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Question 3

$$x + \frac{1}{x} = 2y^2 \tag{1}$$

$$y + \frac{1}{y} = 2z^2 \tag{2}$$

$$z + \frac{1}{z} = 2x^2 \tag{3}$$

Let (x, y, z) be a solution to the system of equations. Note that x, y, z > 0, since if x < 0 then $x + \frac{1}{x} < 0$ and there would be no solution to equation (1) (since $2y^2$ is always non negative). By symmetric argument it follows that y, z > 0.

We can improve this bound to $x, y, z \ge 1$

$$(x-1)^2 \ge 0$$

$$x^2-2x+1\ge 0$$

$$x^2+1\ge 2x$$

$$x+\frac{1}{x}\ge 2$$
 The direction of the inequality is preserved since $x>0$

By (1) it follows that $2y^2 \ge 2$, which along with the fact that y > 0, implies that $y \ge 1$. Again, by symmetric argument, it follows that $x, z \ge 1$. I claim that x = y = z, note that we can rearrange (1) to give us $y = \sqrt{\frac{1}{2}(x + \frac{1}{x})}$

$$x \le x^{2} \qquad \text{since } x \ge 1$$

$$2x \le 2x^{2}$$

$$x + x \le 2x^{2}$$

$$x + \frac{1}{x} \le 2x^{2} \qquad \text{since } \frac{1}{x} \le x$$

$$\frac{1}{2} \left(x + \frac{1}{x} \right) \le x^{2}$$

$$\sqrt{\frac{1}{2} \left(x + \frac{1}{x} \right)} \le x$$

$$y \le x \qquad \text{By eqn (1)}$$

Applying the same argument to eqn (2) and (3) we see that $z \le y$ and $x \le z$, which implies that x = y = z. Now suppose that x > 1, then $x < x^2$, and we can apply the same argument as above to conclude that y < x which is a contradiction to x = y, hence it must be that x = 1 so the only solution to the equations is (x, y, z) = (1, 1, 1).