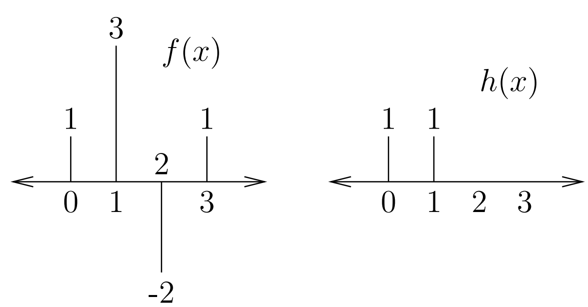
EEL 6562
Image Processing & Computer Vision
Linear and Circular Convolution Example

Linear Convolution

One dimensional linear discrete convolution is defined as:

$$g(x) = \sum_{s=-\infty}^{\infty} f(s)\, h(x-s) = f(x) * h(x)$$

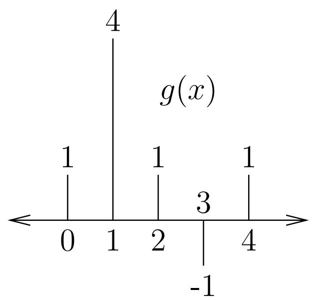
For example, consider the convolution of the following two functions:



This convolution can be performed graphically by reflecting and shifting $h(x)$, as shown in Figure 1. The samples of $f(s)$ and $h(s-x)$ that line up vertically are multiplied and summed:

$$\begin{aligned}
g(0) &= f(-1)h(1) + f(0)h(0) = 0 + 1 = 1 \\
g(1) &= f(0)h(1) + f(1)h(0) = 1 + 3 = 4 \\
g(2) &= f(1)h(1) + f(2)h(0) = 3 + -2 = 1 \\
g(3) &= f(2)h(1) + f(3)h(0) = -2 + 1 = -1 \\
g(4) &= f(3)h(1) + f(4)h(0) = 1 + 0 = 1
\end{aligned}$$

The result of the convolution is as shown below:



Notice that when $f(x)$ is of length 4, and $h(x)$ is of length 2, the linear convolution is of length $4+2-1=5$.

Circular Convolution

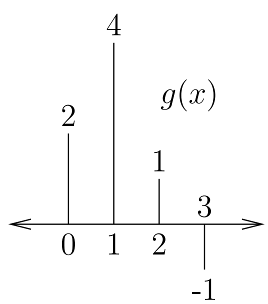
One dimensional circular discrete convolution is defined as:

$$g(x) = \sum_{s=0}^{M-1} f(s)\, h((x-s) \bmod M) = f(x) \circledast h(x)$$

For $M = 4$, the convolution can be performed using circular reflection and shifts of $h(x)$, as shown in Figure 2. The samples of $f(s)$ and $h((s-x) \bmod M)$ that line up vertically are multiplied and summed:

$$\begin{aligned}
g(0) &= f(3)h(1) + f(0)h(0) = 1 + 1 = 2 \\
g(1) &= f(0)h(1) + f(1)h(0) = 1 + 3 = 4 \\
g(2) &= f(1)h(1) + f(2)h(0) = 3 + -2 = 1 \\
g(3) &= f(2)h(1) + f(3)h(0) = -2 + 1 = -1
\end{aligned}$$

The result of the convolution is as shown below:



Notice that $f(x)$ and $h(x)$ are both treated as if they are of length 4, and the circular convolution is also of length 4.

Linear Convolution as Circular Convolution

If $f(x)$ and $g(x)$ are both treated as if they are of length $4+2-1=5$, then the following circular convolution is calculated:

$$\begin{aligned}
g(0) &= f(4)h(1) + f(0)h(0) = 0 + 1 = 1 \\
g(1) &= f(0)h(1) + f(1)h(0) = 1 + 3 = 4 \\
g(2) &= f(1)h(1) + f(2)h(0) = 3 + -2 = 1 \\
g(3) &= f(2)h(1) + f(3)h(0) = -2 + 1 = -1 \\
g(4) &= f(3)h(1) + f(4)h(0) = 1 + 0 = 1
\end{aligned}$$

This procedure is called "zero padding." Notice that this circular convolution matches the linear convolution.

In general, if $f(x)$ has length A, and $h(x)$ has length B, and both $f(x)$ and $h(x)$ are zero padded out to length C, where $C \geq A+B-1$, then the C-point circular convolution matches the linear convolution.

