One-Bit Phase Retrieval: Optimal Rates and Efficient Algorithms

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Outline

1-Bit Compressed Sensing

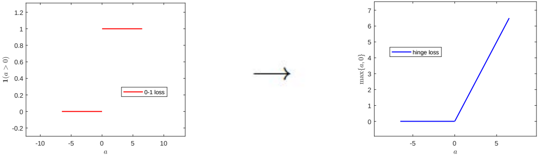
- Compressed sensing: recover k-sparse $\mathbf{x} \in \mathbb{R}^n$ from
$$\mathbf{y} = \mathbf{A}\mathbf{x} + \boldsymbol{\epsilon}, \tag{1}$$
$\mathbf{A} = [\mathbf{a}_1, \cdots, \mathbf{a}_m]^\top \in \mathbb{R}^{m \times n}$ with $m \ll n$.
- 1-bit compressed sensing: recover k-sparse $\mathbf{x} \in \mathbb{S}^{n-1}$ from
$$\mathbf{y} = \mathrm{sign}(\mathbf{A}\mathbf{x}); \tag{2}$$
we assume $\mathbf{A} \sim \mathcal{N}^{m \times n}(0, 1)$
- Optimal ℓ_2 error rate is $\tilde{\Theta}(\frac{k}{m})$ [JLBB13][1] (upper bound achieved by infeasible program). Two downsides:
  - Issue 1: Signal norm recovery is not possible (we assume $\mathbf{x} \in \mathbb{S}^{n-1}$)
  - Issue 2: In general hard to go beyond Gaussian design [ALPV14][2]
- Using dithers $\boldsymbol{\tau} \sim \mathrm{Unif}([-\lambda, \lambda]^m)$ addresses both issues [DM21][3]:
$$\mathbf{y} = \mathrm{sign}(\mathbf{A}\mathbf{x} + \boldsymbol{\tau}) \tag{3}$$
  - Signals with bounded ℓ_2 norm
  - $\mathbf{A}$ has independent sub-Gaussian rows

[1] Robust 1-Bit Compressive Sensing via Binary Stable Embeddings of Sparse Vectors
[2] One-bit compressed sensing with non-Gaussian measurements
[3] Non-Gaussian hyperplane tessellations and robust one-bit compressed sensing

1-Bit Compressed Sensing

- **N**ormalized **B**inary **I**terative **H**ard **T**hresholding — an efficient algorithm to achieve $\tilde{O}(\frac{k}{m})$ [MM24][4]
- Hamming distance loss $\mathcal{L}_{hd}(\mathbf{u}) = \frac{1}{m}\sum_{i=1}^{m} \mathbb{1}(\mathrm{sign}(\mathbf{a}_i^\top \mathbf{u}) \neq y_i)$ $= \frac{1}{m}\sum_{i=1}^{m} \mathbb{1}(-y_i \mathbf{a}_i^\top \mathbf{u} \geq 0) \longrightarrow$ Hinge loss

$$\mathcal{L}(\mathbf{u}) = \frac{1}{2m}\sum_{i=1}^{m}\left(-y_i \mathbf{a}_i^\top \mathbf{u} + |\mathbf{a}_i^\top \mathbf{u}|\right) \tag{4}$$



- NBIHT starts with *arbitrary* $\mathbf{x}^{(0)} \in \mathbb{S}^{n-1}$ and produces

$$\mathbf{x}^{(t+1)} = \frac{\mathsf{T}_{(k)}(\mathbf{x}^{(t)} - \eta \cdot \partial\mathcal{L}(\mathbf{x}^{(t)}))}{\|\mathsf{T}_{(k)}(\mathbf{x}^{(t)} - \eta \cdot \partial\mathcal{L}(\mathbf{x}^{(t)}))\|_2}, \quad t = 0, 1, \cdots \tag{5}$$

where $\partial\mathcal{L}(\mathbf{u}) = \frac{1}{2m}\sum_{i=1}^{m}\big(\mathrm{sign}(\mathbf{a}_i^\top \mathbf{u}) - \mathrm{sign}(\mathbf{a}_i^\top \mathbf{x})\big)\mathbf{a}_i$

[4] Binary iterative hard thresholding converges with optimal number of measurements for 1-bit compressed sensing

Phase Retrieval

- In many applications we only observe the magnitude $|\mathbf{a}_i^\top \mathbf{x}|$ [SECCMS15][5]
- Phase retrieval: the recovery of $\mathbf{x} \in \mathbb{R}^n$ from $\mathbf{y} = |\mathbf{A}\mathbf{x}|$

Similarity to solving linear systems (solve $\mathbf{x} \in \mathbb{R}^n$ from $\mathbf{y} = \mathbf{A}\mathbf{x}$):

- All $\mathbf{x} \in \mathbb{R}^n$ can be exactly recovered to $\{\pm\mathbf{x}\}$ from generic $\mathbf{A} \in \mathbb{R}^{(2n-1)\times n}$[BCE06];[6]
- All $\mathbf{x} \in \mathbb{C}^n$ can be recovered to $\{e^{\mathbf{i}\theta}\mathbf{x} : \theta \in \mathbb{R}\}$ from generic $\mathbb{C}^{(4n-4)\times n}$ [BCE06];
- $\mathbf{x}$ can be recovered from many efficient algorithms such as (truncated) Wirtinger flow [CLS15],[7] [CC17][8] from $O(n)$ *Gaussian* measurements;
- Randomized Kaczmarz also works for phase retrieval [TV19];[9]
- Sparse phase retrieval resembles compressed sensing in terms of sample complexity $\tilde{O}(k \log \frac{n}{k})$ [EM14],[10] with a major difference on sample complexity for efficient algorithm $\tilde{O}(k^2)$

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[5] Phase Retrieval with Application to Optical Imaging: A contemporary overview
[6] On signal reconstruction without phase
[7] Phase retrieval via Wirtinger flow: Theory and algorithms
[8] Solving random quadratic systems of equations is nearly as easy as solving linear systems
[9] Phase retrieval via randomized Kaczmarz: theoretical guarantees
[10] Phase retrieval: Stability and recovery guarantees

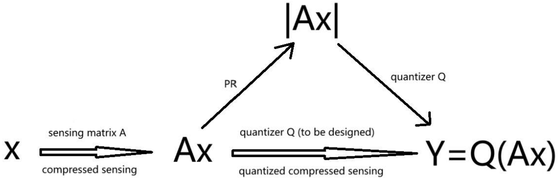
1-Bit Phase Retrieval

Question:

How to achieve phase retrieval from *quantized* measurements?

Why is this interesting?

- The loss of phase and quantization are both ubiquitous;
- Quantized phase retrieval is not theoretically well understood [DB22];[11]
- Is quantized phase retrieval *similar* to quantized compressed sensing in some sense?
- New contributions to the well-developed area of quantized compressed sensing.



[11] Phase Retrieval by Binary Questions: Which Complementary Subspace is Closer?

1-Bit Phase Retrieval

Our problem setup:

- We deal with 1-bit phase retrieval
- $\text{sign}(|\mathbf{a}_i^\top \mathbf{x}|) = 1 \longrightarrow$ no information!
- We use positive quantization threshold $\tau > 0$ and observe

$$\mathbf{y} = \text{sign}(|\mathbf{A}\mathbf{x}| - \tau) \tag{6}$$

- We assume $\mathbf{A} \sim \mathcal{N}^{m \times n}(0, 1)$ and for some $\beta \geq \alpha > 0$:

$$\mathbf{x} \in \mathbb{A}_\alpha^\beta := \{\mathbf{u} \in \mathbb{R}^n : \alpha \leq \|\mathbf{u}\|_2 \leq \beta\} \tag{7}$$

- We study two cases:
  - 1-bit phase retrieval (1bPR): $\mathbf{x}$ is unstructured
  - 1-bit sparse phase retrival (1bSPR): $\mathbf{x}$ is k-sparse

Overview of this Talk

This talk demonstrates that:

> Major findings in 1bCS theory, including *hyperplane tessellation*, *optimal rates* and *efficient algorithms*, can also be established in phase retrieval

In other words,

> In some sense, phase information is inessential for 1bCS

Model	$\mathbf{y}$	$\mathbf{x}$	opti. rate	opti. alg sample
1bCS	$\mathbf{y}=\mathrm{sign}(\mathbf{A}\mathbf{x})$	$\Sigma_k^n\cap\mathbb{S}^{n-1}$	$\tilde{\Theta}(\frac{k}{m})$	$\tilde{O}(k)$
D1bCS	$\mathbf{y}=\mathrm{sign}(\mathbf{A}\mathbf{x}+\boldsymbol{\tau})$	$\Sigma_k^n\cap\mathbb{B}_2^n$	$\tilde{\Theta}(\frac{k}{m})$	$\tilde{O}(k)$
1bPR	$\mathbf{y}=\mathrm{sign}(\lvert\mathbf{A}\mathbf{x}\rvert-\tau)$	$\mathbb{A}^1_{1/2}$	$\tilde{\Theta}(\frac{n}{m})$	$\tilde{O}(n)$
1bSPR	$\mathbf{y}=\mathrm{sign}(\lvert\mathbf{A}\mathbf{x}\rvert-\tau)$	$\Sigma_k^n\cap\mathbb{A}^1_{1/2}$	$\tilde{\Theta}(\frac{k}{m})$	$\tilde{O}(k^2)$

