1. Set.

- collection of things
- identified by what's in $\{A, B, C\} = \{C, B, A\} = \{A, A, B, C, B\}$.

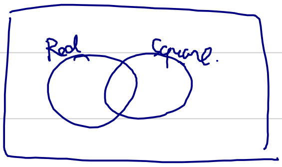
Construction

- Extensive : $\{$ John, Paul, George, Ringo $\}$.
  $\{1, 2, 3 \cdots \}$
- Intensive : $\{x \mid x$ is a member of Beatles $\}$
  : $\{x \mid x$ is a natural number $\}$

(1). Predicate as a set

$\{x \mid \underbrace{x \text{ is Red}}_{Rx}\}$.

$\{x \mid x$ is red & square $\}$



(2). Solutions as a set.

$$\underbrace{\{x \mid x^2 = 4.\}}_{\text{intensive.}} = \underbrace{\{-2, 2\}}_{\text{extensive.}}$$

2. Membership & the empty set

- Membership relation $\in$.   {the crucial primitive!

  - $\text{Paul} \in \{\text{John}, \text{Paul}, \text{George}, \text{Ringo}\}$
  - $3 \in \mathbb{N}$.
  - $\pi \notin \mathbb{N}$.

- Empty set $\{\}$ or ϕ.

  - $\forall x. (x \notin \phi)$
  - $\phi = \{x \mid x \neq x\}$.

- Subsets.

$$A \subseteq B \iff \forall x \, (x \in A \Rightarrow x \in B)$$

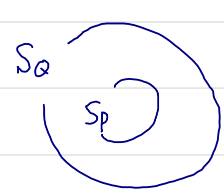
- $A = B \iff A \subseteq B \wedge B \subseteq A$
- $A \subset B \iff A \subseteq B \wedge B \not\subseteq A$.
- $\phi \subseteq A$ for any A. (why?)

cf.

$$S_Q = \{x \mid x \text{ is animal}\}$$

$$\text{UI} \qquad \Uparrow$$

$$S_P = \{x \mid x \text{ is animal} \wedge x \text{ is bipedal}\}$$



$$\frac{P \Rightarrow Q}{S_P \subseteq S_Q} \quad \text{duality}.$$

Don't confuse $\in$ with $\subseteq$!

$$D := \{a, \{a\}, b, c, \{d, e, \{f\}\}, g\}.$$

1. $a \in D$
2. $\{a\} \in D$
3. $\{a\} \subseteq D$
4. $a \subseteq D$ X.
5. $\{b\} \in D$ X
6. $\{f\} \in D$ X
7. $\emptyset \in D$ X
8. $\emptyset \subset D$

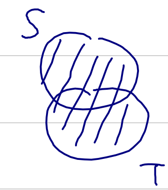
Set Operations

Create a new set from sets

Union

$$S \cup T = \{x \mid x \in S \vee x \in T\}$$

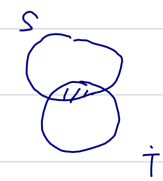
$$\bigcup S_i = \{x \mid x \in S_1 \vee x \in S_2 \vee \cdots\}$$



Intersection

$$S \cap T = \{x \mid x \in S \wedge x \in T\}$$

$$\bigcap S_i = \{x \mid x \in S_1 \wedge x \in S_2 \wedge \cdots\}$$



S & T are disjoint iff $S \cap T = \phi$.

Difference

$$S - T = \{x \mid x \in S \wedge x \notin T\}$$

Set algebra

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B - C) = (A \cap B) - (A \cap C)$$

$$(A \cap B) - C = (A - C) \cap (B - C)$$

$(A \cup B) \cap C = \{x \mid (x \in A \vee x \in B) \wedge x \in C\}$

$= \{x \mid (x \in A \wedge x \in C) \vee (x \in B \wedge x \in C)\}$

$= \{x \mid x \in A \cap B \vee x \in B \cap C\}$

Sets of sets and ranks

Note $\{\{a,b\},\{c,d\}\} \neq \{a,b,c,d\}$.

Ranks.

0 : doesn't contain any set

1 : contains at most Rank 0 sets

2 : " 1 sets

⋮

n : " n−1 sets

what's the rank of

ϕ ? 0.

$\{\{\phi\}\}$? 2

$\{\{\phi\},\{\{\{\phi\}\}\}\}$ 4

Power Sets.

$$\wp(S) = \{x \mid x \subseteq S\}$$

e.g. $\wp(\{A, B\}) = \{\phi, \{A\}, \{B\}, \{A, B\}\}$.

$\wp(\phi) \stackrel{?}{=} \{\phi\}$.

$\wp(\{A, B, C\}) \stackrel{?}{=} \{\phi, \{A\}, \{B\}, \{C\}, \{A, B\}, \{B, C\}, \{C, A\}, \{A, B, C\}\}$

Facts.

- If S has x elements, $\wp(S)$ has 2^x elements.
- $X \in \wp(S)$ iff $X \subseteq S$.
- $\bigcup \wp(S) = S$

13. Sets & Selections

Subset = choice of elements

$T = \{a, b, c, \dots\}$

$$\underbrace{\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ & \vdots & \\ 0 & 0 & 0 \end{array}}_{|T|^2 \text{ rows}} \quad \begin{array}{c} \Rightarrow \\ \\ \\ \\ \Rightarrow \end{array} \quad \underbrace{\begin{array}{l} \{a, b, c\} \\ \{a, b \quad\ \} \\ \{a, \quad c\} \\ \\ \{\ \} \end{array}}_{\in \mathcal{P}(T)}$$

So $\mathcal{P}(T)$ has $|T|^2$ members.

