

Handy Transformations -

$$\mathbb{E}[a + bX] = a + b\mathbb{E}[X]$$
$$\mathbb{E}[X^2] = \text{Var}(X) + (\mathbb{E}[X])^2$$

(ditto but flipped) $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$$\text{Var}(a + bX) = b^2 \text{Var}(X)$$
$$\text{SD}(a + bX) = |b| \text{SD}(X)$$
$$\text{Cov}(a + bX, c + dY) = b \cdot d \cdot \text{Cov}(X, Y)$$
$$\text{Corr}(a + bX, c + dY) = \text{Corr}(X, Y)$$

Uniform distribution facts from HW -

- 1. If $U \sim \text{Unif}[0, 1]$, then for any fixed $a > 0$ and $b \in \mathbb{R}$, we have that $aU + b \sim \text{Unif}[b, a + b]$.
- 2. If $U \sim \text{Unif}[0, 1]$, then $\mathbb{E}[U] = \frac{1}{2}$ and $\text{Var}(U) = \frac{1}{12}$.

When I am not protected from me being me

- Set Algebra:
- **Union** - A or B ; $A \cup B$.
 - **Intersection** - A and B ; $A \cap B$.
 - **Complement** - not A ; A^C .
 - **Difference** - A but not B ; $A \setminus B$
 - **Disjoint Events aka. mutually exclusive** - events A and B are disjoint if they don't share any outcomes in common (i.e., A and $B = \emptyset$).
 - **Subset** - $A \subseteq B$

Trial - a repetition of a random experiment/process. Trials're independent: none gives information about the others; are stable: results could have appeared in any order.

Outcome - a possible result of a trial.

Sample space - the set of all possible outcomes. Often denoted as S .

Event - a set of outcomes of an experiment (i.e., a subset of the sample space).

Probability - is a long run proportion of an outcome in repeated trials.

- Probabilities act as “targets” of estimation
- Proportions based on data “estimate” probabilities. Would approach probabilities if observe infinite trials.

Formally, $A \mapsto \mathbb{P}(A), \mathbb{P}(A) \in [0, 1]$ A probability $\mathbb{P}(\cdot)$ on a sample space S is a function that assigns a number between 0 and 1 to all events, A in the sample space (i.e., any possible subset of the sample space) and subject to three requirements (axioms):

1. $\mathbb{P}(S) = 1$: probability of *something* in the sample space happening is 1
2. $\mathbb{P}(A) \geq 0, \forall A$
3. A and $B = \emptyset$ (A, B disjoint) $\Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

More takeaways

- $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$
- A, B, C are pairwise disjoint $\Rightarrow \mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)$
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B)$

Joint Probability - $\mathbb{P}(A \text{ and } B)$ is the joint probability that events A and B occur.

Conditional Probability - $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}, \mathbb{P}(B) > 0$ the probability of observing event A if (given that) one has observed B . Bear in mind: $\mathbb{P}(A|B) \neq \mathbb{P}(B|A)$

Product Rule - $\mathbb{P}(A \text{ and } B) = \mathbb{P}(B) \cdot \mathbb{P}(A|B)$.

Independent Events - A and B are independent if $\mathbb{P}(A \text{ and } B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$; one event happening doesn't affect the probability of the other event happening. Can easily deduce that $\mathbb{P}(A|B) = \mathbb{P}(A)$ and $\mathbb{P}(B|A) = \mathbb{P}(B)$.

Independence and Disjointness are NOT synonyms.

- Independent $\Rightarrow \mathbb{P}(A \text{ and } B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$
- Disjoint $\Rightarrow \mathbb{P}(A \text{ and } B) = 0$. Disjoint events are extremely dependent: If one event occurs, the other cannot.

Random variable - a numerical function on a sample space with probabilities. (Think as a scoring mechanism.)

