Diversification

Asset return characteristics

Buy an asset (e.g. a stock) at $t = 0$ at price P_0. At time $t = 1$,

- its cash flow (dividend) is D_1, and
- its price is P_1

(both are random variables). The risk-free rate is r_F.

Realized return: $\boxed{r_1 = \frac{D_1 + P_1}{P_0} - 1}$ Returns comes from both dividends and capital gains.

Expected return: $\boxed{E[r_1] = \frac{E[D_1] + E[P_1]}{P_0} - 1}$

Excess return: (realized) $\boxed{r_1 - r_F}$

Risk premium: (expected excess return) $\boxed{E[r_1] - r_F}$

Mean (average) return: $\boxed{\bar{r} = E[r] = \frac{1}{T}\sum_{t=1}^{T} r_t}$

Would be same as the expected return $E[r_t]$ if expected returns are constant for all t.

Estimate Expected Return:

- if have multiple possible scenarios for returns and know the probability of each scenario, use: $\boxed{E[r] = \sum_{i=1}^{N} p_i r_i}$
- if have a time series of past T observations of returns, estimate sample estimate of expected return $\bar{r}$ as: $\boxed{\hat{r} = \frac{1}{T}\sum_{t=1}^{T} r_t}$

Variance: measures the volatility or deviation of returns from the mean.

$\boxed{\mathrm{Var}(r) = \sigma^2 = E\left[(r - E[r])^2\right]}$ If given (past) data sample of T returns, the sample variance is:

$\boxed{\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2}$ where the expected return $\bar{r}$ can be estimated by the sample mean $\bar{r}$ as defined above.

Standard deviation: $\boxed{\sigma = \sqrt{\mathrm{Var}(r)}}$ measures the risk of the asset. Gives a magnitude in percent.

Covariance: measures the degree to which two random variables move together.

$\boxed{\sigma_{ij} = \mathrm{Cov}(r_i, r_j) = E[(r_i - \bar{r_i})(r_j - \bar{r_j})]}$

Estimate Cov: given (past) data sample of T returns, the sample covariance is:

$\boxed{\hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(r_{i,t} - \bar{r_i})(r_{j,t} - \bar{r_j})}$

Covariance can be:

- positive (both variables move in the same direction),
- negative (two variables move in the opposite direction), or
- zero (no relationship).

Variance is a special case of covariance, where the two variables are the same.

$\boxed{\mathrm{Var}(r_i) = \sigma_{ii} = \mathrm{Cov}(r_i, r_i)}$

Correlation: measures the strength of the linear relationship between two random variables. Always between -1 and 1. $\boxed{\mathrm{Corr}(r_i, r_j) = \rho_{ij} = \frac{\mathrm{Cov}(r_i, r_j)}{\sigma_i \sigma_j}}$

Beta: measures the sensitivity of an asset's return to the return of the market portfolio.

$\boxed{\beta_i = \frac{\mathrm{Cov}(r_i, r_m)}{\mathrm{Var}(r_m)}}$ where r_m is the return of the market portfolio.

Other measures of risk:

- **Skewness**: captures the asymmetry of the distribution of returns.
- **Kurtosis**: captures the "tailedness" of the distribution of returns. Higher kurtosis means more extreme values (outliers) in the distribution. i.e. the distribution has "fat tails".

Mean-variance investors: care about the expected return (higher is better) and variance (lower is better) of the return. They are risk-averse with a risk-aversion coefficient of A.

Mean-variance utility function: captures the preferences of mean-variance investors

$\boxed{U(r) = E[r] - \frac{1}{2}A \cdot \mathrm{Var}(r)}$ where $U(r)$ is the utility of the return r.

Indifference curve: is a curve that represents all combinations of expected return and variance that give the same utility to the investor. The slope of the indifference curve is given by: $\boxed{\frac{dE[r]}{d\sigma^2} = -A}$ The slope is negative, meaning that as variance increases, the expected return must also increase to maintain the same utility.

