Net Present Value (NPV)

Discount rate r: you're indifferent between receiving \$1 today and $\$\frac{1}{1+r}$ in one period.

Present Value (PV): $\boxed{PV(CF_t) = \frac{CF_t}{(1+r)^t}}$ how much a cash flow (CF) at time t is worth at time 0 (today). Computing a PV is often called "discounting". **Investors prefer payoffs that are larger, safer, and sooner.**

NPV: $\boxed{NPV = \sum_{t=0}^{T} \frac{CF_t}{(1+r)^t}}$ summs over PVs of all cash flows in a project.

- **Scalability**: $NPV(\alpha CF_1, \ldots, \alpha CF_T) = \alpha NPV(CF_1, \ldots, CF_T)$
- **Additivity**: $NPV(X_1 + Y_1, \ldots, X_T + Y_T) = NPV(X_1, \ldots, X_T) + NPV(Y_1, \ldots, Y_T)$
- **Breaking up by time**: $NPV(CF_1, \ldots, CF_T) = NPV(CF_1, \ldots, CF_j) + NPV(CF_{j+1}, \ldots, CF_T)$

Future Value (FV): $\boxed{FV_T(CF_0) = CF_0(1+r)^T}$ how much a cash flow at time 0 (today) is worth in T periods.

Perpetuity:

- Constant recurring cash flow A forever starting **1 period from now i.e.** $t = 1$: $\boxed{PV = \frac{A}{r}}$
- Growing perpetuity starting **1 period from now i.e.** $t = 1$ with cash flow A, growth rate g: $\boxed{PV = \frac{A}{r-g} (r > g)}$
- To include the cash flow at time 0, add A to the PV formula above.

Annuity:

- Constant recurring cash flow A for T periods starting **1 period from now**: (E.g. a loan)

$$\boxed{PV = \frac{A}{r}\left(1 - \frac{1}{(1+r)^T}\right)}$$

$$\boxed{FV = PV \cdot (1+r)^T = A\frac{(1+r)^T - 1}{r}}$$

- Growing annuity starting **1 period from now** with cash flow A, growth rate g for T periods.
  - If $r \neq g$ $\boxed{PV = \frac{A}{r-g}\left(1 - \frac{(1+g)^T}{(1+r)^T}\right)}$; $\boxed{FV = A\left(\frac{(1+r)^T - (1+g)^T}{r-g}\right)}$
  - If $r = g$ $\boxed{PV = T\left(\frac{A}{1+r}\right)}$; $\boxed{FV = T \cdot A \cdot (1+r)^{T-1}}$

Annual Percentage Rate (APR) & Effective Annual Rate (EAR): $\boxed{(1 + EAR) = \left(1 + \frac{APR}{k}\right)^k = (1+r)^k}$ where k is the number of compounding periods per year and r is the per-period (e.g. monthly) interest rate.

- APR $= r \cdot k$
- EAR i.e. Annual Percentage Yield (APY)

If $APR = EAR$ between two savings accounts, APR (the one with more frequent compounding) is better.

Mortgage-related terms

Principal: the amount of \$ borrowed in a lending agreement. E.g. Buy a \$1,000,000 house with a 20% down payment, the principal is \$800,000.

Interest:

- **Fixed rate**: No matter what happens to interest rates around the world, you would still be charged interest at this same rate.
- **Adjustable rate (ARM)**: E.g. an adjustable rate of 3% above the federal funds rate (the Fed's benchmark rate). If this rate is around 4.5%, you would be charged a 7.5% interest rate. If in the next month the Fed raises to 5%, you would be charged an 8% interest rate.

Amortization schedule: sequence of payments made through the loan's lifetime. A part of the payments goes to reduce (i.e. amortize) the principal owed, and the rest goes to pay the interest on the loan.

Collateral: An asset offered by the borrower as a guarantee in a loan. If you fail to make payments, the bank can take the collateral.

Refinancing: Paying off an existing loan with a new loan that has better terms. E.g. lower interest rate, lower monthly payment, shorter loan term.

E.g.: a 30-year fixed-rate mortgage, APR 9% compounded monthly. Fixed monthly payment = \$3000. First payment will start next month and last until the contract expires in 30 years.

- **How much borrowed when took out the mortgage?**: use the constant annuity formula, where $A = \$3000$, $r = \frac{0.09}{12} = 0.0075$, $T = 30 \times 12 = 360$. $PV = \frac{A}{r}\left(1 - \frac{1}{(1+r)^T}\right) = \frac{3000}{0.0075}\left(1 - \frac{1}{(1+0.0075)^{360}}\right) = \$372,845.60$
- **10 years later, how much must pay back to the bank if sell the house?**: (i.e. NPV of the remaining principal amount as of this future date.) $T = 20 \times 12 = 240$
  $PV = \frac{3000}{0.0075}\left(1 - \frac{1}{(1+0.0075)^{240}}\right) = \$333,434.86$
- **Why the remaining principal is still so high after 10 years' worth of repayments?**: Because the amortization schedule is front-loaded. I.e. Early on in the life of the mortgage, the vast majority of each monthly payment is interest: interest is applied to the remaining principal amount, and this remaining principal is highest in month 1 and decreases over time. Because the majority of each monthly payment in the early years is interest, the principal repayment amounts are small, and the remaining principal de-creases very slowly. (It's only toward the end of the mortgage that interest payments decline enough to repay the principal more quickly.)