Consider function $f : T \longrightarrow \{0, 1\}$ ← call it 2.

assign $0/1$ to each $t \in T$

Then each row above specifies a func. $f_1, f_2 \dots$

$\mathcal{P}(T) \cong$ functions from T to 2, or 2^T.

14. Universe

obj & str

Bourbaki : reduce mathematics to set theory.
"everything is a set".
How to construct "everything" w/t set?
$\longrightarrow$ recursion

V_0	$\{\}$	0
$V_1 = P(V_0)$	$\{\{\}\}$	2^0
$V_2 = P(V_1)$	$\{\{\}, \{\{\}\}\}$	2^1
$V_3 = P(V_2)$	$\{\{\}, \{\{\}\}, \{\{\{\}\}\}, \{\{\}, \{\{\}\}\}\}$	2^2
$\vdots$		$\vdots$
$V_{i+1} = P(V_i)$		2^i

15. Sets & Numbers

Defining numbers w/t sets . . . .

1. Von Neumann.

$$
\begin{aligned}
0 &= \{\} \\
1 &= \{0\} = \{\{\}\} \\
2 &= \{0, 1\} = \{\{\}, \{\{\}\}\} \\
3 &= \{0, 1, 2\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}. \\
&\vdots
\end{aligned}
$$

2. Zermelo

$$
\begin{aligned}
0 &= \{\} \\
1 &= \{0\} = \{\{\}\} \\
2 &= \{1\} = \{\{\{\}\}\} \\
3 &= \{2\} = \{\{\{\{\}\}\}\}
\end{aligned}
$$

Both are good. So there's no such thing as the Number?

(Benacerraf 1965)

$\Downarrow$

Structuralism : Numbers are unique up to isomorphism.

17. Ordered Pairs & tuples

- Sets don't have any <u>structure</u>.

  e.g : No order : $\{a, b\} = \{b, a\}$.

- To express ordered <u>pair</u>.

$$(x, y) = \{\{x\}, \{x, y\}\}. \neq (y, x)$$

$$(x, x) = \{\{x\}, \{x, x\}\}$$
$$= \{\{x\}\} \qquad \neq x.$$

- Tuples.

$$(x, y, z) = ((x, y), z)$$

$$(w, x, y, z) = (((w, x), y), z)$$, and so on

cf.

$$((x,y),z) = \{\{(x,y)\}, \qquad \{(x,y), z\} \qquad \}$$
$$= \{\{\{\{x\}, \{x,y\}\}\}, \qquad \{\{\{x\}, \{x,y\}\}, z\}\}$$

$$\{\{\{x\}, \{x,y\}\}\} \qquad \{\{\{x\}, \{x,y\}\}, z\}$$

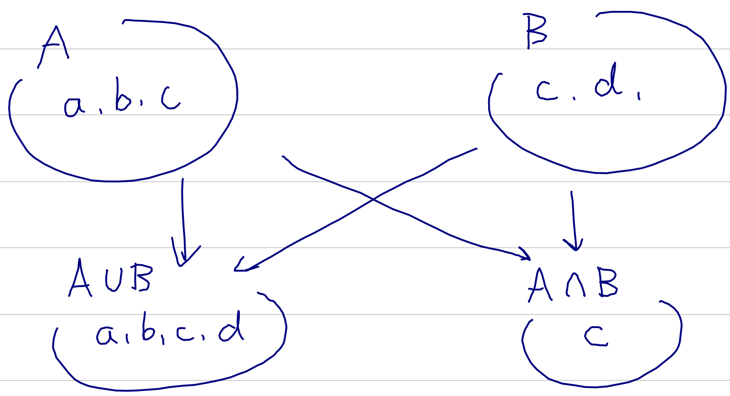
$$\{\{x\}, \{x,y\}\}$$

$$\{x\} \qquad , \qquad \{x,y\}$$

$$x \qquad y \qquad z$$

19. Cartesian Products

- We've seen union & disjunction.



Combine two sets to create another set.

- Product: $A \times B$: Combine sets to create set of <u>pairs</u>.

$A \backslash B$	1	2	3
a	(a, 1)	(a, 2)	(a, 3)
b	(b, 1)	(b, 2)	(b, 3)
c	(c, 1)	(c, 2)	(c, 3)

$A \times B$

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

Likewise.

$$A \times B \times C = \{a, b, c \mid a \in A, b \in B, c \in C\}.$$

Example: coordinate.

Note

$A \times B \neq B \times A$.

But they are <u>isomorphic</u>

Chap 2 Relations

1. Relations

eg. a has been to b $\qquad aBb$

a is no less than b $\qquad a \leq b$.

