Handy Transformations -

$$\begin{aligned}\mathbb{E}[a+bX] &= a+b\mathbb{E}[X]\\ \mathbb{E}\left[X^2\right] &= \mathrm{Var}(X)+(\mathbb{E}[X])^2\\ \textit{(ditto but flipped) }\mathrm{Var}(X) &= \mathbb{E}\left[X^2\right]-(\mathbb{E}[X])^2\\ \mathrm{Var}(a+bX) &= b^2\mathrm{Var}(X)\\ \mathrm{SD}(a+bX) &= |b|\mathrm{SD}(X)\\ \mathrm{Cov}(a+bX,c+dY) &= b\cdot d\cdot\mathrm{Cov}(X,Y)\\ \mathrm{Corr}(a+bX,c+dY) &= \mathrm{Corr}(X,Y)\end{aligned}$$

Uniform distribution facts from HW -

1. If $U\sim\mathrm{Unif}[0,1]$, then for any fixed $a>0$ and $b\in\mathbb{R}$, we have that $aU+b\sim\mathrm{Unif}[b,a+b]$.
2. If $U\sim\mathrm{Unif}[0,1]$, then $\mathbb{E}[U]=\frac{1}{2}$ and $\mathrm{Var}(U)=\frac{1}{12}$.

When I am not protected from me being me

Set Algebra:

- **Union -** A or B; $A\cup B$.
- **Intersection -** A and B; $A\cap B$.
- **Complement -** not A; A^C.
- **Difference -** A but not B; $A\backslash B$
- **Disjoint Events aka. mutually exclusive -** events A and B are disjoint if they don't share any outcomes in common (i.e., A and $B=\varnothing$).
- **Subset -** $A\subseteq B$

Trial - a repetition of a random experiment/process. Trials're independent: none gives information about the others; are stable: reuslts could have appeared in any order.

Outcome - a possible result of a trial.

Sample space - the set of all possible outcomes. Often denoted as S.

Event - a set of outcomes of an experiment (i.e., a subset of the sample space).

Probability - is a long run proportion of an outcome in repeated trials.

- Probabilities act as "targets" of estimation
- Proportions based on data "estimate" probabilities. Would approach probabilities if observe infinite trials.

Formally, $\boxed{A\mapsto\mathbb{P}(A),\mathbb{P}(A)\in[0,1]}$ A probability $\mathbb{P}(\cdot)$ on a sample space S is a function that assigns a snumber between 0 and 1 to all events, A in the sample space (i.e., any possible subset of the sample space) and subject to three requirements (axioms):

1. $\mathbb{P}(S)=1$: probability of *something* in the sample space happening is 1
2. $\mathbb{P}(A)\geq 0,\forall A$
3. A and $B=\varnothing$ (A, B disjoint) $\Rightarrow\mathbb{P}(A\cup B)=\mathbb{P}(A)+\mathbb{P}(B)$

More takeaways

- $A\subseteq B\Rightarrow\mathbb{P}(A)\leq\mathbb{P}(B)$
- A, B, C are pairwise disjoint $\Rightarrow\mathbb{P}(A\cup B\cup C)=\mathbb{P}(A)+\mathbb{P}(B)+\mathbb{P}(C)$
- $\mathbb{P}(A\cup B)=\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\text{ and }B)$

Joint Probability - $\mathbb{P}(A\text{ and }B)$ is the joint probability that events A and B occur.

Conditional Probability - $\mathbb{P}(A|B)=\frac{\mathbb{P}(A\text{ and }B)}{\mathbb{P}(B)},\mathbb{P}(B)>0$ the probability of observing event A if (given that) one has observed B. Bear in mind: $\mathbb{P}(A|B)\neq\mathbb{P}(B|A)$

Product Rule - $\mathbb{P}(A\text{ and }B)=\mathbb{P}(B)\cdot\mathbb{P}(A|B)$.

Independent Events - A and B are independent if $\mathbb{P}(A\text{ and }B)=\mathbb{P}(A)\cdot\mathbb{P}(B)$; one event happening doesn't affect the probability of the other event happening. Can easily deduce that $\mathbb{P}(A|B)=\mathbb{P}(A)$ and $\mathbb{P}(B|A)=\mathbb{P}(B)$.

Independence and Disjointness are **NOT** synonyms.