Inflation i: the change in CPI $\boxed{1 + i_{t+1} = \frac{CPI_{t+1}}{CPI_t}}$

- **"Nominal"**: not adjusted for inflation
- **"Real"**: adjusted for inflation

Real rate of return: $\boxed{r_{\text{real}} = \left(\frac{1+r_{\text{nominal}}}{1+i} - 1\right) \approx (r_{\text{nominal}} - i)}$ If i is too high, don't use the approximation (too off).

Treat inflation consistently for NPV: $\boxed{PV(CF_T) = \frac{CF_{\text{nominal},T}}{(1+r_{\text{nominal},T})^T} = \frac{CF_{\text{real},T}}{(1+r_{\text{real},T})^T}}$ (",T" denotes the cash flow at time T)

E.g. False statement: *If inflation is zero, two projects paying \$100 in different years have the same NPV.* Time value of money still applies; later payments are worth less.

E.g.: Now 25 yo ($t = 0$) and wants to retire at 65 yo ($t = 40$) with \$3 milion. Will make first deposit next year ($t = 1$) and continue until last deposit at $t = 40$. Income expected to increase at 3% per year (nominal) (i.e. g). Savings will earn an annual return of 8% (nominal) (i.e. r).

1. **What should be first deposit X to achieve the goal?**: Perpetuity value of X from today minus perpetuity value of the balance at $t = 41$'s PV should be \$3m's PV.
   $\frac{3 \times 10^6}{(1+r)^{40}} = \frac{X}{r-g} - \frac{X(1+g)^{40}}{r-g} \cdot \frac{1}{(1+r)^{40}} \Rightarrow X = 8124.58$
2. **If inflation is 2% per year. What would be the real value of the savings balance at** $t = 40$: Deflate the nominal future value:
   $FV_{\text{real}} = 3 \times 10^6 \times 1.02^{-40} = 1,358,671.25$. (Alt. calculate $r_{\text{real}} = \frac{1.08}{1.02} - 1 = 5.88\%$ and then $FV_{\text{real}} = 3 \times 10^6 \times \frac{(1+r_{\text{real}})^{40}}{(1+r_{\text{nominal}})^{-40}}$.)
3. **If inflation is instead 3% and there's no change in nominal income growth or expected nominal return on savings, what would the first deposi be to achieve the same real value balance goal at $t = 40$ (i.e. previous question's FV_{real} value)?**: Let Y be the nominal amount at $t = 40$ to achive the same FV_{real}. $Y \times \frac{(1+r'_{\text{real}})^{40}}{(1+r_{\text{nominal}})^{40}} = 1,358,671.25 \Rightarrow$
   $Y \times \frac{(1+(\frac{1.08}{1.03})-1)^{40}}{(1.08)^{40}} = 1,358,671.25 \Rightarrow$
   $Y = 4,432,036.95$. Then, similar to q1: $\frac{Y}{(1.08)^{40}} = \frac{X'}{0.08-0.03} - \frac{X'(1.03)^{40}}{0.08-0.03} \cdot \frac{1}{(1.08)^{40}} \Rightarrow X' = 12,002.75$

Capital Budgeting

To maximize value, take on only projects with positive NPV.

- **Single**: take it only if it has positive NPV.
- **Independent**: take all with positive NPV.
- **Mutually exclusive**: take the one with the highest positive NPV.
- Ignore sunk costs, including opportunity costs.

Cash operating expenses:

- **COGS**: direct costs attributable to the production of the goods sold by a business.
- **R&D**: costs associated with discovering new knowledge or develop new products, processes, and services.
- **SG&A**: costs not directly tied to the production of goods. e.g. "S": advertising and sales commissions. "G": salaries of non-production personnel. "A": legal, accounting, and exec salaries.

Depreciation: non-cash expense that reduces the value of an asset over time. For most finance problems, we want to strip out effects of depreciation to get back to free cash flow. Exception: if depreciation affects free cash flows through taxes.

EBITDA: $\boxed{= \text{(Op. Rev.)} - \text{(All Op. Exp. w/o depreciation)}}$

EBIT: $\boxed{= \text{EBITDA} - \text{Depreciation (\& Amort.)}}$

Cash Flows: from accounting statements

$$\boxed{CF = (1-\tau)(\text{EBITDA}) + \tau(\text{Dep.}) - (\text{CapEx}) - \Delta\text{WC}}$$

$$\boxed{CF = (1-\tau)(\text{EBIT}) + (\text{Dep.}) - (\text{CapEx}) - \Delta\text{WC}}$$

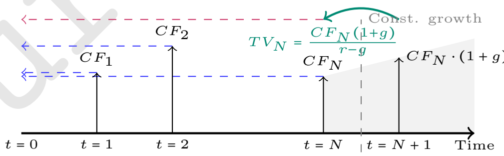
Working Capital (WC): $\boxed{= \text{Inventory} + \text{A/R} - \text{A/P}}$ We are about changes (i.e. Δ) in WC, not levels because if keeping WC constant, no new cash flow required.

Valuing Firms (usual approach)

- Forecasted CF_1 through CF_N
- Assume constant growth at rate g after year N.

Then enterprise value as of today ($t = 0$):

$$\boxed{EV = \left(\sum_{t=1}^{N} \frac{CF_t}{(1+r)^t}\right) + \frac{TV_N}{(1+r)^N}}$$



Hotspur example: $N = 12$ (2019) i.e. year 13 begins the constant growth.

