Diversification

Asset return characteristics

Buy an asset (e.g. a stock) at $t = 0$ at price P_0. At time $t = 1$,

- its cash flow (dividend) is D_1, and
- its price is P_1

(both are random variables). The risk-free rate is r_F.

Realized return: $\boxed{r_1 = \frac{D_1+P_1}{P_0} - 1}$ Returns comes from both dividends and capital gains.

Expected return: $\boxed{E[r_1] = \frac{E[D_1]+E[P_1]}{P_0} - 1}$

Excess return: (realized) $\boxed{r_1 - r_F}$

Risk premium: (expected excess return) $\boxed{E[r_1] - r_F}$

Mean (average) return: $\boxed{\bar{r} = E[r] = \frac{1}{T}\sum_{t=1}^{T} r_t}$

Would be same as the expected return $E[r_t]$ if expected returns are constant for all t.

Estimate Expected Return:

- if have multiple possible scenarios for returns and know the probability of each scenario, use: $\boxed{E[r] = \sum_{i=1}^{N} p_i r_i}$
- if have a time series of past T observations of returns, estimate sample estimate of expected return $\bar{r}$ as: $\boxed{\hat{r} = \frac{1}{T}\sum_{t=1}^{T} r_t}$

Variance: measures the volatility or deviation of returns from the mean.

$\boxed{\text{Var}(r) = \sigma^2 = E\left[(r - E[r])^2\right]}$ If given (past) data sample of T returns, the sample variance is:

$\boxed{\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2}$ where the expected return $\bar{r}$ can be estimated by the sample mean $\bar{r}$ as defined above.

Standard deviation: $\boxed{\sigma = \sqrt{\text{Var}(r)}}$ measures the risk of the asset. Gives a magnitude in percent.

Covariance: measures the degree to which two random variables move together.

$\boxed{\sigma_{ij} = \text{Cov}(r_i, r_j) = E[(r_i - \bar{r_i})(r_j - \bar{r_j})]}$

Estimate Cov: given (past) data sample of T returns, the sample covariance is:

$\boxed{\hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(r_{i,t} - \bar{r_i})(r_{j,t} - \bar{r_j})}$

Covariance can be:

- positive (both variables move in the same direction),
- negative (two variables move in the opposite direction), or
- zero (no relationship).

Variance is a special case of covariance, where the two variables are the same.

$\boxed{\text{Var}(r_i) = \sigma_{ii} = \text{Cov}(r_i, r_i)}$

Correlation: measures the strength of the linear relationship between two random variables. Always between -1 and 1. $\boxed{\text{Corr}(r_i, r_j) = \rho_{ij} = \frac{\text{Cov}(r_i, r_j)}{\sigma_i \sigma_j}}$

Beta: measures the sensitivity of an asset's return to the return of the market portfolio.

$\boxed{\beta_i = \frac{\text{Cov}(r_i, r_m)}{\text{Var}(r_m)}}$ where r_m is the return of the market portfolio.

Other measures of risk:

- **Skewness**: captures the asymmetry of the distribution of returns.
- **Kurtosis**: captures the "tailedness" of the distribution of returns. Higher kurtosis means more extreme values (outliers) in the distribution. i.e. the distribution has "fat tails".

Mean-variance investors: care about the expected return (higher is better) and variance (lower is better) of the return. They are risk-averse with a risk-aversion coefficient of A.

Mean-variance utility function: captures the preferences of mean-variance investors

$\boxed{U(r) = E[r] - \frac{1}{2}A \cdot \text{Var}(r)}$ where $U(r)$ is the utility of the return r.

Indifference curve: is a curve that represents all combinations of expected return and variance that give the same utility to the investor. The slope of the indifference curve is given by: $\boxed{\frac{dE[r]}{d\sigma^2} = -A}$ The slope is negative, meaning that as variance increases, the expected return must also increase to maintain the same utility.

Portfolio

Portfolio: a combination of different assets or securities. Defined by number N_i and the price P_i of each asset i in the portfolio. The total value of the portfolio is: $\boxed{V = \sum_{i=1}^{n} N_i P_i}$

Portfolio weight: of asset i in the portfolio is:

$\boxed{w_i = \frac{N_i P_i}{V}}$ The portfolio weight represents the proportion of the total portfolio value that is invested in asset i. Weights can be positive (long position) or negative (short position). $\sum_{i=1}^{n} w_i = 1$

Mean and variance of a portfolio: with weights: $w_1, \ldots, w_n$ and returns $r_1, \ldots, r_n$:

- **Random variable**: $R_p = \sum_{i=1}^{n} w_i r_i$
- **Expected return**: $\boxed{E[R_p] = \sum_{i=1}^{N} w_i E[r_i]}$
- **Variance**: $\boxed{\text{Var}(R_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \text{Cov}(r_i, r_j)}$

Special case - two assets:

- $E[R_p] = w_1 E[r_1] + w_2 E[r_2]$
- $\text{Var}(R_p) = w_1^2 \text{Var}(r_1) + w_2^2 \text{Var}(r_2) + 2 w_1 w_2 \text{Cov}(r_1, r_2)$ (Note: $w_2 = 1 - w_1$)

Special case - n equally-weighted assets: with returns $r_1, \ldots, r_n$:

- $E[R_p] = \frac{1}{n}\sum_{i=1}^{n} E[r_i]$
- $\text{Var}(R_p) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \text{Cov}(r_i, r_j)$

Portfolio beta: is the sensitivity of the portfolio return to the return of the market portfolio.