Relation is a subset of product $S \times T$.

$$aFb \Leftrightarrow (a,b) \in \{(x,y), x \in S, y \in T \mid x \text{ has been to } y\}$$

$$a \leq b \Leftrightarrow (a,b) \in \{(x,y), x,y \in \mathbb{N} \mid x \text{ is no less than } y\}$$

codomain

S \ T	清水	銀閣	鈴虫
Alice	✓		✓
Bob		✓	✓
Chris	✓		

Domain: {Alice, Bob, Chris}

$$\left\{ \begin{array}{l} (A,k),(A,s), \\ (B,g),(B,s), \\ (C,k) \end{array} \right\} \subseteq S \times T.$$

N \ N	1	2	3
1	✓	✓	✓
2		✓	✓
3			✓

other e.g. solution

$$\boxed{\{(x,y) \mid y = x^2\} \subseteq X \times Y}$$

Arity

- Binary $\subseteq S_1 \times S_2$
- Ternary $\subseteq S_1 \times S_2 \times S_3$
- Quaternary $\subseteq S_1 \times S_2 \times S_3 \times S_4$
- n-ary $\subseteq S_1 \times \cdots \times S_n$

2. Features of relations.

Relations themselves are just subsets of set products.
But some relations have a certain "structure".

- <u>Inverse</u> of relation R.

$$R^{-1} = \{ (y, x) \mid (x, y) \in R \}$$

Specifically, $\{(x, y)\}^{-1} = \{(y, x)\}$

These are <u>axioms</u> about particular relations

- <u>Reflexivity</u> :

R is reflexive $\iff \forall x \in S \, ((x, x) \in R)$

- <u>Symmetry</u>

R is symmetric $\iff \forall x, y \in S \, ((x, y) \in R \iff (y, x) \in R)$

$\iff R = R^{-1}$

R is anti-symmetric $\iff \forall x, y \in S \, ((x, y) \in R \land (y, x) \in R \Rightarrow x = y)$

- <u>Transitivity</u>

R is transitive $\iff \forall x, y, z \in S, ((x, y) \in R \land (y, z) \in R \Rightarrow (x, z) \in R)$

3. Equivalence

Equivalence relation = reflexive, symmetric, transitive.

e.g. "is the same age of"
"is the same color of".

Equi. rel. <u>partitions</u> a set into equivalent classes.

$\{$山形, 福島, 福井, 岡山, 山口, 福岡$\}$.

R_1: next-to $\quad \underbrace{\text{山形, 福島}}_{C_1}, \underbrace{\text{福井}}_{C_2}, \underbrace{\text{岡山}}_{C_3}, \underbrace{\text{山口, 福岡}}_{C_4}$

R_2: same-island-as $\quad \underbrace{\text{山形, 福島, 福井, 岡山, 山口}}_{C_1'}, \underbrace{\text{福岡}}_{C_2'}$

R_3: have same character as $\quad C_1'' \quad C_2''$

- $[x]_R = \{ y \mid xRy \}$.

- $[x]_R \cap [y]_R \neq \phi \iff [x]_R = [y]_R$
- $\bigcup \{ [x] \mid x \in C \} = C$

Equiv. classes are used to represent "types" (from relations)

- $[x] = \{y \mid y \text{ is the same color as } x\}$

  creates abstract types (color) from relation

- $[x] = \{y \mid y \text{ is parallel to } x\}$.

  "direction"

$[x] = \{y \mid \exists a \in \mathbb{N} \, (y = ax)\}$

4. Closures of Relations

ϕ closure of R : transform R into a relation with a property ϕ.

1. Reflexive closure

$$R \cup \{ (x,x) \mid x \in X \}$$

2. Symmetry closure

$$R \cup \underbrace{\{ (y,x) \mid (x,y) \in R \}}_{R^{-1}}$$

3. Transitive closure

$$R^1 = R$$
$$R^2 = R \circ R = \{ (x,y) \mid \exists z ((x,z) \in R \ \& \ (z,y) \in R \}$$
$$R^3 = R \circ R^2 = \{ \ '' \quad \mid \qquad , \qquad \& \qquad \in R^2 \}$$
$$\vdots$$

$$R^+ = \bigcup_{i=1}^{\infty} R^i$$

12. Functions & Operations (as relations)

Let X, Y be sets.

Function $f: X \to Y$ asign one $y \in Y$ to each $x \in X$.

$$\begin{array}{ccc} X & \to & Y \\ \Downarrow & & \Downarrow \\ x & \mapsto & y \end{array}$$

codomain (Y) domain (X)

※ Domain & Codomain can be set products.

Functions are special cases of relations

Domain \ Codomain	Fr	Jr	Sop	Sen
Alice			✓	
Bob		✓		
Chris				✓

f: Student → Year

$\Downarrow$

{Alice, Sop}

{Bob, Jr}

{Chris, Sen}

→ Relation R on $X \times Y$ is a function $f: X \to Y$ iff

H.W!

$$\forall x \in X \{ \exists y ((x,y) \in R) \wedge$$
$$(\forall y, y' ((x,y) \in R \ \& \ (x,y') \in R \to y = y')\}$$

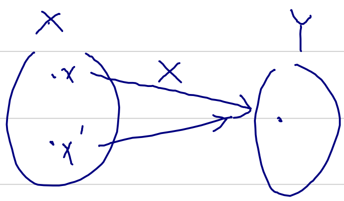
We note $y = f(x)$ iff $x R_f y$, i.e. $\{x, y\} \in f$.

