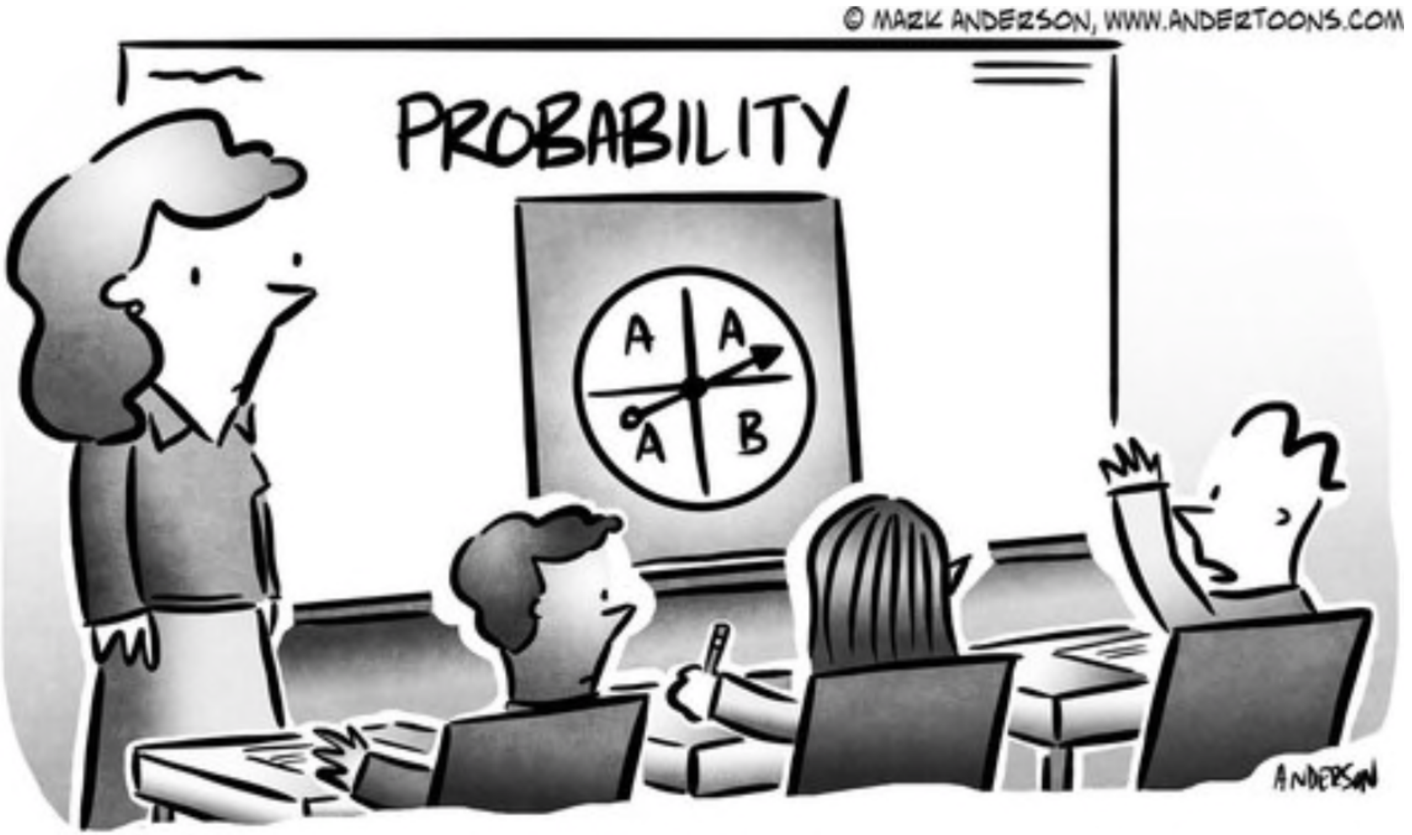
CSCI 3022: Intro to Data Science

Lecture 4: An Introduction to Probability

Rachel Cox

Department of Computer Science



"I know mathematically that A is more likely, but I gotta say, I feel like B wants it more."

What will we learn today?

- ❑ Probability definitions
- ❑ Brief review of Set Notation
- ❑ DeMorgans Laws
- ❑ Probability of Disjoint Events
- ❑ Probability of Non-Disjoint Events
- ❑ Probability of Complementary Events
- ❑ Independence

- ❑ *A Modern Introduction to Probability and Statistics, Chapter 2*

Announcements & Reminders

- HW1 due Friday Jan 31 at 11:59 PM

  .ipynb to Canvas * make sure you really upload this type of file.