	0	1	2	...	11	12	13
EBITDA	5.00	5.85	6.76		17.93	19.49	20.27
- Depreciation	2.20	2.29	2.38		3.39	3.52	3.66
EBIT	2.80	3.56	4.38		14.54	15.97	16.61
EBIT × $(1-\tau)$		2.32	2.85		9.45	10.38	10.79
+ Depreciation		2.29	2.38		3.39	3.52	3.66
- CapEx		-3.43	-3.57		-5.08	-5.28	-5.49
- Δ WC		3.90	4.25		9.22	10.05	4.87
Cash Flow		5.07	5.90		16.98	18.67	13.83

Key fact: everything grows at 4% per year from year 12 (2019) onwards → we have a **perpetuity with growth**, with starting cash flow CF_{2020}

$$V_{2019} = \frac{CF_{2020}}{r-g} = \frac{13.83}{10.25\% - 4\%} = £221.32\text{M}$$

PV of perpetuity with growth
(10.25% discount rate, 4% growth)

(Could also start the perpetuity in 2021 and calculate PV as of 2020…but then need to be sure we include 2020 CF separately for total NPV)

	0	1	2	...	11	12	13	
Cash Flow		5.07	5.90		16.98	18.67	13.83	···
Terminal Value						221.32		Infinite stream
"Total" Cash Flow		5.07	5.90		16.98	239.98		
Discount: $(1+r)^t$		1.1025	1.2155		2.9253	3.2251		
Present Value		4.60	4.85		5.80	74.41		

Also CF in year 0 is not added to the EV.

Enterprise Value (EV): PV of future CFs $EV = \sum_{t=1}^{\infty} PV(CF_t)$ i.e. estimated via discounted CF (DCF) analysis.

From EV to Equity Value: MV of Equity = Value of operations (EV) + Non-operational assets − MV of Debt

For Hotspur, Value of Assets $= EV +$ (Cash & equivalents) + (Investments available for sale)

MV of Debt = (Long-term debt)

Alternatives to NPV

Internal rate of return: discount rate that makes zero NPV. $\boxed{NPV = \sum_{t=0}^{T} \frac{CF_t}{(1+IRR)^t} = 0}$ Determine some fixed IRR^* i.e. "threshold rate":

- **Independent**: take if $IRR > IRR^*$.
- **Mutually exclusive**: take the one with the highest IRR among projects $IRR > IRR^*$.

IRR leads to the same decision as NPV if:

- Cash outflow occurs only at time 0
- Only one project is being considered
- Opportunity cost of capital (discount rate r) remains constant for all periods
- $IRR^* = r$

Shortcomings: no solution, multiple solutions, project size not accounted for, different projects' horizons not fully considered.

E.g. False statement: *If project A's NPV is larger than project B's, then the IRR of investing in project A is always higher than that of project B.* The IRR could yield incongruent predictions in capital budgeting compared to the NPV rule.

E.g. False statement: *XZ Company is considering purchasing a copper mine, which requires an initial investment in the first year, followed by positive cash flows for 30 years, but a large closure cost in year 31. IRR is a good method for analyzing this investment.* Not a good metric when there are negative CFs after time 0, as it may yield multiple IRRs

Payback period: min. length of time k such that sum of CFs from a project is positive.

$\boxed{\sum_{t=1}^{k} CF_t \geq -CF_0 = I_0}$ Determine some fixed threshold k^*:

- **Independent**: take if $k \leq k^*$.
- **Mutually exclusive**: take the one with the minimum k among projects $k \leq k^*$.

Discounted payback period: ditto but discount CFs. $\boxed{\sum_{t=1}^{k} \frac{CF_t}{(1+IRR)^t} \geq -CF_0 = I_0}$

Shortcomings: ignores CFs after k and project size.

Profitability index (PI): ratio of the NPV of future CFs to the initial investment. $\boxed{PI = \frac{NPV}{I_0}}$

- **Independent**: take all $PI > 1$.
- **Mutually exclusive**: take the one with the highest PI and $PI > 1$.

Shortcomings: doesn't account for project size.

Bonds

Face value i.e. "par value" / "principal": the value of a bond that appears on its face and that will be paid to the investor by the issuer at maturity.

Coupon: interest paid on a bond's face value on a periodic basis prior to maturity.

Spot interest rate r_t: the market interest rate for discounting a single risk-free cash flow at horizon t.

Zero coupon bond: a bond that pays no coupons and is sold at a discount to its face value. Price of a \$1 ZC bond with maturity T is $\boxed{P_T = \frac{\$1}{(1+r_T)^T}}$. We can infer spot interest rates from different ZC bond prices. $\boxed{\text{Given } P_T \Rightarrow r_T = \left(\frac{\$1}{P_T}\right)^{\frac{1}{T}} - 1}$.

Coupon bonds: with periodic coupon payments of C_t and face value F. Price of a coupon bond is

$$\boxed{P = \left(\sum_{t=1}^{T} \frac{C_t}{(1+r_t)^t}\right) + \frac{F}{(1+r_T)^T} = \left(\sum_{t=1}^{T} C_t \cdot P_t^{ZC}\right) + F \cdot P_T^{ZC}}$$

Yield to maturity (YTM): the discount rate y that makes the PV of a bond's future cash flows equal to its price. $\boxed{P = \left(\sum_{t=1}^{T} \frac{C_t}{(1+y)^t}\right) + \frac{F}{(1+y)^T}}$ For ZC bonds, the YTM is equal to the spot interest rate $\boxed{y = r_T}$.

