Question 4

(Note: I use the convention that $\mathbb{N}$ does not include 0)

We will use numbers and sequences to represent letters and words, this is to make notation easier to work with. So for example, ‘a’ will be represented by 0, ‘b’ with 1, and ‘z’ with 25. The word *abdc* will be represented by the sequence $(0, 1, 3, 2)$.

We define a factorial number system, where the ith digit has base i, and the place value of the ith digit is $(i-1)!$. A number in the factorial system is written with pipes separating the digits and subscripted with ‘!’. For example, $3|0|2|1|0_! = 3 \times 4! + 0 \times 3! + 2 \times 2! + 1 \times 1! + 0 \times 0! = 77$. The highest digit allowed in the ith digit is $i-1$, hence the first digit in the factorial number system is always zero.

To convert a number from base 10 to the factorial system we may use repeated division, for example to convert 463

- $463 \div 1 = 463$, remainder 0
- $463 \div 2 = 231$, remainder 1
- $231 \div 3 = 77$, remainder 0
- $77 \div 4 = 19$, remainder 1
- $19 \div 5 = 3$, remainder 4
- $3 \div 6 = 0$, remainder 3

So $463 = 3|4|1|0|1|0_!$.

Fix some $n \in \mathbb{N}$. A sequence $(a_0, ..., a_{n-1})$ of natural numbers is a *good* sequence just in case each number $0, 1, ..., n-1$ appears exactly once in the sequence. Let A be the set of good sequences. Note that $|A| = n!$. We may define a total order on A by way of lexicographic order, it is easy to see that the words will be published in lexicographic order. Let $B = \{0, 1, 2, ..., n! - 1\}$ we define a map $f : B \to A$ given by

$$f(x_0|x_1|...|x_{n-1_!}) = (L(S_0)_{x_0}, L(S_1)_{x_1}, ..., L(S_{n-1})_{x_{n-1}})$$

Where $L(S)$ takes a finite set of natural numbers and returns the sequence of natural numbers in S in strictly increasing order, $L(S)_i$ denotes the ith element in the sequence (starting from zero), and $S_0 = \{0, ..., n-1\}$,$S_1 = S_0 \setminus \{L(S_0)_{x_0}\}$, $S_2 = S_1 \setminus \{L(S_1)_{x_1}\}$ etc...

An example will clear things up, take $n = 4$ and consider $f(3) = (a_0, a_1, a_2, a_3)$. Note that $3 = 0|1|1|0_!$.

- $S_0 = \{0, 1, 2, 3\}$, hence $L(S_0) = (0, 1, 2, 3)$, so $a_0 = L(S_0)_0 = (0, 1, 2, 3)_0 = 0$.
- $S_1 = S_0 \setminus a_0 = \{1, 2, 3\}$, hence, $L(S_1) = (1, 2, 3)$, so $a_1 = L(S_1)_1 = (1, 2, 3)_1 = 2$.
- $S_2 = S_1 \setminus a_1 = \{1, 3\}$, hence $L(S_2) = (1, 3)$, so $a_2 = L(S_2)_1 = (1, 3)_1 = 3$.
- $S_3 = S_2 \setminus a_2 = \{1\}$, hence $L(S_3) = (3)$, so $a_3 = L(S_3)_0 = (1)_0 = 1$.

Hence $f(3) = (0, 2, 3, 1)$

It remains to show that for every $x_0|x_1|...|x_{n-1_!} \in B$, $f(x_0|x_1|...|x_{n-1_!}) = (a_0, ..., a_n)$ does give a good sequence, and indexing the sequence $L(S_i)$ with index x_i does not go out of bounds. This is the case since at each i, $S_i = S_0 \setminus \{a_0, ..., a_{i-1}\}$, hence a_i cannot be equal to any of $a_0, ..., a_{i-1}$. The indexing does not go out of bounds since x_i is at the $n-i$th digit and by definition of the factorial number system, $x_i < n - i$, in addition, $|S_i| = n - i$, so $L(S_i)_{x_i}$ is well defined.

Claim: f is strictly increasing

Let $x = x_0|x_1|...|x_{n-1!}$ and $y = y_0|y_1|...|y_{n-1!}$ such that $x < y$. Then there exist an i such that $x_0 = y_0$, $x_1 = y_1$,...,$x_{i-1} = y_{i-1}$ and $x_i < y_i$. Let $f(x) = (a_0, ..., a_{n-1})$ and $f(y) = (b_0, ..., b_{n-1})$ then it follows that $a_0 = b_0$, $a_1 = b_1$,...,$a_{i-1} = b_{i-1}$ (since the last i digits of x and y are the same). Hence $S_i = S_0 \setminus \{a_0, ..., a_{i-1}\} = S_0 \setminus \{b_0, ..., b_{i-1}\}$, now $a_i = L(S_i)_{x_i}$ and $b_i = L(S_i)_{y_i}$, since $L(S_i)$ is and increasing sequence, and $x_i < y_i$, it follows that $a_i < b_i$. And so $(a_0, ..., a_{n-1}) < (b_0, ..., b_{n-1})$ (lexicographically)

Since f is a strictly increasing function between two finite totally ordered sets of the same size, it follows that f is a bijection. Taking $n = 26$ it follows that $f(0)$ corresponds to the first word in the list, $f(1)$, the second, etc...

With the relevant theory established, we return to the question at hand. There are $26!/21$ words in a volume, we wish to find the last word of the first volume, i.e. the $26!/21$th word, which corresponds to $f(26!/21 - 1)$. We will calculate $26!/21$ in the factorial number system then subtract 1. Since $20!$ divides $26!/21$, the first 20 digits of $26!/21$ are 0. The quotient after 20 divisions is $26 \times 25 \times 24 \times 23 \times 22 = 7893600$. Here are the rest of the steps

- $7893600 \div 21 = 375885$, remainder 15
- $375885 \div 22 = 17085$, remainder 15
- $17085 \div 23 = 742$, remainder 19
- $742 \div 24 = 30$, remainder 22
- $30 \div 25 = 1$, remainder 5
- $1 \div 26 = 0$, remainder 1

Hence, $26!/21 = 1|5|22|19|15|15|0|0|0|0|0|0|0|0|0|0|0|0|0|0|0|0|0|0|0|0_!$,
so $26!/21 - 1 = 1|5|22|19|15|14|19|18|17|16|15|14|13|12|11|10|9|8|7|6|5|4|3|2|1|0_!$. Hence

$$\begin{aligned} f(26!/21 - 1) &= (1, 6, 24, 21, 17, 16, 25, 23, 22, 20, 19, 18, 15, 14, 13, 12, 11, 10, 9, 8, 7, 5, 4, 3, 2, 0) \\ &= bgyvrqzxwutsponmlkjihfedca \end{aligned}$$