X(t) has integer valued, non-decreasing sample functions, such as check-in registers, arrival of a customer, arrival of vehicles at a particular point etc.

A single event occurs at a random time. Counting the number of occurrences with time is a poisson process. also called counting process.



A poisson Counting process.

The condition for a poisson process X(t) are

1. $X(0) = 0$
2. Only one event occurs in any instant of time, i.e, in an infinitesimal time interval.
3. The number of events that occurs in any given time interval is independent of the number of events in any other non-overlapping time intervals i.e, X(t) has independent increments.

probability density function :- If the number of occurrences of an event in any finite-interval of time; is described by a poission distribution with the average rate of occurrences is λ, then the probability of exactly k occurrences over a time interval (0, t) is

$$P[X(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \ldots$$

& PDF $\quad f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k)$

Mean value :- $\quad E[X(t)] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k)\, dx$

Since $\int_{-\infty}^{\infty} x\, \delta(x-k)\, dx = k$

From poission density function of a random variable that the mean value is λt So $E[X(t)] = \lambda t$

$$E[X(t)] = \sum_{k=0}^{\infty} \frac{(k \lambda t)^k e^{-\lambda t}}{k!} = \lambda t$$

Second moment

The second moment is

$$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$\int_{-\infty}^{\infty} x^2 \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k)\, dx$$

Since $\int_{-\infty}^{\infty} x^2 \delta(x-k)\, dx = k^2$

$$E[X^2(t)] = \sum_{k=0}^{\infty} \frac{k^2 (\lambda t)^k e^{-\lambda t}}{k!}$$

But we know that the second moment is

$$E[X^2] = \sigma_X^2 + \bar{X}^2$$

$$= \lambda t + (\lambda t)^2 = \lambda t + \lambda^2 t^2$$

$$E[x^2(t)] = \sum_{k=0}^{\infty} \frac{k^2 (\lambda t)^k e^{-\lambda t}}{k!} = \lambda t + \lambda^2 t^2.$$

Ex: Given $\bar{X} = 6$ & $R_{XX}(t, t+T) = 36 + 25\, Exp(-T)$ for a random process $x(t)$. Indicate which of the following statements are true based on what is known, with certainty : $X(t)$.

ⓐ First order stationary
ⓑ Has total average power of 61 W
ⓒ is Ergodic
ⓓ is WSS.
ⓔ Has a periodic component
ⓕ has an AC power of 36 W.

Sol Given the random process $\bar{X} = 6$

$R_{XX}(t, t+T) = 36 + 25\, exp(-T)$.

~~$R_{XX}(t, t+T) = 36 + 25\, Exp(-T)$~~ is independent of t.

ⓐ $\bar{X} = 6$ constant & ~~$R_{XX}(t, t+T)$~~

So $x(t)$ is First order stationary process

(b) Total average power is $P_{avg} = R_{XX}(0)$

$$P_{avg} = 36 + 25\,Exp(-T) = 36 + 25\,Exp(0)$$
$$= 36 + 25$$
$$= 61\,W.$$

$\therefore$ Avg power $= 61\,W$

(c) If $x(t)$ is Ergodic process all its Ensemble averages are Equal to time averages.

$$\therefore A[x(t)] = \bar{X} = 6$$

$$\& \; A[R_{XX}(t, t+T)] = R_{XX}(t, t+T) = 36 + 25\,e^{-|T|}$$

$\therefore$ process is Ergodic

(d) WSS – Conditions are $\bar{X}$ = Constant

$R_{XX}(t, t+T)$ is independent of t.

Mean $\bar{X} = 6$ Constant

$R_{XX}(t, t+T)$ is $36 + 25\,Exp(-T)$ independent of time hence

$x(t)$ is WSS

(e) The process $\because \lim_{T\to\infty} R_{XX}(t, t+T) = E[X]^2$ if no periodic Components

$$\lim_{T\to\infty} R_{XX}(t, t+T) = 36 + 25\,e^{-|T|} = 36$$
$$= 36 + 0 \quad = 36$$

$$\therefore E[X]^2 = 36.$$

$\therefore$ The process has no periodic Components.

(f) The AC power of the process is the Variance of $X(t)$

$$\sigma_X^2 = E[X(t)^2] - E[X(t)]^2$$

$$E[X(t)^2] = R_{XX}(t, t+T)|_{T=0} = R_{XX}(0) = 61$$

$$\sigma_X^2 = 61 - 6^2 = 61 - 36 = 25$$

AC power is 25 watts

Ex • A stationary process has an autocorrelation function is given by (20)

$R(\tau) = \dfrac{25\tau^2+36}{6.25\tau^2+4}$ Find mean value, mean square value &

variance of the process

Sol Given $R(\tau) = \dfrac{25\tau^2+36}{6.25\tau^2+4}$

ⓐ The mean value is

$\bar{X}^2 = \lim_{\tau\to\infty} R(\tau)$

$E[X(t)]^2 = \lim_{\tau\to\infty} \dfrac{25\tau^2+36}{6.25\tau^2+4}$

$= \lim_{\tau\to\infty} \dfrac{\cancel{\tau^2}\left[25+\frac{36}{\tau^2}\right]}{\cancel{\tau^2}\left[6.25+\frac{4}{\tau^2}\right]} = \dfrac{25}{6.25}$

$= \dfrac{2500}{625} = 4.$

$\bar{X}^2 = 4$

$\bar{X} = 2$

ⓑ mean square value

$E[\bar{X}^2] = R(\tau)\Big|_{\tau=0}$

$= \dfrac{25\tau^2+36}{6.25\tau^2+4}\Big|_{\tau=0}$

$E[\bar{X}^2] = \dfrac{36}{4} = 9$

$E[\bar{X}^2] = 9$

ⓒ Variance $\sigma_X^2 = E[x^2] - \bar{x}^2$

$= 9 - 4$

$= 5$

Ex:- Assume that an Ergodic process $x(t)$ has an autocorrelation function

$R_{XY}(\tau) = 18 + \dfrac{2}{6+\tau^2}(1+4\cos(2\tau))$

ⓐ Find $\bar{X}$

$\bar{X}^2 = \lim_{\tau\to\infty} R_{XX}(\tau)$

$= \lim_{\tau\to\infty} 18 + \dfrac{2}{6+\tau^2}(1+4\cos 2\tau)$

$\bar{X}^2 = 18$

$\bar{X} = 4.2426$

ⓑ For a periodic component

$\lim_{\tau\to\infty} R_{XX}(\tau) = \lim_{\tau\to\infty} \left(18+\dfrac{2}{6+\tau^2}\right)(1+4\cos 2\tau)$

$= E[\bar{X}^2]$

$= R(\tau)\big|_{\tau=0}$

$= \left(18+\frac{8}{0}\right)(1+4(\cos 0))$

$= \infty$

So the process has a periodic component.

ⓒ Average power of $X(t)$ is

$E[x^2] = R_{XX}(0)$

$= 18 + \dfrac{2}{6+0}(1+4\cos 0) = 18 + \dfrac{10}{6} = 19.667$ //

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UNIT - IV

Stochastic processes - Spectral characteristics

Introduction :- Consider a random process $X(t)$. The amplitude of the random process, when it varies randomly with time; doesn't satisfy dirichlet's conditions. therefore it not possible to apply FT directly on the random process for a frequency domain analysis. Thus the ACF of WSS RP is used to study spectral characteristics such as power spectral density & power density spectrum.

Definition of Power Spectrum Density :-

① The PSD of a WSS random process $X(t)$ is defined as the fourier Transform of the autocorrelation $R_{XX}(\tau)$ of $X(t)$ can be expressed as

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau.$$

we can obtain $R_{XX}(\tau)$ from $S_{XX}(\omega)$ by taking IFT.

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$$

$\therefore$ PSD $S_{XX}(\omega)$ & $R_{XX}(\tau)$ are Fourier Transform pairs.

