

Exact Thresholds in Noisy Non-Adaptive Group Testing

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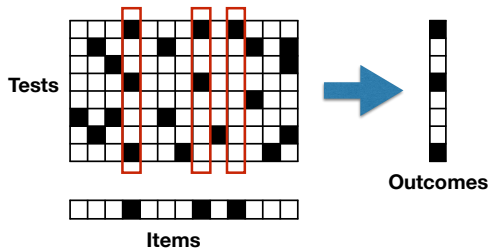
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Joint work with Jonathan Scarlett (NUS)

I. Problem Setup

(Noisy) Group Testing



In this talk, we consider probabilistic group testing:

- ▶ Defective set $S \sim \text{Uniform}(\binom{p}{k})$ (i.e., k out of p items with a uniform prior)
- ▶ Non-adaptive: the test design $\mathbf{X} = (X_{ij}) \in \{0, 1\}^{n \times p}$ is fixed before observing any outcome
- ▶ Noiseless:

$$Y_i = \bigvee_{j \in S} X_{ij} \quad (1)$$

- ▶ Noisy:

$$Y_i = \left(\bigvee_{j \in S} X_{ij} \right) \oplus Z_i \quad (2)$$

with $Z \sim \text{Bernoulli}(\rho)$ for some noise level $\rho \in (0, \frac{1}{2})$.

Recovery Criteria

- ▶ We consider two popular random designs:
 - ▶ **Bernoulli design:** $X_{ij} \stackrel{iid}{\sim} \text{Bernoulli}(\frac{\nu}{k})$; each item is independently placed in each test with probability $\frac{\nu}{k}$ for some $\nu > 0$
 - ▶ **Near-constant weight design:** each item is independently placed in $\Delta = \frac{\nu n}{k}$ tests chosen uniformly at random with replacement for some $\nu > 0$

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- ▶ Given a decoder \hat{S} , we define error probability as

$$P_e := \mathbb{P}[\hat{S} \neq S]$$

taken w.r.t. randomness of $(S, \mathbf{X}, \mathbf{Z})$

- ▶ **Goal:** Conditions on n for $P_e \rightarrow 0$ in the large-system limit
- ▶ Sublinear sparsity: $k = \Theta(p^\theta)$ for $\theta \in (0, 1)$

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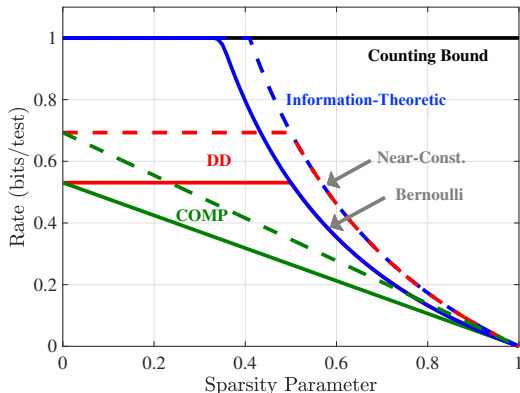
- ▶ **Goal:** Conditions on n for $P_e \rightarrow 0$ in the large-system limit
- ▶ Sublinear sparsity: $k = \Theta(p^\theta)$ for $\theta \in (0, 1)$
- ▶ Our work establishes the exact thresholds $n^* = Ck \log \frac{p}{k}$ with precise constants C for both designs, such that:
 - ▶ (Exact achievability)

When $n > (1 + o(1))n^$, some decoder gives $P_e \rightarrow 0$;*

- ▶ (Exact converse)

When $n < (1 - o(1))n^$, any decoder suffers from $P_e \rightarrow 1$*

Milestones in Noiseless GT ($\text{rate} = \lim_{p \rightarrow \infty} \frac{\log_2 \binom{p}{k}}{n}$)



- ▶ Exact thresholds for bernoulli design ¹ (ensemble tightness) ²
- ▶ Exact thresholds for NCC design and ensemble tightness ³
- ▶ The blue dashed curve is near optimal for arbitrary design ⁴

