Handy Transformations -

$$\mathbb{E}[a+bX] = a + b\mathbb{E}[X]$$
$$\text{Var}(a+bX) = b^2\text{Var}(X)$$
$$\text{SD}(a+bX) = |b|\text{SD}(X)$$
$$\text{Cov}(a+bX, c+dY) = b\cdot d\cdot \text{Cov}(X,Y)$$

Set Algebra:

- **Union** - A or B; $A \cup B$.
- **Intersection** - A and B; $A \cap B$.
- **Complement** - not A; A^C.
- **Difference** - A but not B; $A \backslash B$
- **Disjoint Events aka. mutually exclusive** - events A and B are disjoint if they don't share any outcomes in common (i.e., A and $B = \varnothing$).
- **Subset** - $A \subseteq B$

Trial - a repetition of a random experiment/process. Trials're independent: none gives information about the others; are stable: reuslts could have appeared in any order.

Outcome - a possible result of a trial.

Sample space - the set of all possible outcomes. Often denoted as S.

Event - a set of outcomes of an experiment (i.e., a subset of the sample space).

Probability - is a long run proportion of an outcome in repeated trials.

- Probabilities act as "targets" of estimation
- Proportions based on data "estimate" probabilities. Would approach probabilities if observe infinite trials.

Formally, $\boxed{A \mapsto \mathbb{P}(A), \mathbb{P}(A) \in [0,1]}$ A probability $\mathbb{P}(\cdot)$ on a sample space S is a function that assigns a snumber between 0 and 1 to all events, A in the sample space (i.e., any possible subset of the sample space) and subject to three requirements (axioms):

1. $\mathbb{P}(S) = 1$: probability of *something* in the sample space happening is 1
2. $\mathbb{P}(A) \geq 0, \forall A$
3. A and $B = \varnothing$ (A, B disjoint) $\Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

More takeaways

- $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$
- A, B, C are pairwise disjoint $\Rightarrow \mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)$
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B)$

Joint Probability - $\mathbb{P}(A \text{ and } B)$ is the joint probability that events A and B occur.

Conditional Probability -

$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}, \mathbb{P}(B) > 0$ the probability of observing event A if (given that) one has observed B. Bear in mind: $\mathbb{P}(A|B) \neq \mathbb{P}(B|A)$

Product Rule - $\mathbb{P}(A \text{ and } B) = \mathbb{P}(B)\cdot\mathbb{P}(A|B)$.

Independent Events - A and B are independent if $\mathbb{P}(A \text{ and } B) = \mathbb{P}(A)\cdot\mathbb{P}(B)$; one event happening doesn't affect the probability of the other event happening. Can easily deduce that $\mathbb{P}(A|B) = \mathbb{P}(A)$ and $\mathbb{P}(B|A) = \mathbb{P}(B)$.

Independence and Disjointness are **NOT** synonyms.

- Independent $\Rightarrow \mathbb{P}(A \text{ and } B) = \mathbb{P}(A)\cdot\mathbb{P}(B)$
- Disjoint $\Rightarrow \mathbb{P}(A \text{ and } B) = 0$. Disjoint events are extremely dependent: If one event occurs, the other cannot.

Random variable - a numerical function on a sample space with probabilities. (Think as a scoring mechanism.)

- Input: an outcome in the sample space
- Output: a number

Discrete RVs - only countably many values are possible

Continous RVs - can take on uncountably infinitely many values

Probability Distribution Function (PDF) - $\boxed{\mathbf{p}_X(x) = \mathbb{P}(X = x)}$ is the probability that the random variable X takes on the value x. I really hate $\mathbf{p}_X(x)$ this styling, so only $\mathbb{P}(X = x)$ moving forward.

Properties of PDFs - any function that satisfies the following conditions is a probability distribution function of a Discrete random variable:

1. $\mathbb{P}(X = x) \geq 0, \forall x \in \mathbb{R}$ (for any real number)
2. $\mathbb{P}(X = x) > 0$ for values that the random variable X can actually take on
3. $\mathbb{P}(X = x) = 0$ for values that aren't possible for the random variable X
4. $\sum_x \mathbb{P}(X = x) = 1$

Expected Value - $\boxed{\mathbb{E}[X] = \sum_x x\cdot\mathbb{P}(X = x)}$

Variance - a probability weighted mean of the possible squared deviations.