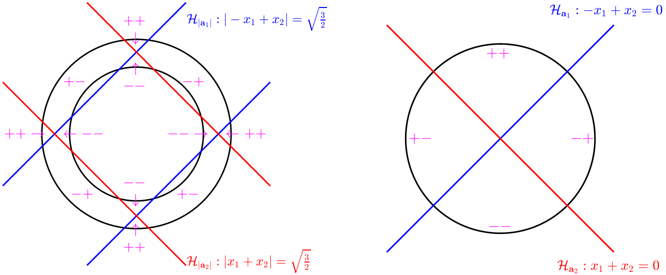
Outline

Ideal Program & Tessellation

- The best program is to minimize hamming distance loss over signal set:

$$\hat{\mathbf{x}}_{hdm} = \arg \min_{\mathbf{u} \in \mathbb{A}_{\alpha}^{\beta} (\cap \Sigma_k^n)} \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\left(\operatorname{sign}(|\mathbf{a}_i^\top \mathbf{u}| - \tau) \neq y_i\right) \tag{8}$$

- In the noiseless case with $y_i = \operatorname{sign}(|\mathbf{a}_i^\top \mathbf{x}| - \tau)$, (8) returns estimates having same measurements as $\mathbf{x}$: $\operatorname{sign}(|\mathbf{A}\hat{\mathbf{x}}_{hdm}| - \tau) = \operatorname{sign}(|\mathbf{A}\mathbf{x}| - \tau)$
- $\mathcal{H}_{\mathbf{a}_i,\tau} := \{\mathbf{u} \in \mathbb{R}^n : \mathbf{a}_i^\top \mathbf{u} = \tau\} \longrightarrow$
  $\mathcal{H}_{|\mathbf{a}_i|,\tau} := \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{a}_i^\top \mathbf{u}| = \tau\} = \mathcal{H}_{\mathbf{a}_i,\tau} \cup \mathcal{H}_{\mathbf{a}_i,-\tau}$
- Geometric interpretation:



Local Tessellation (Local Binary Embedding)

- *Arbitrary* signal set: $\mathcal{K} \subset \mathbb{A}_\alpha^\beta \overset{\text{localize}}{\longrightarrow} \mathcal{K}_{(r)} := (\mathcal{K} - \mathcal{K}) \cap \mathbb{B}_2^n(r)$
- Gaussian width $\omega(\mathcal{K}) := \mathbb{E} \sup_{\mathbf{u} \in \mathcal{K}} |\langle \mathbf{g}, \mathbf{u} \rangle|$ where $\mathbf{g} \sim \mathcal{N}(0, \mathbf{I}_n)$
- Covering number $\mathscr{N}(\mathcal{K}, r)$; metric entropy $\mathscr{H}(\mathcal{K}, r) = \log \mathscr{N}(\mathcal{K}, r)$
- $\text{dist}(\mathbf{u}, \mathbf{v}) = \min\{\|\mathbf{u} - \mathbf{v}\|_2, \|\mathbf{u} + \mathbf{v}\|_2\}$

Theorem 2.1: Phaseless Gaussian Hyperplane Tessellation

Under Gaussian design and any positive $\beta \ge \alpha$ and τ, for small enough $r > 0$ we let $r' = \frac{c_1 r}{\log^{1/2}(r^{-1})}$ (for some small c_1). If

$$m \gtrsim \frac{\omega^2(\mathcal{K}_{(3r'/2)})}{r^3} + \frac{\log \mathscr{N}(\mathcal{K}, r')}{r} \tag{9}$$

then w.p. $\ge 1 - \exp(-\Omega(rm))$ we have:

- Any $\mathbf{u}, \mathbf{v} \in \mathcal{K}$ obeying $\text{dist}(\mathbf{u}, \mathbf{v}) \le \frac{r'}{2}$ satisfy

$$m^{-1} d_H\big(\text{sign}(|\mathbf{A}\mathbf{u}| - \tau), \text{sign}(|\mathbf{A}\mathbf{v}| - \tau)\big) \le C_2 r \tag{10}$$

- Any $\mathbf{u}, \mathbf{v} \in \mathcal{K}$ obeying $\text{dist}(\mathbf{u}, \mathbf{v}) \ge 2r$ satisfy

$$m^{-1} d_H\big(\text{sign}(|\mathbf{A}\mathbf{u}| - \tau), \text{sign}(|\mathbf{A}\mathbf{v}| - \tau)\big) \ge c_3 \text{dist}(\mathbf{u}, \mathbf{v}) \tag{11}$$

Implications

Information-theoretic recovery guarantees:

- If

$$m \gtrsim \frac{\omega^2(\mathcal{K}_{(3r'/2)})}{r^3} + \frac{\log \mathscr{N}(\mathcal{K}, r')}{r}, \tag{12}$$

  then

$$\mathrm{dist}(\hat{\mathbf{x}}_{hdm}, \mathbf{x}) < 2r, \quad \forall \mathbf{x} \in \mathcal{K} \tag{13}$$

- If $\mathcal{K} \subset \mathcal{C}$ for a cone $\mathcal{C}$,

$$m = \tilde{O}\left(\frac{\omega^2((\mathcal{C} - \mathcal{C}) \cap \mathbb{B}_2^n) + \log \mathscr{N}(\mathcal{K}, r')}{r}\right) \tag{14}$$

  implies uniform recovery accuracy of $2r$.
- (1bPR) $\mathcal{C} = \mathbb{R}^n, \mathcal{K} = \mathbb{A}_\alpha^\beta \longrightarrow r = \tilde{O}(\frac{n}{m})$
- (1bSPR) $\mathcal{C} = \Sigma_k^n, \mathcal{K} = \Sigma_k^n \cap \mathbb{A}_\alpha^\beta \longrightarrow r = \tilde{O}(\frac{k}{m})$

Proof Sketch

- Similar results appeared in 1bCS literature [OR15],[12] [DM21], built upon a covering argument along with the well-known probabilistic observation ($\forall \mathbf{u}, \mathbf{v} \in \mathbb{S}^{n-1}$)

$$\mathbb{P}\Big(\mathrm{sign}(\mathbf{a}_i^\top \mathbf{u}) \neq \mathrm{sign}(\mathbf{a}_i^\top \mathbf{v})\Big) = \frac{\arccos(\langle \mathbf{u}, \mathbf{v}\rangle)}{\pi} \asymp \|\mathbf{u}-\mathbf{v}\|_2. \tag{15}$$

- We largely follow their arguments but need a novel relation ($\forall \mathbf{u}, \mathbf{v} \in \mathbb{A}_\alpha^\beta$)

$$\mathsf{P}_{\mathbf{u},\mathbf{v}} := \mathbb{P}\Big(\mathrm{sign}(|\mathbf{a}_i^\top \mathbf{u}| - \tau) \neq \mathrm{sign}(|\mathbf{a}_i^\top \mathbf{v}| - \tau)\Big) \asymp \mathrm{dist}(\mathbf{u}, \mathbf{v}) \tag{16}$$

- Actually, to get similar results under sub-Gaussian design, we only need

$$\mathsf{P}_{\mathbf{u},\mathbf{v}} \gtrsim \mathrm{dist}(\mathbf{u}, \mathbf{v}), \tag{17}$$

$$\mathbb{P}(||\mathbf{a}_i^\top \mathbf{u}| - \tau| \leq r) \lesssim r, \tag{18}$$

see the unified framework in [CY24b][13]

[12] Near-optimal bounds for binary embeddings of arbitrary sets

[13] Optimal quantized compressed sensinig via projected gradient descent

Lower Bounds

Is the upper bounds $\tilde{O}(\frac{n}{m})$ and $\tilde{O}(\frac{k}{m})$ tight? Yes — up to log!