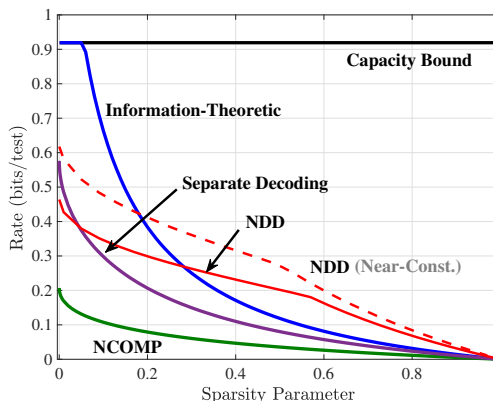
¹ Phase transitions in group testing, J. Scarlett and V. Cevher, 16 SODA

² The capacity of Bernoulli nonadaptive group testing, M. Aldridge, 17 T-IT

³ Information-theoretic and algorithmic thresholds for group testing, A. Coja-Oghlan et al., 20 T-IT

⁴ Optimal group testing, Coja-Oghlan et al., 20 COLT

Noisy GT Bounds Before Our Work ($\rho = 0.01$)

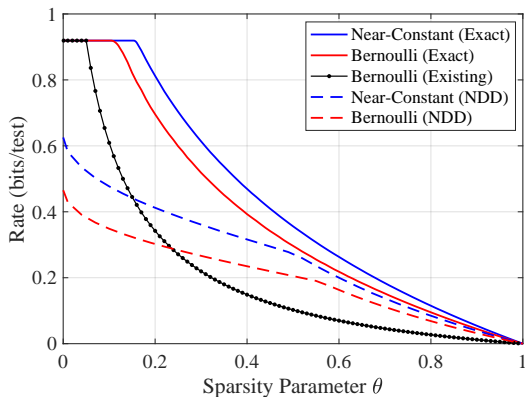


- ▶ Information-theoretic upper bounds that are tight for very small values of θ [SC16]
- ▶ The information-theoretic upper bound is an attempt to exact thresholds under Bernoulli design ⁵
- ▶ Compared to the noiseless case, the prior work is much less complete!

⁵Noisy non-adaptive group testing: A (near-)definite defectives approach, J. Scarlett and O. Johnson, 20 T-IT

II. Exact Thresholds

Graphical Illustration of Our Exact Thresholds ($\rho = 0.01$)



- The best existing efficient algorithms fall short of information theoretic thresholds

Preliminaries

Notation:

$$a \star b = ab + (1 - a)(1 - b) \quad (3)$$

$$D(a\|b) = a \log \left(\frac{a}{b} \right) + (1 - a) \log \left(\frac{1 - a}{1 - b} \right) \quad (4)$$

$$H_2(a) = a \log \left(\frac{1}{a} \right) + (1 - a) \log \left(\frac{1}{1 - a} \right) \quad (5)$$

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Technical Lemma:

Consider $X \sim \text{Bin}(N, q)$, then we have

► Chernoff bound:

$$\mathbb{P}(X \leq k) \leq \exp \left(-N \cdot D\left(\frac{k}{N} \| q\right) \right), \quad \text{if } k \leq Nq \quad (6)$$

$$\mathbb{P}(X \geq k) \leq \exp \left(-N \cdot D\left(\frac{k}{N} \| q\right) \right), \quad \text{if } k \geq Nq \quad (7)$$

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► Anti-concentration:

$$\mathbb{P}(X = k) \geq \underbrace{\frac{1}{2\sqrt{2k(1 - \frac{k}{N})}} \exp \left(-N \cdot D\left(\frac{k}{N} \| q\right) \right)}_{\text{often} = \exp \left(-N \cdot \left[D\left(\frac{k}{N} \| q\right) + o(1) \right] \right)}, \quad k = 1, 2, \dots, N - 1 \quad (8)$$