Efficient frontier and the tangency portfolio:

- **Mean-variance frontier portfolio**: minimizes risk (measured by variance) for a given expected return.
- **Efficient frontier**: is the set of portfolios that offer the highest expected return for a given level of risk. (Upper part of the mean-variance frontier)
- **Sharpe ratio**: $\boxed{= \frac{E[R_p] - r_f}{\sigma_p}}$ is the slope of the line from the risk-free rate to the portfolio. It measures the risk-adjusted return of the portfolio.
- **Capital Market Line**: represents the risk-return trade-off of efficient portfolios. It starts at the risk-free rate and is tangent to the efficient frontier. The slope of the CML is equal to the Sharpe ratio of the tangency portfolio.

Options

Options: Derivative contracts specifying **a right** to **buy (call option)** or **sell (put option)** an underlying asset at a specified price K (the **strike/exercise price**) on or before a specified date T (the **expiration/maturity date**).

- **Call option**:: right to buy the underlying asset at the strike price.
- **Put option**:: right to sell the underlying asset at the strike price.

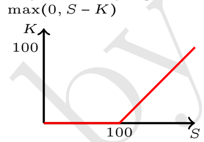
Exercise style:

- **American option**:: can be exercised at any time before expiration.
- **European option**:: can only be exercised at expiration.

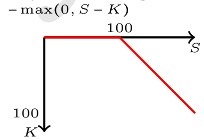
Option Payoff curves

- S: Price of the underlying asset at expiration
- K: Strike price of option
- Payoff ≠ Profit. To get profit (net payoff), need to subtract the option's cost.

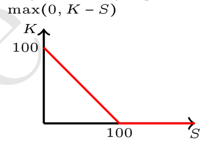
Payoff of buying a Call
$\max(0, S - K)$



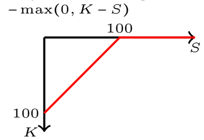
Payoff of selling a Call
$-\max(0, S - K)$



Payoff of buying a Put
$\max(0, K - S)$

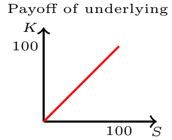


Payoff of selling a Put
$-\max(0, K - S)$

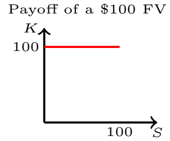


Payoff curves of other assets that can be used with options:

Payoff of underlying asset



Payoff of a \$100 FV ZC Bond



Option payoff and profit

- r: Risk-free interest rate (EAR)
- C: Call option price
- P: Put option price

Call option:

	$S < K$	$S = K$	$S > K$
Payoff	0	0	$S - K$
Profit	$-C(1+r)^T$	$-C(1+r)^T$	$S - K - C(1+r)^T$

Put option:

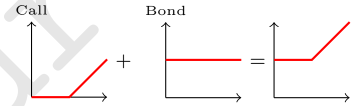
	$S < K$	$S = K$	$S > K$
Payoff	$K - S$	0	0
Profit	$K - S - P(1+r)^T$	$-P(1+r)^T$	$-P(1+r)^T$

An option is

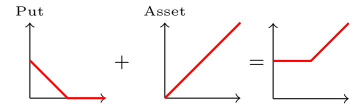
- **in-the-money**: if it has positive payoff at expiration. A call option is in-the-money if $S > K$, and a put option is in-the-money if $S < K$.
- **out-of-the-money**: if it has zero payoff at expiration. A call option is out-of-the-money if $S < K$, and a put option is out-of-the-money if $S > K$.
- **at-the-money**: if it has zero payoff at expiration. A call option is at-the-money if $S = K$, and a put option is at-the-money if $S = K$.

Put-call parity: following portfolios have the same payoff at expiration:

1. **Long call with strike price K + Bond with face value K**:



2. **Long put with strike price K + Underlying asset**:



Given their identical payoffs, under no-arbitrage, they should have the same price:

- $P_{call} + P_{bond} = P_{put} + P_{asset}$
- $C + K(1+r)^{-T} = P + S$

Binomial option pricing model: Iterative approach to price options that makes the following simplifications:

- Discrete periods, in which stock price can either go up or down.
- We find the option price by a *no arbitrage* argument. Price is equal to the cost of purchasing a *replicating portfolio* whose payoffs match the option payoff in each state. E.g., for a call, we solve: $\begin{cases} aS_u + bB_u &= C_u \\ aS_d + bB_d &= C_d \end{cases}$ where a is the number of shares of stock, b is the number of bonds, S_u and S_d are the stock prices if it goes up or down, and C_u and C_d are the call option prices if stock goes up or down.
- Under the binomial assumptions, the probability of a stock moves up or down is irrelevant, and the price of options can be determined solely using:
  - Current stock price S_0, interest rate r, strike price K and time to maturity T;
  - Magnitude of possible future changes of stock price (volatility), captured implicitly by the possible values the stock can take S_u and S_d.

Black-Scholes formula: Taking the limit of binomial model as the number of periods gets large, we obtain the B-S formula for the price of a European call option without dividends:

$\boxed{C(S, K, T, r, \sigma) = S \cdot N(x) - K(1+r)^{-T} \cdot N(x - \sigma\sqrt{T})}$

- S: current value of the underlying asset (in \$)
- K: strike price of the option (in \$)
- T: option maturity (in years)
- r: annual risk-free interest rate
- σ: annualized standard deviation of the underlying asset's return (volatility)
- $N(\cdot)$: cumulative normal distribution function (`NORM.S.DIST(x, TRUE)`). These $N(\cdot)$ terms capture the replicating portfolio weights.
- x: $\boxed{x = \frac{\ln\left(\frac{S}{K(1+r)^{-T}}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}}$

Real options

Option-like payoffs appear in many contexts outside of financial markets. Management can be thought of as the act of creating and optimally exercising real options. **E.g.**: follow-on products, R&D investments, delaying product launches, abandoning projects, etc.