Theorem 2.2: Lower Bounds for 1-Bit (Sparse) PR

For arbitrary known $(\mathbf{A}, \tau)$ we have the following:

- Any estimator $\hat{\mathbf{x}}$ for recovering $\mathbf{x} \in \mathbb{A}_1^2$ from $\mathrm{sign}(|\mathbf{A}\mathbf{x}| - \tau)$ obeys $\sup_{\mathbf{x} \in \mathbb{A}_1^2} \ \mathrm{dist}(\hat{\mathbf{x}}, \mathbf{x}) \gtrsim \frac{n}{m}$
- Any estimator $\hat{\mathbf{x}}$ for recovering $\mathbf{x} \in \Sigma_k^n \cap \mathbb{A}_1^2$ from $\mathrm{sign}(|\mathbf{A}\mathbf{x}| - \tau)$ obeys $\sup_{\mathbf{x} \in \Sigma_k^n \cap \mathbb{A}_1^2} \ \mathrm{dist}(\hat{\mathbf{x}}, \mathbf{x}) \gtrsim \frac{k}{m}$

Counting argument: let V_d be a d-dimensional space in $\mathbb{R}^n$

- Number of $\mathbf{y}$: $|\{\mathrm{sign}(|\mathbf{A}\mathbf{x}| - \tau) : \mathbf{x} \in \mathsf{V}_d\}|$ $\leq |\{\mathrm{sign}(\mathbf{A}\mathbf{x} - \tau) : \mathbf{x} \in \mathsf{V}_d\}| + |\{\mathrm{sign}(\mathbf{A}\mathbf{x} + \tau) : \mathbf{x} \in \mathsf{V}_d\}| \leq 2(\frac{em}{d})^d \ll 2^m$
- An ϵ-packing of $\mathsf{V}_d \cap \mathbb{A}_1^2$ with cardinality greater than $(\frac{2}{\epsilon})^d$
- Thus

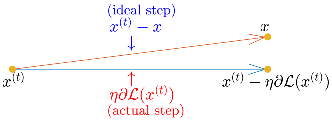
$$2\left(\frac{em}{l}\right)^d \geq \left(\frac{2}{\epsilon}\right)^d \quad \longrightarrow \quad \epsilon \gtrsim \frac{d}{m}. \tag{19}$$

Outline

NBIHT for 1bCS [MM24]

- Hinge loss $\mathcal{L}(\mathbf{u}) = \frac{1}{2m}\sum_{i=1}^{m}(-y_i \mathbf{a}_i^\top \mathbf{u} + |\mathbf{a}_i^\top \mathbf{u}|)$ with (sub-)gradient $\partial\mathcal{L}(\mathbf{u}) = \frac{1}{2m}\sum_{1}^{m}\big(\mathrm{sign}(\mathbf{a}_i^\top \mathbf{u}) - \mathrm{sign}(\mathbf{a}_i^\top \mathbf{x})\big)\mathbf{a}_i := \mathbf{h}(\mathbf{u},\mathbf{x})$
- NBIHT: $\tilde{\mathbf{x}}^{(t+1)} = \mathsf{T}_{(k)}(\mathbf{x}^{(t)} - \eta\cdot\partial\mathcal{L}(\mathbf{x}^{(t)})),\ \ \mathbf{x}^{(t+1)} = \tilde{\mathbf{x}}^{(t+1)}/\|\tilde{\mathbf{x}}^{(t+1)}\|_2$
- Optimization: $\|\mathbf{x}^{(t+1)} - \mathbf{x}\|_2 \le 4\|\mathbf{x}^{(t)} - \mathbf{x} - \eta\cdot\mathbf{h}(\mathbf{x}^{(t)},\mathbf{x})\|_{(\Sigma_{2k}^{n,*})^\circ}$
- HD Probability $\longrightarrow$ **R**estricted **A**pproximate **I**nvertibility **C**ondition (RAIC) [FJPY21],[14] [MM24], $\forall \mathbf{u},\mathbf{v}\in\Sigma_k^{n,*}$,

$$\|\mathbf{u} - \mathbf{v} - \eta\cdot\partial\mathbf{h}(\mathbf{u},\mathbf{v})\|_{(\Sigma_{2k}^{n,*})^\circ} \le \tilde{O}(\frac{k}{m}) + \sqrt{\tilde{O}(\frac{k}{m})\|\mathbf{u}-\mathbf{v}\|_2} \qquad (20)$$



- Optimization: $\|\mathbf{x}^{(t+1)} - \mathbf{x}\|_2 \le \tilde{O}(\frac{k}{m}) + \sqrt{\tilde{O}(\frac{k}{m})\|\mathbf{x}^{(t)} - \mathbf{x}\|_2}$
  $\longrightarrow$ fast *quadratic* convergence taking $O(\log(\log(m/k)))$ steps

[14] NBIHT: An efficient algorithm for 1-bit compressed sensing with optimal error decay rate

Our Algorithm

- Hamming distance loss: $\mathcal{L}_{hd}(\mathbf{u}) = \frac{1}{m}\sum_{i=1}^m \mathbb{1}(\mathrm{sign}(|\mathbf{a}_i^\top \mathbf{u}| - \tau) \neq y_i)$ $= \frac{1}{m}\sum_{i=1}^m \mathbb{1}(-y_i(|\mathbf{a}_i^\top \mathbf{u}| - \tau) \geq 0)$
- Use the same idea $\mathbb{1}(u \geq 0) \longrightarrow \max\{u, 0\} = \frac{u+|u|}{2}$ to get (nonconvex) Hinge loss $\mathcal{L}(\mathbf{u}) = \frac{1}{2m}\sum_{i=1}^m \left[\left||\mathbf{a}_i^\top \mathbf{u}| - \tau\right| - y_i(|\mathbf{a}_i^\top \mathbf{u}| - \tau)\right]$, with

$$\partial\mathcal{L}(\mathbf{u}) = \frac{1}{2m}\sum_{i=1}^m \big(\mathrm{sign}(|\mathbf{a}_i^\top \mathbf{u}| - \tau) - \mathrm{sign}(|\mathbf{a}_i^\top \mathbf{x}| - \tau)\big)\mathrm{sign}(\mathbf{a}_i^\top \mathbf{u})\mathbf{a}_i$$

$$\mathbf{h}(\mathbf{u}, \mathbf{v}) := \frac{1}{2m}\sum_{i=1}^m \big(\mathrm{sign}(|\mathbf{a}_i^\top \mathbf{u}| - \tau) - \mathrm{sign}(|\mathbf{a}_i^\top \mathbf{v}| - \tau)\big)\mathrm{sign}(\mathbf{a}_i^\top \mathbf{u})\mathbf{a}_i$$

- 1bPR:
  - Spectral initialization $\mathbf{x}^{(0)}$: leading eigenvector of $\hat{\mathbf{S}} = \frac{1}{m}\sum_{i=1}^m y_i \mathbf{a}_i \mathbf{a}_i^\top$
  - GD: $\mathbf{x}^{(t)} = \mathbf{x}^{(t-1)} - \eta \cdot \partial\mathcal{L}(\mathbf{x}^{(t-1)}), \quad t = 1, 2, 3, \cdots$
- 1bSPR:
  - Spectral initialization $\mathbf{x}^{(0)}$: leading eigenvector of a submatrix of $\hat{\mathbf{S}}$
  - PGD: $\mathbf{x}^{(t)} = \mathcal{T}_{(k)}(\mathbf{x}^{(t-1)} - \eta \cdot \partial\mathcal{L}(\mathbf{x}^{(t-1)})), \quad t = 1, 2, 3, \cdots$

Optimal Guarantees

Theorem 3.1: GD is Optimal for 1bPR

If $m \gtrsim n$, then w.h.p., running GD with spectral initialization and $\eta = \sqrt{\frac{\pi e}{2}}\tau$ uniformly recovers all $\mathbf{x} \in \mathbb{A}_\alpha^\beta$ to

$$\operatorname{dist}(\mathbf{x}^{(t)}, \mathbf{x}) \lesssim \frac{n}{m}\log^2\left(\frac{m}{n}\right), \quad \forall t \gtrsim \log\left(\frac{m}{n}\right). \tag{21}$$

Theorem 3.2: PGD is Optimal for 1bSPR

If $m \gtrsim k^2 \log(n) \log^2(\frac{m}{k})$, $\frac{\tau}{\alpha} \leq C_1$, $\frac{\beta}{\tau} \leq C_2$, then w.h.p., running PGD with spectral initialization and $\eta = \sqrt{\frac{\pi e}{2}}\tau$ recovers a $\mathbf{x} \in \Sigma_k^n \cap \mathbb{A}_\alpha^\beta$ to

$$\operatorname{dist}(\mathbf{x}^{(t)}, \mathbf{x}) \lesssim \frac{k}{m}\log\left(\frac{mn}{k^2}\right)\log\left(\frac{m}{k}\right), \quad \forall t \gtrsim \log\left(\frac{m}{k}\right). \tag{22}$$

- $\tilde{O}(k^2)$ in sparse case is needed in initialization (a widely existing gap)
- Need $\tilde{O}(k^3)$ to ensure uniform recovery

Proof: What to Bound

- Spectral method $\longrightarrow \|\mathbf{x}^{(0)} - \mathbf{x}\|_2 \leq \delta_4$
- Per-iterate analysis:
  - 1bPR:

$$\begin{aligned}\|\mathbf{x}^{(t)} - \mathbf{x}\|_2 &= \|\mathbf{x}^{(t-1)} - \mathbf{x} - \eta \cdot \partial\mathcal{L}(\mathbf{x}^{(t-1)})\|_2 \\ &= \|\mathbf{x}^{(t-1)} - \mathbf{x} - \eta \cdot \mathbf{h}(\mathbf{x}^{(t-1)}, \mathbf{x})\|_2\end{aligned}$$