- Input: an outcome in the sample space
- Output: a number

Discrete RVs - only countably many values are possible

Continous RVs - can take on uncountably infinitely many values

Probability Distribution Function (PDF) - $\mathbf{p}_X(x) = \mathbb{P}(X = x)$ is the probability that the random variable X takes on the value x . I really hate $\mathbf{p}_X(x)$ this styling, so only $\mathbb{P}(X = x)$ moving forward.

Properties of PDFs - any function that satisfies the following conditions is a probability distribution function of a Discrete random variable:

1. $\mathbb{P}(X = x) \geq 0, \forall x \in \mathbb{R}$ (for any real number)
2. $\mathbb{P}(X = x) > 0$ for values that the random variable X can actually take on
3. $\mathbb{P}(X = x) = 0$ for values that aren't possible for the random variable X
4. $\sum_x \mathbb{P}(X = x) = 1$

Expected Value - $\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x)$

Variance - a probability weighted mean of the possible squared deviations.

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$
$$= \sum_x (x - \mathbb{E}[X])^2 \cdot \mathbb{P}(X = x)$$
$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Standard Deviation - $\text{SD}(X) = \sqrt{\text{Var}(X)}$

Given $Y = g(X)$ and X 's PDF -

$$\mathbb{E}[Y] = \sum_x g(x) \cdot \mathbb{P}(X = x)$$
$$\text{Var}(Y) = \sum_x (g(x) - \mathbb{E}[g(X)])^2 \cdot \mathbb{P}(X = x)$$

RVs with only 2 outcomes - (not necessarily Bernoullis yet) Suppose RV X 's PDF is:

$$\begin{cases} \mathbb{P}(X = a) &= p \\ \mathbb{P}(X = b) &= 1 - p, \text{ then:} \\ \mathbb{P}(X = \text{all other values}) &= 0 \end{cases}$$

$$\begin{aligned} \mathbb{E}[X] &= ap + b(1 - p) \\ \text{Var}(X) &= (a - b)^2 p(1 - p) \\ \text{SD}(X) &= |a - b| \sqrt{p(1 - p)} \end{aligned}$$

Bernoulli Random Variable - aforementioned

when $\begin{cases} a &= 1 \\ b &= 0 \end{cases}$. If $X \sim \text{Bern}(p)$, then:

$$\begin{aligned} \mathbb{E}[X] &= p \\ \text{Var}(X) &= p(1 - p) \\ \text{SD}(X) &= \sqrt{p(1 - p)} \end{aligned}$$

- **Variance maximized when $p = 0.5$**
- **Variance minimized when $p = 0$ or 1**

Useful for tracking how many successes happen in n independent trials.

Binomial Random Variable - If $X \sim \text{Binom}(n, p)$, then:

$$\begin{aligned} \mathbb{E}[X] &= n \cdot p \\ \text{Var}(X) &= n \cdot p(1 - p) \\ \text{SD}(X) &= \sqrt{n \cdot p(1 - p)} \end{aligned}$$

Binomial Problems - following must hold:

1. Constant success probability p and failure probability $(1 - p)$.
2. Fixed total number of trials: n
3. trials are **independent**
4. Only two outcomes of interest (success or failure) on each trial
5. **Want to find the probability of observing k successes among the total number of n trials.** (Order doesn't matter.)

$$\mathbb{P}(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k (1 - p)^{n - k}$$

Combination - how many ways to choose k out of n : $\binom{n}{k} = \frac{n!}{k!(n - k)!}$.

Binomial Distribution - $X \sim \text{Binom}(n, p)$

where X is an RV tracking the number of successes in n independent trials with success probability p . X 's PDF:

$$\forall k \in \{0, \dots, n\}, \mathbb{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n - k}$$

Attn: X here is not for one single trial!!

- A **Bernoulli RV**: useful for one trial's success/failure.
- A **Binomial RV**: useful for total number of successes.

