Exact Thresholds in Noisy Non-Adaptive Group Testing

Junren Chen
The University of Hong Kong

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Joint work with Jonathan Scarlett (NUS)

I. Problem Setup

(Noisy) Group Testing

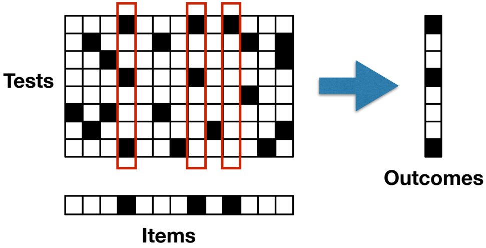

Figure: Noiseless GT

In this talk, we consider probabilistic group testing:

- Defective set $S \sim \mathrm{Uniform}\binom{p}{k}$ (i.e., k out of p items with a uniform prior)
- Non-adaptive: the test design $\mathbf{X} = (X_{ij}) \in \{0,1\}^{n \times p}$ is fixed before observing any outcome
- Noiseless:

$$Y_i = \bigvee_{j \in S} X_{ij} \tag{1}$$

- Noisy:

$$Y_i = \Big(\bigvee_{j \in S} X_{ij}\Big) \oplus Z_i \tag{2}$$

with $Z \sim \mathrm{Bernoulli}(\rho)$ for some noise level $\rho \in (0, \frac{1}{2})$

Recovery Criteria

- We consider two popular random designs:
  - Bernoulli design: $X_{ij} \overset{iid}{\sim} \mathrm{Bernoulli}(\frac{\nu}{k})$; each item is independently placed in each test with probability $\frac{\nu}{k}$ for some $\nu > 0$
  - Near-constant weight design: each item is independently placed in $\Delta = \frac{\nu n}{k}$ tests chosen uniformly at random with replacement for some $\nu > 0$

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- Given a decoder $\widehat{S}$, we define error probability as

$$P_{\mathrm{e}} := \mathbb{P}[\widehat{S} \neq S]$$

  taken w.r.t. randomness of $(S, \mathbf{X}, \mathbf{Z})$

- **Goal**: Conditions on n for $P_{\mathrm{e}} \to 0$ in the large-system limit
- Sublinear sparsity: $k = \Theta(p^{\theta})$ for $\theta \in (0, 1)$

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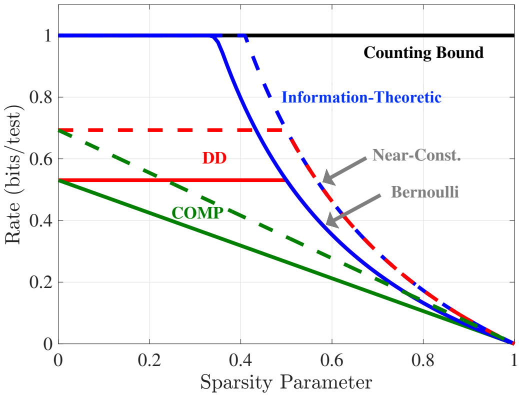
- **Goal**: Conditions on n for $P_{\mathrm{e}} \to 0$ in the large-system limit
- Sublinear sparsity: $k = \Theta(p^{\theta})$ for $\theta \in (0,1)$
- Our work establishes the exact thresholds $n^* = Ck \log \frac{p}{k}$ with precise constants C for both designs, such that:
  - (Exact achievability)

    When $n > (1 + o(1))n^*$, *some decoder gives* $P_e \to 0$;

  - (Exact converse)

    When $n < (1 - o(1))n^*$, *any decoder suffers from* $P_e \to 1$

Milestones in Noiseless GT ($rate = \lim_{p\to\infty} \frac{\log_2 \binom{p}{k}}{n}$)



- Exact thresholds for bernoulli design [1] (ensemble tightness) [2]
- Exact thresholds for NCC design and ensemble tightness [3]
- The blue dashed curve is near optimal for arbitrary design [4]