Various types of function

- <u>one-to-one</u> (injection)

  Asigns <u>different</u> $y, y' \in Y$ to different $x, x' \in X$.

  i.e. $f(x) \neq f(x')$ if $x \neq x'$



- <u>onto</u> (surjection)

  every $y \in Y$ get asigned w/T at least one x

  i.e. $Y = \{ \underbrace{y \mid \exists x\, (y = f(x))}_{\text{image of } f} \}$

- <u>one-to-one correspondence</u> (bijection)

  iff its injection & surjection

- <u>inverse</u>

  $f^{-1} = \{ (y, x) \mid (x, y) \in f \}$

  f^{-1} is a function iff f is bijection n-ary functions $f : X_1 \times X_2 \times \cdots \times X_n \longrightarrow Y$

$$\begin{array}{ccc} X_1 \times X_2 \times \cdots \times X_n & \longrightarrow & Y \\ \cup & & \cup \\ (x_1, x_2, \cdots, x_n) & \longmapsto & y \end{array}$$

· e.g. arithmetic functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

$x + y = z \quad \cdots \quad f_+ : (x, y) \longmapsto z$.

$x \cdot y = z \quad \cdots \quad f_\times : (x, y) \longmapsto z$.

Set operators $\quad P(S) \times P(S) \longrightarrow P(S)$

$\cap : (S, T) \longmapsto S \cap T$

$\cup : (S, T) \longmapsto S \cup T$

$- : (S, T) \longmapsto S - T$

Charasteric function $f_T : S \longrightarrow \underbrace{\{0, 1\}}_{2}$. (seen before)

$$f_T(s) = \begin{cases} 1 & \text{if } s \in T \\ 0 & \text{if } s \notin T \end{cases}$$ $\rightarrow$ Characterises subset T.

$\rightarrow$ Each $T \in P(S)$ can be identified w/ char. func f_T

$\rightarrow$ $P(S) \cong \{ f \mid f : S \rightarrow 2 \}$

$= 2^S$.

§1. Structured sets.

Set equipped w/t relations $\langle S, \underbrace{R_1, R_2 \cdots}_{\text{satisfy certain axioms}} \rangle$

Satisfy certain axioms

Preorder $\langle S, R_1 \rangle$

where R_1 is reflexive & transitive

- e.g. "is-at-least-as-old-as" (後輩でない)

Partial order $\langle S, R_2 \rangle$

where R_2 is above + anti-symmetric

e.g. ancestor

Total order $\langle S, \leq \rangle$

where $\leq$ is above + connex

$\forall x, y \, (x \leq y \vee y \leq x)$

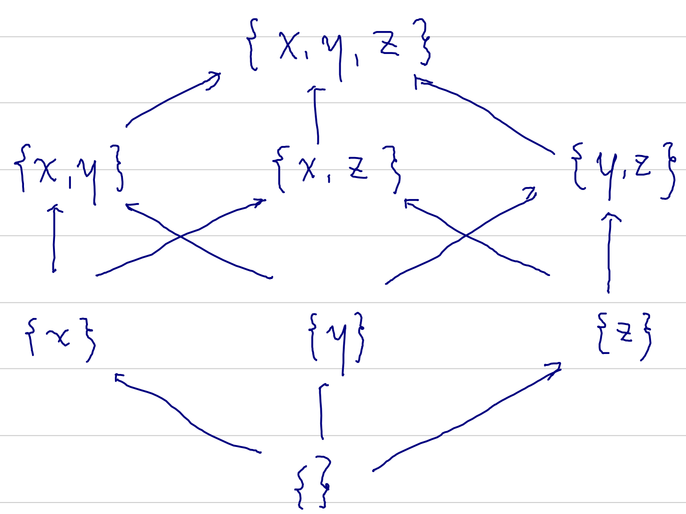
e.g. $\mathbb{N}, \mathbb{Q}, \mathbb{R}$.

Lattice $\langle L, \leq \rangle$

is a preorder where any $S \subseteq L$ has a join & meet

- For $S \subseteq L$, $u \in L$ is an upperbound of S if $\forall s \in S \, (s \leq u)$
  An upper bound u of S is its least upper bound / join
  if $u \leq x$ for all upper bounds of S
- " $\ell \in L$ is a lower bound of S if $\forall s \in S \, (\ell \leq s)$
  A lowerbound ℓ of S is its greatest lower bound / meet
  if $x \leq \ell$ for all lower bounds of S.

- e.g. $\langle \wp(S), \subseteq \rangle$



Group $\langle G, \circ, e, {}^{-1} \rangle$

$\circ$: binary function $G \circ G \to G$
that satisfies associativity $(g \circ h) \circ i = g \circ (h \circ i)$

$e \in G$: identity element s.t $g \circ e = e \circ g = g$ for $\forall g \in G$

${}^{-1}$: unary function $g \circ g^{-1} = g^{-1} \circ g = e$ for $\forall g \in G$

- e.g $\langle \mathbb{Z}, +, 0, - \rangle$
  $\langle \mathbb{Q}/\{0\}, \times, 1, {}^{-1} \rangle$

14. Homomorphism / Isomorphism

Let (A, R) & (B, S) structured sets.

