

On Subsample Size of Quantile-Based Randomized Kaczmarz

Jian-Feng Cai*

Junren Chen[†]

Anna Ma[‡]

Tong Wu[§]

July 19, 2025

Abstract

Quantile-based randomized Kaczmarz (QRK) was recently introduced to efficiently solve sparsely corrupted linear systems $\mathbf{Ax}^* + \epsilon = \mathbf{b}$ [*SIAM J. Matrix Anal. Appl.*, 43(2), 605-637], where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and ϵ is an arbitrary (βm) -sparse corruption. However, all existing theoretical guarantees for QRK require quantiles to be computed using all m samples (or a subsample of the same order), thus negating the computational advantage of Kaczmarz-type methods. This paper overcomes the bottleneck. We analyze a subsampling QRK, which computes quantiles from D uniformly chosen samples at each iteration. Under some standard scaling assumptions on the coefficient matrix, we show that QRK with subsample size $D \geq \frac{C \log(T)}{\log(1/\beta)}$ linearly converges over the first T iterations with high probability, where C is some absolute constant. This subsample size is a substantial reduction from $O(m)$ in prior results. For instance, it translates into $O(\log(n))$ even if an approximation error of $\exp(-n^2)$ is desired. Intriguingly, our subsample size is also tight up to a multiplicative constant: if $D \leq \frac{c \log(T)}{\log(1/\beta)}$ for some constant c , the error of the T -th iterate could be arbitrarily large with high probability. Numerical results are provided to corroborate our theory.

1 Introduction

Solving large-scale systems of linear equations is a fundamental task with widespread applications in scientific computing, data science, engineering, and related fields. In practice, the right-hand side of the linear system is often corrupted due to sensor failures, transmission errors, or adversarial attacks. As a consequence, efficient algorithms for solving corrupted linear systems have been highly sought after by applied mathematicians. This problem can be formulated as finding \mathbf{x}^* from

$$\mathbf{Ax}^* + \epsilon = \mathbf{b}, \tag{1}$$

where $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_m]^\top \in \mathbb{R}^{m \times n}$ denotes the coefficient matrix with rows $\mathbf{a}_j \in \mathbb{R}^n$, $\mathbf{x}^* \in \mathbb{R}^n$ denotes the desired solution, $\mathbf{b} = [b_1, \dots, b_m]^\top$ is the measurement vector that is corrupted by (βm) -sparse $\epsilon \in \mathbb{R}^m$ whose nonzero entries can take *arbitrary* values, and $\beta \in (0, 1)$ is small to ensure the well-posedness of the problem.

Storing the entire linear system may not be possible in the highly overdetermined case with an extremely large m . Kaczmarz methods have proven particularly advantageous in this regime as they only require one row per iteration. In particular, the Kaczmarz algorithm iteratively projects the iterate onto the solution

*Department of Mathematics, Hong Kong University of Science and Technology. email: jfcai@ust.hk

[†]Department of Mathematics, The University of Hong Kong. email: chenjr58@connect.hku.hk

[‡]Department of Mathematics, University of California, Irvine. email: anna.ma@uci.edu

[§]Department of Mathematics, Hong Kong University of Science and Technology. email: twubi@connect.ust.hk

hyperplane of a chosen row: suppose \mathbf{a}_j is chosen for updating the current iterate \mathbf{x}_k , one simply projects \mathbf{x}_k to $\{\mathbf{u} \in \mathbb{R}^n : \mathbf{a}_j^\top \mathbf{u} = b_j\}$, i.e.,

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \frac{(\mathbf{a}_j^\top \mathbf{x}_k - b_j)\mathbf{a}_j}{\|\mathbf{a}_j\|^2}. \quad (2)$$

The rate of convergence of the Kaczmarz algorithm heavily relies on the ordering of the rows that the iterates are projected onto [8]. As a consequence, only convergence to the solution (for consistent systems) was guaranteed in its initial study [9].

The randomized Kaczmarz (RK) method was introduced by Strohmer and Vershynin, who proved that RK converges linearly in expectation [16] when the linear system is consistent and rows are selected randomly with probability proportional to each row norm. Unfortunately, when the system is noisy and inconsistent, the best one can hope for is convergence in expectation to a ball containing the least squares solution, where the radius of the ball depends on the norm of the noise [12, 20, 18]. When noise is very large and rows are severely corrupted, this convergence error horizon grows with it, even if only one row is corrupted.

Recently, Haddock, Needell, Rebroya, and Swartworth [7] developed a quantile-based randomized Kaczmarz (QRK) that can solve *sparsely* corrupted linear systems. The core idea is to avoid projections onto corrupted rows by using quantile statistics. To update \mathbf{x}_k via the j -th row, the quantile of the residuals (or a randomly chosen subset) $\{|\mathbf{a}_i^\top \mathbf{x}_k - b_i|\}_{i=1}^m$ is computed and the projection is only made when $|\mathbf{a}_j^\top \mathbf{x}_k - b_j|$ is below the quantile. To make this precise while simplifying notation, we assume the rows of \mathbf{A} have unit Euclidean norm and introduce the QRK algorithm from [7] in Algorithm 1. In Algorithm 1, $\{i_j^{(k+1)}\}_{j=1}^D$ is the set of D uniformly chosen row indices for computing the quantile (referred to as *quantile subsample*) and r_{k+1} is the index of the row for potential projection (referred to as *update sample*).

Algorithm 1: QuantileRK(q) [7]

Input: \mathbf{A} , \mathbf{b} , \mathbf{x}_0 , $q \in (0, 1)$, quantile subsample size D , iteration number T

for $k = 0$ **to** $T - 1$ **do**

sample $i_1^{(k+1)}, \dots, i_D^{(k+1)} \sim \text{Uniform}(1, \dots, m)$;
sample $r_{k+1} \sim \text{Uniform}(1, \dots, m)$;
Compute $h_{r_{k+1}} := \mathbf{a}_{r_{k+1}}^\top \mathbf{x}_k - b_{r_{k+1}}$;
if $|h_{r_{k+1}}| \leq q\text{-quantile}\left(\{|\mathbf{a}_{i_j^{(k+1)}}^\top \mathbf{x}_k - b_{i_j^{(k+1)}}|\}_{j=1}^D\right)$ **then**
 $\mathbf{x}_{k+1} = \mathbf{x}_k - h_{r_{k+1}} \mathbf{a}_{r_{k+1}};$
else
 $\mathbf{x}_{k+1} = \mathbf{x}_k;$

return \mathbf{x}_T

Under a class of random coefficient matrices (e.g., \mathbf{A} with i.i.d. rows uniformly drawn from the unit sphere), the authors of [7] established linear convergence to the solution \mathbf{x}^* of the corrupt system for QRK, similar to [16]. Later, Steinerberger [15] provided an analysis over deterministic \mathbf{A} . However, both these guarantees are only valid for QRK using *full samples*, meaning that the quantile over a random subsample of size D in Algorithm 1 is replaced by the quantile of $\{|\mathbf{a}_i^\top \mathbf{x}_k - b_i|\}_{i=1}^m$. Consequently, existing theoretical guarantees for QRK need to access the entire matrix \mathbf{A} at each iteration. This negates the major computational advantage of Kaczmarz methods, i.e., requiring only one row per iteration, and a small part of the system overall.

On the other hand, Algorithm 1 with $D \ll m$ works well in numerical simulations (e.g., see [7]). To the best of our knowledge, providing a theoretical guarantee to this small quantile subsample regime remains

an important open question. Indeed, there has only been one work [6] that analyzed a slightly different subsampled QRK. In particular, [7] established linear convergence using a quantile subsample of size αm for some $\alpha \geq \max\{\frac{\beta}{q}, \frac{\beta}{1-q}\}$. However, accessing a constant fraction of the system remains computationally demanding.

The goal of this paper is to identify the subsample size required for Algorithm 1 to succeed. Unlike previously established guarantees [16, 7, 15, 6] which allow for infinite iterations, we restrict our attention to a given finite number of iterations (say, T). This is somewhat necessary: a bad step which projects the iterate onto a corrupted row will happen if sufficiently many iterations are run, and one such bad step can make the estimate arbitrarily distant from \mathbf{x}^* . While this might appear as a limitation, in practice, one always runs finite iterations to achieve a desired approximation error. We now formally introduce the question that we focus on.

Question. *Given a positive integer T , what is the size of the quantile subsample needed for Algorithm 1 to linearly converge over the first T iterations?*

We denote the size of the quantile subsample (also called subsample size) by D throughout the paper. Our main contributions are to provide upper and lower bounds, which match up to a multiplicative constant, for D . For simplicity, we assume that the rows of \mathbf{A} are uniformly distributed over the unit Euclidean sphere.¹ We first provide an upper bound showing that a uniformly chosen quantile subsample of size $D = O(\frac{\log T}{\log(1/\beta)})$ is sufficient to guarantee linear convergence. Let us provide an informal statement here; see Theorem 1 for the complete statement.

Theorem (Informal). *There exist some constants C_1, c_2 such that for any $T \in \mathbb{Z}_+$ and for arbitrary (βm) -sparse ϵ , Algorithm 1 with $D \geq \frac{C_1 \log T}{\log(1/\beta)}$ satisfies $\|\mathbf{x}_T - \mathbf{x}^*\|^2 \leq (1 - \frac{c_2}{n})^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2$ with $1 - o(1)$ probability.*

In the most interesting regime where β is some positive constant, we need a quantile subsample of size $O(\log T)$ to run T iterations. As a consequence, to achieve a squared ℓ_2 approximation error of $\varepsilon \|\mathbf{x}_0 - \mathbf{x}^*\|^2$, we can set $T = \frac{n \log(1/\varepsilon)}{c_2}$ and hence require a subsample size of

$$D = O(\log n + \log \log(1/\varepsilon)).$$

Such subsample size reduces to $D = O(\log n)$ that is substantially smaller than $O(m)$ even under $\varepsilon = \exp(-n^2)$, an approximation error that is more than sufficient for most practical applications.²

It is interesting to note that the subsample size can be further reduced if $\beta = o(1)$. Let us consider a sublinear number of corruptions, i.e., $\beta = \Theta(m^{-\xi})$ for some $\xi \in (0, 1]$. To achieve a squared ℓ_2 approximation error of $\varepsilon \|\mathbf{x}_0 - \mathbf{x}^*\|^2$, we only require a subsample size of $D = O(\frac{\log n + \log \log \varepsilon^{-1}}{\xi \log m})$. If additionally $\varepsilon \geq \exp(-n^2)$, then we only require $D = O(\frac{\log n}{\xi \log m}) = O(\xi^{-1})$, meaning that a constant subsample size (depending only on ξ) suffices under any $\xi \in (0, 1]$.