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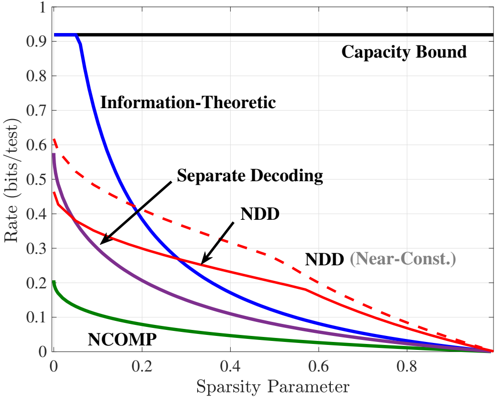
[1] Phase transitions in group testing, J. Scarlett and V. Cevher, 16 SODA
[2] The capacity of Bernoulli nonadaptive group testing, M. Aldridge, 17 T-IT
[3] Information-theoretic and algorithmic thresholds for group testing, A. Coja-Oghlan et al., 20 T-IT
[4] Optimal group testing, Coja-Oghlan et al., 20 COLT

Noisy GT Bounds Before Our Work ($\rho = 0.01$)

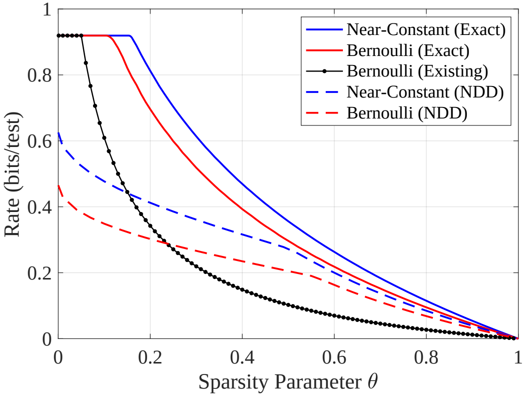


- Information-theoretic upper bounds that are tight for very small values of θ [SC16]
- The information-theoretic upper bound is an attempt to exact thresholds under Bernoulli design [5]
- Compared to the noiseless case, the prior work is much less complete!

[5] Noisy non-adaptive group testing: A (near-)definite defectives approach, J. Scarlett and O. Johnson, 20 T-IT

II. Exact Thresholds

Graphical Illustration of Our Exact Thresholds ($\rho = 0.01$)



- The best exisiting efficient algorithms fall short of information theoretic thresholds

Preliminaries

Notation:

$$a \star b = ab + (1-a)(1-b) \tag{3}$$

$$D(a\|b) = a\log\left(\frac{a}{b}\right) + (1-a)\log\left(\frac{1-a}{1-b}\right) \tag{4}$$

$$H_2(a) = a\log\left(\frac{1}{a}\right) + (1-a)\log\left(\frac{1}{1-a}\right) \tag{5}$$

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Technical Lemma: Tight chernoff bound for binomial variable

Consider $X \sim \mathrm{Bin}(N, q)$, then we have

- Chernoff bound:

$$\mathbb{P}(X \le k) \le \exp\Big(-N \cdot D(\frac{k}{N}\|q)\Big), \quad \text{if } k \le Nq \tag{6}$$

$$\mathbb{P}(X \ge k) \le \exp\Big(-N \cdot D(\frac{k}{N}\|q)\Big), \quad \text{if } k \ge Nq \tag{7}$$

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- Anti-concentration:

$$\mathbb{P}(X = k) \ge \underbrace{\frac{1}{2\sqrt{2k(1-\frac{k}{N})}}\exp\Big(-N \cdot D\Big(\frac{k}{N}\|q\Big)\Big)}_{\text{often } = \exp\left(-N\cdot\left[D\left(\frac{k}{N}\|q\right)+o(1)\right]\right)}, \quad k = 1, 2, \ldots, N-1 \tag{8}$$

Thresholds for Bernoulli Designs

Thresholds for Bernoulli design with i.i.d. $\text{Bernoulli}(\frac{\nu}{k})$ entries:

$$n^*_{\text{Bern}} = \max\left\{ \frac{k \log \frac{p}{k}}{H_2(e^{-\nu} \star \rho) - H_2(\rho)}, \quad \textcolor{red}{\text{(first branch)}} \right.$$

$$\left. \frac{k \log \frac{p}{k}}{(1-\theta)\nu e^{-\nu} \min_{\substack{C>0 \\ \zeta \in (0,1)}} \max\{\frac{1}{\theta} f_1^{\text{Bern}}(C,\zeta,\rho), f_2^{\text{Bern}}(C,\zeta,\rho)\}} \quad \textcolor{red}{\text{(second branch)}} \right\},$$