② The PSD can also be defined as

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

where $X_T(\omega)$ - Fourier Transform of $x(t)$ in the interval $[-T, T]$

Average power of the random process :-

The average power P_{XX} of a WSS random process $X(t)$ is defined as the time average of its second moment / autocorrelation function at $\tau = 0$

Mathematically $P_{XX} = A\{E[X^2(t)]\}$

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[X^2(t)]\, dt$$

or $P_{XX} = R_{XX}(\tau)\big|_{\tau=0}$

we know that $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$

At $\tau = 0$

$$P_{XX} = R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{0}\, d\omega$$

Average power $$\boxed{P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega.}$$

Proof :- Let us consider a wss random process $x(t)$. To obtain FT let $X_T(t)$ be defined in the interval $[-T, T]$

$$X_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

$$\text{FT of } X_T(t) = X_T(\omega) = \int_{-\infty}^{\infty} X_T(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$$

Now from parseval's theorem; the Energy contained in $X_T(t)$ is

$$\text{Energy} = \int_{-T}^{T} X_T^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(\omega)|^2\, d\omega.$$

The average power over the interval $[-T, T]$ is

$$P(T) = \frac{1}{2T}[\text{Energy}] = \frac{1}{2T}\int_{-T}^{T} (X_T^2(t))\, dt = \frac{1}{2T}\frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(\omega)|^2\, d\omega$$

$$P(T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\, d\omega$$

The average power of the random process $X(t)$ is the entire time interval $-\infty < T < \infty$ is equal to the mean value of $P(T)$ as $T \to \infty$

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[X^2(t)]\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-T}^{T} \lim_{T\to\infty} \int_{-T}^{T} \frac{E[|X_T(\omega)|^2]}{2T}\, d\omega$$

$$P_{XX} = A\{E[X^2(t)]\}.$$

Thus the average power P_{XX} is the time average of its second moment

Also $$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}\, d\omega$$

∴ power spectral density is defined as

$$\boxed{S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}}$$

$$\boxed{P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega}$$

$$\therefore P_{XX} = R_{XX}(0).$$

The average power is autocorrelation of $x(t)$ at the origin.

Properties of PSD :-

The properties of WSS random process are

1. $S_{XX}(\omega) \geq 0$ - Always non negative

Proof: From definition $S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$ is always non negative function $E[|X_T(\omega)|^2]$ is always non negative $S_{XX}(\omega) \geq 0$.

Expected value of a non negative function is always non negative. Hence $S_{XX}(\omega)$ is always non negative.

2. The PSD at zero frequency is equal to the area under the curve of the autocorrelation $R_{XX}(\tau)$.

$$S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, d\tau$$

Proof: $$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau.$$

At $\omega = 0$ $$S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{0}\, d\tau = \int_{-\infty}^{\infty} R_{XX}(\tau)\, d\tau$$

3. The PSD of a real process $X(t)$ is an even function

$$S_{XX}(\omega) = S_{XX}(-\omega) \quad X(t) \text{ is real}$$

Proof: $$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{j\omega\tau}\, d\tau$$

sub $\tau = -\tau$

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(-\tau) e^{-j\omega\tau} d\tau$$

$x(t)$ is real, from properties of ACF we know that

$$R_{XX}(\tau) = R_{XX}(-\tau)$$

So $$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$\therefore \quad S_{XX}(\omega) = S_{XX}(-\omega).$$

$\therefore$ $S_{XX}(\omega)$ is always a real function

4. $S_{XX}(\omega)$ is always a real function

proof:- $$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$|X_T(\omega)|^2$ is always real function hence $S_{XX}(\omega)$ also always real.

5. If $S_{XX}(\omega)$ is a psd of WSS random process $x(t)$ then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = A\{E[X^2(t)]\} = R_{XX}(0)$$

(or)

Time average of the mean square value of a WSS RP equal the area under the curve of the psd.

proof

$$R_{XX}(\tau) = A\{E[X(t+\tau)X(t)]\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

Now at $\tau = 0$

$$R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{0} d\omega$$

$$R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = \text{Area under the curve of}$$

$S_{XX}(\omega)$.

6. If $x(t)$ is a WSS random process with psd $S_{XX}(\omega)$. Then the psd of the derivative of $x(t)$ is equal to ω^2 times of the psd $S_{XX}(\omega)$

$$S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$$

Proof :-

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{E[|X_T(\omega)|^2]}{2T} \qquad \because X_T(\omega) = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$$

$$\dot{X}_T(\omega) = \frac{dX_T(\omega)}{dt} = \frac{d}{d\omega}\int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T}^{T} x(t) \frac{d}{d\omega} e^{-j\omega t}\, dt$$

$$= \int_{-T}^{T} x(t)(-j\omega) e^{-j\omega t}\, dt$$

$$= (-j\omega)\int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$$

$$\dot{X}_T(\omega) = (-j\omega) X_T(\omega)$$

$$S_{\dot{X}\dot{X}}(\omega) = \underset{T\to\infty}{Lt} \frac{E[|\dot{X}_T(\omega)|^2]}{2T}$$

$$= \underset{T\to\infty}{Lt} \frac{E[|(-j\omega) X_T(\omega)|^2]}{2T}$$

$$= \underset{T\to\infty}{Lt} \frac{\omega^2 E[|X_T(\omega)|^2]}{2T}$$

$$= \omega^2 \underset{T\to\infty}{Lt} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$\boxed{S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)}$$

7. The psd and the time average of the ACF form a Fourier transform pair.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega.$$

Proof :- Consider a WSS random process $X(t)$. The time average of autocorrelation function is

$$A[R_{XX}(t, t+\tau)] = A\{E[X(t) \cdot X(t+\tau)]\}$$

For WSS $\quad A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)$

PSD $\quad S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{E[|X_T(\omega)|^2]}{2T}$

$$|X_T(\omega)|^2 = X_T(\omega)\, X_T^*(\omega)$$

$$X_T(\omega) = \int_{-T}^{T} x(t_1)\, e^{-j\omega t_1}\, dt_1 \qquad X_T^*(\omega) = \int_{-T}^{T} x(t)\, e^{j\omega t}\, dt$$

$$S_{XX}(\omega) = \lim_{T\to\infty} E\left[\frac{1}{2T}\int_{-T}^{T} x(t)\, e^{j\omega t}\, dt \int_{-T}^{T} x(t_1)\, e^{-j\omega t_1}\, dt_1\right]$$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[x(t_1)\, x(t)]\, e^{-j\omega(t_1-t)}\, dt\, dt_1$$

we know $E[x(t_1)\, x(t)] = R_{XX}(t, t_1)$

$$S_{XX}(\omega) = \lim_{T\to\infty} E\left[\frac{1}{2T}\int_{-T}^{T} x(t)\, e^{j\omega t}\, dt \int_{-T}^{T} x(t_1)\, e^{-j\omega t_1}\, dt_1\right]$$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[x(t_1)\, x(t)]\, e^{-j\omega(t_1-t)}\, dt\, dt_1$$

we know $E[x(t_1)\, x(t)] = R_{XX}(t, t_1)$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t, t_1)\, e^{-j\omega(t_1-t)}\, dt_1\, dt$$

By taking Fourier transform.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t, t_1)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} e^{j\omega\tau}\, e^{-j\omega(t_1-t)}\, d\omega\, dt\, dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t, t_1)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} e^{-j\omega(t_1-t-\tau)}\, d\omega\, dt\, dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t, t_1)\, \delta(t_1-t-\tau)\, dt\, dt_1$$

$$\int_{-\infty}^{\infty} e^{-j\omega(t_1-t-\tau)}\, d\omega = 2\pi\delta(t_1-t-\tau)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{+j\omega\tau}\, d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XX}(t, t_1)\underbrace{\int_{-T}^{T} \delta(t_1-t-\tau)\, dt_1}_{=1}\, dt$$

let $t_1 = t+\tau$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XX}(t, t+\tau)\, dt$$

$$= A[R_{XX}(t, t+\tau)]$$

$$A[R_{XX}(t, t+T)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega T}\, dT$$

Taking FT

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A[X_X(t, t+T)]\, e^{-j\omega T}\, dT$$

For WSS $\quad R_{XX}(t, t+T) = R_{XX}(T)$

$\therefore R_{XX}(T)$ & $S_{XX}(\omega)$ are a fourier Transform pair.

Ex:- Determine which of the following functions are valid power spectrum density & why?

(a) $\dfrac{\cos 8(\omega)}{2+\omega^4}$ $\qquad$ (b) $e^{-(\omega-1)^2}$ $\qquad$ (c) $\dfrac{\omega^2}{\omega^6+3\omega^2+3}$

(a) Soln $S_{XX}(\omega) = \dfrac{\cos 8(\omega)}{2+\omega^4}$

$S_{XX}(-\omega) = \dfrac{\cos 8(-\omega)}{2+(-\omega)^4}$

$= \dfrac{\cos 8\omega}{2+\omega^4}$

$S_{XX}(\omega) = S_{XX}(-\omega)$
and $S_{XX}(\omega)$ is real & +ve

$\therefore$ Given function is valid psd.