We also provide a converse result which indicates that a subsample size at the order of $\frac{\log T}{\log(1/\beta)}$ is necessary for Algorithm 1 to converge linearly over the first T iterations. The following is an informal version of Theorem 2.

Theorem (Informal). *Under some mild scaling assumptions on (β, T) , there exists an absolute constant c_1 such that for some (βm) -sparse corruption ϵ , Algorithm 1 with $D \leq \max\{\frac{c_1 \log T}{\log(1/\beta)}, 1\}$ returns \mathbf{x}_T with arbitrarily large $\|\mathbf{x}_T - \mathbf{x}^*\|^2$ with $1 - o(1)$ probability.*

¹As we shall see, our results only require certain properties on \mathbf{A} that are satisfied with high probability by the class of random matrices considered in [7, Assumptions 1–2]; see our Assumption 1–Assumption 2. Moreover, it is not hard to adapt our results to a deterministic manner, as done in [15].

²In fact, our $D = O(\log n + \log \log(1/\varepsilon))$ improves on $O(m)$ so long as $\varepsilon \geq \exp(-\exp(m))$.

Our proof techniques for the upper bound depart from existing analysis [7, 15, 6] in many ways. In particular, we first develop two-sided probabilistic bounds on the subsampled quantile. We then rely on the upper bound to control the impact of accepting a corrupted row, and utilize the lower bound to establish the contraction after accepting an uncorrupted row. Both of these two steps require a number of new ideas, especially in iterating an appropriate one-step contraction and constructing a sufficiently large set of acceptable uncorrupted rows. See Section 3.1.1 for details. On the other hand, we build the lower bound by ensuring the projection onto some corrupted row in some step and then showing the subsequent iterations cannot reduce the approximation error too much.

The paper is organized as follows. We introduce the notation and some useful technical tools in Section 2. We present our main theoretical results and their proofs in Section 3. We also provide a set of numerical examples in Section 4 which support our theoretical findings. Lastly, we provide concluding remarks in Section 5.

2 Preliminaries

In this section, we present notation that will be used throughout for easy referencing. In addition to notation, we present key technical tools and summarize preliminary observations from prior works when Algorithm 1 chooses a corrupted row in Section 2.1 and a non-corrupted row in Section 2.2. Some properties of the coefficient matrix \mathbf{A} are discussed in Section 2.3.

We adopt the convention $[m] = \{1, \dots, m\}$ and for simplicity, we assume that the proportions of m (such as $qm, \beta m, \alpha m$, where $\alpha, \beta, q \in [0, 1]$) and qD are positive integers. If not, rounding can be applied without significantly affecting any results. The inner product between $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ is defined as $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^\top \mathbf{b}$, and the norm $\|\mathbf{a}\| := \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle}$. We work with natural logarithm $\log(\cdot)$ with base e . The Kullback–Leibler (KL) divergence between two Bernoulli distributions with parameters $p, q \in [0, 1]$, is defined as

$$D_{\text{KL}}(p\|q) = p \log \left(\frac{p}{q} \right) + (1 - p) \log \left(\frac{1 - p}{1 - q} \right).$$

We denote absolute constants by C, C_i, c, c_i whose values may vary from line to line. We will adopt standard complexity notation, by writing $I_1 = O(I_2)$ if $I_1 \leq CI_2$ for some absolute constant C , $I_1 = \Omega(I_2)$ if $I_1 \geq cI_2$ for some absolute constant c , and $I_1 = \Theta(I_2)$ if $I_1 = O(I_2)$ and $I_1 = \Omega(I_2)$ simultaneously hold. We will generically use $o(1)$ to denote quantities that tend to 0 when $m, n, T \rightarrow \infty$.

We assume the rows of $\mathbf{A} \in \mathbb{R}^{m \times n}$ are independent random vectors uniformly distributed on the unit sphere \mathbb{S}^{n-1} . The Frobenius norm of \mathbf{A} is denoted by $\|\mathbf{A}\|_F$. The smallest non-zero singular value and the largest singular value of \mathbf{A} are $\sigma_{\min}(\mathbf{A})$ and $\sigma_{\max}(\mathbf{A})$, respectively. For any $S \subset [m]$, $\mathbf{A}_S \in \mathbb{R}^{|S| \times n}$ denotes the submatrix of \mathbf{A} with rows indexed by S .

The q -quantile of a *multi-set* $\{z_1, z_2, \dots, z_N\}$ (in which $z_i = z_j$ is possible for $i \neq j$) is defined as $z_{[qN]}^*$ where we let $z_1^* \leq z_2^* \leq \dots \leq z_N^*$ be the non-decreasing rearrangement of the N elements. (If $qN < 1$, we will instead define the q -quantile as z_1^* .) Let $B = \{i : \epsilon_i \neq 0\}$ be the index set of the corrupted measurements in the right-hand side \mathbf{b} , and let $B^c := [m] \setminus B$ be its complement. At runtime, we can only use the possibly corrupted \mathbf{b} to compute the quantile, where the quantile is denoted by

$$Q_q(\mathbf{x}, S) := q\text{-quantile of } \{|b_i - \langle \mathbf{a}_i, \mathbf{x} \rangle| : i \in S\},$$

for a given multi-set S . We shall denote the full-sample quantile by

$$Q_q(\mathbf{x}) := Q_q(\mathbf{x}, [m]).$$

If b_i is uncorrupted, then $b_i = \langle \mathbf{a}_i, \mathbf{x}^* \rangle$ and hence the absolute residual for the i th row can be expressed as $|\langle \mathbf{a}_i, \mathbf{x} - \mathbf{x}^* \rangle|$. Hence, quantiles computed from uncorrupted measurements are denoted

$$\tilde{Q}_q(\mathbf{x}, S) := q\text{-quantile of } \{|\langle \mathbf{x} - \mathbf{x}^*, \mathbf{a}_i \rangle| : i \in S\}.$$

Similarly, the full-sample uncorrupted quantile will be written as

$$\tilde{Q}_q(\mathbf{x}) := \tilde{Q}_q(\mathbf{x}, [m]).$$

We denote by \mathbf{X}_k the QRK iterate after k steps when we want to emphasize its randomness. We may also write it as \mathbf{x}_k when working with a deterministic realization of \mathbf{X}_k . The error vector at iteration k is defined as $\mathbf{e}_k := \mathbf{x}_k - \mathbf{x}^*$.

We now proceed to introduce the technical tools used in our paper. At the center of our analysis are various binomial variables; hence, our major technical tool is the following tight Chernoff bound. We emphasize that the Chernoff bound is tighter than the one used in [7, 15], and such tightness is essential for capturing the role of β in the subsample size.

Lemma 1. *Let $X \sim \text{Bin}(N, q)$, then we have the Chernoff bound: for $k \leq Nq$,*

$$\mathbb{P}(X \leq k) \leq \exp\left(-N \cdot \text{D}_{KL}\left(\frac{k}{N} \| q\right)\right).$$

Similarly, for $k \geq Nq$, $\mathbb{P}(X \geq k) \leq \exp\left(-N \cdot \text{D}_{KL}\left(\frac{k}{N} \| q\right)\right).$

2.1 Projection onto Corrupted Row

In Algorithm 1, it is possible that a corrupted row is selected for updating the iterate. However, since the subsampled quantile bounds the residual, the resulting error increase can be effectively controlled, as shown in Lemma 2.

Lemma 2. *Given \mathbf{x}_k , the update sample $r_{k+1} = i$, and the quantile subsample $\{i_j^{(k+1)}\}_{j=1}^D = \{(i')_j^{(k+1)}\}_{j=1}^D$. If $Q(\mathbf{x}_k, \{(i')_j^{(k+1)}\}_{j=1}^D) \leq m_Q$, then*

$$\|\mathbf{e}_{k+1}\|^2 \leq \|\mathbf{e}_k\|^2 + m_Q^2 + 2m_Q |\langle \mathbf{e}_k, \mathbf{a}_i \rangle|. \quad (3)$$

Proof. In Algorithm 1, we can bound the residual of the update sample $r_{k+1} = i$ as $|h_i| = |\langle \mathbf{x}_k, \mathbf{a}_i \rangle - b_i| \leq Q(\mathbf{x}_k, \{(i')_j^{(k+1)}\}_{j=1}^D) \leq m_Q$. Thus, we have

$$\begin{aligned} \|\mathbf{e}_{k+1}\|^2 &= \|\mathbf{e}_k - h_i \mathbf{a}_i\|^2 \\ &\begin{cases} \leq \left(\|\mathbf{e}_k\|^2 + Q\left(\mathbf{x}_k, \{(i')_j^{(k+1)}\}_{j=1}^D\right)^2 + 2Q\left(\mathbf{x}_k, \{(i')_j^{(k+1)}\}_{j=1}^D\right) |\langle \mathbf{e}_k, \mathbf{a}_i \rangle| \right) & \text{if accepted} \\ = \|\mathbf{e}_k\|^2 & \text{if not accepted} \end{cases} \\ &\leq \|\mathbf{e}_k\|^2 + m_Q^2 + 2m_Q |\langle \mathbf{e}_k, \mathbf{a}_i \rangle|. \end{aligned}$$

□

Lemma 3 relates the realized quantiles Q_q with the uncorrupted quantiles \tilde{Q}_q in a full sample regime. (We will derive an analogous lemma for the subsampled regime in Section 3.)

Lemma 3 (Lemma 3.6 in [7]). *Let \mathbf{x}_k be a fixed vector. Under (βm) -sparse corruption, we have*

$$\tilde{Q}_{q-\beta}(\mathbf{x}_k) \leq \tilde{Q}_{\frac{q-\beta}{1-\beta}}(\mathbf{x}_k, B^c) \leq Q_q(\mathbf{x}_k) \leq \tilde{Q}_{q+\beta}(\mathbf{x}_k).$$

While the uncorrupted quantiles are not accessible during runtime, they are more technically amenable and can be bounded by Lemma 4.

Lemma 4 ([15], Lemma 1). *For any $q' \in (0, 1)$, the uncorrupted quantile satisfies:*

$$\tilde{Q}_{q'}(\mathbf{x}_k) \leq \Phi_{q'} \frac{\|\mathbf{x}_k - \mathbf{x}^*\|}{\sqrt{n}} \quad \text{where} \quad \Phi_{q'} := \frac{\sigma_{\max}(\mathbf{A})\sqrt{n}}{\sqrt{m}\sqrt{1-q'}}. \quad (4)$$

Lemma 4 is a straightforward consequence of the following lemma.

