Empirical Orthogonal Function Analysis (EOF)

Empirical Orthogonal Function Analysis

- Empirical Orthogonal Function Analysis (EOF) is a commonly used tool for data compression or dimension reduction in meteorology and oceanography (Monahan et al. 2009). It is often used to extract dominant modes of climate variability in climate analysis.
- EOF analysis is also known as principal component analysis (PCA). The technique became popular in meteorology following the paper by Lorenz (1956).
- EOF attempts to represent a dataset containing a large number (M) of variables with a relative small number (K) of new variables. These new variables are linear combinations of the original ones, and these linear combinations are chosen to represent the maximum possible fractional variance of the original data. EOF is most effective when this data compression can be achieved with K << M.

Wilks 2011

EOF Analysis of a Geophysical Field

- The dataset containing a large number of variables can be a geophysical field, and the data point at each grid point can be regarded as a variable.
- Since geophysical fields often have strong spatial coherence, a geophysical field of M grid points can often be well represented by K new variables, where K << M and the new variables each represent a recurrent spatial pattern.
  - M: spatial dimension (number of grid points for the geophysical field)
  - K: number of EOF modes to retain
  - N: temporal dimension (number of observations)

What Does EOF Analysis do?

- In meteorological application, EOF analysis uses a set of orthogonal functions (EOFs) to represent the time series of a field variable Z (x, y, t).

$$Z(x,y,t) = \sum_{k=1}^{M} PC_k(t) \cdot EOF_k(x,y)$$

- EOFs(x, y) are the spatial structures that can account for the temporal variations of Z ("new variables").
  - The first EOF explains most fractional variance of Z. The subsequent EOFs explain the largest possible fraction of the remaining variance, subject to the condition that they are mutually uncorrelated or orthogonal to the lower-order EOFs.
  - If Z contains M grid points, there are M EOFs available, which together explain the total variance of Z without any loss of information, but we don't need to retain all EOFs since the high-order EOFs explain very little variance of Z.
- PC(t) is the principal component, and each EOF has a corresponding PC(t).
  - If Z contains N observations, each PC(t) is a time series of the length of N.
  - PC(t) determines how the amplitude of the corresponding EOF varies with time.

Wilks 2011

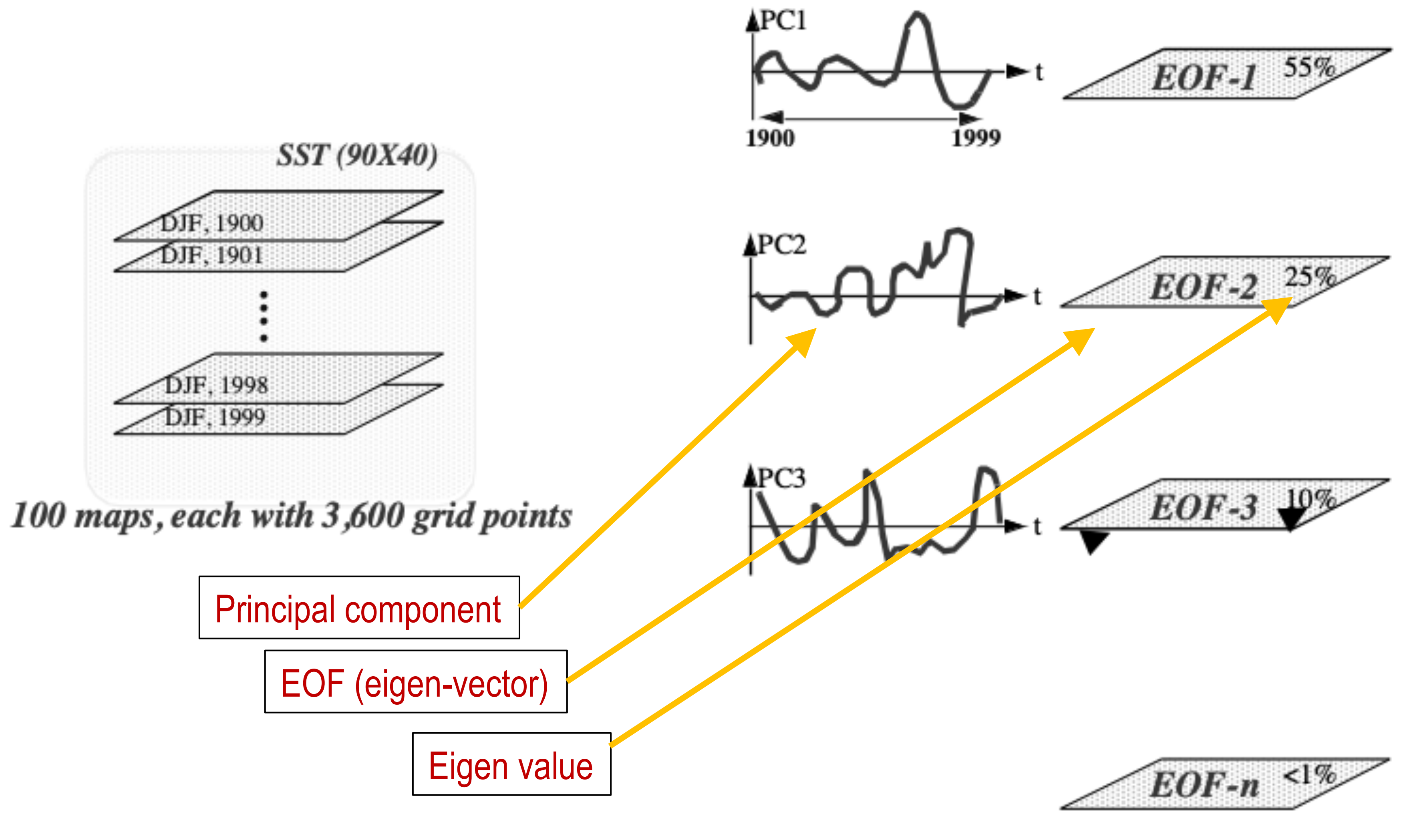
There are three key output variables from the EOF analysis:

- EOF modes (or eigenvectors), $EOF_k(x,y)$
- The corresponding $PC_k(t)$
- The corresponding eigen values.

In this example:

- The SST field has 3,600 grid points and covers 100 years from 1900-1999.
- Each EOF mode represents a SST pattern.
- Each PC is a time series with the length of 100. The large magnitude of PC(t) in a year indicates a strong EOF pattern in that year.
- The eigen values indicate the fraction of variance explained by the corresponding EOF mode. We can use the first three EOF modes to explain 90% of the variance, which substantially reduces the data dimension (3,600 → 3).

What do we get from EOF?



Covariance Matrix

- The EOF analysis can be based on the covariance matrix.
- For a geophysical field of the spatial dimension M, the covariance matrix has the shape of M * M.
- X(i,j) represents the covariance of Z at grid point i and grid point j.

$$\begin{vmatrix} X_{11} & X_{12} & X_{13} & \bullet\bullet\bullet\bullet\bullet & X_{1,M-1} & X_{1M} \\ X_{21} & X_{22} & X_{23} & \bullet\bullet\bullet\bullet\bullet & X_{2,M-1} & X_{2,M} \\ X_{31} & X_{32} & X_{33} & \bullet\bullet\bullet\bullet\bullet & X_{3,M-1} & X_{3,M} \\ X_{M-1,1} & X_{M-1,2} & X_{M-1,3} & \bullet\bullet\bullet\bullet\bullet & X_{M-1,M-1} & X_{M-1,M} \\ X_{M,1} & X_{M,2} & X_{M,3} & \bullet\bullet\bullet\bullet\bullet & X_{M,M-1} & X_{M,M} \end{vmatrix}$$

Here M= 3,600

$$X_{i,j} = cov(Z_i, Z_j) = \frac{1}{N}\sum_{t=1}^{N}(Z_i(t) - \bar{Z}_i)(Z_j(t) - \bar{Z}_j)$$

Correlation Matrix

- We can also use the correlation matrix for EOF analysis.
- The EOFs obtained from the covariance matrix will be different from the EOFs obtained from the correlation matrix.
    - In the case of the covariance matrix formulation, the grid points with larger variances will be weighted more heavily.
    - With the correlation matrix, all grid points receive the same weight and only the structure, not the amplitude, will influence the principal components.