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \sum_x (x - \mathbb{E}[X])^2 \cdot \mathbb{P}(X = x) \\ &= \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2\end{aligned}$$

Standard Deviation - $\boxed{\text{SD}(X) = \sqrt{\text{Var}(X)}}$

Given $Y = g(X)$ and X's PDF -

$$\mathbb{E}[Y] = \sum_x g(x)\cdot\mathbb{P}(X = x)$$
$$\text{Var}(Y) = \sum_x (g(x) - \mathbb{E}[g(X)])^2 \cdot \mathbb{P}(X = x)$$

RVs with only 2 outcomes - (not necessarily Bernoullis yet) Suppose RV X's PDF is:

$$\begin{cases}\mathbb{P}(X = a) & = p \\ \mathbb{P}(X = b) & = 1 - p \text{, then:} \\ \mathbb{P}(X = \text{all other values}) & = 0\end{cases}$$

$$\mathbb{E}[X] = ap + b(1-p)$$
$$\text{Var}(X) = (a-b)^2 p(1-p)$$
$$\text{SD}(X) = |a - b|\sqrt{p(1-p)}$$

Bernoulli Random Variable - aforementioned when $\begin{cases}a & = 1 \\ b & = 0\end{cases}$. If $\boxed{X \sim \text{Bern(p)}}$, then:

$$\mathbb{E}[X] = p$$
$$\text{Var}(X) = p(1-p)$$
$$\text{SD}(X) = \sqrt{p(1-p)}$$

- Variance maximized when $p = 0.5$
- Variance minimized when $p = 0$ or 1

Useful for tracking how many successes happen in n independent trials.

Binomial Problems - following must hold:

1. Constant success probability p and failure probability $(1-p)$.
2. Fixed total number of trials: n
3. trials are **independent**
4. Only two outcomes of interest (success or failure) on each trial
5. Want to find the probability of observing k successes among the total number of n trials. (Order doesn't matter.)

$\boxed{\mathbb{P}(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k (1-p)^{n-k}}$

Combination - how many ways to choose k out of n: $\boxed{\binom{n}{k} = \frac{n!}{k!(n-k!)}}$.

Binomial Distribution - $\boxed{X \sim \text{Binom}(n,p)}$ where X is an RV tracking the number of successes in n independent trials with success probability p. X's PDF:

$\boxed{\forall k \in \{0,\ldots,n\}, \mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}}$.

Attn: X here is not for one single trial!.

- A **Bernoulli RV**: useful for one trial's success/failure.
- A **Binomial RV**: useful for total number of successes.

Binomial as the Sum of Bernoullis - n independent Bernoulli RVs each with the same success probability p: $\forall i \in 1,\ldots,n, X_i \sim \text{Bern}(p)$.

Define $\boxed{S_n = \sum_{i=1}^n X_i}$, then denote

$\boxed{S_n \sim \text{Binom}(n,p)}$. $\text{Binom}(1,p) = \text{Bern}(p)$

Joint Distribution of 2 RVs - the probability that 2 RVs simultaneously take on 2 values.

$\forall x \in X, \forall y \in Y$ $\boxed{\mathbb{P}(X = x, Y = y)}$.

Marginal probability distribution - can be found given with the joint PDF:

$\boxed{\mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y)}$

Z-Score of a Random Variable X -

$\boxed{Z(X) = \frac{X - \mathbb{E}[X]}{SD(X)}}$

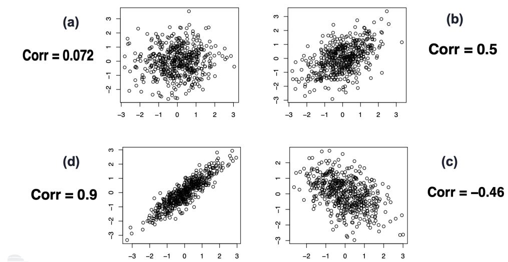
$\mathbb{E}[Z(X)] = 0$ and $SD(Z(X)) = 1$

Correlation Between RVs X and Y - "average of the product of z-scores"

$$\begin{aligned}\text{Corr}(X,Y) &= \mathbb{E}[Z(X)\cdot Z(Y)] \\ &= \frac{\text{Cov(X, Y)}}{SD(X)\cdot SD(Y)}\end{aligned}$$

- $\text{Corr}(X,Y)$ is unit-free.
- $\text{Corr}(X,Y)$ doesn't exist if either $SD(X) = 0$ or $SD(Y) = 0$ (can't divide by 0!).
- Correlation is guaranteed to lie between $+1$ (perfect positive correlation) and -1 (perfect negative correlation). Hence Corr is more commonly used than Covariance.