Lemma 5 ([7], Lemma 3.7). *Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. For all unit vectors $\mathbf{x} \in \mathbb{R}^n$ and every subset $S \subseteq [m]$, the following holds:*

$$\sum_{i \in S} |\langle \mathbf{x}, \mathbf{a}_i \rangle| \leq \sqrt{|S|} \cdot \left(\sum_{i \in S} |\langle \mathbf{x}, \mathbf{a}_i \rangle|^2 \right)^{1/2} \leq \sigma_{\max}(\mathbf{A}) \sqrt{\frac{n}{m}} \cdot \sqrt{\frac{m|S|}{n}}. \quad (5)$$

2.2 Projection onto Uncorrupted Row

In this scenario, the error is non-expansive since

$$\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 = \left\| \left(\mathbf{I} - \frac{\mathbf{a}_{r_{k+1}} \mathbf{a}_{r_{k+1}}^\top}{\|\mathbf{a}_{r_{k+1}}\|^2} \right) \mathbf{e}_k \right\|^2 \leq \|\mathbf{e}_k\|^2. \quad (6)$$

In expectation, one actually has an error contraction captured by the following Strohmer-Vershynin bound.

Lemma 6 ([16]). *Let $\mathbf{A}\mathbf{x} = \mathbf{b}$ be a consistent linear system. The iterates \mathbf{X}_{k+1} generated by RK (2) satisfy*

$$\mathbb{E} \|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \leq \left(1 - \frac{\sigma_{\min}^2(\mathbf{A})}{\|\mathbf{A}\|_F^2} \right) \|\mathbf{e}_k\|^2, \quad (7)$$

where $\mathbf{a}_{r_{k+1}}$ is selected with probability proportional to $\|\mathbf{a}_{r_{k+1}}\|^2$.

In the analysis of QRK, the quantity

$$\sigma_{\min, \alpha} := \inf_{S \subseteq [m]: |S| \geq \alpha m} \sigma_{\min}(\mathbf{A}_S) \sqrt{\frac{n}{m}} \quad \alpha \in (0, 1), \quad (8)$$

plays an important role in our application of Lemma 6. The reason is that accepting an uncorrupted update sample in QRK can be treated as running one iteration of standard RK for an uncorrupted linear subsystem with a number of rows greater than αm .

2.3 Coefficient Matrix \mathbf{A}

When \mathbf{A} have rows sampled i.i.d. uniformly from \mathbb{S}^{n-1} , the constants $\sigma_{\min, \alpha}$ in (8) and $\Phi_{q'}$ in (4) are bounded with high probability when $m = \Omega(n)$.

Lemma 7 ([7], Lemma 3.7 & Proposition 3.4). *Let $\alpha_0 \in (0, 1)$, there exists absolute constants C_1, c_2, c, C such that if $m \geq \max \{C_1 \frac{1}{\alpha_0} \log(\frac{\sqrt{2/\pi}}{\alpha_0})n, n\}$, then with probability at least $1 - 3 \exp(-c_2 \alpha_0 m) - 2 \exp(-cm)$, we have:*

$$\sigma_{\min, \alpha_0} \geq \frac{\sqrt{2\pi} \alpha_0^{3/2}}{24}, \quad \frac{\sigma_{\max}(\mathbf{A}) \sqrt{n}}{\sqrt{m}} \leq C, \quad \Phi_{q'} \leq \frac{C}{\sqrt{1-q'}} \quad (\forall q' \in (0, 1)). \quad (9)$$

3 Main Results

We present our main theoretical results. First, we introduce an upper bound on the subsample size in Section 3.1 and present its proof in Section 3.2. Then, we provide the lower bound in Section 3.3 and prove it in Section 3.4 to complement the tightness of our upper bound.

3.1 Upper Bound

In our upper bound, we make the following assumption on \mathbf{A} .

Assumption 1. \mathbf{A} has rows in \mathbb{S}^{n-1} and satisfies all the spectral properties in (9) with $\alpha_0 = \frac{q}{2}$.

When \mathbf{A} has i.i.d. rows uniformly sampled from \mathbb{S}^{n-1} , these properties hold with high probability under $m \geq C_q n$, where C_q is some constant depending only on q ; see Lemma 7.

We now formally present the upper bound on the subsample size D .

Theorem 1. *Consider problem (1) with arbitrary (βm) -sparse ϵ and suppose Assumption 1 holds. Let $q \in (0, 1)$, $T > 1$ be the number of iterations run, and D be the subsample size used in Algorithm 1. There exist constants c_1, C_2, c_3, c_4, c_5 depending only on q , such that for any $\beta \in (0, c_1)$, if the positive integer D satisfies*

$$D \geq \frac{C_2 \log T}{\log(1/\beta)},$$

then we have the following guarantees on iterate \mathbf{X}_T of Algorithm 1:

- (Convergence in expectation) There exists a failure event Ω satisfying $\mathbb{P}(\Omega) \leq T^{-5} \exp(-\frac{1-q}{4} \log(1/\beta))$. $D) \leq \frac{1}{2}$ such that

$$\mathbb{E} \left(\|\mathbf{X}_T - \mathbf{x}^*\|^2 \mathbf{1}_{\Omega^c} \right) \leq \left(1 - \frac{c_3}{n} \right)^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2. \quad (10)$$

- (Convergence in probability) We have

$$\mathbb{P} \left(\|\mathbf{X}_T - \mathbf{x}^*\|^2 \leq \left(1 - \frac{c_4}{n} \right)^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2 \right) \geq 1 - T^{-5} - 2 \exp \left(-\frac{c_5 T}{n} \right). \quad (11)$$

We mention that the above (11) is what we claimed as the first informal theorem in Section 1. Due to Markov's inequality, (11) is indeed an immediate consequence of the expected convergence (10).

Proposition 1. *In Theorem 1, the convergence in expectation (10) implies the high-probability one (11).*

Proof. Since $\mathbb{P}(\Omega^c) \geq \frac{1}{2}$, we have

$$\mathbb{E}(\|\mathbf{X}_T - \mathbf{x}^*\|^2 | \Omega^c) = \mathbb{P}(\Omega^c) \cdot \mathbb{E}(\|\mathbf{X}_T - \mathbf{x}^*\|^2 | \Omega^c) \geq \frac{1}{2} \mathbb{E}(\|\mathbf{X}_T - \mathbf{x}^*\|^2 | \Omega^c),$$

which gives

$$\mathbb{E}(\|\mathbf{X}_T - \mathbf{x}^*\|^2 | \Omega^c) \leq 2 \left(1 - \frac{c_2}{n}\right)^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2.$$

Given $\epsilon > 0$, by Markov's inequality we have

$$\mathbb{P}\left(\|\mathbf{X}_T - \mathbf{x}^*\|^2 \geq \epsilon \|\mathbf{x}_0 - \mathbf{x}^*\|^2 \mid \Omega^c\right) \leq \frac{2(1 - \frac{c_2}{n})^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2}{\epsilon \|\mathbf{x}_0 - \mathbf{x}^*\|^2} = \frac{2(1 - \frac{c_2}{n})^T}{\epsilon},$$

and hence

$$\begin{aligned} & \mathbb{P}\left(\|\mathbf{X}_T - \mathbf{x}^*\|^2 \geq \epsilon \|\mathbf{x}_0 - \mathbf{x}^*\|^2\right) \\ & \leq \mathbb{P}(\Omega^c) \mathbb{P}\left(\|\mathbf{X}_T - \mathbf{x}^*\|^2 \geq \epsilon \|\mathbf{x}_0 - \mathbf{x}^*\|^2 \mid \Omega^c\right) + \mathbb{P}(\Omega) \\ & \leq \frac{2(1 - \frac{c_2}{n})^T}{\epsilon} + \mathbb{P}(\Omega). \end{aligned}$$

To obtain the desired result (11), we set $\epsilon = (1 - \frac{c_2}{2n})^T$ and notice

$$\frac{2(1 - c_2/n)^T}{\epsilon} \leq 2 \left(\frac{1 - c_2/n}{1 - c_2/(2n)} \right)^T \leq 2 \left(1 - \frac{c_2}{2n}\right)^T \leq 2 \exp\left(-\frac{c_2 T}{2n}\right),$$

where the last step uses $\log(1 - x) \leq -x$. Setting $c_4 = c_2/2$ and $c_5 = c_2/2$ yields the result in (11). \square

Therefore, all that remains is to prove the convergence in expectation in (10). To that end, we require some new ideas beyond [7, 6, 15]. We shall especially compare to the proof of [7, Thm. 1.1] for an algorithm nearly identical to Algorithm 1, with the only difference being the replacement of our subsampled quantile by the full-sample one $Q_q(\mathbf{x}_k)$.³ Before presenting the proof of Theorem 1 in the next subsection, we shall pause to summarize these new ideas as our technical contributions.

3.1.1 New Ideas for Proving (10)

Prior works relied on Lemma 3 to relate the full-sample quantile to the uncorrupted quantile \tilde{Q} (which can be further bounded by Lemma 4). Such a relation no longer holds arbitrarily for quantiles computed from D samples with $D \ll \beta m$, as the entire quantile subsample can be corrupted and the subsampled quantile can take an arbitrary value. Fortunately, we can still bound our subsampled quantile from both sides by the uncorrupted quantile, with high probability, using the Chernoff bound.

Lemma 8. *Given \mathbf{x}_k , $q \in (0, 1)$, $\epsilon_l \in (0, q)$, $\epsilon_u \in (0, 1 - q)$ and $\beta \in (0, \min\{q - \epsilon_l, 1 - q - \epsilon_u\})$, with respect to the randomness in the selection of subsample $\{i_j^{(k+1)}\}_{j=1}^D$, we have the following statements:*

- (Upper bound) *With probability at least $1 - \exp(-D_{KL}(q \| q + \epsilon_u)D)$, we have*

$$Q_q\left(\mathbf{x}_k, \{i_j^{(k+1)}\}_{j=1}^D\right) \leq \tilde{Q}_{q+\beta+\epsilon_u}(\mathbf{x}_k) \quad (12)$$

³In contrast, [15, 6] analyzed algorithms which select the update sample from acceptable rows to avoid the potential rejection of some update.