  - 1bSPR:

$$\begin{aligned}\|\mathbf{x}^{(t)} - \mathbf{x}\|_2 &\leq 2\|\mathcal{T}_{(2k)}(\mathbf{x}^{(t-1)} - \mathbf{x} - \eta \cdot \partial\mathcal{L}(\mathbf{x}^{(t-1)}))\|_2 \\ &= 2\big\|\mathcal{T}_{(2k)}(\mathbf{x}^{(t-1)} - \mathbf{x} - \eta \cdot \mathbf{h}(\mathbf{x}^{(t-1)}, \mathbf{x}))\big\|_2\end{aligned}$$

- For cone $\mathcal{C}$ with $\mathcal{C}_- = \mathcal{C} - \mathcal{C}$, we want to bound

$$\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{u} - \mathbf{v} - \eta \cdot \mathbf{h}(\mathbf{u}, \mathbf{v}))\|_2, \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{C}_{\alpha,\beta} := \mathcal{C} \cap \mathbb{A}_\alpha^\beta,$$

Proof: Phaseless Local AIC

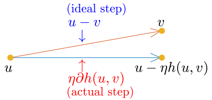
Definition 3.1: Phaseless Local AIC (PLL-AIC)

Given $\beta_1 \geq \alpha_1 > 0$ and $\tau > 0$, $\mathbf{A} = [\mathbf{a}_1^\top, \cdots, \mathbf{a}_m^\top]^\top \in \mathbb{R}^{m\times n}$, a cone $\mathcal{C}$, a step size η, and certain non-negative scalars $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3, \delta_4)^\top$, we say $(\mathbf{A}, \tau, \mathcal{C}, \eta)$ respects $(\alpha_1, \beta_1, \boldsymbol{\delta})$-PLL-AIC if

$$\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{u} - \mathbf{v} - \eta \cdot \mathbf{h}(\mathbf{u}, \mathbf{v}))\|_2 \leq \delta_1 \|\mathbf{u} - \mathbf{v}\|_2 + \sqrt{\delta_2 \cdot \|\mathbf{u} - \mathbf{v}\|_2} + \delta_3,$$
$$\forall \mathbf{u}, \mathbf{v} \in \mathcal{C}_{\alpha_1, \beta_1} \text{ obeying } \|\mathbf{u} - \mathbf{v}\|_2 \leq \delta_4,$$

where $\mathbf{h}(\mathbf{u}, \mathbf{v})$ denotes the subgradient at $\mathbf{u}$ when $\mathbf{v}$ is underlying signal: $\mathbf{h}(\mathbf{u}, \mathbf{v}) = \frac{1}{2m} \sum_{i=1}^m \big(\mathrm{sign}(|\mathbf{a}_i^\top \mathbf{u}| - \tau) - \mathrm{sign}(|\mathbf{a}_i^\top \mathbf{v}| - \tau)\big)\mathrm{sign}(\mathbf{a}_i^\top \mathbf{u})\mathbf{a}_i$

- The linear term '$\delta_1 \|\mathbf{u} - \mathbf{v}\|$' is necessary if $\mathbf{x} \in \mathbb{A}_\alpha^\beta$ with $\beta > \alpha$
- Local: $\|\mathbf{u} - \mathbf{v}\|_2 \leq \delta_4 \longleftarrow$ spectral method;
- Meaning: $\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{u} - \mathbf{v} - \eta \mathbf{h}(\mathbf{u}, \mathbf{v}))\|_2 = \|\mathbf{u} - \mathbf{v} - \eta \mathbf{h}(\mathbf{u}, \mathbf{v})\|_{(\mathcal{C}_- \cap \mathbb{S}^{n-1})^\circ}$



- Phaseless: it holds for $\mathbf{v}$ $\iff$ it holds for $-\mathbf{v}$

Proof: PLL-AIC $\longrightarrow$ Convergence

Why is AIC useful? Prove $\delta_2, \delta_3 = \tilde{O}(\text{optimal rate})$, $\delta_1 \approx F(\eta)$, $\delta_4 \approx \frac{1}{\sqrt{\log *}}$

- $\|\mathbf{x}^{(0)} - \mathbf{x}\|_2 \leq \delta_4$ ensured by spectral method
- 1bPR ($\mathcal{C} = \mathbb{R}^n$): if $\|\mathbf{x}^{(t-1)} - \mathbf{x}\| \gg \tilde{O}(n/m)$

$$\|\mathbf{x}^{(t)} - \mathbf{x}\|_2 \overset{raic}{\leq} \delta_1 \|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2 + \sqrt{\tilde{O}(n/m)\|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2} + \tilde{O}(n/m)$$
$$\leq (\delta_1 + \epsilon_1)\|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2 \leq (1 - \epsilon_2)\|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2$$

- 1bSPR ($\mathcal{C} = \Sigma^n_k$): if $\|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2 \gg \tilde{O}(k/m)$

$$\|\mathbf{x}^{(t)} - \mathbf{x}\|_2 \overset{raic}{\leq} 2\delta_1 \|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2 + \sqrt{\tilde{O}(k/m)\|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2} + \tilde{O}(k/m)$$
$$\leq (2\delta_1 + \epsilon_1)\|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2 \leq (1 - \epsilon_2)\|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2$$

- We obtain (at least) linear convergence to optimal error rates

Proof: Gaussian $\mathbf{A}$ Respects RAIC

Theorem 3.3: Gaussian $\mathbf{A}$ Respects PLL-AIC

Suppose $\mathbf{A} \sim \mathcal{N}^{m\times n}(0,1)$, $\beta \geq \alpha > 0, \tau > 0$, $\mathcal{C}$ is a cone. For some constants c_i's and C_i's depending on (α,β,τ), if $r \in (0, c_1)$,

$$m \geq \frac{C_2[\mathscr{H}(\mathcal{C}_{\alpha,\beta}, r) + \omega^2(\mathcal{C}_{(1)})]}{r}, \tag{23}$$

then with probability at least $1 - \exp(-c_3\mathscr{H}(\mathcal{C}_{\alpha,\beta}, r))$, $(\mathbf{A}, \tau, \mathcal{C}, \eta)$ respects $(\alpha, \beta, \boldsymbol{\delta})$-PLL-AIC with

$$\delta_1 = \sup_{a^2+b^2\in[\alpha^2,\beta^2]} \sqrt{|1-\eta g_\eta(a,b)|^2 + |\eta h_\eta(a,b)|^2} + c_3 \log^{-1/8}(r^{-1})$$

$$\delta_2 = C_4 r,\ \ \delta_3 = C_5 r\log(r^{-1}),\ \ \delta_4 = \frac{c_5}{\log^{1/2}(r^{-1})}$$

where $g_\eta(a,b) = \sqrt{\frac{2}{\pi}}\exp\Big(-\frac{\tau^2}{2(a^2+b^2)}\Big)\frac{\tau^2a^2+b^2(a^2+b^2)}{(a^2+b^2)^{5/2}}$ and $h_\eta(a,b) = \sqrt{\frac{2}{\pi}}\exp\Big(-\frac{\tau^2}{2(a^2+b^2)}\Big)\frac{ab(a^2+b^2-\tau^2)}{(a^2+b^2)^{5/2}}$.

Proof: Covering Framework

The goal is to bound $\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{u}-\mathbf{v}-\eta\cdot\mathbf{h}(\mathbf{u},\mathbf{v}))\|_2$ *for all* $\mathbf{u},\mathbf{v}\in\mathcal{C}_{\alpha_1,\beta_1}$ obeying $\|\mathbf{u}-\mathbf{v}\|_2\le\delta_4$. We use a covering argument:

- Let $\mathcal{N}_r$ be a minimal r-net of $\mathcal{C}_{\alpha,\beta}$
- $\mathbf{u}_1,\mathbf{v}_1\in\mathcal{N}_r$ closest to $\mathbf{u},\mathbf{v}$, respectively, $\|\mathbf{u}-\mathbf{u}_1\|_2,\|\mathbf{v}-\mathbf{v}_1\|_2\le r$
- $\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{u}-\mathbf{v}-\eta\mathbf{h}(\mathbf{u},\mathbf{v}))\|_2\le 2r+\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{u}_1-\mathbf{v}_1-\eta\mathbf{h}(\mathbf{u},\mathbf{v}))\|_2$
- Large-distance regime ($\|\mathbf{u}_1-\mathbf{v}_1\|_2\ge r$):

$$\begin{aligned}&\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{u}_1-\mathbf{v}_1-\eta\mathbf{h}(\mathbf{u},\mathbf{v}))\|_2 \\ &\le \underbrace{\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{u}_1-\mathbf{v}_1-\eta\mathbf{h}(\mathbf{u}_1,\mathbf{v}_1))\|_2}_{\text{discrete AIC}}+\eta\underbrace{\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}(\mathbf{u},\mathbf{v})-\mathbf{h}(\mathbf{u}_1,\mathbf{v}_1))\|_2}_{\text{gradient mismatch}}\end{aligned} \tag{24}$$

