On Subsample Size of Quantile-Based Randomized Kaczmarz

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Joint Mathematics Meetings

January 4, 2026

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Outline

1. Motivation and Setup
2. Main Results
3. Proof Overview
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Motivation and Setup

Motivation and Model

Corrupted linear systems.

- Large linear systems arise in scientific computing, ML, imaging.
- In practice, some measurements are arbitrarily corrupted (sensor failures, transmission errors, adversarial attacks ...).

Sparse corruption model.

$$\underbrace{\mathbf{A}}_{\substack{\mathbb{R}^{m\times n}\\ \text{(known)}}} \underbrace{\mathbf{x}^*}_{\substack{\text{to be}\\ \text{exactly}\\ \text{recovered}}} + \underbrace{\epsilon}_{\substack{\beta m-\text{sparse}\\ \text{corruption}\\ 0<\beta<c}} = \underbrace{\mathbf{b}}_{\substack{\text{corrupted}\\ \text{observations}}} .$$

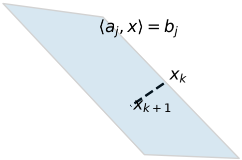
- In highly over-determined case $m \gg n$: we hope to access one or only a few measurements in each iteration

From RK to QuantileRK

Randomized Kaczmarz (RK) for solving $\mathbf{Ax} = \mathbf{b}$

- Uses one row per iteration

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \frac{\langle \mathbf{a}_j, \mathbf{x}_k \rangle - b_j}{\|\mathbf{a}_j\|^2} \mathbf{a}_j \quad \text{for some randomly chosen } j$$



- (Strohmer and Vershynin, 2009): linear convergence for consistent systems.
- Noisy linear system $\mathbf{Ax} + \underbrace{\epsilon}_{\text{noise vector}} = \mathbf{b}$: converges only to an error depending on noise level (Needell, 2010).
- Corrupted linear system: one bad row can destroy convergence

QuantileRK Algorithm

QuantileRK (QRK) for solving $\mathbf{Ax} + \epsilon = \mathbf{b}$ (Haddock et al., 2022)

- Idea: use residual quantiles to avoid corrupted rows.
- The q-quantile of a *multi-set* $\{z_1, z_2, \dots, z_N\}$ is defined as $z^*_{\lfloor qN \rfloor}$, where we let
$$z_1^* \le z_2^* \le \cdots \le z_N^*$$
be the non-decreasing rearrangement of the N elements.
- At iteration k, compute the quantile of $\{\, |\mathbf{a}_i^\top \mathbf{x}_k - b_i| \,:\, i \in [m]\}$.
- Only update if the chosen row has residual below this quantile.

Quantile RK Algorithm

We assume the rows of **A** are iid uniformly distributed over S^{n-1}

QuantileRK(q) (Haddock et al., 2022)

Input: A, b, x_0, $q \in (0,1)$, quantile subsample size D, iteration number T
for $k = 0$ to $T-1$ **do**
  Sample $i_1^{(k+1)}, \dots, i_D^{(k+1)} \sim \text{Uniform}(1,\dots,m) \longrightarrow$ quantile subsample
  Compute $Q := q\text{-quantile}\left(\{|\mathbf{a}_{i_j^{(k+1)}}^\top \mathbf{x}_k - b_{i_j^{(k+1)}}|\}_{j=1}^D\right)$
  Sample $r_{k+1} \sim \text{Uniform}(1,\dots,m) \longrightarrow$ update sample
  Compute $h_{r_{k+1}} := \mathbf{a}_{r_{k+1}}^\top \mathbf{x}_k - b_{r_{k+1}}$
  if $|h_{r_{k+1}}| \le Q$ **then**
    $\mathbf{x}_{k+1} \leftarrow \mathbf{x}_k - h_{r_{k+1}} \mathbf{a}_{k+1} \longrightarrow$ accept & update
  else
    $\mathbf{x}_{k+1} \leftarrow \mathbf{x}_k \longrightarrow$ reject
  end if
end for
Output: $\mathbf{x}_T$

Main Results

Main Question

Gap in the literature:

- **Theory:** quantile computed from all m rows every iteration.
- **Practice:** quantile computed from a small subsample $D \ll m$.

> How large the subsample size D need to be to guarantee linear convergence of the first T iterations of QRK?

Main Theorems: Upper and Lower Bounds

Upper bound (sufficiency).

If $D \ \geq \ C\,\frac{\log T}{\log(1/\beta)}$, then with high probability
$\|\mathbf{x}_T - \mathbf{x}^*\|^2 \ \leq \ \left(1 - \frac{c}{n}\right)^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2.$

Lower bound (necessity).

If $D \ \leq \ c\,\frac{\log T}{\log(1/\beta)}$, then there exists (βm)-sparse corruption such that $\|\mathbf{x}_T - \mathbf{x}^*\|$ is arbitrarily large

Conclusion. $D \asymp \frac{\log T}{\log(1/\beta)}$ is the *minimal* subsample size, up to constant.