$$f_1^{\text{Bern}}(C,\zeta,\rho) = C \log C - C + C \cdot D(\zeta \| \rho) + 1,$$

$$f_2^{\text{Bern}}(C,\zeta,\rho) = \min_{d \geq \max\{0, C(1-2\zeta)/\rho\}} g^{\text{Bern}}(C,\zeta,d,\rho),$$

$$g^{\text{Bern}}(C,\zeta,d,\rho) = \rho d \log d + \big(\rho d - C(1-2\zeta)\big) \log\left(\frac{\rho d - C(1-2\zeta)}{1-\rho}\right) + 1 - 2\rho d + C(1-2\zeta)$$

recall some notation:

- p items, k defectives, $k \sim p^\theta$
- ν: design parameter
- ρ: noise level
- C, ζ: optimization parameters

Thresholds for Near-Constant Weight Designs

Thresholds for near-constant weight design with $\Delta = \frac{\nu n}{k}$ placements per item:

$$n^*_{\mathrm{NC}} = \max\Bigg\{ \frac{k \log \frac{p}{k}}{H_2(e^{-\nu} \star \rho) - H_2(\rho)}, \quad \textcolor{red}{\text{(first branch)}}$$

$$\frac{k \log \frac{p}{k}}{(1-\theta)\nu e^{-\nu} \min_{C \in (0, e^{\nu}), \zeta \in (0,1)} \max\{\frac{1}{\theta} f_1^{\mathrm{NC}}(C, \zeta, \rho, \nu), f_2^{\mathrm{NC}}(C, \zeta, \rho, \nu)\}} \quad \textcolor{red}{\text{(second branch)}} \Bigg\},$$

$$f_1^{\mathrm{NC}}(C, \zeta, \rho, \nu) = e^{\nu} D(Ce^{-\nu} \| e^{-\nu}) + C \cdot D(\zeta \| \rho),$$

$$f_2^{\mathrm{NC}}(C, \zeta, \rho, \nu) = \min_{d \,:\, |C(1-2\zeta)| \le d \le e^{\nu}} g^{\mathrm{NC}}(C, \zeta, d, \rho, \nu),$$

$$g^{\mathrm{NC}}(C, \zeta, d, \rho, \nu) = e^{\nu} \cdot D\big(de^{-\nu} \| e^{-\nu}\big) + d \cdot D\Big(\frac{1}{2} + \frac{C(1-2\zeta)}{2d} \| \rho\Big).$$

High-level Intuitions

Two branches appear in the final thresholds:

1. The common first branch $\frac{k\log(p/k)}{H_2(e^{-\nu}\star\rho)-H_2(\rho)}$ is related to the Shannon capacity of the binary symmetric channel.
   - Established by analyzing $\ell = |\widehat{S} \setminus S| = k$ (high ℓ, low overlap)

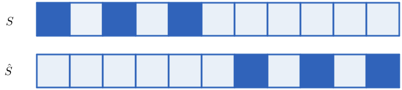

Figure: $|\widehat{S} \setminus S| = k$

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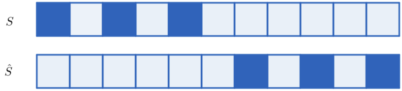

Figure: $|\widehat{S}\setminus S| = k$

2. The more complicated second branches involve f_1 and f_2:
   - Established by analyzing $\ell = |\widehat{S}\setminus S| = 1$ (low ℓ, high overlap)
   - The optimization constants (C, ζ, d) that are introduced to characterize certain quantities in the error events.

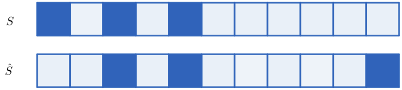

Figure: $|\widehat{S}\setminus S| = 1$

III. Proofs for Converse

The First Branch $\frac{k\log(p/k)}{H_2(e^{-\nu}\star\rho)-H_2(\rho)}$

Intuition:

- The test has probability about $e^{-\nu}$ of containing no defectives;
- (Roughly) $e^{-\nu}\star\rho$ of being positive;
- Thus, each test can only reveal $H_2(e^{-\nu}\star\rho) - H_2(\rho)$ bits of information;
- With $\binom{p}{k}$ possible defective sets, we need (roughly) $\log\binom{p}{k} \sim k\log\frac{p}{k}$ bits; comparing them gives the capacity branch.

