MIT Linear Algebra Lecture 1: The Geometry of Linear Equations 11/19/25

n linear equations, n unknowns

Row Picture → The picture of one equation at a time
* Column Picture → The column of a matrix
Matrix Form → Algebra way to look at problem

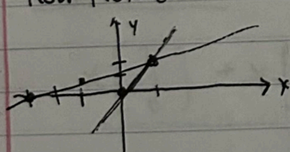
A matrix → rectangular array of #'s

Ex. $2x-y=0$ Intro of 2 eq. + 2 un.
$-x+2y=3$

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{\substack{\text{coefficients} \\ A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\substack{\text{unknowns} \\ x}} = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{\substack{\text{RHS} \\ b}}$$

Matrix Form of Matrix A

Row Picture



For $2x-y=0$

Start with horizontal axis, when $y=0$... $x=0$
$(0,0)$ is on the line and solves the equation
Suppose $x=1$, then $y=2$ → $(1,2)$ also solves

Now for $-x+2y=3$

$(0,0)$ won't work because of 3, it isn't 0
if $y=0$ → $x=-3$
if $x=-1$ → $y=1$

Now, the point that crosses both equations should be $(1,2)$

$\boxed{\begin{matrix} x=1 \\ y=2 \end{matrix}}$ Solves both equations

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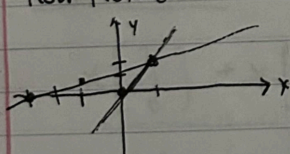

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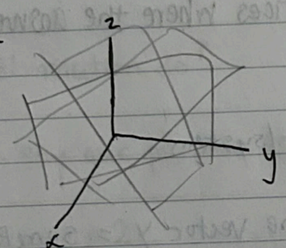
3 equations and 3 unknowns

$$
\begin{aligned}
2x - y &= 0 \\
-x + 2y - z &= -1 \\
-3y + 4z &= 4
\end{aligned}
$$

Matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$

<u>Row picture</u>



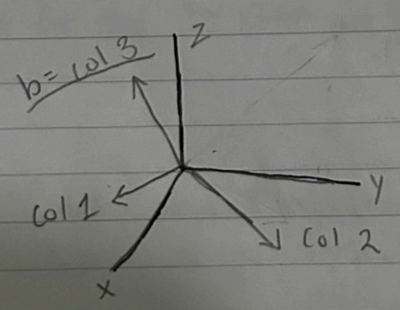
Each row in a 3x3 will give us a plane of solutions

So 3 planes and they will meet at one point which is our solution

<u>Column Picture</u>

$$\underbrace{x\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

LHS : Now we have a linear combination of 3 vectors



$z\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$ matches what we want

$\boxed{x=0,\ y=0,\ z=1}$

we only want z

Matrix $\uparrow$ vector

Can I solve $\underline{A} \underline{x} = b$ for every b?

Is there a solution for every RHS?

Do the linear combinations of the columns fill 3-D space?

For this previous matrix, the ans is yes.
Because it is a non-singular matrix,
an invertible matrix

There will be other matrices where the answer is no.
↳ ex. same plane columns.

The matrix form of my equation /system

Some matrix A, (times) some vector y = some RHS B
$Ax = b$
So how do you multiply a matrix by a vector?

$Ax = b$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

column way, but can also do by rows

MIT Lecture 2 - Elimination with Matrices 11/19/25

Good matrix = elimination works, what happens for it to not work?

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y + z = 2$$

$Ax = b$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

What does Elimination do?

Let's say our purpose is to eliminate x. from eqn 2

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$

$$\begin{matrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{matrix} \xrightarrow{\times 3} \begin{matrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{matrix}$$

1st pivot

What abt the RHS?

After cleaning Row 2 Column 1 we should clean R3C1 but it's already 0

Now for the 2nd pivot $\rightarrow$ $\begin{matrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{matrix} \rightarrow \begin{matrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{matrix}$ The 3 pivots

1st

2nd

matrix u

The whole point of elimination was to get from A to u

temp. & perm. failure

Can run into trouble if $\rightarrow$ 0 in position of pivot and cannot exchange

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Back Substitution

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}}_{b} \longrightarrow \begin{matrix} \boxed{1} & 2 & 1 & & 2 \\ 0 & \boxed{2} & -2 & & 6 \\ 0 & 4 & 1 & & 2 \end{matrix} \longrightarrow \begin{matrix} \boxed{1} & 2 & 1 & & 2 \\ 0 & \boxed{2} & -2 & & 6 \\ 0 & 0 & \boxed{5} & & -10 \end{matrix}$$

u — matrix

c — vector

u is what happens to A

c is what happens to b

Final Equations

$$x + 2y + z = 2$$
$$2y - 2z = 6$$
$$5z = -10$$

$ux = c$

Now solve for z first

$z = -2$

$\hookrightarrow y = 1$

$\hookrightarrow x = 2$

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Matrices

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix}3\\4\\5\end{bmatrix} = \begin{matrix} 3\times \text{col } 1 \\ + \\ 4\times \text{col } 2 \\ + \\ 5\times \text{col } 3 \end{matrix}$$