Thresholds for Bernoulli Designs

Thresholds for Bernoulli design with i.i.d. Bernoulli($\frac{\nu}{k}$) entries:

$$n_{\text{Bern}}^* = \max \left\{ \frac{k \log \frac{p}{k}}{H_2(e^{-\nu} \star \rho) - H_2(\rho)}, \quad (\text{first branch}) \right. \\ \left. \frac{k \log \frac{p}{k}}{(1 - \theta)\nu e^{-\nu} \min_{\zeta \in (0,1)} \max\{\frac{1}{\theta} f_1^{\text{Bern}}(C, \zeta, \rho), f_2^{\text{Bern}}(C, \zeta, \rho)\}} \quad (\text{second branch}) \right\},$$
$$f_1^{\text{Bern}}(C, \zeta, \rho) = C \log C - C + C \cdot D(\zeta \parallel \rho) + 1,$$
$$f_2^{\text{Bern}}(C, \zeta, \rho) = \min_{d \geq \max\{0, C(1-2\zeta)/\rho\}} g^{\text{Bern}}(C, \zeta, d, \rho),$$
$$g^{\text{Bern}}(C, \zeta, d, \rho) = \rho d \log d + (\rho d - C(1 - 2\zeta)) \log \left(\frac{\rho d - C(1 - 2\zeta)}{1 - \rho} \right) + 1 - 2\rho d + C(1 - 2\zeta)$$

recall some notation:

- ▶ p items, k defectives, $k \sim p^\theta$
- ▶ ν : design parameter
- ▶ ρ : noise level
- ▶ C, ζ : optimization parameters

Thresholds for Near-Constant Weight Designs

Thresholds for near-constant weight design with $\Delta = \frac{\nu n}{k}$ placements per item:

$$n_{\text{NC}}^* = \max \left\{ \frac{k \log \frac{p}{k}}{H_2(e^{-\nu} \star \rho) - H_2(\rho)}, \quad (\text{first branch}) \right. \\ \left. \frac{k \log \frac{p}{k}}{(1 - \theta)\nu e^{-\nu} \min_{C \in (0, e^\nu), \zeta \in (0, 1)} \max \{ \frac{1}{\theta} f_1^{\text{NC}}(C, \zeta, \rho, \nu), f_2^{\text{NC}}(C, \zeta, \rho, \nu) \}} \quad (\text{second branch}) \right\},$$

$$f_1^{\text{NC}}(C, \zeta, \rho, \nu) = e^\nu D(Ce^{-\nu} \| e^{-\nu}) + C \cdot D(\zeta \| \rho),$$

$$f_2^{\text{NC}}(C, \zeta, \rho, \nu) = \min_{d: |C(1-2\zeta)| \leq d \leq e^\nu} g^{\text{NC}}(C, \zeta, d, \rho, \nu),$$

$$g^{\text{NC}}(C, \zeta, d, \rho, \nu) = e^\nu \cdot D(de^{-\nu} \| e^{-\nu}) + d \cdot D\left(\frac{1}{2} + \frac{C(1-2\zeta)}{2d} \| \rho\right).$$

High-level Intuitions

Two branches appear in the final thresholds:

1. The common first branch $\frac{k \log(p/k)}{H_2(e^{-\nu} \star \rho) - H_2(\rho)}$ is related to the Shannon capacity of the binary symmetric channel.
 - Established by analyzing $\ell = |\hat{S} \setminus S| = k$ (high ℓ , low overlap)

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 - ▶ Established by analyzing $\ell = |\hat{S} \setminus S| = k$ (high ℓ , low overlap)
2. The more complicated second branches involve f_1 and f_2 :
 - ▶ Established by analyzing $\ell = |\hat{S} \setminus S| = 1$ (low ℓ , high overlap)
 - ▶ The optimization constants (C, ζ, d) that are introduced to characterize certain quantities in the error events.

III. Proofs for Converse

The First Branch $\frac{k \log(p/k)}{H_2(e^{-\nu} \star \rho) - H_2(\rho)}$

Intuition:

- ▶ The test has probability about $e^{-\nu}$ of containing no defectives;
- ▶ (Roughly) $e^{-\nu} \star \rho$ of being positive;
- ▶ Thus, each test can only reveal $H_2(e^{-\nu} \star \rho) - H_2(\rho)$ bits of information;
- ▶ With $\binom{p}{k}$ possible defective sets, we need (roughly) $\log \binom{p}{k} \sim k \log \frac{p}{k}$ bits; comparing them gives the capacity branch.