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Sketch of technical argument:

- For any $\delta_1 > 0$, we have [SC16]

$$P_e \geq \mathbb{P}\left(\imath^n(\mathbf{X}_s, \mathbf{Y}) \leq \log\left(\delta_1 \binom{p}{k} \right) \right) - \delta_1 \tag{9}$$

$$\approx \mathbb{P}\left(\imath^n(\mathbf{X}_s, \mathbf{Y}) \leq k \log \binom{p}{k} \right) \tag{10}$$

  where $\imath^n(\mathbf{X}_s, \mathbf{Y}) = \log \frac{\mathbb{P}(\mathbf{Y}|\mathbf{X}_s)}{\mathbb{P}(\mathbf{Y})} = \log \mathbb{P}(\mathbf{Y}|\mathbf{X}_s) - \log \mathbb{P}(\mathbf{Y})$.
- Establish *upper concentration bound* for $\imath^n(\mathbf{X}_s, \mathbf{Y})$ by separately analyzing $\log \mathbb{P}(\mathbf{Y}|\mathbf{X}_s)$ and $\log \mathbb{P}(\mathbf{Y})$.

The Second Branch – Failure of MLE

Challenge:

- This kind of terms appeared in thresholds for noiseless case, based on such a central idea:
  if a defective item is *masked* (Definition: every test it is in also contains at least one other defective), then even an optimal decoder will be unable to identify it.

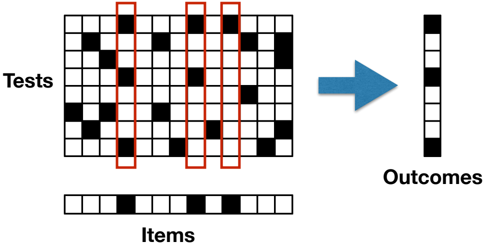

Figure: The last two defectives are *masked*

- However, this is no longer the dominant error event in the noisy case, thus cannot be used to derive the exact/tight converse bounds

The Second Branch – Failure of MLE

Ideas:

- MLE is the optimal decoding strategy, and we only need to show MLE fails when n is below n^*;
- Given $(\mathbf{X}, \mathbf{Y})$, the likelihood of an estimate s is

$$\mathcal{L}_{\mathbf{X},\mathbf{Y}}(s) = \rho^{N_{\mathbf{X},\mathbf{Y}}(s)}(1-\rho)^{n-N_{\mathbf{X},\mathbf{Y}}(s)} \tag{11}$$

  where $N_{\mathbf{X},\mathbf{Y}}(s)$ denotes the number of "correct tests"
- Error event: for some defective j and nondefective j' it holds that have

$$N_{\mathbf{X},\mathbf{Y}}\big(\underbrace{(S \setminus \{j\}) \cup \{j'\}}_{:=\widehat{S}}\big) > N_{\mathbf{X},\mathbf{Y}}\big(S\big) \iff N_{\mathbf{X},\mathbf{Y}}(\widehat{S}) - N_{\mathbf{X},\mathbf{Y}}(S) > 0 \tag{12}$$

The Second Branch – Failure of MLE

$$\widehat{S} = (S \setminus \{j\}) \cup \{j'\}$$

Counting:

- Only two types of tests contribute to $N_{\mathbf{X},\mathbf{Y}}(\widehat{S})$ and $N_{\mathbf{X},\mathbf{Y}}(S)$ differently:

$$\text{contain } j \text{ as the only defective but not contain } j' \begin{cases} \text{positive } (I_1) : N_{\mathbf{X},\mathbf{Y}}(S) \leftarrow N_{\mathbf{X},\mathbf{Y}}(S) + 1 \\ \text{negative } (I_2) : N_{\mathbf{X},\mathbf{Y}}(\widehat{S}) \leftarrow N_{\mathbf{X},\mathbf{Y}}(\widehat{S}) + 1 \end{cases} \tag{13}$$

$$\text{contain no defective but contain } j' \begin{cases} \text{positive } (I_3) : N_{\mathbf{X},\mathbf{Y}}(\widehat{S}) \leftarrow N_{\mathbf{X},\mathbf{Y}}(\widehat{S}) + 1 \\ \text{negative } (I_4) : N_{\mathbf{X},\mathbf{Y}}(S) \leftarrow N_{\mathbf{X},\mathbf{Y}}(S) + 1 \end{cases} \tag{14}$$