Columns → matrix * column is a column

$$\begin{bmatrix}1 & 2 & 7\end{bmatrix}\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = \begin{matrix} 1 * \text{row } 1 \\ + \\ 2 * \text{row } 2 \\ + \\ 7 * \text{row } 3 \end{matrix}$$

rows → matrix * row is a row

1×3 multi. 3×3

Let's say: $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$

(1) and we subtract 3 * row 1 from row 2

E for Elementary Matrix ← 21 because fixing Row 2 col 1 posit

so $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = identity matrix

does nothing

What matrix will do it?

step (2)

and we subtract 2 * row 2 from row 3

E_{32} → $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$

Matrix Notation

$$E_{32}(E_{21}A) = U$$

Suppose I want ___ A = U, what matrix will take me straight from A to U?

$$(E_{32}E_{21})A = U$$

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Permutation Matrix

Let's say we want to find the matrix that exchanges rows 1 & 2

$$\underset{P}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Let's say we wanted to exchange columns

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

can't do it on the left

have to do it on the right

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

<u>In short</u>

Column operation → multiply on the right

Row operation → multiply on the left

In terms of matrices $\left[a \cdot b \neq b \cdot a \right]$

Inverse Matrix (Inverses)

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow$$ we want to find the matrix that undoes this step

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Identity}}$$

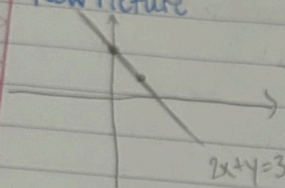
Solve $\begin{cases} 2x+y=3 \\ x-2y=-1 \end{cases}$

$x = -1 + 2y = 2y - 1$ $\quad 2(2y-1)+y=3$

$4y-\cancel{2}+y=3$

$3y = 5 \quad y=1 \quad x=1$

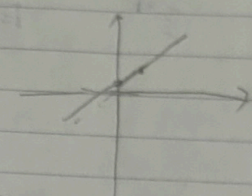
Row Picture



$2x+y=3$ is a line.

$x=0 \rightarrow y=3$

$x=1 \rightarrow y=1$

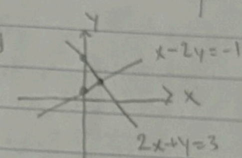


$x-2y=-1$ is also line

$x=0 \rightarrow y=\frac{1}{2}$

$x=1 \rightarrow y=1$

Combined



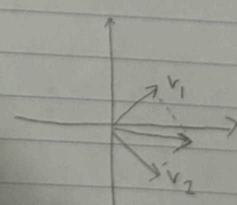
Intersect $(1,1)$

Solution is $(1,1)$

Column Picture

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \end{bmatrix} y = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$v_1 \qquad v_2$



$v_1 + v_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Matrix Form

$A \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$ax=b$

$x = \frac{b}{a} = a^{-1}b$

$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Elimination Practice

$x - y - z + u = 0$
$2x + 2z = 8$
$-y - 2z = -8$
$3x - 3y - 2z + 4u = 7$

$$\begin{array}{c} (-2) \\ \\ (-3) \end{array} \left[\begin{array}{cccc|c} \boxed{1} & -1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{array}\right]$$

$$\tfrac{1}{2} \left[\begin{array}{cccc|c} \boxed{1} & -1 & -1 & 1 & 0 \\ 0 & \boxed{2} & 4 & -2 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right]$$

$$\text{swap} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \boxed{1} & -1 & -1 & 1 & 0 \\ 0 & \boxed{2} & 4 & -2 & 8 \\ 0 & 0 & \boxed{1} & 1 & 7 \\ 0 & 0 & 0 & \boxed{-1} & -4 \end{array}\right]$$

$-1(u) = -4$

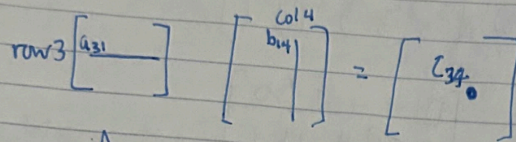
$z + u = 7$
$2y + 4z - 2u = 8$
$x - 2 - 3 + 4 = 0$