- (Lower bound) With probability at least $1 - \exp(-D_{KL}(q\|q - \epsilon_l)D)$, we have

$$Q_q(\mathbf{x}_k, \{i_j^{(k+1)}\}_{j=1}^D) \geq \tilde{Q}_{\frac{q-\beta-\epsilon_l}{1-\beta}}(\mathbf{x}_k, B^c) \geq \tilde{Q}_{q-\beta-\epsilon_l}(\mathbf{x}_k) \quad (13)$$

Proof. For the lower bound of $Q_q(\mathbf{x}_k, \{i_j^{(k+1)}\}_{j=1}^D)$, it suffices that more than $(1-q)D$ indices in $\{i_j^{(k+1)}\}_{j=1}^D$ fall in the set

$$S_l := \left\{ i \in B^c : |\langle \mathbf{x}_k - \mathbf{x}^*, \mathbf{a}_i \rangle| \geq \tilde{Q}_{\frac{q-\beta-\epsilon_l}{1-\beta}}(\mathbf{x}_k, B^c) \right\},$$

which contains at least $(1-q+\epsilon_l+\beta)\frac{|B^c|}{1-\beta} - \beta m = (1-q+\epsilon_l)m$ indices. Let Y_j indicate whether $i_j^{(k+1)}$ belongs to these $(1-q+\epsilon_l)m$ indices, then $\sum_{j=1}^D Y_j \sim \text{Bin}(D, 1-q+\epsilon_l)$. By Lemma 1, we have

$$\mathbb{P}\left(\sum_{j=1}^D Y_j \leq (1-q)D\right) \leq \exp(-D_{KL}(1-q\|1-q+\epsilon_l)D), \quad D_{KL}(1-q\|1-q+\epsilon_l) = D_{KL}(q\|q-\epsilon_l) > 0.$$

Moreover, observe that $\tilde{Q}_{q-\beta-\epsilon_l}(\mathbf{x}_k) \leq \tilde{Q}_{\frac{q-\beta-\epsilon_l}{1-\beta}}(\mathbf{x}_k, B^c)$, since the $(q-\beta-\epsilon_l)m$ -th smallest ideal residual among all indices $[m]$ cannot exceed the $(q-\beta-\epsilon_l)m$ -th smallest among the uncorrupted set B^c .

For the upper bound, it suffices that at least qD indices in $\{i_j^{(k+1)}\}_{j=1}^D$ fall in

$$S_u := \left\{ i \in [m] : |\langle \mathbf{x}_k - \mathbf{x}^*, \mathbf{a}_i \rangle| \leq \tilde{Q}_{q+\beta+\epsilon_u}(\mathbf{x}_k) \right\},$$

which has cardinality at least $(q+\beta+\epsilon_u)m - \beta m = (q+\epsilon_u)m$. Let Y_j indicate whether $i_j^{(k+1)}$ belongs to these $(q+\epsilon_u)m$ indices, then $\sum_{j=1}^D Y_j \sim \text{Bin}(D, q+\epsilon_u)$. By Lemma 1, we have

$$\mathbb{P}\left(\sum_{j=1}^D Y_j \leq qD\right) \leq \exp(-D_{KL}(q\|q+\epsilon_u)D).$$

The proof is complete. \square

Our overall approach for proving (10) remains similar to prior works, in that the two central steps are controlling the impact of selecting a corrupted row and utilizing the expected contraction induced by accepting an uncorrupted row (due to the Strohmer-Vershynin bound). Nonetheless, each of these two steps involves new technicalities, as we present below.

The key to controlling the impact of the update sample being corrupted is an appropriate upper bound on the subsampled quantile. As mentioned, this does not hold arbitrarily, and we shall rely on the probabilistic upper bound (12) from Lemma 8. Focusing on the first T iterations, we advocate to work with a failure event Ω defined as (12) fails for some $k < T$, whose probability is bounded by a union bound; see Lemma 9.

Then we seek to bound $\mathbb{E}(\|\mathbf{X}_T - \mathbf{x}^*\|^2 \mathbf{1}_{\Omega^c})$. Due to the indicator function, it is unclear what per-iteration contraction we can use to obtain the desired bound. Specifically, previous works [7, 6, 15] showed $\mathbb{E}(\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2) \leq (1 - \frac{c}{n})\|\mathbf{X}_k - \mathbf{x}^*\|^2$ and iterated this bound. Unfortunately, this is no longer feasible in our subsampled setting.⁴ Our remedy is to define an event S_k for the k -th iteration in (15) and a stopping

⁴To see this, one may use Lemma 11–Lemma 12 to reach the lower bound

$$\mathbb{E}(\|\mathbf{X}_{T+1} - \mathbf{x}^*\|^2) \geq \left(\frac{\beta}{2}\right)^{D+1} \min_{i \in B} |\epsilon_i|^2$$

and then set the corruption large enough.

time τ in (16), and then relate these to Ω^c in (17). Further using some concepts from stochastic process theory [5], we observe that the one-step contraction bound on $\mathbb{E}[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{S_{k+1}}]$ in (18) can be iterated to achieve our goal; see Lemma 10. Similar arguments were employed in [17] to address a similar issue in phase retrieval—the error contraction was only established within a neighborhood of the true signal, yet the randomized Kaczmarz iterates do not reside in this neighborhood arbitrarily.

Therefore, it remains to establish (18) in which the contraction stems from an application of Strohmer-Vershynin bound to the case of accepting an uncorrupted row. Note that exactly qm rows are acceptable under full-sample quantile, the authors of [7] could easily find a set of acceptable uncorrupted rows of size $(q - 2\beta)m$ (called I_2 therein) and then apply Strohmer-Vershynin bound to the uncorrupted linear system constituted by the rows in I_2 . However, this becomes more intricate under our subsampled quantile, since it is much less clear which rows will be accepted. Our idea is to identify *a large enough uncorrupted subset of the acceptable rows*. To find such a subset, we first establish a lower bound on the subsampled quantile by (13) from Lemma 8 and then consider the uncorrupted rows with residuals smaller than such a lower bound; see the lower bound event in (31) and the set S in (32). We manage to ensure $|S| = \Omega(m)$ by carefully setting the relevant parameters.

3.2 Proof of Theorem 1

We define the failure event Ω as

$$\Omega := \{ \text{(12) fails for some } 0 \leq k \leq T-1 \}. \quad (14)$$

Applying a union bound over T iterations, we see that the probability of Ω can be made arbitrarily small by choosing D sufficiently large.

Lemma 9. *We have*

$$\mathbb{P}(\Omega) \leq T \exp(-D_{KL}(q\|q + \epsilon_u)D),$$

where ϵ_u is a constant as in Lemma 8.

Proof.

$$\begin{aligned} \mathbb{P}(\Omega) &= \mathbb{P}(\cup_{k=0}^{T-1} \{ \text{(12) fails at step } k \}) \\ &\leq \sum_{k=0}^{T-1} \mathbb{P}(\{ \text{(12) fails at step } k \}) \\ &\stackrel{(a)}{\leq} T \exp(-D_{KL}(q\|q + \epsilon_u)D), \end{aligned}$$

where (a) follows by Lemma 8. □

We define the event that (12) holds for step k as

$$S_{k+1} := \left\{ Q_q(\mathbf{X}_k, \{i_j^{(k+1)}\}_{j=1}^D) \leq \tilde{Q}_{q+\beta+\epsilon_u}(\mathbf{X}_k) \right\}, \quad (15)$$

as well as introduce the stopping time as

$$\tau := \inf_{t \geq 1} \left\{ t : Q_q(\mathbf{X}_{t-1}, \{i_j^{(t)}\}_{j=1}^D) > \tilde{Q}_{q+\beta+\epsilon_u}(\mathbf{X}_{t-1}) \right\}. \quad (16)$$

These conventions can further characterize Ω in (14) as

$$\{\tau > k+1\} = \{\tau > k, S_{k+1} \text{ holds}\} \subset \{\tau > k\} \quad \text{and} \quad \Omega^c = \{\tau > T\}. \quad (17)$$

These observations allow us to bound $\mathbb{E}(\|\mathbf{x}_T - \mathbf{x}^*\|^2 1_{\Omega^c})$ by induction. To formalize this, we introduce the standard filtration from stochastic processes [5]. For each k , let \mathcal{F}_k denote the σ -algebra capturing the history up to \mathbf{X}_k , generated by all update and quantile sample selections:

$$r_1, i_1^{(1)}, \dots, i_D^{(1)}; r_2, i_1^{(2)}, \dots, i_D^{(2)}; \dots; r_k, i_1^{(k)}, \dots, i_D^{(k)},$$

where these random variables are sampled uniformly from $\{1, \dots, m\}$. We then have:

Lemma 10. *If there exists a constant $c_2 > 0$ such that*

$$\mathbb{E}[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{S_{k+1}} \mid \mathbf{X}_k = \mathbf{x}_k] \leq \left(1 - \frac{c_2}{n}\right) \|\mathbf{x}_k - \mathbf{x}^*\|^2 \quad (18)$$

holds for all fixed \mathbf{x}_k and $0 \leq k \leq T - 1$, then we have

$$\mathbb{E}(\|\mathbf{X}_T - \mathbf{x}^*\|^2 1_{\Omega^c}) \leq \left(1 - \frac{c_2}{n}\right)^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2.$$

Proof. For any $0 \leq k \leq T - 1$, we have

$$\begin{aligned} & \mathbb{E}[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{\tau > k+1}] \\ & \stackrel{(a)}{=} \mathbb{E}[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{\tau > k} 1_{S_{k+1}}] \\ & \stackrel{(b)}{\leq} \mathbb{E}[\mathbb{E}[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{S_{k+1}} \mid \mathcal{F}_k] 1_{\tau > k}] \\ & \stackrel{(c)}{\leq} \mathbb{E}[\mathbb{E}[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{S_{k+1}} \mid \mathbf{X}_k] 1_{\tau > k}] \\ & \stackrel{(d)}{\leq} \left(1 - \frac{c_2}{n}\right) \mathbb{E}[\|\mathbf{X}_k - \mathbf{x}^*\|^2 1_{\tau > k}], \end{aligned} \quad (19)$$

where (a) follows from (17), (b) follows from the fact that $1_{\tau > k}$ is measurable with respect to \mathcal{F}_k , (c) follows from the Markov property of $\{\mathbf{X}_k\}$ (i.e., \mathbf{X}_{k+1} depends on \mathcal{F}_k only through \mathbf{X}_k), (d) follows from (18). Now further using $\Omega^c = \{\tau > T\}$, we can iterate (19) to arrive at

$$\begin{aligned} \mathbb{E}(\|\mathbf{X}_T - \mathbf{x}^*\|^2 1_{\Omega^c}) &= \mathbb{E}(\|\mathbf{X}_T - \mathbf{x}^*\|^2 1_{\tau > T}) \leq \left(1 - \frac{c_2}{n}\right) \mathbb{E}(\|\mathbf{X}_{T-1} - \mathbf{x}^*\|^2 1_{\tau > T-1}) \\ &\leq \left(1 - \frac{c_2}{n}\right)^2 \mathbb{E}(\|\mathbf{X}_{T-2} - \mathbf{x}^*\|^2 1_{\tau > T-2}) \leq \dots \leq \left(1 - \frac{c_2}{n}\right)^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2, \end{aligned}$$

as desired. \square

With the above preparations, we are in a position to prove [Theorem 1](#).

Proof of Theorem 1. Before proceeding to the main arguments, we pause to define some constants involved in our proof.