- Failure condition:

$$N_{\mathbf{X},\mathbf{Y}}(\widehat{S}) - N_{\mathbf{X},\mathbf{Y}}(S) > 0 \Longrightarrow I_2 + I_3 > I_1 + I_4 \tag{15}$$

The Second Branch – Failure of MLE

Analytical formulation:

- Notation:
  - $\mathcal{M}_j$: tests in which j is the only defective
  - $\mathcal{N}_0$: tests containing no defective
  - $\mathcal{M}_{j1}$ ($\mathcal{M}_{j0}$): the positive (negative) tests in $\mathcal{M}_j$
  - $\mathcal{N}_{01}$ ($\mathcal{N}_{00}$): the positive (negative) tests in $\mathcal{N}_0$
  - $G_{j,j',1}$: number of tests in $\mathcal{N}_{01} \cup \mathcal{M}_{j1}$ that contain j'
  - $G_{j,j',2}$: number of tests in $\mathcal{N}_{00} \cup \mathcal{M}_{j0}$ that contain j'

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  - $G_{j,j',2}$: number of tests in $\mathcal{N}_{00} \cup \mathcal{M}_{j0}$ that contain j'
- For some $(C, \zeta) \in (0, \infty) \times (0, 1)$ such that $\frac{Cn\nu e^{-\nu}}{k}, \frac{\zeta \cdot Cn\nu e^{-\nu}}{k} \in \mathbb{Z}$ we have
  - **(C1)** There exists some $j \in S$ such that

$$M_j = |\mathcal{M}_j| = \frac{Cn\nu e^{-\nu}}{k} \tag{16}$$

$$M_{j0} = |\mathcal{M}_{j0}| = \zeta \cdot M_j \tag{17}$$

  - **(C2)** Failure condition: For some $j' \in [p] \setminus S$,

$$I_2 + I_3 > I_1 + I_4 \Longrightarrow G_{j,j',1} - G_{j,j',2} > (1 - 2\zeta)\frac{Cn\nu e^{-\nu}}{k} \tag{18}$$

The Second Branch – Failure of MLE

The second branch takes the form $\frac{k\log(p/k)}{(1-\theta)\nu e^{-\nu}\min_{C,\zeta}\max\{\frac{f_1}{\theta},f_2\}}$

Step I. Ensuring (C1) leads to f_1:

- **(C1)** for some $j \in S$, $M_j = \frac{Cn\nu e^{-\nu}}{k}$ and $M_{j0} = \zeta \cdot \frac{Cn\nu e^{-\nu}}{k}$
- Challenge lies on M_j
- (Bernoulli) Poisson approximation for the multinomial distribution

$$\left(M_1, M_2, \cdots, M_{k\xi}\right) \tag{19}$$

- (Near-Constant)
  - Work with a surrogate of M_j— M_j': the number of tests in which j is the only defective and *is placed exactly once*
  - Interpret the placements of items into tests as edges in a bipartite graph [CGHL20], and use symmetry to show $(M_1', M_2', \cdots, M_{k\xi}')$ obeys

$$\begin{aligned} M_1' &\sim \mathrm{Hg}(k\Delta, e^{-\nu}k\Delta, \Delta) \\ M_2'|M_1' &\sim \mathrm{Hg}(k\Delta, e^{-\nu}k\Delta, \Delta) \\ &\cdots \\ M_{k\xi}'\left|(M_1', M_2', \cdots, M_{k\xi-1}')\right. &\sim \mathrm{Hg}(k\Delta, e^{-\nu}k\Delta, \Delta) \end{aligned} \tag{20}$$

The Second Branch – Failure of MLE

The second branch takes the form $\frac{k \log(p/k)}{(1-\theta)\nu e^{-\nu} \min_{C,\zeta} \max\{\frac{f_1}{\theta}, f_2\}}$

Step II. Ensuring (C2) leads to f_2:

- **(C2)**: $G_{j,j',1} - G_{j,j',2} > (1 - 2\zeta)\frac{Cn\nu e^{-\nu}}{k}$
- This comes down to the analysis of the difference of *two independent binomial random variables*, and can be handled by anti-concentration
- Many technical challenges/details omitted ...