Portfolio

Portfolio: a combination of different assets or securities. Defined by number N_i and the price P_i of each asset i in the portfolio. The total value of the portfolio is: $\boxed{V = \sum_{i=1}^{n} N_i P_i}$

Portfolio weight: of asset i in the portfolio is:

$\boxed{w_i = \frac{N_i P_i}{V}}$ The portfolio weight represents the proportion of the total portfolio value that is invested in asset i. Weights can be positive (long position) or negative (short position). $\sum_{i=1}^{n} w_i = 1$

Mean and variance of a portfolio: with weights: $w_1, \ldots, w_n$ and returns $r_1, \ldots, r_n$:

- **Random variable**: $R_p = \sum_{i=1}^{n} w_i r_i$
- **Expected return**: $\boxed{E[R_p] = \sum_{i=1}^{N} w_i E[r_i]}$
- **Variance**: $\boxed{\mathrm{Var}(R_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{Cov}(r_i, r_j)}$

Special case - two assets:

- $E[R_p] = w_1 E[r_1] + w_2 E[r_2]$
- $\mathrm{Var}(R_p) = w_1^2 \mathrm{Var}(r_1) + w_2^2 \mathrm{Var}(r_2) + 2 w_1 w_2 \mathrm{Cov}(r_1, r_2)$ (Note: $w_2 = 1 - w_1$)

Special case - n equally-weighted assets: with returns $r_1, \ldots, r_n$:

- $E[R_p] = \frac{1}{n}\sum_{i=1}^{n} E[r_i]$
- $\mathrm{Var}(R_p) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(r_i, r_j)$

Portfolio beta: is the sensitivity of the portfolio return to the return of the market portfolio.

Efficient frontier and the tangency portfolio:

- **Mean-variance frontier portfolio**: minimizes risk (measured by variance) for a given expected return.
- **Efficient frontier**: is the set of portfolios that offer the highest expected return for a given level of risk. (Upper part of the mean-variance frontier)
- **Sharpe ratio**: $\boxed{= \frac{E[R_p] - r_f}{\sigma_p}}$ is the slope of the line from the risk-free rate to the portfolio. It measures the risk-adjusted return of the portfolio.
- **Capital Market Line**: represents the risk-return trade-off of efficient portfolios. It starts at the risk-free rate and is tangent to the efficient frontier. The slope of the CML is equal to the Sharpe ratio of the tangency portfolio.

Risk and Return

Basic Principles:

- Investors prefer higher expected returns and lower risk
- **Idiosyncratic (diversifiable)**: risks specific to that asset; can be eliminated through proper diversification.
- **Systematic (non-diversifiable)**: risks inherent in the entire market; cannot be diversified away.

CAPM implication: investors are only compensated for exposure to systematic risk.

CAPM implies that we can measure systematic risk using β.

Estimate β: as the slope coefficient when running this regression equation:

$\boxed{r_{it} - r_{ft} = \alpha_i + \beta_i(r_{mt} - r_{ft}) + \epsilon_{it}}$ We can use β_i to calculate the expected return of stock i implied by the CAPM: $\boxed{E(r_i) = r_f + \beta_i(r_M - r_f)}$ where

- r_f is the risk-free rate
- r_M (also denoted $\bar{r}_M$) is the **expected return** on the market (here, $r_M = E[r_m]$)
- $(\bar{r}_M - r_f)$ is the **expected excess return**, aka **"market risk premium"**.

Market portfolio: According to the CAPM, since all investors hold risky assets in the same proportions, the tangency portfolio is the market portfolio – a value weighted index of all investors' risky positions (i.e. it excludes risk-free borrowing/lending). Market portfolio's $\beta = 1$.

Risk-free portfolio: Since the risk-free asset has no risk, it cannot remove any other assets.

Therefore: $\boxed{\beta_f = 0 \Rightarrow \bar{r}_f = r_f = 0(\bar{r}_m - r_f)}$

CAPM formula for a portfolio of assets:

$\boxed{E[R_p] = R_f + \beta_p \cdot (E[R_m] - R_f)}$ The beta of a portfolio of assets is equal to the weighted average of its constituents' β's: $\beta_p = w_1\beta_1 + \ldots w_n\beta_n$.

Unlevering Betas: a special case of the formula for the beta of portfolio is frequently used to get estimates of the cost of capital for an entire firm, whose assets can be viewed as a portfolio of debt and equity.

- We usually estimate firm-level equity betas (β_E) using the stock returns of the firm.
- Recover asset betas (β_A, aka unlevered betas β_U) using the following formula:

Options

Options: Derivative contracts specifying **a right** to **buy (call option)** or **sell (put option)** an underlying asset at a specified price K (the **strike/exercise price**) on or before a specified date T (the **expiration/maturity date**).

- **Call option**:: right to buy the underlying asset at the strike price.
- **Put option**:: right to sell the underlying asset at the strike price.

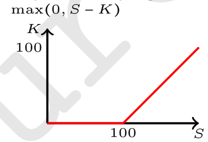
Exercise style:

- **American option**:: can be exercised at any time before expiration.
- **European option**:: can only be exercised at expiration.