E.g. Bond Arbitrage: construct a trading strategy based on a mixture of bonds to generate an arbitrage profit.

A. 1-year ZC bond: FV: \$1, price \$0.8
B. 2-year ZC bond: FV: \$105, price \$100
C. 2-year 5% bond: FV: \$100, price \$106

1. Infer spot rates from ZCBs:

Maturity	Spot Rate
1	$\frac{1}{0.8} - 1 = 0.25$
2	$\left(\frac{105}{100}\right)^{\frac{1}{2}} - 1 \approx 0.0247$

2. Calculate the no-arbitrage price of the coupon bond using spot rates: $P = \frac{5}{(1+0.25)} + \frac{105}{(1+0.0247)^2} \approx 103.999$ which is less than the market price of \$106.
3. Intuition: The coupon bond is overpriced, so sell it and buy the ZC bonds.
4. Goal: Use ZC bonds to replicate the coupon bond's future cash flows.

Assume Buying	Bond	CF_1	CF_2
x	1-year ZC	1	0
y	2-year ZC	0	105
1	5% bond	5	105

5. Solve for x and y: $\begin{cases} 1x + 0y & = 5 \\ 0x + 105y & = 105 \end{cases} \Rightarrow \begin{cases} x = 5 \\ y = 1 \end{cases}$
6. At $t = 0$, buy 5 1-year ZC bonds and 1 2-year ZC bond (costs $5 \times \$0.8 + 1 \times \$100 = 104$) and sell 1 coupon bond (earning \$106). This generates \$2 arbitrage profit.

Expectations hypothesis (EH): the yield curve reflects the market's expectations of future interest rates. I.e. Long-term interest rates are equal to the geometric average of short-term interest rates until maturity. Intuition: investors indifferent between long bond and rolling over short bond. **E.g. 2-year rate should be given by**:

$\boxed{(1 + r_{t,2})^2 = (1 + r_{t,1})\,(1 + E_t(r_{t+1,1}))}$, where

- $r_{t,1}$: the 1-year rate at time t (e.g. today)
- $r_{t,2}$: the 2-year rate at time t (e.g. today)
- $E_t(r_{t+1,1})$: expectation at time t (e.g. today) of the 1-year rate that will exist at time $t+1$

More generally: yields reflect a component related to expectations about future short-term interest rates plus an additional term reflecting a risk premium reflecting the uncertainty about fluctuations in future interest rates:

$\boxed{(1 + r_{t,2})^2 = (1 + r_{t,1})\,(1 + E_t(r_{t+1,1})) + \text{Risk Premium}}$,

E.g. bond risks: interest rate changes, inflation, credit/default

Duration

(Macaulay) Duration: the weighted average time to receive the bond's cash flows. Tells how long you have to wait to receive the average dollar from owning that bond in PV-weighted terms.

$$\boxed{D = \frac{1}{P}\sum_{t=1}^{T} t \cdot PV(CF_t) = \frac{1}{P}\sum_{t=1}^{T} t \cdot \frac{CF_t}{(1+y)^t}}$$

Zero-coupon bonds' duration is equal to their maturity. All else equal, coupon bond has shorter duration than zero-coupon bond; higher coupon means lower duration.

Modified duration: $\boxed{MD = \frac{D}{(1+y)}}$ the sensitivity of a bond's price to changes in interest rates.

Price change estimate: accurate for small changes in y $\boxed{\frac{\Delta P}{P} \approx -MD \times \Delta y}$ which shows that duration measures bond exposure to interest-rate risk. The higher the (modified) duration of a bond, the more its price moves (ΔP) given a small change in interest rates (Δy). Longer duration ⇒ more sensitive to interest rate changes.

Duration matching (i.e. immunization): matching the duration of assets and liabilities to eliminate interest rate risk.

$\boxed{\frac{D^{\text{Assets}} \cdot \Delta y}{1+y} P^{\text{Assets}} \approx \frac{D^{\text{Liab.}} \cdot \Delta y}{1+y} P^{\text{Liab.}}}$ If the value of assets (P^A) equal to value of liabilities (P^L) (usually the case), then matching durations of two immunizes portfolio from interest-rate risk.

- $\boxed{\text{If } P^A \neq P^L, \text{ we need to set } D^A P^A = D^L P^L}$ to immunize.
- Value of portfolio unchanged for small movements in y ⇒ hedged against interest-rate risk.
- $D^A > D^L$ bets rates will stay constant or decrease
- $D^A < D^L$ bets rates will increase.
- This is only a local approximation. If interest rates change, need to rebalance the portfolio.

E.g. False statement: *You're uncertain about future interest rates, but your expectation is that they will stay unchanged relative to today. It's therefore risk-free to issue short-term bonds and invest the proceeds in long-term bonds with higher yields, and no bank that follows this strategy will ever go bankrupt.* **Think SVB.** If the duration of assets exceeds the duration of liabilities, then the bank is exposed to interest-rate risk. If interest rates increase, then the value of assets will decrease more than the value of liabilities, and the bank may not be able to repay its short-term debt.