Step III. Optimizing (C, ζ)

- **(C1) and (C2)**
- Optimizing (C, ζ) to establish the strongest converse bound

IV. Proofs for Achievability

Information density Decoder in [SC16]

Existing Information density decoder (Scarlett & Cevher, 16 SODA):

- We assume $S = s$ is the defective set
- We consider partitioning s into $(s_{\text{dif}}, s_{\text{eq}})$ with $s_{\text{dif}} \neq \emptyset$, and define for each s_{dif} the information density as

$$\imath^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}}) := \log \frac{\mathbb{P}(\mathbf{Y} | \mathbf{X}_{s_{\text{dif}}}, \mathbf{X}_{s_{\text{eq}}})}{\mathbb{P}(\mathbf{Y} | \mathbf{X}_{s_{\text{eq}}})}. \tag{21}$$

- Its expectation depends only on $\ell := |s_{\text{dif}}|$ and is defined as

$$\mathbb{E}\left[\imath^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}})\right] := I(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}}) := I_\ell^n \tag{22}$$

- **Information density decoder**:
  - Fix the constants $\{\gamma_\ell\}_{\ell=1}^k$, and search for a set s of cardinality k such that

$$\imath^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}}) \geq \gamma_{|s_{\text{dif}}|}, \quad \forall (s_{\text{dif}}, s_{\text{eq}}) \text{ such that } |s_{\text{dif}}| \neq 0. \tag{23}$$

- **Intuition:** $\imath^n(\mathbf{X}_{s_{\text{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\text{eq}}})$ tends to be high for the actual defective set s;
- **Limitation:** Analyzing this decoder under small ℓ leads to sub-optimal threshold for noisy GT.

Our hybrid decoding rule

- We resort to MLE for low-ℓ case
- **Hybrid Decoder:** We search for a set $\hat{s}$ of cardinality k that satisfies
  - (Low ℓ: MLE) It holds that

$$\mathcal{L}_{\mathbf{X},\mathbf{Y}}(\hat{s}) > \mathcal{L}_{\mathbf{X},\mathbf{Y}}(s'), \quad \forall s' \text{ such that } 1 \leq |\hat{s} \setminus s'| \leq \frac{k}{\log k}, \tag{24}$$

    where we implicitly also constrain s' to have cardinality k.
  - (High-ℓ: information density) For suitably chosen $\{\gamma_\ell\}_{\frac{k}{\log k} < \ell \leq k}$, it holds that

$$\imath^n(\mathbf{X}_{s_{\mathrm{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\mathrm{eq}}}) \geq \gamma_{|s_{\mathrm{dif}}|}, \quad \forall (s_{\mathrm{dif}}, s_{\mathrm{eq}}) \text{ such that } |s_{\mathrm{dif}}| > \frac{k}{\log k}, \tag{25}$$

    where $(s_{\mathrm{dif}}, s_{\mathrm{eq}})$ is a disjoint partition of $\hat{s}$.
- **Success conditions**:
  - Success condition for Low-ℓ:

$$(24) \text{ holds for } \hat{s} = s \tag{26}$$

  - Success condition for high-ℓ:

$$(25) \text{ holds for } \hat{s} = s \tag{27}$$

$$\forall \tilde{s} \text{ with } |\tilde{s}| = k, \; |s \setminus \tilde{s}| > \frac{k}{\log k}, \text{ it holds that } \imath^n(\mathbf{X}_{\tilde{s}\setminus s}; \mathbf{Y} | \mathbf{X}_{\tilde{s}\cap s}) < \gamma_{|s\setminus\hat{s}|} \tag{28}$$

