Net Present Value (NPV)

Discount rate r: you're indifferent between receiving \$1 today and $\$\frac{1}{1+r}$ in one period.

Present Value (PV): $\boxed{PV(CF_t) = \frac{CF_t}{(1+r)^t}}$ how much a cash flow (CF) at time t is worth at time 0 (today). Computing a PV is often called "discounting". **Investors prefer payoffs that are larger, safer, and sooner.**

NPV: $\boxed{NPV = \sum_{t=0}^{T} \frac{CF_t}{(1+r)^t}}$ summs over PVs of all cash flows in a project.

- **Scalability**: $NPV(\alpha CF_1, \ldots, \alpha CF_T) = \alpha NPV(CF_1, \ldots, CF_T)$
- **Additivity**: $NPV(X_1 + Y_1, \ldots, X_T + Y_T) = NPV(X_1, \ldots, X_T) + NPV(Y_1, \ldots, Y_T)$
- **Breaking up by time**: $NPV(CF_1, \ldots, CF_T) = NPV(CF_1, \ldots, CF_j) + NPV(CF_{j+1}, \ldots, CF_T)$

Future Value (FV): $\boxed{FV_T(CF_0) = CF_0(1+r)^T}$ how much a cash flow at time 0 (today) is worth in T periods.

Perpetuity:
- Constant recurring cash flow A forever starting **1 period from now i.e.** $t = 1$: $\boxed{PV = \frac{A}{r}}$
- Growing perpetuity starting **1 period from now i.e.** $t = 1$ with cash flow A, growth rate g: $\boxed{PV = \frac{A}{r-g} (r > g)}$
- To include the cash flow at time 0, add A to the PV formula above.

Annuity:
- Constant recurring cash flow A for T periods starting **1 period from now**: (E.g. a loan)
  $\boxed{PV = \frac{A}{r}\left(1 - \frac{1}{(1+r)^T}\right)}$
  $\boxed{FV = PV \cdot (1+r)^T = A\frac{(1+r)^T - 1}{r}}$
- Growing annuity starting **1 period from now** with cash flow A, growth rate g for T periods.
  - If $r \neq g$ $\boxed{PV = \frac{A}{r-g}\left(1 - \frac{(1+g)^T}{(1+r)^T}\right)}$;
    $\boxed{FV = A\left(\frac{(1+r)^T - (1+g)^T}{r-g}\right)}$
  - If $r = g$ $\boxed{PV = T\left(\frac{A}{1+r}\right)}$;
    $\boxed{FV = T \cdot A \cdot (1+r)^{T-1}}$

Annual Percentage Rate (APR) & Effective Annual Rate (EAR):

$\boxed{(1 + EAR) = \left(1 + \frac{APR}{k}\right)^k = (1+r)^k}$ where k is the number of compounding periods per year and r is the per-period (e.g. monthly) interest rate.

- APR $= r \cdot k$
- EAR i.e. Annual Percentage Yield (APY)

If $APR = EAR$ between two savings accounts, APR (the one with more frequent compounding) is better.

Mortgage-related terms

Principal: the amount of \$ borrowed in a lending agreement. E.g. Buy a \$1,000,000 house with a 20% down payment, the principal is \$800,000.

Interest:
- **Fixed rate**: No matter what happens to interest rates around the world, you would still be charged interest at this same rate.
- **Adjustable rate (ARM)**: E.g. an adjustable rate of 3% above the federal funds rate (the Fed's benchmark rate). If this rate is around 4.5%, you would be charged a 7.5% interest rate. If in the next month the Fed raises to 5%, you would be charged an 8% interest rate.

Amortization schedule: sequence of payments made through the loan's lifetime. A part of the payments goes to reduce (i.e. amortize) the principal owed, and the rest goes to pay the interest on the loan.

Collateral: An asset offered by the borrower as a guarantee in a loan. If you fail to make payments, the bank can take the collateral.