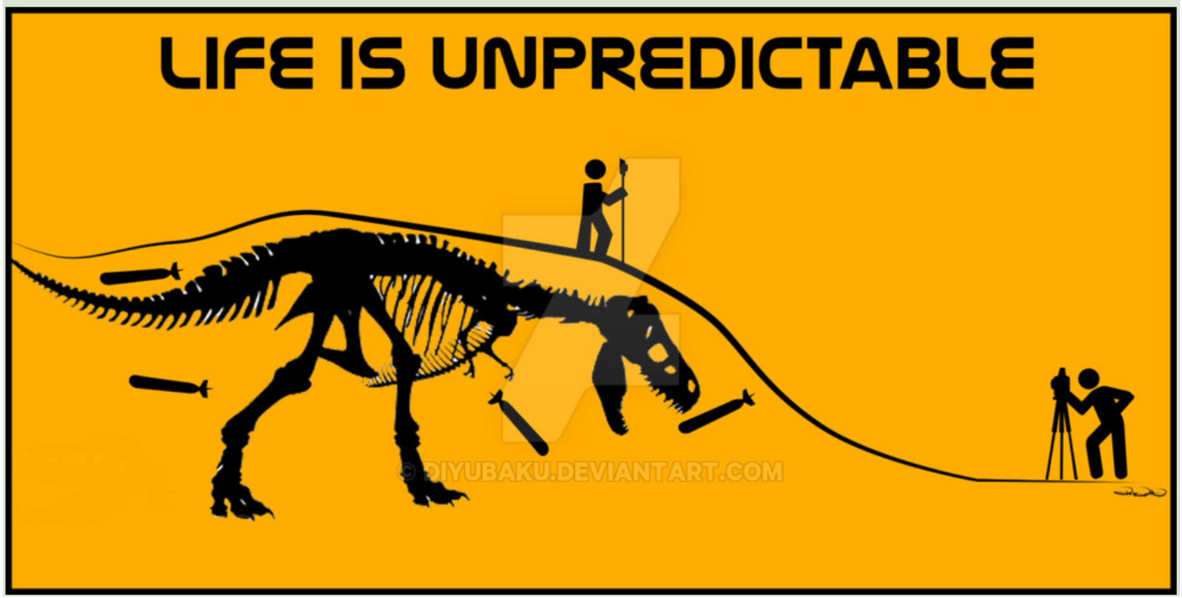
- Quizlet 02 next Monday – Wednesday

Probability

Aspects of the world **seem** random and unpredictable.

- Are we short or tall?
- Do we have Mom’s eyes or Dad’s?
- Is the eye of the hurricane going to pass over New Orleans?
- How long will it take to drive to the airport?
- Which grocery store line should I get in?

❖ Probability is a way of thinking about unpredictable phenomena as if they were each generated from some random process.



Probability – Basic Definitions

The **sample space** $\boldsymbol{\Omega}$ is the set of all possible outcomes of the experiment.

- Think of a random process as a trial or experiment.

Example: If we flip a fair coin a single time, what is the sample space?

$$\Omega = \{H, T\}$$

Example: If we are doing a poll, and ask each person their birth month, what is the sample space?

$$\Omega = \{Jan, Feb, Mar, \ldots, Nov, Dec\}$$

- Observation: These are ***discrete*** sample spaces because there are a finite number of outcomes.

Probability – Discrete vs. Continuous

There is continuous math, like derivatives and integrals. Or the flow of water out of a faucet.

Then there is discrete math, like counting, sorting, and enumeration. Or an individual droplet of water.

Discrete Probability – Basic Definitions

The **cardinality** of a set A is the number of elements in that set; denoted |A|.

For each event A in Ω the **probability** is calculated by $P(A) = \frac{|A|}{|\Omega|}$ ★

For each event in Ω the **probability** is a measure between 0 and 1 of how likely it is for the event to occur.

Observation: The sum of the probability of each distinct outcome in Ω is 1. Why?

Something has to occur

e.g. Consider Rolling a fair die.

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$p(1) + p(2) + p(3) + p(4) + p(5) + p(6)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$

Set Notation – Quick Review

What is the difference between the following:

$\{0, 1\}$ denotes the set containing the elements 0, 1

$[0, 1]$ all Real numbers from 0 to 1 inclusive

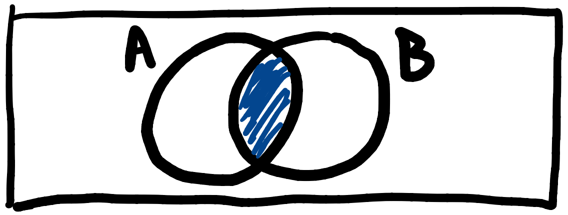
$(0, 1)$ all Real number from 0 to 1, not including 0, 1

Set Operations

The **intersection** of two events is the subset of outcomes in **both** events.