Defining the Constants: By [Assumption 1](#), it holds for $\alpha_0 = q/2$ that

$$\frac{\sigma_{\max}(\mathbf{A})\sqrt{n}}{\sqrt{m}} \leq C \quad \text{and} \quad \sigma_{\min, \alpha_0} \geq \frac{\sqrt{2\pi}(q/2)^{3/2}}{24} > 0, \quad (20)$$

where C is an absolute constant. Choose $\epsilon_l = q/4 \in (0, q - \alpha_0)$, and define

$$p_l := \exp(-D_{KL}(q \| q - \epsilon_l)D), \quad p_l^c := 1 - p_l \geq 1 - \exp(-D_{KL}(q \| q - \epsilon_l)) > 0, \quad (21)$$

where $D_{KL}(q\|q - \epsilon_l) = D_{KL}(q\|\frac{3}{4}q) > 0$. We assume

$$\beta < q - \alpha_0 - \epsilon_l = q/4, \quad (22)$$

and note that this leads to

$$\alpha := q - \beta - \epsilon_l > \alpha_0, \quad \sigma_{\min, \alpha} \geq \sigma_{\min, \alpha_0} > 0. \quad (23)$$

We also define

$$C_1 := \max \left\{ \frac{(\sigma_{\max}(\mathbf{A})\sqrt{n/m})^2}{\frac{1}{4}p_l^c \sigma_{\min, \alpha_0}^2}, \left(\frac{2\sigma_{\max}(\mathbf{A})\sqrt{n/m}}{\frac{1}{4}p_l^c \sigma_{\min, \alpha_0}^2} \right)^2 \right\} \leq C'_1 \quad \text{and} \quad \alpha' := C'_1 \beta, \quad (24)$$

where C'_1 can be a constant depending only on q from (20) and (21). Assume β is small enough such that

$$\beta < \frac{1 - q}{C'_1 + 1}, \quad (25)$$

then we have

$$\epsilon_u := 1 - q - \beta - \alpha' > 0, \quad \beta + \alpha' = (1 + C'_1)\beta = 1 - q - \epsilon_u < 1 - q. \quad (26)$$

Bounding $\mathbb{P}(\Omega)$: The failure event Ω , S_k , and τ can be defined based on the constants q, β, ϵ_u defined above, as in the beginning of Section 3.2. $\mathbb{P}(\Omega)$ can be upper bounded by Lemma 9 and by definition of $D_{KL}(q\|q + \epsilon_u)$:

$$\begin{aligned} \mathbb{P}(\Omega) &\leq T \exp(-D_{KL}(q\|q + \epsilon_u)D) \\ &\stackrel{(a)}{=} T \exp\left(-\left((1 - q) \log \frac{1 - q}{(1 + C'_1)\beta} + q \log \frac{q}{1 - (1 + C'_1)\beta}\right) D\right) \\ &\stackrel{(b)}{=} T \exp\left(-\left((1 - q) \log \frac{1}{\beta} - C_2\right) D\right) \\ &\stackrel{(c)}{\leq} T \exp\left(-\left(\frac{1}{2}(1 - q) \log \frac{1}{\beta}\right) D\right), \end{aligned}$$

where (a) uses (26), (b) defines $C_2 := -(1 - q) \log(\frac{1 - q}{1 + C'_1}) - q \log(\frac{q}{1 - (1 + C'_1)\beta})$, and (c) notes that for some $c_q > 0$, for all

$$\beta < c_q, \quad (27)$$

we have $(1 - q) \log \frac{1}{\beta} - C_2 > \frac{1}{2}(1 - q) \log \frac{1}{\beta} > 0$. Moreover, there exists a constant $C_1 = \frac{6}{(1 - q)/4} > 0$ such that if $D \geq C_1 \frac{\log T}{\log(1/\beta)}$ and $T > 1$, then

$$\mathbb{P}(\Omega) \leq T^{-5} \exp\left(-\frac{1 - q}{4} \log(1/\beta) \cdot D\right) \leq \frac{1}{2}. \quad (28)$$

Bounding $\mathbb{E}(\|\mathbf{X}_T - \mathbf{x}^*\|^2 1_{\Omega^c})$: By Lemma 10, it suffices to show a one-step contraction:

$$\mathbb{E}[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{S_{k+1}} \mid \mathbf{X}_k = \mathbf{x}_k] \leq \left(1 - \frac{c_2}{n}\right) \|\mathbf{x}_k - \mathbf{x}^*\|^2$$

for some $c_2 > 0$. We proceed by conditioning on whether the update sample r_{k+1} is corrupted:

$$\begin{aligned} &\mathbb{E}[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{S_{k+1}} \mid \mathbf{X}_k = \mathbf{x}_k] \\ &= \beta \mathbb{E}[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{S_{k+1}} \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in B] \\ &\quad + (1 - \beta) \mathbb{E}[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{S_{k+1}} \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in B^c] \\ &:= \beta I_1 + (1 - \beta) I_2. \end{aligned} \quad (29)$$

Case I: Corrupted Update Sample $r_{k+1} \in B$

With the aid of $1_{S_{k+1}}$, we can proceed as follows:

$$\begin{aligned}
I_1 &= \mathbb{E} \left[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{S_{k+1}} \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in B \right] \\
&= \frac{1}{\beta m^{D+1}} \sum_{i \in B} \sum_{\{(i')_j^{(k+1)}\}_{j=1}^D} \mathbb{E} \left[\|\mathbf{e}_{k+1}\|^2 1_{S_{k+1}} \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} = i, \{i_j^{(k+1)}\}_{j=1}^D = \{(i')_j^{(k+1)}\}_{j=1}^D \right] \\
&\stackrel{(a)}{\leq} \sum_{i \in B} \sum_{\{(i')_j^{(k+1)}\}_{j=1}^D} \left(\|\mathbf{e}_k\|^2 + \tilde{Q}_{q+\beta+\epsilon_u}(\mathbf{x}_k)^2 + 2\tilde{Q}_{q+\beta+\epsilon_u}(\mathbf{x}_k) |\langle \mathbf{e}_k, \mathbf{a}_i \rangle| \right) \frac{1}{\beta m^{D+1}} \\
&\stackrel{(b)}{\leq} \sum_{i \in B} \left(\|\mathbf{e}_k\|^2 + \frac{(\Phi_{1-\alpha'})^2}{n} \|\mathbf{e}_k\|^2 + \frac{2(\Phi_{1-\alpha'})}{\sqrt{n}} \|\mathbf{e}_k\| |\langle \mathbf{a}_i, \mathbf{e}_k \rangle| \right) \frac{1}{\beta m} \\
&\stackrel{(c)}{\leq} \left(1 + \frac{\Phi_{1-\alpha'}^2 + 2\Phi_{1-\alpha'} c_B}{n} \right) \|\mathbf{e}_k\|^2, \tag{30}
\end{aligned}$$

where the sum $\sum_{\{(i')_j^{(k+1)}\}_{j=1}^D}$ is over all possible quantile subsamples of size D from $[m]$ (i.e., m^D terms).

Step (a) uses the definition of S_{k+1} , which ensures the subsampled quantile is upper-bounded by $\tilde{Q}_{q+\beta+\epsilon_u}(\mathbf{x}_k)$, allowing application of [Lemma 2](#). Step (b) applies [Lemma 4](#) with $q' = q + \beta + \epsilon_u = 1 - \alpha'$ and $\Phi_{1-\alpha'} = \sigma_{\max}(\mathbf{A})\sqrt{n/m}/\sqrt{\alpha'}$. Step (c) uses the bound $\frac{1}{\beta m} \sum_{i \in B} |\langle \mathbf{e}_k, \mathbf{a}_i \rangle| \leq \frac{\sigma_{\max}(\mathbf{A})\sqrt{n/m}}{\sqrt{n\beta}} \|\mathbf{e}_k\|$ (by [Lemma 5](#)), and defines $c_B := \sigma_{\max}(\mathbf{A})\sqrt{n/m}/\sqrt{\beta}$.

Case II: Uncorrupted Update Sample $r_{k+1} \in B^c$

In this case, the error is non-expansive. To ensure contraction, we need to identify *a large enough uncorrupted subset of the acceptable rows* for applying [Lemma 6](#). According to [Algorithm 1](#), acceptable rows have residuals smaller than the subsampled quantile. To facilitate analysis, we define the event S_{k+1}^* , where the subsampled quantile exceeds $\tilde{Q}_{\frac{q-\beta-\epsilon_l}{1-\beta}}(\mathbf{x}_k, B^c)$. By the lower bound in [Lemma 8](#), this event occurs with probability at least $1 - p_l$ (p_l defined in [\(21\)](#)):

$$S_{k+1}^* := \left\{ \{i_j^{(k+1)}\}_{j=1}^D : Q_q(\mathbf{x}_k, \{i_j^{(k+1)}\}_{j=1}^D) \geq \tilde{Q}_{\frac{q-\beta-\epsilon_l}{1-\beta}}(\mathbf{x}_k, B^c) \right\}, \quad \mathbb{P}((S_{k+1}^*)^c) \leq p_l. \tag{31}$$

And we define S as the set of indices in B^c that are guaranteed to be accepted when S_{k+1}^* holds:

$$S := \left\{ i \in B^c : |\langle \mathbf{x}_k - \mathbf{x}^*, \mathbf{a}_i \rangle| \leq \tilde{Q}_{\frac{q-\beta-\epsilon_l}{1-\beta}}(\mathbf{x}_k, B^c) \right\}, \tag{32}$$

where $|S| = (q - \beta - \epsilon_l)m = \alpha m \geq \alpha_0 m = qm/2$ (assumed to be integer for simplicity). Its complement, $S^c = B^c \setminus S$, consists of uncorrupted indices with large residuals. Thus, the expectation can be decomposed

as follows:

$$\begin{aligned}
I_2 &= \mathbb{E} \left[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{S_{k+1}} \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in B^c \right] \\
&\leq \mathbb{E} \left[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in B^c \right] \\
&= \mathbb{E} \left[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S \right] \frac{\alpha}{1-\beta} \\
&\quad + \mathbb{E} \left[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S^c \right] \left(1 - \frac{\alpha}{1-\beta} \right), \\
&\stackrel{(a)}{\leq} \mathbb{E} \left[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S \right] + \|\mathbf{e}_k\|^2 \left(1 - \frac{\alpha}{1-\beta} \right), \tag{33}
\end{aligned}$$

where (a) uses that the error is non-expansive for $r_{k+1} \in S^c \subset B^c$. Then we have:

$$\begin{aligned}
&\mathbb{E} \left[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S \right] \\
&\stackrel{(a)}{=} \mathbb{E} \left[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S, \{i_j^{(k+1)}\}_{j=1}^D \in S_{k+1}^* \right] \mathbb{P}(S_{k+1}^*) \\
&\quad + \mathbb{E} \left[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S, \{i_j^{(k+1)}\}_{j=1}^D \in (S_{k+1}^*)^c \right] \mathbb{P}((S_{k+1}^*)^c) \\
&\stackrel{(b)}{\leq} \mathbb{E} \left[\|\mathbf{e}_k - \langle \mathbf{e}_k, \mathbf{a}_{r_{k+1}} \rangle \mathbf{a}_{r_{k+1}}\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S \right] \mathbb{P}(S_{k+1}^*) \\
&\quad + \mathbb{E} \left[\|\mathbf{e}_k\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S \right] \mathbb{P}((S_{k+1}^*)^c) \\
&\stackrel{(c)}{\leq} \mathbb{E} \left[(1 - p_l) \|\mathbf{e}_k - \langle \mathbf{e}_k, \mathbf{a}_{r_{k+1}} \rangle \mathbf{a}_{r_{k+1}}\|^2 + p_l \|\mathbf{e}_k\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S \right],
\end{aligned}$$

where (a) uses the independence of $\{i_j^{(k+1)}\}_{j=1}^D$ and r_{k+1} . In (b), when $r_{k+1} \in S$ and S_{k+1}^* holds, the update is accepted and the error becomes $\|\mathbf{e}_{k+1}\|^2 = \|\mathbf{e}_k - \langle \mathbf{e}_k, \mathbf{a}_{r_{k+1}} \rangle \mathbf{a}_{r_{k+1}}\|^2$; otherwise, the error is non-expansive since $r_{k+1} \in B^c$. In (c), we use (31) and $\|\mathbf{e}_k - \langle \mathbf{e}_k, \mathbf{a}_{r_{k+1}} \rangle \mathbf{a}_{r_{k+1}}\|^2 \leq \|\mathbf{e}_k\|^2$. Substituting the final estimation into (33) yields:

$$\begin{aligned}
I_2 &\leq \mathbb{E} \left[\|\mathbf{e}_k - \langle \mathbf{e}_k, \mathbf{a}_{r_{k+1}} \rangle \mathbf{a}_{r_{k+1}}\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S \right] \frac{\alpha}{1-\beta} p_l^c \\
&\quad + \mathbb{E} \left[\|\mathbf{e}_k\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S \right] \frac{\alpha}{1-\beta} p_l \\
&\quad + \mathbb{E} \left[\|\mathbf{e}_k\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S^c \right] \left(1 - \frac{\alpha}{1-\beta} \right) \\
&\leq \left(1 - \frac{\alpha}{1-\beta} p_l^c \right) \|\mathbf{e}_k\|^2 + \underbrace{\mathbb{E} \left[\|\mathbf{e}_k - \langle \mathbf{e}_k, \mathbf{a}_{r_{k+1}} \rangle \mathbf{a}_{r_{k+1}}\|^2 \mid \mathbf{X}_k = \mathbf{x}_k, r_{k+1} \in S \right]}_{\leq \left(1 - \frac{\sigma_{\min}^2(\mathbf{A}_S)}{\|\mathbf{A}_S\|_F^2} \right) \|\mathbf{e}_k\|^2} \frac{\alpha}{1-\beta} p_l^c,
\end{aligned}$$

where we apply the Strohmer-Vershynin bound in Lemma 6 to the submatrix \mathbf{A}_S (with $|S| = \alpha m$) to estimate the contraction in the last term. By definition (8) and the lower bound in (23), we have $\sigma_{\min}(\mathbf{A}_S) \geq$

$\sigma_{\min, \alpha} \sqrt{\frac{m}{n}} \geq \sigma_{\min, \alpha_0} \sqrt{\frac{m}{n}} > 0$. Therefore,

$$\begin{aligned}
I_2 &\leq \left(1 - \frac{\alpha}{1-\beta} p_l^c\right) \|\mathbf{e}_k\|^2 + \frac{\alpha}{1-\beta} p_l^c \left(1 - \frac{\sigma_{\min, \alpha}^2}{\alpha} \frac{1}{n}\right) \|\mathbf{e}_k\|^2 \\
&\leq \left(1 - p_l^c \frac{\alpha}{1-\beta} \frac{\sigma_{\min, \alpha}^2}{\alpha} \frac{1}{n}\right) \|\mathbf{e}_k\|^2 \\
&= \left(1 - p_l^c \frac{\sigma_{\min, \alpha}^2}{1-\beta} \frac{1}{n}\right) \|\mathbf{e}_k\|^2.
\end{aligned} \tag{34}$$

Combing Cases I and II: Substituting (30) and (34) into (29), we have

$$\begin{aligned}
&\mathbb{E} \left[\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 1_{S_{k+1}} \mid \mathbf{X}_k = \mathbf{x}_k \right] \\
&\leq \beta \left(\|\mathbf{e}_k\|^2 + (\Phi_{1-\alpha'}^2 + 2\Phi_{1-\alpha'} c_B) \frac{\|\mathbf{e}_k\|^2}{n} \right) + (1-\beta) \left(1 - p_l^c \frac{\sigma_{\min, \alpha}^2}{1-\beta} \frac{1}{n} \right) \|\mathbf{e}_k\|^2 \\
&= \left(1 - \frac{p_l^c \sigma_{\min, \alpha}^2 - \beta (\Phi_{1-\alpha'}^2 + 2\Phi_{1-\alpha'} c_B)}{n} \right) \|\mathbf{e}_k\|^2
\end{aligned}$$

Combining the upper bounds for β from (22), (25) and (27), define $c_1 := \min\{q/4, \frac{1-q}{C_1'+1}, c_q\}$. Let $c_2 := \frac{1}{2}(1 - \exp(-D_{KL}(q\|\frac{3}{4}q))\sigma_{\min, \alpha_0}^2) > 0$. Both c_1 and c_2 are positive constants depending only on q . For $\beta \in (0, c_1)$, the following lower bound holds:

$$p_l^c \sigma_{\min, \alpha}^2 - \beta (\Phi_{1-\alpha'}^2 + 2\Phi_{1-\alpha'} c_B) > p_l^c \sigma_{\min, \alpha}^2 - p_l^c \frac{1}{2} \sigma_{\min, \alpha_0}^2 \geq c_2 > 0,$$

where the first inequality follows from the definition of C_1' and α' in (24), and the second inequality follows from (21) and (23). This finishes the proof of (10) by Lemma 10, and the proof of (11) follows from the upper bound on the failure probability that we established in (28). \square

3.3 Lower Bound

We assume the following on **A**. With high probability, it is satisfied by **A** with rows being i.i.d. uniformly distributed over \mathbb{S}^{n-1} when $m \geq C_1 n$ for large enough C_1 ; see Lemma 7.

Assumption 2. **A** has rows in \mathbb{S}^{n-1} and satisfies

$$\Phi_{q'} = \frac{\sigma_{\max}(\mathbf{A})\sqrt{n}}{\sqrt{m}\sqrt{1-q'}} \leq \frac{C_0}{\sqrt{1-q'}}, \quad \forall q' \in (0, 1) \tag{35}$$

for some absolute constant C_0 .

We now provide the formal theorem regarding the lower bound on D .

Theorem 2. Consider problem (1) with arbitrary (βm) -sparse ϵ and suppose Assumption 2 holds. Let $q \in (0, 1)$, and the number of iterations T be such that $T \geq \lceil \frac{n}{\log n} \rceil$. For any $c_0 \in (0, 1)$, if the subsample size D is a positive integer satisfying

$$D \leq \max \left\{ \frac{c_0 \log T}{\log(2/\beta)}, 1 \right\},$$

then with probability at least $1 - \frac{C_1}{\log n} - \exp(-\frac{\beta^2 n}{4 \log n}) - \exp(-\frac{\beta}{2} \frac{n}{T^{c_0 \log n}})$ we have

$$\|\mathbf{X}_T - \mathbf{x}^*\|^2 \geq \left(\frac{1}{2}\right)^{\lfloor \frac{n}{\log n} \rfloor} \min_{i \in B} |\epsilon_i|^2.$$

As a consequence, under large enough $\min_{i \in B} |\epsilon_i|$, $\|\mathbf{X}_T - \mathbf{x}^*\|^2$ can be made arbitrarily large with $1 - o(1)$ probability as long as

$$\frac{\log n}{\beta^2 n} = o(1) \quad \text{and} \quad \frac{T^{c_0} \log n}{\beta n} = o(1). \quad (36)$$

Note that such scaling assumptions are mild when $c_0 \in (0, 1)$ is small. For instance, if β is a given positive constant, then ((36)) holds as long as $T \leq n^\xi$ for some $\xi < c_0^{-1}$, where c_0^{-1} is a large constant. It also encompasses some settings with $\beta = o(1)$, with a specific example being

$$\beta = \Theta(n^{-\xi}) \quad \text{and} \quad T \leq n^{\xi_1}$$

for some positive ξ, ξ_1 satisfying $\xi < \frac{1}{2}$ and $\xi_1 < \frac{1-\xi}{c_0}$. In addition, when $\frac{\log T}{\log(2/\beta)}$ is small enough, our result ensures the failure of QRK under $D = 1$ (recall that when $qD < 1$ we will let the subsampled quantile be the smallest residual in the subsample).

3.4 Proof of Theorem 2

The approach to proving Theorem 2 is first to identify a step at which the iterate is projected onto a corrupted row (we will use the last one over the first T iterations) and then demonstrate that the subsequent iterations cannot significantly reduce the approximation error. The following lemma describes the minimum impact of projecting onto a corrupted row.

Lemma 11. *If \mathbf{X}_{k+1} is obtained by projecting \mathbf{x}_k onto a corrupted row, which is equivalent to*

$$r_{k+1} \in B \quad \text{and} \quad |\langle \mathbf{a}_{r_{k+1}}, \mathbf{x}_k \rangle - b_{r_{k+1}}| \leq Q \left(\mathbf{x}_k, \{i_j^{(k+1)}\}_{j=1}^D \right), \quad (37)$$

then:

$$\|\mathbf{X}_{k+1} - \mathbf{x}^*\| \geq \min_{i \in B} |\epsilon_i|. \quad (38)$$

Proof. Under the condition, \mathbf{X}_{k+1} is projected onto the corrupted row with $\langle \mathbf{a}_{r_{k+1}}, \mathbf{X}_{k+1} \rangle = b_{r_{k+1}}$, and we have

$$\|\mathbf{X}_{k+1} - \mathbf{x}^*\| \geq |\langle \mathbf{a}_{r_{k+1}}, \mathbf{X}_{k+1} - \mathbf{x}^* \rangle| = |\epsilon_{r_{k+1}}| \geq \min_{i \in B} |\epsilon_i|,$$

as desired. \square

Further, such event occurs with probability lower bounded as follows.