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Sketch of technical argument:

- ▶ For any $\delta_1 > 0$, we have [SC16]

$$P_e \geq \mathbb{P} \left(i^n(\mathbf{X}_s, \mathbf{Y}) \leq \log \left(\delta_1 \binom{p}{k} \right) \right) - \delta_1 \quad (9)$$

$$\approx \mathbb{P} \left(i^n(\mathbf{X}_s, \mathbf{Y}) \leq k \log \binom{p}{k} \right) \quad (10)$$

where $i^n(\mathbf{X}_s, \mathbf{Y}) = \log \frac{\mathbb{P}(\mathbf{Y}|\mathbf{X}_s)}{\mathbb{P}(\mathbf{Y})} = \log \mathbb{P}(\mathbf{Y}|\mathbf{X}_s) - \log \mathbb{P}(\mathbf{Y})$.

- ▶ Establish *upper concentration bound* for $i^n(\mathbf{X}_s, \mathbf{Y})$ by separately analyzing $\log \mathbb{P}(\mathbf{Y}|\mathbf{X}_s)$ and $\log \mathbb{P}(\mathbf{Y})$.

The Second Branch – Failure of MLE

Challenge:

- ▶ This kind of terms appeared in thresholds for noiseless case, based on such a central idea:
if a defective item is *masked* (i.e., every test it is in also contains at least one other defective), then even an optimal decoder will be unable to identify it.
- ▶ However, this is no longer the dominant error event in the noisy case, thus cannot be used to derive the exact/tight converse bounds

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Ideas:

- ▶ MLE is the optimal decoding strategy, and we only need to show **MLE fails when n is below n^*** ;
- ▶ Given (\mathbf{X}, \mathbf{Y}) , the likelihood of an estimate s is

$$\mathcal{L}_{\mathbf{X}, \mathbf{Y}}(s) = \rho^{N_{\mathbf{X}, \mathbf{Y}}(s)} (1 - \rho)^{n - N_{\mathbf{X}, \mathbf{Y}}(s)} \quad (11)$$

where $N_{\mathbf{X}, \mathbf{Y}}(s)$ denotes the number of “correct tests”

- ▶ **Error event:** for some defective j and nondefective j' it holds that have

$$N_{\mathbf{X}, \mathbf{Y}}(\underbrace{(S \setminus \{j\}) \cup \{j'\}}_{:=\hat{S}}) > N_{\mathbf{X}, \mathbf{Y}}(S) \iff N_{\mathbf{X}, \mathbf{Y}}(\hat{S}) - N_{\mathbf{X}, \mathbf{Y}}(S) > 0 \quad (12)$$

The Second Branch – Failure of MLE

$$\widehat{S} = (S \setminus \{j\}) \cup \{j'\}$$

Counting:

- Only two types of tests contribute to $N_{\mathbf{X}, \mathbf{Y}}(\widehat{S})$ and $N_{\mathbf{X}, \mathbf{Y}}(S)$ differently:

$$\text{contain } j \text{ as the only defective but not contain } j' \begin{cases} \text{positive } (I_1) : N_{\mathbf{X}, \mathbf{Y}}(S) \leftarrow N_{\mathbf{X}, \mathbf{Y}}(S) + 1 \\ \text{negative } (I_2) : N_{\mathbf{X}, \mathbf{Y}}(\widehat{S}) \leftarrow N_{\mathbf{X}, \mathbf{Y}}(\widehat{S}) + 1 \end{cases} \quad (13)$$

$$\text{contain no defective but contain } j' \begin{cases} \text{positive } (I_3) : N_{\mathbf{X}, \mathbf{Y}}(\widehat{S}) \leftarrow N_{\mathbf{X}, \mathbf{Y}}(\widehat{S}) + 1 \\ \text{negative } (I_4) : N_{\mathbf{X}, \mathbf{Y}}(S) \leftarrow N_{\mathbf{X}, \mathbf{Y}}(S) + 1 \end{cases} \quad (14)$$

