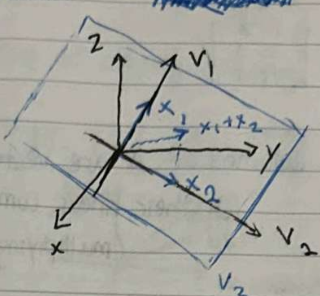
Subspaces of Three Dimensional Space - Recitation

not drawn to scale



$x_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ $x_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$

1. Find v_1 = subspace generated by x_1
   v_2 = subspace generated by x_2
   Describe $v_1 \cap v_2$

generated = the smallest subspace that contains $x_1 \rightarrow v_1$ same with v_2

v_1 = line , v_2 = line $\rightarrow$ can only multiply with scalars

maybe just $\{0\}$ is good

$\underline{v_1 \cap v_2}$ $\rightarrow$ is at $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ because of the zeroes in x_1 & x_2

they should never intersect elsewhere

2. Find V_3 = subspace generated by $\{x_1, x_2\}$
   Is V_3 equal to $V_1 \cup V_2$?
   Find a subspace S of V_3 such that $x_1 \notin S$, $x_2 \notin S$

Let's say $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$ $\rightarrow$ linear combination not on original line

$\boxed{V_3 = \text{the entire plane formed by } x_1, x_2}$

$\boxed{V_3 \neq V_1 \cup V_2}$ $\boxed{x_1 + x_2 = S}$

$\downarrow$ choice is not unique, this isn't the only answer

S of V_3 must mean it's in V_3

↳ need to draw 3rd line (S) on same paper that isn't x_1 or x_2

3. What is $V_3 \cap \{xy \text{ plane}\}$?

It is just V_2 because the intersection between V_3 and xy plane is just when $z = 0$ (z-component = 0) which is just the subspace generated by x_2

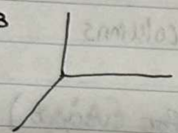
Column Space and Nullspace - Lecture 6

12/11/25

Vector Space Requirements

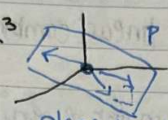
↳ $v + w$ and $c \cdot v$ are in the space
↳ all combinations $cv + dw$ are in the space

R^3



is a vector space

R^3 P



plane through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is a subspace of R^3

R^3 L

line through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is a subspace of R^3

2 subspaces : P and L

$P \cup L$ = all vectors in P or L
↳ Is this a subspace? No, because $v+w$ is not in the space

$P \cap L$ = all vectors in both P and L
↳ Is this a subspace? Yes, the only common vector is $\begin{bmatrix}0\\0\\0\end{bmatrix}$

General Question

Subspaces S and T

intersection $S \cap T$ IS a subspace

Column Space of A

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

Vectors in R^4, we're in 4-dimensional space
subspace of R^4

Column Space $C(A)$ of A = all linear combinations of columns

Does $Ax=b$ always have a solution for every RHS? (for every b)
<u>No</u>, because $Ax=b \rightarrow$ 4 eqns, 3 unknowns

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Can't always do it
Combinations of columns cannot fill 4D space

So which vectors b, allow this system to be solved?

→ If b_1 to b_4 are all 0's ($Ax=0$)

→ Solve for columns (solution for col 1 = $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
col 2 = $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, col 3 = $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$)

→ Can solve for $Ax=b$ exactly when the RHS b ~~is~~ a vector in the column space

[I can solve $Ax=b$ exactly when b is in $C(A)$]

Are all the columns needed for A? Which ones are not needed?
Col 3 is not needs because adding col 1 + 2 = col 3
We don't get anything more from col 3
col 1 and 2 are pivot columns, col 3 is not
But in all honesty, we can remove col 1 as well

The column space of this matrix is 2D subspace of R^4

The Null Space

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$N(A)$

Null space of A = contains all solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax = 0$

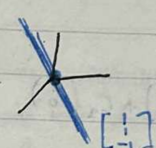
→ subspace of R^3 because we have 3 - x components

$N(A)$ contains $\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$

→ so it's a multiple of $\begin{bmatrix} c \\ c \\ -c \end{bmatrix}$ which includes the zero vector

$N(A)$ contains $c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

→ The Nullspace is a line in R^3



How do we know that nullspace is a vector space?

Check that the solutions to $Ax=0$ always give a subspace

In order to use space

if $Ax=0$ and $Ax^*=0$

(if $Av=0$ and $Aw=0$)

→ $A(v+w)=0$

→ $\boxed{Av+Aw=0}$

then $A(12v)=0$

→ the multiples will also be 0

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

> Two natural ways to tell what's in subspace
> → give a few vectors and fill it, or system of eqns that x has to satisfy

Solutions = $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

is not a subspace because doesn't go through origin

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not a solution

It's like a line or plane that doesn't go through origin

Subspaces must go through origin

Vector Supspaces (Recitation) 12/13/25

Which are subspaces of $R^3 = \left\{ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right\}$

1) $b_1 + b_2 - b_3 = 0$

This is a subspace because the zero vector holds true, the shape of this should be a plane that passes through the origin.

2) $b_1 b_2 - b_3 = 0$

This is not a subspace because even though the zero vector holds true, scalar multiplication proves this equation as false.

ex. $b_1 = 1 \quad b_2 = 1$

$1(1) - 1 = 0 \rightsquigarrow 2(2) - 2 \neq 0$

3) $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Solving for 0 vector,

$0 = 1 + c_1 + c_2$

$0 = 0 + 0 + 0$

$0 = 0 - c_1 + c_2$

$\Rightarrow c_1 = -0.5, \quad c_2 = -0.5$

This means that this is a subspace because the origin included

4) $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$b_2 \Rightarrow 0 = 1 + c_1(0) + c_2(0)$

$0 = 1 + 0 + 0$

This doesn't work, so there is no solution for $b_2 = 0$ which means this doesn't pass the origin. This is not a subspace.

Subspace Checklist

- Zero Vector Test
- Linearity Test
- Does the vector in parametric form shift from origin?

Lecture #7

Solving $Ax=0$: Pivot Variables, Special Solutions

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

At first looks col 2 is multiple of col 1 and row 1+2 = row 3

↳ Elimination

$$A = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

(not quite upper △)

upper echelon?

Amount of pivots in $A = 2 \Rightarrow$ rank of $A = 2$

rank of A = number of pivots = 2

$$\begin{array}{cccc} & & & \\ \end{array}\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

2 pivot columns

2 free columns

You can assign any variable to x_2 and x_4, 2nd and 4th column to solve for the other 2 variables. Hence free column.

of Free Variables $= \overset{\text{columns}}{n} - \underset{\text{ran}}{r}$

The 2 eqns :

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$2x_3 + 4x_4 = 0$$

Let's say $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ($x_2 = 1$, $x_4 = 0$ since we are free)

Then the x_1 and x_3 that solve the 2 eqns are $x_1 = -2$, $x_3 = 0$

$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ → a vector in the nullspace

↳ To get more just get multiples of this one

But this is just one iteration of free variables

Let's say $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ($x_2 = 0$, $x_4 = 1$) $\Rightarrow x = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

$\boxed{\text{algo} : \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

So the solution to $Ux = 0$ (null space) is all the combinations of special solutions : $x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$