Lemma 12. *Given \mathbf{x}_k , the event in (38) occurs with probability at least $(\frac{\beta}{2})^{D+1}$.*

Proof. By Lemma 11, it suffices to show that (37) occurs with probability at least $(\frac{\beta}{2})^{D+1}$. Given \mathbf{x}_k , we can always find $B_1, B_2 \subset B$ such that $|B_1| \geq \frac{\beta m}{2}$, $|B_2| \geq \frac{\beta m}{2}$, and the residuals in B_2 are uniformly larger than those in B_1 :

$$|\langle \mathbf{a}_{i_1}, \mathbf{x}_k \rangle - b_{i_1}| \leq |\langle \mathbf{a}_{i_2}, \mathbf{x}_k \rangle - b_{i_2}|, \quad \forall i_1 \in B_1, i_2 \in B_2.$$

A sufficient condition for (37) is that the update index r_{k+1} is selected from B_1 , and all indices in the quantile subsample $\{i_j^{(k+1)}\}_{j=1}^D$ are from B_2 . Since r_{k+1} and $\{i_j^{(k+1)}\}_{j=1}^D$ are chosen independently from $\text{Uniform}(1, \dots, m)$, this occurs with probability at least $(\frac{\beta}{2})^{D+1}$. \square

After (38) happens, the error may decrease in subsequent iterations when uncorrupted rows are accepted. Hence, we need to study the overall error reduction in these iterations. This amounts to analyzing the performance limit of standard RK, and we note that a lower bound $\mathbb{E}\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \geq (1 - \frac{c}{n})\mathbb{E}\|\mathbf{X}_k - \mathbf{x}^*\|^2$ was established in [16, Thm. 3]; see also [14]. However, this does not immediately transfer to a probabilistic statement. To that end, we introduce an additional parameter κ and develop the following lemma.

Lemma 13. *Given \mathbf{x}_k and assume that \mathbf{X}_{k+1} is obtained from \mathbf{x}_k by projecting onto an uncorrupted row. For $\kappa \in (0, 1)$, the error satisfies*

$$\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \geq \left(1 - \frac{C}{\kappa n}\right) \|\mathbf{e}_k\|^2, \quad (39)$$

with probability at least $1 - \kappa$, where $C > 0$ is an absolute constant.

Proof. Since \mathbf{X}_{k+1} is obtained from projecting \mathbf{x}_k onto an uncorrupted row, we have

$$\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 = \|\mathbf{e}_k - \langle \mathbf{a}_{r_{k+1}}, \mathbf{e}_k \rangle \mathbf{a}_{r_{k+1}}\|^2 = \|\mathbf{e}_k\|^2 - |\langle \mathbf{a}_{r_{k+1}}, \mathbf{e}_k \rangle|^2,$$

i.e., that the reduction in error at each step depends on $|\langle \mathbf{a}_{r_{k+1}}, \mathbf{e}_k \rangle|$. For any $\kappa > 0$, the probability of the event $\{|\langle \mathbf{a}_{r_{k+1}}, \mathbf{e}_k \rangle| \leq \tilde{Q}_{1-\kappa}(\mathbf{x}_k)\}$ is $1 - \kappa$, and we assume such event holds. Therefore, we have

$$\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \geq \|\mathbf{e}_k\|^2 - \tilde{Q}_{1-\kappa}(\mathbf{X}_k)^2 \stackrel{(a)}{\geq} \left(1 - \frac{\Phi_{1-\kappa}^2}{n}\right) \|\mathbf{e}_k\|^2 \stackrel{(b)}{\geq} \left(1 - \frac{C}{\kappa n}\right) \|\mathbf{e}_k\|^2, \quad (40)$$

where (a) follows from Lemma 4 and (b) follows from (35) in Assumption 2. \square

We are now ready to prove Theorem 2.

Proof of Theorem 2. We let k^* be the largest $k \leq T - 1$ such that \mathbf{X}_{k^*+1} is obtained from \mathbf{X}_{k^*} by projecting onto some corrupted row. We first show that $k^* \geq T - \lceil \frac{n}{\log n} \rceil$ holds with high probability. To see this, we notice that $k^* < T - \lceil \frac{n}{\log n} \rceil$ implies that the last $\lceil \frac{n}{\log n} \rceil$ iterations (among the T iterations) are not projections onto corrupted row, hence by Lemma 12 we have

$$\mathbb{P}\left(k^* < T - \left\lceil \frac{n}{\log n} \right\rceil\right) \leq \left(1 - \left(\frac{\beta}{2}\right)^{D+1}\right)^{\frac{n}{\log n}} \leq \exp\left(-\left(\frac{\beta}{2}\right)^{D+1} \frac{n}{\log n}\right).$$

Further using $D \leq \max\{\frac{c_0 \log T}{\log(2/\beta)}, 1\}$, which gives $(\frac{\beta}{2})^D \geq \min\{T^{-c_0}, \frac{\beta}{2}\}$, we reach

$$\mathbb{P}\left(k^* < T - \left\lceil \frac{n}{\log n} \right\rceil\right) \leq \exp\left(-\frac{\beta^2 n}{4 \log n}\right) + \exp\left(-\frac{\beta}{2} \frac{n}{T^{c_0} \log n}\right).$$

Hence, we can proceed on the event $k^* \geq T - \lceil \frac{n}{\log n} \rceil$ by ruling out the probability of $\exp(-\frac{\beta^2 n}{4 \log n}) + \exp(-\frac{\beta}{2} \frac{n}{T^{c_0} \log n})$. By Lemma 11, the error of \mathbf{X}_{k^*+1} is lower bounded by $\|\mathbf{X}_{k^*+1} - \mathbf{x}^*\|^2 \geq \min_{i \in B} |\epsilon_i|^2$.

We further show the subsequent steps can not reduce such error too much. We set $\kappa = \frac{2C}{n}$ in Lemma 13 to yield that for any k , if \mathbf{X}_{k+1} is obtained from \mathbf{x}_k by projecting onto an uncorrupted row, the error satisfies $\|\mathbf{X}_{k+1} - \mathbf{x}^*\|^2 \geq \frac{1}{2} \|\mathbf{x}_k - \mathbf{x}^*\|^2$ with probability at least $1 - \frac{2C}{n}$ for some absolute constant C . Starting from \mathbf{X}_{k^*+1} , we will run no more than $T - k^* - 1 \leq \lfloor \frac{n}{\log n} \rfloor$ iterations to obtain \mathbf{X}_T . By the definition of k^* , none of these iterations performs a projection onto a corrupted row. Hence, these iterations either do not change the iterate or perform projection onto an uncorrupted row. We only need to consider the projections

onto uncorrupted rows that are “nontrivial,” and the number of such iterations is bounded by $\lfloor \frac{n}{\log n} \rfloor$. By a union bound, with probability at least $1 - \frac{2C}{n} \cdot \lfloor \frac{n}{\log n} \rfloor \geq 1 - \frac{2C}{\log n}$, each of these projections (onto some uncorrupted row) can not reduce the squared ℓ_2 error by a factor beyond $\frac{1}{2}$. Overall, we will have

$$\|\mathbf{X}_T - \mathbf{x}^*\|^2 \geq \left(\frac{1}{2}\right)^{\lfloor \frac{n}{\log n} \rfloor} \|\mathbf{X}_{k^*+1} - \mathbf{x}^*\|^2 \geq \left(\frac{1}{2}\right)^{\lfloor \frac{n}{\log n} \rfloor} \min_{i \in B} |\epsilon_i|^2,$$

with probability at least $1 - \exp(-\frac{\beta^2 n}{4 \log n}) - \exp(-\frac{\beta}{2} \frac{n}{T^{c_0} \log n}) - \frac{2C}{\log n}$. \square

4 Numerical Simulations

The experiments presented in this section were conducted using MATLAB R2022b on an iMac (2023) featuring an Apple M3 chip (8-core CPU) and 24 GB of unified memory. In our simulations, we generate the rows of the matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m = 50000$ and $n = 100$ as i.i.d. samples uniformly distributed on the unit sphere \mathbb{S}^{n-1} . The target vector $\mathbf{x}^* \in \mathbb{R}^n$ is also sampled uniformly from \mathbb{S}^{n-1} , while the initial iterate \mathbf{x}_0 is set to the zero vector. For a given corruption level $\beta \in (0, 1)$, the corruption vector ϵ is generated to corrupt the first βm measurements and its nonzero entries are i.i.d. sampled uniformly from the interval $[-5, 5]$. All experiments are run for $200n = 20000$ iterations with $q = 0.5$. The error $\|\mathbf{X}_k - \mathbf{x}^*\|$ is recorded at each iteration k and averaged over 10 independent trials. The MATLAB codes for reproducing our simulations are available online.⁵

4.1 Convergence under $D = \log(T)$, αm , m

We fix $\beta = 0.01$ and compare approximation error versus iteration and runtime for subsample sizes $D \in \{4, 40, 5000, 50000\}$ in Figure 1. Here, $D \in \{4, 40\}$ simulates the logarithmically small subsample size in our theory, while $D = 5000$ simulates the $\Theta(m)$ subsampled size in [6], and $D = 50000$ simulates the full-sample quantiles in [7]. Across all D , the contraction rates per iteration are similar, but the computational cost per iteration grows with D , and hence regarding the runtime QRK with $D = 5000, 50000$ is much slower. While our Algorithm 1 draws the quantile subsample with replacement (as with [7]), we also tested QRK that draws the subsample without replacement (as adopted by [6]). We find that both versions have similar numerical performance; see Figure 1 (a).

4.2 Dependence of D on (T, β)

We numerically simulate the impact of T, β on the subsample size. To this end, we plot the “error versus iteration” curves for varying $(\beta, D) \in \{0.01, 0.06, 0.11\} \times \{4, 8, 12\}$. When \mathbf{x}_{k+1} is projected onto a corrupted row, the error can significantly increase and this leads to “jumps” in the curves. While we use corruption bounded by 5 in our experiment, in general, such jump can ruin the entire estimation procedure when the corruption is set arbitrarily large (see Lemma 11). Therefore, *one has to avoid the occurrence of such jump over the desired number of iterations*. In light of this, our experimental results in Figure 2 lead to the following conclusions that are consistent with our theory:

- Under a fixed $T = 20000$, we compare Figure 2 (a)–(c) and find that increasing the corruption level β necessitates a larger subsample size D to avoid the occurrence of a jump in error before the completion of T iterations. For instance, under $D = 8$, the error decreases over the first T iterations when $\beta = 0.06$, while a jump occurs when $\beta = 0.11$, and we have to use larger D (such as $D = 12$) to avoid the jump. This corroborates the $\frac{1}{\log(1/\beta)}$ dependence of D on β .

