11/29/25

$$\overset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}$$

want to operate with Elementary Matrix E

$$\overset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \overset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

E_{21} because we want a 0 in the 21 position

Now : $A = LU$

$$\overset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

What's L? Inverse of E

U stands for upper triangular, then L stands ~~for~~... Lower triangular

$E_{32} E_{31} E_{21} A = U$ (no row exchanges)

$$A = \underbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}_{} U$$

$= LU$

Suppose :

$$\overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}$$

$\Rightarrow E$ (left of A)

$-2 \to -5 \to 10$

$\Rightarrow EA = U$

Inverses (Reverse order)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L$$

$+2 \to +5$

Matrix $L \Rightarrow$ left of U

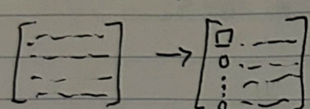
$A = LU$

$A = LU$

If no row exchanges, the multipliers go directly into L

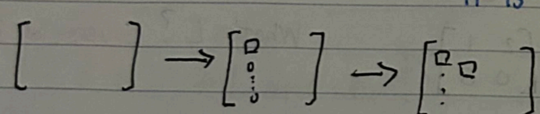
How many operations
on a $n \times n$ matrix A? (Multiplication & Subtraction)

Let's say $n = 100 \Rightarrow 100 \times 100$ matrix
how many steps?



What is the cost?
n? n^2? n^3? $n!$?

For the first step (operating on 1st row)
it is about 100^2 in cost (n^2)



2nd step is about 99^2

Pattern continues until 1^2

So the total cost
$\to n^3$
at the worst
it is n^3

Count: $n^2 + \ldots + 1^2$

$\approx \boxed{\frac{1}{3} n^3}$ on A, this is the cost

For vector B, the cost is a lot less $\to \boxed{n^2}$ RHS cost

The most fundamental algorithm for systems of equations

What happens when there are row exchanges?

↳ row exchanges happen when there are 0's in the pivot

Permutations

Say we have 3×3 matrices :

6 total ↗ group of 6 matrices

$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$, $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $P_{23} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

↑ Identity Matrix (Does nothing)

↑ Permutation matrix that exchanges row 1 and 2

↑ switches rows 1 and 3

↑ switches rows 2 and 3

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

↑ switches all 3 rows

↑ also switches all 3

$P^{-1} = P^{T}$ ⇒ permutation matrices, their inverse is the transpose

What about for 4×4 matrices? → 24 P's

Find the LU-decomposition of the Matrix 12/1/25

$A = \begin{pmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{pmatrix}$ when it exists.

For which real numbers a and b does it exist?

$a \neq 0$,

$A = LU$

$\begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix}$

↳ to upper Δ

$\begin{array}{c} \\ \times a \\ \times \frac{b}{a} \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a+b \end{bmatrix}$

U

$A = LU$

$\underset{A}{\begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix}} = \underset{L}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a+b \end{bmatrix}}$

$EA = U$

$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -b/a & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & b/a & 1 \end{bmatrix}}$

$EA = U$

$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & b/a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a+b \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & 0 & 1 \end{bmatrix}^{-1} \underset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b/a & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & b/a & 1 \end{bmatrix} = L$

X

$A = LU$

$\begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & b/a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a+b \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix}$$

we want to do elimination

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ b & b & a \end{bmatrix}$$

got here through $E_{21} \rightarrow \begin{bmatrix} 1 & & \\ -a & 1 & \\ 0 & 0 & 1 \end{bmatrix}$ 2nd row minus 1st row * a

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & b & a-b \end{bmatrix}$$

got here through $E_{31} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -b & 0 & 1 \end{bmatrix}$ 3rd row minus 1st row *

Assume $a \neq 0$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{bmatrix}$$

got here through $E_{32} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -b/a & 1 \end{bmatrix}$ 3rd row minus 1st row × a/b