(b) Sol $S_{XX}(\omega) = e^{-(\omega-1)^2}$

$S_{XX}(-\omega) = e^{-(-\omega-1)^2}$

$= e^{-(\omega+1)^2}$

$S_{XX}(\omega) \neq S_{XX}(-\omega)$

$\therefore$ Not valid psd

(c) Soln $S_{XX}(\omega) = \dfrac{\omega^2}{\omega^6+3\omega^2+3}$

$S_{XX}(-\omega) = \dfrac{(-\omega)^2}{(-\omega)^6+3(-\omega)^2+3}$

$= \dfrac{\omega^2}{\omega^6+3\omega^2+3}$

$S_{XX}(\omega) = S_{XX}(-\omega)$

Valid psd

Base band process RMS bandwidth $\quad W_{rms}^2 = \dfrac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega}$

Bandpass process Rms B.W $\quad W_{rms}^2 = \dfrac{4\int_{0}^{\infty} (\omega-\bar{\omega}_0)^2 S_{XX}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega}$

(P): If $R_{XX}(\tau) = a e^{-b|\tau|}$; $H(\omega) = 1/2+j\omega$ show that PSD is $\frac{2ab}{b^2+\omega^2}$

Sol: Given $R_{XX}(\tau) = a e^{-b|\tau|}$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} a e^{-b|\tau|} e^{-j\omega\tau}\, d\tau$$

$$= a \int_{-\infty}^{\infty} e^{-b|\tau|} e^{-j\omega\tau}\, d\tau$$

$$= a \left[\int_{-\infty}^{0} e^{+(b-j\omega)\tau}\, d\tau + \int_{0}^{\infty} e^{-(b+j\omega)\tau}\, d\tau \right]$$

$$= a \left[\frac{e^{(b-j\omega)\tau}}{b-j\omega} \Big|_{-\infty}^{0} + \frac{e^{-(b+j\omega)\tau}}{-(b+j\omega)} \Big|_{0}^{\infty} \right]$$

$$= a \left\{ \left[\frac{e^{0}}{b-j\omega} - \left(\frac{e^{-\infty}}{b-j\omega} \right) \right] + \left[\frac{e^{-\infty}}{-(b+j\omega)} - \left(\frac{e^{0}}{-b+j\omega} \right) \right] \right\}$$

$$= a \left\{ \left[\frac{1}{b-j\omega} - 0 \right] + \left[0 + \frac{1}{b+j\omega} \right] \right\}$$

$$= a \left[\frac{(b+j\omega)+(b-j\omega)}{b^2-(j\omega)^2} \right]$$

$$= a \frac{2b}{b^2+\omega^2}$$

$$S_{XX}(\omega) = \frac{2ab}{b^2+\omega^2}$$

Cross power spectral density :-

* **Relation b/w cross ~~correlation~~ power spectral density & cross correlation function :-**

Let $x(t)$ and $y(t)$ are two random processes $x(t)$ & $y(t)$ are sample functions & $x_T(t)$, $y_T(t)$ are truncated portions in the interval $(-T,T)$.

$X_T(\omega)$, $Y_T(\omega)$ are fourier Transforms of $x_T(t)$, $y_T(t)$

$$X_T(\omega) = \int_{-T}^{T} x(t_1) e^{-j\omega t_1} dt_1 \qquad Y_T(\omega) = \int_{-T}^{T} y(t_2) e^{-j\omega t_2} dt_2$$

$$X_T^*(\omega) = \int_{-T}^{T} x(t_1) e^{j\omega t_1} dt_1$$

we know

$$S_{xy}(\omega) = \lim_{T\to\infty} \frac{E[X_T^*(\omega) Y_T(\omega)]}{2T}$$

$$= \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} x(t_1) e^{j\omega t_1} dt_1 \cdot \int_{-T}^{T} y(t_2) e^{-j\omega t_2} dt_2\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\int_{-T}^{T}\int_{-T}^{T} E[x(t_1) y(t_2)] e^{-j\omega(t_2-t_1)} dt_2\, dt_1\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{XY}(t_1, t_2) e^{-j\omega(t_2-t_1)} dt_2\, dt_1$$

Taking inverse ~~laplace~~ Transform

$$IFT[S_{xy}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xy}(t_1, t_2) e^{-j\omega(t_2-t_1)} e^{j\omega\tau} dt_2\, dt_1\, d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{XY}(t_1, t_2)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(t_2-(t_1+\tau))} d\omega\right] dt_2\, dt_1$$

$$IFT[S_{xy}(\omega)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xy}(t_1, t_2)\, \delta(t_2 - (t_1 + \tau))\, dt_2\, dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xy}(t_1, t_1 + \tau)\, dt_1$$

$$IFT[S_{xy}(\omega)] = A[R_{xy}(t_1, t_1 + \tau)]$$

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} A[R_{xy}(t_1, t_1+\tau)]\, e^{-j\omega\tau}\, d\tau$$

$$\boxed{\therefore \quad S_{xy}(\omega) \overset{FT}{\longleftrightarrow} A[R_{xy}(t_1, t_1+\tau)]}$$

For WSS
$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau.$$

$$R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau}\, d\omega$$

||ly
$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau)\, e^{-j\omega\tau}\, d\tau.$$

$$R_{yx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yx}(\omega)\, e^{j\omega\tau}\, d\omega.$$

* Expression for cross power spectral density :-

Let us consider two random process $x(t)$ & $y(t)$ whose random functions are $x(t)$ & $y(t)$ truncated portions $x_T(t)$, $y_T(t)$ in interval $(-T, T)$.

i.e.
$$x_T(t) = \begin{cases} x(t) & \text{for } -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

$$y_T(t) = \begin{cases} y(t) & \text{for } -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

$$X_T(t) \xleftrightarrow{FT} X_T(\omega) \qquad Y_T(t) \xleftrightarrow{FT} Y_T(\omega)$$

Power in truncated portions

$$P_{XY}(T) = \frac{1}{2T}\int_{-T}^{T} X_T(t)\, Y_T(t)\, dt$$

By parsevals theorem

$$P_{XY}(T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X_T^*(\omega)\, Y_T(\omega)}{2T}\, d\omega$$

Let $T \to \infty$ and apply Expected value

Total average power

$$P_{XY} = \lim_{t\to\infty} E\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X_T^*(\omega)\, Y_T(\omega)\, d\omega}{2T}\right]$$

$$= \lim_{t\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E[X_T^*(\omega)\, Y_T(\omega)]}{2T}\, d\omega$$

$$= \cancel{\lim_{t\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(t,t)\, d\omega}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[X_T^*(\omega)\, Y_T(\omega)]}{2T}\, d\omega$$

total cross power

$$\boxed{P_{XY} = P_{YX}^*}$$

$$P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, d\omega$$

where

$$\boxed{\begin{aligned} S_{XY}(\omega) &= \lim_{T\to\infty} \frac{E[X_T^*(\omega)\, Y_T(\omega)]}{2T} \\ \text{Illy}\quad S_{YX}(\omega) &= \lim_{T\to\infty} \frac{E[Y_T^*(\omega)\, X_T(\omega)]}{2T} \end{aligned}}$$

Properties:

1. $S_{XY}(\omega) = S_{YX}(-\omega) = S^{*}_{YX}(\omega)$

2. $Re[S_{XY}(\omega)]$ and $Re[S_{YX}(\omega)]$ are even function of ω.

3. $Im[S_{XY}(\omega)]$ & $Im[S_{YX}(\omega)]$ are odd function of ω.

4. $S_{XY}(\omega) = 0$ & $S_{YX}(\omega) = 0$ if $X(t)$ & $Y(t)$ are orthogonal.

5. If $x(t)$ and $y(t)$ are uncorrelated & have central means $\bar{X}$ & $\bar{Y}$. $S_{XY}(\omega) = S_{YX}(\omega) = 2\pi \bar{X}\bar{Y}\delta(\omega)$

6. $A[R_{XY}(t, t+\tau)] \longleftrightarrow S_{XY}(\omega)$

$A[R_{YX}(t, t+\tau)] \longleftrightarrow S_{YX}(\omega)$.