Binomial as the Sum of Bernoullis - n independent Bernoulli RVs each with the same success probability p : $\forall i \in 1, \dots, n, X_i \sim \text{Bern}(p)$.

Define $S_n = \sum_{i=1}^n X_i$, then denote

$$S_n \sim \text{Binom}(n, p).$$
 Binom(1, p) = Bern(p)

Joint Distribution of 2 RVs - the probability that 2 RVs simultaneously take on 2 values.

$$\forall x \in X, \forall y \in Y \quad \mathbb{P}(X = x, Y = y).$$

Marginal probability distribution - can be found given with the joint PDF:

$$\mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y)$$

Z-Score of a Random Variable X -

$$Z(X) = \frac{X - \mathbb{E}[X]}{\text{SD}(X)}$$

$$\mathbb{E}[Z(X)] = 0 \text{ and } \text{SD}(Z(X)) = 1$$

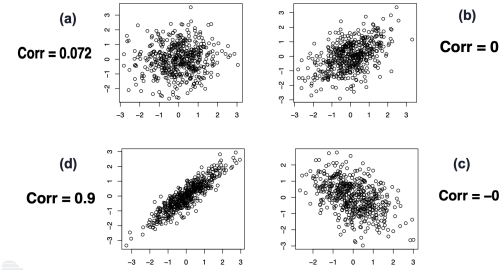
Correlation and Covariance

Correlation Between RVs X and Y - “average of the product of z-scores”

$$\begin{aligned} \text{Corr}(X, Y) &= \mathbb{E}[Z(X) \cdot Z(Y)] \\ &= \frac{\text{Cov}(X, Y)}{\text{SD}(X) \cdot \text{SD}(Y)} \end{aligned}$$

- $\text{Corr}(X, Y)$ is unit-free.
- **$\text{Corr}(X, Y)$ doesn't exist if either $\text{SD}(X) = 0$ or $\text{SD}(Y) = 0$** (can't divide by 0!).
- Correlation is guaranteed to **lie between +1 (perfect positive correlation) and -1 (perfect negative correlation)**. Hence Corr is more commonly used than Covariance.

Correlation Examples



Covariance Between RVs X and Y - “average of the product of the centered variables”. Necessary for assessing variability of sums of RVs (e.g. portfolios).

$$\begin{aligned} \text{Cov}(X, Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\ &= \text{Corr}(X, Y) \cdot \text{SD}(X) \cdot \text{SD}(Y) \end{aligned}$$

- $\text{Cov}(X, Y)$ has funny units: product of the X and Y units.
- **$\text{Cov}(X, Y)$ always exists.** If SDs are 0, $\text{Cov}(X, Y) = 0$
- $\text{Cov}(X, X) = V(X)$
- **If $\text{SD}(X) > 0$ and $\text{SD}(Y) > 0$, then $\text{Corr}(X, Y)$ and $\text{Cov}(X, Y)$ have the same sign.**

Expected Value of RVs summed - is regardless of RVs' joint distribution:

$$\begin{aligned}\mathbb{E}[X + Y] &= \mathbb{E}[X] + \mathbb{E}[Y] \\ \mathbb{E}[X + Y + W] &= \mathbb{E}[X] + \mathbb{E}[Y] + \mathbb{E}[W] \\ \mathbb{E}\left[\sum_{i=1}^n X_i\right] &= \sum_{i=1}^n \mathbb{E}[X_i] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n]\end{aligned}$$

Variance of of RVs summed -

$$\begin{aligned}\text{Var}(X + Y) &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) \\ \text{Var}(X + Y + W) &= \text{Var}(X) + \text{Var}(Y) + \text{Var}(W) \\ &\quad + 2\text{Cov}(X, Y) + 2\text{Cov}(X, W) + 2\text{Cov}(Y, W) \\ \text{Var}\left(\sum_{i=1}^n X_i\right) &= \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j)\end{aligned}$$

Have to consider the covariance of all possible pairs: X_i and X_j .