- Small-distance regime ($\|\mathbf{u}_1-\mathbf{v}_1\|_2< r$):

$$\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{u}_1-\mathbf{v}_1-\eta\mathbf{h}(\mathbf{u},\mathbf{v}))\|_2\le r+\eta\cdot\underbrace{\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}(\mathbf{u},\mathbf{v}))\|_2}_{\text{gradient}} \tag{25}$$

Proof: Simplify the Gradient

$$\mathbf{h}(\mathbf{u},\mathbf{v}) = \frac{1}{2m}\sum_{i=1}^{m}\left(\operatorname{sign}(|\mathbf{a}_i^\top \mathbf{u}| - \tau) - \operatorname{sign}(|\mathbf{a}_i^\top \mathbf{v}| - \tau)\right)\operatorname{sign}(\mathbf{a}_i^\top \mathbf{u})\mathbf{a}_i$$

- Introduce two index sets

$$\mathbf{R}_{\mathbf{p},\mathbf{q}} = \left\{i \in [m] : \operatorname{sign}(|\mathbf{a}_i^\top \mathbf{p}| - \tau) \neq \operatorname{sign}(|\mathbf{a}_i^\top \mathbf{q}| - \tau)\right\} \tag{26}$$

$$\mathbf{L}_{\mathbf{p},\mathbf{q}} = \left\{i \in [m] : \operatorname{sign}(\mathbf{a}_i^\top \mathbf{p}) \neq \operatorname{sign}(\mathbf{a}_i^\top \mathbf{q})\right\} \tag{27}$$

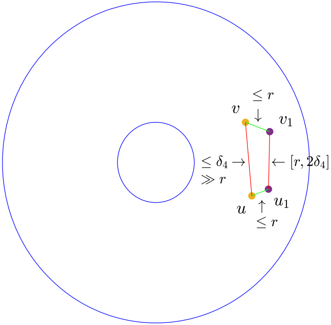
- Then we find $\mathbf{h}(\mathbf{p},\mathbf{q}) = \mathbf{h}_1(\mathbf{p},\mathbf{q}) + \mathbf{h}_2(\mathbf{p},\mathbf{q})$ where

$$\mathbf{h}_1(\mathbf{p},\mathbf{q}) = \frac{1}{m}\sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}} \operatorname{sign}(\mathbf{a}_i^\top(\mathbf{p}-\mathbf{q}))\mathbf{a}_i, \tag{28}$$

$$\mathbf{h}_2(\mathbf{p},\mathbf{q}) = \frac{1}{m}\sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}\cap\mathbf{L}_{\mathbf{p},\mathbf{q}}} \left[\operatorname{sign}(\mathbf{a}_i^\top(\mathbf{p}+\mathbf{q})) - \operatorname{sign}(\mathbf{a}_i^\top(\mathbf{p}-\mathbf{q}))\right]\mathbf{a}_i \tag{29}$$

- $\mathbf{h}_1(\mathbf{p},\mathbf{q})$ is the main term and close to 1bCS gradient $\frac{1}{m}\sum_{i\in\mathbf{L}_{\mathbf{p},\mathbf{q}}}\operatorname{sign}(\mathbf{a}_i^\top(\mathbf{p}-\mathbf{q}))\mathbf{a}_i$
- $\mathbf{h}_2(\mathbf{p},\mathbf{q})$ is a negligible higher-order term

Proof: Large-distance Regime



From (24), we need to bound

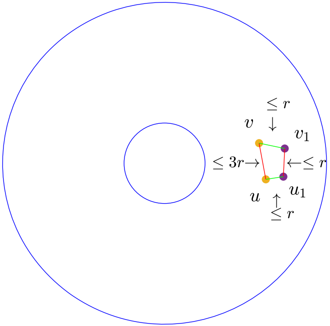
- $\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{u}_1 - \mathbf{v}_1 - \eta \mathbf{h}(\mathbf{u}_1, \mathbf{v}_1))\|_2$ uniformly over $\mathcal{N}^{(2)}_{r,\delta_4} := \{(\mathbf{p}, \mathbf{q}) \in \mathcal{N}_r \times \mathcal{N}_r : \|\mathbf{p} - \mathbf{q}\|_2 \in [r, 2\delta_4]\}$, and by $\mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2$ we only need to bound

Term1: $\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{u}_1 - \mathbf{v}_1 - \eta \mathbf{h}_1(\mathbf{u}_1, \mathbf{v}_1))\|_2, \quad (\mathbf{u}_1, \mathbf{v}_1) \in \mathcal{N}^{(2)}_{r,\delta_4}$

Term2: $\eta\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}_2(\mathbf{u}_1, \mathbf{v}_1))\|_2, \quad (\mathbf{u}_1, \mathbf{v}_1) \in \mathcal{N}^{(2)}_{r,\delta_4}$

- Term3: $\eta\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}(\mathbf{u}, \mathbf{v}) - \mathbf{h}(\mathbf{u}_1, \mathbf{v}_1))\|_2$

Proof: Small-distance Regime



From (25) we need to bound

- Term4: $\eta \|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}(\mathbf{u}, \mathbf{v}))\|_2$ uniformly over all $\mathbf{u}, \mathbf{v} \in \mathcal{C}_{\alpha,\beta}$ obeying $\|\mathbf{u} - \mathbf{v}\|_2 \leq 3r$.

Proof: Bounding Term 1

Bound $\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{p}-\mathbf{q}-\eta\mathbf{h}_1(\mathbf{p},\mathbf{q}))\|_2$ for all $(\mathbf{p},\mathbf{q})\in\mathcal{N}^{(2)}_{r,\delta_4}$:

- $|\mathcal{N}^{(2)}_{r,\delta_4}| \le |\mathcal{N}_r|^2 = [\mathscr{N}(\mathcal{C}_{\alpha,\beta}, r)]^2$
- Only need to bound it for fixed $(\mathbf{p},\mathbf{q})$ — followed by union bound

Orthogonal decomposition:

- Useful parameterization: we can find orthonormal $\boldsymbol{\beta}_1 = \frac{\mathbf{u}-\mathbf{v}}{\|\mathbf{u}-\mathbf{v}\|_2}$ and $\boldsymbol{\beta}_2$ such that
$$\mathbf{p} = u_1\boldsymbol{\beta}_1 + u_2\boldsymbol{\beta}_2, \quad \mathbf{q} = v_1\boldsymbol{\beta}_1 + u_2\boldsymbol{\beta}_2$$
for some u_1, u_2, v_1 obeying $u_1 > v_1$ and $u_2 \ge 0$. Then we have
$$\begin{aligned}\mathbf{h}_1(\mathbf{p},\mathbf{q}) &= \langle \mathbf{h}_1(\mathbf{p},\mathbf{q}), \boldsymbol{\beta}_1\rangle\boldsymbol{\beta}_1 + \langle \mathbf{h}_1(\mathbf{p},\mathbf{q}), \boldsymbol{\beta}_2\rangle\boldsymbol{\beta}_2 \\ &\quad + \underbrace{\left\{\mathbf{h}_1(\mathbf{p},\mathbf{q}) - \langle \mathbf{h}_1(\mathbf{p},\mathbf{q}), \boldsymbol{\beta}_1\rangle\boldsymbol{\beta}_1 - \langle \mathbf{h}_1(\mathbf{p},\mathbf{q}), \boldsymbol{\beta}_2\rangle\boldsymbol{\beta}_2\right\}}_{:=\mathbf{h}_1^{\perp}(\mathbf{p},\mathbf{q})}.\end{aligned}$$

- $\langle \mathbf{h}_1(\mathbf{p},\mathbf{q}), \boldsymbol{\beta}_1\rangle\boldsymbol{\beta}_1$ is the main term to cancel out $\mathbf{p}-\mathbf{q}$
- We need to control the effect of $\langle \mathbf{h}_1(\mathbf{p},\mathbf{q}), \boldsymbol{\beta}_2\rangle\boldsymbol{\beta}_2$ and $\mathbf{h}_1^{\perp}(\mathbf{p},\mathbf{q})$