For WSS

$R_{XY}(\tau) \longleftrightarrow S_{XY}(\omega)$

$R_{YX}(\tau) \longleftrightarrow S_{YX}(\omega)$

Properties of CPSD

1. $S_{XY}(\omega) = S_{YX}(-\omega) = S^{*}_{YX}(\omega)$

proof :- $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$

$\tau = -\tau$

$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(-\tau) e^{-j\omega(-\tau)} d\tau$

we know $R_{XY}(-\tau) = R_{YX}(+\tau)$

$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{j\omega\tau} d\tau$

$S_{XY}(\omega) = S_{YX}(-\omega) = S^{*}_{XY}(\omega)$

2. Real part of $S_{XY}(\omega)$ & $S_{YX}(\omega)$ are even functions of ω.

proof :-

$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau}$

$e^{-j\omega\tau} = \cos\omega\tau - j\sin\omega\tau$

$Re[S_{XY}(\omega)] = \int_{-\infty}^{\infty} R_{XY} \cos\omega\tau \, d\tau$

$\cos\omega\tau$ is even function means

$\cos(-\omega\tau) = \cos\omega\tau$

$Re[S_{XY}(-\omega)] = \int_{-\infty}^{\infty} R_{XY}(\tau) \cos(-\omega\tau) d\tau$

$= \int_{-\infty}^{\infty} R_{XY}(\tau) \cos\omega\tau \, d\tau$

$Re[S_{XY}(-\omega)] = Re[S_{XY}(\omega)]$ //

llar $Re[S_{YX}(-\omega)] = Re[S_{YX}(\omega)]$ //

3. Imaginary part of $S_{XY}(\omega)$ & $S_{YX}(\omega)$ are odd functions.

proof :- $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$

$e^{-j\omega\tau} = \cos\omega\tau - j\sin\omega\tau$

$Im[S_{XY}(\omega)] = \int_{-\infty}^{\infty} R_{XY}(\tau)[-\sin\omega\tau] d\tau$

$= -\int_{-\infty}^{\infty} R_{XY}(\tau_1) \sin\omega\tau \, d\tau$

$\sin\omega\tau$ is odd function means

$\sin(-\omega\tau) = -\sin\omega\tau$

$Im[S_{XY}(-\omega)] = \int_{-\infty}^{\infty} R_{XY}(\tau_1) \sin(-\omega\tau) d\tau$

$= -\int_{-\infty}^{\infty} R_{XY}(\tau_1) \sin\omega\tau \, d\tau$

$Im[S_{XY}(-\omega)] = -Im[S_{XY}(\omega)]$

llar $Im[S_{YX}(-\omega)] = -Im[S_{YX}(\omega)]$ /

4. $S_{XY}(\omega) = 0$ & $S_{YX}(\omega) = 0$ if $X(t)$ & $Y(t)$ are orthogonal.

proof :- $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$

if $X(t)$ & $Y(t)$ are orthogonal

then $R_{XY}(\tau) = R_{YX}(\tau) = 0$

then $S_{XY}(\omega) = 0$ & $S_{YX}(\omega) = 0$.

5. if $X(t)$ & $Y(t)$ are uncorrelated & have constant mean values $\bar{X}, \bar{Y}$ then

$S_{XY}(\omega) = 2\pi \bar{X}\bar{Y} \delta(\omega)$

proof :- $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$

$= \int_{-\infty}^{\infty} E[X(t) Y(t+\tau)] e^{-j\omega\tau} d\tau$

$= \int_{-\infty}^{\infty} E[X(t)] E[Y(t+\tau)] e^{-j\omega\tau} d\tau$

$= \bar{X}\bar{Y} \int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau$ $\quad \boxed{\int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau = 2\pi\delta(\omega)}$

$\boxed{S_{XY}(\omega) = \bar{X}\bar{Y}\, 2\pi\, \delta(\omega)}$

(P) The cross psd is given as $S_{XY}(\omega) = \frac{1}{(a+j\omega)^2}$, $a>0$, a is constant. Find cross correlation function.

Sol

$$S_{XY}(\omega) = \frac{1}{(a+j\omega)^2}$$

$$R_{XY}(\omega) = F^{-1}[S_{XY}(\omega)] = F^{-1}\left[\frac{1}{(a+j\omega)^2}\right] = te^{-at}u(t)$$

(P) Consider the random process $X(t) = A\cos(\omega t+\theta)$ where A & ω are real ~~constants~~ & θ is random variable uniformly distributed over $[0, 2\pi]$. Find average power P_{XX}.

Sol

$$P_{XX} = A\{E[X^2(t)]\}$$

Uniformly distributed function: $f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0<\theta<2\pi \\ 0 & \text{otherwise} \end{cases}$

$\left[\because \frac{1}{b-a} = \frac{1}{2\pi - 0} = \frac{1}{2\pi}\right]$

Now $X(t) = A\cos(\omega t+\theta)$

$$E[X^2(t)] = \int_{-\infty}^{\infty} x^2(t) f_\theta(\theta)\, d\theta$$

$$= \int_0^{2\pi} A^2 \cos^2(\omega t+\theta) \frac{1}{2\pi}\, d\theta$$

$$= \frac{A^2}{2\pi}\int_0^{2\pi}\left[\frac{1+\cos 2(\omega t+\theta)}{2}\right] d\theta$$

$$= \frac{A^2}{4\pi}\left\{[\theta]_0^{2\pi} + \left[\frac{-\sin 2(\omega t+\theta)}{2}\right]_0^{2\pi}\right\}$$

$$= \frac{A^2}{4\pi}\left\{2\pi \mp \frac{1}{2}\left[\sin(4\pi+2\omega t) - \sin 2\omega t\right]\right\}$$

$$= \frac{A^2}{4\pi}\left\{2\pi - \frac{1}{2}\left[+\cancel{\sin 2\omega t} - \cancel{\sin 2\omega t}\right]\right\}$$

$$= \frac{A^2}{4\pi}\left[2\pi \mp 0\, \cancel{\sin 2\omega t}\right]$$

$$= \frac{A^2}{2}$$

$\left[\because \sin(4\pi+2\omega t) = \sin 4\pi \cos 2\omega t + \cos 4\pi \sin 2\omega t = 0 + \cos 4\pi \sin 2\omega t = \sin 2\omega t\right]$

Time average power $P_{XX} = A[E[X^2(t)]]$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{A^2}{2}\, dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\frac{A^2}{2}[t]_{-T}^{T}$$

$$P_{XX} = \frac{1}{2T}\cdot\frac{A^2}{2}\cdot(T+(-T)) = \frac{A^2}{2}$$

If $R_{XX}(\tau) = \frac{A_0^2}{2} \sin w_0\tau$ find $S_{XX}(w)$

sol

$$S_{XX}(w) = FT[R_{XX}(\tau)]$$

$$= \int_{-\infty}^{\infty} \frac{A_0^2}{2} \sin w_0\tau \, e^{-jw\tau} \, d\tau$$

$$\boxed{S_{XX}(w) = j\frac{A_0^2}{2}\pi[\delta(w+w_0) - \delta(w-w_0)]}$$

(P) If $R_{XX}(\tau) = \frac{A_0^2}{2} \cos w_0\tau$ find $S_{XX}(w)$

sol

$$S_{XX}(w) = FT[R_{XX}(\tau)] = \int_{-\infty}^{\infty} \frac{A_0^2}{2} \cos w_0\tau \, e^{-jw\tau} d\tau$$

$$= \frac{A_0^2}{2}\pi[\delta(w+w_0) + \delta(w-w_0)]$$

(P) The cross sponse spectral density is given as $S_{XY}(w) = w^2/w^2+1$. find out cross correlation function

sol

Given $S_{XY}(w) = \frac{w^2}{w^2+1}$

$$R_{XY}(\tau) = F^{-1}[S_{XY}(w)] = F^{-1}\left[\frac{w^2}{w^2+1}\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{w^2}{w^2+1} \, dw \quad \left[\because \int \frac{x^2}{a^2+x^2}dx = \frac{-x}{2(a^2+x^2)} + \frac{1}{2a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{w^2}{w^2+1} dw$$

$$= \frac{1}{2\pi}\left[\frac{-w}{2(w^2+1)} + \frac{1}{2a}\tan^{-1}\left(\frac{x}{a}\right)\right]_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi}[\,\cdot$$

$$R_{XY}(\tau) = F^{-1}\left[\frac{w^2}{w^2+1}\right]$$

$$= F^{-1}\left[\frac{w^2+1-1}{w^2+1}\right]$$

$$= F^{-1}\left[\frac{w^2+1}{w^2+1} - \frac{1}{w^2+1}\right]$$

$$= F^{-1}\left[1 - \frac{1}{w^2+1}\right]$$

$$= F^{-1}[1] - F^{-1}\left[\frac{1}{w^2+1}\right]$$

$$F^{-1}[1] = \sqrt{2\pi}\,\delta(t)$$

$$F^{-1}\left[\frac{1}{w^2+1}\right] = \sqrt{\frac{\pi}{2}}\,e^{-|t|}.$$

$$R_{XY}(\tau) = \sqrt{2\pi}\,\delta(t) - \sqrt{\frac{\pi}{2}}\,e^{-|t|}$$

(P) The autocorrelation function of a random process $R_{XX}(\tau) = 4\cos\omega_0\tau$ where ω_0 is a constant obtain its power spectral density.