Default risk: a debt issuer fails to make interest or principal payments when due. Consider a ZCB. Let q be the probability of the bond pays the promised payoff (i.e. $(1-q)$ it defaults), the price of this bond: $\boxed{P = \frac{q \times \$1 + (1-q) \times 0}{1+r}}$.

Ratings: assess the probability of default of bonds, by agencies like Moody's, S&P, Fitch.

E.g. default-free bonds, pay coupons annually:

Bond	Coupon	YTM	Price	Face Value
A	0%	1	\$95.238	\$100
B	4%	2	\$90.045	\$100

1. **What's the current 2-yr spot interest rate?**: Bond A to get 1-yr spot rate r_1: $\$95.238 = \frac{\$100}{(1+r_1)^1} \Rightarrow r_1 = 5\%$. Then, express bond B: $\$90.045 = \frac{4\% \cdot \$100}{(1+5\%)^1} + \frac{(1+4\%) \cdot \$100}{(1+r_2)^2} \Rightarrow r_2 = 4.5\%$.
2. **What's the no arbitrage price of a ZC default-free w/ FV \$100 maturing 2yrs from now?**: $P = \frac{\$100}{(1+r_2)^2} = 91.573$.

Stocks

Common stocks: equity or ownership shares in a corporation. Give right to payments; uncertain in timing and magnitude.

- **Dividends**: periodic payments to shareholders.
- **Residual claim**: to assets and cash flows after all other creditors (e.g. debtholders) have been paid.
- **Voting rights**: elect board of directors, approve major corporate actions.
- **Limited liability**: shareholders not personally liable for company's debts.

Primary market: where firms issue securities.

- **Underwriting**: in an IPO, investment bank buys securities from issuer and resells to public while taking the underwriting spread as profits.
- **Venture capital**: is raised from investment partnerships, investment institutions, or wealthy individuals.
- **Secondary (seasoned) offering**: when a public company issues additional shares to the public.

Secondary market: where investors trade previously issued securities.

- **Exchanges**: e.g. NYSE, NASDAQ
- **Over-the-counter (OTC)**: markets

Stock Valuation

Discounted Dividend Model (DDM): price equals discounted future dividends.

$$\boxed{P_0 = \sum_{t=1}^{\infty} \frac{D_t}{(1+r_t)^t}}$$

Gordon growth model: a special case of DDM where dividends grow at a constant rate g.

$\boxed{P_0 = \frac{D_0 \cdot (1+g)}{r-g} = \frac{D_1}{r-g} = \frac{ROE-g}{r-g} \cdot BV_0}$ Current stock price depends on the firm's BV, ROE, growth rate, and discount rate.

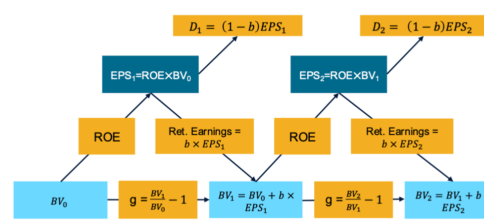
- If $ROE = r$, then the price of the stock is exactly equal to the book value of equity.
- If $ROE > r$, reinvesting more increases shareholder value. Optimal to reinvest i.e. investment opportunities that earn expected returns higher than the cost of capital. More common early in the life cycle of the firm.
- If $ROE < r$, reinvesting more destroys shareholder value. Optimal not to reinvest i.e. return excess cash to shareholders. More common later in the life cycle of the firm where higher growth can be costly.

If stock's growth pattern is forecasted to change over time, typically compute early-stage PV explicitly and then terminal value.

Forcasting dividends D_t: firm investment determines dividend growth. Always a trade-off between (1) growning the firm and (2) paying out dividends.

- **Earnings Per Share (EPS)**: net profit after taxes $\boxed{EPS = BV \times ROE}$
- **Dividend Payout Ratio**: $\boxed{p = \frac{\text{Dividends}}{\text{Net Income}} = 1 - b}$
- **Retained earnings**: $\boxed{\text{Net Income} - \text{Dividends}}$
- **Plowback Ratio**: $\boxed{b = \frac{\text{Retained Earnings}}{\text{Net Income}}}$
- **Book value of Equity**: cumulative retained earnings $\boxed{\text{BV} = \text{Assets} - \text{Liabilities}}$
- **Return on Equity (ROE)**: how efficiently firm uses equity to generate profits $\boxed{\frac{\text{Net Income}}{\text{BV}}}$
- **Dividend growth rate**: BV grows from R.E. If ROE and b are constant, g is constant. $\boxed{g = b \cdot ROE = \frac{\Delta BV}{BV} = \frac{\text{Retained Earnings}}{\text{Net Income}} \times \frac{\text{Net Income}}{BV}}$.
- **Change in book value**: $\boxed{\Delta\text{BV} = b \cdot ROE \cdot \text{BV}}$

$$\begin{aligned}
\text{Dividends} &= \text{Net Income} - \text{Retained Earnings} \\
&= BV \cdot ROE - \Delta BV \\
&= BV \cdot ROE - b \cdot ROE \cdot BV \\
&= (1-b) \cdot ROE \cdot BV \\
&= p \cdot ROE \cdot BV
\end{aligned}$$



E.g. What should be Firm ABC's stock price today: (last day of year 0)

- Book value of \$37.80 (per share) at the end of year 0.
- Growth plan is to reinvest 80% of its earnings next year.
- After that, it will focus on returning cash to shareholder beginning in year 2, paying out 60% of its earnings as dividends.
- From year 3 onwards, ABC will pay out 80% of earnings and plans to maintain this payout policy forever.
- Assume that the market is expecting an ROE of 10% for years 1 and 2, and an ROE of 8% from year 3 onwards,
- Assume a constant discount rate of 8%.