Implication (β is a small constant)

$D \asymp \frac{\log T}{\log(1/\beta)}$ is the *minimal* subsample size

Setting 1: β is a small constant

- $D = O(\log T)$ is minimal
- $\|\mathbf{x}_T - \mathbf{x}^*\|^2 \ \leq \ \left(1 - \frac{c}{n}\right)^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2$
  $\implies T \asymp n \log(1/\varepsilon)$ to achieve approximation error $\varepsilon\|\mathbf{x}_0 - \mathbf{x}^*\|_2^2$
  $\implies$
  $$D = O(\log n + \log\log\varepsilon^{-1}) \overset{\varepsilon \geq \exp(-n)}{=} O(\log n)$$
- Previous theory:
  - $D = m$ (Haddock et al., 2022; Steinerberger, 2023)
  - $D = \alpha m$ with $\alpha \geq 2\beta$ (Haddock et al., 2023)
- We achieve massive reduction: $\Theta(m) \ \longrightarrow \ O(\log n)$

Implication ($\beta = o(1)$)

$D \asymp \frac{\log T}{\log(1/\beta)}$ is the *minimal* subsample size

Setting 2: $\beta = o(1)$

- The subsample size can be further reduced by a factor of $\frac{1}{\log(1/\beta)}$!
- (Sublinear number of corruptions) If

$$\beta = \Theta(m^{-\xi}), \qquad \text{for some } \xi \in (0,1),$$

together with $\varepsilon \geq \exp(-n)$, then we can use

$$D = O\Big(\frac{\log n}{\xi \log m}\Big) = O\Big(\frac{1}{\xi}\Big)$$

Proof Overview

Preliminary: Chernoff Bound

- Tight Chernoff bounds for binomial variable

$$\mathbb{P}(\mathrm{Bin}(N,q) \le k) \le \exp\Big(-N\cdot D_{KL}(\frac{k}{N}\|q)\Big), \quad k \le Nq$$
$$\mathbb{P}(\mathrm{Bin}(N,q) \ge k) \le \exp\Big(-N\cdot D_{KL}(\frac{k}{N}\|q)\Big), \quad k \ge Nq$$

where $D_{KL}(p\|q) = p\log\frac{p}{q} + (1-p)\log\frac{1-p}{1-q}$

- We cannot capture $O(\frac{1}{\log(1/\beta)})$ if we stick with the weakened Chernoff bound in Haddock et al. (2022):

$$\mathbb{P}\big(\mathrm{Bin}(N,q) \ge (1+\delta)Nq\big) \le \exp\Big(-\frac{\delta^2 Nq}{3}\Big), \quad \forall\delta\in(0,1)$$

Preliminary: Quantiles

- Runtime quantile:

$$Q_q(\mathbf{x}, S) = q\text{-quantile of } \{|b_i - \langle \mathbf{a}_i, \mathbf{x}\rangle| : i \in S\}.$$

- Ideal quantile: $(b_i \longrightarrow \langle \mathbf{a}_i, \mathbf{x}^*\rangle)$

$$\tilde{Q}_q(\mathbf{x}) = q\text{-quantile of } \{|\langle \mathbf{x} - \mathbf{x}^*, \mathbf{a}_i\rangle| : i \in [m]\}.$$

which satisfies

$$\tilde{Q}_q(\mathbf{x}) \le \frac{\sigma_{\max}(\mathbf{A})\|\mathbf{x} - \mathbf{x}^*\|}{\sqrt{(1-q)m}}$$

Proof Overview (Upper Bound)

1. By Markov's inequality, for $\mathbb{P}(\Omega) \le \epsilon$,

$$\mathbb{E}\Big(\|\mathbf{x}_T - \mathbf{x}^*\|^2 1_{\Omega^c}\Big) \le (1 - \frac{c}{n})^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2 \tag{3.1}$$

$$\Longrightarrow \|\mathbf{x}_T - \mathbf{x}^*\|^2 \le (1 - \frac{c'}{n})^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2 , \quad \text{w.h.p.} \tag{3.2}$$

Question: How to choose the failure event Ω?