Correlation Examples



Covariance Between RVs X and Y - "average of the product of the centered variables". Necessary for assessing variability of sums of RVs (e.g. portfolios).

$$\begin{aligned}\text{Cov}(X,Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\ &= \text{Corr}(X,Y)\cdot SD(X)\cdot SD(Y)\end{aligned}$$

- $\text{Cov}(X,Y)$ has funny units: product of the X and Y units.
- $\text{Cov}(X,Y)$ always exists. If SDs are 0, $\text{Cov}(X,Y) = 0$
- $\text{Cov}(X,X) = V(X)$
- If $SD(X) > 0$ and $SD(Y) > 0$, then $\text{Corr}(X,Y)$ and $\text{Cov}(X,Y)$ have the same sign.

Expected Value of RVs summed - is regardless of RVs' joint distribution:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$
$$\mathbb{E}[X+Y+W] = \mathbb{E}[X] + \mathbb{E}[Y] + \mathbb{E}[W]$$
$$\mathbb{E}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{E}[X_i] = \mathbb{E}[X_i] + \cdots + \mathbb{E}[X_n]$$

Variance of of RVs summed -

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)$$
$$\text{Var}(X+Y+W) = \text{Var}(X) + \text{Var}(Y) + \text{Var}(W)$$
$$+ 2\text{Cov}(X,Y) + 2\text{Cov}(X,W) + 2\text{Cov}(Y,W)$$
$$\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i) + 2\sum_{i<j} \text{Cov}(X_i, X_j)$$

Have to consider the covariance of all possible pairs: X_i and X_j.

- If $\text{Corr}(X,Y)$ increases, then $\text{Var}(X+Y)$ increases.
- If $V(X) = V(Y)$, then $\text{Var}(X+Y)$ is maximized when $\text{Cov}(X,Y)$ is maximized.

Uncorrelated RVs - if $\text{Corr}(X, Y) = 0$. Equivalently, they are uncorrelated if $\text{Cov}(X, Y) = 0, SD(X) > 0, SD(Y) > 0$

Variance of of Uncorrelated or Independent RVs summed -

$$\boxed{\begin{gathered}\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \\ \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i)\end{gathered}}$$

No change in expected value's formula.

Independent RVs -

$$\boxed{\forall x, y, \mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)}$$

Independence implies uncorrelatedness: if two RVs X and Y are independent, then they are uncorrelated.

$$\boxed{\text{Independence} \Rightarrow \text{Corr}(X, Y) = 0 = \text{Cov}(X, Y)}$$

But uncorrelated RVs can be dependent!

***(iid)* Independent and Identically Distributed RVs -** for a collection of *iid* RVs $\{X_1, \ldots, X_n\}$: $\forall i \in \{1, \ldots, n\}$

$$\boxed{\begin{gathered}\mathbb{E}[X_i] = \mu \\ \text{Var}(X_i) = \sigma^2 \\ \text{SD}(X_i) = \sigma\end{gathered}}$$

Sum of *iid* RVs - $S_n = X_1 + \cdots + X_n$:

$$\boxed{\begin{gathered}\mathbb{E}[S_n] = n \cdot \mu \\ \text{Var}(S_n) = n \cdot \sigma^2 \\ \text{SD}(X_i) = \sqrt{n} \cdot \sigma\end{gathered}}$$

Central Limit Theorem (CLT)

If $\{X_1, \ldots, X_n\}$ are *iid* with expected value $\mathbb{E}[X_i] = \mu$ and variance $\text{Var}(X_i) = \sigma^2 < \infty$, then as $n \to \infty$:

- $S_n \sim \mathcal{N}(n \cdot \mu, \sqrt{n} \cdot \sigma)$
- $\text{Mean}_n \sim \mathcal{N}(\mu, \frac{\sigma}{\sqrt{n}})$

If n is large enough (heuristic: $n > 30$), we can calculate probabilities for the sum and mean of RVs by using the normal distribution.

Emperical Rules under CLT -

- 50% of the time,
  - S_n will fall within $n\mu \pm \frac{2}{3}\sqrt{n}\sigma$
  - M_n will fall within $\mu \pm \frac{2}{3}\frac{\sigma}{\sqrt{n}}$
- 68% of the time,
  - S_n will fall within $n\mu \pm \sqrt{n}\sigma$
  - M_n will fall within $\mu \pm \frac{\sigma}{\sqrt{n}}$
- 95% of the time,
  - S_n will fall within $n\mu \pm 2\sqrt{n}\sigma$
  - M_n will fall within $\mu \pm 2\frac{\sigma}{\sqrt{n}}$
- 99.7% of the time,
  - S_n will fall within $n\mu \pm 3\sqrt{n}\sigma$
  - M_n will fall within $\mu \pm 3\frac{\sigma}{\sqrt{n}}$