	0	1	2	3
ROE		10%	10%	8%
EPS		\$3.78	\$4.08	\$3.40
Begin BV		\$37.80	\$40.28	\$42.46
Payout p		20%	60%	60%
Plowback b		80%	40%	40%
End BV	\$37.80	\$40.82	\$42.46	\$43.14
Div./share		\$0.76	\$2.45	\$2.72
Term. Val.			\$42.46	
$\sum$ CFs		\$0.76	\$44.91	
Discnt Fac		1.08	1.17	
$PV(CFs)$		\$0.70	\$38.50	
Stock Price				

Diversification

Asset return characteristics

Buy an asset (e.g. a stock) at $t = 0$ at price P_0. At time $t = 1$,
- its cash flow (dividend) is D_1, and
- its price is P_1

(both are random variables). The risk-free rate is r_F.

Realized return: $\boxed{r_1 = \frac{D_1+P_1}{P_0} - 1}$ Returns comes from both dividends and capital gains.

Expected return: $\boxed{E[r_1] = \frac{E[D_1]+E[P_1]}{P_0} - 1}$

Excess return: (realized) $\boxed{r_1 - r_F}$

Risk premium: (expected excess return)

$\boxed{E[r_1] - r_F}$

Mean (average) return: $\boxed{\bar{r} = E[r] = \frac{1}{T}\sum_{t=1}^{T} r_t}$

Would be same as the expected return $E[r_t]$ if expected returns are constant for all t.

Estimate Expected Return:
- if have multiple possible scenarios for returns and know the probability of each scenario, use:
  $\boxed{E[r] = \sum_{i=1}^{N} p_i r_i}$
- if have a time series of past T observations of returns, estimate sample estimate of expected return $\bar{r}$ as: $\boxed{\hat{r} = \frac{1}{T}\sum_{t=1}^{T} r_t}$

Variance: measures the volatility or deviation of returns from the mean.

$\boxed{\text{Var}(r) = \sigma^2 = E\left[(r - E[r])^2\right]}$ If given (past) data sample of T returns, the sample variance is:

$\boxed{\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2}$ where the expected return $\bar{r}$ can be estimated by the sample mean $\bar{r}$ as defined above.

Standard deviation: $\boxed{\sigma = \sqrt{\text{Var}(r)}}$ measures the risk of the asset. Gives a magnitude in percent.

Covariance: measures the degree to which two random variables move together.

$\boxed{\sigma_{ij} = \text{Cov}(r_i, r_j) = E[(r_i - \bar{r_i})(r_j - \bar{r_j})]}$

Estimate Cov: given (past) data sample of T returns, the sample covariance is:

$\boxed{\hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(r_{i,t} - \bar{r_i})(r_{j,t} - \bar{r_j})}$

Covariance can be:
- positive (both variables move in the same direction),
- negative (two variables move in the opposite direction), or
- zero (no relationship).

Variance is a special case of covariance, where the two variables are the same.

$\boxed{\text{Var}(r_i) = \sigma_{ii} = \text{Cov}(r_i, r_i)}$

Correlation: measures the strength of the linear relationship between two random variables. Always between -1 and 1. $\boxed{\text{Corr}(r_i, r_j) = \rho_{ij} = \frac{\text{Cov}(r_i,r_j)}{\sigma_i\sigma_j}}$

Beta: measures the sensitivity of an asset's return to the return of the market portfolio.

$\boxed{\beta_i = \frac{\text{Cov}(r_i, r_m)}{\text{Var}(r_m)}}$ where r_m is the return of the market portfolio.

Other measures of risk:
- **Skewness**: captures the asymmetry of the distribution of returns.
- **Kurtosis**: captures the "tailedness" of the distribution of returns. Higher kurtosis means more extreme values (outliers) in the distribution. i.e. the distribution has "fat tails".

Mean-variance investors: care about the expected return (higher is better) and variance (lower is better) of the return. They are risk-averse with a risk-aversion coefficient of A.

Mean-variance utility function: captures the preferences of mean-variance investors

$\boxed{U(r) = E[r] - \frac{1}{2}A \cdot \text{Var}(r)}$ where $U(r)$ is the utility of the return r.

Indifference curve: is a curve that represents all combinations of expected return and variance that give the same utility to the investor. The slope of the indifference curve is given by: $\boxed{\frac{dE[r]}{d\sigma^2} = -A}$ The slope is negative, meaning that as variance increases, the expected return must also increase to maintain the same utility.