Refinancing: Paying off an existing loan with a new loan that has better terms. E.g. lower interest rate, lower monthly payment, shorter loan term.

E.g.: a 30-year fixed-rate mortgage, APR 9% compounded monthly. Fixed monthly payment = \$3000. First payment will start next month and last until the contract expires in 30 years.

- **How much borrowed when took out the mortgage?**: use the constant annuity formula, where $A = \$3000$, $r = \frac{0.09}{12} = 0.0075$, $T = 30 \times 12 = 360$. $PV = \frac{A}{r}\left(1 - \frac{1}{(1+r)^T}\right) = \frac{3000}{0.0075}\left(1 - \frac{1}{(1+0.0075)^{360}}\right) = \$372,845.60$
- **10 years later, how much must pay back to the bank if sell the house?**: (i.e. NPV of the remaining principal amount as of this future date.) $T = 20 \times 12 = 240$
  $PV = \frac{3000}{0.0075}\left(1 - \frac{1}{(1+0.0075)^{240}}\right) = \$333,434.86$
- **Why the remaining principal is still so high after 10 years' worth of repayments?**: Because the amortization schedule is front-loaded. I.e. Early on in the life of the mortgage, the vast majority of each monthly payment is interest: interest is applied to the remaining principal amount, and this remaining principal is highest in month 1 and decreases over time. Because the majority of each monthly payment in the early years is interest, the principal repayment amounts are small, and the remaining principal de-creases very slowly. (It's only toward the end of the mortgage that interest payments decline enough to repay the principal more quickly.)

Inflation i: the change in CPI $\boxed{1 + i_{t+1} = \frac{CPI_{t+1}}{CPI_t}}$

- **"Nominal"**: not adjusted for inflation
- **"Real"**: adjusted for inflation

Real rate of return:

$\boxed{r_{\text{real}} = \left(\frac{1 + r_{\text{nominal}}}{1+i} - 1\right) \approx (r_{\text{nominal}} - i)}$ If i is too high, don't use the approximation (too off).

Treat inflation consistently for NPV:

$\boxed{PV(CF_T) = \frac{CF_{\text{nominal},T}}{(1 + r_{\text{nominal},T})^T} = \frac{CF_{\text{real},T}}{(1 + r_{\text{real},T})^T}}$ ("$,T$" denotes the cash flow at time T)

E.g. False statement: *If inflation is zero, two projects paying \$100 in different years have the same NPV.* Time value of money still applies; later payments are worth less.

Capital Budgeting

To maximize value, take on only projects with positive NPV.

- **Single**: take it only if it has positive NPV.
- **Independent**: take all with positive NPV.
- **Mutually exclusive**: take the one with the highest positive NPV.
- Ignore sunk costs, including opportunity costs.

Cash operating expenses:
- **COGS**: direct costs attributable to the production of the goods sold by a business.
- **R&D**: costs associated with discovering new knowledge or develop new products, processes, and services.
- **SG&A**: costs not directly tied to the production of goods. e.g. "S": advertising and sales commissions. "G": salaries of non-production personnel. "A": legal, accounting, and exec salaries.

Depreciation: non-cash expense that reduces the value of an asset over time. For most finance problems, we want to strip out effects of depreciation to get back to free cash flow. Exception: if depreciation affects free cash flows through taxes.

EBITDA:
$\boxed{= \text{(Op. Rev.)} - \text{(All Op. Exp. w/o depreciation)}}$

EBIT: $\boxed{= \text{EBITDA} - \text{Depreciation (\& Amort.)}}$

Cash Flows: from accounting statements
$\boxed{CF = (1-\tau)(\text{EBITDA}) + \tau(\text{Dep.}) - (\text{CapEx}) - \Delta\text{WC}}$
$\boxed{CF = (1-\tau)(\text{EBIT}) + (\text{Dep.}) - (\text{CapEx}) - \Delta\text{WC}}$

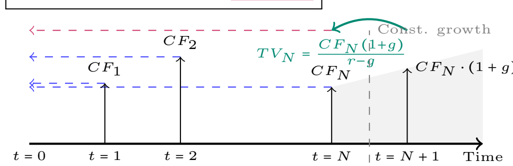
Working Capital (WC):
$\boxed{= \text{Inventory} + \text{A/R} - \text{A/P}}$ We are about changes (i.e. Δ) in WC, not levels because if keeping WC constant, no new cash flow required.