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{bmatrix} \rightarrow \text{Now find } L$$

$$\underbrace{E_{32}E_{31}}E_{21}A = U$$

$\Rightarrow$ move elimination matrices to other side for L

$$\Rightarrow A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$$

$$A = \begin{bmatrix} 1 & & \\ a & 1 & \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ 0 & 1 & \\ b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & b/a & 1 \end{bmatrix} U$$

$$\boxed{L = \begin{bmatrix} 1 & & \\ a & 1 & \\ 0 & b/a & 1 \end{bmatrix}}$$

It exists when $a \neq 0$

My mistakes the first time, wrong + - signs on pivot operation $\rightarrow$ missed the E_{32} elimination matrix mw

Transposes, Permutations, spaces R^n 12/3/25

Permutations $P \rightarrow$ execute row exchanges

$A = LU = \begin{bmatrix} 1 & & 0 \\ \equiv & 1 & \\ \equiv & & 1 \end{bmatrix} \begin{bmatrix} \diagdown & \equiv \\ 0 & \diagdown \end{bmatrix}$ ← this assumes we don't have row exchanges

↓

$\boxed{PA = LU}$ → the description of elimination with row exchanges

↳ for any invertible A

Permutations : P = identity matrix with ordered rows

How many possibilities? **n!** possibilities

counts reorderings

counts all n×n permutations

$\underline{P^{-1} = P^T}$ $\underline{P^TP = I}$

Transposing

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

Transpose $(A^T)_{ij} = A_{ji}$

12/4/25

Symmetric Matrices

↳ When transposing doesn't change the matrix

$$A^T = A$$

Ex. $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

$R^T R$ is always symmetric

$$\underset{R^T}{\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$$

Why? Take transpose to see if it didn't change (symmetric)

$$(R^T R)^T = R^T R^{TT} = \underline{R^T R}$$

we got back $R^T R$ again

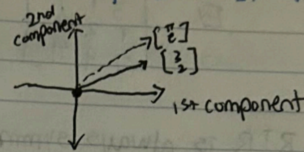
Vector Spaces 12/5/25

↓

space of vectors that allow operations of adding, scaling, lin. comb.

Examples : $R^2 \rightarrow$ all 2D vectors (real) → ex. $\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}, \ldots$

↳ the plane, "the x-y plane"



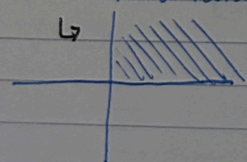
R^3 = all vectors with 3 components

↳ ex. $\begin{bmatrix}3\\2\\0\end{bmatrix}$

R^n = all vectors with n components with real numbers

what is not a vector space

↳



← 1/4 (quarter plane)

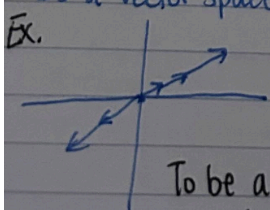
There are scalars that will take me out of this quarter plane so it is not closed

Subspaces of R^2
1. all of R^2
2. any line through $\begin{bmatrix}0\\0\end{bmatrix}$
3. zero vector alone (Z for 0)

A vector space has to be closed by addition & multiplication of vectors (linear combinations)

Is a vector space inside R^2 (SUBSPACE of R^2) Example of a subspace

Ex.



have to be able to multiply or add by any # to that vector

To be a subspace, the line in R^2 must go through the zero vector it has to contain zero because I have to be allowed to multiply by zero

12/6/25

$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$

Columns of A are vectors in R^3 (cols in R^3)

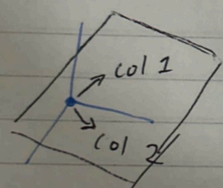
all their linear combinations form a subspace called the column space $C(A)$

(multiplying & adding)

Lesson of Today's Lecture

- Got a few vectors
- not satisfied → we want a space of vectors
- vectors are in R^3 so our space of vectors will be in R^3

(but we have to be able to take their combinations)



What's in the whole $C(A)$?

How to draw the column space?

When we take all the combinations of $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ we fill out the whole plane