Portfolio

Portfolio: a combination of different assets or securities. Defined by number N_i and the price P_i of each asset i in the portfolio. The total value of the portfolio is: $\boxed{V = \sum_{i=1}^{n} N_i P_i}$

Portfolio weight: of asset i in the portfolio is:

$\boxed{w_i = \frac{N_i P_i}{V}}$ The portfolio weight represents the proportion of the total portfolio value that is invested in asset i. Weights can be positive (long position) or negative (short position). $\sum_{i=1}^{n} w_i = 1$

Mean and variance of a portfolio: with weights: $w_1, \ldots, w_n$ and returns $r_1, \ldots, r_n$:
- **Random variable**: $R_p = \sum_{i=1}^{n} w_i r_i$
- **Expected return**: $\boxed{E[R_p] = \sum_{i=1}^{N} w_i E[r_i]}$
- **Variance**:
  $\boxed{\text{Var}(R_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \text{Cov}(r_i, r_j)}$

Special case - two assets:
- $E[R_p] = w_1 E[r_1] + w_2 E[r_2]$
- $\text{Var}(R_p) = w_1^2\text{Var}(r_1) + w_2^2\text{Var}(r_2) + 2w_1w_2\text{Cov}(r_1, r_2)$ (Note: $w_2 = 1 - w_1$)

Special case - n equally-weighted assets: with returns $r_1, \ldots, r_n$:
- $E[R_p] = \frac{1}{n}\sum_{i=1}^{n} E[r_i]$
- $\text{Var}(R_p) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\text{Cov}(r_i, r_j)$

Portfolio beta: is the sensitivity of the portfolio return to the return of the market portfolio.

Efficient frontier and the tangency portfolio:
- **Mean-variance frontier portfolio**: minimizes risk (measured by variance) for a given expected return.
- **Efficient frontier**: is the set of portfolios that offer the highest expected return for a given level of risk. (Upper part of the mean-variance frontier)
- **Sharpe ratio**: $\boxed{= \frac{E[R_p]-r_f}{\sigma_p}}$ is the slope of the line from the risk-free rate to the portfolio. It measures the risk-adjusted return of the portfolio.
- **Capital Market Line**: represents the risk-return trade-off of efficient portfolios. It starts at the risk-free rate and is tangent to the efficient frontier. The slope of the CML is equal to the Sharpe ratio of the tangency portfolio.

Risk and Return

Basic Principles:
- Investors prefer higher expected returns and lower risk
- **Idiosyncratic (diversifiable)**: risks specific to that asset; can be eliminated through proper diversification.
- **Systematic (non-diversifiable)**: risks inherent in the entire market; cannot be diversified away.

Capital Asset Pricing Model (CAPM): describes the relationship between risk and expected return. It states that the expected return of an asset is equal to the risk-free rate plus a risk premium that is proportional to the asset's beta.

CAPM implications:
- investors are only compensated for exposure to systematic risk.
- we can measure systematic risk using β.
- **Alpha α**: (intercept in the regression equation used to estimate β) should be zero because β alone should explain all differences in expected returns among assets.

CAPM assumptions:
- No asymmetric information (everyone has the same information), and everyone agrees on the distribution of all assets' returns.
- Investors are rational and risk-averse.
- There exists a risk-free asset for both borrowing and investing.

Estimate β: as the slope coefficient when running this regression equation:

$\boxed{r_{it} - r_{ft} = \alpha_i + \beta_i(r_{mt} - r_{ft}) + \epsilon_{it}}$ We can use β_i to calculate the expected return of stock i implied by the CAPM: $\boxed{E(r_i) = r_f + \beta_i(r_M - r_f)}$ where
- r_f is the risk-free rate
- r_M (also denoted $\bar{r}_M$) is the **expected return** on the market (here, $r_M = E[r_m]$)
- $(\bar{r}_M - r_f)$ is the **expected excess return**, aka **"market risk premium"**.

Market portfolio: According to the CAPM, since all investors hold risky assets in the same proportions, the tangency portfolio is the market portfolio – a value weighted index of all investors' risky positions (i.e. it excludes risk-free borrowing/lending). Market portfolio's $\beta = 1$.

Risk-free portfolio: Since the risk-free asset has no risk, it cannot remove any other assets. Therefore: $\boxed{\beta_f = 0 \Rightarrow \bar{r}_f = r_f = 0(\bar{r}_m - r_f)}$

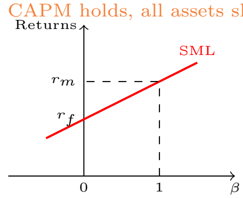
CAPM formula for a portfolio of assets:

$\boxed{E[R_p] = R_f + \beta_p \cdot (E[R_m] - R_f)}$ The beta of a portfolio of assets is equal to the weighted average of its constituents' β's: $\beta_p = w_1\beta_1 + \ldots w_n\beta_n$.

Unlevering Betas: a special case of the formula for the beta of portfolio is frequently used to get estimates of the cost of capital for an entire firm, whose assets can be viewed as a portfolio of debt and equity.

- Usually estimate firm-level equity betas (β_E) using the stock returns of the firm.
- Recover asset betas (β_A, aka unlevered betas β_U) using the following formula:
  $\boxed{\beta_U = \beta_A = \frac{D}{D+E}\beta_D + \frac{E}{D+E}\beta_E}$
- Investment-grade debt (rated > BBB): assume $\beta_D = 0$, cost of debt = risk-free rate.

Security Market Line (SML): CAPM's graphical representation. Plots returns (Y) on β (X). If CAPM holds, all assets should lie on the SML.



- Slope = market risk premium $(E[r_m] - r_f)$.
- An asset with a β of 0 has an expected return = the risk-free rate r_f.
- An asset with a β of 1 has an expected return = the market return $E[r_m]$.