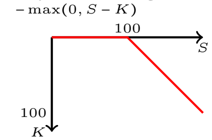
Option Payoff curves

- S: Price of the underlying asset at expiration
- K: Strike price of option
- Payoff ≠ Profit. To get profit (net payoff), need to subtract the option's cost.

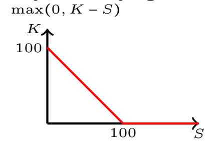
Payoff of buying a Call
$\max(0, S - K)$



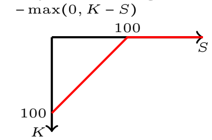
Payoff of selling a Call
$-\max(0, S - K)$



Payoff of buying a Put
$\max(0, K - S)$

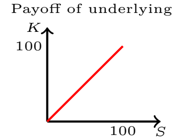


Payoff of selling a Put
$-\max(0, K - S)$

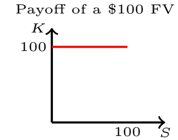


Payoff curves of other assets that can be used with options:

Payoff of underlying asset



Payoff of a $100 FV ZC Bond



Option payoff and profit

- r: Risk-free interest rate (EAR)
- C: Call option price
- P: Put option price

Call option:

	$S < K$	$S = K$	$S > K$
Payoff	0	0	$S - K$
Profit	$-C(1+r)^T$	$-C(1+r)^T$	$S - K - C(1+r)$

Put option:

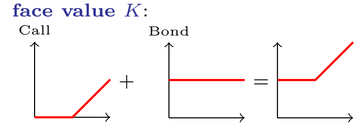
	$S < K$	$S = K$	$S > K$
Payoff	$K - S$	0	0
Profit	$K - S - P(1+r)^T$	$-P(1+r)^T$	$-P(1+r)$

An option is

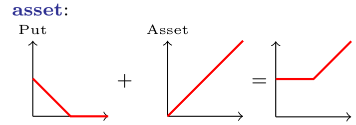
- **in-the-money**: if it has positive payoff at expiration. A call option is in-the-money if $S > K$, and a put option is in-the-money if $S < K$.
- **out-of-the-money**: if it has zero payoff at expiration. A call option is out-of-the-money if $S < K$, and a put option is out-of-the-money if $S > K$.
- **at-the-money**: if it has zero payoff at expiration. A call option is at-the-money if $S = K$, and a put option is at-the-money if $S = K$.

Put-call parity: following portfolios have the same payoff at expiration:

1. **Long call with strike price K + Bond with face value K**:



2. **Long put with strike price K + Underlying asset**:



Given their identical payoffs, under no-arbitrage, they should have the same price:

- $P_{call} + P_{bond} = P_{put} + P_{asset}$

- $C + K(1+r)^{-T} = P + S$

Binomial option pricing model: Iterative approach to price options that makes the following simplifications:

- Discrete periods, in which stock price can either go up or down.
- We find the option price by a *no arbitrage* argument. Price is equal to the cost of purchasing a *replicating portfolio* whose payoffs match the option payoff in each state. E.g., for a call, we solve: $\begin{cases} aS_u + bB_u &= C_u \\ aS_d + bB_d &= C_d \end{cases}$ where a is the number of shares of stock, b is the number of bonds, S_u and S_d are the stock prices if it goes up or down, and C_u and C_d are the call option prices if stock goes up or down.
- Under the binomial assumptions, the probability of a stock moves up or down is irrelevant, and the price of options can be determined solely using:
  - Current stock price S_0, interest rate r, strike price K and time to maturity T;
  - Magnitude of possible future changes of stock price (volatility), captured implicitly by the possible values the stock can take S_u and S_d.

Black-Scholes formula: Taking the limit of binomial model as the number of periods gets large, we obtain the B-S formula for the price of a European call option without dividends:

$$\boxed{C(S,K,T,r,\sigma) = S \cdot N(x) - K(1+r)^{-T} \cdot N(x - \sigma\sqrt{T})}$$

- S: current value of the underlying asset (in $)
- K: strike price of the option (in $)
- T: option maturity (in years)
- r: annual risk-free interest rate
- σ: annualized standard deviation of the underlying asset's return (volatility)
- $N(\cdot)$: cumulative normal distribution function (`NORM.S.DIST(x, TRUE)`). These $N(\cdot)$ terms capture the replicating portfolio weights.
- x: $\boxed{x = \frac{\ln\left(\frac{S}{K(1+r)^{-T}}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}}$

Real options

Option-like payoffs appear in many contexts outside of financial markets. Management can be thought of as the act of creating and optimally exercising real options. **E.g.**: follow-on products, R&D investments, delaying product launches, abandoning projects, etc.