2. Upper bound on quantile is important to control the worst case, but it does not arbitrarily if $D \ll \beta m$
(possibility of $i_1, ..., i_D \in B$)

3. Fortunately, we can prove for one step that

$$\mathbb{P}\Big(\underbrace{Q_q(\mathbf{x}_k, \{i_j^{(k+1)}\}_{j=1}^D) \le \tilde{Q}_{q+\beta+\epsilon}(\mathbf{x}_k)}_{\substack{:= S_{k+1} \\ \text{(quantile being upper bounded)}}}\Big) \ge 1 - \exp(-D \cdot D_{KL}(q \| q + \epsilon))$$

Proof Overview (Upper Bound)

4. We work with $\Omega := \{S_{k+1} \text{ fails for some } 0 \le k \le T-1\}$ satisfying

$$\mathbb{P}(\Omega) \le T \exp(-D \cdot D_{KL}(q \| q + \epsilon)) \tag{3.3}$$
$$\le \epsilon \qquad \text{(so long as } D \text{ is suitably large)} \tag{3.4}$$

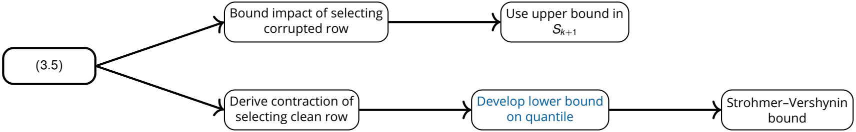
5. A stopping time argument (Tan and Vershynin, 2019) shows

$$\mathbb{E}\left[\|\mathbf{x}_{k+1} - \mathbf{x}\|^2 1_{S_{k+1}}\right] \le (1 - \frac{c}{n})\|\mathbf{x}_k - \mathbf{x}^*\|^2 \tag{3.5}$$
$$\Longrightarrow \mathbb{E}\left(\|\mathbf{x}_T - \mathbf{x}^*\|^2 1_{\Omega^c}\right) \le (1 - \frac{c}{n})^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2$$

Proof Overview (Upper Bound)

6. We discuss if the update sample r_{k+1} is corrupted or not:

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{x}_{k+1}-\mathbf{x}\|^2 1_{S_{k+1}}\right] &= \overbrace{\beta\cdot\mathbb{E}\left[\|\mathbf{x}_{k+1}-\mathbf{x}\|^2 1_{S_{k+1}}\,\middle|\, r_{k+1}\in B\right]}^{\text{select corrupted row}} && (3.6)\\
&\quad+\underbrace{(1-\beta)\mathbb{E}\left[\|\mathbf{x}_{k+1}-\mathbf{x}\|^2 1_{S_{k+1}}\,\middle|\, r_{k+1}\in B^c\right]}_{\text{select clean row}} && (3.7)\\
&= \cdots\cdots
\end{aligned}$$



$$\mathbb{P}\Big(Q_q(\mathbf{x}_k,\{i_j^{(k+1)}\}_{j=1}^D)\geq \tilde{Q}_{q-\beta-\epsilon}(\mathbf{x}_k)\Big)\geq 1-\exp(-D_{KL}(q\|q+\epsilon)\cdot D)$$

Proof Overview (Lower Bound)

1. Key idea: A bad step — which projects the iterate onto a corrupted row — makes the error arbitrarily large. Formally,

$$\|\mathbf{x}_{k,\mathrm{bad}} - \mathbf{x}^*\| \geq \min_{i \in B} |\epsilon_i| \tag{3.8}$$

2. Lower bounding the prob. of a bad step:

$$\mathbb{P}(\text{Bad step}) \geq \mathbb{P}\left(\substack{i_1,\ldots,i_D \in B^* \\ r \in B_*}\right) \geq \left(\frac{\beta}{2}\right)^{D+1} \tag{3.9}$$

(Argument: Equally divide $B = B^* \cup B_*$ such that residuals of B^* are uniformly higher than those of B_*)

3. Brief outline:

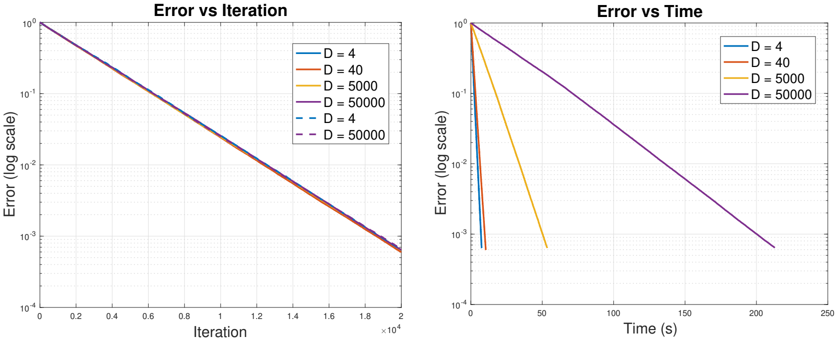
Find the last bad step in first T iterations → Show the remaining steps cannot reduce error too much

Numerical Example

Effect of Subsample Size D

Experiment. $(m = 50000,\ n = 100,\ T = 20000,\ \beta = 0.01)$

- Compare $D \in \{4,\ 40,\ 5000,\ 50000\}$.



- Similar convergence rate & much faster

"On Subsample Size of Quantile-Based Randomized Kaczmarz"

Available on arXiv: https://arxiv.org/abs/2507.15185

Under revision at SIAM Journal on Matrix Analysis and Applications

Thank You!

Questions?

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