- $f: A \to B$ is a homomorphism iff $\forall a_1 \cdots a_n \in A$,

$$R(a_1 \cdots a_n) \iff S(f(a_1), \cdots f(a_n))$$

- f is isomorphism if it is also bijective.

e.g. $(\mathbb{Z}, +)$ is homomorphic w/t $(\mathbb{R}^+, \cdot)$ w/t $f = \exp$.

$$z_1 + z_2 = z_3 \iff \exp(z_1) \cdot \exp(z_2) = \exp(z_3)$$

i.e. $\exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2)$

16. Sequences. Cardinality

- Sequence is a function : $\mathbb{N} \to S$.

- Cardinality.

  A & B have the same cardinality $\longleftrightarrow$ there's a bijection $A \to B$.

  $|A| = |B|$

- Set A is <u>countable</u> iff $|A| = |\mathbb{N}|$

18. Russel's paradox

$$R = \{ x \mid x \notin x \}$$

Q. $R \in R$?

· Suppose $R \notin R$. Then R satisfies the condition.
Thus $R \in R$.

· Suppose $R \in R$. But then from the condition $R \notin R$.

Solution : Class.

- proper class X : there's no class Y s.t $X \in Y$.

Exercises

1. Show that ϕ is a subset of any set. (p.17)

2. Prove set algebra in §.9 ?

3. For tuple (x, y, z), make a fig like 1.2, and write down elements

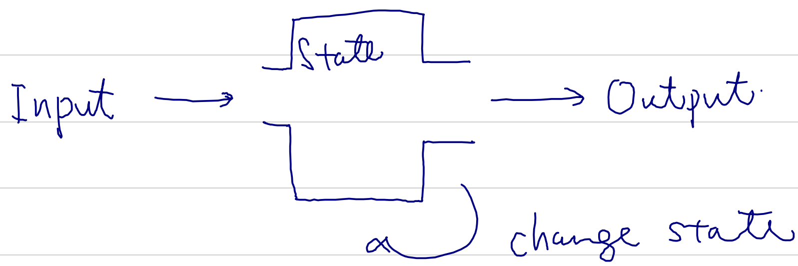
- Find an example of partial order that is not complete.

- Complete the def of function

- Find an ex. of group.. Specify what amounts to identity, inverse.

Ch.3 Machines

Machine



program = tuple (I, S, O, F, G) where

start w/ (S, F)
then add I, O, G?

I : a sets of inputs

S : " states

O : " outputs

$F : (I \times S) \xrightarrow{rel} S$: transition relation / function

$G : (I \times S) \to O$: output relation / function

State-transition network

- node : S
- arrows : F.

Configuration Space

$C_M = \{P\} \times I \times S \times O$.

at any time, machine's in $\in C_M$.

Let $x = (p_x, i_x, S_x, O_x)$ & $y = (p_y, i_y, S_y, O_y) \in C_M$.

x is a successor of y iff

- $p_x = p_y$
- $S_x = F_{py}(i_y, S_y)$
- $O_x = F_{py}(i_y, S_y)$

Career / trajectory

$t : \mathbb{N} \to C_M$ st.

$t(i+1)$ is a successor of $t(i)$, $\forall i \in \mathbb{N}$.

$t(0)$ is the initial condition.

Utility function

$u : S \to \mathbb{R}$

$\longrightarrow$ cf. goal-directedness.

3. The game of life

$N \times N$ grid of machines, M_L

Def: neighbor $N(m_{ij})$ of $m_{i,j} \in M_L$

$$\{m_{\ell,k} \mid i-1 \le \ell \le i+1,\ j-1 \le k \le j+1\} - \{m_{i,j}\}.$$

Each $m \in M_L$ run the same program $L = (I, S, O, F, G)$ s.t

$$\begin{cases} S = \{0, 1\} \\ I = \sum_{n \in N(m)} S(n) \\ O = G = \phi \end{cases}$$

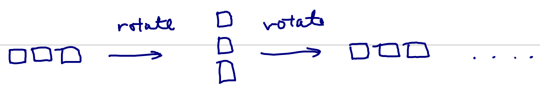
· Transition function

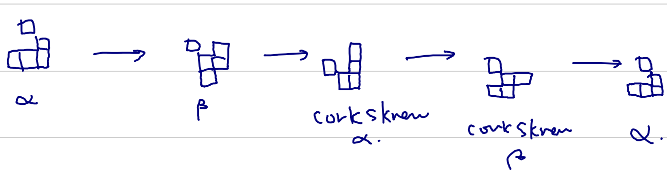
s \ i	0	1	2	3	4	5	6	7	8
0	0	0	0	1	0	0	0	0	0
1	0	0	1	1	0	0	0	0	0

$\leftarrow F(i, s)$

$\Longrightarrow$ produce various patterns. (= have own essence (program)).

How to express these patterns?





4 Turing Machines

- Finite State Machie + Tape

  Inputs = {0, 1}

  States = { Start, Moving, Check, Halt }

  Actions (Outputs) = { print, erase, move right, move left }

  Transition function : Inputs × States → Action × States.

  → See text for an example.

- Computation by TM.