Valuing Firms (usual approach)

- Forecasted CF_1 through CF_N
- Assume constant growth at rate g after year N.

Then enterprise value as of today ($t = 0$):

$$\boxed{EV = \left(\sum_{t=1}^{N} \frac{CF_t}{(1+r)^t}\right) + \frac{TV_N}{(1+r)^N}}$$



Hotspur example: $N = 12$ (2019) i.e. year 13 begins the constant growth.

	0	1	2	...	11	12	13
EBITDA	5.00	5.85	6.76		17.93	19.49	20.27
- Depreciation	2.20	2.29	2.38		3.39	3.52	3.66
EBIT	2.80	3.56	4.38		14.54	15.97	16.61
EBIT × $(1-\tau)$		2.32	2.85		9.45	10.38	10.79
+ Depreciation		2.29	2.38		3.39	3.52	3.66
- CapEx		-3.43	-3.57		-5.08	-5.28	-5.49
- Δ WC		3.90	4.25		9.22	10.05	4.87
Cash Flow		5.07	5.90		16.98	18.67	13.83

Key fact: everything grows at 4% per year from year 12 (2019) onwards → we have a **perpetuity with growth**, with starting cash flow CF_{2020}

$V_{2019} = \frac{CF_{2020}}{r-g} = \frac{13.83}{10.25\% - 4\%} = £221.32\text{M}$

PV of perpetuity with growth (10.25% discount rate, 4% growth)

(Could also start the perpetuity in 2021 and calculate PV as of 2020…but then need to be sure we include 2020 CF separately for total NPV)

	0	1	2	...	11	12	13	
Cash Flow		5.07	5.90		16.98	18.67	13.83	...
Terminal Value						221.32		Infinite stream
"Total" Cash Flow		5.07	5.90		16.98	239.98		
Discount: $(1+r)^t$		1.1025	1.2155		2.9253	3.2251		
Present Value		4.60	4.85		5.80	74.41		

Also CF in year 0 is not added to the EV.

Enterprise Value (EV): PV of future CFs $EV = \sum_{t=1}^{\infty} PV(CF_t)$ i.e. estimated via discounted CF (DCF) analysis.

From EV to Equity Value: MV of Equity = Value of operations (EV) + Non-operational assets − MV of Debt

For Hotspur,

Value of Assets $= EV +$ (Cash & equivalents) + (Investments available for sale)

MV of Debt = (Long-term debt)

Alternatives to NPV

Internal rate of return: discount rate that makes zero NPV. $\boxed{NPV = \sum_{t=0}^{T} \frac{CF_t}{(1+IRR)^t} = 0}$ Determine some fixed IRR^* i.e. "threshold rate":

- **Independent**: take if $IRR > IRR^*$.
- **Mutually exclusive**: take the one with the highest IRR among projects $IRR > IRR^*$.

IRR leads to the same decision as NPV if:
- Cash outflow occurs only at time 0
- Only one project is being considered
- Opportunity cost of capital (discount rate r) remains constant for all periods
- $IRR^* = r$

Shortcomings: no solution, multiple solutions, project size not accounted for, different projects' horizons not fully considered.

E.g. False statement: *If project A's NPV is larger than project B's, then the IRR of investing in project A is always higher than that of project B.* The IRR could yield incongruent predictions in capital budgeting compared to the NPV rule.

E.g. False statement: *XZ Company is considering purchasing a copper mine, which requires an initial investment in the first year, followed by positive cash flows for 30 years, but a large closure cost in year 31. IRR is a good method for analyzing this investment.* Not a good metric when there are negative CFs after time 0, as it may yield multiple IRRs

Payback period: min. length of time k such that sum of CFs from a project is positive.