Proof: Bounding Term 1

$$
\begin{aligned}
&\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{p}-\mathbf{q}-\eta\cdot\mathbf{h}_1(\mathbf{p},\mathbf{q}))\|_2 \\
&\leq \left\|\mathbf{p}-\mathbf{q}-\eta\cdot\langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\boldsymbol{\beta}_1\rangle\boldsymbol{\beta}_1-\eta\cdot\langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\boldsymbol{\beta}_2\rangle\boldsymbol{\beta}_2\right\|_2+\eta\cdot\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}_1^{\perp}(\mathbf{p},\mathbf{q}))\|_2 \\
&\leq \left(\left|\|\mathbf{p}-\mathbf{q}\|_2-\eta\cdot\left\langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\frac{\mathbf{p}-\mathbf{q}}{\|\mathbf{p}-\mathbf{q}\|_2}\right\rangle\right|^2+\eta^2\cdot|\langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\boldsymbol{\beta}_2\rangle|^2\right)^{1/2} \\
&\qquad +\eta\cdot\left\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}_1^{\perp}(\mathbf{p},\mathbf{q}))\right\|_2 \qquad (30)\\
&:= ((T_1^{\mathbf{p},\mathbf{q}})^2+\eta^2\cdot|T_2^{\mathbf{p},\mathbf{q}}|^2)^{1/2}+\eta\cdot T_3^{\mathbf{p},\mathbf{q}},
\end{aligned}
$$

where

$$
T_1^{\mathbf{p},\mathbf{q}} := \left|\|\mathbf{p}-\mathbf{q}\|_2-\eta\cdot\left\langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\frac{\mathbf{p}-\mathbf{q}}{\|\mathbf{p}-\mathbf{q}\|_2}\right\rangle\right|, \qquad (31)
$$

$$
T_2^{\mathbf{p},\mathbf{q}} := \langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\boldsymbol{\beta}_2\rangle, \quad T_3^{\mathbf{p},\mathbf{q}} := \|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}_1^{\perp}(\mathbf{p},\mathbf{q}))\|_2 \qquad (32)
$$

- Need to separately bound $T_1^{\mathbf{p},\mathbf{q}}, T_2^{\mathbf{p},\mathbf{q}}, T_3^{\mathbf{p},\mathbf{q}}$

Proof: Bounding Term 1 (Example: Bound $T_1^{\mathbf{p},\mathbf{q}}$)

The ideas in bounding $T_i^{\mathbf{p},\mathbf{q}}$, $i = 1, 2, 3$ are similar. Use $T_1^{\mathbf{p},\mathbf{q}}$ as an example:

$$T_1^{\mathbf{p},\mathbf{q}} = \left| \|\mathbf{p} - \mathbf{q}\|_2 - \frac{\eta}{m} \sum_{i \in \mathbf{R}_{\mathbf{p},\mathbf{q}}} |\mathbf{a}_i^\top \boldsymbol{\beta}_1| \right|$$

$$\leq \underbrace{\eta \left| \frac{1}{m} \sum_{i \in \mathbf{R}_{\mathbf{p},\mathbf{q}}} |\mathbf{a}_i^\top \boldsymbol{\beta}_1| - \mathbb{E}\left[\mathbb{1}(i \in \mathbf{R}_{\mathbf{p},\mathbf{q}}) |\mathbf{a}_i^\top \boldsymbol{\beta}_1| \right] \right|}_{\text{Concentration term}} + \underbrace{\left| \|\mathbf{p} - \mathbf{q}\|_2 - \eta \mathbb{E}\left[\mathbb{1}(i \in \mathbf{R}_{\mathbf{p},\mathbf{q}}) |\mathbf{a}_i^\top \boldsymbol{\beta}_1| \right] \right|}_{\text{Deviation}}$$

- Careful calculation shows: Deviation $= \|\mathbf{p} - \mathbf{q}\|_2 (1 - \eta f(\mathbf{p}, \mathbf{q}) + o(1))$
- Conditioning on $\mathbf{R}_{\mathbf{p},\mathbf{q}}$ with cardinality $r_{\mathbf{p},\mathbf{q}}$, we have

$$\frac{1}{m} \sum_{i \in \mathbf{R}_{\mathbf{p},\mathbf{q}}} |\mathbf{a}_i^\top \boldsymbol{\beta}_1| \sim \frac{1}{m} \sum_{i=1}^{r_{\mathbf{p},\mathbf{q}}} Z_i^{\mathbf{p},\mathbf{q}} \tag{33}$$

where we let $a_1, a_2 \overset{iid}{\sim} \mathcal{N}(0, 1)$

$$\begin{aligned} Z_i^{\mathbf{p},\mathbf{q}} &\overset{iid}{\sim} |\mathbf{a}_i^\top \boldsymbol{\beta}_1| \big| \{\mathrm{sign}(|\mathbf{a}_i^\top \mathbf{p}| - \tau) \neq \mathrm{sign}(|\mathbf{a}_i^\top \mathbf{q}| - \tau)\} \\ &\sim |\mathbf{a}_i^\top \boldsymbol{\beta}_1| \big| \{\mathrm{sign}(|u_1 \mathbf{a}_i^\top \boldsymbol{\beta}_1 + u_2 \mathbf{a}_i^\top \boldsymbol{\beta}_2| - \tau) \neq \mathrm{sign}(|v_1 \mathbf{a}_i^\top \boldsymbol{\beta}_1 + u_2 \mathbf{a}_i^\top \boldsymbol{\beta}_2| - \tau)\} \\ &\sim |a_1| \big| \{\mathrm{sign}(|u_1 a_1 + u_2 a_2| - \tau) \neq \mathrm{sign}(|v_1 a_1 + u_2 a_2| - \tau)\} \end{aligned}$$

Proof: Bounding Term 1 (Example: Bound $T_1^{\mathbf{p},\mathbf{q}}$)

- Show that $Z_i^{\mathbf{p},\mathbf{q}}$ are sub-Gaussian:
  - Write down the P.D.F. of $Z_i^{\mathbf{p},\mathbf{q}}$;
  - Show the tail of P.D.F. is bounded by some Gaussian tail (tedious!);
- This shows conditional concentration: conditioning on $\{|\mathbf{R}_{\mathbf{p},\mathbf{q}}| = r_{\mathbf{p},\mathbf{q}}\}$, with prob. $\geq 1 - 2\exp(-4\log \mathscr{H}(\mathcal{C}_{\alpha,\beta}, r))$,

$$\text{concentration term} \leq \frac{|r_{\mathbf{p},\mathbf{q}} - m\mathsf{P}_{\mathbf{p},\mathbf{q}}| + \sqrt{r_{\mathbf{p},\mathbf{q}}\mathscr{H}(\mathcal{C}_{\alpha,\beta}, r)}}{m}$$

- Remains to analyze $|\mathbf{R}_{\mathbf{p},\mathbf{q}}| \sim \mathsf{Bin}(m, \mathsf{P}_{\mathbf{p},\mathbf{q}})$. By Chernoff bound, with prob. $\geq 1 - 2\exp(-4\log \mathscr{H}(\mathcal{C}_{\alpha,\beta}, r))$,

$$\big||\mathbf{R}_{\mathbf{p},\mathbf{q}}| - m\mathsf{P}_{\mathbf{p},\mathbf{q}}\big| \leq \sqrt{12m\mathsf{P}_{\mathbf{p},\mathbf{q}}\mathscr{H}(\mathcal{C}_{\alpha,\beta}, r)}$$

- Final bound: Concentration term $\lesssim \sqrt{\frac{\|\mathbf{p}-\mathbf{q}\|_2 \mathscr{H}(\mathcal{C}_{\alpha,\beta}, r)}{m}}$

Proof: Bounding Terms 2, 3, 4

$$\text{Term 2:}\quad \|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}_2(\mathbf{p},\mathbf{q}))\|_2, \quad \forall(\mathbf{p},\mathbf{q}) \in \mathcal{N}^{(2)}_{r,\delta_4}$$

$$\text{Term 3:}\quad \|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}(\mathbf{u},\mathbf{v}) - \mathbf{h}(\mathbf{u}_1,\mathbf{v}_1))\|_2, \quad \|\mathbf{u}-\mathbf{u}_1\|_2, \|\mathbf{v}-\mathbf{v}_1\|_2 \le r$$

$$\text{Term 4:}\quad \|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}(\mathbf{u},\mathbf{v}))\|_2, \quad \|\mathbf{u}-\mathbf{v}\|_2 \le 3r,$$

$$\text{where}\quad \mathbf{h}(\mathbf{u},\mathbf{v}) = \frac{1}{2m}\sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}} \big(\mathrm{sign}(|\mathbf{a}_i^\top\mathbf{u}| - \tau) - \mathrm{sign}(|\mathbf{a}_i^\top\mathbf{v}| - \tau)\big)\mathrm{sign}(\mathbf{a}_i^\top\mathbf{u})\mathbf{a}_i$$

$$\mathbf{h}_2(\mathbf{p},\mathbf{q}) = \frac{1}{m}\sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}\cap\mathbf{L}_{\mathbf{p},\mathbf{q}}} \big[\mathrm{sign}(\mathbf{a}_i^\top(\mathbf{p}+\mathbf{q})) - \mathrm{sign}(\mathbf{a}_i^\top(\mathbf{p}-\mathbf{q}))\big]\mathbf{a}_i$$