- Independent $\Rightarrow\mathbb{P}(A\text{ and }B)=\mathbb{P}(A)\cdot\mathbb{P}(B)$
- Disjoint $\Rightarrow\mathbb{P}(A\text{ and }B)=0$. Disjoint events are extremely dependent: If one event occurs, the other cannot.

Random variable - a numerical function on a sample space with probabilities. (Think as a scoring mechanism.)

- Input: an outcome in the sample space
- Output: a number

Discrete RVs - only countably many values are possible

Continous RVs - can take on uncountably infinitely many values

Probability Distribution Function (PDF) - $\boxed{\mathbf{p}_X(x)=\mathbb{P}(X=x)}$ is the probability that the random variable X takes on the value x. I really hate $\mathbf{p}_X(x)$ this styling, so only $\mathbb{P}(X=x)$ moving forward.

Properties of PDFs - any function that satisfies the following conditions is a probability distribution function of a Discrete random variable:

1. $\mathbb{P}(X=x)\geq 0,\forall x\in\mathbb{R}$ (for any real number)
2. $\mathbb{P}(X=x)>0$ for values that the random variable X can actually take on
3. $\mathbb{P}(X=x)=0$ for values that aren't possible for the random variable X
4. $\sum_x\mathbb{P}(X=x)=1$

Expected Value - $\boxed{\mathbb{E}[X]=\sum_x x\cdot\mathbb{P}(X=x)}$

Variance - a probability weighted mean of the possible squared deviations.

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}\left[(X-\mathbb{E}[X])^2\right]\\ &= \sum_x(x-\mathbb{E}[X])^2\cdot\mathbb{P}(X=x)\\ &= \mathbb{E}\left[X^2\right]-(\mathbb{E}[X])^2\end{aligned}$$

Standard Deviation - $\boxed{\mathrm{SD}(X)=\sqrt{\mathrm{Var}(X)}}$

Given $Y=g(X)$ and X's PDF -

$$\begin{aligned}\mathbb{E}[Y] &= \sum_x g(x)\cdot\mathbb{P}(X=x)\\ \mathrm{Var}(Y) &= \sum_x(g(x)-\mathbb{E}[g(X)])^2\cdot\mathbb{P}(X=x)\end{aligned}$$

RVs with only 2 outcomes - (not necessarily Bernoullis yet) Suppose RV X's PDF is:

$$\begin{cases}\mathbb{P}(X=a) & =p\\ \mathbb{P}(X=b) & =1-p\text{, then:}\\ \mathbb{P}(X=\text{all other values}) & =0\end{cases}$$

$$\begin{aligned}\mathbb{E}[X] &= ap+b(1-p)\\ \mathrm{Var}(X) &= (a-b)^2p(1-p)\\ \mathrm{SD}(X) &= |a-b|\sqrt{p(1-p)}\end{aligned}$$

Bernoulli Random Variable - aforementioned when $\begin{cases}a & =1\\ b & =0\end{cases}$. If $\boxed{X\sim\mathrm{Bern}(p)}$, then:

$$\begin{aligned}\mathbb{E}[X] &= p\\ \mathrm{Var}(X) &= p(1-p)\\ \mathrm{SD}(X) &= \sqrt{p(1-p)}\end{aligned}$$

- Variance maximized when $p=0.5$
- Variance minimized when $p=0$ or 1

Useful for tracking how many successes happen in n independent trials.

Binomial Random Variable - If $\boxed{X\sim\mathrm{Binom}(n,p)}$, then:

$$\begin{aligned}\mathbb{E}[X] &= n\cdot p\\ \mathrm{Var}(X) &= n\cdot p(1-p)\\ \mathrm{SD}(X) &= \sqrt{n\cdot p(1-p)}\end{aligned}$$

Binomial Problems - following must hold:

1. Constant success probability p and failure probability $(1-p)$.
2. Fixed total number of trials: n
3. trials are **independent**
4. Only two outcomes of interest (success or failure) on each trial
5. Want to find the probability of observing k successes among the total number of n trials. (Order doesn't matter.)

$\boxed{\mathbb{P}(k\text{ successes in }n\text{ trials})=\binom{n}{k}p^k(1-p)^{n-k}}$

Combination - how many ways to choose k out of n: $\boxed{\binom{n}{k}=\frac{n!}{k!(n-k!)}}$.