The First Branch $\frac{k\log(p/k)}{H_2(e^{-\nu}\star\rho)-H_2(\rho)}$

- **Starting point:** [SC16] for any $\delta_1>0$, $\mathbb{P}\big((27)\&(28)\text{ fail}\big)\le$

$$\mathbb{P}\Bigg[\bigcup_{(s_{\text{dif}},s_{\text{eq}}):|s_{\text{dif}}|\ge\ell_{\min}}\left\{\imath^n(\mathbf{X}_{s_{\text{dif}}};\mathbf{Y}|\mathbf{X}_{s_{\text{eq}}})\le\log\binom{p-k}{|s_{\text{dif}}|}+\log\left(\frac{k}{\delta_1}\binom{k}{|s_{\text{dif}}|}\right)\right\}\Bigg]+\delta_1$$

$$\stackrel{\delta_1\to 0}{\approx}\mathbb{P}\Bigg[\bigcup_{(s_{\text{dif}},s_{\text{eq}}):|s_{\text{dif}}|\ge\ell_{\min}}\left\{\imath^n(\mathbf{X}_{s_{\text{dif}}};\mathbf{Y}|\mathbf{X}_{s_{\text{eq}}})\le(1+o(1))\ell\log\left(\frac{p}{k}\right)\right\}\Bigg]\quad(29)$$

- **Concentration bound:** for any $\delta_2\in(0,1)$,

$$\mathbb{P}\big[\imath^n(\mathbf{X}_{s_{\text{dif}}};\mathbf{Y}|\mathbf{X}_{s_{\text{eq}}})\le(1-\delta_2)I^n_\ell\big]\le\psi_\ell(n,\delta_2),\quad(30)$$

- Therefore, if it holds that

$$\max_{\ell>\frac{k}{\log k}}\frac{(1+o(1))\ell\log(\frac{p}{k})}{I^n_\ell(1-\delta_2)}\le 1,\quad(31)$$

we are able to enforce

$$\mathbb{P}\big((27)\&(28)\text{ fail}\big)\le\sum_{\ell=\ell_{\min}}^{k}\binom{k}{\ell}\psi_\ell(n,\delta_2)\to 0\quad(32)$$

- Combining with the asymptotic of scaling of I^n_ℓ, (31) yields $n\ge$ the first branch

The Second Branch – Success of MLE

- The second branch, $\frac{k\log(p/k)}{(1-\theta)\nu e^{-\nu}\min_{C,\zeta}\max\{\frac{f_1}{\theta},f_2\}}$, is used to ensure

$$\mathcal{L}_{\mathbf{X},\mathbf{Y}}(s) > \mathcal{L}_{\mathbf{X},\mathbf{Y}}(s'), \quad \forall s' \text{ such that } 1 \le |s \setminus s'| \le \frac{k}{\log k} \tag{33}$$

  so that the MLE part succeeds.
- Similar arguments with nontrivial generalizations

Preparations:

- $\mathcal{L}_{\mathbf{X},\mathbf{Y}}(s) > \mathcal{L}_{\mathbf{X},\mathbf{Y}}(s') \iff N_{\mathbf{X},\mathbf{Y}}(s) > N_{\mathbf{X},\mathbf{Y}}(s')$
- $\mathcal{J} = s \setminus s'$ (defective) and $\mathcal{J}' = s' \setminus s$ (non-defective)
- Only two types of tests matter

$$\text{only contain defectives in } \mathcal{J} \text{ but not contain items in } \mathcal{J}' \begin{cases} \text{positive } (I_1): N_{\mathbf{X},\mathbf{Y}}(s)++ \\ \text{negative } (I_2): N_{\mathbf{X},\mathbf{Y}}(s')++ \end{cases} \tag{34}$$

$$\text{contain no defective but contain items from } \mathcal{J} \begin{cases} \text{positive } (I_3): N_{\mathbf{X},\mathbf{Y}}(s')++ \\ \text{negative } (I_4): N_{\mathbf{X},\mathbf{Y}}(s)++ \end{cases} \tag{35}$$

- Success condition: $N_{\mathbf{X},\mathbf{Y}}(s) > N_{\mathbf{X},\mathbf{Y}}(s') \Longrightarrow I_1 + I_4 > I_2 + I_3$

The Second Branch – Success of MLE

Analytical formulation:

- Notation:
  - $\mathcal{M}_{\mathcal{J}}$: tests in which items in $\mathcal{J}$ are the only defectives
  - $\mathcal{M}_{\mathcal{J}1}(\mathcal{M}_{\mathcal{J}0})$: the positive (negative) tests in $\mathcal{M}_{\mathcal{J}}$
  - $\mathcal{N}_0, \mathcal{N}_{00}, \mathcal{N}_{01}$: as before
  - $G_{\mathcal{J},\mathcal{J}',1}$: number of tests in $\mathcal{N}_{01} \cup \mathcal{M}_{\mathcal{J}1}$ that contain some item from $\mathcal{J}'$
  - $G_{\mathcal{J},\mathcal{J}',2}$: number of tests in $\mathcal{N}_{00} \cup \mathcal{M}_{\mathcal{J}0}$ that contain some item from $\mathcal{J}'$
- For any $\ell \leq \frac{k}{\log k}$ and any pairs of $(C, \zeta) \in [0, \infty) \times [0, 1]$ such that $\frac{Cn\nu e^{-\nu}\ell}{k}, \frac{\zeta \cdot Cn\nu e^{-\nu}\ell}{k} \in \mathbb{Z}$, one of the following conditions hold:
  - **(C1)** $\mathcal{K}_{\ell,C,\zeta} = \varnothing$ where

$$\mathcal{K}_{\ell,C,\zeta} = \left\{\mathcal{J} \subset s \,:\, |\mathcal{J}| = \ell, M_{\mathcal{J}} = \frac{Cn\nu e^{-\nu}\ell}{k},\ M_{\mathcal{J}0} = \frac{\zeta \cdot Cn\nu e^{-\nu}\ell}{k}\right\} \tag{36}$$

  - **(C2)** If $\mathcal{K}_{\ell,C,\zeta} \neq \varnothing$, then for any $\mathcal{J} \in \mathcal{K}_{\ell,C,\zeta}$ and $\mathcal{J}' \subset [p] \setminus s$ and $|\mathcal{J}'| = \ell$,

$$I_1 + I_4 > I_2 + I_3 \iff G_{\mathcal{J},\mathcal{J}',1} - G_{\mathcal{J},\mathcal{J}',2} < (1 - 2\zeta)\frac{Cn\nu e^{-\nu}\ell}{k} \tag{37}$$

The Second Branch – Success of MLE

The second branch takes the form $\frac{k\log(p/k)}{(1-\theta)\nu e^{-\nu}\min_{C,\zeta}\max\{\frac{f_1}{\theta},f_2\}}$

Overview:

- **Step 1.** Ensuring **(C1)** yields the f_1 part of the second branch
  - Unlike in the converse, we utilize a first-order method which first bounds

$$\mathbb{E}|\mathcal{K}_{\ell,C,\zeta}| = \binom{k}{\ell}\mathbb{P}\left(\text{for fixed } \mathcal{J}\subset s \text{ with } |\mathcal{J}|=\ell,\ M_{\mathcal{J}} = \frac{Cn\nu e^{-\nu}\ell}{k},\ M_{\mathcal{J}0}=\zeta M_{\mathcal{J}}\right) \tag{38}$$

  and then utilize Markov's inequality to enforce $|\mathcal{K}_{\ell,C,\zeta}|=0$

- **Step 2.** Ensuring **(C2)** yields the f_2 part of the second branch. More complicated than the converse as we need to handle all (ℓ, C, ζ)
  - Working with the sum/difference of two independent binomial variables
- **Step 3.** Optimizing over (C,ζ)

Summary & Future Directions

Summary:

- We established exact thresholds for noisy group testing with Bernoulli design and near-constant weight design
- For converse analysis, the main innovation is to identify a novel set of dominant error events
- For achievablity analysis, we introduce a hybrid decoder that combines the exsiting information density approach and MLE

Future Directions:

1. **Efficient and Optimal Algorithm:** Devise an *efficient* algorithm to achieve the exact thresholds for near-constant weight design.
   - A concurrent work solved this problem via spatial coupling designs.[6]
2. **Converse for Arbitrary Design:** Investigate whether the n^*_{NC} is the general converse for arbitrary design.
   - This is true in the noiseless case [CGHL20]

Thank You

[6] Noisy group testing via spatial coupling, Coja-Oghlan et al., Comb. Probab. Comput., 2024