  4 steps.
  1. find & fill the middle "0"
  2. find the end "0"
  3. erase the last "1"
  4. End.

  e.g. 3 + 2 = 5

  111011 → 11111

  func

State	input	Action	State
S_1	1	MR	S_1
S_1	0	Print	S_2
S_2	1	MR	S_2
S_2	0	ML	S_3
S_3	1	Erase	S_4
S_4 : end			

  Computable = expressed by a finite algorithm

  Algorithm / TM can be expressed by a number / code

- Church-Turing's thesis.

  - TM can imitate any machine (e.g. life game, register machine) calculate any function.

  ⇒ Computable = computable by a TM. (C-T's thesis)

- Halting problem

  Hilbert's decision problem : Is there a math problem that can't be solved by a finite algorithm?

  Yes : A machine that check wheather a given TM halts in finite steps or not.

  proof . Call such a TM H.

  $$H(M) \begin{cases} \text{Yes if } M \text{ stops} \\ \text{No if not} \end{cases}$$

  Modify H to get

  $$H'(M) \begin{cases} \text{Doesn't stop if } M \text{ stops} \\ \text{No if } M \text{ doesn't stop.} \end{cases}$$

  Then

  $$H'(H')\ ? \rightarrow \begin{cases} \text{Doesn't stop if } H' \text{ stops} \\ \text{No if } H' \text{ doesn't stop.} \end{cases} \Rightarrow \text{Contradiction}$$

- Universal Turing Machine

  "Simulate" individual TMs

  Tape: encoded TM | input/output | UTM's working space

- Philosophical implications
  - Turing test
    - → Functionalism
    - → Searle's chinese room

Chap. 4 Semantics.

1. Semantics : assigns meaning to a sentence by reference function f.

f: Sentence $\longrightarrow$ State of affair.

Q: How to construct f?

NAME . f(prop. name) = individual (element)

NOUN - f(nouns) = a set of indivs. (subset)

ADJ. f(adj.) = "

VERB. f(verb) = a set of pairs / tupples.

e.g. f("loves") = { (Alice, Bob). (Bob, Cindy) ... }.

f("weighs") = { (Alice, 60). (Bob, 70) ... }

· Model

· language = $\begin{cases} \text{vocaburary } V \\ \text{ref. function. } f \end{cases}$

· Model = $\{ f(v) \mid v \in V \}$

- Sentences = claims about inclusion relationship in a model / set

→ Its truth value depends $\begin{cases} \cdot \text{ interpretation } f \\ \cdot \text{ model } M \text{ (how it is structured)} \end{cases}$

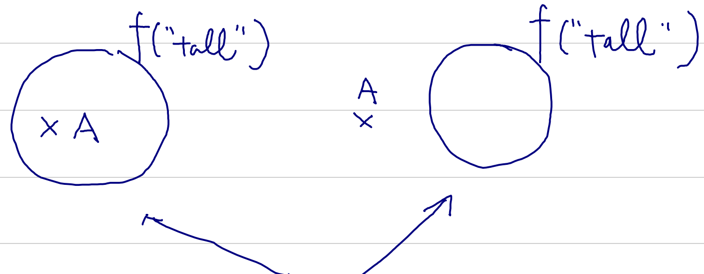
e.g. "Alice is tall" is true given f iff

$f(\text{"Alice"}) \in f(\text{"tall"})$

"Every person has a mother" is true given f iff

$\forall x \, (x \in f(\text{"person"}) \Rightarrow \exists y \, ((y, x) \in f(\text{"is-the-mother-of"})))$

- Note. the same sentence may be true or not depending models.



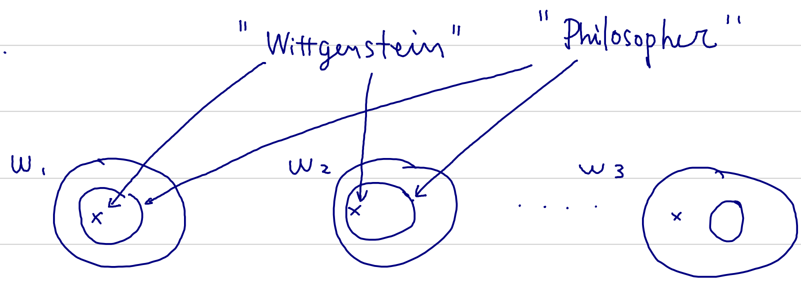
different worlds / possibilities

2. Simple Modal Semantics

modal structure : (V, W, I, f)

- V : Vocaburary
- W : set of possible worlds
- I : Individuals (possible)
- f : ref. function $V \times W \longrightarrow P(I)$.

e.g.



- Proper name is a <u>rigid designator</u>. $f(p, w) = f(p, w') \quad \forall w, w' \in W$.

- Truth value of a sentence depends on evaluating world.

"Witt. is a philosopher" is true at w_1 given f iff

$f(\text{"Witt"}, w_1) \subseteq f(\text{"philosopher"}, w_1)$.

"Philosophers are disciples of Plato" is true wrt $w, \& f$ iff

$f(\text{"Philosopher"}) \subseteq f(\text{"discipler of Plato"})$

- Modalities

necessity : true at every $w \in W$
possibility : " some $w \in W$

De dicto about sentence

$\longrightarrow$ necessary / possibly true by the sentential form.