- If $\text{Corr}(X, Y)$ increases, then $\text{Var}(X + Y)$ increases.
- If $V(X) = V(Y)$, then $\text{Var}(X + Y)$ is maximized when $\text{Cov}(X, Y)$ is maximized.

Uncorrelated RVs - if $\text{Corr}(X, Y) = 0$. Equivalently, they are uncorrelated if $\text{Cov}(X, Y) = 0, SD(X) > 0, SD(Y) > 0$

Variance of of Uncorrelated or Independent RVs summed -

$$\begin{aligned}\text{Var}(X + Y) &= \text{Var}(X) + \text{Var}(Y) \\ \text{Var}\left(\sum_{i=1}^n X_i\right) &= \sum_{i=1}^n \text{Var}(X_i)\end{aligned}$$

No change in expected value's formula. **Independent RVs** -

$$\forall x, y, \mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

Independence implies uncorrelatedness: if two RVs X and Y are independent, then they are uncorrelated.

$$\text{Independence} \Rightarrow \text{Corr}(X, Y) = 0 = \text{Cov}(X, Y)$$

But uncorrelated RVs can be dependent! **(iid) Independent and Identically Distributed RVs** - for a collection of iid RVs $\{X_1, \dots, X_n\}$: $\forall i \in \{1, \dots, n\}$

$$\begin{aligned}\mathbb{E}[X_i] &= \mu \\ \text{Var}(X_i) &= \sigma^2 \\ \text{SD}(X_i) &= \sigma\end{aligned}$$

Sum of iid RVs - $S_n = X_1 + \dots + X_n$:

$$\begin{aligned}\mathbb{E}[S_n] &= n \cdot \mu \\ \text{Var}(S_n) &= n \cdot \sigma^2 \\ \text{SD}(X_i) &= \sqrt{n} \cdot \sigma\end{aligned}$$

Central Limit Theorem (CLT)

If $\{X_1, \dots, X_n\}$ are iid with expected value $\mathbb{E}[X_i] = \mu$ and variance $\text{Var}(X_i) = \sigma^2 < \infty$, then as $n \rightarrow \infty$:

- $S_n \sim \mathcal{N}(n \cdot \mu, \sqrt{n} \cdot \sigma)$
 - $\text{Mean}_n \sim \mathcal{N}(\mu, \frac{\sigma}{\sqrt{n}})$
- If n is large enough (heuristic: $n > 30$), we can calculate probabilities for the sum and mean of RVs by using the normal distribution.

Empirical Rules under CLT -

- 50% of the time,
 - S_n will fall within $n\mu \pm \frac{2}{3}\sqrt{n}\sigma$
 - M_n will fall within $\mu \pm \frac{2}{3}\frac{\sigma}{\sqrt{n}}$
- 68% of the time,
 - S_n will fall within $n\mu \pm \sqrt{n}\sigma$
 - M_n will fall within $\mu \pm \frac{\sigma}{\sqrt{n}}$
- 95% of the time,
 - S_n will fall within $n\mu \pm 2\sqrt{n}\sigma$
 - M_n will fall within $\mu \pm 2\frac{\sigma}{\sqrt{n}}$
- 99.7% of the time,
 - S_n will fall within $n\mu \pm 3\sqrt{n}\sigma$
 - M_n will fall within $\mu \pm 3\frac{\sigma}{\sqrt{n}}$

Sampling and Confidence Intervals

Confidence Interval - contains an unknown (population) quantity at some specified sampling frequency.

- Confidence intervals **do not** depend on population size, but only on sample size.
- For a given sample size, can be very precise with low confidence or very imprecise with high confidence.
- 2x the precision requires 4x sample size; 3x the precision requires 9x sample size.

Wording matters...