$$\begin{vmatrix} X_{11} & X_{12} & X_{13} & \bullet\bullet\bullet\bullet\bullet & X_{1,M-1} & X_{1M} \\ X_{21} & X_{22} & X_{23} & \bullet\bullet\bullet\bullet\bullet & X_{2,M-1} & X_{2,M} \\ X_{31} & X_{32} & X_{33} & \bullet\bullet\bullet\bullet\bullet & X_{3,M-1} & X_{3,M} \\ X_{M-1,1} & X_{M-1,2} & X_{M-1,3} & \bullet\bullet\bullet\bullet\bullet & X_{M-1,M-1} & X_{M-1,M} \\ X_{M,1} & X_{M,2} & X_{M,3} & \bullet\bullet\bullet\bullet\bullet & X_{M,M-1} & X_{M,M} \end{vmatrix}$$

Here $M= 3{,}600$

$$X_{i,j} = corr(Z_i, Z_j) = cov(Z_i, Z_j)/(S_iS_j) = \frac{1}{N}\sum_{t=1}^{N}(Z_i(t) - \overline{Z_i})(Z_j(t) - \overline{Z_j})/(S_iS_j)$$

where Si and Sj represent the standard deviation of Z at grid point I and j, respectively.

Correlation Matrix (cont'd)

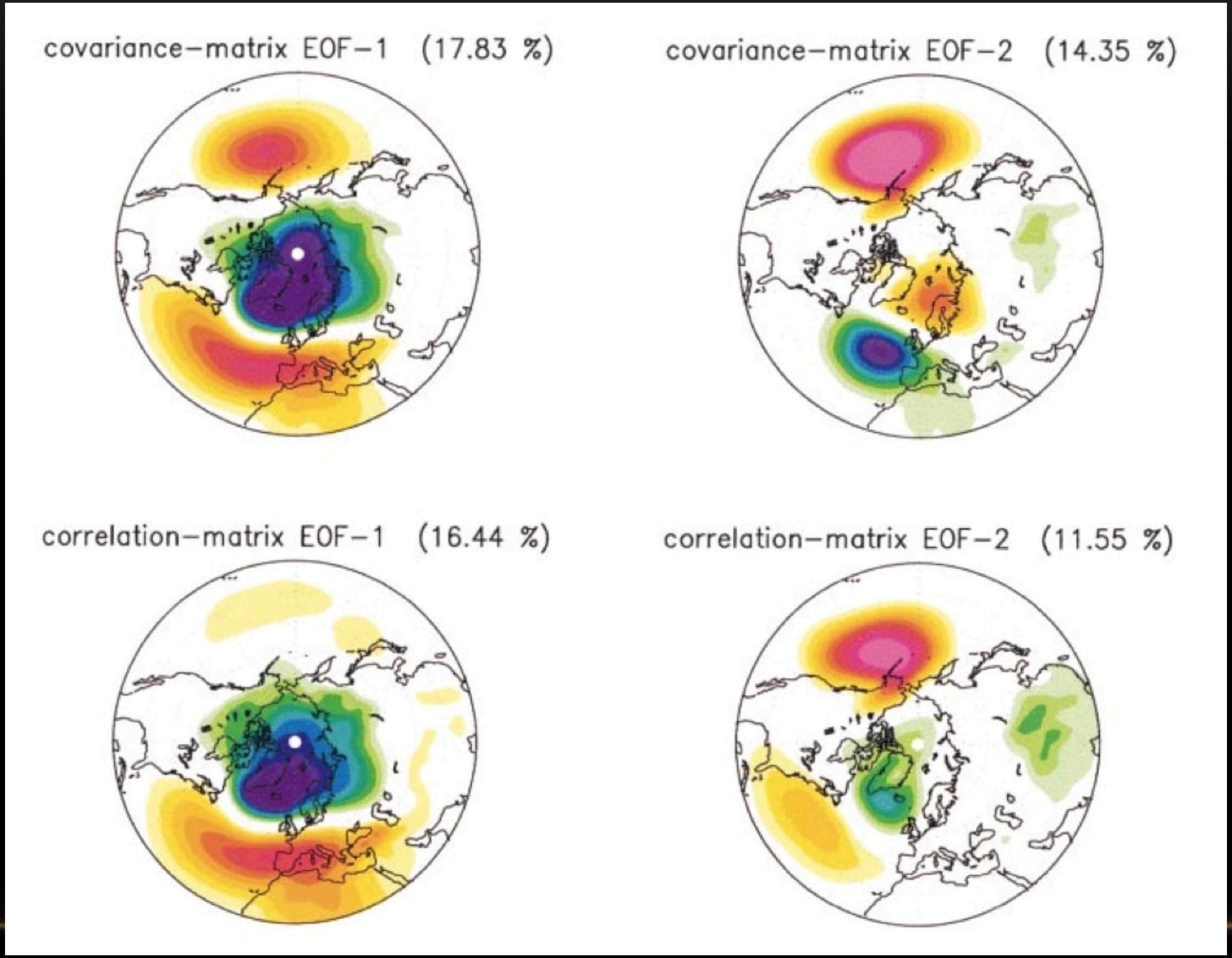
The correlation matrix should be used for the following two cases:

- The state vector is a combination of things with different units.
- The variance of the state vector varies strongly from grid point to grid point so much that this distorts the patterns in the data.

Would you choose the correlation matrix or covariance matrix of 200-hPa geopotential height (H200) if you want to extract a teleconnection pattern spanning from the tropics to the subpolar region?

- The correlation matrix will be a better choice than the covariance matrix since you want to identify the pattern in the tropics, where H200 has relatively weak variability
- Streamfunction will be a better choice than geopotential height.
- You should also use the square root of the cosine of the latitude to compute the area weight, as the grid cell area decreases poleward.

EOF based on Covariance vs. Correlation Matrix

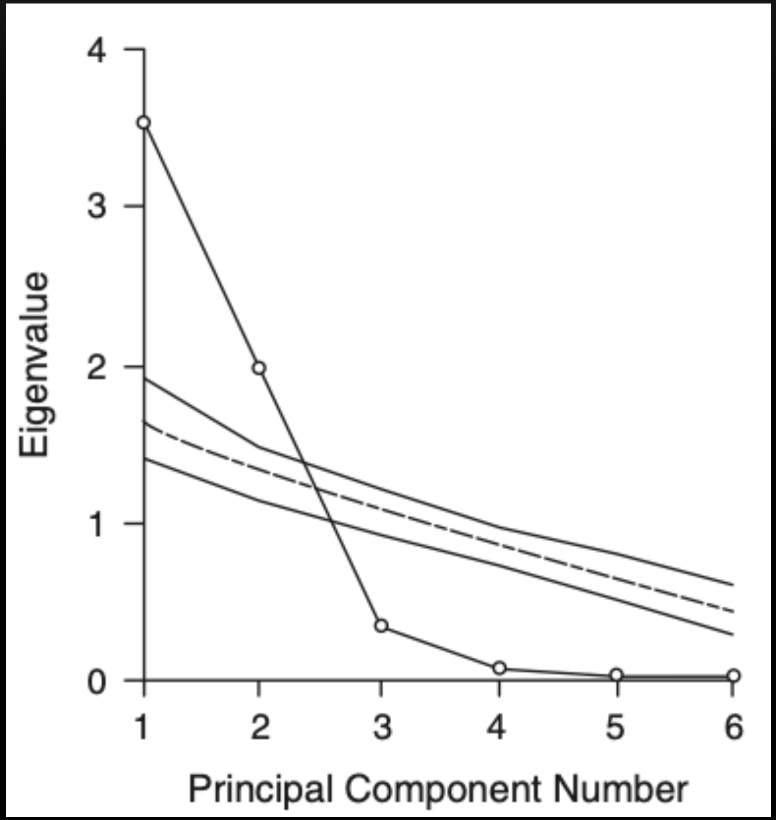


The leading EOFs of monthly mean winter time (Nov– Apr) SLP in the Northern Hemisphere. (Dommengate and Latif 2002, © American Meteorological Society. Used with permission)

A big question about the EOF is How many EOF modes we should retain?

The criteria to truncate EOFs are often subjective. There are two commonly used approaches.
1. One will retain enough principal components to represent a sufficient fraction of the variances of the original data. The fraction often ranges between 70-90%, and the specific threshold is subjective.

2. A second method is known as the scree test. It is based on the slope of the eigenvalue spectrum: look for the turning point where the spectrum curve levels off. In this example, the turning point occurs at k=3 and we will keep the first 3 EOF modes.



Eigenvalue magnitudes as a function of the principal component number (heavier lines connecting circled points), for a K = 6 dimensional analysis.

