Question 1

We first prove a more simple inequality, $\forall a, b \in \mathbb{R}: a^2 - ab + b^2 \geq 0$, with equality iff $a = b = 0$. If a and b have different signs, then $-ab \geq 0$, and so $a^2 - ab + b^2 \geq 0$. So suppose a and b have the same sign, note the following,

$$\begin{aligned}(a-b)^2 &\geq 0\\ a^2 - 2ab + b^2 &\geq 0\\ a^2 - ab + b^2 &\geq ab \qquad (1)\end{aligned}$$

If a and b have the same sign, then $ab \geq 0$, so (1) implies that $a^2 - ab + b^2 \geq 0$.

We now show that $a^2 - ab + b^2 = 0$ iff $a = b = 0$. The reverse direction is clear, so suppose that $a^2 - ab + b^2 = 0$. Now it cannot be the case that $a > 0$ and $b < 0$, since if that were the case, then $-ab > 0$, implying that $a^2 - ab + b^2 > 0$. By symmetric argument, it cannot be that $a < 0$ and $b > 0$. Hence either $a, b \geq 0$ or $a, b \leq 0$, then it follows that $ab \geq 0$, also (1) implies that $ab \leq a^2 - ab + b^2 = 0$. So it must be that $ab = 0$, hence $a = 0$ or $b = 0$, without loss of generality, assume $a = 0$, then $a^2 - ab + b^2 = 0$ implies that $b^2 = 0$ so $b = 0$.

With this inequality established, let $a = x - 1$ and $b = y - 1$, then,

$$\begin{aligned}(x-1)^2 - (x-1)(y-1) + (y-1)^2 &\geq 0\\ (x^2 - 2x + 1) - (xy - x - y + 1) + (y^2 - 2y + 1) &\geq 0\\ x^2 + y^2 - xy - x - y + 1 &\geq 0\\ x^2 + y^2 + 2 - (xy + x + y + 1) &\geq 0\\ x^2 + y^2 + 2 - (x+1)(y+1) &\geq 0\end{aligned}$$

Which implies that $x^2 + y^2 + 2 \geq (x+1)(y+1)$, as required. We have equality iff $a = x - 1 = 0$ and $b = y - 1 = 0$. ie iff $x = y = 1$