Sol:

$$S_{XX}(\omega) = FT[R_{XX}(\tau)]$$

$$= 4\int_{-\infty}^{\infty} \cos\omega_0\tau \, e^{-j\omega\tau} d\tau$$

$$= 4\int_{-\infty}^{\infty} \left(\frac{e^{j\omega_0\tau} + e^{-j\omega_0\tau}}{2}\right) e^{-j\omega\tau} d\tau$$

$$= \frac{4}{2}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)\tau} + e^{-j(\omega+\omega_0)\tau} d\tau$$

$$= 2\left[\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)\tau} d\tau + \int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)\tau} d\tau\right]$$

$$= 2\left[2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right]$$

$$S_{XX}(\omega) = 4\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] \;//$$

(P) If $R_{XX}(\tau) = \frac{A_0^2}{2}\sin\omega_0\tau$ find $S_{XX}(\omega)$

Sol:

$$S_{XX}(\omega) = FT[R_{XX}(\tau)]$$

$$= FT\left[\frac{A_0^2}{2}\sin\omega_0\tau\right]$$

$$= \frac{A_0^2}{2}\int_{-\infty}^{\infty} \sin\omega_0\tau \, e^{-j\omega\tau} d\tau$$

$$= \frac{A_0^2}{2}\int_{-\infty}^{\infty}\left[\frac{e^{j\omega_0\tau} - e^{-j\omega_0\tau}}{2j}\right] e^{-j\omega\tau} d\tau$$

$$= \frac{A_0^2}{4j}\int_{-\infty}^{\infty}\left[e^{-j(\omega-\omega_0)\tau} - e^{-j(\omega+\omega_0)\tau}\right] d\tau$$

$$= \frac{A_0^2}{4j}\left[\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)\tau} d\tau - \int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)\tau} d\tau\right]$$

$$= \frac{A_0^2}{4j}\left[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right]$$

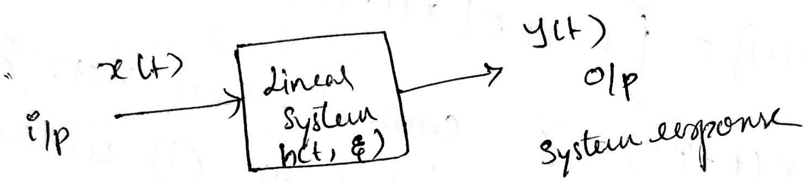
$$= \frac{A_0^2 \, 2\pi}{2\cdot 4j}\left[-\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right)\right] = \frac{+A_0^2\pi}{2} j\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

UNIT - 5 : Lineal System Response to Random inputs

Lineal System :- A system is called lineal if it has two mathematical properties : homogeneity & additivity.

LTI :- (~~Lineal~~ Time invariant System) - if the o/p char doesn't change with time.

→ Lineal systems whose o/ps for a lineal combination of i/p are same as a lineal combination of individual responses to those i/ps.



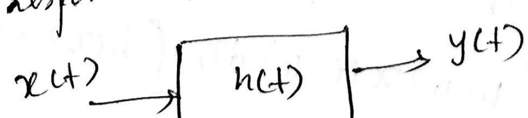
System response $y(t) = L\{x(t)\}$

where L - operator.

→ LTI System - The system that are both lineal & time - invariant are called LTI systems.

System response :- Consider a continuous LTI system with impulse response h(t). Assume that system is always causal & stable. When continuous random process x(t) is applied on this system the o/p response is also continuous time random process y(t). If x(t) & y(t) are discrete time ~~Random~~ random processes, the the lineal system is called discrete ~~time~~ time system.

Let a random process x(t) be applied to a continuous LTI system whose impulse is h(t) as shown in figure. Then the o/p response Y(t) is also a random process.



The o/p response is convolution integral $Y(t) = h(t) * X(t)$

$$= \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau$$

Mean value of o/p response :- Consider a random process x(t) is WSS random process.

Mean value of o/p response $= E[Y(t)]$

then $E[Y(t)] = E[h(t) * x(t)]$

$$= E\left[\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau\right]$$

$$E[Y(t)] = \int_{-\infty}^{\infty} h(\tau)\, E[x(t-\tau)]\, d\tau \rightarrow (1)$$

But $E[X(t-\tau)] = \bar{X}$ = constant, since $x(t)$ is WSS

$E[Y(t)] = \bar{Y}$. So above eqn (1) becomes as below

$$\bar{Y} = \bar{X}\int_{-\infty}^{\infty} h(\tau)\, d\tau \rightarrow (2)$$

$$FT[h(\tau)] = H(\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau \rightarrow (3)$$

At $\omega = 0$ $H(0) = \int_{-\infty}^{\infty} h(t)\, dt$ is called zero frequency response of system.

$$\bar{Y} = \bar{X}\int_{-\infty}^{\infty} h(\tau)\, d\tau$$

Thus the mean value of a o/p response $Y(t)$ of a WSS random process is equal to the product of the mean value of i/p process & zero frequency response of the system.

Mean Square value of o/p response :-

$$E[Y^2(t)] = E[(h(t) * x(t))^2]$$

$$= E[(h(t) * x(t))\,(h(t) * x(t))]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, x(t-\tau_2)\, d\tau_2\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t-\tau_1)\, X(t-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2\right]$$

$$E[y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[x(t-\tau_1)\, x(t-\tau_2)]\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2$$

where τ_1 & τ_2 are shifts in time intervals. if $x(t)$ is a WSS RP

then $E[x(t-\tau_1)\, x(t-\tau_2)] = R_{xx}(\tau_1 - \tau_2)$

Therefore $E[y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\tau_1-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2$

This Expression is independent of time t.
it represents o/p power.

$E\{x(t)\, x(t+\tau)\}$
$= R_{xx}(\tau)$
$= R(t+\tau-t)$

Autocorrelation function of o/p response of

$R_{yy}(\tau_1, \tau_2) = E[y(t_1)\, y(t_2)]$

$= E[(h(t_1) * x(t_1))\, (h(t_2) * x(t_2))]$

$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t_1-\tau_1)\, d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2)\, x(t_2-\tau_2)\, d\tau_2\right]$

$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1-\tau_1)\, x(t_2-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2\right]$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[x(t_1-\tau_1)\, x(t_2-\tau_2)]\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2$

We know that $E[x(t_1-\tau_1)\, x(t_2-\tau_2)] = R_{xx}(t_2-t_1+\tau_1-\tau_2)$, let the time difference

if $x(t)$ is a WSS random process

$\tau = t_1 - t_2$ & $t = t_1$ Then

$t_2 = \tau + t_1$
$= \tau + t$

$E[x(t-\tau_1)\, x(t+\tau-\tau_2)] = R_{xx}(\tau+\tau_1-\tau_2)$ then

$\left[\begin{matrix} R_{xx}(t+\tau-\tau_2-t+\tau_1) \\ R_{xx}(\tau+\tau_1-\tau_2)\end{matrix}\right]$

$R_{yy}(t, t+\tau) = R_{yy}(t, \tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\tau+\tau_1-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2$

if $= R_{xx}(\tau)$ is the auto correlation function is a function of only τ.

Hence o/p random process $y(t)$ is also WSS random process.

$R_{xx}(\tau) * h(\tau) * h(-\tau) = R_{yy}(t, t+\tau) = R_{yy}(\tau)$

ACF is two fold convolution of applied input & impulse response

Cross correlation function of o/p response :-

If the i/p $x(t)$ is WSS random process, then the cross correlation function of i/p $x(t)$ and o/p $y(t)$ is

$$R_{xy}(t, t+\tau) = E[x(t)\, y(t+\tau)]$$

$$= E\left[x(t) \int_{-\infty}^{\infty} h(\tau_1)\, x(t+\tau-\tau_1)\, d\tau_1\right]$$

$$= \int_{-\infty}^{\infty} E\left[x(t)\, x(t+\tau-\tau_1)\right] h(\tau_1)\, d\tau_1$$

$$R_{xy}(t, t+\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau-\tau_1)\, h(\tau_1)\, d\tau_1$$

which is the convolution of $R_{xx}(\tau)$ & $h(\tau)$

$\therefore$ $R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$ similarly ~~we can show that~~

Ilar $R_{yx}(\tau) = R_{xx}(\tau) * h(-\tau)$

This shows that $x(t)$ and $y(t)$ are jointly WSS. And we can also relate the auto correlation functions & cross correlation functions as

$$R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$$

$$R_{yy}(\tau) = R_{yx}(\tau) * h(\tau)$$

Spectral characteristics of a System Response :-

Consider that the random process $x(t)$ is a WSS random process with autocorrelation function $R_{xx}(\tau)$ applied through an LTI system. It is noted that o/p response $y(t)$ is also a WSS & processes $x(t)$ & $y(t)$ are jointly WSS. We can obtain power spectral characteristics of the o/p process $y(t)$ by taking the Fourier transform of the correlation functions