- Failure condition:

$$N_{\mathbf{X}, \mathbf{Y}}(\widehat{S}) - N_{\mathbf{X}, \mathbf{Y}}(S) > 0 \implies I_2 + I_3 > I_1 + I_4 \quad (15)$$

The Second Branch – Failure of MLE

Analytical formulation:

► Notation:

- \mathcal{M}_j : tests in which j is the only defective
- \mathcal{N}_0 : tests containing no defective
- \mathcal{M}_{j1} (\mathcal{M}_{j0}): the positive (negative) tests in \mathcal{M}_j
- \mathcal{N}_{01} (\mathcal{N}_{00}): the positive (negative) tests in \mathcal{N}_0
- $G_{j,j',1}$: number of tests in $\mathcal{N}_{01} \cup \mathcal{M}_{j1}$ that contain j'
- $G_{j,j',2}$: number of tests in $\mathcal{N}_{00} \cup \mathcal{M}_{j0}$ that contain j'

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► For some $(C, \zeta) \in (0, \infty) \times (0, 1)$ such that $\frac{Cn\nu e^{-\nu}}{k}, \frac{\zeta \cdot Cn\nu e^{-\nu}}{k} \in \mathbb{Z}$ we have

► (C1) There exists some $j \in S$ such that

$$M_j = |\mathcal{M}_j| = \frac{Cn\nu e^{-\nu}}{k} \quad (16)$$

$$M_{j0} = |\mathcal{M}_{j0}| = \zeta \cdot M_j \quad (17)$$

► (C2) Failure condition: For some $j' \in [p] \setminus S$,

$$I_2 + I_3 > I_1 + I_4 \implies G_{j,j',1} - G_{j,j',2} > (1 - 2\zeta) \frac{Cn\nu e^{-\nu}}{k} \quad (18)$$

The Second Branch – Failure of MLE

The second branch takes the form $\frac{k \log(p/k)}{(1-\theta)\nu e^{-\nu} \min_{C,\zeta} \max\{\frac{f_1}{\theta}, f_2\}}$

Step I. Ensuring (C1) leads to f_1 :

- ▶ **(C1)** for some $j \in S$, $M_j = \frac{Cn\nu e^{-\nu}}{k}$ and $M_{j0} = \zeta \cdot \frac{Cn\nu e^{-\nu}}{k}$
- ▶ Challenge lies on M_j
- ▶ (Bernoulli) Poisson approximation for the multinomial distribution

$$(M_1, M_2, \dots, M_{k\xi}) \quad (19)$$

▶ (Near-Constant)

- ▶ Work with a surrogate of M_j —
 M'_j : the number of tests in which j is the only defective and *is placed exactly once*
- ▶ Interpret the placements of items into tests as edges in a bipartite graph [CGHL20], and use symmetry to show $(M'_1, M'_2, \dots, M'_{k\xi})$ obeys

$$\begin{aligned} M'_1 &\sim \text{Hg}(k\Delta, e^{-\nu}k\Delta, \Delta) \\ M'_2 | M'_1 &\sim \text{Hg}(k\Delta, e^{-\nu}k\Delta, \Delta) \\ &\dots \\ M'_{k\xi} | (M'_1, M'_2, \dots, M'_{k\xi-1}) &\sim \text{Hg}(k\Delta, e^{-\nu}k\Delta, \Delta) \end{aligned} \quad (20)$$

The Second Branch – Failure of MLE

The second branch takes the form $\frac{k \log(p/k)}{(1-\theta)\nu e^{-\nu} \min_{C,\zeta} \max\{\frac{f_1}{\theta}, f_2\}}$

Step II. Ensuring (C2) leads to f_2 :

- ▶ **(C2):** $G_{j,j',1} - G_{j,j',2} > (1 - 2\zeta) \frac{Cn\nu e^{-\nu}}{k}$
- ▶ This comes down to the analysis of the difference of *two independent binomial random variables*, and can be handled by anti-concentration
- ▶ Many technical challenges/details omitted ...