⁵<https://github.com/wtree101/matlab-rk-analysis>

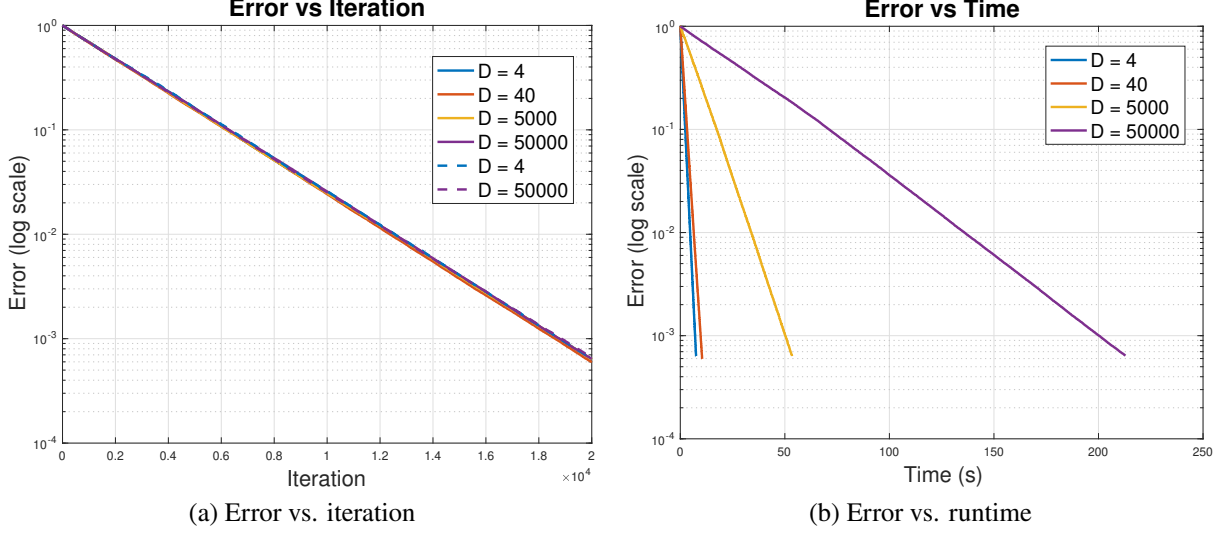


Figure 1: Empirical convergence of QRK for $\beta = 0.01$ and $q = 0.5$ with varying D . (a) Error vs. iteration; (b) Error vs. runtime. Solid lines: subsampling with replacement; dashed lines: without replacement.

- Under a fixed corruption level β , we compare the curves within Figure 2 (b) or (c) and find that increasing T will require larger D . Specifically, in Figure 2 (c) with $\beta = 0.11$, the error curve of $D = 8$ decreases over the first $T_1 = 10000$ and then encounters the first jump, while the curve of $D = 12$ decreases over the $T = 20000$ iterations. This is consistent with the $\log(T)$ dependence of D on T .

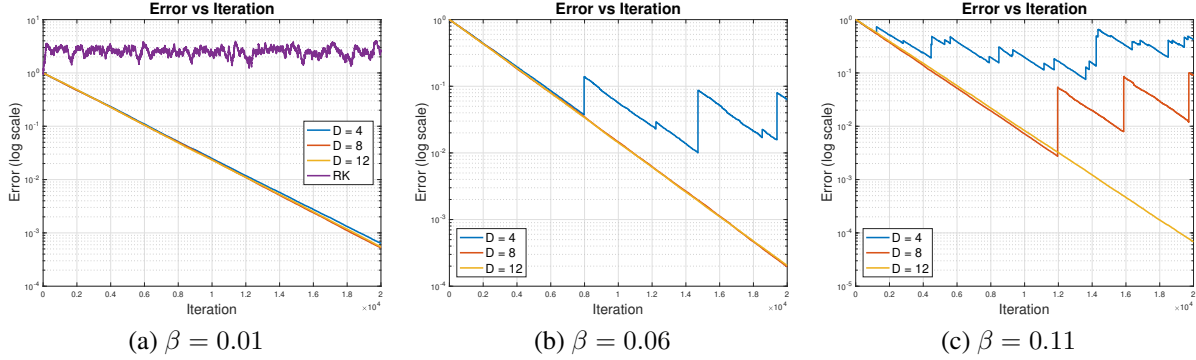


Figure 2: Error v.s. iteration for QRK under different corruption levels β and subsample sizes D .

5 Concluding Remarks

In this work, we provided a theoretical analysis on the subsample size of the quantile-based randomized Kaczmarz (QRK) method. Our main contribution is to identify (the order of) the minimal subsample size D required for QRK to converge over T iterations. In particular, we show that $D = \Theta\left(\frac{\log T}{\log(1/\beta)}\right)$ is both sufficient and necessary, and note that such a subsample size is typically a massive reduction on the previously known $D = \Theta(m)$. Our work thus bridges the gap between the previous theoretical guarantees for QRK, which require full-sample quantiles, and its practical implementations, which only use a very small subsample to

compute the quantiles. Numerical experiments corroborate our theoretical findings and confirm that a fairly small subsample size is sufficient for the applications of QRK to solving large corrupted linear systems.

In order to prove our results, we introduced a number of new technical ingredients. These include a two-sided probabilistic bounds on the subsampled quantiles, a stopping time to control $\mathbb{E}(\|\mathbf{X}_T - \mathbf{x}^*\|^2 1_{\Omega^c})$, and a probabilistic statement on the performance limit of standard RK. We believe that these techniques are of interests to the analysis of (Q)RK and other randomized algorithms. In fact, since its introduction [7], QRK has attracted a lot of subsequent research interests such as block QRK [3, 13], reverse QRK [1], subspace constrained QRK [10], sparse QRK [19], QRK for corrupted tensor linear system [2, 11], QRK under time-varying noise and corruption [4], among others. Note that all of these works require quantiles computed from the full sample, and we believe our techniques can be similarly used to substantially reduce the subsample size in these QRK variants. We leave these promising directions for future work.

In addition to these future directions, we note that while the order of the minimal subsample size was found in our work, multiplicative constants remain unspecified. Another research direction is to derive explicit constants for the upper bound and lower bound on D . Particularly, for a fixed small enough $\beta > 0$, it is of great interest to find explicit constants $c_* < C^*$ which are ideally close, such that

$$\text{QRK with } D \geq C^* \log T \text{ converges over the first } T \text{ iterations, with high probability;} \quad (41)$$

$$\text{QRK with } D \leq c_* \log T \text{ returns a bad estimate } \mathbf{x}_T, \text{ with high probability.} \quad (42)$$

A further question will be to explore a potential phase transition phenomenon in QRK, that is, whether there exists a pair (c_*, C^*) satisfying $\frac{C^*}{c_*} = 1 + o(1)$ such that (41)–(42) hold.

References

- [1] Emeric Battaglia and Anna Ma. Reverse quantile-rk and its application to quantile-rk. *Numerical Linear Algebra with Applications*, 32(3):e70024, 2025.
- [2] Alejandra Castillo, Jamie Haddock, Iryna Hartsock, Paulina Hoyos, Lara Kassab, Alona Kryshchenko, Kamila Larripa, Deanna Needell, Shambhavi Suryanarayanan, and Karamatou Yacoubou Djima. Quantile-based randomized kaczmarz for corrupted tensor linear systems. *arXiv preprint arXiv:2503.18190*, 2025.
- [3] Lu Cheng, Benjamin Jarman, Deanna Needell, and Elizaveta Rebrova. On block accelerations of quantile randomized kaczmarz for corrupted systems of linear equations. *Inverse Problems*, 39(2):024002, 2022.
- [4] Nestor Coria, Jamie Haddock, and Jaime Pacheco. On quantile randomized kaczmarz for linear systems with time-varying noise and corruption. *arXiv preprint arXiv:2403.19874*, 2024.
- [5] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.
- [6] Jamie Haddock, Anna Ma, and Elizaveta Rebrova. On subsampled quantile randomized kaczmarz. In *2023 59th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 1–8. IEEE, 2023.
- [7] Jamie Haddock, Deanna Needell, Elizaveta Rebrova, and William Swartworth. Quantile-Based Iterative Methods for Corrupted Systems of Linear Equations. *SIAM Journal on Matrix Analysis and Applications*, 43(2):605–637, June 2022.

- [8] Gabor T Herman and Lorraine B Meyer. Algebraic reconstruction techniques can be made computationally efficient (positron emission tomography application). *IEEE transactions on medical imaging*, 12(3):600–609, 1993.
- [9] Stefan Kaczmarz. Angenaherte auflösung von systemen linearer gleichungen. *Bull. Int. Acad. Pol. Sic. Let., Cl. Sci. Math. Nat.*, pages 355–357, 1937.
- [10] Jackie Lok and Elizaveta Rebrova. A subspace constrained randomized kaczmarz method for structure or external knowledge exploitation. *Linear Algebra and its Applications*, 698:220–260, 2024.
- [11] Anna Ma and Denali Molitor. Randomized kaczmarz for tensor linear systems. *BIT Numerical Mathematics*, 62(1):171–194, 2022.
- [12] Deanna Needell. Randomized kaczmarz solver for noisy linear systems. *BIT Numerical Mathematics*, 50:395–403, 2010.
- [13] Deanna Needell and Joel A Tropp. Paved with good intentions: analysis of a randomized block kaczmarz method. *Linear Algebra and its Applications*, 441:199–221, 2014.
- [14] Stefan Steinerberger. Randomized kaczmarz converges along small singular vectors. *SIAM Journal on Matrix Analysis and Applications*, 42(2):608–615, 2021.
- [15] Stefan Steinerberger. Quantile-based random kaczmarz for corrupted linear systems of equations. *Information and Inference: A Journal of the IMA*, 12(1):448–465, 2023.
- [16] Thomas Strohmer and Roman Vershynin. A randomized kaczmarz algorithm with exponential convergence. *Journal of Fourier Analysis and Applications*, 15(2):262–278, 2009.
- [17] Yan Shuo Tan and Roman Vershynin. Phase retrieval via randomized Kaczmarz: theoretical guarantees. *Information and Inference: A Journal of the IMA*, 8(1):97–123, March 2019.
- [18] Chuang Wang, Ameya Agaskar, and Yue M Lu. Randomized kaczmarz algorithm for inconsistent linear systems: An exact mse analysis. In *2015 International Conference on Sampling Theory and Applications (SampTA)*, pages 498–502. IEEE, 2015.
- [19] Lu Zhang, Hongxia Wang, and Hui Zhang. Quantile-based random sparse kaczmarz for corrupted and noisy linear systems. *Numerical Algorithms*, 98(3):1575–1610, 2025.
- [20] Anastasios Zouzias and Nikolaos M Freris. Randomized extended kaczmarz for solving least squares. *SIAM Journal on Matrix Analysis and Applications*, 34(2):773–793, 2013.