PSD of response :- Consider that a random process $x(t)$ is applied on an LTI system having a transfer function $H(\omega)$. The o/p response is $y(t)$. If the power spectrum of the i/p process is $S_{xx}(\omega)$, then the power spectrum of the o/p response is given by

$$S_{yy}(\omega) = |H(\omega)|^2\, S_{xx}(\omega)$$

Proof: Let $R_{YY}(\tau)$ be the autocorrelation of the o/p response $Y(t)$. Then the power spectrum of the response is the FT of $R_{YY}(\tau)$

Therefore $S_{YY}(\omega) = F[S_{YY}(\omega)] = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-j\omega\tau} d\tau$

We know that $R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2) h(\tau_1) h(\tau_2) d\tau_1 d\tau_2$

Then $S_{YY}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2) h(\tau_1) h(\tau_2) d\tau_1 d\tau_2 e^{-j\omega\tau} d\tau$

$= \int_{-\infty}^{\infty} h(\tau_1) \int_{-\infty}^{\infty} h(\tau_2) \int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2) e^{-j\omega(t-(\tau_1-\tau_2))} d\tau d\tau_2 d\tau_1$

$= \int_{-\infty}^{\infty} h(\tau_1) e^{+j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2) e^{-j\omega\tau_2} \int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2) e^{-j\omega t} e^{j\omega\tau_1} e^{-j\omega\tau_2} d\tau d\tau_2 d\tau$

Let $\tau+\tau_1-\tau_2 = t$; $d\tau = dt$; $\tau = t-(\tau_1-\tau_2)$

$\therefore S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\tau_1) e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2) e^{-j\omega\tau_2} d\tau_2 \int_{-\infty}^{\infty} R_{XX}(t) e^{-j\omega t} dt$

We know that $H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega t} dt$; $H^*(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega t}$

$\therefore S_{YY}(\omega) = H^*(\omega) H(\omega) S_{XX}(\omega) = H(-\omega) H(\omega) S_{XX}(\omega)$

$$\therefore \boxed{S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)}$$

$$\boxed{\text{Average power } P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega}$$

Hence proved

Illy we can prove that the cross PSD function is

$S_{XY}(\omega) = S_{XX}(\omega) H(\omega)$ & $S_{YX}(\omega) = S_{XX}(\omega) H(-\omega)$

Spectrum Bandwidth: The spectral density is mostly concentrated at a certain frequency value. It decreases at other frequencies. The BW of spectrum is the range of frequencies having significant values. It is defined as the measure of spread of spectral density and is also called RMS B.W & normalized B.W. It is given by

$$\boxed{W_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega}}$$

Types of random processes :-

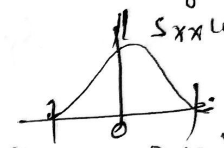
Depending upon frequency components random processes are classified into 4 types.

1. Lowpass random process
2. Band pass random process
3. Band limited random process
4. Narrow Band random process

— A RP $x(t)$ is called lowpass RP if its $S_{XX}(\omega)$ have significant component within the frequency

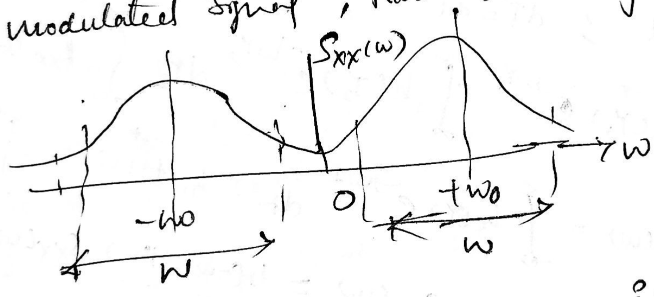
Baseband signals.

Power spectrum of LP

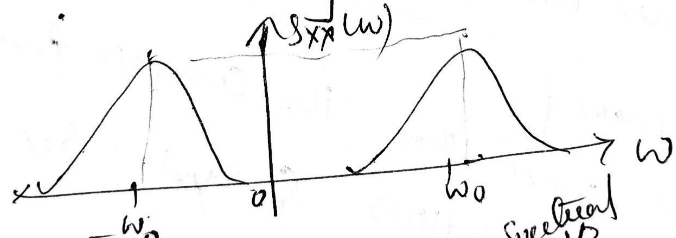


1. Low pass RP — Ex - Speech, Image, Video.
2. Band pass RP — $S_{XX}(\omega)$ have significant components within a B.W 'W' that doesn't include $\omega=0$. BW practically small amount of power spectrum at $\omega=0$

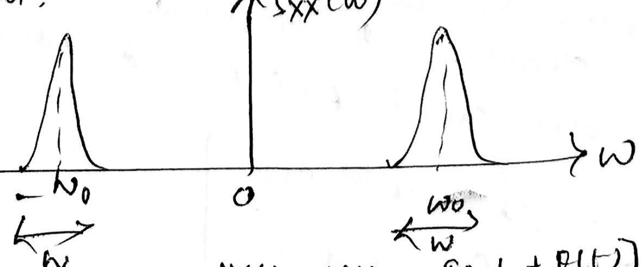
Ex: modulated signal, Noise transmitting over comm channel



3. Band limited random process — if power spectrum components that doesn't include zero outside the frequency band of width W that doesn't include $\omega=0$.



4. Narrow Band Random process — Spectral B.W is very small compared to the band centre frequency. i.e, $W \ll \omega_0$ where W - Band width and ω_0 is frequency at which the power spectrum is maximum.



$N(t) = A(t)\cos[\omega_0 t + \theta(t)]$

$A(t)$ - Amp & $\theta(t)$ phase of RP

Representation of a narrow Band process :-

For any arbitrary WSS R.P. $N(t)$ the quadrature form of narrow band process can be represented as $N(t) = X(t)\cos\omega_0 t - Y(t)\sin\omega_0 t$.
where $X(t)$, $Y(t)$ are respectively called the in-phase & quadrature phase components of $N(t)$. They can be expressed as

$$x(t) = A(t)\cos(\theta(t)).$$

$$Y(t) = A(t)\sin(\theta(t))$$ and relation b/w the processes $A(t)$ & $\theta(t)$

are given by $$A(t) = \sqrt{x^2(t) + y^2(t)}$$

$$\theta(t) = \tan^{-1}\left(\frac{Y(t)}{X(t)}\right)$$

Properties of Band limited Random process :-

Let $N(t)$ is Band limited WSS random process with zero mean value and power spectral density $S_{NN}(\omega)$. $N(t) = X(t)\cos\omega_0 t - Y(t)\sin\omega_0 t$

1. If $N(t)$ is WSS then $X(t)$ & $Y(t)$ are jointly WSS
2. If $N(t)$ has zero mean i.e. $E[N(t)]=0$, then $E[X(t)] = E[Y(t)] = 0$
3. MSV $E[N^2(t)] = E[X^2(t)] = E[Y^2(t)]$
4. $X(t)$ & $Y(t)$ have same Autocorrelation function. $R_{XX}(\tau) = R_{YY}(\tau)$.
5. The Cross correlation function $R_{YX}(\tau) = -R_{YX}(\tau)$.
   if processes are orthogonal, then $R_{YX}(\tau) = R_{YX}(\tau) = 0$.
6. Both $X(t)$, $Y(t)$ have same PSD's

$$S_{YY}(\omega) = S_{XX}(\omega) = \begin{cases} S_N(\omega-\omega_0) + S_N(\omega+\omega_0) & \text{for } |\omega| \le \omega_0 \\ 0 & \text{otherwise} \end{cases}$$

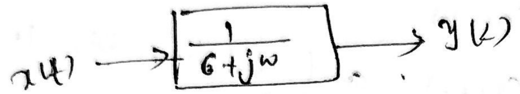
7. CPSD $S_{XY}(\omega) = -S_{YX}(\omega)$
8. $N(t)$ is a Gaussian random process, then $X(t)$ & $Y(t)$ are jointly gaussian
9. The relation b/w autocorrelation & power spectrum $S_{NN}(\omega)$ is

$$R_{XX}(\tau) = \frac{1}{\pi}\int_0^\infty [S_{NN}(\omega)\cos(\omega-\omega_0)\tau]\, d\omega.$$

$$R_{YY}(\tau) = \frac{1}{\pi}\int_0^\infty [S_{NN}(\omega)\cos(\omega-\omega_0)\tau]\, d\omega$$

10. If $N(t)$ is zero mean gaussian and its psd, $S_N(\omega)$ is ~~system~~ symmetric about $\pm\omega_0$ then $X(t)$ & $Y(t)$ are statistically independent.