Step III. Optimizing (C, ζ)

- ▶ **(C1) and (C2)**
- ▶ Optimizing (C, ζ) to establish the strongest converse bound

IV. Proofs for Achievability

Information density Decoder in [SC16]

Existing Information density decoder (Scarlett & Cevher, 16 SODA):

- ▶ We assume $S = s$ is the defective set
- ▶ We consider partitioning s into $(s_{\text{dif}}, s_{\text{eq}})$ with $s_{\text{dif}} \neq \emptyset$, and define for each s_{dif} the information density as

$$i^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}}) := \log \frac{\mathbb{P}(\mathbf{Y} | \mathbf{X}_{s_{\text{dif}}}, \mathbf{X}_{s_{\text{eq}}})}{\mathbb{P}(\mathbf{Y} | \mathbf{X}_{s_{\text{eq}}})}. \quad (21)$$

- ▶ Its expectation depends only on $\ell := |s_{\text{dif}}|$ and is defined as

$$\mathbb{E}[i^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}})] := I(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}}) := I_\ell^n \quad (22)$$

- ▶ **Information density decoder:**

- ▶ Fix the constants $\{\gamma_\ell\}_{\ell=1}^k$, and search for a set s of cardinality k such that

$$i^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}}) \geq \gamma_{|s_{\text{dif}}|}, \quad \forall (s_{\text{dif}}, s_{\text{eq}}) \text{ such that } |s_{\text{dif}}| \neq 0. \quad (23)$$

- ▶ **Intuition:** $i^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}})$ tends to be high for the actual defective set s ;
- ▶ **Limitation:** Analyzing this decoder **under small ℓ** leads to sub-optimal threshold for noisy GT.

Our hybrid decoding rule

- ▶ We resort to MLE for low- ℓ case
- ▶ **Hybrid Decoder:** We search for a set \hat{s} of cardinality k that satisfies
 - ▶ (Low ℓ : MLE) It holds that

$$\mathcal{L}_{\mathbf{X}, \mathbf{Y}}(\hat{s}) > \mathcal{L}_{\mathbf{X}, \mathbf{Y}}(s'), \quad \forall s' \text{ such that } 1 \leq |\hat{s} \setminus s'| \leq \frac{k}{\log k}, \quad (24)$$

where we implicitly also constrain s' to have cardinality k .

- ▶ (High- ℓ : information density) For suitably chosen $\{\gamma_\ell\}_{\frac{k}{\log k} < \ell \leq k}$, it holds that

$$i^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}}) \geq \gamma_{|s_{\text{dif}}|}, \quad \forall (s_{\text{dif}}, s_{\text{eq}}) \text{ such that } |s_{\text{dif}}| > \frac{k}{\log k}, \quad (25)$$

where $(s_{\text{dif}}, s_{\text{eq}})$ is a disjoint partition of \hat{s} .

- ▶ **Success conditions:**
 - ▶ Success condition for Low- ℓ :

$$(24) \text{ holds for } \hat{s} = s \quad (26)$$

- ▶ Success condition for high- ℓ :

$$(25) \text{ holds for } \hat{s} = s \quad (27)$$

$$\forall \tilde{s} \text{ with } |\tilde{s}| = k, \quad |s \setminus \tilde{s}| > \frac{k}{\log k}, \text{ it holds that } i^n(\mathbf{X}_{\tilde{s} \setminus s}; \mathbf{Y} | \mathbf{X}_{\tilde{s} \cap s}) < \gamma_{|s \setminus \tilde{s}|} \quad (28)$$