" It is necessary that A is B " : at w given f

$= \forall w' \in W. (\underbrace{\text{A is B at } w'}_{\text{evaluated in each world.}})$ given f

e.g. " It is necessary that the inventor of instant ramen invented instant ramen " at w given f.

$= \forall w' \in W \, (f(\text{"the inventor of I.R."}, w') \subseteq f(\text{"invent I.R."}, w'))$

$=$ True.

De re about individuals.

$\longrightarrow$ necessarily / possibly true by their nature

" A necessarily is B " at w given f.

$= \forall w' \in W$ (A, evaluated at w, is B at w') given f.

e.g " The inventor of IR necessarily invented I.R " at w given f.

$= \forall w' \in W \, (f \underbrace{(\text{"the inventor of I.R."}, w)}_{\text{安藤 百福}} \subseteq f(\text{"invent I.R"}, w'))$

$=$ False.

Summary

- De dicto : Evaluate referent at each world.
  If the sentence is true at every world.
  → its truth value doesn't depend on the semantic content of the term, but on the sentential form.

- De re : Fix the referent (of the subject) at one world.
  If the sentence is true at every world,
  → it's b/c of the nature of the referent

· Modal claim about a proper name is de re.

- Intensions : $i : V \longrightarrow$ function : $W \longrightarrow \mathcal{P}(I)$

  $\quad\quad\quad\quad\quad\quad\quad\quad\quad \in \quad\quad\quad\quad\quad\quad\quad\quad\quad \in$

  "the philosopher" $\mapsto \{(w_1, \text{Aristotle}), (w_2, \text{Plato}), (w_3, \cdots) \ldots\}$

  "POTUS" $\mapsto \{(w_1, \{\cdots\}), (w_2, \{\cdots\cdots\}), \ldots\}$

- Intentions give a "meaning" of a word

  (Know the meaning of POTUS = Know how to determine its referents at each world.

  = know the func $W \longrightarrow P(I)$)

- Intention of a proper name = constant func.

- $\underbrace{f(\text{"x"}, w) = in(\text{"x"})(w)}$

  Duality of $\dfrac{V \times W \longrightarrow \mathcal{P}(I)}{V \longrightarrow \mathcal{P}(I)^W}$

· Propositions = $\underbrace{\text{the meaning}}_{\substack{\uparrow \\ \text{intension}}}$ of a sentence.

(Know the meaning of "Sky is blue" = Know how to evaluate its truth value at each world

= Know the func. $W \to \{0,1\}$)

Thus proposition = intension of a sentence.

function : Sentence $\times$ $W \longrightarrow \overset{\text{Truth value}}{\{0,1\}}$

or

intention : Sentence $\longrightarrow \underbrace{f : W \to \{0,1\}}_{\text{Subsets } P(W)}$.

$\Downarrow$ $\Downarrow$

a sentence S $\longmapsto$ a set of world in which S is true

Chap 5. Probability

Probability = measure the size of a subset.

- Sample space S: contains all possible outcomes.

  eg. $\{1, 2, 3, 4, 5, 6\}$,

  $\{HH, HT, TH, TT\}$.

- Events Space $E \subseteq P(S)$: subsets of the sample space

  e.g. $\{2, 4, 6\} = \{x \mid x \text{ is even}\}$

  $\{HH, TT\}$

  Some requirements. (σ-field / algebra).

  - S itself is an event
  - If E is an event, E^c is. (closed under compliment)
  - If $E_1, E_2 \cdots E_n$ are events, so is $\bigcup_i^n E_i$ (∵ contable union)

- Possible world interpretation

  - Each element in S = possible realization = possible world

  Recall

  - Proposition of a sentence = the set of possible worlds in which the sentence is true.

    = event.

- Probability function $P : \mathcal{P}(S) \longrightarrow \mathbb{R}^+$

$$\begin{array}{ccc} \mathcal{P}(S) & \longrightarrow & \mathbb{R}^+ \\ \cup\!\!| & & \cup\!\!| \\ \text{event} & \longmapsto & \text{the prob of that event} \end{array}$$

satisfying the following Axioms. (Kolmogorov)

$$\begin{cases} \text{A1.} & \forall E \in \mathcal{P}(S), \quad 0 \le P(E) \le 1 \\ \text{A2.} & P(S) = 1 \\ \text{A3.} & \forall E, E' \, (P(E \cup E') = P(E) + P(E') - P(E \cap E')) \end{cases}$$

A1~3 justify P as measuring the size of subsets / events

interpreted as likelihood/chance of the events.

- a prob. function P assigns a prob. to a sentence

=

as the size of its proposition

=

proportion of the possible worlds in which the sentence is true.

⇒ You don't know which world you are in, but can guess which is likely

Two issues about probability

- Epistemic: How to determine P?

→ One approach: equiprobable

$P(E) := |E| / |S|$. (works only for finite S).

e.g. $P(\{x \mid x \text{ is even}\}) = \frac{|\{2,4,6\}|}{|\{1,2,3,4,5,6\}|} = \frac{3}{6} = \frac{1}{2}$.

→ check that this difinition satisfies A1 ~ 3 above.

note that this is just <u>one</u> way to define P.

- Semantic: How to interpret P?

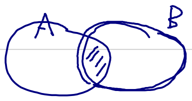
1. Objective / frequentist
   - Frequency of events

2. Subjective / Bayesian ← Focus on this
   - degree of belief.

- Important definitions.