Fig. 11.8 from Wilks 2011

Rotated EOF

- Subsequent eigenvectors must be orthogonal to previously determined eigenvectors in both space and time, regardless of the nature of the physical processes.

- The orthogonality constraint on the eigenvectors can lead to problems with the interpretations, especially for the high-order eigenvectors.

- The figure on the right shows the first four EOF patterns for a certain field. We can see that the high-order eigenvectors have increasingly small scale due to the orthogonality constraint.

- Therefore, we want to relax the spatial orthogonality constraint on eigenvectors.

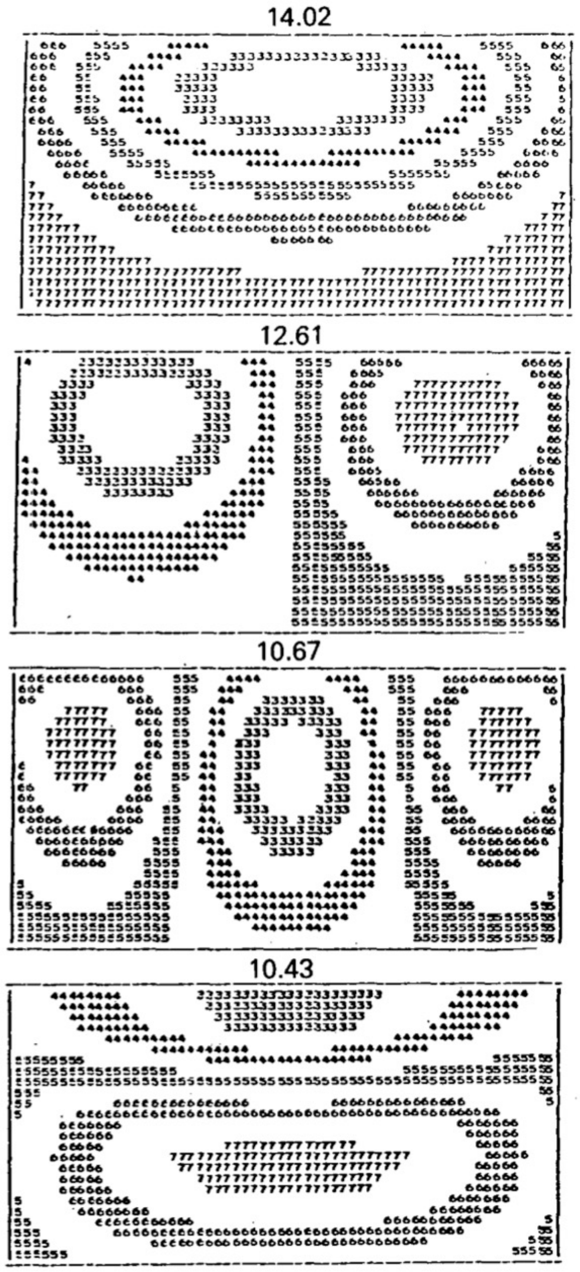


FIG. 2. The first four exact EOF patterns for the field (27) restricted to a patch on the sphere (20 to 60°N and 100° of width in longitude). The number above each contour map is the exact eigenvalue associated with the corresponding EOF pattern.

North et al. (1982) © American Meteorological Society. Used with permission

Rotated EOF (cont'd)

- To assist physical interpretation, it is often desirable to rotate a subset of the initial eigenvectors to a second set of new coordinate vectors, which are called rotated EOFs.
  - First perform the regular EOF analysis.
  - keep a few leading EOF modes for the rotation.
  - These selected EOFs are rotated to form new EOFs.
  - The VARIMAX method is often used to rotate EOFs.