The First Branch $\frac{k \log(p/k)}{H_2(e^{-\nu} \star \rho) - H_2(\rho)}$

- ▶ **Starting point:** [SC16] for any $\delta_1 > 0$, $\mathbb{P}((27)\&(28) \text{ fail}) \leq$

$$\mathbb{P}\left[\bigcup_{(s_{\text{dif}}, s_{\text{eq}}) : |s_{\text{dif}}| \geq \ell_{\min}} \left\{ i^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}}) \leq \log \binom{p-k}{|s_{\text{dif}}|} + \log \left(\frac{k}{\delta_1} \binom{k}{|s_{\text{dif}}|} \right) \right\} \right] + \delta_1$$

$$\stackrel{\delta_1 \rightarrow 0}{\approx} \mathbb{P}\left[\bigcup_{(s_{\text{dif}}, s_{\text{eq}}) : |s_{\text{dif}}| \geq \ell_{\min}} \left\{ i^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}}) \leq (1 + o(1)) \ell \log \left(\frac{p}{k} \right) \right\} \right] \quad (29)$$

- **Concentration bound:** for any $\delta_2 \in (0, 1)$,

$$\mathbb{P}[i^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}}) \leq (1 - \delta_2) I_\ell^n] \leq \psi_\ell(n, \delta_2), \quad (30)$$

- Therefore, if it holds that

$$\max_{\ell > \frac{k}{\log k}} \frac{(1 + o(1))\ell \log(\frac{p}{k})}{I_\ell^n(1 - \delta_2)} \leq 1, \quad (31)$$

we are able to enforce

$$\mathbb{P}((27)\&(28) \text{ fail}) \leq \sum_{\ell=\ell_{\min}}^k \binom{k}{\ell} \psi_{\ell}(n, \delta_2) \rightarrow 0 \quad (32)$$

- Combining with the asymptotic of scaling of I_ρ^n , (31) yields $n \geq$ the first branch

The Second Branch – Success of MLE

- ▶ The second branch, $\frac{k \log(p/k)}{(1-\theta)^\nu e^{-\nu} \min_{C,\zeta} \max\{\frac{f_1}{\theta}, f_2\}}$, is used to ensure

$$\mathcal{L}_{\mathbf{X},\mathbf{Y}}(s) > \mathcal{L}_{\mathbf{X},\mathbf{Y}}(s'), \quad \forall s' \text{ such that } 1 \leq |s \setminus s'| \leq \frac{k}{\log k} \quad (33)$$

so that the MLE part succeeds.

- ▶ Similar arguments with nontrivial generalizations

Preparations:

- ▶ $\mathcal{L}_{\mathbf{X},\mathbf{Y}}(s) > \mathcal{L}_{\mathbf{X},\mathbf{Y}}(s') \iff N_{\mathbf{X},\mathbf{Y}}(s) > N_{\mathbf{X},\mathbf{Y}}(s')$
- ▶ $\mathcal{J} = s \setminus s'$ (defective) and $\mathcal{J}' = s' \setminus s$ (non-defective)
- ▶ Only two types of tests matter

$$\text{only contain defectives in } \mathcal{J} \text{ but not contain items in } \mathcal{J}' \begin{cases} \text{positive } (I_1) : N_{\mathbf{X},\mathbf{Y}}(s) ++ \\ \text{negative } (I_2) : N_{\mathbf{X},\mathbf{Y}}(s') ++ \end{cases} \quad (34)$$

$$\text{contain no defective but contain items from } \mathcal{J} \begin{cases} \text{positive } (I_3) : N_{\mathbf{X},\mathbf{Y}}(s') ++ \\ \text{negative } (I_4) : N_{\mathbf{X},\mathbf{Y}}(s) ++ \end{cases} \quad (35)$$

- ▶ Success condition: $N_{\mathbf{X},\mathbf{Y}}(s) > N_{\mathbf{X},\mathbf{Y}}(s') \implies I_1 + I_4 > I_2 + I_3$

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Analytical formulation:

- ▶ Notation:
 - ▶ $\mathcal{M}_{\mathcal{J}}$: tests in which items in \mathcal{J} are the only defectives
 - ▶ $\mathcal{M}_{\mathcal{J}^1}(\mathcal{M}_{\mathcal{J}^0})$: the positive (negative) tests in $\mathcal{M}_{\mathcal{J}}$
 - ▶ $\mathcal{N}_0, \mathcal{N}_{00}, \mathcal{N}_{01}$: as before
 - ▶ $G_{\mathcal{J}, \mathcal{J}', 1}$: number of tests in $\mathcal{N}_{01} \cup \mathcal{M}_{\mathcal{J}^1}$ that contain some item from \mathcal{J}'
 - ▶ $G_{\mathcal{J}, \mathcal{J}', 2}$: number of tests in $\mathcal{N}_{00} \cup \mathcal{M}_{\mathcal{J}^0}$ that contain some item from \mathcal{J}'

- For any $\ell \leq \frac{k}{\log k}$ and any pairs of $(C, \zeta) \in [0, \infty) \times [0, 1]$ such that

$\frac{Cnve^{-\nu\ell}}{k}, \frac{\zeta \cdot Cnve^{-\nu\ell}}{k} \in \mathbb{Z}$, one of the following conditions hold:

- **(C1)** $\mathcal{K}_{\ell,C,\zeta} = \emptyset$ where

$$\mathcal{K}_{\ell, C, \zeta} = \left\{ \mathcal{J} \subset s : |\mathcal{J}| = \ell, M_{\mathcal{J}} = \frac{C n \nu e^{-\nu} \ell}{k}, M_{\mathcal{J}^c} = \frac{\zeta \cdot C n \nu e^{-\nu} \ell}{k} \right\} \quad (36)$$

- **(C2)** If $\mathcal{K}_{\ell, C, \zeta} \neq \emptyset$, then for any $\mathcal{J} \in \mathcal{K}_{\ell, C, \zeta}$ and $\mathcal{J}' \subset [p] \setminus s$ and $|\mathcal{J}'| = \ell$,

$$I_1 + I_4 > I_2 + I_3 \iff G_{\mathcal{J}, \mathcal{J}', 1} - G_{\mathcal{J}, \mathcal{J}', 2} < (1 - 2\zeta) \frac{C n \nu e^{-\nu \ell}}{k} \quad (37)$$

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The second branch takes the form $\frac{k \log(p/k)}{(1-\theta)\nu e^{-\nu} \min_{C,\zeta} \max\{\frac{f_1}{\theta}, f_2\}}$

Overview:

- ▶ **Step 1.** Ensuring **(C1)** yields the f_1 part of the second branch
 - ▶ Unlike in the converse, we utilize a first-order method which first bounds

$$\mathbb{E}|\mathcal{K}_{\ell,C,\zeta}| = \binom{k}{\ell} \mathbb{P} \left(\text{for fixed } \mathcal{J} \subset s \text{ with } |\mathcal{J}| = \ell, M_{\mathcal{J}} = \frac{Cn\nu e^{-\nu} \ell}{k}, M_{\mathcal{J}^c} = \zeta M_{\mathcal{J}} \right) \quad (38)$$

and then utilize Markov's inequality to enforce $|\mathcal{K}_{\ell,C,\zeta}| = 0$

- ▶ **Step 2.** Ensuring **(C2)** yields the f_2 part of the second branch. More complicated than the converse as we need to handle all (ℓ, C, ζ)
 - ▶ Working with the sum/difference of two independent binomial variables
- ▶ **Step 3.** Optimizing over (C, ζ)

Summary & Future Directions

Summary:

- ▶ We established exact thresholds for noisy group testing with Bernoulli design and near-constant weight design
- ▶ For converse analysis, the main innovation is to identify a novel set of dominant error events
- ▶ For achievability analysis, we introduce a hybrid decoder that combines the existing information density approach and MLE

Future Directions:

1. **Efficient and Optimal Algorithm:** Devise an *efficient* algorithm to achieve the exact thresholds for near-constant weight design.
 - ▶ A concurrent work solved this problem via spatial coupling designs.⁶
2. **Converse for Arbitrary Design:** Investigate whether the n_{NC}^* is the general converse for arbitrary design.
 - ▶ This is true in the noiseless case [\[CGHL20\]](#)

Thank You

⁶Noisy group testing via spatial coupling, Coja-Oghlan et al., Comb. Probab. Comput., 2024