Q) Consider a linear system shown below. $x(t)$ is i/p, $y(t)$ is o/p. If ACF of $x(t)$ is $R_{xx}(\tau) = 3\delta(t)$ find PSD, ACF & MS value of the o/p $y(t)$



Sol Given $H(\omega) = \frac{1}{6+j\omega}$

(i) PSD $= S_{xx}(\omega)\,|H(\omega)|^2$

$$H(\omega) = \frac{1}{6+j\omega}$$

$$|H(\omega)| = \left|\frac{1}{6+j\omega}\right|$$

$$|H(\omega)| = \frac{1}{\sqrt{36+\omega^2}}$$

$$S_{xx}(\omega) = FT[R_{xx}(\tau)]$$

$$= F\{3\delta(t)\}$$

$$= 3(1) = 3.$$

$$PSD = 3\left(\frac{1}{\sqrt{36+\omega^2}}\right)^2 = \frac{3}{36+\omega^2}$$

(ii) $R_{yy}(\tau) = F^{-1}\{S_{yy}(\omega)\}$

$$= F^{-1}\left\{\frac{3}{6^2+\omega^2}\right\} =$$

$$F^{-1}\left\{\frac{2ab}{b^2+\omega^2}\right\} = ae^{-t}.$$

$$t = \frac{1}{6^2+0} = \frac{1}{12}$$

$$= \frac{3}{12} e^{-6|\tau|}$$

$$= \frac{1}{4} e^{-6|\tau|}$$

(iii) $E[Y^2(t)] = R_{yy}(0) = P$

$$= \frac{1}{4} e^{-6|0|}$$

$$= \frac{1}{4} = 0.25$$

② $x(t)$ is WSS $E[x(t)]=0$ $R_{xx}(\tau) = e^{-2|\tau|}$; $H(\omega) = \frac{1}{2+j\omega}$

find $S_{YY}(\omega)$ & $E[Y(t)]$

sol 1) $S_{YY}(\omega) = S_{XX}(\omega)\ |H(\omega)|^2$

$S_{XX}(\omega) = FT[R_{XX}(\tau)] = FT[e^{-2|\tau|}] = \frac{2(1)(2)}{2^2+\omega^2} = \frac{4}{4+\omega^2}$

$|H(\omega)|^2 = \left|\frac{1}{2+j\omega}\right|^2 = \left(\frac{1}{\sqrt{4+\omega^2}}\right)^2 = \frac{1}{4+\omega^2}$

$S_{YY}(\omega) = \frac{4}{4+\omega^2} \cdot \frac{1}{4+\omega^2} = \frac{4}{(4+\omega^2)^2}$ //

2) $P_{XX} = R_{XX}(0)$

$\cancel{R_{XX}}(\tau) = IFT[\cancel{S_{XX}(\tau)}] = IFT\left[\frac{4}{4+\omega^2}\right] =$

2) $E[Y(t)] = \bar{X}\ H(0)$

$= E[X(t)] \cdot H(0)$

$E[Y(t)] = 0 \cdot \frac{1}{2} = 0$ //

③ If $X(t)$ is WSS, $E[X(t)] = 2$; $R_{XX}(T) = 3e^{-2|\tau|}$, $h(t) = e^{-4t}$

find $S_{YY}(\omega)$, $E[Y(t)]$

sol ① $S_{YY}(\omega) = S_{XX}(\omega)\ |H(\omega)|^2$

$= FT[R_{XX}(\tau)] \left[\frac{1}{j\omega+4}\right]^2$

$= FT[3e^{-2|\tau|}] \left|\frac{1}{j\omega+4}\right|^2$

$= 3\left(\frac{4}{4+\omega^2}\right)\left(\frac{1}{\omega^2+16}\right) = \frac{12}{(\omega^2+4)(\omega^2+16)}$ //

② $E[Y(t)] = E[X(t)]\ H(0)$

$= 2\left(\frac{1}{0+4}\right) = 2/4 = 1/2 = 0.5$ //

① A random ~~process~~ noise ~~X(t)~~ having power spectrum $S_{XX}(\omega) = \frac{3}{49+\omega^2}$ is applied to a n/w for which $h(t) = t^2 e^{-7|t|}$. The n/w response is $y(t)$.

a) Avg power of $x(t)$

b) Find power spectrum of $y(t)$

c) Find Average power of $y(t)$.

Sol :- ① $P_{XX} = R_{XX}(0)$

$$R_{XX}(\tau) = F^{-1}\{S_{XX}(\tau)\} = F^{-1}\left\{\frac{3}{49+\omega^2}\right\} = F^{-1}\left\{\frac{3\times 2\times 7}{7^2+\omega^2}\right\}\frac{1}{14}$$

$$= \frac{3}{14} e^{-7|\tau|}$$

$P_{XX} = R_{XX}(0)$

$$\boxed{P_{XX} = \frac{3}{14} e^{-7(0)} = \frac{3}{14} \text{ watts}}$$

② $S_{YY}(\omega) = S_{XX}(\omega)\, |H(\omega)|^2$

$S_{XX}(\omega) = \frac{3}{49+\omega^2}$, $\{H(\omega)\} = F\{h(t)\}$

$$= F\{t^2 e^{-7|t|}\}$$

$$= \frac{2}{(s+7)^3} = \frac{2}{(j\omega+7)^3}$$

$$|H(\omega)| = \left|\frac{2}{(j\omega+7)^3}\right| = \frac{2}{\left(\sqrt{7^2+\omega^2}\right)^3}$$

$S_{YY}(\omega) = S_{XX}(\omega)\, |H(\omega)|^2$ $\qquad |H(\omega)|^2 = \frac{4}{(7^2+\omega^2)^3}$

$$= \frac{3}{49+\omega^2} \cdot \frac{4}{(49+\omega^2)^3}$$

$$\boxed{S_{YY}(\omega) = \frac{12}{(49+\omega^2)^4}}$$

③ $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{12}{(49+\omega^2)^4}\, d\omega$ $\qquad \left[\because \int_{-\infty}^{\infty}\frac{1}{(a^2+\omega^2)^4}\,d\omega = \frac{5\pi}{16a^7}\right]$

$$= \frac{12}{2\pi}\left[\int_{-\infty}^{\infty}\frac{1}{(49+\omega^2)^4}\,d\omega\right]$$

$$= \frac{12}{2\cancel{\pi}}\,\frac{5\cancel{\pi}}{16(7)^7} = 2.276\times 10^{6}\ \mu\omega$$ //

Average power $P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, |H(\omega)|^2\, d\omega$$

<u>Cross power density spectrum of i/p & o/p of system</u> :-

$$S_{XY}(\omega) = FT\{R_{XY}(\tau)\} = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau) * h(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau-\tau_1)\, h(\tau_1)\, e^{-j\omega\tau}\, d\tau\, d\tau_1$$

Let $\tau - \tau_1 = t$ $\qquad d\tau = dt$

$\tau = t + \tau_1$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(t)\, h(\tau_1)\, e^{-j\omega(t+\tau_1)}\, dt\, d\tau_1$$

$$= \int_{-\infty}^{\infty} R_{XX}(t)\, e^{-j\omega t}\, dt \int_{-\infty}^{\infty} h(\tau_1)\, e^{-j\omega\tau_1}\, d\tau_1$$

$$= FT[R_{XX}(\tau)]\, FT[h(\tau)]$$

$$\boxed{S_{XY}(\omega) = S_{XX}(\omega)\, H(\omega)}$$

$|||^{ly}$ $S_{YX}(\omega) = S_{XX}(\omega)\, H(-\omega)$

$$= S_{XX}(\omega)\, H^*(\omega)$$

Cross power $\cdot$ $P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, d\omega$ $\qquad$ $P_{YX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)\, d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, H(\omega)\, d\omega \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, H^*(\omega)\, d\omega$$

Total power $P = P_{XY} + P_{YX}$ = Total average power.