- Idea: *If the number of contributors is small enough, then we can get tight enough bound* $\longrightarrow$ we shall look at the number of summands
- Challenge: Terms $3, 4$ involve infinitely many points $\mathbf{u}$ and $\mathbf{v}$
- Remedy: local binary embedding! [OR15], [DM21]; see also (10)

Proof: Bounding Terms 2, 3, 4

Lemma 3.1: Uniform Bound on Partial Sum of Squares (e.g., [DM21])

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be independent random vectors in $\mathbb{R}^n$ satisfying $\mathbb{E}(\mathbf{a}_i \mathbf{a}_i^\top) = \mathbf{I}_n$ and $\max_i \|\mathbf{a}_i\|_{\psi_2} \le L$. For some given given $\mathcal{W} \subset \mathbb{R}^n$ and $1 \le \ell \le m$, there exist constants C_1, c_2 depending only on L such that the event

$$\sup_{\mathbf{x} \in \mathcal{W}} \max_{\substack{I \subset [m] \\ |I| \le \ell}} \left(\frac{1}{\ell} \sum_{i \in I} |\langle \mathbf{a}_i, \mathbf{x} \rangle|^2 \right)^{1/2} \le C_1 \left(\frac{\omega(\mathcal{W})}{\sqrt{\ell}} + \mathrm{rad}(\mathcal{W}) \sqrt{\log\left(\frac{em}{\ell}\right)} \right)$$

holds with probability at least $1 - 2\exp(-c_2 \ell \log(\frac{em}{\ell}))$.

By the above Lemma, it suffices to show the number of summands in Terms 2,3,4 are fewer than $\tilde{O}(mr)$, for instance:

$$\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}_2(\mathbf{p}, \mathbf{q}))\|_2 = \sup_{\mathbf{w} \in \mathcal{C}_- \cap \mathbb{B}_2^n} \langle \mathbf{w}, \mathbf{h}_2(\mathbf{p}, \mathbf{q}) \rangle$$

$$= \sup_{\mathbf{w} \in \mathcal{C}_- \cap \mathbb{B}_2^n} \frac{1}{m} \sum_{i \in \mathbf{R}_{\mathbf{p},\mathbf{q}} \cap \mathbf{L}_{\mathbf{p},\mathbf{q}}} \left[\mathrm{sign}(\mathbf{a}_i^\top(\mathbf{p}+\mathbf{q})) - \mathrm{sign}(\mathbf{a}_i^\top(\mathbf{p}-\mathbf{q}))\right] \mathbf{a}_i^\top \mathbf{w}$$

$$\le \sup_{\mathbf{w} \in \mathcal{C}_- \cap \mathbb{B}_2^n} \max_{\substack{\mathcal{S} \subset [m] \\ |\mathcal{S}| = \tilde{O}(mr)}} \frac{2|\mathbf{a}_i^\top \mathbf{w}|}{m}$$

- number of summands is uniformly small

$$= \tilde{O}(r)$$

- By Lemma 3.1

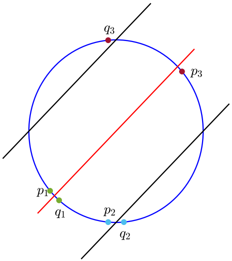
Number of Summands in Term 2: $\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}_2(\mathbf{p},\mathbf{q}))\|_2, \quad (\mathbf{p},\mathbf{q}) \in \mathcal{N}_{r,\delta_4}^{(2)}$

- Control $|\mathbf{R}_{\mathbf{p},\mathbf{q}} \cap \mathbf{L}_{\mathbf{p},\mathbf{q}}|$ over $\mathcal{N}_{r,\delta_4}^{(2)}$
- $|\mathbf{R}_{\mathbf{p},\mathbf{q}} \cap \mathbf{L}_{\mathbf{p},\mathbf{q}}| \sim \mathsf{Bin}(m, \mathsf{P}_{\mathbf{p},\mathbf{q}}^{(2)})$, where

$$\mathsf{P}_{\mathbf{p},\mathbf{q}}^{(2)} := \mathbb{P}(i \in \mathbf{R}_{\mathbf{p},\mathbf{q}} \cap \mathbf{L}_{\mathbf{p},\mathbf{q}}) = \mathbb{P}\begin{pmatrix} \mathrm{sign}(|\mathbf{a}_i^\top \mathbf{p}| - \tau) \neq \mathrm{sign}(|\mathbf{a}_i^\top \mathbf{q}| - \tau) \\ \mathrm{sign}(\mathbf{a}_i^\top \mathbf{p}) \neq \mathrm{sign}(\mathbf{a}_i^\top \mathbf{q}) \end{pmatrix}$$

- $\mathsf{P}_{\mathbf{p},\mathbf{q}}^{(2)} \leq 4\exp(-\frac{\tau^2}{2\|\mathbf{p}-\mathbf{q}\|_2^2})$, in stark contrast to:
  - $\mathsf{P}_{\mathbf{p},\mathbf{q}} = \mathbb{P}(\mathrm{sign}(|\mathbf{a}_i^\top \mathbf{p}| - \tau) \neq \mathrm{sign}(|\mathbf{a}_i^\top \mathbf{q}| - \tau)) \asymp \mathrm{dist}(\mathbf{p},\mathbf{q}) = \|\mathbf{p}-\mathbf{q}\|_2$
  - $\mathbb{P}(i \in \mathbf{L}_{\mathbf{p},\mathbf{q}}) = \mathbb{P}(\mathrm{sign}(\mathbf{a}_i^\top \mathbf{p}) \neq \mathrm{sign}(\mathbf{a}_i^\top \mathbf{q})) \asymp \|\mathbf{p}-\mathbf{q}\|_2$
- $\mathsf{P}_{\mathbf{p},\mathbf{q}}^{(2)} \ll \mathsf{P}_{\mathbf{p},\mathbf{q}}$ as $\|\mathbf{p}-\mathbf{q}\|_2 \leq \delta_4 \asymp \frac{1}{\log^{1/2}(r^{-1})} = o(1)$

  $\longrightarrow |\mathbf{R}_{\mathbf{p},\mathbf{q}} \cap \mathbf{L}_{\mathbf{p},\mathbf{q}}| \lesssim \frac{mr}{\log^{1/2}(r^{-1})}, \ \forall (\mathbf{p},\mathbf{q}) \in \mathcal{N}_{r,\delta_4}^{(2)}$



$$\mathbf{a}_i \in \mathbf{L}_{\mathbf{p}_1,\mathbf{q}_1}, \ \mathbf{a}_i \notin \mathbf{R}_{\mathbf{p}_1,\mathbf{q}_1}$$
$$\mathbf{a}_i \notin \mathbf{L}_{\mathbf{p}_2,\mathbf{q}_2}, \ \mathbf{a}_i \in \mathbf{R}_{\mathbf{p}_2,\mathbf{q}_2}$$
$$\mathbf{a}_i \in \mathbf{L}_{\mathbf{p}_3,\mathbf{q}_3}, \ \mathbf{a}_i \in \mathbf{R}_{\mathbf{p}_3,\mathbf{q}_3}$$

Double separation is much more stringent for small $\|\mathbf{p}-\mathbf{q}\|_2$

Number of Summands in Terms 3, 4

- Recall that Term 3 is

$$\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}(\mathbf{u},\mathbf{v}) - \mathbf{h}(\mathbf{u}_1,\mathbf{v}_1))\|_2, \quad (\|\mathbf{u}-\mathbf{u}_1\|_2, \|\mathbf{v}-\mathbf{v}_1\|_2 \le r)$$

  Term 4 is

$$\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}(\mathbf{u},\mathbf{v}))\|_2, \quad (\|\mathbf{u}-\mathbf{v}\|_2 \le 3r)$$

- How can we bound number of separations over infinite set?
  $\longrightarrow$ Local binary embedding! [OR15], [DM21]

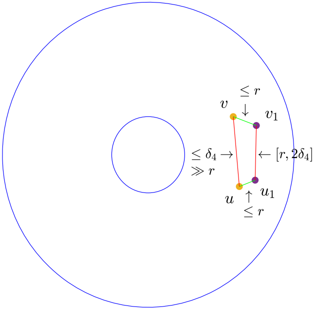
Theorem 3.4: Local Binary Embedding

For small enough $r > 0$ and $r' = \frac{c_1 r}{\log^{1/2}(r^{-1})}$ for some small c_1. If $m \gtrsim \frac{\omega^2(\mathcal{K}_{(3r'/2)})}{r^3} + \frac{\log \mathscr{N}(\mathcal{K}, r')}{r}$, then with prob. $\ge 1 - \exp(-\Omega(rm))$ we have:

- (1bPR embeding; This work) Any $\mathbf{u},\mathbf{v} \in \mathcal{K} \subset \mathbb{A}_\alpha^\beta$ obeying $\mathrm{dist}(\mathbf{u},\mathbf{v}) \le \frac{r'}{2}$ satisfy $|\mathbf{R}_{\mathbf{u},\mathbf{v}}| \lesssim mr$
- (1bCS embeding; [OR15]) Any $\mathbf{u},\mathbf{v} \in \mathcal{K} \subset \mathbb{S}^{n-1}$ obeying $\|\mathbf{u}-\mathbf{v}\|_2 \le r'$ satisfy $|\mathbf{L}_{\mathbf{u},\mathbf{v}}| \lesssim mr$

Number of Summands in Terms 3, 4

- It directly works out for Term 4:
  - No more than $|\mathbf{R}_{\mathbf{p},\mathbf{q}}|$ summands; $\|\mathbf{u}-\mathbf{v}\|_2 \leq 3r$
- Issue with Term 3 — $\|\mathcal{P}_{\mathcal{C}_-}(\mathbf{h}(\mathbf{u},\mathbf{v}) - \mathbf{h}(\mathbf{u}_1,\mathbf{v}_1))\|_2$:
  - No more than $|\mathbf{R}_{\mathbf{u},\mathbf{v}}| + |\mathbf{R}_{\mathbf{u}_1,\mathbf{v}_1}|$ summands
  - However, we do not have tight enough bound on $|\mathbf{R}_{\mathbf{u},\mathbf{v}}|$ and $|\mathbf{R}_{\mathbf{u}_1,\mathbf{v}_1}|$, as $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u}_1$ and $\mathbf{v}_1$, are not close enough.
  - More precisely, $\|\mathbf{u}-\mathbf{v}\|_2$ and $\|\mathbf{u}_1-\mathbf{v}_1\|_2$ are not on a scale of $\tilde{O}(r)$



Large-distance regime: $\|\mathbf{u}_1-\mathbf{v}_1\|_2 \geq r$
$\|\mathbf{u}-\mathbf{v}\|_2 \leq \delta_4$, most likely $\|\mathbf{u}-\mathbf{v}\|_2 \gg r$
so most likely, $\|\mathbf{u}_1-\mathbf{v}_1\|_2 \gg r$

Number of Summands in Terms 3, 4

- We need a rearrangement of $\mathbf{h}(\mathbf{u},\mathbf{v}) - \mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)$ to get tighter bound

$$
\begin{aligned}
&\mathbf{h}(\mathbf{u}_1,\mathbf{v}_1) - \mathbf{h}(\mathbf{u},\mathbf{v}) \\
&= \frac{1}{2m}\sum_{i=1}^{m}\left[\mathrm{sign}(|\mathbf{a}_i^\top \mathbf{v}| - \tau) - \mathrm{sign}(|\mathbf{a}_i^\top \mathbf{v}_1| - \tau)\right]\mathrm{sign}(\mathbf{a}_i^\top \mathbf{u}_1)\mathbf{a}_i \\
&+ \frac{1}{2m}\sum_{i=1}^{m}\left[\mathrm{sign}(|\mathbf{a}_i^\top \mathbf{u}_1| - \tau) - \mathrm{sign}(|\mathbf{a}_i^\top \mathbf{u}| - \tau)\right]\mathrm{sign}(\mathbf{a}_i^\top \mathbf{u}_1)\mathbf{a}_i \\
&+ \frac{1}{2m}\sum_{i=1}^{m}\left[\mathrm{sign}(\mathbf{a}_i^\top \mathbf{u}) - \mathrm{sign}(\mathbf{a}_i^\top \mathbf{u}_1)\right]\left[\mathrm{sign}(|\mathbf{a}_i^\top \mathbf{v}| - \tau) - \mathrm{sign}(|\mathbf{a}_i^\top \mathbf{u}| - \tau)\right]\mathbf{a}_i,
\end{aligned}
$$

- No more than $|\mathbf{R}_{\mathbf{v},\mathbf{v}_1}| + |\mathbf{R}_{\mathbf{u},\mathbf{u}_1}| + |\mathbf{L}_{\mathbf{u},\mathbf{u}_1}|$ summands
- $\|\mathbf{u}-\mathbf{u}_1\|_2, \|\mathbf{v}-\mathbf{v}_1\|_2 \le r \longrightarrow$ no more than $\tilde{O}(mr)$ summands

Outline

Synthetic Data

We test the case of $\|\mathbf{x}\|_2 = 1$:

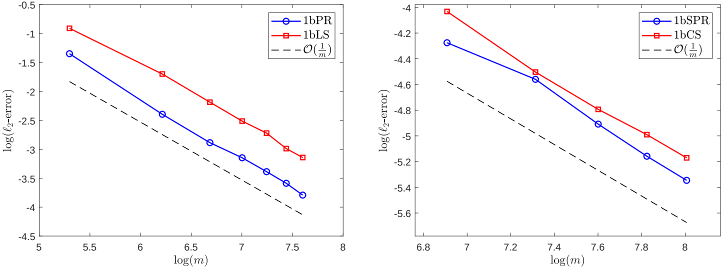


Figure: *Phases are non-essential in solving 1-bit linear system (Left; $\mathbf{x} \in \mathbb{S}^{29}$) and in 1-bit compressed sensing (Right; $\mathbf{x} \in \Sigma_3^{500,*}$).*

Synthetic Data

We test the case of $\mathbf{x} \in \mathbb{A}_{\alpha}^{\beta}$ $(\beta \geq \alpha)$:

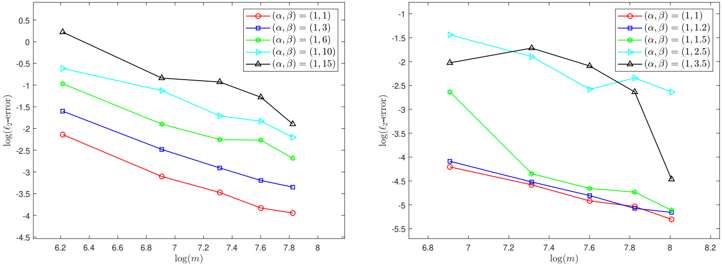


Figure: *Full Signal Reconstruction over $\mathbb{A}_{\alpha,\beta}$ in 1-bit phase retrieval (Left; $\mathbf{x} \in \mathbb{R}^{30}$) and 1-bit sparse phase retrieval (Right; $\mathbf{x} \in \Sigma_3^{500}$).*

Real Images

(a) Original image: Milky Way Galaxy.

(b) Recovered image after SI-1bPR ($L = 64$): relative error = 0.270, PSNR = 25.14.

(c) Recovered image after GD-1bPR ($L = 64$): relative error = 0.029, PSNR = 44.65.

Figure: *Recovering the* $1080 \times 1980 \times 3$ *Milky Way Galaxy image from phaseless bits produced by CDP with* $L = 64$ *random patterns.*

Outline

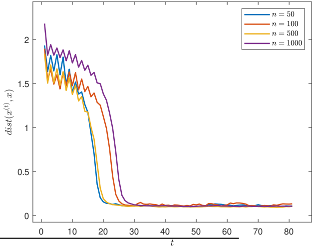
Random Initialization

Question:

For 1bPR, can gradient descent start from *random initialization*?

Literature (phase retrieval):

- Optimization landscape (no polynomial time algorithm): [SQW18][15]
- No sample splitting: [CCFM19][16]
- Sample splitting with sharp rate [CPD23][17]
- Stochastic GD: [TV23][18]



Simulations: $m = 10n$

Start with a snower convergence [CCFM19]

[15] A geometric analysis of phase retrieval

[16] Gradient descent with random initialization: Fast global convergence for nonconvex phase retrieval

[17] Sharp global convergence guarantees for iterative nonconvex optimization with random data

[18] Online stochastic gradient descent with arbitrary initialization solves non-smooth, non-convex phase retrieval

Other Questions

- Can we extend to complex case $\mathbf{y} = \mathrm{sign}(|\mathbf{\Phi x}| - \tau)$ where $\mathbf{\Phi} \in \mathbb{C}^{m\times n}$ and $\mathbf{x} \in \mathbb{C}^n$?
  - Randomized Kaczmarz;
  - Random initialization;
- Can we go beyond Gaussian design?
  - Sub-Gaussian matrix [KL17];[19]
  - Structured sensing matrix;
- Can we extend the results to multi-bit?
  - This relies on *dithering* in compressed sensing [XJ20].[20]
- Can we precisely compare the errors in 1-bit sensing and 1-bit phase retrieval?
  - Precise bounds are lacking in nonlinear structured problems;
  - See [CPD23] for unstrctured case with sample splitting.
- Can we develop some practical applications?

Thank You

[19] Phase retrieval without small-ball probability assumptions

[20] Quantized compressive sensing with rip matrices: The benefit of dithering

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