$\boxed{\sum_{t=1}^{k} CF_t \geq -CF_0 = I_0}$ Determine some fixed threshold k^*:

- **Independent**: take if $k \leq k^*$.
- **Mutually exclusive**: take the one with the minnimum k among projects $k \leq k^*$.

Discounted payback period: ditto but discount CFs. $\boxed{\sum_{t=1}^{k} \frac{CF_t}{(1+IRR)^t} \geq -CF_0 = I_0}$

Shortcomings: ignores CFs after k and project size.

Profitability index (PI): ratio of the NPV of future CFs to the initial investment. $\boxed{PI = \frac{NPV}{I_0}}$

- **Independent**: take all $PI > 1$.
- **Mutually exclusive**: take the one with the highest PI and $PI > 1$.

Shortcomings: doesn't account for project size.

Bonds

Face value i.e. "par value" / "principal": the value of a bond that appears on its face and that will be paid to the investor by the issuer at maturity.

Coupon: interest paid on a bond's face value on a periodic basis prior to maturity.

Spot interest rate r_t: the market interest rate for discounting a signle risk-free cash flow at horizon t.

Zero coupon bond: a bond that pays no coupons and is sold at a discount to its face value. Price of a \$1 ZC bond with maturity T is $\boxed{P_T = \frac{\$1}{(1+r_T)^T}}$. We can infer spot interest rates from different ZC bond prices. $\boxed{\text{Given } P_T \Rightarrow r_T = \left(\frac{\$1}{P_T}\right)^{\frac{1}{T}} - 1}$.

Coupon bonds: with periodic coupon payments of C_t and face value F. Price of a coupon bond is

$$\boxed{P = \left(\sum_{t=1}^{T} \frac{C_t}{(1+r_t)^t}\right) + \frac{F}{(1+r_T)^T} = \left(\sum_{t=1}^{T} C_t \cdot P_t^{ZC}\right) + F \cdot P_T^{ZC}}$$

Yield to maturity (YTM): the discount rate y that makes the PV of a bond's future cash flows equal to its price. $\boxed{P = \left(\sum_{t=1}^{T} \frac{C_t}{(1+y)^t}\right) + \frac{F}{(1+y)^T}}$ For ZC bonds, the YTM is equal to the spot interest rate $\boxed{y = r_T}$.

E.g. Bond Arbitrage: construct a trading strategy based on a mixture of bonds to generate an arbitrage profit.

A. 1-year ZC bond: FV: \$1, price \$0.8
B. 2-year ZC bond: FV: \$105, price \$100
C. 2-year 5% bond: FV: \$100, price \$106

1. Infer spot rates from ZCBs:

Maturity	Spot Rate
1	$\frac{1}{0.8} - 1 = 0.25$
2	$\left(\frac{105}{100}\right)^{\frac{1}{2}} - 1 \approx 0.0247$

2. Calculate the no-arbitrage price of the coupon bond using spot rates: $P = \frac{5}{(1+0.25)} + \frac{105}{(1+0.0247)^2} \approx 103.999$ which is less than the market price of \$106.
3. Intuition: The coupon bond is overpriced, so sell it and buy the ZC bonds.
4. Goal: Use ZC bonds to replicate the coupon bond's future cash flows.

Assume Buying	Bond	CF_1	CF_2
x	1-year ZC	1	0
y	2-year ZC	0	105
1	5% bond	5	105

5. Solve for x and y: $\begin{cases} 1x + 0y & = 5 \\ 0x + 105y & = 105 \end{cases} \Rightarrow \begin{cases} x = 5 \\ y = 1 \end{cases}$
6. At $t = 0$, buy 5 1-year ZC bonds and 1 2-year ZC bond (costs $5 \times \$0.8 + 1 \times \$100 = 104$) and sell 1 coupon bond (earning \$106). This generates \$2 arbitrage profit.