Conventional vs. Rotated EOFs

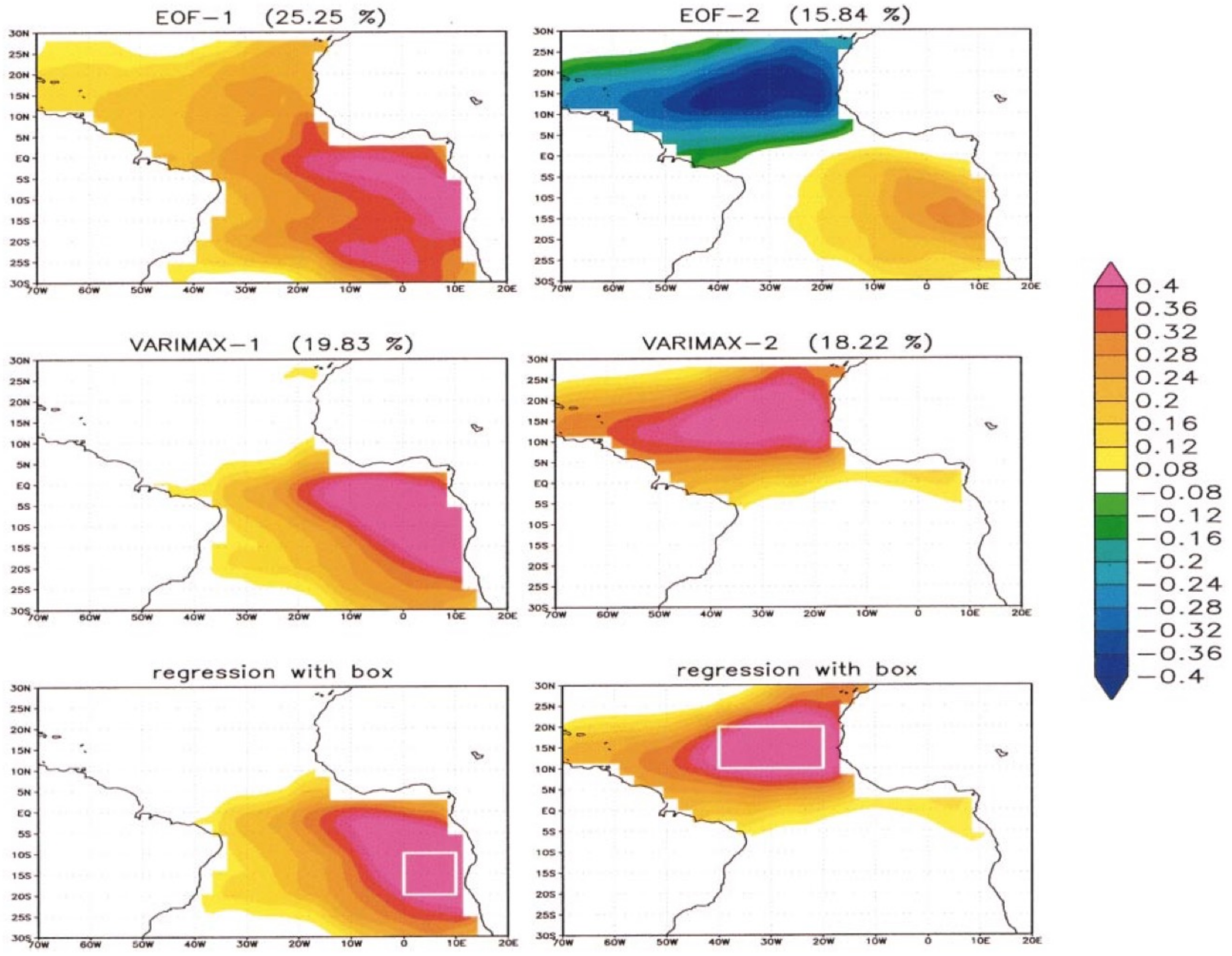


FIG. 1. The EOFs, VARIMAX patterns, and regressions of box averaged monthly mean SST in the tropical Atlantic Ocean. The amplitudes are in kelvins.

Dommengate and Latif 2002© American Meteorological Society. Used with permission

"A Cautionary Note on the Interpretation of EOFs" by Dommengate and Latif 2002

From Dommengate and Latif 2002

- *The teleconnection patterns derived from the orthogonal analysis cannot necessarily be interpreted as teleconnections that are associated with a potential physical process.*
- *The centers of action derived from the EOF or VARIMAX methods do not need to be the centers of action of the real physical modes.*
- *The PCs of the dominant patterns are often a super- position of many different modes that are uncorrelated in time and that are often modes of remote regions that have no influence on the region in which the pattern of this PC has its center of action.*