(P) ~~A~~ A WSS RP $X(t)$ is applied to i/p of an LTI system whose impulse response is $5te^{-2t}$ the mean of $X(t)$ is 3 find mean of system response

Sol: Given $h(t) = 5te^{-2t}$

$$E[X(t)] = \bar{X} = 3$$

$$L[e^{-at}] = \frac{1}{s+a}$$

$$E[Y(t)] = E[X(t)]H(0)$$

$$L[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}}$$

$$= \bar{X}H(0)$$

$$H(\omega) = \frac{5(1)}{(s+2)^2} = \frac{5}{(s+2)^2} = \frac{5}{(j\omega+2)^2}$$

$$H(0) = \frac{5}{(0+2)^2} = \frac{5}{4}$$

$$E[Y(t)] = \bar{X}H(0)$$

$$= 3\left(\frac{5}{4}\right) = \frac{15}{4} = 3.75$$

(P) $\bar{X} = 3$; $h(t) = 5t^2 e^{-2t}$ mean of system

Sol

$$H(\omega) = 5\left[\frac{2!}{(s+2)^3}\right] = \frac{10}{(s+2)^3} = \frac{10}{(j\omega+2)^3}$$

$$H(0) = \frac{10}{(0+2)^3} = \frac{10}{8} = 5/4 = 1.25$$

$$E[Y(t)] = \bar{X}\cdot H(0)$$

$$= 3 \times 1.25 = 3.75.$$

(P) $\bar{X} = 2$; $h(t) = 2e^{-2t}u(t)$

Sol:

$$H(\omega) = FT\{h(t)\} = FT\{2e^{-2t}u(t)\}$$

$$= \int_{-\infty}^{\infty} 2e^{-2t}u(t)\, e^{-j\omega t}\, dt$$

$u(t)$ - Unit step

$$= \int_0^{\infty} 2e^{-(2+j\omega)t}\, dt$$

$$= 2\left.\frac{e^{-(2+j\omega)t}}{-(2+j\omega)}\right|_0^{\infty}$$

$$H(\omega) = \frac{2}{-(2+j\omega)}\left[e^{-\infty} - 1\right] = \frac{2}{2+j\omega}$$

$$H(0) = \frac{2}{2} = 1 \quad ; \quad \bar{X} = 2 \quad ; \quad E[Y(t)] = \bar{X}H(0) = 2(1) = 2$$

Random process: Spectral characteristics:

(1) power spectrum density (or) power density spectrum:

Definition (1):-

The power spectral density of a wide sense stationary (WSS) random process $X(t)$ is defined as the fourier Transform of the autocorrelation function $[R_{xx}(\tau)]$ of $X(t)$. It can be expressed as

$$\boxed{S_{xx}(\omega) = FT[R_{xx}(\tau)]}$$

$$\therefore \boxed{S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau}$$

$\therefore$ we can obtain the autocorrelation function from the PSD by taking Inverse fourier Transform

$$\boxed{R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega.}$$

$\therefore$ Power density spectrum $S_{xx}(\omega)$ and the auto correlation function $R_{xx}(\tau)$ are Fourier Transform pairs.

Definition (2) :- The power spectral density can also be defined as

$$S_{XX}(\omega) = \underset{T \to \infty}{Lt} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

where $|X_T(\omega)|$ is a Fourier transform of $x(t)$ in the interval $[-T, T]$.

VVimp
sem (2)

Relationship betwenn power spectral density & autocorrelation function of random process

(or)

weiner khinchin relation

Ans:- The power spectral density and the time average of the autocorrelation function form a fourier transform pair.

$$\therefore \boxed{S_{XX}(\omega) \underset{IFT}{\overset{FT}{\rightleftarrows}} R_{XX}(\tau)}$$

proof: we have power spectral density

$$S_{XX}(\omega) = \underset{T \to \infty}{Lt} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

we know that, we can write

$$\boxed{|X_T(\omega)|^2 = X_T^*(\omega)\, X_T(\omega)}$$

$$\therefore \quad S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{E[X_T^*(\omega)\, X_T(\omega)]}{2T} \quad \text{—} \; ①$$

here

$$X_T(\omega) = \int_{-T}^{T} x(t_2)\, e^{-j\omega t_2}\, dt_2$$

$$X_T^*(\omega) = \int_{-T}^{T} x(t_1)\, e^{j\omega t_1}\, dt_1$$

$\therefore$ eq ① can be written as

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} E\left[\int_{-T}^{T} x(t_1) e^{j\omega t_1} dt_1 \cdot \int_{-T}^{T} x(t_2) e^{-j\omega t_2} dt_2\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} E\left[\int_{-T}^{T}\int_{-T}^{T} x(t_1)x(t_2)\, e^{-j\omega(t_2-t_1)}\, dt_1 . dt_2\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\left[\int_{-T}^{T}\int_{-T}^{T} E[x(t_1)x(t_2)] . e^{-j\omega(t_2-t_1)}\, dt_1 . dt_2\right]$$

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\, e^{-j\omega(t_2-t_1)}\, dt_1 . dt_2$$

By taking Inverse fourier transform on both sides

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\underset{T\to\infty}{Lt}\ \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1,t_2)\, e^{-j\omega(t_2-t_1)}\, dt_1.dt_2\right] e^{j\omega\tau}\, d\omega$$

$$= \underset{T\to\infty}{Lt}\ \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1,t_2)\ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(t_2-t_1)} . e^{j\omega\tau}\, d\omega.dt_1.dt_2.$$

$$= \underset{T\to\infty}{Lt}\ \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1,t_2)\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau-t_2+t_1)}\, d\omega.dt_1.dt_2$$

we have

$$F[\delta(t)] = 1$$

$$\delta(t) = F^{-1}[1]$$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\cdot e^{j\omega t}\, d\omega$$

from the above eqn, we can write

$$\therefore\ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau-t_2+t_1)}\, d\omega = \delta(\tau-t_2+t_1)$$

$\therefore$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\cdot\delta(\tau - t_2 + t_1)\, dt_1\, dt_2$$

we know that

$$\boxed{\int_{-T}^{T} \delta(\tau - t_2 + t_1)\, dt_2 = 1 \text{ at } \tau - t_2 + t_1 = 0}$$

i.e $\boxed{t_2 = \tau + t_1}$

$$\therefore F^{-1}[S_{XX}(\omega)] = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} R_{XX}(t_1, \tau + t_1)\, dt_1$$

Let $t_1 = t$, $dt_1 = dt$

$$\therefore F^{-1}[S_{XX}(\omega)] = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} R_{XX}(t, \tau + t)\, dt$$

$\therefore$ The RHS of the above equation is the time average of the autocorrelation function.

Thus the time average of the auto correlation function and the power spectral density form a Fourier Transform pair, if the process is stationary, the time average of autocorrelation $R_{XX}(t, t+\tau)$ will be $R_{XX}(\tau)$. Since the autocorrelation function of stationary process is independent of time.

So $\bar{F}^{1}[S_{xx}(\omega)] = R_{xx}(\tau)$

$$\boxed{\therefore \quad R_{xx}(\tau) \underset{FT}{\overset{IFT}{\rightleftarrows}} S_{xx}(\omega)}$$

hence proved.

Note :- This equation also known as weiner khinchin relation.

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③ properties of the power density spectrum:-

property ① $S_{xx}(\omega) \geq 0$ i.e PSD is non negative.

proof: → from the definition ② of PSD is

$$S_{xx}(\omega) = \underset{T \to \infty}{Lt} \frac{E[|X_T(\omega)|^2]}{2T}$$

→ from the above equation, the expected value of a non negative function $E[|X_T(\omega)|^2]$ is always non negative.

Hense $\boxed{S_{xx}(\omega) \geq 0}$

Property (2) The power spectral density at zero frequency is equal to the area under the curve of the autocorrelation $R_{xx}(\tau)$

i.e $$\boxed{S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, d\tau}$$

proof:- from the definition we know that

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

At $\omega = 0$, then

$$\boxed{S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, d\tau}$$ hence proved.

Property (3) The power spectral density of a real process $x(t)$ is an even function

i.e $$\boxed{S_{xx}(-\omega) = S_{xx}(\omega)}$$

proof:- we know that

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

Also

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{j\omega\tau}\, d\tau$$

Now Substitute $\tau = -\tau$ then

$$\therefore\ S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(-\tau)\, e^{-j\omega\tau}\, d\tau$$

from the property of Auto correlation

$$\boxed{R_{XX}(-\tau) = R_{XX}(\tau)}$$

$$\therefore\ S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$\therefore\ \boxed{S_{XX}(-\omega) = S_{XX}(\omega)}$$

$\therefore$ $S_{XX}(\omega)$ is an even function.

Property (4) $S_{XX}(\omega)$ is always real function.

Proof: from the def (2) of PSD, we know that

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{E\left[\,|X_T(\omega)|^2\right]}{2T}$$

since the function $|X_T(\omega)|$ is a real function

so, $S_{XX}(\omega)$ is always real.

Property (5) :- If $x(t)$ is a WSS random process with power spectral density $S_{xx}(\omega)$, then the PSD of the derivative of $x(t)$ is equal to ω^2 times the PSD $S_{xx}(\omega)$.

i.e $$\boxed{S_{\dot{x}\dot{x}}(\omega) = \omega^2 S_{xx}(\omega)}$$