- OK: "I am 95% confident the interval [a,b] contains the true population proportion"
- OK: "There is a 95% probability the interval [a,b] contains the true population proportion"
- Not OK: "There is a 95% probability the true population proportion lies in the interval [a,b]"

Confidence Level (L) to c -

$$c = \text{qnorm}(p = \frac{(1+L)}{2}, \mu = 0, \sigma = 1)$$

(find the value c such that the area under $\mathcal{N}(\mu, \sigma)$ is p)

confidence level	L	c
90%	0.9	1.65
95%	0.95	1.96
99%	0.99	2.58

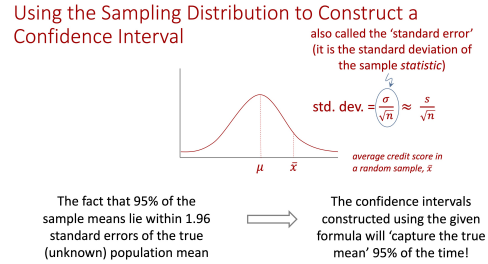
For a population mean - Given sample size n , sample average \bar{x} , and sample standard deviation s , we are X% confident the true population mean lies in the interval: $\bar{x} \pm \left(c \frac{s}{\sqrt{n}}\right)$ "MOE": $\left(c \frac{s}{\sqrt{n}}\right)$

For a population proportion - Given sample size n , sample proportion \bar{p} , and standard deviation in the population to be 0.5, we are X% confident the true population mean lies in the interval: $\bar{p} \pm \left(c \frac{0.5}{\sqrt{n}}\right)$ "MOE": $\left(c \frac{0.5}{\sqrt{n}}\right)$

Important assumptions:

- Sample is random
- Sample is large enough ($n > 30$) for CLT
- The worst possible standard deviation in the population to be 0.5

Sampling Distribution - is well approximated by $\mathcal{N}(\mu = \text{true population parameter}, \sigma = \frac{\text{population SD}}{\sqrt{n}})$, based on CLT.



Take away: assuming a conservative confidence interval based on 0.5 is not the only way! Can estimate standard error using the surveyed proportion too.

Similar Analysis for Proportions

- When outcomes are binary, the standard deviation of observations in the population equals $\sqrt{p(1-p)}$. *the 'standard error'*
- We get the standard error by dividing by \sqrt{n} . The exact interval is $\bar{p} \pm 1.96 \frac{\sqrt{p(1-p)}}{\sqrt{n}}$. *0.5*
- But since we don't know p (if we did we won't be sampling would we?!):
 - We can take a conservative approach and assume $p = 0.5$. *0.5*
 - Or, an approximate approach by assuming $p = \bar{p}$ to get the practical alternative, $\bar{p} \pm 1.96 \frac{\sqrt{\bar{p}(1-\bar{p})}}{\sqrt{n}}$. *0.5*

Sampling Errors - the sample-to-sample variations due to pure chance. MOE and confidence intervals quantify this uncertainty well.

Non-Sampling Errors - (some examples)

- Selection Bias: happens when each member of the population does not have the same chance of being selected.
- Response/Non-response Bias: happens when some fraction of the individuals surveyed don't respond for reasons related to what's being asked in the survey

R Distribution Functions

- **p** ("probability"): cumulative distribution function ("what is the probability above or below a cutoff?")
- **q** ("quantile"): inverse CDF ("what value do we find at, say, 80% of the way to the maximal value?")
- **d** ("density"): density function (gives us the "height" or y-value of distribution for a particular z-score - mainly useful in plotting)

Exactly k successes in n trials given success probability p
dbinom(k , $\text{size}=n$, $p=p$)
k or more successes in n trials given success probability p
sum(**dbinom**($k:n$, $\text{size}=n$, $p=p$))

Probability of this value or less
pnorm(value , mean , sd)
Probability of this value or greater
pnorm(value , mean , sd , **lower.tail**=**FALSE**)
Highest value associated with a given percentile
qnorm(percentile , mean , sd)