Binomial Distribution - $\boxed{X\sim\mathrm{Binom}(n,p)}$ where X is an RV tracking the number of successes in n independent trials with success probability p. X's PDF:

$\boxed{\forall k\in\{0,\ldots,n\},\mathbb{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}}$.

Attn: X here is not for one single trial!.

- A **Bernoulli RV**: useful for one trial's success/failure.
- A **Binomial RV**: useful for total number of successes.

Binomial as the Sum of Bernoullis - n independent Bernoulli RVs each with the same success probability p: $\forall i\in 1,\ldots,n,X_i\sim\mathrm{Bern}(p)$.

Define $\boxed{S_n=\sum_{i=1}^n X_i}$, then denote $\boxed{S_n\sim\mathrm{Binom}(n,p)}$. $\mathrm{Binom}(1,p)=\mathrm{Bern}(p)$

Joint Distribution of 2 RVs - the probability that 2 RVs simultaneously take on 2 values. $\forall x\in X,\forall y\in Y$ $\boxed{\mathbb{P}(X=x,Y=y)}$.

Marginal probability distribution - can be found given with the joint PDF: $\boxed{\mathbb{P}(X=x)=\sum_y\mathbb{P}(X=x,Y=y)}$

Z-Score of a Random Variable X - $\boxed{Z(X)=\frac{X-\mathbb{E}[X]}{SD(X)}}$

$\mathbb{E}[Z(X)]=0$ and $SD(Z(X))=1$

Correlation and Covariance

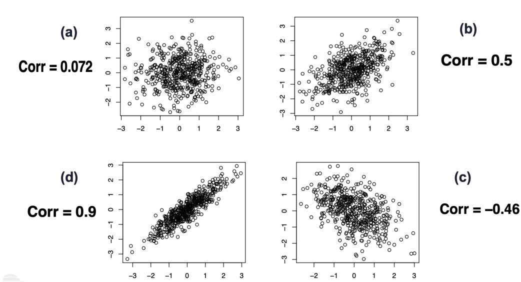
Correlation Between RVs X and Y - "average of the product of z-scores"

$$\begin{aligned}\mathrm{Corr}(X,Y) &= \mathbb{E}[Z(X)\cdot Z(Y)]\\ &= \frac{\mathrm{Cov(X,\ Y)}}{SD(X)\cdot SD(Y)}\end{aligned}$$

- $\mathrm{Corr}(X,Y)$ is unit-free.
- $\mathrm{Corr}(X,Y)$ doesn't exist if either $SD(X)=0$ or $SD(Y)=0$ (can't divide by 0!).
- Correlation is guaranteed to lie between +1 (perfect positive correlation) and −1 (perfect negative correlation). Hence Corr is more commonly used than Covariance.

$\mathrm{Corr}(X,Y)$ here quantifies the strength and direction of the **linear relationship** between two variables. Therefore, if two variables have a strong but non-linear relationship, $\mathrm{Corr}(X,Y)\approx 0$, indicating no linear correlation, even though a strong non-linear relationship exists.

Correlation Examples



Covariance Between RVs X and Y - "average of the product of the centered variables". Necessary for assessing variability of sums of RVs (e.g. portfolios).

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]\\ &= \mathrm{Corr}(X,Y)\cdot SD(X)\cdot SD(Y)\end{aligned}$$

- $\mathrm{Cov}(X,Y)$ has funny units: product of the X and Y units.
- $\mathrm{Cov}(X,Y)$ always exists. If SDs are 0, $\mathrm{Cov}(X,Y)=0$
- $\mathrm{Cov}(X,X)=V(X)$
- If $SD(X)>0$ and $SD(Y)>0$, then $\mathrm{Corr}(X,Y)$ and $\mathrm{Cov}(X,Y)$ have the same sign.

Expected Value of RVs summed - is regardless of RVs' joint distribution:

$$\boxed{\begin{array}{l}\mathbb{E}[X+Y]=\mathbb{E}[X]+\mathbb{E}[Y]\\ \mathbb{E}[X+Y+W]=\mathbb{E}[X]+\mathbb{E}[Y]+\mathbb{E}[W]\\ \mathbb{E}\left[\sum_{i=1}^{n}X_i\right]=\sum_{i=1}^{n}\mathbb{E}[X_i]=\mathbb{E}[X_1]+\cdots+\mathbb{E}[X_n]\end{array}}$$