Conditional Prob. $P(A|B) := \dfrac{P(A \cap B)}{P(B)}$.

→ The probability of A given that B occurs.

(Proportion of A part within B) 

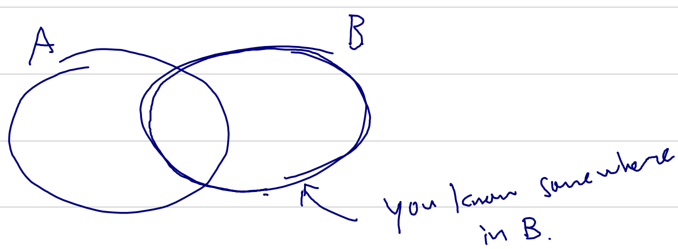
Conditionalization (in Subjective interpret.)

$P(A|B)$ is the degree of belief in A given B is true

(If you observe B, You have to update your degree of belief in A to $P(A|B)$)

⇑

In terms of possible world ...



Independence

- A & B are independent iff $P(A|B) = P(A)$.

  check this is same as $P(B|A) = P(B)$,

  $P(A \cap B) = P(A)P(B)$.

  → In this case, A (B) gives no info about B (A).

- " positively correlated iff $P(A|B) > P(A)$

- " negatively " $P(A|B) < P(A)$.

· Bayes theorem

$$\underbrace{P(H \mid E)}_{\text{posterior}} = \underbrace{\frac{P(E \mid H)}{P(E)}}_{\text{likelihood.}} \cdot \underbrace{P(H)}_{\text{prior}}.$$

$$= \frac{P(E \mid H) \cdot P(H)}{P(E \mid H) P(H) + P(E \mid \neg H) P(\neg H)}$$

→ Calculate the (posterior) prob of H after the obs. of evidence E

- Likelihood $P(E \mid H)$: Degree to whic H implies E.
  - if 1 ($H \Rightarrow E$), ⇒ $P(H \mid \sim E) = 0$. (modus tolens)
  - if > 0.5 (positive. con). $P(H \mid E) > P(H)$ (confirm)
  - if < 0.5 (negative ") $P(H \mid E) < P(H)$ (dis ").

- Even if H gets high support from E, $P(H \mid E)$ is low if $P(H)$ is
  (the base rate fallacy)

  so the probabilistic modus tolens

  $P(E \mid H) \approx 0$, $E \Rightarrow P(H) \approx 0$ is not sound.

Inductive reasoning : Bayes thm makes sense of some apparently valid inductions that are not valid deductively

< Enumerative induction >

$$\frac{Wa_1,\ Wa_2,\ Wa_3 \cdots}{\forall x\, Wx.}$$

not valid in deductive logic (not truth preserving)

In inductive logic...

let $H: \forall x\, Wx.$

$E_i: Wa_i.$

Then $P(E_i|H) = 1$

$$P(H|E_1) = \frac{P(E_1|H)}{P(E)} \cdot P(H) = \frac{P(H)}{P(E)} \geq P(H)$$

So more we observe Wa_i, more likely H becomes.

< Abduction >

$$\frac{\begin{array}{c}A \Rightarrow B\\ B\end{array}}{A}$$

Affirming the consequent

But in inductive logic, if $P(B|A) = 1$ ...

$$P(A|B) = \frac{P(B|A)}{P(B)} P(A) = \frac{P(A)}{P(B)} \geq P(A)$$

So the antecedent becomes more likely.

↳ But: base-rate fallacy (if $P(A) \approx 0$, $\frac{P(A)}{P(B)} \approx 0$)

Deductive vs Inductive logic

Deductive reasoning.

$$\begin{array}{c} H \Rightarrow E \\ \neg E \\ \hline \neg H \end{array} \qquad \left.\begin{array}{l} V(H \Rightarrow E) = 1 \\ V(E) = 0 \\ V(H) = 0 \end{array}\right\}$$

consistent assignment of truth values

$\Rightarrow$ Violation = contradiction

Inductive reasoning

$$\left.\begin{array}{l} P(E|H) = \cdots \\ P(\neg E) = \cdots \\ P(H) = \cdots \\ \hline P(H|E) = \cdots \end{array}\right\}$$

Bayes theorem assures a consistent assingment of probability values

$\Rightarrow$ Violation = of Kolmogorov's axiom

Implications to epistemology.

- Web of belief : beliefs form a justificatory str/. network.
- Coherence : (degree of) belief must be consistent to each other.
- Issue : How this network relate to the world?
  $\rightarrow$ Determine V or P.

Application

Entropy $\quad H(A) = -\sum_{a \in A} p(a) \cdot \log_2 p(a)$

・nコから1コ生じる：情報量 $= \log_2 n$. ← 何回分ければよいか

・確率 p で生じる event E の info. let $p = k/n$.

we have n things, $\mathcal{E}$ has k things

$$\text{Info.}\left(\text{one out of } n\right) = \text{info}(E) + \text{info}(\text{one out of } k)$$

$$\log n = I + \log k \longrightarrow I = -\log \frac{k}{n} = -p.$$

Expected utility $\quad U(a) = \sum_{x} v(x) \cdot p(x \mid a)$