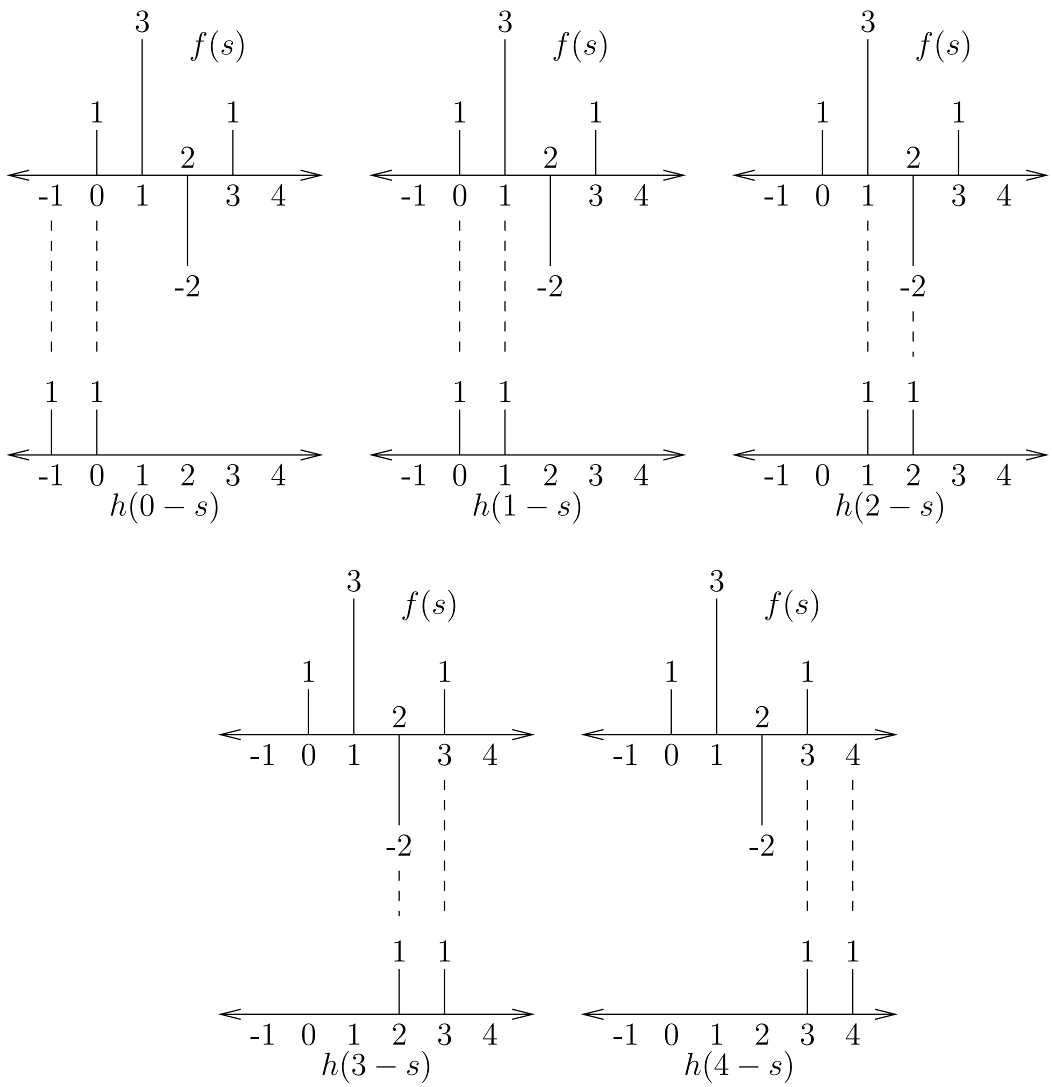


Figure 1: Linear convolution by the graphical method.

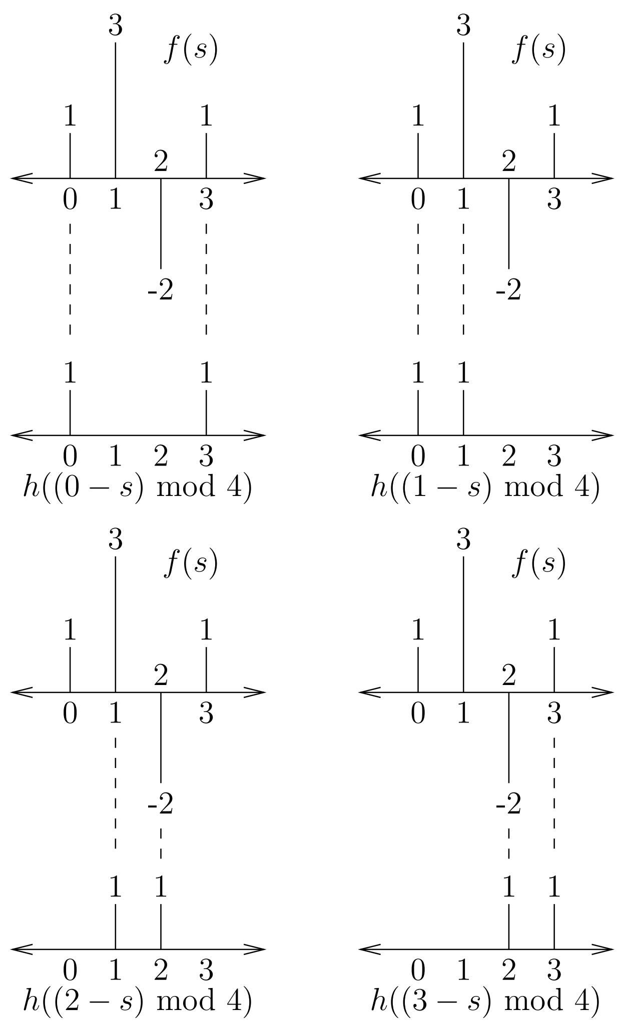


Figure 2: Circular convolution using the graphical method.