Expectations hypothesis (EH): the yield curve reflects the market's expectations of future interest rates. I.e. Long-term interest rates are equal to the geometric average of short-term interest rates until maturity. Intuition: investors indifferent between long bond and rolling over short bond. **E.g. 2-year rate should be given by**:

$\boxed{(1 + r_{t,2})^2 = (1 + r_{t,1})(1 + E_t(r_{t+1,1}))}$, where

- $r_{t,1}$: the 1-year rate at time t (e.g. today)
- $r_{t,2}$: the 2-year rate at time t (e.g. today)
- $E_t(r_{t+1,1})$: expectation at time t (e.g. today) of the 1-year rate that will exist at time $t+1$

More generally: yields reflect a component related to expectations about future short-term interest rates plus an additional term reflecting a risk premium reflecting the uncertainty about fluctuations in future interest rates:

$$\boxed{(1 + r_{t,2})^2 = (1 + r_{t,1})(1 + E_t(r_{t+1,1})) + \text{Risk Premium}}$$

E.g. bond risks: interest rate changes, inflation, credit/default

Duration

(Macaulay) Duration: the weighted average time to receive the bond's cash flows. Tells how long you have to wait to receive the average dollar from owning that bond in PV-weighted terms.

$$\boxed{D = \frac{1}{P}\sum_{t=1}^{T} t \cdot PV(CF_t) = \frac{1}{P}\sum_{t=1}^{T} t \cdot \frac{CF_t}{(1+y)^t}}$$

Zero-coupon bonds' duration is equal to their maturity. All else equal, coupon bond has shorter duration than zero-coupon bond; higher coupon means lower duration.

Modified duration: $\boxed{MD = \frac{D}{(1+y)}}$ the sensitivity of a bond's price to changes in interest rates.

Price change estimate: accurate for small changes in y $\boxed{\frac{\Delta P}{P} \approx -MD \times \Delta y}$ which shows that duration measures bond exposure to interest-rate risk. The higher the (modified) duration of a bond, the more its price moves (ΔP) given a small change in interest rates (Δy). Longer duration $\Rightarrow$ more sensitive to interest rate changes.

Duration matching (i.e. immunization): matching the duration of assets and liabilities to eliminate interest rate risk. $\boxed{\frac{D^{\text{Assets}} \cdot \Delta y}{1+y} P^{\text{Assets}} \approx \frac{D^{\text{Liab.}} \cdot \Delta y}{1+y} P^{\text{Liab.}}}$ If the value of assets (P^A) equal to value of liabilities (P^L) (usually the case), then matching durations of two immunizes portfolio from interest-rate risk.

- $\boxed{\text{If } P^A \neq P^L \text{, we need to set } D^A P^A = D^L P^L}$ to immunize.
- Value of portfolio unchanged for small movements in $y \Rightarrow$ hedged against interest-rate risk.
- $D^A > D^L$ bets rates will stay constant or decrease
- $D^A < D^L$ bets rates will increase.
- This is only a local approximation. If interest rates change, need to rebalance the portfolio.

E.g. False statement: *You're uncertain about future interest rates, but your expectation is that they will stay unchanged relative to today. It's therefore risk-free to issue short-term bonds and invest the proceeds in long-term bonds with higher yields, and no bank that follows this strategy will ever go bankrupt.* **Think SVB.** If the duration of assets exceeds the duration of liabilities, then the bank is exposed to interest-rate risk. If interest rates increase, then the value of assets will decrease more than the value of liabilities, and the bank may not be able to repay its short-term debt.

Default risk: a debt issuer fails to make interest or principal payments when due. Consider a ZCB. Let q be the probability of the bond pays the promised payoff (i.e. $(1-q)$ it defaults), the price of this bond: $\boxed{P = \frac{q \times \$1 + (1-q) \times 0}{1+r}}$. **Ratings**: assess the probability of default of bonds, by agencies like Moody's, S&P, Fitch.