CML	**SML**
Plots $E[r]$ vs σ	$E[r]$ vs asset β
Passes thru risk-free asset & tangency portfolio	
Slope = Maximum Sharpe ratio	Market risk prem
All individual risky assets lie below	lie on

Options

Options: Derivative contracts specifying **a right** to **buy (call option)** or **sell (put option)** an underlying asset at a specified price K (the **strike/exercise price**) on or before a specified date T (the **expiration/maturity date**).

- **Call option**:: right to buy the underlying asset at the strike price.
- **Put option**:: right to sell the underlying asset at the strike price.

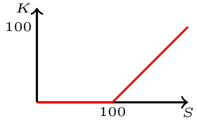
Exercise style:

- **American option**:: can be exercised at any time before expiration.
- **European option**:: can only be exercised at expiration.

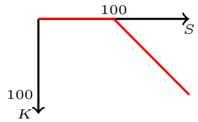
Option Payoff curves

- S: Price of the underlying asset at expiration
- K: Strike price of option
- Payoff ≠ Profit. To get profit (net payoff), need to subtract the option's cost.

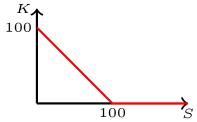
Payoff of buying a Call
$\max(0, S-K)$



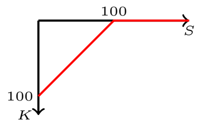
Payoff of selling a Call
$-\max(0, S-K)$



Payoff of buying a Put
$\max(0, K-S)$

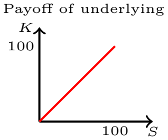


Payoff of selling a Put
$-\max(0, K-S)$

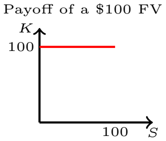


Payoff curves of other assets that can be used with options:

Payoff of underlying asset



Payoff of a \$100 FV ZC Bond



Option payoff and profit

- r: Risk-free interest rate (EAR)
- C: Call option price
- P: Put option price

Call option:

	$S<K$	$S=K$	$S>K$
Payoff	0	0	$S-K$
Profit	$-C(1+r)^T$	$-C(1+r)^T$	$S-K-C(1+r)^T$

Put option:

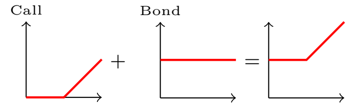
	$S<K$	$S=K$	$S>K$
Payoff	$K-S$	0	0
Profit	$K-S-P(1+r)^T$	$-P(1+r)^T$	$-P(1+r)^T$

An option is

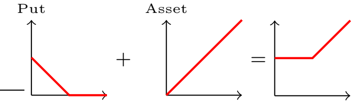
- **in-the-money**: if it has positive payoff at expiration. A call option is in-the-money if $S>K$, and a put option is in-the-money if $S<K$.
- **out-of-the-money**: if it has zero payoff at expiration. A call option is out-of-the-money if $S<K$, and a put option is out-of-the-money if $S>K$.
- **at-the-money**: if it has zero payoff at expiration. A call option is at-the-money if $S=K$, and a put option is at-the-money if $S=K$.

Put-call parity: following portfolios have the same payoff at expiration:

1. **Long call with strike price K + Bond with face value K**:



2. **Long put with strike price K + Underlying asset**:



Given their identical payoffs, under no-arbitrage, they should have the same price:

- $P_{call} + P_{bond} = P_{put} + P_{asset}$
- $C + K(1+r)^{-T} = P + S$

Binomial option pricing model: Iterative approach to price options that makes the following simplifications:

- Discrete periods, in which stock price can either go up or down.
- We find the option price by a *no arbitrage* argument. Price is equal to the cost of purchasing a *replicating portfolio* whose payoffs match the option payoff in each state. E.g., for a call, we solve: $\begin{cases} aS_u + bB_u & = C_u \\ aS_d + bB_d & = C_d \end{cases}$ where a is the number of shares of stock, b is the number of bonds, S_u and S_d are the stock prices if it goes up or down, and C_u and C_d are the call option prices if stock goes up or down.
- Under the binomial assumptions, the probability of a stock moves up or down is irrelevant, and the price of options can be determined solely using:
  - Current stock price S_0, interest rate r, strike price K and time to maturity T;
  - Magnitude of possible future changes of stock price (volatility), captured implicitly by the possible values the stock can take S_u and S_d.

Black-Scholes formula: Taking the limit of binomial model as the number of periods gets large, we obtain the B-S formula for the price of a European call option without dividends:

$$\boxed{C(S,K,T,r,\sigma) = S \cdot N(x) - K(1+r)^{-T} \cdot N(x - \sigma\sqrt{T})}$$

- S: current value of the underlying asset (in \$)
- K: strike price of the option (in \$)
- T: option maturity (in years)
- r: annual risk-free interest rate
- σ: annualized standard deviation of the underlying asset's return (volatility)
- $N(\cdot)$: cumulative normal distribution function (`NORM.S.DIST(x, TRUE)`). These $N(\cdot)$ terms capture the replicating portfolio weights.
- x: $\boxed{x = \frac{\ln\left(\frac{S}{K(1+r)^{-T}}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}}$

Real options

Option-like payoffs appear in many contexts outside of financial markets. Management can be thought of as the act of creating and optimally exercising real options. **E.g.**: follow-on products, R&D investments, delaying product launches, abandoning projects, etc.