$$\boxed{\begin{aligned} u &= 4 \\ z &= 3 \\ y &= 2 \\ x &= -1 \end{aligned}}$$

Lecture #3 : Multiplication and Inverse Matrices 11/23/25

Suppose Matrix $A \cdot B = C$

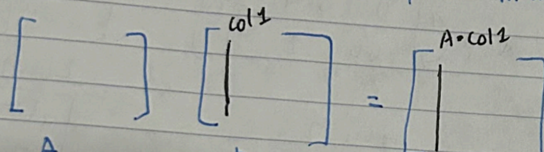
row x column



A $(m \times n)$ B $(n \times p)$ $C = AB$ $(m \times p)$

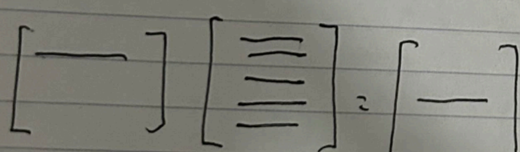
$c_{34} = (\text{row 3 of } A) \cdot (\text{col 4 of } B)$
$= a_{31}b_{14} + a_{32}b_{24} + \dots$
$= \sum_{k=1}^{n} a_{3k} b_{k4}$

When are we allowed to multiply these matrices?



$A_{(m \times n)}$ $B_{(n \times p)}$ $C_{(m \times p)}$

these columns of C are combinations of columns of A



columns of C are combinations of ~~rows~~ rows of B

column x row

Column of A x row of B

What shape? → matrix

A → (col) $m \times 1$

B → $1 \times p$ (row)

ex.

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

<u>4th way</u>

AB = sum of (cols of A) x (rows of B)

ex.

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix Multiplication by Block

11/24/25

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} A_1B_1 + A_2B_3 & \\ \hline & \end{array}\right]$$

A B

Inverses

Not all matrixes have inverses.

Is it a square? Is it invertible?

If this exists → A^{-1}

when does it exist? How do we find it?

these good matrices are called invertible, non-singular

← Identity matrix

$A^{-1} A = I$

$A^{-1}A = I = A A^{-1}$

For square matrices, the left inverse is also the right inverse (if it has an inverse)

Singular Case (no inverse)

$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ → Why isn't this invertible?

You can find a vector x with $Ax = 0$ (that's not $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$)

This is because if you multiply the inverse, you can't escape 0.

Non-Singular

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} & \\ & \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

→ $A \times$(col j of A^{-1}) = col j of I

→

Gauss-Jordan (Solve 2 eqn at once) 11/25/25

$$\underbrace{\begin{bmatrix}1&3\\2&7\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}a&c\\b&d\end{bmatrix}}_{A^{-1}}=\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}_{I}$$

eqn 1 $\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$

eqn 2 $\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$

$A \times$ col j of A^{-1} = col j of I

$$\underbrace{\begin{bmatrix}1&3\\2&7\end{bmatrix}}_{A}\rightarrow\left[\begin{array}{cc|cc}1&3&1&0\\2&7&0&1\end{array}\right]\xrightarrow{\text{take 2 of row 1}}\left[\begin{array}{cc|cc}1&3&1&0\\0&1&-2&1\end{array}\right]$$

(under the second matrix: A I)

don't stop even if upper Δ form

↓ take 3 of row 2

$$\left[\begin{array}{cc|cc}1&0&7&-3\\0&1&-2&1\end{array}\right]$$

(under: I A^{-1})

Make sure to check answer

$$\underbrace{\begin{bmatrix}1&3\\2&7\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}7&-3\\-2&1\end{bmatrix}}_{A^{-1}}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\ \checkmark$$

Elimination Matrix

$E[A\ I]=[I\ A^{-1}]$ ← The statement of Gaussian-Jordan Elimination

$EA=I$ tells us $E=A^{-1}$

Elimination of solving N equations at the same time

Recitation

Find the conditions on a and b that make the matrix A invertible, and find A^{-1} when it exists

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

$a \neq b \quad a \neq 0$

1. Only square matrices can have inverses
2. If you have columns or rows of $0 \to$ not invertible
3. Two columns/rows that are the same $\to$ not invertible

How to find inverse of a matrix?

$$[A \mid I] \xrightarrow{\dots} [I \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ a & a & b & 0 & 1 & 0 \\ a & a & a & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & a-b & a-b & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right]$$

$a \neq 0$
$a - b \neq 0$

$\downarrow$

$$\begin{array}{l} \text{divided by } a \to \\ \text{divided by } a-b \to \\ \text{divided by } a-b \to \end{array} \left[\begin{array}{ccc|ccc} 1 & b/a & b/a & 1/a & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{a-b} & \frac{1}{a-b} \end{array}\right]$$

$\downarrow$

elim. b/a and b/a for I

row 1 $- \frac{b}{a}$(row 2 + row 3)

$\downarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a-b} & 0 & -\frac{b}{a(a-b)} \\ 0 & 1 & 0 & -\frac{1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{a-b} & \frac{1}{a-b} \end{array}\right]$$

$a \neq 0$
$a \neq b$

$$\boxed{A^{-1} = \frac{1}{a-b} \begin{bmatrix} 1 & 0 & -\frac{b}{a} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}$$