Variance of of RVs summed -

$$\boxed{\begin{array}{l}\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)+2\mathrm{Cov}(X,Y)\\ \mathrm{Var}(X+Y+W)=\mathrm{Var}(X)+\mathrm{Var}(Y)+\mathrm{Var}(W)\\ +2\mathrm{Cov}(X,Y)+2\mathrm{Cov}(X,W)+2\mathrm{Cov}(Y,W)\\ \mathrm{Var}\left(\sum_{i=1}^{n}X_i\right)=\sum_{i=1}^{n}\mathrm{Var}(X_i)+2\sum_{i<j}\mathrm{Cov}(X_i,X_j)\end{array}}$$

Have to consider the covariance of all possible pairs: X_i and X_j.

- If $\mathrm{Corr}(X,Y)$ increases, then $\mathrm{Var}(X+Y)$ increases.
- If $V(X)=V(Y)$, then $\mathrm{Var}(X+Y)$ is maximized when $\mathrm{Cov}(X,Y)$ is maximized.

Uncorrelated RVs - if $\mathrm{Corr}(X,Y)=0$. Equivalently, they are uncorrelated if $\mathrm{Cov}(X,Y)=0, SD(X)>0, SD(Y)>0$

Variance of of Uncorrelated or Independent RVs summed -

$$\boxed{\begin{array}{l}\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)\\ \mathrm{Var}\left(\sum_{i=1}^{n}X_i\right)=\sum_{i=1}^{n}\mathrm{Var}(X_i)\end{array}}$$

No change in expected value's formula.

Independent RVs -

$$\boxed{\forall x,y,\mathbb{P}(X=x,Y=y)=\mathbb{P}(X=x)\mathbb{P}(Y=y)}$$

Independence implies uncorrelatedness: if two RVs X and Y are independent, then they are uncorrelated.

$$\boxed{\text{Independence}\Rightarrow\mathrm{Corr}(X,Y)=0=\mathrm{Cov}(X,Y)}$$

But uncorrelated RVs can be dependent!

***(iid)* Independent and Identically Distributed RVs -** for a collection of *iid* RVs $\{X_1,\ldots,X_n\}$: $\forall i\in\{1,\ldots,n\}$

$$\boxed{\begin{array}{c}\mathbb{E}[X_i]=\mu\\ \mathrm{Var}(X_i)=\sigma^2\\ \mathrm{SD}(X_i)=\sigma\end{array}}$$

Sum of *iid* RVs - $S_n=X_1+\cdots+X_n$:

$$\boxed{\begin{array}{c}\mathbb{E}[S_n]=n\cdot\mu\\ \mathrm{Var}(S_n)=n\cdot\sigma^2\\ \mathrm{SD}(X_i)=\sqrt{n}\cdot\sigma\end{array}}$$

Central Limit Theorem (CLT)

If $\{X_1,\ldots,X_n\}$ are *iid* with expected value $\mathbb{E}[X_i]=\mu$ and variance $\mathrm{Var}(X_i)=\sigma^2<\infty$, then as $n\to\infty$:

- $S_n\sim\mathcal{N}(n\cdot\mu,\sqrt{n}\cdot\sigma)$
- $\mathrm{Mean}_n\sim\mathcal{N}(\mu,\frac{\sigma}{\sqrt{n}})$

If n is large enough (heuristic: $n>30$), we can calculate probabilities for the sum and mean of RVs by using the normal distribution.

Emperical Rules under CLT -

- 50% of the time,
  - S_n will fall within $n\mu\pm\frac{2}{3}\sqrt{n}\sigma$
  - M_n will fall within $\mu\pm\frac{2}{3}\frac{\sigma}{\sqrt{n}}$
- 68% of the time,
  - S_n will fall within $n\mu\pm\sqrt{n}\sigma$
  - M_n will fall within $\mu\pm\frac{\sigma}{\sqrt{n}}$
- 95% of the time,
  - S_n will fall within $n\mu\pm2\sqrt{n}\sigma$
  - M_n will fall within $\mu\pm2\frac{\sigma}{\sqrt{n}}$
- 99.7% of the time,
  - S_n will fall within $n\mu\pm3\sqrt{n}\sigma$
  - M_n will fall within $\mu\pm3\frac{\sigma}{\sqrt{n}}$

Sampling and Confidence Intervals

Confidence Interval - contains an unknown (population) quantity at some specified sampling frequency.