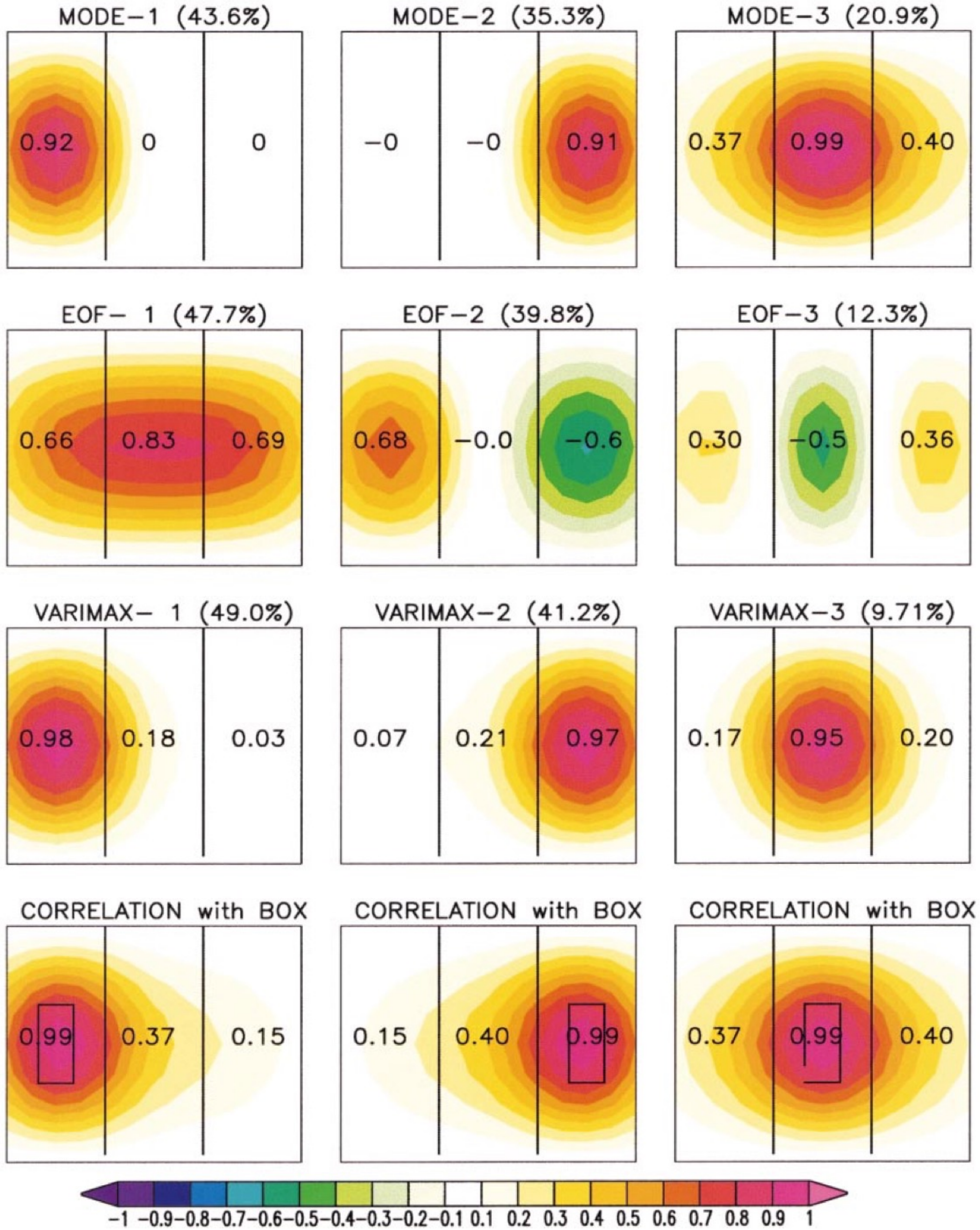


FIG. 5. Same as in Fig. 4, but all analysis are based on the correlation matrix and the values are in terms of correlation.

Dommengate and Latif 2002© American Meteorological Society. Used with permission

References

- Dommenget, D., & Latif, M. (2002). A Cautionary Note on the Interpretation of EOFs, *Journal of Climate*, *15*(2), 216-225.
- North, G. R., T. L. Bell, R. F. Cahalan, and F. J. Moeng, 1982: Sampling errors in the estimation of empirical orthogonal functions. *Mon. Wea. Rev.,* **110,** 699–706.
- *Wilks, D., 2011: Statistical Methods in the Atmospheric Sciences . 3rd ed. Oxford ; Academic Press, Chapter 11.*