Intersection = "and"

Box Represents Ω

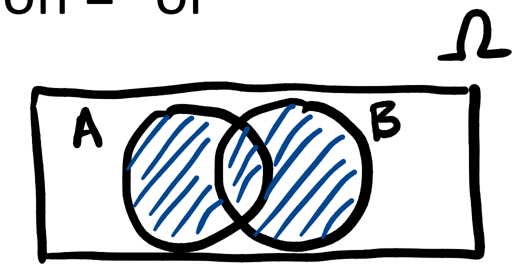


Question: What is $P(A \cap B)$?

$$= \frac{|A \cap B|}{|\Omega|}$$

The **union** of two events is the subset of outcomes in **one or both** events.

union = "or"

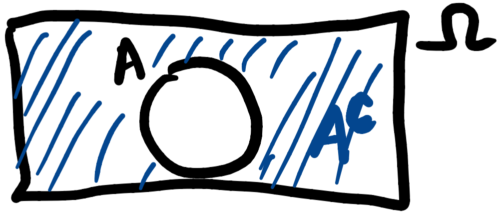


Question: What is $P(A \cup B)$?

$$= \frac{|A \cup B|}{|\Omega|}$$ *

Set Operations

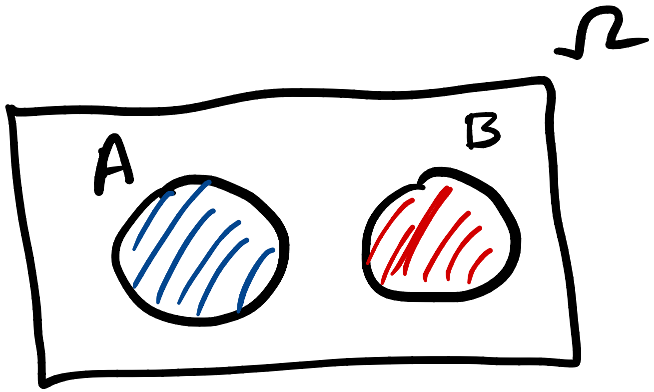
The **complement** of an event A is the set of all outcomes in Ω that are **not in** A



Question: What is $P(A^c)$?

$P(\Omega) = 1$ $\quad$ $P(A) + P(A^c) = 1$

When the intersection of two events is empty, we call those two events **disjoint** or **mutually exclusive**.



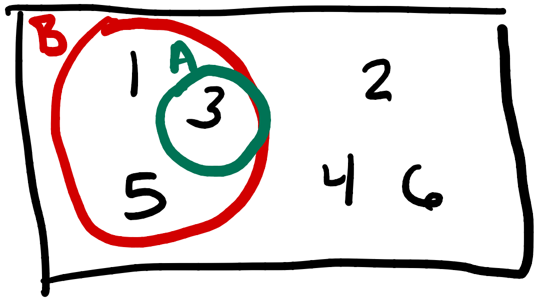
- Outcome space aka Sample space
  - set of all possible Results
- An event is a subset of the sample space.

Set Operations

If all outcomes of event A are also outcomes of event B, we say that A is a **subset** of B.

e.g. B = Rolling an odd number on a die. $\Omega = \{1, 2, 3, 4, 5, 6\}$

A = Rolling a 3



* $A \subseteq B$

$A \subset B$

"proper subset"

$X \leq 3$

$X < 3$

Notation:

A^c

\bar{A}

Complement: A^c or $\bar{A}$ or $\sim A$

Intersection: $A \cap B$ \cap

Union: $A \cup B$ \cup

Disjoint: $A \cap B = \emptyset$

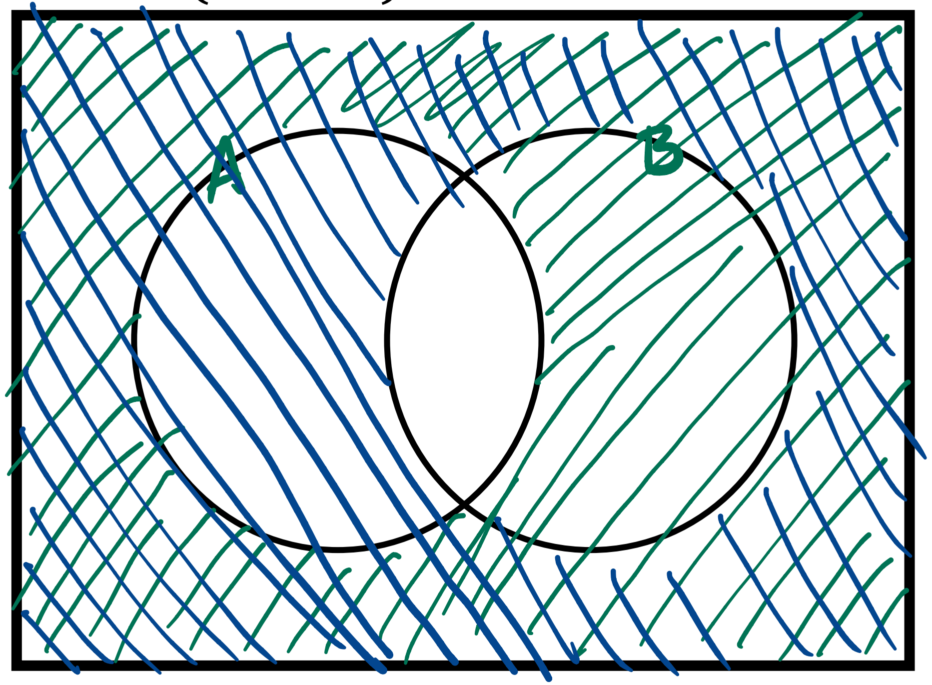
Subset: $A \subseteq B$, $A \subset B$

De Morgan's Law for Sets

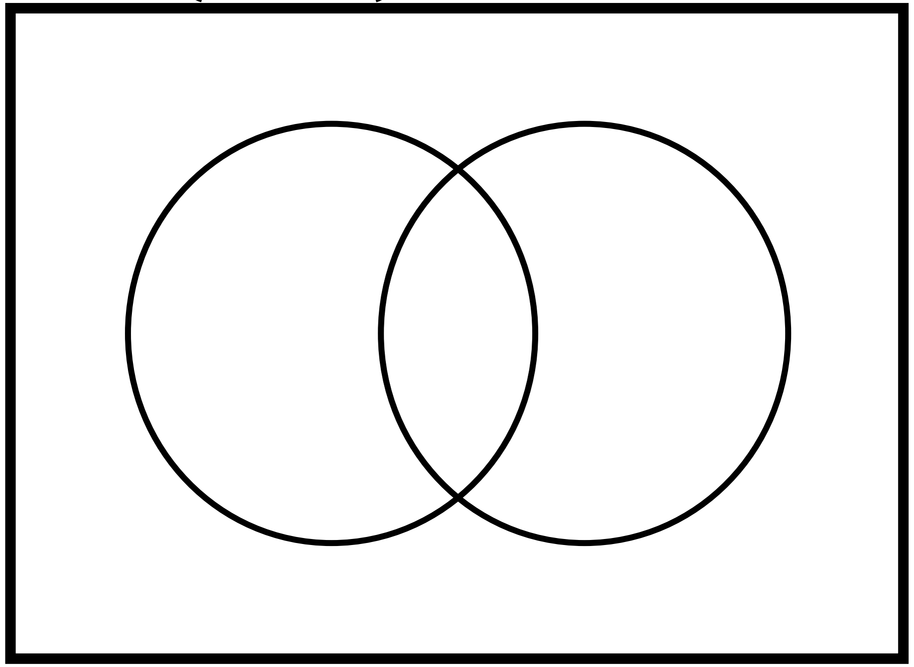
Complement of a Union: $(A \cup B)^c = A^c \cap B^c$

Complement of an Intersection: $(A \cap B)^c = A^c \cup B^c$

$(A \cup B)^c = A^c \cap B^c$



$(A \cap B)^c = A^c \cup B^c$



Probability

Example: A biased coin is a coin with a modified probability function. Instead of $P(\{H, T\}) = \{½, ½\}$, a biased coin's probability function is $P(\{H, T\}) = \{p, q\}$. What can we say about q?

$\Omega = \{H, T\}$

$\boxed{P(\{H, T\}) = \{p, q\}} \rightarrow p(H) = p \qquad p(T) = q$

we know that $p(H) + p(T) = 1$

$p + q = 1$

$\boxed{p(\{H, T\}) = \{p, 1-p\}} \qquad \boxed{q = 1 - p}$

Probability Functions

A random process with two outcomes with fixed probabilities assigned to each outcome is called a **Bernoulli Trial**.