- Confidence intervals do not depend on population size, but only on sample size.
- For a given sample size, can be very precise with low confidence or very imprecise with high confidence.
- 2x the precision requires 4x sample size; 3x the precision requires 9x sample size.

Wording matters...

- OK: "I am 95% confident the interval [a,b] contains the true population proportion"
- OK: "There is a 95% probability the interval [a,b] contains the true population proportion"
- Not OK: "There is a 95% probability the true population proportion lies in the interval [a,b]"

Confidence Level (L) to c - $\boxed{c=\texttt{qnorm}(p=\frac{(1+L)}{2},\mu=0,\sigma=1)}$ (find the value c such that the area under $\mathcal{N}(\mu,\sigma)$ is p)

confidence level	L	c
90%	0.9	1.65
95%	0.95	1.96
99%	0.99	2.58

For a population mean - Given sample size n, sample average $\bar{x}$, and sample standard deviation s, we are X% confident the true population mean lies in the interval: $\boxed{\bar{x}\pm\left(c\frac{s}{\sqrt{n}}\right)}$ "MOE": $\left(c\frac{s}{\sqrt{n}}\right)$

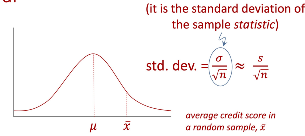
For a population proportion - Given sample size n, sample proportion $\bar{p}$, and standard deviation in the population to be 0.5, we are X% confident the true population mean lies in the interval: $\boxed{\bar{p}\pm\left(c\frac{0.5}{\sqrt{n}}\right)}$ "MOE": $\left(c\frac{0.5}{\sqrt{n}}\right)$

Important assumptions:

- Sample is random
- Sample is large enough $(n>30)$ for CLT
- The worst possible standard deviation in the population to be 0.5

Sampling Distribution - is well approximated by $\mathcal{N}(\mu=\text{true population parameter},\sigma=\frac{\text{population SD}}{\sqrt{n}})$, based on CLT.

Using the Sampling Distribution to Construct a Confidence Interval



The fact that 95% of the sample means lie within 1.96 standard errors of the true (unknown) population mean ⇒ The confidence intervals constructed using the given formula will 'capture the true mean' 95% of the time!

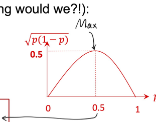
Take away: assuming a conservative confidence interval based on 0.5 is not the only way! Can estimate standard error using the surveyed proportion too.

Similar Analysis for Proportions

- When outcomes are binary, the standard deviation of observations in the populatio[n] equals $\sqrt{p(1-p)}$.
- We get the standard error by dividing by $\sqrt{n}$. The exact interval is $\bar{p}\pm1.96\sqrt{\frac{p\,(1-p)}{n}}$.

  the 'standard erro[r]'

  p (not $\bar{p}$) Theoretical real value
- But since we don't know p (if we did we won't be sampling would we?!):
  - We can take a conservative approach and assume p = 0.5, $\boxed{\bar{p}\pm1.96\frac{0.5}{\sqrt{n}}}$
  - Or, an approximate approach by assuming p = $\bar{p}$ to get the practical alternative, $\boxed{\bar{p}\pm1.96\sqrt{\frac{\bar{p}\,(1-\bar{p})}{n}}}$



Sampling Errors - the sample-to-sample variations due to pure chance. MOE and confidence intervals quantify this uncertainty well.

Non-Sampling Errors - (some examples)

- Selection Bias: happens when each member of the population does not have the same chance of being selected.
- Response/Non-response Bias: happens when some fraction of the individuals surveyed don't respond for reasons related to what's being asked in the survey

R Distribution Functions

- `p` ("probability"): cumulative distribution function ("what is the probability above or below a cutoff?")
- `q` ("quantile"): inverse CDF ("what value do we find at, say, 80% of the way to the maximal value?")
- `d` ("density"): density function (gives us the "height" or y-value of distribution for a particular z-score - mainly useful in plotting)

```
# Exactly k successes in n trials
  given success probability p
dbinom(k, size=n, p=p)
# k or more successes in n trials
  given success probability p
sum(dbinom(k:n, size=n, p=p))

# Probability of this value or less
pnorm(value, mean, sd)
# Probability of this value or
  greater
pnorm(value, mean, sd, lower.tail=
  FALSE)
# Highest value associated with a
  given percentile
qnorm(percentile, mean, sd)
```