Question 3

$$x + \frac{1}{x} = 2y^2 \tag{1}$$
$$y + \frac{1}{y} = 2z^2 \tag{2}$$
$$z + \frac{1}{z} = 2x^2 \tag{3}$$

Let (x, y, z) be a solution to the system of equations. Note that $x, y, z > 0$, since if $x < 0$ then $x + \frac{1}{x} < 0$ and there would be no solution to equation (1) (since $2y^2$ is always non negative). By symmetric argument it follows that $y, z > 0$.

We can improve this bound to $x, y, z \geq 1$

$$\begin{aligned}
(x-1)^2 &\geq 0 \\
x^2 - 2x + 1 &\geq 0 \\
x^2 + 1 &\geq 2x \\
x + \frac{1}{x} &\geq 2 && \text{The direction of the inequality is preserved since } x > 0
\end{aligned}$$

By (1) it follows that $2y^2 \geq 2$, which along with the fact that $y > 0$, implies that $y \geq 1$. Again, by symmetric argument, it follows that $x, z \geq 1$. I claim that $x = y = z$, note that we can rearrange (1) to give us $y = \sqrt{\frac{1}{2}\left(x + \frac{1}{x}\right)}$

$$\begin{aligned}
x &\leq x^2 && \text{since } x \geq 1 \\
2x &\leq 2x^2 \\
x + x &\leq 2x^2 \\
x + \frac{1}{x} &\leq 2x^2 && \text{since } \frac{1}{x} \leq x \\
\frac{1}{2}\left(x + \frac{1}{x}\right) &\leq x^2 \\
\sqrt{\frac{1}{2}\left(x + \frac{1}{x}\right)} &\leq x \\
y &\leq x && \text{By eqn (1)}
\end{aligned}$$

Applying the same argument to eqn (2) and (3) we see that $z \leq y$ and $x \leq z$, which implies that $x = y = z$. Now suppose that $x > 1$, then $x < x^2$, and we can apply the same argument as above to conclude that $y < x$ which is a contradiction to $x = y$, hence it must be that $x = 1$ so the only solution to the equations is $(x, y, z) = (1, 1, 1)$.