A **probability function** P assigns to each event A a number $P(A)$ in $[0, 1]$ such that

1) $P(\Omega) = 1$
2) $P(A \cup B) = P(A) + P(B)$ if A and B are disjoint events.

Note that a probability function has two key properties:

1) The probability of the entire sample space is 1.

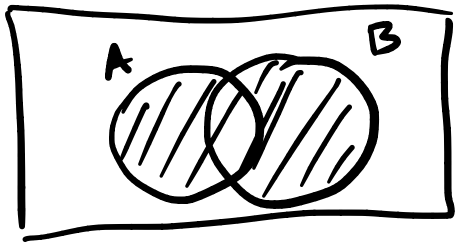
2) The probability of the union of disjoint events is the sum of the probability of each event.

Recall : $\Omega = \{1, 2, 3, 4, 5, 6\}$

$P(1 \cup 3) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

Probability of Non–Disjoint Events

Question: What is the probability of the union of events A and B if they are not disjoint?



set theory : $|A \cup B| = |A| + |B| - |A \cap B|$

$$\frac{|A \cup B|}{|\Omega|} = \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} - \frac{|A \cap B|}{|\Omega|}$$

$$\boxed{p(A \cup B) = p(A) + p(B) - P(A \cap B)}$$

Probability of Non–Disjoint Events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example: Consider rolling a fair die. Let Ω be the sample space. Let $A = \{2, 4, 6\}$ (the event of rolling an even number) and $B = \{1, 3, 6\}$ (the event of rolling a 1, 3, or 6). What is $P(A \cup B)$?

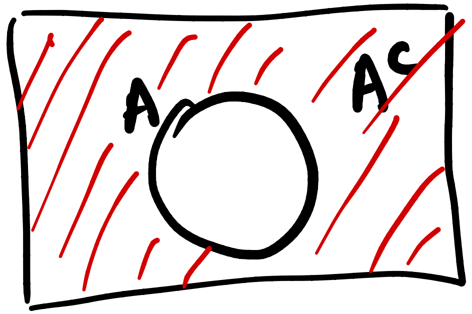
$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

$$= \frac{3}{6} + \frac{3}{6} - \frac{1}{6}$$

$$= \frac{5}{6}$$

Probability of Non–Disjoint Events

Question: What is the probability of the complement of an event A?



$$p(\Omega) = 1$$

$$p(A) + p(A^c) = 1$$

$$p(A^c) = 1 - p(A)$$

Probability of the Complement

$$P(A^c) = 1 - P(A)$$

Example: $A = \{2, 4\}$, $\Omega = \{1, 2, 3, 4, 5, 6\}$. Find $P(A^c)$.

$A^c = \{1, 3, 5, 6\}$

$P(A^c) = \frac{4}{6} = \frac{2}{3}$

$P(A) = \frac{2}{6} = \frac{1}{3}$

$\boxed{P(A^c) = 1 - \frac{1}{3} = \frac{2}{3}}$

More complicated coin questions

Question: What is the probability that if a biased coin is flipped twice that both flips come up heads?

$$p(\{H,T\}) = \{p, 1-p\}$$

$$\Omega = \{HH, TT, TH, HT\}$$

$$p(HH) = p \cdot p = p^2$$

Sanity check: Does the result of the first flip affect the result of the second flip?

nope!

Probability – Independence

When two (or more) trials do not affect each other, we say they are **independent**.

- When two events are independent, we can multiply their probabilities to find the probability of both occurring.

Example: What is the probability of flipping a biased coin twice and getting one H and one T?

$p(H) = p \qquad p(T) = 1-p$

$\Omega = \{HH, TT, TH, HT\}$

disjoint events!

$$p(TH \text{ or } HT) = p(TH) + p(HT)$$
$$= (1-p)\cdot p + p(1-p) = \boxed{2p(1-p)}$$

Probability – Coin Flips

Example: What is the probability that 5 biased coins are flipped and only one lands as a head?

All 5: $p(H) = p$

$p(T) = 1-p$

event: HTTTT, THTTT, TTHTT, TTTHT, TTTTH

$p(1-p)(1-p)(1-p)(1-p)$, $(1-p)\cdot p(1-p)^3$ . . . .

$p(\text{one head}) = p(1-p)^4 + p(1-p)^4 + p(1-p)^4 + p(1-p)^4 + p(1-p)^4$

$\boxed{= 5p(1-p)^4}$

Next Time:

More Probability!